Calculus(II) Exam 3, June 24, 2020

Please show <u>all work (80%)</u> and <u>simplify solutions (20%).</u>

1. (10 points) Find the limit $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$, if it exists, or show that the limit does not exist.

2. (10 points) Use polar coordinates to find the limit $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

   [If (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, note that $r \longrightarrow 0^+$ as $(x,y) \to (0,0)$.]

3. (8 points) Given $f(x,y) = x^3y + e^{xy^2}$, find each of the following derivatives.

   (a) $f_x = ?$ (b) $f_y = ?$ (c) $f_{xy} = ?$ (d) $f_{yx} = ?$

4. (10 points) Given $f(x,y,z) = x^2y^3 - 4xz$, find $D_{\vec{v}}f(x,y,z)$ in the direction of $\vec{v} = \langle -1,2,0 \rangle$.

5. (12 points) Find the point on the surface $y^2 = 9 + xz$, which is closest to the origin.

6. (10 points) Evaluate the iterated integral $\int_0^1 \int_x^1 \frac{1}{1+y^2}\, dy\, dx$.

7. (12 points) Evaluate the double integral $\iint_R e^{\frac{y}{x}}\, dA$, $R = \{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

8. (12 points) Evaluate the iterated integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ via converting it to polar coordinates.

9. (16 points) Find the mass and center of mass of the lamina that occupies the region D which is bounded by $y = 1 - x^2$, $y = 0$, and has the given density function $p(x,y) = ky$.

10. (14 points) Use a triple integral to find the volume of the given solid that is enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

11. (12 points) Use cylindrical coordinates to evaluate $\iiint_E z\, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

12. (14 points) Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Calculus(II) Exam 3 Answer, June 24, 2020

1. (10%) Find the limit $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$, if it exists, or show that the limit does not exist.

Check

(1) $\lim_{x\to 0} \frac{x\times 0}{x^2+0} = 0.$

(2) $\lim_{y\to 0} \frac{0\times y^4}{0+y^4} = 0.$

(3) $y = mx \Rightarrow \lim_{x\to 0} \frac{x\times(mx)^4}{x^2+(mx)^8} = \lim_{x\to 0} \frac{m^4x^5}{x^2+m^8x^8} \overset{同\div x^2}{=} \lim_{x\to 0} \frac{m^4x^3}{1+m^8x^6} = 0.$

(4) $x = y^4 \Rightarrow \lim_{y\to 0} \frac{y^4\times y^4}{(y^4)^2+y^8} = \frac{y^8}{2y^8} = \frac{1}{2} \neq 0.$

2. (10%) Use polar coordinates to find the limit $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.[If (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, note that $r \longrightarrow 0^+$ as $(x,y) \to (0,0)$.]

$\boxed{x = r\cos\theta,\ y = r\sin\theta,\ x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2 \quad \text{極座標轉換}}$

$$\lim_{r\to 0^+} r^2\ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{r^{-2}} \Rightarrow \frac{\infty}{\infty}$$

$$= \lim_{r\to 0^+} \frac{\frac{1}{r^2}\times 2r}{-2r^{-3}} = \lim_{r\to 0^+} \frac{\frac{2}{r}}{\frac{-2}{r^3}} = \lim_{r\to 0^+} \frac{r^2}{-1} = 0.$$

3. (8%) Given $f(x,y) = x^3y + e^{xy^2}$, find each of the following derivatives.

(a) $f_x = ?$

See y as a constant

$3x^2y + e^{xy^2}\cdot y^2.$

(b) $f_y = ?$

See x as a constant

$x^3 + e^{xy^2}\cdot 2xy.$

(c) $f_{xy} = ?$

$f_x = 3x^2y + e^{xy^2}\cdot y^2$

$f_{xy} = 3x^2 + e^{xy^2}\cdot 2yx\cdot y^2 + e^{xy^2}\cdot 2y$

$= 3x^2 + 2xy^3e^{xy^2} + 2ye^{xy^2}.$

(d) $f_{yx} = ?$

$f_y = x^3 + e^{xy^2}\cdot 2xy$

$f_{yx} = 3x^2 + e^{xy^2}\cdot y^2\cdot 2xy + e^{xy^2}\cdot 2y$

$= 3x^2 + 2xy^3e^{xy^2} + 2ye^{xy^2}.$

4. (10%) Given $f(x,y,z) = x^2y^3 - 4xz$, find $D_{\vec{u}}f(x,y,z)$ in the direction of $\vec{v} = \langle -1,2,0 \rangle$.

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle = \langle 2xy^3 - 4z, 3x^2y^2, -4x \rangle$$

$$u = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \rangle \longrightarrow |\vec{v}| = \sqrt{(-1)^2 + 2^2 + 0^2} = \sqrt{5}$$

$$\Rightarrow D_{\vec{u}}f(x,y,z) = \nabla f(x,y,z) \cdot u = \langle 2xy^3 - 4z, 3x^2y^2, -4x \rangle \cdot \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \rangle$$

$$= \tfrac{-1}{\sqrt{5}}(2xy^3 - 4z) + \tfrac{2}{\sqrt{5}}(3x^2y^2) = \tfrac{-2}{\sqrt{5}}\ xy^3 + \tfrac{4}{\sqrt{5}}z + \tfrac{6}{\sqrt{5}}x^2y^2.$$

5. (12%) Find the point on the surface $y^2 = 9 + xz$, which is closest to the origin.

$\Rightarrow$ Distance formula: $d^2 = x^2 + y^2 + z^2$

$$\Rightarrow d^2 = f = x^2 + 9 + xz + z^2 \Rightarrow \begin{array}{l} f_x = 2x + z \\ f_z = x + 2z \end{array}$$

$$\text{Set } f_x = 0 \Rightarrow \begin{array}{l} 0 = 2x + z \\ z = -2x \end{array}, \text{ Set } f_z = 0 \Rightarrow \begin{array}{l} 0 = x + 2z \\ x = -2z \end{array}$$

$$\Rightarrow \begin{array}{l} z = -2(-2z) \\ z = 4z \\ -3z = 0 \end{array} \Rightarrow \begin{array}{l} z = 0 \\ x = 0 \end{array} \Rightarrow y^2 = 9 \Rightarrow \mathrm{y} = \pm 3$$

The closest points are $\begin{array}{l} (0,3,0) \\ (0,-3,0) \end{array}$

6. (10%) Evaluate the iterated integral $\int_0^1 \int_x^1 \frac{1}{1+y^2}\, dy\, dx$.

Sol 1:

類似課本 12-2 Ex5，要用到順序對調

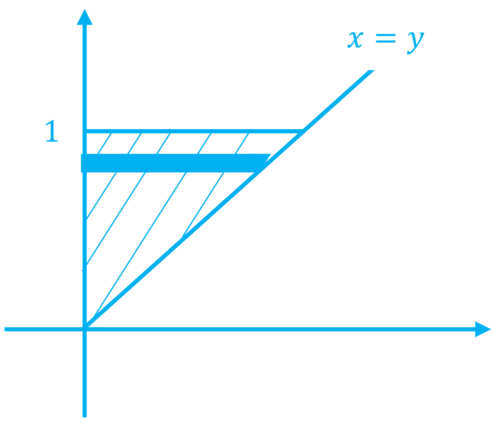
$$\int_0^1 \int_x^1 \frac{1}{1+y^2}\, dy\, dx$$

$$= \int_0^1 \frac{1}{1+y^2} x \Big|_0^y dy = \int_0^1 \frac{y}{1+y^2} dy$$

令 $u = y^2$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+y^2} dy^2 = \frac{1}{2}\int_0^1 \frac{1}{1+u} dy^2$$

$$= \tfrac{1}{2} \cdot \ln(1+u) \Big|_0^1 = \tfrac{1}{2} \cdot \ln 2.$$



Sol 2:

直接算

$$\int_0^1 \int_x^1 \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \tan^{-1} y \Big|_x^1 dx = \int_0^1 \left(\frac{\pi}{4} - \tan^{-1} x\right) dx$$

$$= \frac{\pi}{4} - \int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \left\{x \tan^{-1} x - \frac{1}{2}\ln|1+x^2|\right\}_0^1$$

$$= \frac{1}{2}\ln 2.$$

7. (12%) Evaluate the double integral $\iint_R e^{\frac{y}{x}}\,dA,\ R = \{(x,y)|0 \le x \le 1, 0 \le y \le x^2\}$.

$$\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx = \int_0^1 x e^{\frac{y}{x}} \Big|_0^{x^2} dx = \left(xe^x - e^x - \frac{e^x}{2}\right)_0^1$$

$$= e - e - \frac{1}{2} - (0 - 1 - 0) = \frac{1}{2}.$$

8. (12%) Evaluate the iterated integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$ via converting it to polar coordinates.

令 $x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2,\ dxdy = rdrd\theta$

$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 e^{r^2}\,rdrd\theta = \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 e^{r^2}\,rdr = \left(\frac{\pi}{2} - 0\right) \cdot \frac{1}{2} \int_0^1 e^{r^2}\,dr^2$$

$$= \frac{\pi}{4} e^{r^2}\Big|_0^1 = \frac{\pi}{4}(e-1).$$

9. (16%) Find the mass and center of mass of the lamina that occupies the region D which is bounded by $y = 1 - x^2$, $y = 0$, and has the given density function $p(x, y) = ky$.

When $1 - x^2 = 0$ then $x = -1$ or 1, so intersection point are $(-1,0)$ and $(1,0)$ define D as $(x, y) \in D | -1 \le x \le 1, 0 \le y \le 1 - x^2$

$\Rightarrow$ mass lamina is $m = \iint_D \rho(x,y)\, dA = \int_{-1}^{1} \int_0^{1-x^2} ky\, dy\, dx = \int_{-1}^{1} k \left[\frac{y^2}{2}\right]_0^{1-x^2} dx$

$$= \frac{k}{2} \int_{-1}^{1} (x^2 - 1)^2\, dy = \frac{k}{2} \int_{-1}^{1} x^4 - 2x^2 + 1\, dx = \frac{k}{2} \left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right]_{-1}^{1}$$

$$= k\left[\frac{1}{5} - \frac{2}{3} + 1\right] = \frac{8k}{15}$$

$\Rightarrow \bar{x} = \frac{My}{m}$ $My = \iint_D x\rho(x,y)\, dA = \int_{-1}^{1} \int_0^{1-x^2} kxy\, dy\, dx = \frac{k}{2}\left[\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^2}{2}\right]_{-1}^{1} = 0.$

$\Rightarrow \bar{x} = \frac{My}{m} = \frac{0}{\frac{8k}{15}} = 0.$

$\bar{y} = \frac{Mx}{m}$ $Mx = \iint_D y\rho(x,y)\, dA = \int_{-1}^{1} \int_0^{1-x^2} ky^2\, dy\, dx = \frac{k}{3} \int_{-1}^{1} (1 - x^2)^3 dx$

$$= \frac{k}{3} \int_{-1}^{1} 1^3 + (-x^2)^3 + 3(1)^2(-x^2) + 3(1)(-x^2)^2\, dx$$

$$= \frac{2k}{3}\left[\frac{-5+21}{35}\right] = \frac{32k}{105}.$$

$\bar{y} = \frac{Mx}{m} = \frac{\frac{32k}{105}}{\frac{8k}{15}} = \frac{4}{7}.$

$\Longrightarrow$ mass of the lamina: $\frac{8k}{15}$

center mass: $\left(0, \frac{4}{7}\right)$

10. (14%) Use a triple integral to find the volume of the given solid that is enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

Planes $y + z = 1$ and $z = 0$, so intersect in xy plane along the line $y = 1$

$\Rightarrow$ region E: $\{(x, y, z) \in E | 0 \le z \le 1 - y, x^2 \le y \le 1, -1 \le x \le 1\}$

$$\Rightarrow V = \iiint_E dV = \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} dz\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} 1 - y\, dy\, dx$$

$$= \int_{-1}^{1} \frac{1}{2} - x^2 + \frac{x^4}{2} dx = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{8}{15}.$$

11. (12%) Use cylindrical coordinates to evaluate $\iiint_E z\,dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

$$\iiint_E z\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} z\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r\right]_{r^2}^{4} dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[8r - \frac{1}{2}r^5\right] dr\,d\theta$$

$$= \int_0^{2\pi}\left[4r^2 - \frac{1}{12}r^6\right]_0^2 d\theta = \int_0^{2\pi}\left[16 - \frac{16}{3}\right] d\theta = \int_0^{2\pi}\frac{32}{3}\,d\theta = \left[\frac{32\theta}{3}\right]_0^{2\pi} = \frac{64\pi}{3}.$$

12. (14%) Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

$$x^2 + y^2 = 1 \Rightarrow r^2 = 1,\ x^2 + y^2 + z^2 = 4 \Rightarrow r^2 + z^2 = 4 \Rightarrow z = \pm\sqrt{4 - r^2}$$

$$\Rightarrow V = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{4 - r^2}\,dr\,d\theta$$

$\Rightarrow$ 令 $\begin{array}{l} u = 4 - r^2 \\ du = -2r dr \end{array}$ $\Rightarrow$ $\begin{array}{l} V = \int_0^{2\pi}\int_4^3 -\sqrt{u}\,du\,d\theta = \int_0^{2\pi}\int_3^4 \sqrt{u}\,du\,d\theta = \int_0^{2\pi}\frac{16}{3} - 2\sqrt{3}\,d\theta \\ V = \int_0^{2\pi}\frac{16}{3} - 2\sqrt{3}\,d\theta = \frac{32}{3}\pi - 4\pi\sqrt{3} \end{array}$