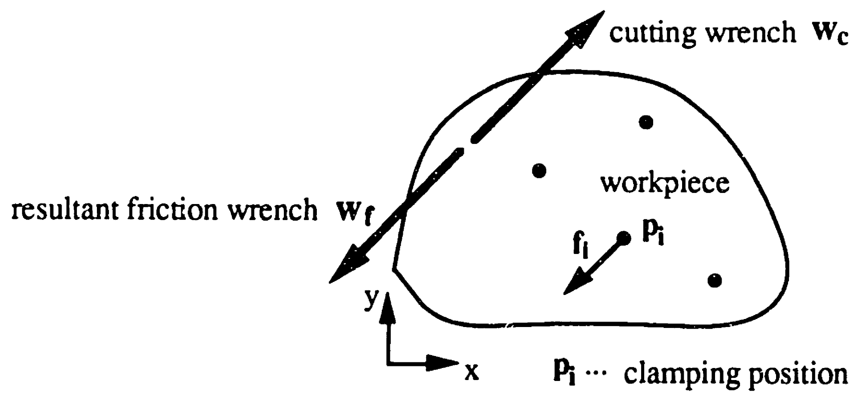

Figure 4.2.6 Two dimensional model of down-holding clamping

4.2.3 Necessary Clamping Forces to Resist a Cutting Wrench

4.2.3.1 Static Analysis

An attempt was made to calculate the necessary clamping forces with force and moment equilibrium analysis. Let f_{ci} denote the magnitude of the clamping force of the ith clamp, $\mathbf{p}_i = [x_i \ \ y_i\ ]^T$ denote the ith clamping position, $\mathbf{f}_i = [f_{ix} \ \ f_{iy}\ ]^T$ denote the friction force at the ith clamping position, $\mathbf{p}_c = [x_c \ \ y_c\ ]^T$ denote the position of the cutting force, $\mathbf{f}_c = [f_{cx} \ \ f_{cy}\ ]^T$ denote the cutting force, m_c denote the cutting torque, $\mathbf{w}_c = [w_{cx} \ \ w_{cy} \ \ w_{cm}\ ]^T = [f_{cx} \ \ f_{cy} \ \ x_c f_{cy} - y_c f_{cx} + m_c]^T$ denote the resulting cutting wrench resolved around the origin and n denote the number of clamps. See Figure 4.2.6. Now, with these notations, the force and moment equilibrium is expressed as:

$$\begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \\ -y_1 & x_1 & \cdots & -y_n & x_n \end{bmatrix} \begin{bmatrix} f_{1x} \\ f_{1y} \\ \vdots \\ f_{nx} \\ f_{ny} \end{bmatrix} = - \begin{bmatrix} w_{cx} \\ w_{cy} \\ w_{cm} \end{bmatrix} \tag{4.2.9}$$

With matrix and vector notations, equation (4.2.9) is:

$$\mathbf{W}\ \mathbf{f} = -\mathbf{w}_c \tag{4.2.10}$$

If this equation is solved, the necessary clamping forces are:

$$f_{ci} = \frac{\sqrt{f^2_{ix} + f^2_{iy}}}{\mu} \tag{4.2.11}$$

where μ is the friction coefficient.

But the problem is statically indeterminate and the solution has $q = 2n - 3$ free variables and is expressed as follows:

$$\mathbf{f} = \mathbf{f}_p + \sum_{i=1}^{q} \lambda_i \mathbf{v}_i \tag{4.2.12}$$

where $\mathbf{f}_p$ is a particular solution, λ_i is a free variable and $\mathbf{v}_i$ is a basis of the null space of $\mathbf{W}$. Making the problem determinate requires elasticity. To avoid this, quasi-static analysis is introduced in the next sections.

4.2.3.2 Quasi-Static Analysis

In the previous section, clamping forces to balance the cutting wrench were sought by solving static equilibrium equations for the cutting wrench. In this section, the problem is reversed. When and how does the workpiece start slipping if the magnitude of the cutting wrench is increased gradually ? Once the workpiece starts slipping, the instantaneous velocity and the resulting friction force at each clamping position are uniquely related by Coulomb's law. The Coulomb's law of dry friction, when the normal force $N \geq 0$ is treated as known, states a relation between the friction force $\mathbf{f}$ and sliding velocity v formulated as follows :

There exists $\mu \geq 0$, the friction coefficient, such that

$$\text{if } \mathbf{v} = 0: |\mathbf{f}| \leq \mu N \tag{4.2.13}$$

$$\text{if } \mathbf{v} \neq 0: |\mathbf{f}| = \mu N \tag{4.2.14}$$

and the vectors **f** and **v** are parallel with opposite directions.

Since we are interested in when the workpiece starts slipping, we will use static friction coefficient instead of kinetic friction coefficient.

To avoid dynamics getting involved, the quasi-static assumption is made here. Quasi-static means that the inertia of the workpiece is negligible. When the cutting force is barely large enough to move the workpiece, the acceleration of the workpiece is very small. By this assumption, the static equilibrium equation (4.2.10) remains valid when the workpiece starts slipping. Now how can we use equation (4.2.14) to solve equation (4.2.10) ? One way is to assume the center of rotation. Then the friction force at each clamping position can be expressed with the coordinates of the center of rotation based on equation (4.2.14). Then equation (4.2.10) can be solved for the coordinates of the center of rotation.

Let $\mathbf{p_0} = [x_0 \;\; y_0]^T$ be the center of rotation when the workpiece starts moving and let's assume a clockwise rotation. Then the friction force at the ith clamping position is,

$$\begin{bmatrix} f_{ix} \\ f_{iy} \end{bmatrix} = \frac{\mu f_c}{|\mathbf{p_i} - \mathbf{p_0}|} \begin{bmatrix} (y_i - y_0) \\ -(x_i - x_0) \end{bmatrix} \tag{4.2.15}$$

where f_c is the clamping force common to the all the clamps. Then equation (4.2.10) is expressed as,

$$\sum_{i=1}^{n} \frac{y_i - y_0}{|\mathbf{p_i} - \mathbf{p_0}|} (\mu f_c) = -w_{cx}$$

$$\sum_{i=1}^{n} \frac{-(x_i - x_0)}{|\mathbf{p_i} - \mathbf{p_0}|} (\mu f_c) = -w_{cy} \tag{4.2.16}$$

$$\sum_{i=1}^{n} \frac{-x_i(x_i - x_0) - y_i(y_i - y_0)}{|\mathbf{p}_i - \mathbf{p}_0|} (\mu f_c) = -w_{cm}$$

Here unknowns are x_0, y_0 and f_c. Since these equations contain lots of square root terms, a numerical method like Newton-Raphson's method must be used and the solution can not be reliably obtained.

Furthermore, when the center of rotation coincides with one of the clamping positions, equations (4.2.16) is invalid since the friction force at that clamping position is different from the one given by equation (4.2.15). So, before solving equation (4.2.16), the following equations must be solved for every clamping position p_j.

$$\sum_{i=1}^{n} \frac{y_i - y_j}{|\mathbf{p}_i - \mathbf{p}_j|} (\mu f_c) + f_{jx} = -w_{cx} \qquad i \neq j$$

$$\sum_{i=1}^{n} \frac{-(x_i - x_j)}{|\mathbf{p}_i - \mathbf{p}_j|} (\mu f_c) + f_{jy} = -w_{cy} \qquad i \neq j \tag{4.2.17}$$

$$\sum_{i=1}^{n} \frac{-x_i(x_i - x_j) - y_i(y_i - y_j)}{|\mathbf{p}_i - \mathbf{p}_j|} (\mu f_c) + (x_j f_{jy} - y_j f_{jx}) = -w_{cm} \qquad i \neq j$$

$$f_{jx}^2 + f_{jy}^2 - (\mu f_c)^2 \leq 0$$

The unknowns are f_{jx}, f_{jy} and f_c. If the jth clamping position coincides with the center of rotation, the above equations (4.2.17) have a solution.

Also, the workpiece may translate. Conceptually, a translation can be thought as a rotation around infinity. But numerically, it causes a trouble. So, it must be checked if the cutting wrench goes through the center of friction before solving equations (4.2.16).

Mason tried to find out the center of rotation when a planar object is pushed by the method described here and had convergence problem [Mason 85]. So, I did not try to solve these nasty non-linear equations.

4.2.3.3 Maximum Friction Wrench Theorem

The method applied here to solve the problem is based on a theorem about the friction of the multiple point contact found and proved in this research. The proof of the theorem is presented in the next section. When the theorem is applied to the current clamping problem, it means:

<u>When the workpiece starts slipping, the magnitude of the resultant friction wrench is the maximum.</u>

Because of the quasi-static assumption, the direction of the friction wrench is opposite to that of the cutting wrench and its magnitude is the same as that of the cutting wrench when the workpiece starts slipping. This relation is shown in equation (4.2.10) where $\mathbf{W}\,\mathbf{f}$ is the friction wrench and $\mathbf{w}_c$ is the cutting wrench.

A friction wrench whose direction is opposite to that of the cutting wrench is expressed as:

$$\mathbf{W}\,\mathbf{f} = -\mathbf{w}\,I \tag{4.2.18}$$

where I is the magnitude of the wrench and $\mathbf{w} = [w_x \ \ w_y \ \ w_m\,]^T$ is the direction of the cutting wrench and is given by the following:

$$\mathbf{w} = \frac{\mathbf{w}_c}{|\mathbf{w}_c|} = \frac{1}{\sqrt{f_{cx}^2 + f_{cy}^2 + \left(x\,f_{cy} - y\,f_{cx} + m_c\right)^2}} \begin{bmatrix} f_{cx} \\ f_{cy} \\ x\,f_{cy} - y\,f_{cx} + m_c \end{bmatrix} \tag{4.2.19}$$

Now the theorem is represented as the following maximization problem.

$$\begin{aligned} &\text{Maximize} && I = -\,\mathbf{w} \cdot (\mathbf{W}\mathbf{f}) \\ &\text{subject to} && \mathbf{w} \times (\mathbf{W}\mathbf{f}) = \mathbf{0} \\ & && f_{ix}^2 + f_{iy}^2 \le (\mu f_{ci})^2 \qquad \text{for } i = 1,\cdots,n \end{aligned} \tag{4.2.20}$$

The equality constraint assures that the friction wrench is parallel to the cutting wrench. Since the objective function and inequality constraints are convex functions and equality constraints are linear, optimization theory guarantees that the maximum obtained is global

[Avriel 76]. When the maximum magnitude is obtained, friction forces are also obtained at the same time. Then the necessary clamping force for each clamp is obtained as:

$$f_{necessary} = f_{ci}\frac{I_c}{I_{max}} \tag{4.2.21}$$

where I_c is the magnitude of the cutting wrench and I_{max} is the maximum magnitude obtained from the maximization (4.2.20). f_{ci} represents the ratio of the magnitude among the clamping forces. If the magnitude of the clamping forces is the same, f_{ci} is one. Any two dimensional movement can be expressed as a rotation around a point. The center of rotation of the workpiece can be easily obtained by drawing lines perpendicular to the friction forces at the clamping positions. The intersection of these lines is the center of rotation.

In implementing this method, Linear Programming was used again. Since the inequality constraints in maximization (4.2.20) are circles, they are easily approximated with regular polygons and Linear Programming becomes applicable.

The method was implemented on IRIS 4D graphics workstation. In the case where four clamps are used and each inequality constraint is approximated with 36 facets, the obtained friction forces were correct to the third digit and the computing time required was less than a second. Figure 4.2.7 shows an example. The clamping positions and the position and the direction of the cutting force are shown in Figure 4.2.7(a). The equation (4.2.18) and maximization (4.2.20) are written as:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 2 & -1 & 0 \end{bmatrix}}_{\mathbf{W}} \underbrace{\begin{bmatrix} f_{1x} \\ f_{1y} \\ \vdots \\ f_{3y} \end{bmatrix}}_{\mathbf{f}} = - \underbrace{\begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix}}_{\mathbf{w}} I$$

Maximize $\quad I = -0.707(f_{1x} - f_{1y} + f_{2x} + 3f_{2y} - f_{3x} + f_{3y})$

subject to $\quad f_{1x} + f_{2x} + f_{3x} = 0$

$$-f_{1x} + 3f_{1y} - f_{2x} - f_{2y} + f_{3x} + f_{3y} = 0$$

$$f_{ix}^2 + f_{iy}^2 \leq 1 \qquad \text{for } i = 1,2,3$$

The result is shown in Figure 4.2.7(b). I_{max} is only 2.267 even though there are three clamps. The center of rotation of the workpiece coincides with one of the clamping positions.

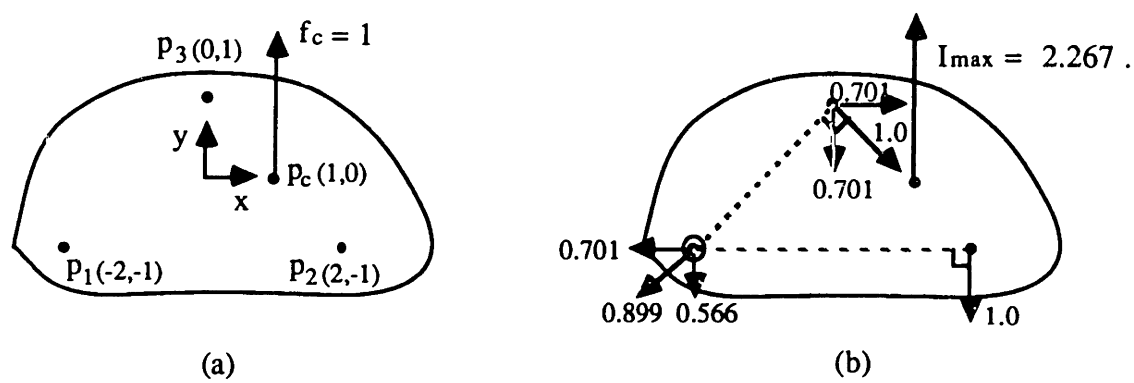

Figure 4.2.7 An example of maximum magnitude method

Though the method can be applied only to multiple point contact cases as in the example, there is no limitation to the number of contact points. I guess that by approximating a pressure distribution with many contact points, we can estimate clamping forces satisfactorily accurately for any pressure distribution. The remaining problem is that there is no way to find the pressure distribution itself.

Cutkosky is the only one who considered friction force in automating fixture design [Cutkosky 89]. He checked if a given set of clamping forces can prevent the workpiece from slipping in the following way. His idea is to check if a cutting wrench lies inside of the limit surface of the given clamping forces. A three dimensional limit surface in force and moment space is a concept developed by Goyal [Goyal 89]. It is a closed surface and represents the frictional load-motion relation of a planar slider based on friction laws. A vector from the origin to a point on the limit surface represents the friction wrench $\mathbf{w} = [f_x\ f_y\ m]^T$ generated by the slider and the normal at the point represents the direction of the corresponding motion of the slider $\mathbf{v} = [v_x\ v_y\ \omega]^T$. The problem with this is that limit surface is simply a concept and it cannot be expressed analytically. So Cutkosky calculated points on the limit surface for assumed centers of rotation. Then he approximated the limit surface with an ellipsoid using the calculated points on the limit surface. He says that a friction force is not exactly predictable anyway, this approximation is acceptable.

The method used in this work directly calculates the point on the limit surface for the direction of a given cutting wrench. There is no approximation at all and CPU time required is minimal. Also this method is useful in predicting the motion of a planar slider as presented in Appendix B.

The method can be applied to three dimensional cases only if the cutting wrench is perpendicular to the clamping forces and the forces exerted by the clamps can remain constant. But in three dimensional cases, usually they do not remain constant and it requires elasticity to calculate the forces exerted by the clamps. A simple example in Figure 4.2.8 shows this difficulty. In Figure 4.2.8(a), a workpiece is located and clamped with a locator and a clamp. The reaction force f_l of the locator is equal to the clamping force f_c. Now in Figure 4.2.8(b), a cutting force f_{cut} is applied. What are the forces f_l' and f_c' exerted by the locator and the clamp ? We need elasticity to answer this question.

Figure 4.2.8 Static indetermination

4.2.4 Proof of Maximum Friction Wrench Theorem

In this section, the maximum friction wrench theorem that is used in calculating necessary clamping forces is proved. Figure 4.2.9 shows an object supported at multiple points and sliding on a horizontal plane. The maximum friction wrench theorem is:

(1) When the magnitude of the friction wrench is the maximum under Coulomb's law, the object is sliding.

(2) When the object is sliding, the magnitude of the friction wrench is the maximum under Coulomb's law.

or in simpler words, it means that when you try to move an object lying on a plane, it resist you as much as possible.

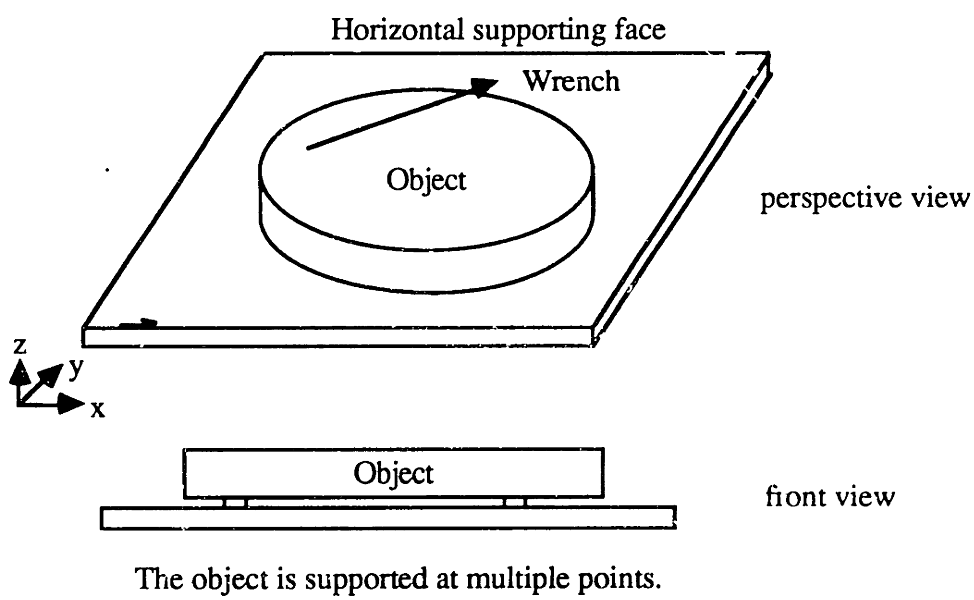

Figure 4.2.9 An object sliding on a horizontal plane

Before starting the proof, Kuhn and Tucker's optimality conditions for convex program is reviewed [Avriel 76].

Kuhn and Tucker's optimality conditions for convex program

Given a constrained optimization problem of the form:

$$\text{(CP)} \quad \text{Minimize} \quad F(\mathbf{x})$$

$$\text{subject to} \quad g_i(\mathbf{x}) \le 0 \qquad i=1,\cdots,m \tag{P.1}$$

$$h_j(\mathbf{x}) = 0 \qquad j=1,\cdots,p \tag{P.2}$$

$$\mathbf{x} \in R^n$$

where $F(\mathbf{x})$ and $g_i(\mathbf{x})$ are convex functions and $h_j(\mathbf{x})$ are linear.

The followings are the necessary and sufficient conditions for the global optimality.

There exist vectors $\mathbf{x}^*$, λ^* and μ^*, with $\mathbf{x}^*$ satisfying (P.1) and (P.2), and

$$\text{(KT)} \quad \nabla F(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_i^* \nabla h_j(\mathbf{x}^*) = 0$$

$$\lambda_i^* g_i(\mathbf{x}^*) = 0 \qquad i=1,\cdots,m$$

$$\lambda^* \ge \mathbf{0}$$

$F(\mathbf{x}^*)$ is the global minimum of (CP).

The maximization problem (4.2.20) is changed into the following minimization problem by simply multiplying the objective function by -1.

$$\text{Minimize} \quad F(\mathbf{f}) = -I = w_x \sum_{i=1}^{n} f_{ix} + w_y \sum_{i=1}^{n} f_{iy} + w_m \sum_{i=1}^{n} (x_i f_{iy} - y_i f_{ix}) \tag{4.2.22}$$

$$\text{subject to} \quad g_i(\mathbf{f}) = f_{ix}^2 + f_{iy}^2 - (\mu f_{ci})^2 \le 0 \quad \text{for } i = 1,\cdots,n \tag{4.2.23}$$

$$h_1(\mathbf{f}) = w_y \sum_{i=1}^{n} (x_i f_{iy} - y_i f_{ix}) - w_m \sum_{i=1}^{n} f_{iy} = 0 \tag{4.2.24}$$

$$h_2(\mathbf{f}) = w_m \sum_{i=1}^{n} f_{ix} - w_x \sum_{i=1}^{n} (x_i f_{iy} - y_i f_{ix}) = 0 \tag{4.2.25}$$

$$h_3(\mathbf{f}) = w_x \sum_{i=1}^{n} f_{iy} - w_y \sum_{i=1}^{n} f_{ix} = 0 \tag{4.2.26}$$

$$\mathbf{f} = [f_{ix} \;\; f_{iy} \;\; \cdots \;\; f_{nx} \; f_{ny} \;]^T$$

Since $F(\mathbf{f})$ is linear and hence convex, g_i are convex and h_1, h_2 and h_3 are linear, this optimization is a convex program and Kuhn and Tucker's optimality conditions for convex program are applicable.

The necessary and sufficient conditions for the global optimality applied to this problem are:

There exist vectors $\mathbf{f}^*$, λ^*, μ^*, with $\mathbf{f}^*$ satisfying (4.2.23) through (4.2.26), and

$$\nabla F(\mathbf{f}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{f}^*) + \sum_{j=1}^{p} \mu_i^* \nabla h_j(\mathbf{f}^*) = 0 \tag{KT.1}$$

$$\lambda_i^* g_i(\mathbf{f}^*) = 0 \qquad i=1,\cdots,m \tag{KT.2}$$

$$\lambda^* \geq \mathbf{0} \tag{KT.3}$$

Now we are ready to prove the theorem. To avoid the notations get meaninglessly confusing, * is omitted.

(1) <u>Proof that when the magnitude of the friction wrench is the maximum under Coulomb's law, the object is sliding</u>

When the magnitude of the friction wrench is maximum, Kuhn and Tucker's conditions hold true. Among the equations in (KT.1), only two of them contain f_{ix} and f_{iy}. They are: