Changing Variables in Recurrences

Hsing-Kuo Pao

Department of Computer Science & Information Engineering
National Taiwan University of Science & Technology
pao@mail.ntust.edu.tw

1 When zero is not a solution

The master theorem states:

Theorem 1 (Master theorem) *Let $a \geq 1$, $b > 1$, $n \in \mathbf{N} \cup \{0\}$, and let $f(n)$ be a nonnegative function. Let $T(n)$ be defined by the recurrence equation $T(n) = aT(n/b) + f(n)$, then $T(n)$ can be bounded asymptotically as follows.*
Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$,
Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$,
Case 3: If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

The master method is surely not enough to apply to many recurrences. We have discussed two cases already:

1. When function $f(n)$ falls in between the cases, such as the function is "smaller" than $n^{\log_b a}$ and "bigger" than $n^{\log_b a - \epsilon}$, e.g., $f(n) = n^{\log_b a} / \ln n$. None of the cases can be applied.

2. The regularity condition in Case 3 is not satisfied.

However, we show that we can still deal with some other recurrences that are not in the form of Theorem 1. More discussion follows.

Example 1

$$T(n) = 2T(n/4 + 3)\,. \tag{1}$$

Sol. *The original recurrence equation does not go through the origin. The form*

$$T(n) = 2T(n/4)\,,$$

however will go through the origin with $T(0) = 0$ and we know how to apply the master method to this case. Let us substitute variables and make the recurrence to be the one that can go through the origin. We start by letting $n = m + c$ and the recurrence in Eq. 1 becomes

$$T(m + c) = 2T(m/4 + c/4 + 3)\,.$$

Let us choose c so that $c = c/4 + 3$ which gives us $c = 4$. That is, we choose $c = 4$ and we have:

$$T(m + 4) = 2T(m/4 + 4)\,. \tag{2}$$

Now we observe that the left and the right has the identical form and let $S(m) = T(m + 4)$ and the recurrence Eq. 2 becomes

$$S(m) = 2S(m/4)\,, \tag{3}$$

and it is now a recurrence that goes through the origin. We can apply the master method to find the answer. That is, we can apply case 1 with $a = 2$ and $b = 4$ and we have $S(m) = \Theta(\sqrt{m})$. In fact, $S(m)$'s exact solution is given by $S(m) = \sqrt{m}$ if $S(1) = 1$. We now find the solution for $T(n)$ as follows.

$$T(n) = T(m + 4) = S(m) = \sqrt{m} = \sqrt{n - 4} = \Theta(n)\,.$$

The exact solution is given by $T(n) = \sqrt{n-4}$ if we choose $T(5) = 1$. We have $T(5) = 1 = \sqrt{5-4}, T(8) = 2T(5) = 2 = \sqrt{8-4}, T(20) = 2T(8) = 4 = \sqrt{20-4}$, etc.

□

We have the solution identical to the case when we aim to solve $T(n) = 2T(n/4)$. The constant term 3 in Eq. 1 does not make much difference if only the asymptotical behavior is all that we care.

2 Other variable changing methods

Example 2

$$T(n) = 2T(\sqrt{n}) + \lg n\,. \tag{4}$$

Let $m = \lg n$, we have $T(2^m) = 2T(2^{m/2}) + m$. We consider a function substitution by $S(m) = T(2^m)$ and the recurrence becomes

$$S(m) = 2S(m/2) + m\,. \tag{5}$$

The solution to Eq. 5 is known to be $S(m) = \Theta(m \lg m)$ when we deal with mergesort. The exact solution to Eq. 5 is given by $S(m) = m \lg m$ if we choose $S(1) = 1$. Now we have

$$T(n) = T(2^m) = S(m) = m \lg m = \lg n \lg \lg n\,. \tag{6}$$

It is the exact solution when we choose $T(2) = 0$. We have $T(4) = 2T(2) + \lg 4 = 2 = \lg 4 \lg \lg 4, T(16) = 2T(4) + \lg 16 = 8 = \lg 16 \cdot \lg \lg 16, T(256) = 2T(16) + \lg 256 = 24 = \lg 256 \cdot \lg \lg 256$, etc. It is a case of changing variable to the logarithmic scale. □

Let us consider another example which needs some modification before we can apply the variable changing and the master theorem.

Example 3

$$T(n) = \sqrt{n}T(\sqrt{n}) + n\,. \tag{7}$$

Let us divide the recurrence by n and the recurrence becomes

$$T(n)/n = T(\sqrt{n})/\sqrt{n} + 1\,. \tag{8}$$

We substitute the function $T(n)$ by another function as $S(n) = T(n)/n$ and now the recurrence becomes

$$S(n) = S(\sqrt{n}) + 1\,.$$

The rest of the story is similar to Example 2. Let $m = \lg n$ and we have

$$S(2^m) = S(2^{m/2}) + 1\,.$$

Let us substitute $S(2^m)$ by $U(m) = S(2^m)$ and the recurrence is now

$$U(m) = U(m/2) + 1\,. \tag{9}$$

We can now solve Eq. 9 by applying the case 2 of the master theorem and conclude $U(m) = \Theta(m^{\log_b a}\lg m) = \Theta(\lg m)$. In the end, we have

$$T(n) = nS(n) = 2^m S(2^m) = 2^m U(m) = 2^m \Theta(\lg m) = \Theta(n \lg\lg n)\,.$$

The exact solution is $T(n) = n\lg\lg n$ if we choose $T(2) = 0$. We also have

$$\begin{aligned}
T(4) &= \sqrt{4}T(2) + 4 = 4 = \quad 4\lg\lg 4\,,\\
T(16) &= \sqrt{16}T(\sqrt{16}) + 16 = 32 = \quad 16\lg\lg 16\,,\\
T(256) &= \sqrt{256}T(\sqrt{256}) + 256 = 16\cdot 32 + 256 = 512 = \quad 256\lg\lg 256\,,\\
&\vdots
\end{aligned}$$

Now we confirm the exact solution.

□