Fibonacci Numbers

Hsing-Kuo Pao

Department of Computer Science & Information Engineering
National Taiwan University of Science & Technology
pao@mail.ntust.edu.tw

We show a way to compute Fibonacci numbers given the background of *linear algebra*. First, the definition of Fibonacci numbers is given as follows:

$$F_{n+2} = F_{n+1} + F_n\,,\ F_0 = 0, F_1 = 1\,. \tag{1}$$

We can find out the leading terms of the Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, \ldots$. Apparently, we can easily compute Fibonacci numbers using a small piece of codes. However, we can also use mathematics to obtain the result.

Using the matrix form, we can write down the relation of Fibonacci numbers as:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}, \tag{2}$$

where the matrix M denotes the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now to compute Fibonacci numbers, we can rewrite Eq. 2 as

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = M \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M^2 \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \cdots = M^{n+1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = M^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3}$$

Let us diagonalize M as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = M = PDP^{-1}\,, \tag{4}$$

where D is obtained by finding the roots of the characteristic polynomial

$$\det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}\,, \tag{5}$$

and

$$D = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix}, \tag{6}$$

Where we use two constants to denote the two eigenvalues as

$$\varphi = \frac{1+\sqrt{5}}{2}$$

and

$$\psi = \frac{1-\sqrt{5}}{2}.$$

Now, let us rewrite Eq. 3 as

$$\begin{aligned} \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} &= M^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = PD^{n+1}P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ \psi & -\varphi \end{bmatrix} \begin{bmatrix} \varphi^{n+1} & 0 \\ 0 & \psi^{n+1} \end{bmatrix} \begin{bmatrix} -\varphi/\sqrt{5} & -1/\sqrt{5} \\ -\psi\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi} \\ \dfrac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} \end{bmatrix}. \end{aligned} \tag{7}$$

Now we have the closed-form solution given as:

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}, \tag{8}$$

if we want to compute F_n. Given the formula, a computer scientist like us should think about which way is easier to obtain the answer, either using the formula Eq. 8 or from the defintion eq. 1 with a few lines of codes.