Calculus(I) Exam 1, Oct. 17, 2019

Please show <u>all work (80%)</u> and <u>simplify solutions (20%).</u>

1. (14 points) Determine whether the statement is true or false.
   (a) If f is differentiable at a, then f is continuous at a.
   (b) If the function $f(x)$ is continuous, then $f(x)$ must be differentiable.
   (c) Suppose that $f(x)$ is a continuous function on a closed interval $[a, b]$ and $f(a)f(b) < 0$, Then there is at least one $c \in (a, b)$ such that $f(c) = 0$.
   (d) If a function $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ and $g(x) = f(x^2 - 1)$, then $g(x)$ is continuous at a point $x = 0$.
   (e) If f and g are two functions, then $f \circ g = g \circ f$.
   (f) If $f(x)$ is a continuous function and $g(x)$ is a discontinuous function, then both $f(x) + g(x)$ and $f(x) \cdot g(x)$ must be discontinuous functions.
   (g) $f(x) = \lim\limits_{n\to\infty} \frac{1+x}{1+x^{2n}}$ is continuous everywhere.
2. (10 points) Evaluate each of the following derivatives.
   (a) Given $f(x) = \frac{(x-1)(x-2)(x-3)(x-5)}{x-4}$, find $f'(1)$.
   (b) Given $g(x) = \frac{x(1+x)(2+x)\ldots(n+x)}{(1-x)(2-x)\ldots(n-x)}$, find $g'(0)$.
3. (12 points) Find each of the following limits:
   (a) $\lim\limits_{x\to0} \frac{\tan^3 3x}{x^3}$ (b) $\lim\limits_{y\to81} \frac{y-81}{3-y^{\frac{1}{4}}}$
   (c) $\lim\limits_{n\to\infty} \frac{n!}{n^n}$ (using squeeze theorem) (d) $\lim\limits_{x\to\infty} \sqrt{9x^2 + x} - 3x$.
4. (10 points) Find $f \circ g \circ h$ and its domain.
   (a) $f(x) = x + \sqrt{5 - x}$, $g(x) = [\![x]\!]$, and $h(x) = x - 1$. (Hint: $[\![\quad]\!]$ is "The greatest integer function.")
   (b) $f(x) = \sin x$, $g(x) = \frac{x}{x-1}$, and $h(x) = \sqrt[3]{x}$.
5. (12 points) There is a function $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x \neq 0 \end{cases}$ (Hint: $f'^{(x)} = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ when $x \neq 0$).
   Evaluate each of the following limits:
   (a) $\lim\limits_{x\to\infty} f(x)$; (b) $\lim\limits_{x\to0} f(x)$; (c) $f'(0)$;
   (d) Determine whether $f'(x)$ is continuous at point $x = 0$ or not?
6. (12 points) Let $f(x) = \sqrt[3]{x-3}$.
   (a) If $a \neq 3$, use the formula $f'(a) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$ to find $f'(a)$.
   (b) Show that $f'(3)$ does not exist.
7. (12 points) Evaluate the limit and justify each step by indicating the appropriate Limit Laws.
   Given $g(x) = \frac{x^2+x-6}{|x-2|}$, prove that $\lim\limits_{x\to2} g(x)$ exists or not?
8. (18 points) If $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{2x^{2n+1}-x^{2n}+ax^2+bx}{x^{2n}+1}$, find a and b to let $f(x)$ be continuous.

Calculus(I) Exam 1 Answer, Oct. 17, 2019

1. (14 points) Determine whether the statement is true or false.
   - (a) True
   - (b) False
   - (c) True
   - (d) True
   - (e) False
   - (f) False
   - (g) False
2. (10 points)
   - (a) Sol:

$$f'(1)=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{(x-2)(x-3)(x-5)}{x-4}=\frac{(1-2)(1-3)(1-5)}{1-4}=\frac{8}{3}.$$

   - (b) Sol:

$$g'(0)=\lim_{x\to0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to0}\frac{(1+x)(2+x)\dots(n+x)}{(1-x)(2-x)\dots(n-x)}=\frac{1\cdot2\cdot\dots\cdot n}{1\cdot2\cdot\dots\cdot n}=1.$$

3. (12 points)
   - (a) Sol:

$$\lim_{x\to0}\frac{\tan^3 3x}{x^3}=\lim_{x\to0}\frac{1}{x^3}\times\frac{\sin^3 3x}{\cos^3 3x}=\lim_{x\to0}\frac{27}{\cos^3 3x}\times\left(\frac{\sin^3 3x}{3x}\right)^3=27.$$

   - (b) Sol:

$$\lim_{y\to81}\frac{(\sqrt{y}+9)(\sqrt{y}-9)}{3-y^{\frac{1}{4}}}=\lim_{y\to81}\frac{(\sqrt{y}+9)\left(y^{\frac{1}{4}}+3\right)\left(y^{\frac{1}{4}}-3\right)}{\left(y^{\frac{1}{4}}-3\right)(-1)}=18\times6\times(-1)=-108.$$

   - (c) Sol:

$$0\le\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\dots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)\le\frac{1}{n}\cdot\frac{n}{n}\cdot1,$$

$$\because\ 0<\frac{2}{n}\cdot\frac{3}{n}\cdot\dots\cdot\frac{n-1}{n}<1\ \text{ and }\ \lim_{n\to\infty}\frac{1}{n}=0$$

by squeeze theorem

$$\therefore\ \lim_{n\to\infty}\frac{n!}{n^n}=0.$$

   - (d) Sol:

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\times\left(\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\right)=\lim_{x\to\infty}\frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3}=\frac{1}{6}.$$

4. (10 points)
   - (a) Sol:

$$(f\circ g\circ h)(x)=f\left(g\left(h(x)\right)\right)=f\left(g(x-1)\right)=f([\![x-1]\!])=[\![x-1]\!]+\sqrt{5-[\![x-1]\!]}$$

$$\because\ 5-[\![x-1]\!]\ge0$$

$$\therefore\quad [\![x-1]\!]\le5\ \Rightarrow x-1<6\ \Longrightarrow x<7$$

$(f\circ g\circ h)(x)$ *domain* $\Longrightarrow\{x|x<7,x\in\mathbb{R}\}$ or $(-\infty,7)$

(b) Sol:

$$(f \circ g \circ h)(x) = f\left(g\big(h(x)\big)\right) = f\left(g\big(\sqrt[3]{x}\big)\right) = f\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right) = \sin\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right).$$

$\because \sqrt[3]{x} - 1 \neq 0$

$\therefore x \neq 1$

$(f \circ g \circ h)(x)\ domain \Longrightarrow \{x | x \neq 1, x \in \mathbb{R}\}$ or $(-\infty, 1) \cup (1, \infty)$

5. (12 points)

(a) Sol:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x \sin\tfrac{1}{x}. \text{ Let } \theta = \tfrac{1}{x},\ x \to \infty \sim \theta \to 0.\ \lim_{\theta\to 0} \tfrac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\tfrac{\sin\theta}{\theta} = 1.$$

(b) Sol:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin\frac{1}{x}$$

$$\because -1 \leq \sin\frac{1}{x} \leq 1 \Rightarrow -x \leq x\sin\frac{1}{x} \leq x$$

$$\because \lim_{x\to 0} -x = 0 \text{ and } \lim_{x\to 0} x = 0.$$

$$\therefore \text{ by the squeeze theorem } \lim_{x\to 0} x\sin\tfrac{1}{x} = 0.$$

(c) Sol:

$$f'(0) = \lim_{x\to 0}\tfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\tfrac{f(x)}{x} = \lim_{x\to 0}\sin\tfrac{1}{x} \Rightarrow \text{ does not exist.}$$

(d) Sol:

$\because f'(0)$ does not exist

$\therefore f'(x)$ is discontinuous at point $x = 0$.

6. (12 points)

(a) Sol:

$$f'(a) = \lim_{x\to a}\frac{(x-3)^{\frac{1}{3}} - (a-3)^{\frac{1}{3}}}{(x-3)-(a-3)}$$

$$= \lim_{x\to a}\frac{(x-3)^{\frac{1}{3}} - (a-3)^{\frac{1}{3}}}{((x-3)^{\frac{1}{3}} - (a-3)^{\frac{1}{3}})((x-3)^{\frac{2}{3}} + (x-3)^{\frac{1}{3}}(a-3)^{\frac{1}{3}} + (a-3)^{\frac{2}{3}})}$$

$$= \lim_{x\to a}\tfrac{1}{(x-3)^{\frac{2}{3}}+(x-3)^{\frac{1}{3}}(a-3)^{\frac{1}{3}}+(a-3)^{\frac{2}{3}}}$$

$$x = a \text{ 代入 } \Rightarrow \tfrac{1}{(a-3)^{\frac{2}{3}}+(a-3)^{\frac{1}{3}}(a-3)^{\frac{1}{3}}+(a-3)^{\frac{2}{3}}} = \tfrac{1}{3}(a-3)^{\frac{-2}{3}}.$$

(b) Sol:

$$f'(3) = \lim_{x\to 3}\tfrac{f(x)-f(a)}{x-a} = \lim_{x\to 3}\tfrac{f(x)-f(3)}{x-3} = \lim_{x\to 3}(x-3)^{\frac{-2}{3}} = \lim_{x\to 3}\tfrac{1}{(x-3)^{\frac{2}{3}}}.$$

$$\because \lim_{x\to 3}\frac{1}{(x-3)^{\frac{2}{3}}} \to \infty$$

$\therefore f'(3)$ does not exist.

7. (12 points)

Sol:

$\lim_{x\to 2^+} g(x) = 5$

$\lim_{x\to 2^-} g(x) = -5$

$\because \lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

$\therefore \lim_{x\to 2} g(x)$ does not exist.

8. (18 points)

Sol:

I. $|x| < 1, \lim_{n\to\infty} x^{2n} = 0, \lim_{n\to\infty} x^{2n+1} = 0 \ \therefore f(x) = ax^2 + bx$

II. $x = 1,\ f(1) = \frac{2-1+a+b}{2} = \frac{1+a+b}{2}$

III. $x = -1,\ f(-1) = \frac{-2-1+a-b}{2} = \frac{-3+a-b}{2}$

IV. $|x| > 1,\ \frac{1}{|x|} < 1, \lim_{n\to\infty} \frac{1}{x^{2n}} = 0,\ f(x) = \lim_{n\to\infty} \frac{2x-1+\frac{a}{x^{2n-2}}+\frac{b}{x^{2n-1}}}{1+\frac{1}{x^{2n}}} = 2x - 1$

In conclusion $f(x) = \begin{cases} ax^2 + bx , & |x| < 1 \\ \frac{1+a+b}{2} , & x = 1 \\ \frac{-3+a-b}{2} , & x = -1 \\ 2x - 1 , & |x| > 1 \end{cases}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x - 1) = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax^2 + bx) = a + b$

$\lim_{x\to 1} f(x) = f(1)$, s.t. $x = 1,\ f(x)$ is continuous.

$\therefore 1 = a + b = \frac{1+a+b}{2} \ \Rightarrow\ a + b = 1$ --(A)

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2x - 1) = -3$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (ax^2 + bx) = a - b$

$\lim_{x\to -1} f(x) = f(-1),\ x = -1,\ f(x)$ is continuous.

$\therefore -3 = a - b = \frac{-3+a-b}{2} \Rightarrow a - b = -3$ --------------------------------------(B)

$\begin{cases} a + b = 1 \\ a - b = -3 \end{cases} \ \Rightarrow\ a = -1,\ b = 2.$