Calculus(I) Exam 2, Nov. 28, 2019

Please show <u>all work (80%)</u> and <u>simplify solutions (20%).</u>

1. (20 points) Find $\frac{dy}{dx}$ for each of the following functions.

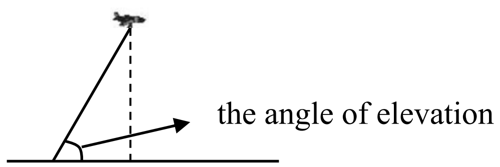
   (a) $y = 5x^4 - 3\sqrt{x}$ (b) $y = x^3 - x^2 \sin x$

   (c) $y = \sqrt{1 + \sqrt{1 + x^2}}$ (d) $\tan(x - y) = \frac{y}{1+x^2}$ at $(x, y) = (\pi, 0)$.

2. (10 points) Let $x = \sqrt{t}, y = (3t + 1)^2$, and $\frac{d^2y}{dx^2} = at + b, (a, b) \in \mathbb{Z}$. Find a and b.

3. (10 points) A water tank has the shape of an inverted circular cone with base radius $2\ m$ and height $4\ m$. If water is being pumped into the tank at a rate $2\ m^3/min$, find the rate at which the water level is rising when the water is $3\ m$ deep. (Hint: $V = \frac{1}{3}\pi r^2 h$)

4. (10 points) A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane travelling at that time? (Hint: $(\cot\theta)' = -\csc^2\theta$.)



5. (10 points) The $f(x) = x^3 - 3x^2 - 24x + 5$ is defined on $[-3, 1]$. Determine the absolute maximum and the absolute minimum where they occur in the function within the given interval.

6. (10 points) Let $f(x) = 3x^5 - 5x^3 + 3$. Identify the intervals where the function is increasing or decreasing and the intervals where the function is concave up or concave down.

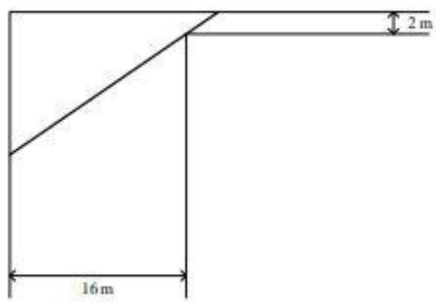
7. (10 points) Use Newton's method with the specified initial approximation x_1 to find x_3 that is the third approximation to the root of the given equation. (Reply your answer to four decimal places.)

   (a) $x^3 + 2x - 4 = 0,\ x_1 = 1$

   (b) $x^7 + 4 = 0,\ x_1 = -1$

8. (10 points) A box with a square base and an open top must have a volume of $32{,}000\ cm^3$. Find the dimensions of the box that minimize the amount of material used.

9. (10 points) A steel pipe is carried down a hallway 16 meter wide. At the end of the hall, there is a right angled turn into a narrower hallway 2 meter wide. What is the length of the longest pipe that can be carried horizontally around the corner?



Calculus(I) Exam 2 Answer, Nov. 28, 2019

1. (15 points) Find $\frac{dy}{dx}$ for each of the following functions.

   (a) $20x^3 - \frac{3}{2\sqrt{x}}$. (b) $3x^2 - 2x\sin x - x^2\cos x$. (c) $\frac{1}{2\sqrt{1+\sqrt{1+x^2}}} \cdot \frac{x}{\sqrt{1+x^2}}$

   (d) $\frac{d}{dx}(\tan(x-y)) = \frac{d}{dx}(\frac{y}{1+x^2})$

   $\sec^2(x-y)\left(1-\frac{dy}{dx}\right) = \frac{\frac{dy}{dx}(1+x^2)-y(2x)}{(1+x^2)^2} \Rightarrow 1\left(1-\frac{dy}{dx}\right) = \frac{\frac{dy}{dx}(1+\pi^2)}{(1+\pi^2)^2} \Rightarrow \frac{dy}{dx} = \frac{1+\pi^2}{2+\pi^2}$.

2. (10 points) If $x = \sqrt{t}, y = (3t+1)^2$, and $\frac{d^2y}{dx^2} = at + b, (a, b) \in \mathbb{Z}$. Find a, b.

   Sol:

   $$\frac{dx}{dt} = \frac{1}{2}t^{\frac{-1}{2}} = \frac{1}{2t^{\frac{1}{2}}}, \frac{dy}{dt} = 2(3t+1)\times 3 = 18t+6$$

   $$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = (18t+6)\left(2t^{\frac{1}{2}}\right) = 36t^{\frac{3}{2}} + 12t^{\frac{1}{2}}$$

   $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(36t^{\frac{3}{2}} + 12t^{\frac{1}{2}}\right) = \left(36\times\frac{3}{2}\times t^{\frac{1}{2}}\times\frac{dt}{dx}\right) + (12\times\frac{1}{2}\times t^{\frac{-1}{2}}\times\frac{dt}{dx})$$

   $$= \left(54\times t^{\frac{1}{2}}\times 2t^{\frac{1}{2}}\right) + \left(6\times t^{-\frac{1}{2}}\times 2t^{\frac{1}{2}}\right) = 108t + 12 \to a = 108, b = 12$$

3. (10 points) A water tank has the shape of an inverted circular cone with base radius $2\ m$ and height $4\ m$. If water is being pumped into the tank at a rate $2\ m^3/min$, find the rate at which the water level is rising when the water is $3\ m$ deep. (Hint: $V = \frac{1}{3}\pi r^2 h$)

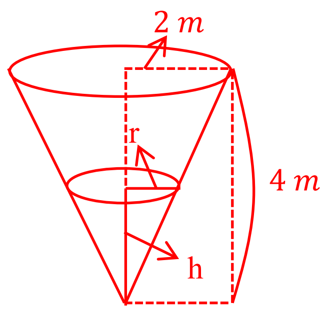
   Sol:

   If $\frac{dV}{dt} = 2\ m^3/min$, find $\frac{dh}{dt} =?$

   $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{h}{4} = \frac{r}{2}(by\ similar\ triangles) \Rightarrow r = \frac{h}{2}$,

   $V = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{1}{12}\pi h^3$,

   $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} \Rightarrow 2 = \frac{1}{4}\pi 3^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{8}{9\pi}(m/min)$.



4. (10 points) A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane travelling at that time? (Hint: $(\cot\theta)' = -\csc^2\theta$.)

Sol:

Let the angle of elevation be $\theta(t)$, and the horizontal displacement of the plane from the tracking telescope be $x(t)$, then from the figure we have

$$\tan\theta(t) = \frac{5}{x(t)}, \text{ or equivalently, } x(t) = 5\cot\theta(t)$$

And, we are given that

$$\left.\frac{d}{dt}\theta(t)\right|_{\theta=\frac{\pi}{3}} = -\frac{\pi}{6}$$

Therefore, the velocity of the plane is

$$\left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{3}} = 5\times(-\csc^2\theta)\times\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{3}} = 5\times\left(-\left(\frac{2}{\sqrt{3}}\right)^2\right)\times\left(-\frac{\pi}{6}\right) = \frac{10\pi}{9}\ (km/min).$$

(The other way)

$$\tan\theta(t) = \frac{5}{x(t)} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt} \Rightarrow 2^2\left(-\frac{\pi}{6}\right) = -\frac{5}{\left(\frac{5}{\sqrt{3}}\right)^2}\left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{3}} \Rightarrow \left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{3}} = \frac{10\pi}{9}\ (km/min).$$

5. (10 points) The $f(x) = x^3 - 3x^2 - 24x + 5$ is defined on $[-3, 1]$. Determine the absolute maximum and the absolute minimum where they occur in the function within the given interval.

Sol:

$f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x-4)(x+2)$

If $f'(x) = 0 \Rightarrow x = 4, -2.$ $\because 4 \notin [-3, 1]$ $\therefore x = -2$

x	-3	-2	1
$f(x)$	23	33	-21

$\therefore$ When $x = -2, f(x) = 33$; when $x = 1, f(x) = -21$.

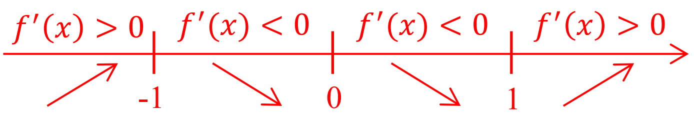
6. (10 points) Let $f(x) = 3x^5 - 5x^3 + 3$. Identify the intervals where the function is increasing or decreasing and the intervals where the function is concave up or concave down.

Sol:

$f(x) = 3x^5 - 5x^3 + 3$

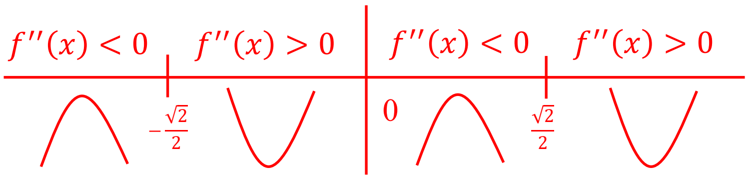
$f'(x) = 15x^4 - 15x^2$

If $f'(x) = 0$, then $15x^2 = (x^2 - 1) = 0 \Rightarrow x = 0, 1, -1$ critical number



$f''(x) = 15 \times 4 \times x^3 - 15 \times 2 \times x = 60x^3 - 30x$

If $f''(x) = 0$, then $30x(2x^2 - 1) \Rightarrow x = 0, \pm\sqrt{\frac{1}{2}} \longrightarrow \pm\frac{\sqrt{2}}{2}$



increasing: $(-\infty, -1), (1, \infty)$ concare up: $\left(-\frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, \infty\right)$.

decreasing: $(-1, 0), (1, 0)$ concare down: $\left(-\infty, -\frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}\right)$.

7. (10 points) Use Newton's method with the specified initial approximation x_1 to find x_3 that is the third approximation to the root of the given equation. (Reply your answer to four decimal places.)

(a) $x^3 + 2x - 4 = 0,\ x_1 = 1$

Sol:

$f(x) = x^3 + 2x - 4$

$f'(x) = 3x^2 + 2$

$\because x_1 = 1;\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{-1}{5} = \frac{6}{5}$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^3 + 2\left(\frac{6}{5}\right) - 4}{3\left(\frac{6}{5}\right)^2 + 2} = \frac{6}{5} - \frac{\frac{516}{126} - 4}{\frac{158}{25}} = \frac{6}{5} - \frac{16}{5 \times 158} = \frac{932}{790} \fallingdotseq 1.1797.$

(b) $x^7 + 4 = 0,\ x_1 = -1$

Sol:

$f(x) = x^7 + 4$

$f'(x) = 7x^6$

$\because x_1 = -1;\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{(-1)^7 + 4}{7(-1)^6} = -1 - \frac{3}{7} = \frac{-10}{7}$

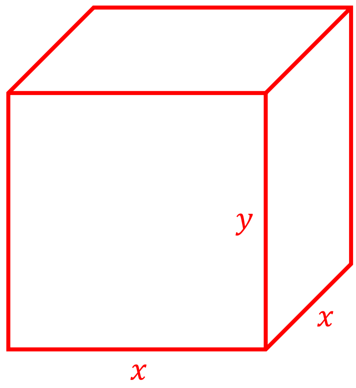
$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{-10}{7} - \frac{\left(\frac{-10}{7}\right)^7 + 4}{7\left(\frac{-10}{7}\right)^6} \fallingdotseq -1.4286 - \frac{(-1.4286)^7 + 4}{7(-1.4286)^6} \fallingdotseq 1.2917.$

8. (10 points) A box with a square base and an open top must have a volume of $32{,}000\ cm^3$. Find the dimensions of the box that minimize the amount of material used.

Sol:

$V = 32000\ cm^3 \Rightarrow x^2 y = 32000 \rightarrow y = \frac{32000}{x^2}$

$Suface\ Area = 4xy + x^2$ ⟶ target: minimize this
(without top)



$S.A = 4x \times \frac{32000}{x^2} + x^2 = \frac{128000}{x} + x^2$

$(S.A)' = \frac{-128000}{x^2} + 2x$

$\because extremum \rightarrow (S.A)' = 0$ (fermat's theorem)

$\therefore \frac{-128000}{x^2} + 2x = 0$

$\frac{-128000}{x^2} \times x^2 + 2x \times x^3 = 0 \times x^2 \rightarrow -128000 + 2x^3 = 0$

$x = \sqrt[3]{64000} = 40\ cm \rightarrow y = \frac{32000}{x^2} = \frac{32000}{40^2} = 20\ cm$

$\therefore$ minimize $S.A = 4xy + x^2 = 4 \times 40 \times 20 + 40^2 = 3200 + 1600 = 4800\ cm^2$.

The box is $40 \times 40 \times 20$.

9. (10 points) A steel pipe is carried down a hallway 16 meter wide. At the end of the hall, there is a right angled turn into a narrower hallway 2 meter wide. What is the length of the longest pipe that can be carried horizontally around the corner?

Sol:

Let the width of aisle is $l(\theta)$ where $\theta \in (0, \pi)$ is the angle between the pipe and the horizontal line. Therefore, we can have $l(\theta) = \frac{16}{\cos(\theta)} + \frac{2}{\sin(\theta)}$. We need to find the minimum of the $l(\theta)$; in this way we can find the length of the longest pipe.

$$l'(\theta) = 16 \sec(\theta)\tan(\theta) - 2\csc(\theta)\cot(\theta)$$

To find the minimum of the $l(\theta)$, we should solve $l'(\theta) = 0$.

$$l'(\theta) = 16\sec(\theta)\tan(\theta) - 2\csc(\theta)\cot(\theta) = 16\frac{\sin\theta}{\cos^2\theta} - 2\frac{\cos\theta}{\sin^2\theta} = \frac{16\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$$

$$\Rightarrow 16\sin^3\theta - 2\cos^3\theta = 0 \Rightarrow \tan^3\theta = \frac{1}{8} \Rightarrow \tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2}$$

We can check that $l''\left(\tan^{-1}\frac{1}{2}\right) > 0$. Therefore, the minimum of the $l(\theta)$ happens at $\theta = \tan^{-1}\frac{1}{2}$ and we can compute the answer $l\left(\tan^{-1}\frac{1}{2}\right) = 10\sqrt{5}(m)$.