Wear, *143* (1991) 331–352

Planar sliding with dry friction
Part 2. Dynamics of motion

Suresh Goyal*[a] and **Andy Ruina[b]**
[a]*Mechanical Engineering and Computer Science,* [b]*Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853 (U.S.A.)*

Jim Papadopoulos
Mechanical Engineering, Northern Illinois University, DeKalb, IL 60115 (U.S.A.)

(Received March 27, 1990; accepted August 15, 1990)

Abstract

Some problems in the dynamics of sliding of planar rigid bodies are treated by geometric methods based on the limit surface description of friction (S. Goyal, A. Ruina and J. Papadopoulos, *Wear*, *143* (1991) 307–330). The problems we consider, where the normal force is known *a priori*, have unique solutions although the friction force (and torque) may be a discontinuous function of the direction of motion. When a freely sliding object comes to rest it always does so with one of several definite ratios of translation to rotation. These special generalized velocity directions, termed eigen-directions, depend on the friction law used, the contact pressure distribution and the mass distribution. The eigen-directions correspond to local extrema of the generalized frictional load $|\boldsymbol{P}|$ on the limit surface, *i.e.* to directions in load space where $\boldsymbol{P}$ is parallel to the generalized direction of motion $\boldsymbol{q}$. For most objects, if the contact region is sufficiently smaller than the radius of gyration, final motion is always pure rotation about the center of mass; if the contact region is sufficiently spread out final motion is a pure translation. A simple model of a car with locked rear wheels shows the effect of speed and orientation on skid stability at finite speeds. Sliders have a propensity to rotate about points of support.

1. Introduction

Ishlinskii *et al.* [1] and (apparently independently) Vøyenli and Eriksen [2] noticed the special motions of a sliding ring and a sliding disk as they come to rest during free sliding on a horizontal plane under the influence of gravity and dry friction. In particular, a rigid ring, with all contact pressure and mass concentrated on a circle, ends up by rotating about a point on the ring just before stopping, no matter what the initial condition (unless the initial condition is pure rotation or pure translation). Similarly, a disk with uniform pressure and mass distribution, comes to rest by rotation about a point 0.653 of the disk radius from the center, irrespective of the initial condition (again, unless the initial condition is a pure rotation or a pure

*Present address: AT&T Bell Laboratories, Murray Hill, NJ 07974, U.S.A.

0043-1648/91/$3.50 © Elsevier Sequoia/Printed in The Netherlands

translation). Additionally, Wittenburg [3] and Ishlinskii *et al.* [1] find that a bar with symmetric point masses and supports at its ends, stops with rotation about one of its points of support for all initial conditions, except pure rotation about its center or pure translation. The general tendency of objects to rotate about points of support has also been noted by earlier investigators, such as Jellet [4], Prescott [5], and MacMillan [6].

These results led us to look into the dynamics of sliders. The limit surface description from refs. 7–9 allows us to simply reproduce the results named above as well as to generalize them. Other aspects of free sliding motion are also illuminated with the limit surface.

The order of the material presented in the rest of the paper is as follows. The equations of motion for a rigid planar slider subject to applied loads and friction are reviewed. The uniqueness of dynamic solutions is then discussed and contrasted with the non-uniqueness of quasi-static solutions. That final motions are eigen-directions is then shown, and the stability of motion in an eigen-direction is discussed. The effect of mass distribution on this motion is demonstrated. The example of the freely sliding bar, supported at its ends, is presented in some detail. The stability of orientation at high slip speeds is illustrated by use of a model of a braked sliding car.

Many of the results in this paper are presented with more detail in ref. 7.

2. Governing laws

2.1. Principle of maximum power and limit surface

The constitutive law relating the motion of the slider to the friction force and torque is expressed by the limit surface for the slider and the load-motion inequality (essentially the principle of maximum plastic work from classical plasticity), both of which are discussed at length in refs. 7–9. Briefly, the limit surface is a closed convex surface in the three-dimensional space of possible friction loads $\boldsymbol{P} \equiv [F_x, F_y, M]$. The motion is described by the generalized velocity $\boldsymbol{Q} \equiv [V_x, V_y, \omega]$, with components which are the translation velocity of a reference point O on the slider, and the angular velocity of the slider; and the motion vector $\boldsymbol{q} \equiv [q_x, q_y, q_\omega] \equiv \boldsymbol{Q}/|\boldsymbol{Q}|$. If m is the mass of the slider, r_g is its radius of gyration about its center of mass, and g is the acceleration due to gravity, then all lengths are assumed to have been normalized with the reference length r_g, all forces with the reference force mg (weight of the slider), all moments by the reference moment mgr_g, linear velocities by the reference linear velocity $(gr_g)^{1/2}$, angular velocities by the reference angular velocity $(g/r_g)^{1/2}$, and time with the reference $(r_g/g)^{1/2}$. $\boldsymbol{P}$ and $\boldsymbol{Q}$ are related by the load–motion inequality

$$(\boldsymbol{P}-\boldsymbol{P}^*)\boldsymbol{Q} \geqslant 0 \tag{1}$$

which must hold for all $\boldsymbol{P}^*$ on or inside the given limit surface. When $\boldsymbol{Q}=0$, inequality (1) does not restrict $\boldsymbol{P}$ so it may be any load on or in the limit surface. Otherwise (1) restricts $\boldsymbol{P}$ to be on the limit surface at a point with

normal $\boldsymbol{q}$. At a vertex (a point or curve of slope discontinuity on the limit surface) we use the word "normal" to describe any vector normal to the vertex if it was locally blunted. At a vertex a range of motions $\boldsymbol{q}$ all have a common load $\boldsymbol{P}$. If the limit surface is locally flat (*i.e.* appears as a plane or cylinder) $\boldsymbol{P}$ is not uniquely determined for a given $\boldsymbol{q}$.

Friction laws for which this formalism applies include ordinary isotropic Coulomb friction and also contact mediated by wheels (possibly with bearing friction and/or a bearing ratchet).

2.2. *Convection of the limit surface*

With a given pressure distribution on the slip surface of the slider, the shape of the limit surface also depends on the position of the slider if the support plane has inhomogeneous properties, and on the orientation of the slider if it is anisotropic or if the support plane has anisotropic properties.

If the supporting surface is completely isotropic and homogeneous (in frictional properties), the limit surface for the slider has a fixed shape that rotates with the slider and is most conveniently defined in a slider-fixed axis system. Additionally if the limit surface is a surface of revolution about the M axis, as it is for an axisymmetric slider, then it does not change at all in either body- or space-fixed coordinates.

On the other hand, if the slider is completely isotropic and axisymmetric (in support and frictional properties) and the supporting surface is homogeneous but anisotropic, the limit surface is fixed in shape and orientation in the frame of the fixed support surface, but not fixed in the frame of the body. An example would be the limit surface for a smooth circular disk sliding on a bed of infinitely fine parallel rollers.

2.3. *Momentum balance*

The motion of a planar slider is governed by the balance of linear and angular momentum, expressed compactly as

$$\dot{\boldsymbol{Q}}|_{\text{inertial}} = \boldsymbol{P}^{\text{app}} - \boldsymbol{P} \tag{2}$$

where $\boldsymbol{P}^{\text{app}}$ is the externally applied load, and $\dot{(\,)} = \mathrm{d}(\,)/\mathrm{d}t$. We take the x and y axis to be fixed in the slider (with origin at O) so care must be taken when differentiating $\boldsymbol{Q}$. By $\dot{\boldsymbol{Q}}|_{\text{inertial}}$, specified by $[A_x, A_y, \dot{\omega}]$, is meant a list of three numbers representing the absolute acceleration and angular acceleration of the slider. On the other hand, it is convenient to differentiate the components of $\boldsymbol{Q}$ as seen by the sliding body. So we define $\dot{\boldsymbol{Q}}|_{\text{body}} \equiv [\dot{V}_x, \dot{V}_y, \dot{\omega}]$. The difference between the two is a simple convection term.

$$\dot{\boldsymbol{Q}}|_{\text{body}} = \dot{\boldsymbol{Q}}|_{\text{inertial}} - \omega \times \boldsymbol{Q} \tag{3}$$

or

$$[\dot{V}_x, \dot{V}_y, \dot{\omega}] = [A_x + \omega V_y, A_y - \omega V_x, \dot{\omega}]$$

2.4. *Body fixed equations of motion*

Most of the reasoning which follows assumes that the limit surface, though not necessarily axisymmetric, is of fixed shape and orientation in

the slider coordinates. Rather than express rules for the evolution of the limit surface in inertial space, we write the equations of motion in a body-fixed frame:

$$\dot{\boldsymbol{Q}}|_{\text{body}} = \boldsymbol{P}^{\text{eff}} - \boldsymbol{P} \tag{4}$$

where we define the effective load $\boldsymbol{P}^{\text{eff}} \equiv \boldsymbol{P}^{\text{app}} - \omega \times \boldsymbol{Q}$.

3. Uniqueness of solutions

The problem of determining the quasi-static motion of a frictional slider often does not have a unique solution. That is, if the generalized load on a slider is gradually increased until $\boldsymbol{P}$ is on the limit surface at a vertex, the incipient motion direction can have a range of values. On the other hand Mason states [10] without explanation, that, with the inclusion of inertia, slip motion problems have unique solutions. Also, Vøyenli and Eriksen claim [2] that the conditions of uniqueness are satisfied up until the slider comes to rest (after which the equations of motion obviously cannot be integrated backwards in time in a unique way). However, they only consider sliders for which the frictional load is continuous during the motion. In contrast, we consider sliders for which the frictional force may be discontinuous in time (because of facets on the limit surface).

For given initial conditions and a given applied load $\boldsymbol{P}^{\text{app}}$ the right hand side of eqn. (2) or eqn. (4) is usually fully determined and solution proceeds by direct integration with no ambiguity in the value of $\dot{\boldsymbol{Q}}$ at any time, and with $\dot{\boldsymbol{Q}}|_{\text{body}}$ being sufficiently smooth in time that theorems of uniqueness for ordinary differential equations apply. The two possible exceptional cases are when motion first starts and when the instantaneous value of $\boldsymbol{q}$ (or $\boldsymbol{Q}$) is normal to a flat region on the limit surface.

3.1. Uniqueness of first motion

Assume that at time zero a finite load $\boldsymbol{P}^{\text{app}}$ is applied to the slider and that velocity Q equals 0. Either

(1) $\boldsymbol{P}^{\text{app}}$ is inside the limit surface then the only consistent solution is $\dot{\boldsymbol{Q}}|_{\text{body}} = 0$ (imagined small perturbations from no slip decay immediately), and $\boldsymbol{P} = \boldsymbol{P}^{\text{app}}$; or

(2) $\boldsymbol{P}^{\text{app}}$ is outside the limit surface and some motion must occur. At first motion (time $= 0^+$) L'Hopital's rule or the mean value theorem can be used to show that $\dot{\boldsymbol{Q}}|_{\text{body}}$ must be parallel to $\boldsymbol{Q}$. So, from the equation of motion (4) $\boldsymbol{P}^{\text{eff}} - \boldsymbol{P}$ must be parallel to $\boldsymbol{Q}$. But the load–motion inequality (1) forces $\boldsymbol{Q}$ to be normal to the limit surface. So $\boldsymbol{P}^{\text{eff}} - \boldsymbol{P}$ must, in turn, be normal to the limit surface at $\boldsymbol{P}$. Since the limit surface is convex, $\boldsymbol{P}$ is the (unique) point on the limit surface which is closest to $\boldsymbol{P}^{\text{eff}}$.

Thus the right-hand side of eqn. (4) is fully determined at first motion.

At first motion $\omega \times \boldsymbol{Q}$ is of second order so $\boldsymbol{P}^{\text{eff}}$ can be replaced by $\boldsymbol{P}^{\text{app}}$. If $\boldsymbol{P}^{\text{app}}$ is much larger than $\boldsymbol{P}$ on the limit surface $\dot{\boldsymbol{Q}}|_{\text{body}}$ will be nearly parallel

to $\boldsymbol{P}^{\text{app}}$ as one would intuitively expect. If $\boldsymbol{P}^{\text{app}}$ is just outside the limit surface, however, the shape of the limit surface naturally affects the acceleration direction.

3.2. Uniqueness of $\boldsymbol{P}$ when $\boldsymbol{Q}$ is normal to a facet

Once slip is underway eqn. (1) fully determines $\boldsymbol{P}$ from $\boldsymbol{Q}$ unless $\boldsymbol{Q}$ corresponds to a flat region on the limit surface. In $\boldsymbol{q}$ space flat regions on the limit surface occupy only selected points and thus one might want to neglect them altogether as only occurring with zero probability. But during motion that is not fully constrained, flat sections on the limit surface often attract and/or sustain motions (as will be discussed) and so cannot be neglected.

To understand the evolution of $\boldsymbol{P}$ on and near a facet, it is useful to look at the components of eqn. (4) along the direction of motion $\boldsymbol{q}$ and in a plane perpendicular to it. This yields the following equations as a replacement for eqn. (4):

$$\dot{Q} = (\boldsymbol{P}^{\text{eff}} - \boldsymbol{P})\boldsymbol{q} \tag{5}$$

$$Q\dot{\boldsymbol{q}}|_{\text{body}} \equiv \dot{\boldsymbol{Q}}_{\text{tan}}|_{\text{body}} = \boldsymbol{P}^{\text{eff}}_{\text{tan}} - \boldsymbol{P}_{\text{tan}} \tag{6}$$

where $\boldsymbol{P}^{\text{eff}}_{\text{tan}} = \boldsymbol{P}^{\text{eff}} - (\boldsymbol{P}^{\text{eff}}\boldsymbol{q})\boldsymbol{q}$, and $\boldsymbol{P}_{\text{tan}} = \boldsymbol{P} - (\boldsymbol{P}\boldsymbol{q})\boldsymbol{q}$. Since $\boldsymbol{P}\boldsymbol{q}$ is the same for all candidate $\boldsymbol{P}$ lying on the facet, eqn. (5) uniquely determines the change in the magnitude of the velocity $\dot{Q}$.

On and near a facet there are the following two mutually exclusive possibilities for eqn. (6).

(1) $\boldsymbol{P}^{\text{eff}}$ is within the infinite cylinder defined by the normals of the facet. Then the only consistent solution is for $\boldsymbol{Q}$ to remain normal to the facet for finite time with $\dot{\boldsymbol{q}} = 0$. $\boldsymbol{P}$ is that point on the facet closest to $\boldsymbol{P}^{\text{eff}}$.

(2) $\boldsymbol{P}^{\text{eff}}$ is outside the infinite cylinder defined by the normals of the facet and some tangential motion must occur. $\dot{\boldsymbol{q}}|_{\text{body}} \neq 0$ and $\boldsymbol{P}$ passes off the facet. Now $\boldsymbol{P}$ can be found by a method like that used for the case of first motion. Analogous to $\boldsymbol{Q}$ in first motion is $\boldsymbol{Q}_{\text{tan}}$, the part (departing from zero) of $\boldsymbol{Q}$ which is tangential to the facet. Also $\dot{\boldsymbol{Q}}_{\text{tan}}|_{\text{body}} = Q\dot{\boldsymbol{q}}|_{\text{body}}$. At first motion L'Hopital's rule or the mean value theorem can be used to show that $\dot{\boldsymbol{Q}}_{\text{tan}}|_{\text{body}}$ must be parallel to $\boldsymbol{Q}_{\text{tan}}$. So, from the equation of motion (6) $\boldsymbol{P}^{\text{eff}}_{\text{tan}} - \boldsymbol{P}_{\text{tan}}$ must be parallel to $\boldsymbol{Q}_{\text{tan}}$. But the load–motion inequality (1), with the local geometry of the convex limit surface near the facet, forces $\boldsymbol{Q}_{\text{tan}}$ to be normal to the boundary of the facet. So $\boldsymbol{P}^{\text{eff}}_{\text{tan}} - \boldsymbol{P}_{\text{tan}}$ must, in turn, be normal to the boundary of the facet. This determines that $\boldsymbol{P}$ is the (unique) point on the limit surface which is closest to $\boldsymbol{P}^{\text{eff}}$.

Thus the right hand side of eqn. (4) is, again, fully determined.

3.3. Determination of $\boldsymbol{P}$ given $\boldsymbol{Q}$ and $\boldsymbol{P}^{app}$

The unique determination of $\boldsymbol{P}$ given $\boldsymbol{Q}$ and $\boldsymbol{P}^{\text{app}}$ is given by the following statement:

Of all $\boldsymbol{P}$ consistent, through eqn. (1) and the limit surface, with a given $\boldsymbol{Q}$, the actual $\boldsymbol{P}$ is that which is closest to $\boldsymbol{P}^{\text{eff}}$.

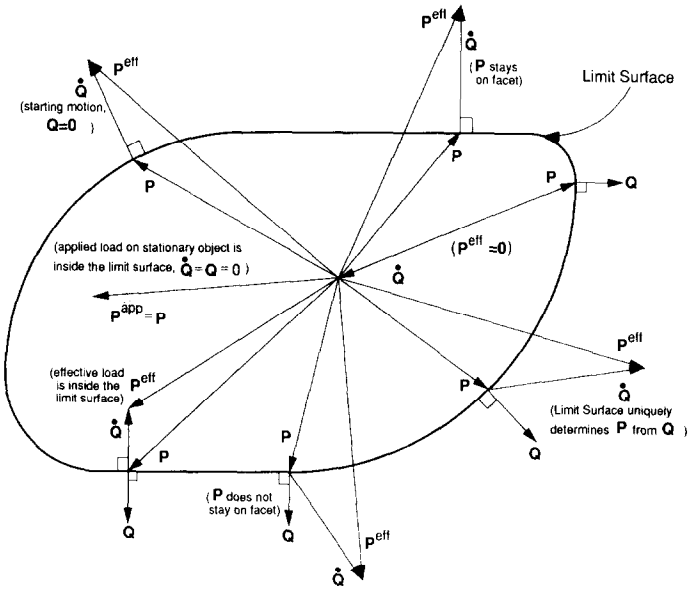


Fig. 1. Determination of $\boldsymbol{P}$ given $\boldsymbol{Q}$ and $\boldsymbol{P}^{\text{eff}}$. Several possible cases illustrate that of all $\boldsymbol{P}$ consistent, through the load–motion inequality and the limit surface, with a given $\boldsymbol{Q}$, the actual $\boldsymbol{P}$ is that which is closest to $\boldsymbol{P}^{\text{eff}}$.

So, assuming that the limit surface is known, the right-hand side of eqn. (4) is always fully determined. At points of frictional force discontinuity the above reasoning leads to well-defined jump conditions and thus, we believe, to the unique solution (at least integrating forwards in time) of the equations of motion for any given initial condition and applied load. The determination of $\boldsymbol{P}$ in various cases is shown schematically in Fig. 1 which shows several cases of $\boldsymbol{P}^{\text{eff}}$ and $\boldsymbol{Q}$.

4. Eigen-directions as final motions

Free sliding is the motion of the slider under the action of gravity and dry friction only. During free sliding, $\boldsymbol{P}^{\text{app}} = 0$ in eqn. (4), and the body-fixed equation of motion becomes

$$\dot{\boldsymbol{Q}}|_{\text{body}} = -\boldsymbol{P} - \omega \times \boldsymbol{Q} \tag{7}$$

As the slider comes to rest owing to frictional dissipation alone, both ω and $\boldsymbol{Q}$ vanish, so their product may be neglected compared with $\boldsymbol{P}$, whose magnitude is unaffected by sliding speed. So, as the slider asymptotically approaches zero velocity, its equation of motion (7) becomes

$$\dot{\boldsymbol{Q}}|_{\text{body}} = -\boldsymbol{P} \tag{8}$$

That is, the convection term introduced in eqn. (3) may be neglected. (At final motion the space-fixed (2) and body-fixed (4) equations are identical.)

When the slider comes to rest its acceleration is finite and non-zero. Assuming that acceleration is continuous (until it vanishes at final stopping), the mean value theorem or L'Hopital's rule implies that the generalized velocity must be parallel to the generalized acceleration just as the slider comes to rest. Applying this fact to eqn. (8) leads us to the conclusion that, when the slider comes to rest during free sliding, its motion $\boldsymbol{Q}$ is parallel to the frictional load $\boldsymbol{P}$. This implies that the velocity vector $\boldsymbol{Q}$ comes to rest at only those special points on the limit surface for the slider where the direction of $\boldsymbol{P}$ is parallel to the normal $\boldsymbol{q}$ at that point. We call these special directions in load–motion space, eigen-directions; they include all extrema of $|\boldsymbol{P}|$ on the limit surface.

$$\boldsymbol{Q} \| \boldsymbol{P} \text{ at final motion} \tag{9}$$

4.1. Trajectory on the limit surface during final motion

Integration of one of the eqns. (2), (4), (7), (8) determines the trajectory of the slider in generalized velocity space $\boldsymbol{Q}$, and $\boldsymbol{P}$ is determined as an intermediate variable in this calculation. One can obtain useful information about slip motion, however, by formulating equations for the evolution of $\boldsymbol{P}$.

In order to construct such equations a local description of the limit surface is required. Here we assume sufficient smoothness for a well-defined curvature tensor $[\boldsymbol{\kappa}]$ and its inverse $[\boldsymbol{\rho}]$ [11] for points on the limit surface. For our purposes these should be thought of as symmetric 3×3 matrices that have one principal direction orthogonal to the limit surface and two principle directions along the directions of principal curvature of the surface. The eigenvalues of $[\boldsymbol{\rho}]$ are the two principal radii of curvature and a third number arbitrarily taken to equal unity. If $\boldsymbol{q}$ is the unit normal associated with $\boldsymbol{P}$ and $[\boldsymbol{\rho}]$ is the inverse curvature tensor at that point, we have:

$$\dot{\boldsymbol{P}} = [\boldsymbol{\rho}]\dot{\boldsymbol{q}} \tag{10}$$

Applying eqn. (10) to the body-fixed equation of free final motion (8), the normality relation in the friction law, the definition $\boldsymbol{q} = \boldsymbol{Q}/|\boldsymbol{Q}|$, and the fact that when $[\boldsymbol{\rho}]$ acts on vectors tangent to the limit surface the result is a vector tangent to the limit surface:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{P}| = \frac{-\boldsymbol{P}_{\tan}[\boldsymbol{\rho}]\boldsymbol{P}_{\tan}}{|\boldsymbol{P}||\boldsymbol{Q}|} \tag{11}$$

where vector transpose is assumed as needed. Because the limit surface is convex both radii of curvature are non-negative and the tensor $[\boldsymbol{\rho}]$ is positive semi-definite (*i.e.* $\boldsymbol{X}[\boldsymbol{\rho}]\boldsymbol{X} \geqslant 0$ for all three-vectors $\boldsymbol{X}$). The right side of eqn. (11) is a negative quantity except at eigen-directions, where $\boldsymbol{P}_{\tan} = 0$ (and also possibly at vertices on the limit surface). Since $|\boldsymbol{P}|$ is the radius of the limit surface we have the geometric description of final motion as being such that the net frictional load decreases in magnitude with time until $\boldsymbol{P}$ reaches an eigen-direction.

5. Two-dimensional motion

For simplicity consider problems of pure translation, where rotation is not an issue. Such could be the case, for example, if a slider were symmetrically supported with a set of points of identically anisotropic friction and given no initial rotations. Here $\boldsymbol{P}$ space and $\boldsymbol{Q}$ space are essentially two dimensional and we need not think about the slider's rotation or the convection of the limit surface.

The evolution of $\boldsymbol{P}$ and $\boldsymbol{Q}$ in one time increment is shown in Fig. 2. The motion is understood using the equation of motion $\mathrm{d}\boldsymbol{Q}/\mathrm{d}t = -\boldsymbol{P}$, and the facts that $\boldsymbol{Q}$ is always normal to the limit curve and that the limit curve is convex (with the origin in load space inside the limit curve). The evolution proceeds as $\boldsymbol{P}$ decreases in magnitude and $\boldsymbol{Q}$ also decreases in magnitude (5).

5.1. Eigen-directions in two-dimensional slip

Figure 3 shows three examples of eigen-directions for two-dimensional motion (or motion of an axisymmetric object). As one can easily verify from the three cases drawn in Fig. 3, small deviations from motion in an eigen-direction either grow or decay depending on the radius and curvature of the limit surface in the neighborhood of the eigen-direction, *i.e.* $|\boldsymbol{P}-\boldsymbol{P}_{\mathrm{eigen}}|$ and $|\boldsymbol{q}-\boldsymbol{q}_{\mathrm{eigen}}|$ either grow or decay as time progresses.

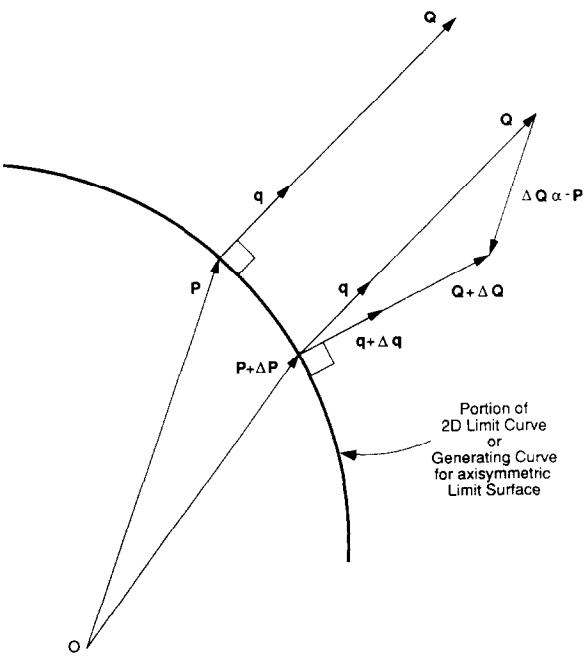


Fig. 2. A portion of a two-dimensional limit surface (a curve). The curve may also represent a generating curve for an axisymmetric limit surface. A friction load $\boldsymbol{P}$ is marked as is its associated $\boldsymbol{Q}$. Since $\dot{\boldsymbol{Q}} = -\boldsymbol{P}$, $\boldsymbol{P}$ changes in time so as to decrease in magnitude.

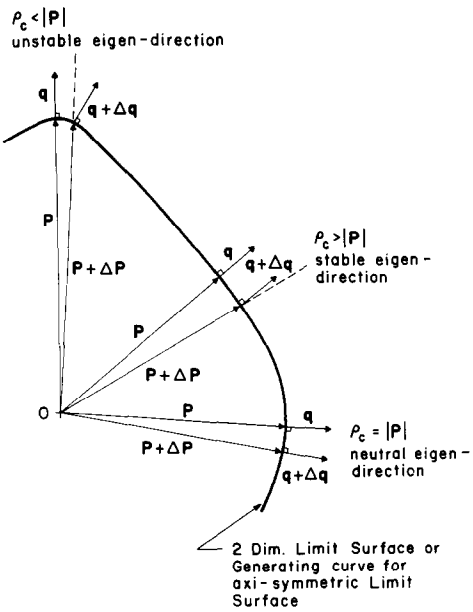


Fig. 3. Figure shows eigen-direction $\boldsymbol{P}$ (with motion $\boldsymbol{q}$) and a perturbed motion $\boldsymbol{q}+\Delta\boldsymbol{q}$ (with an associated load $\boldsymbol{P}+\Delta\boldsymbol{P}$ in its vicinity) on a limit surface. The three different types of eigen-directions, unstable, stable, and neutral, are shown.

The result from eqn. (11) that $|\boldsymbol{P}|$ decreases in time implies (since the set of $\boldsymbol{P}$ with constant magnitude is a circle) that the directional stability of an eigen-direction is determined by the radius of curvature ρ_c of the limit surface and the magnitude of the frictional load $|\boldsymbol{P}|$ [7]. In summary:

(1) $\rho_c > |\boldsymbol{P}| \Rightarrow |\boldsymbol{P}|$ is a local minimum and $\boldsymbol{q}$ is locally attracted to the eigen-direction. Many initial conditions will have this eigen-direction as a final motion.

(2) $\rho_c < |\boldsymbol{P}| \Rightarrow |\boldsymbol{P}|$ is a local maximum and $\boldsymbol{q}$ is locally repelled by the eigen-direction. Only special initial conditions will have this final motion.

(3) $\rho_c = |\boldsymbol{P}| \Rightarrow \boldsymbol{q}$ is neither repelled nor attracted by its neighboring eigen-direction. To first order, there is a small neighborhood of eigen-directions.

So the critical condition for directional stability of an eigen-direction is that a circle centered at the origin of load space has second-order tangency with the limit surface:

$$\rho_c = |\boldsymbol{P}| \tag{12}$$

5.2. *Axisymmetric sliders have two-dimensional behavior*

The reasoning associated with Fig. 2 is also applicable to axisymmetric bodies. In this case $\boldsymbol{P}$ in eqn. (1) is fully represented by a generating curve of the axisymmetric limit surface. The vertical axis then represents moment and rotation rate and the horizontal axis represents translation and force in the initial direction of motion.

5.3. Eigen-directions for axisymmetric sliders

The center of mass of a freely sliding axisymmetric body always travels in a straight line in inertial space [2]. If $\boldsymbol{q}$ is constant (in space-fixed coordinates), the locus of instantaneous centers of rotation is a straight line in physical space and a circle on the body. So constant $\boldsymbol{q}$ motion is rolling of a circle attached to the slider on a line fixed in inertial space. This is the final constant $\boldsymbol{q}$ eigen-motion of an axisymmetric slider. The radius of the rolling circle is determined by the eigen-direction $\boldsymbol{q}$. Final motions for a uniformly supported ring and a uniformly supported disk, both with isotropic friction, are found by Ishlinskii *et al.* [1] and by Vøyenli and Eriksen [2]. Although our results generalize the results in these papers, we consider the same two examples for illustration.

5.4. Hollow ring with isotropic friction

For the uniformly supported ring of isotropic support (radius R, radius of gyration $r_g = 1$, unit weight and unit coefficient of friction), a sliding hula hoop, the equations parameterizing the generating curve of its limit surface can be given as:

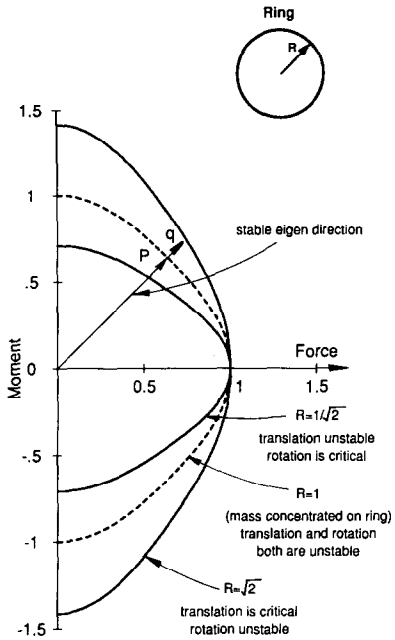
$$F_y(r_c) = \frac{1}{\pi}\int_0^{\pi} \frac{\cos\theta - r_c}{\{1 + r_c^2 - 2r_c\cos\theta\}^{1/2}}\,\mathrm{d}\theta$$
$$M(r_c) = \frac{R}{\pi}\int_0^{\pi} \frac{1 - r_c\cos\theta}{\{1 + r_c^2 - 2r_c\cos\theta\}^{1/2}}\,\mathrm{d}\theta \tag{13}$$

where motion is in the y direction. This curve, parameterized through the radius of the circle of centers of rotation r_c for $0 < r_c < \infty$, is shown by the dotted curve in Fig. 4. The attracting eigen-direction for the ring, when $R = 1$, as shown in Fig. 4 ($\boldsymbol{P} = [0,\ 2/\pi,\ 2/\pi]$ and $\boldsymbol{q} = [0,\ 1/\sqrt{2},\ 1/\sqrt{2}]$), is rotation about a point on the ring itself. Curiously, the generating curve for the limit surface of the ring is symmetric about a 45° line on Fig. 4.

5.5. Disk of isotropic friction

For the uniformly supported disk of isotropic support (radius R, unit weight, radius of gyration $r_g = 1$, and unit coefficient of friction) the generating curve is shown in Fig. 5 where motion is again in the y direction [7]. The attracting motion for the disk, when $R = \sqrt{2}$ depicted on the dotted generating curve in Fig. 5 ($\boldsymbol{P} \approx [0,\ 0.616,\ 0.667]$ and $\boldsymbol{q} \approx [0,\ 0.678,\ 0.735]$), is rotation about points on a circle of radius approximately 0.923 drawn on the disk. This is equal to $0.653 \times R$, the same value as that obtained in ref. 2. (Ishlinskii *et al.* [1] report the radius of this circle incorrectly as $0.71 \times R$).

For both the ring and the disk, pure rotation about the center and pure translations are unstable eigen-directions (*i.e.* small perturbations from these motions grow in time) and they persist only if started precisely.



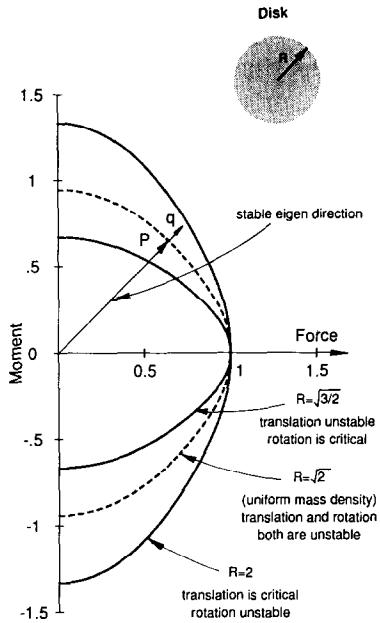


Fig. 4. The generating curves for the axisymmetric limit surface (axis of symmetry is the M axis) for a uniformly supported ring with isotropic friction, for different radii R.

Fig. 5. The generating curves for the axisymmetric limit surface (axis of symmetry is the M axis) for a uniformly supported disk with isotropic friction, for different radii R.

6. Dependence of eigen-directions on mass distribution

As mentioned earlier, the radius of gyration r_g of the slider is used as the reference length for non-dimensionalization of moment M and velocity V. Hence the shape of the limit surface for a slider with a fixed contact pressure distribution can be changed by altering its mass distribution (which determines r_g) with respect to the radius of the pressure distribution. For a fixed contact pressure distribution, distributing the mass further away from the center (increasing r_g) effectively compresses the limit surface along the M axis, and concentrating the mass closer to the center (decreasing r_g) stretches it. For either of these cases the limit surface remains fixed along the force axes, but suffers a change in aspect ratio (shape). This can lead to changed eigen-directions. Also eigen-directions that are preserved by the distortion of the limit surface (on the M axis or the F axis) can have altered stability and hence changed "free sliding" behavior.

In particular, for the uniformly supported ring of isotropic support, we might imagine gluing a massless board on top and distributing mass at an arbitrary radius and thereby changing the non-dimensional radius R of the ring. Remembering the critical condition (12) for the directional stability of an eigen-direction, the critical radius R for having pure translation of the ring as its only stable motion can be obtained by equating the radius of

curvature of the limit surface (as obtained from eqn. (13)) at the F_y axis to $F_y(r_c = \infty)$. Similarly for determining the critical radius R for having pure rotation of the ring as its only stable motion, we equate the radius of curvature of the limit surface (again as obtained from eqn. (13)) at the M axis to $M(r_c = 0)$. Thus it can be shown that [7]:

(1) For $R > \sqrt{2}$, all initial motions (except pure rotation) of the freely sliding ring end as pure translations.

(2) For $R > 1/\sqrt{2}$ all initial motions (except pure translations) of the ring end as pure rotations about its center.

(3) For $1/\sqrt{2} \leqslant R \leqslant \sqrt{2}$, the final attracting motion depends on the value of R and ranges from rotations about the center to pure translation.

Hence any desired radius of "rolling-like motion" can be made its final attracting motion by choosing an appropriate mass distribution.

A similar result holds for the uniformly supported disk [7]. By varying its radius R between $\sqrt{3/2}$ and 2 and keeping its support pressure distribution unaltered, any desired radius of rolling can be made the final attracting motion. Figures 4 and 5 show the original generating curves for the ring and the disk respectively with their shapes at the bounding values of R. For both sliders, if R is sufficiently large, translation is the only stable eigen-direction. If the radius of gyration is large, rotation is the only stable eigen-direction. These latter properties hold for all axisymmetric sliders with smooth, facetless limit surfaces.

7. Three degrees of freedom motion

With three degrees of freedom (two translations and a rotation) the situation is somewhat more complex, except in the special case of axisymmetry just discussed. $\boldsymbol{P}$ in inequality (1) requires a three-dimensional representation and eqn. (4) contains a convective term.

But at final motion the convective term in eqn. (4) can be neglected and the motion can again be pictured simply. The direction of $\dot{\boldsymbol{Q}}|_{\text{body}}$ depends only on $\boldsymbol{P}$, $\boldsymbol{q}$, and the local shape of the limit surface, and not on the magnitude of $\boldsymbol{Q}$. So the final trajectories in $\boldsymbol{P}$ and $\boldsymbol{q}$ space are intrinsic to the limit surface.

As for the two-dimensional case, three-dimensional geometry and the equations of motion for the three-dimensional limit surface show that if $\boldsymbol{P}$ is an eigen-direction, with associated principal radii of curvature ρ_1 and ρ_2 (in load space) then:

(1) $\rho_1, \rho_2 > |\boldsymbol{P}|$ implies that $|\boldsymbol{P}|$ is a local minimum and the eigen-direction attracts all motions in its local vicinity. $\boldsymbol{P}$ is a stable eigen-direction.

(2) ρ_1 or $\rho_2 < |\boldsymbol{P}|$ implies that $|\boldsymbol{P}|$ is a local maximum (for some tangent directions on the limit surface) and repels motions in its vicinity. $\boldsymbol{P}$ is an unstable eigen-direction.

(3) $\rho_1 = \rho_2 = |\boldsymbol{P}|$ implies that $|\boldsymbol{P}|$ is stationary and motions are neither attracted nor repelled (to first order) in its local vicinity. $\boldsymbol{P}$ is a neutrally stable eigen-direction.

So the critical condition for directional stability of an eigen-direction is that a sphere of radius $|\boldsymbol{P}|$ centered at the origin of load space lies locally inside the limit surface.

This directly implies that eigen-directions at vertices on the limit surface are completely unstable and that eigen-directions on flat regions of the limit surface are stable. The fact that the limit surface is closed, convex and encloses the origin implies that it has to have at least two eigen-directions, the minimum and the maximum load respectively. The minimum load is either a stable or a neutrally stable eigen-direction. Similarly, the maximum load is either an unstable or a neutrally stable eigen-direction.

As noted with regard to axisymmetric sliders, the mass distribution affects the eigen-directions and their stability. For all sliders, if mass is broadly distributed compared with the contact region, pure rotation will become a stable eigen-direction (unless the limit surface has a vertex on the M axis, in which case something very close to pure rotation will be a stable eigen-direction). If mass is quite central, pure translation will become a stable eigen-direction (unless the limit surface has a vertex on the F plane, in which case something very close to pure translation will be a stable eigen-direction).

7.1. Bead analogy

The motion of $\boldsymbol{P}$ on the limit surface during free slip is partially analogous to the motion of a tiny massless bead moving on a viscously coated, closed, convex shell in the physical shape of the limit surface. This bead is attracted towards the center of the shell by a spring. The equilibrium positions for the bead on this surface are those where the direction of the spring force (direction of $-\boldsymbol{P}$) is parallel to the normal of the surface (the direction of $\boldsymbol{Q}$), so that there is no component of the spring force in the tangent plant to the surface trying to move the bead along it. These equilibrium positions, local extrema of the radius vector from the center to the surface of the shell, are eigen-directions. Additionally, local minima are stable equilibrium positions for the bead and maxima are unstable equilibrium positions.

8. Rotation about points of isotropic support

Investigators such as Jellett [4], Prescott [5] and MacMillan [6] have noted that sliders with isotropic friction show a propensity towards rotating about points of support. In terms of the limit surface formalism, we can explain this phenomenon roughly in the following way.

(1) Point supports on a slider with isotropic friction lead to flat regions on its limit surface that might occupy a substantial area of the total surface. Hence, by increasing any applied load until it lies on the limit surface there is a high probability that the corresponding first motion of the slider will be a rotation about a point of support. This is exemplified by the limit surface of the symmetrically supported bar of Fig. 6 [9].

(2) Since point supports lead to flat facets and the corresponding principal radii of curvature are $\rho_1 = \rho_2 = \infty$, an eigen-direction on the facet must be stable. For the symmetrically supported bar (below), rotation about one of the points of support is the final attracting motion for all initial motions except pure rotations and translations. However, not all flat regions need have an eigen-direction, as can be seen from the example of the asymmetrically supported bar discussed later.

(3) From manipulation of eqn. (6) it can also be shown that, besides pure rotations and translations, the only motion direction $\boldsymbol{q}$ that can be sustained for a finite amount of time corresponds to the normal of a flat region on the limit surface. Motions about points of isotropic support satisfy this criterion and can in fact be sustained for a finite time. If such motion is an eigen-motion then, if it occurs at all, it is sustained until the slider comes to rest.

8.1. Free sliding of symmetrically supported bar

To verify and illustrate the above conclusions, we present results of the numerical simulation of the free sliding of a symmetrically supported bar with isotropic friction. The bar has unit weight with point masses concentrated at its ends, half length of the bar is unity, and the coefficient of friction is unity. The bar and its limit surface are presented in Fig. 6. Motion of similar objects was studied numerically by Ishlinskii *et al.* [1] and analytically by Wittenburg [3].

From the limit surface of the bar (Fig. 6), one can see that instantaneous rotations about its points of support are final attracting motions for all initial

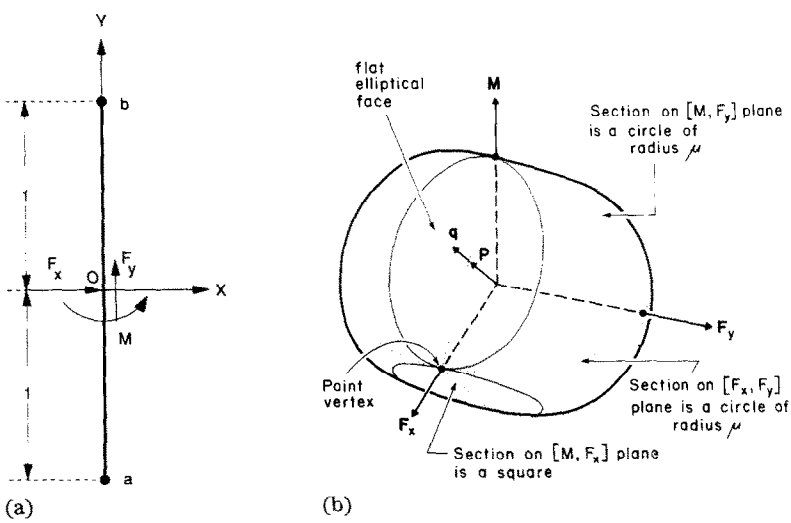


Fig. 6. Bar supported symmetrically at its ends by isotropic friction, and its limit surface. A stable eigen-direction is shown.

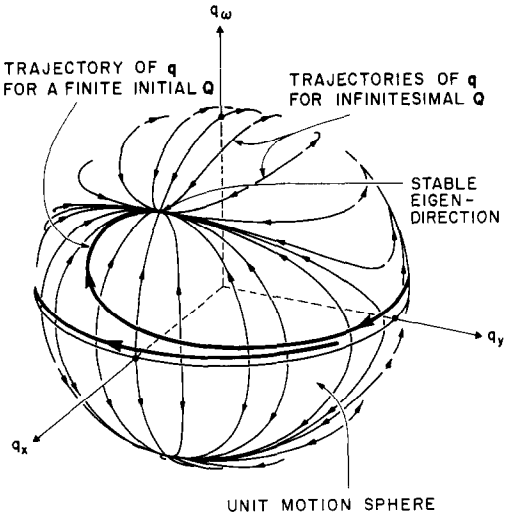


Fig. 7. Results of the numerical simulation of the free sliding of the bar of Fig. 6 as seen in a bar-fixed reference system, depicted on the unit motion sphere. The light curves on the unit motion sphere depict the flow field, along which various motions of almost zero magnitude approach the stable eigen-direction. The dark line shows one trajectory of evolution of a finite non-zero motion of the bar as it comes to rest along the stable eigen-direction $\boldsymbol{P}=[0.5,\ 0,\ 0.5]$ and associated $\boldsymbol{q}=[1/\sqrt{2},\ 0,\ 1/\sqrt{2}]$.

motions of the slider except for pure rotations and translations. Also, the slider rotates about a point of support for a finite amount of time before it comes to rest, for a time period which depends on initial conditions. The facet on the limit surface generally leads to discontinuity in the frictional load $\boldsymbol{P}$ during the "free sliding". In general, the load jumps from the edge of the flat facet to some point on its interior.

"Free-sliding" of the bar was numerically simulated, and the results depicted on the unit motion sphere of Fig. 7. This figure shows the evolution of motion as seen in a bar-fixed axes system. The light curves on the unit motion sphere depict the flow field, along which various motions of almost zero magnitude (governed by eqn. (8)) approach the stable eigen-direction. The dark line shows the trajectory of a finite non-zero motion of the bar (governed by eqn. (7)) as it approaches sleep along the stable eigen-direction $\boldsymbol{P}=[0.5,\ 0,\ 0.5]$ and associated $\boldsymbol{q}=[1/\sqrt{2},\ 0,\ 1/\sqrt{2}]$. Not only do all initial motions come to rest along a stable eigen-direction, but also their direction of final approach towards this $\boldsymbol{q}$ is unique. It corresponds to a slight translation along the bar superposed on the rotation about one end point. Its apparent physical origin is the centrifugal pull of the last moving point mass on the end point which is about to be the center of rotation.

8.2. Other examples of free sliding with point supports

Flat regions on the limit surface need not represent an eigen-direction. Consider a variation on the bar mentioned in the previous example that is

supported by friction strengths of magnitudes two and one, at distances 0.25 and 0.5 respectively on either side of its center of mass (note that the mass is not concentrated at the points of support). Then the section of its limit surface on the $[F_x, M]$ plane is as shown in Fig. 8. As can be seen from this figure only one set of flat regions has a stable eigen-direction, corresponding to instantaneous rotation about the stronger point of support. Rotation about the weaker point of support is no longer an eigen-direction and this particular motion cannot be sustained even for finite time.

Another interesting aspect of rotation about a point of support is that it can in some circumstances be sustained for a finite time, even if not associated with an eigen-direction and final motion. This can be illustrated with the example of the strangely supported plate below of Fig. 9. The plate has all its mass m concentrated very close to O, its center of mass. It is supported at various points with spatially non-uniform isotropic friction (the center of the contact pressure distribution coincides with the center of mass), but only at two points, labeled a and b in the figure, is the coefficient of friction non-zero. Also the frictional strength at b is larger than at a. For this plate, rotation about a is not an eigen-direction, although such a motion

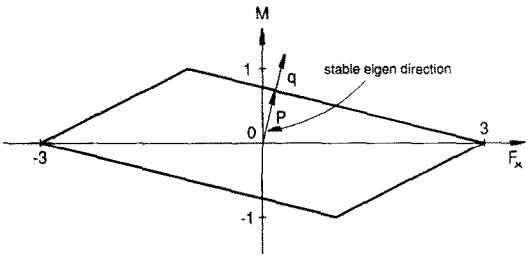


Fig. 8. Section on the $[F_x, M]$ plane of the limit surface of a bar supported asymmetrically by isotropic friction, so that rotation about the lighter point of support is not an eigen-direction. Rotation about the heavier point of support is the final attracting motion, as shown.

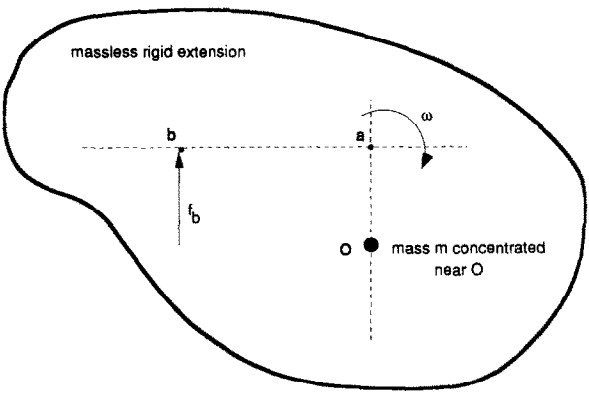


Fig. 9. Plate supported by isotropic but spatially non-uniform friction. All its mass is concentrated near O. The friction force is non-zero only at two points of support a and b and $f_b > f_a$. Rotation about a is not an eigen-direction, although this rotation can be sustained for a finite amount of time at finite rotation rates.

can be sustained for a finite amount of time. Depending on the angular velocity ω with which the plate is rotating about a, force balance can be attained between the friction forces f_a and f_b, the centrifugal force $mr\omega^2$ due to the rotating point mass m, and other inertial forces, such that this motion is sustained for a while. But as the plate slows down owing to frictional dissipation, ω and hence $mr\omega^2$ decrease in magnitude and eventually a value of ω is attained when f_a is no longer enough to counteract the other forces and the plate breaks loose at a.

9. Effect of convective terms on orientation stability

It is well known in the automotive industry, though possibly counter-intuitive, that if the rear wheels of a vehicle are locked while the front wheels roll freely the vehicle will lose directional stability [12]. Keeping this phenomenon in mind, we consider the limit surface description of an elementary model of a car with locked rear wheels.

We collapse the car laterally (making it effectively unidimensional) and represent it as a bar supported at its ends, the front end being supported by a wheel with its axle perpendicular to the car, and the back end by a point of isotropic support (a locked wheel), each end being unit distance away from the center of the car. Equal point masses are located at each end of the car and their coefficients of friction μ are equal. Both the car and its limit surface are shown in Fig. 10. The limit surface for the car is an elliptical cylinder and it is rather singular in that it is made up entirely of vertices and flat (at least in one direction) regions. The elliptical facets correspond to rotation about the point of isotropic support (the locked rear wheel), the sides correspond to pure rolling of the front wheel (CORs with the coordinate $y_c = 1$), and each point on the elliptical edge has a set of CORs lying on a line (specific to that point) passing through the point of isotropic support. Observe also that the limit surface has only one plane of symmetry, the M–F_x plane.

It can be seen that of the many eigen-directions there are only four which are stable. These are forward and backward translations of the car (in the direction of the car itself) and clockwise and counter-clockwise rotations about the point of isotropic support. Thus at final motion the car is necessarily moving either frontwards, backwards or is rotating about its braked wheel.

One can also discuss directional stability at finite velocities. At all velocities, straight ahead motion and straight backwards motion persist till the car stops, if the car is started precisely with these motions. But the directional stability is different for the two cases. In particular, the symmetry of the limit surface for forwards translation ($\boldsymbol{q} = [0, 1, 0]$) and backwards translation ($\boldsymbol{q} = [0, -1, 0]$) is not manifested by identical equations of motion for the two cases because of the convective term. Calculation shows (Appendix A) that forward translation is directionally unstable at speeds of $\sqrt{\mu}$ and above, and stable at less than $\sqrt{\mu}$. Backwards translation (or equivalently forwards

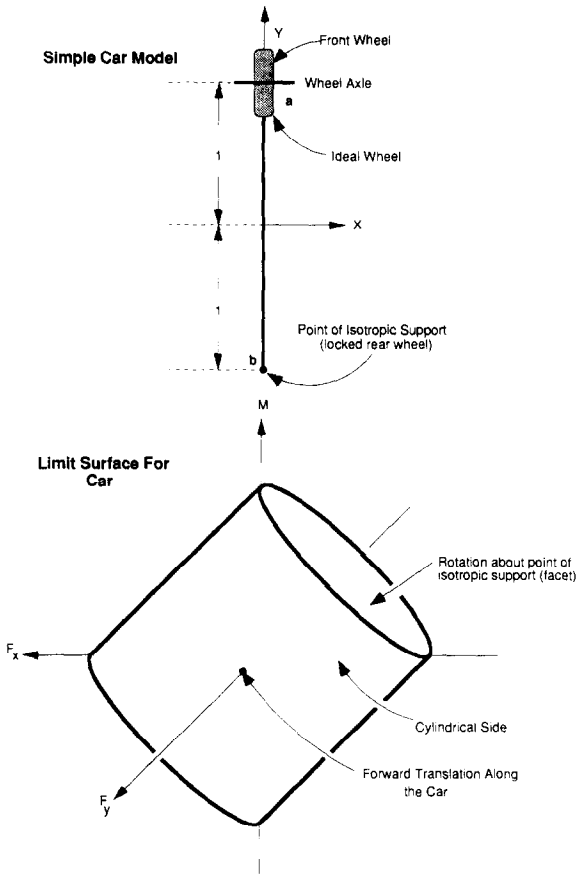


Fig. 10. Simple model of a car with locked rear wheels (bar supported at its ends by an ideal wheel and a point of isotropic support) and its limit surface (an elliptical cylinder). This limit surface has only one plane of symmetry, the F_x–M plane. From a first-order stability analysis of this surface it can be seen that forward and backward translations of the bar (in the direction of the bar itself), and clockwise and counter-clockwise rotations about the point of isotropic support are all stable eigen-directions. For high velocities, forward translation is unstable even though it is a stable eigen-direction.

translation with front wheels locked and rear wheels free) is always directionally stable.

10. Free sliding of sliders with a single coefficient of isotropic friction

For reference we summarize here some properties of free slip for a slider that has a single coefficient of friction acting at each of the contact points, and has all mass concentrated in the slip plane. Most have been mentioned by other authors and can be directly demonstrated without the use of the limit surface description. Most of them also do not depend on

the pressure distribution and so are insensitive to the large variations in pressure distribution that one expects during slip of real nearly flat hard objects in contact.

$|Q|$ *decreases in time.* $\boldsymbol{QQ}$ is proportional to the kinetic energy, which is naturally expected to decrease in time. Direct application of inequality (1) and eqn. (4), together with the fact that the limit surface is convex and encloses the origin, so $\boldsymbol{PQ} > 0$, shows that $\dot{\boldsymbol{Q}}\boldsymbol{Q} < 0$. This is true, of course, for all sliders and does not depend on the isotropy of friction.

$|P|$ *sometimes decreases in time.* When the convection term in eqn. (3) can be neglected, $|\boldsymbol{P}|$ is non-increasing. The convection terms are zero for pure translation and pure rotation, and asymptotically zero as motion comes to a stop.

ω *decreases in time.* The limit surfaces under discussion intersect the x–y plane at a circle (independent of the symmetry of the pressure distribution). The normals to the limit surface on this circle lie in the x–y plane. Since the limit surface is convex, all $\boldsymbol{Q}$ with positive ω correspond to $\boldsymbol{P}$ in the upper half-space where M is positive and, by eqn. (4), $\dot{\omega} < 0$ (note that the convection terms make no contribution to $\dot{\omega}$).

Note that, even if pure translation corresponds to an unstable eigen-direction and that small perturbations from pure translation grow (as seen by the changes in $\boldsymbol{P}$ and $\boldsymbol{q}$), the absolute rotation rate is always decreasing. When the ring described earlier is started in nearly pure translation and ends by rotating and translating at the same rate, the rotation rate is always decreasing (the translation rate decreases even faster).

When anisotropic friction is allowed it is possible not just for the direction in motion space to lead to increased relative rotation, but for the absolute rotation rate to increase in time. The braked car considered above is an example.

Pure rotation and pure translation are eigen-directions. Sliders with uniform coefficient of friction and identical centers of pressure and gravity have horizontal limit surfaces on the M axis and intersect the x–y plane at a circle with the tangent to the limit surface being vertical at the intersection. Both pure rotation and pure translation are eigen-directions.

Additionally the convective term in eqn. (2) is identically zero for these motions so they persist for finite time once initiated. An object set in translation will continue translating in a straight line until it stops. An object set in pure rotation will rotate without translation until it stops.

An axisymmetric object moves in a straight line. A non-axisymmetric body does not have to travel in a straight line, however. The center of gravity of the sliding dumbell does not travel on a straight line if the rotation rate is non-zero.

11. Motion planning

The details of the pressure distribution between hard, nearly flat objects is notoriously unpredictable. Generically one expects support to be governed

by three point-like contacts of unknown location and one expects a six-faceted limit surface with facets in unknown orientations. So the overall friction relation is also highly indeterminate, as Mason and Salisbury [10] and Peshkin and Sanderson [13] have emphasized. One approach to control is to overwhelm the indeterminate friction forces with large, accurately controlled externally applied loads.

Understanding the transition from indeterminacy at small applied loads to determinacy at high loads might be enhanced by reference to the earlier discussion of uniqueness. During slip the friction load is that point on the limit surface which is closest to the applied load. If the applied load is not much larger than the frictional resistance, there is high probability of the closest point on the limit surface being on a facet (if the limit surface has facets) or other non-spherical feature of the limit surface and for the first motion to be that corresponding to the normal at that point. If the applied load is large, however, $\boldsymbol{Q}$ will be nearly parallel to the load, and therefore less dependent on the details of the frictional interface.

12. Conclusions

The limit surface description of the load–motion relation for planar sliders is a useful tool for understanding slip and its control for both slipping and wheeled objects. The frictional load during dynamic slip is unambiguous, even if the friction law is in some senses multivalued. When sliders come to rest during free slip they do so with special motions called eigen-directions, where $\boldsymbol{P}$ is parallel to $\boldsymbol{q}$ on the limit surface. For motions of very small magnitude the "stability" of eigen-directions is governed by the principle radii of curvature and the load vector at the eigen-direction. The change in stability aspects of eigen-directions for motions of higher magnitude, due to non-symmetries of the governing equations, is illustrated by a partially braked car. The relation between mass distribution and eigen-directions is such that with most objects, if the mass is sufficiently spread compared with the contact region, the object will have final motion which is pure rotation about the center of mass, and if the mass is sufficiently central it will have final motion that is pure translation. The tendency of objects to rotate about points of support can be explained in terms of the finite surface area for flat regions (corresponding to point supports) on the limit surface and their propensity to attract and sustain motions for a finite amount of time.

Acknowledgments

Thanks to all those acknowledged in part one of this paper [9], to Z. Mroz for refs. 1 and 3, to J. Burns for pointing out ref. 2, to K. Vøyenli and E. Eriksen themselves for pointing out a calculation error, and to Les Schaeffer for a hockey rink observation.

References

1 A. Yu. Ishlinskii, B. N. Sokolov and F. L. Chernousko, Motion of plane bodies with dry friction, *Izv. Akad. Nauk. SSSR Mekh. Tverd. Tela, 16* (4) (1981) 17–28.
2 K. Vøyenli and E. Eriksen, On the motion of an ice hockey puck, *Am. J. Phys., 53* (12) (1985) 1149–1153.
3 J. Wittenburg, Ebene Bewegungen Bei Flächenhaft Verteilten Reibungskräften, *Z. Angew. Math. Mech., 5* (1970) 637–640 (in German).
4 J. H. Jellet, *A treatise on the theory of friction* (Hodges, Foster, and Co.) 1872.
5 J. Prescott, *Mechanics of particles and rigid bodies*, Longmans, London, 1923.
6 W. D. MacMillan, *Dynamics of rigid bodies*, Dover Publications, New York, 1936.
7 S. Goyal, Planar sliding of a rigid body with dry friction: limit surfaces and dynamics of motion, *Ph.D. Thesis*, Cornell University, Ithaca, 1989.
8 S. Goyal, A. Ruina and J. Papadopoulos, Limit surface and moment function descriptions of planar sliding, *Proc. IEEE Int. Conf. Robotics Automation, May 1989*, Vol. 2, IEEE Computer Society Press, Washington, D.C., pp. 794–799.
9 S. Goyal, A. Ruina and J. Papadopoulos, Planar sliding with dry friction. Part 1. Limit surface and moment function, *Wear, 143* (1991) 307–330.
10 M. T. Mason and J. K. Salisbury, *Robot hands and the mechanics of manipulation*, Massachusetts Institute of Technology Press, Cambridge, MA, 1985.
11 B. O'Neill, *Elementary differential geometry*, Academic Press, New York, 1966.
12 J. Y. Wong, *Theory of ground vehicles*, Wiley, New York, 1978.
13 M. A. Peshkin and A. C. Sanderson, *Planning Robotic Manipulation Strategies for Workpieces that Slide*, IEEE J. Robotics & Automation, 4 (5) (1988) 524–531.

Appendix A: analysis of braked car

Except for the elliptical caps, the limit surface for the braked car is given in terms of q_x, q_y, q_ω and the parameter A as:

$$F_x = \frac{\mu(q_x+q_\omega)}{2(1+2q_xq_\omega)^{1/2}} + \frac{\mu A}{2} \qquad F_y = \frac{\mu q_y}{2(1+2q_xq_\omega)^{1/2}}$$
$$M = \frac{\mu(q_x+q_\omega)}{2(1+2q_xq_\omega)^{1/2}} - \frac{\mu A}{2} \tag{A1}$$

where the value of A is given by the following.

(1) If $q_x = q_\omega \Rightarrow -1 \leqslant A \leqslant 1$, *i.e.* the frictional load is on the sides of the cylinder. (However a unique value can be found from dynamical considerations, given as $A = |\boldsymbol{Q}|^2 q_y q_\omega/\mu$).

(2) If $q_x > q_\omega \Rightarrow A = 1$ (one elliptical cap).

(3) If $q_x < q_\omega \Rightarrow A = -1$ (the other elliptical cap).

To investigate the stability of forward and backward translation $[q_x, q_y, q_\omega] = [0, \pm 1, 0]$ we manipulate eqns. (4) and (A1) to get:

$$|\boldsymbol{Q}|\dot{\boldsymbol{q}}|_{\text{body}} = \begin{bmatrix} -\dfrac{\mu q_\omega(1-2q_x{}^2)}{2(1+2q_xq_\omega)^{1/2}} + \dfrac{\mu}{2}A(q_x{}^2 - q_xq_\omega - 1) + |\boldsymbol{Q}|^2 q_y q_\omega \\ \dfrac{\mu q_x q_\omega q_y}{(1+2q_xq_\omega)^{1/2}} + \dfrac{\mu}{2}Aq_y(q_x - q_\omega) - |\boldsymbol{Q}|^2 q_x q_\omega \\ -\dfrac{\mu q_x(1-2q_\omega{}^2)}{2(1+2q_xq_\omega)^{1/2}} + \dfrac{\mu}{2}A(q_\omega q_x - q_\omega{}^2 + 1) \end{bmatrix} \tag{A2}$$

The only perturbations from straight motion that do not decay immediately are those where the free front wheel keeps rolling and the locked rear wheel acquires a small sideways motion. This group of perturbations is given by $q_x = q_\omega \ll 1$, $q_y \approx \pm 1$, and consequently eqn. (A2) reduces to:

$$|\boldsymbol{Q}|\dot{\boldsymbol{q}}|_{\text{body}} \approx \left(\frac{|\boldsymbol{Q}|^2}{2} - \frac{\mu q_y}{2}\right)\begin{bmatrix} q_xq_y \\ -2q_x{}^2 \\ q_\omega q_y \end{bmatrix} \tag{A3}$$

Examination of eqn. (A3) shows that for all velocity magnitudes $|\boldsymbol{Q}|$ of the bar, $\boldsymbol{q} = [0,\ -1,\ 0]$ is an attracting motion, *i.e.* motions in its vicinity are attracted to it. But $\boldsymbol{q} = [0,\ 1,\ 0]$ is an attracter only for $|\boldsymbol{Q}| < \sqrt{\mu}$, for higher velocity magnitudes all perturbations are attracted to $\boldsymbol{q} = [0,\ -1,\ 0]$.

Redimensionalizing and using a friction coefficient $\mu = 1$, gravitational acceleration $g = 9.8$ m s^{-2}, and radius of gyration $r_g = 1$ m (for a 2 m car), we get a critical speed of $v = \sqrt{\mu}\sqrt{gr_g} \approx 11.3$ km h^{-1}. At speeds less than this, rear braking is safe. In fact a braked car is stable with its front wheels locked for much the same reason that the feather end of an arrow stably follows the head in flight.