Calculus(I) Exam 3, Jan. 08, 2020

Please show <u>all work (80%)</u> and <u>simplify solutions (20%).</u>

If the score exceeds 100, it is counted as 100.

1. (30 points) Determine whether the statement is true or false.
   (a) Some continuous functions do not have antiderivatives.
   (b) The left Riemann sum of a continuous function $f(x)$ is always less than or equal to its right Riemann sum.
   (c) A function is one-to-one if and only if no horizontal line intersects its graph more than once.
   (d) If f is continuous on $[a, b]$, then $\int_a^b xf(x)dx = x\int_a^b f(x)dx$.
   (e) If $0 < a < b$, then $\ln a < \ln b$.
   (f) $\sinh 2x = 2\sinh x \cosh x$.
   (g) $\cosh x + \sinh x = e^{2x}$.
   (h) We know that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, and we can evaluate the integral $\int \frac{1+x}{1+x^2}dx$ is equal to $\tan^{-1}(x) + \frac{\ln|1+x^2|}{2} + c$.
   (i) Because $x^2 + 1 = (x)^2 + (1)^2$, we can know there is an angle θ on the triangle that has three edges are $\sqrt{x^2+1}$, x and 1. Hence, we can get $\sin\tan^{-1} x$ is equal to $\frac{\sqrt{x^2+1}}{x}$.
   (j) By L' Hospital's Rule, we can know that $\lim_{x\to\infty}\frac{e^x}{x^2} = \infty$.
2. (15 points) Evaluate each of the following integrals:
   (a) $\int \frac{e^x}{1+e^x}dx$ (b) $\int \frac{1}{1+e^x}dx$ (c) $\int \frac{1}{(1+e^x)^2}dx$
3. (15 points) Evaluate each of the following questions.
   (a) (7 points) If $f(x) = 2x^3 + 3x^2 + 7x + 4$, $a = 4$, find $(f^{-1})'(a)$.
   (b) (8 points) If $f(x) = \frac{4x-1}{2x+3}$, find a formula for the inverse of the function.
4. (20 points) Find each of the following limits:
   (a) $\lim_{x\to\infty}\frac{\ln(3x)}{x^2}$ (b) $\lim_{\theta\to 0}\frac{\theta - \sin\theta}{\theta^3}$ (c) $\lim_{x\to\infty}\frac{2^x}{e^{x^2}}$
   (d) $\lim_{x\to\infty}(\cosh^{-1} x - \ln x)$.
5. (15 points) Evaluate each of the following differentials.
   (a) (9 points) $\frac{d}{dx}(a^{a^x} + a^{x^a} + x^{x^a})$.
   (b) (6 points) $\frac{d}{dx}\log_{10}(\sec x + \tan x)$.
6. (15 points) Evaluate the following integrals:
   (a) $\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{\sin x}{x^2+2}dx$ (b) $\int_0^{\frac{1}{2}}\frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$ (c) $\int_0^1 10^{3x}\,dx$.
7. (10 points) Suppose I have \$1,000 in savings.
   (a) How much will I have in 10 years if the interest rate is 5%?
      (i) Simple interest.
      (ii) Compounded annually.
      (iii) Compounded monthly.

      (Note that for (ii)&(iii) the answer is written in this format: $1000 \times \square^{\square}$.)
   (b) How long should I take to double my savings under:
      (i) Simple interest.
      (ii) Compounded annually interest.

      (Note that the answer is rounded to the nearest integer.)

   (Hint: $\ln 0.95 = -0.051, \ln 1.05 \fallingdotseq 0.049, \ln 1.95 \fallingdotseq 0.668$, $\ln 2 \fallingdotseq 0.693, \ln 2.05 \fallingdotseq 0.718$)

Calculus(I) Exam 3 Answer, Jan. 08, 2020

1. (30 points) Determine whether the statement is true or false.

F (a) Some continuous functions do not have antiderivatives.

F (b) The left Riemann sum of a continuous function $f(x)$ is always less than or equal to its right Riemann sum.

T (c) A function is one-to-one if and only if no horizontal line intersects its graph more than once.

F (d) If f is continuous on $[a, b]$, then $\int_a^b xf(x)dx = x\int_a^b f(x)dx$.

T (e) If $0 < a < b$, then $\ln a < \ln b$.

T (f) $\sinh 2x = 2\sinh x \cosh x$

F (g) $\cosh x + \sinh x = e^{2x}$.

T (h) When we know that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and we can evaluate the integral $\int \frac{1+x}{1+x^2}dx$ is equal to $\tan^{-1}(x) + \frac{\ln|1+x^2|}{2} + c$.

F (i) Because $x^2 + 1 = (x)^2 + (1)^2$, we can know there is an angle θ on the triangle that has three edges are $\sqrt{x^2+1}$, x and 1. Hence, we can get $\sin\tan^{-1} x$ is equal to $\frac{\sqrt{x^2+1}}{x}$.

T (j) By L' Hospital's Rule, we can know that $\lim_{x\to\infty}\frac{e^x}{x^2} = \infty$.

2. (15 points) Evaluate each of the following integrals:

(a) Let $u = 1 + e^x$, and $du = e^x dx$,

then $\int \frac{e^x}{1+e^x}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|1 + e^x| + c$.

(b) $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx = \int \frac{1+e^x}{1+e^x}dx - \int \frac{e^x}{1+e^x}dx = \int 1\,dx - \int \frac{e^x}{1+e^x}dx = x - \ln|1 + e^x| + c$.

(c) $\int \frac{1}{(1+e^x)^2}dx = \int \frac{1+e^x-e^x}{(1+e^x)^2}dx = \int \frac{1+e^x}{(1+e^x)^2}dx - \int \frac{e^x}{(1+e^x)^2}dx = \int \frac{1}{1+e^x}dx - \int \frac{e^x}{(1+e^x)^2}dx$

$$\boxed{\begin{array}{l}\displaystyle\int \frac{e^x}{(1+e^x)^2}dx \\ \Rightarrow \text{Let } u = 1 + e^x, du = e^x dx, \text{ then } \int \frac{e^x}{(1+e^x)^2}dx = \int \frac{1}{u^2}du = -u^{-1} + c = -\frac{1}{1+e^x} + c\end{array}}$$

$\because \int \frac{1}{1+e^x}dx = x - \ln|1 + e^x| + c$, and $\int \frac{e^x}{(1+e^x)^2}dx = -\frac{1}{1+e^x} + c$

$\therefore \int \frac{1}{(1+e^x)^2}dx = x - \ln|1 + e^x| - (-\frac{1}{1+e^x}) + C = x - \ln|1 + e^x| + \frac{1}{1+e^x} + C$.

3. (15 points) If $f(x) = 2x^3 + 3x^2 + 7x + 4$, $a = 4$, find $(f^{-1})'(a)$

(a) Sol:

$\because f'(x) = 6x^2 + 6x + 7. > 0 \Rightarrow$ One-to-one increasing function.

$\because f(0) = 4 \quad \therefore f^{-1}(4) = 0$

$\because f'(f^{-1}(4)) = f'(0) = 7 \neq 0$

$\therefore$ By Theorem 7 $\Rightarrow (f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{7}$.

(b) Sol:

According textbook 5-1 $\boxed{5}$

$$y=\frac{4x-1}{2x+3}\Rightarrow(2x+3)y=4x-1\Rightarrow 3y+1=x(4-2y)\Rightarrow x=\frac{3y+1}{4-2y}$$

Interchange x and y $\Rightarrow y=\frac{3x+1}{4-2x}$ $\therefore f^{-1}(x)=\frac{3x+1}{4-2x}$, $x\neq 2$.

4. (20 points) Find each of the following limits:

(a) Sol:

$$\because \lim_{x\to\infty}\frac{\ln(3x)}{x^2}\to\frac{\infty}{\infty}\quad\therefore \text{ By L' Hospital's Rule }\Rightarrow \lim_{x\to\infty}\frac{\ln(3x)}{x^2}=\lim_{x\to\infty}\frac{\frac{1}{3x}\times 3}{2x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=0.$$

(b) Sol:

$$\because \lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}\to\frac{0}{0}\quad\therefore \text{ By L' Hospital's Rule }\Rightarrow \lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=\lim_{\theta\to 0}\frac{1-\cos\theta}{3\theta^2}=\frac{1}{3}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$$

$$\Rightarrow\ \because \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\to\frac{0}{0}\quad\therefore \text{ By L' Hospital's Rule }\Rightarrow \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\lim_{\theta\to 0}\frac{\sin\theta}{2\theta}=\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.$$

$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\to\frac{0}{0}\quad\therefore \text{ By L' Hospital's Rule }\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\lim_{\theta\to 0}\frac{\cos\theta}{1}=1$$

$$\therefore \lim_{\theta\to 0}\frac{\theta-\sin\theta}{\theta^3}=\frac{1}{3}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\frac{1}{3}\times\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{1}{3}\times\frac{1}{2}\times 1=\frac{1}{6}.$$

(c) Sol:

This question should not use L' Hospital's Rule.

Because $\lim_{x\to\infty}\frac{2^x}{e^{x^2}}\to\frac{\infty}{\infty}\overset{L}{=}\lim_{x\to\infty}\frac{2^x\ln 2}{(2x)(e^{x^2})}\to\frac{\infty}{\infty}$ …….. will endless L' Hospital's.

$$\lim_{x\to\infty}\frac{2^x}{e^{x^2}}=\lim_{x\to\infty}\frac{e^{\ln 2^x}}{e^{x^2}}=\lim_{x\to\infty}\frac{e^{x\ln 2}}{e^{x^2}}=\lim_{x\to\infty}e^{x\ln 2-x^2}=\lim_{x\to\infty}e^{x(\ln 2-x)}\to e^{-\infty}=0.$$

(d) Sol:

$$\because \cosh^{-1}x=\ln(x+\sqrt{x^2-1})$$

(ch 5-7 $\boxed{4}$)

$$\therefore \lim_{x\to\infty}(\cosh^{-1}x-\ln x)=\lim_{x\to\infty}\left(\ln\left(x+\sqrt{x^2-1}\right)-\ln x\right)=\lim_{x\to\infty}\ln\frac{x+\sqrt{x^2-1}}{x}$$

$$=\lim_{x\to\infty}\ln\left|1+\sqrt{1-\frac{1}{x^2}}\right|=\ln 2.$$

Prove: $\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$

pf. By definition of the Hyperbolic function

$$\cosh x=\frac{e^x+e^{-x}}{2}$$

Let $y=\frac{e^x+e^{-x}}{2}\Rightarrow 2y=e^x+\frac{1}{e^x}\Rightarrow(e^x)^2-2ye^x+1=0$

$e^x=\frac{2y\pm\sqrt{4y^2-4}}{2}\Rightarrow e^x=y\pm\sqrt{y^2-1}$ $(e^x>0)$

x,y change $\Rightarrow e^y=x+\sqrt{x^2-1}\Rightarrow y=\ln(x+\sqrt{x^2-1})=\cosh^{-1}x$

5. (15 points) Evaluate each of the following differentials.

(a) Sol:

$$\Rightarrow \frac{d}{dx}a^{a^x} = \frac{d}{dx}e^{\ln a^{a^x}} = \frac{d}{dx}e^{a^x \ln a} = e^{a^x \ln a} \times \frac{d}{dx}a^x \ln a = a^{a^x} \times \ln a \times \frac{d}{dx}a^x$$
$$= a^{a^x} \times \ln a \times a^x \times \ln a = a^{a^x} \times a^x \times (\ln a)^2.$$

$$\Rightarrow \frac{d}{dx}a^{x^a} = \frac{d}{dx}e^{\ln a^{x^a}} = e^{\ln a^{x^a}} \times \frac{d}{dx}\left(\ln a^{x^a}\right) = a^{x^a} \times \frac{d}{dx}(x^a \ln a) = a^{x^a} \times ax^{a-1} \times \ln a.$$

$$\Rightarrow \frac{d}{dx}x^{x^a} = \frac{d}{dx}e^{\ln x^{x^a}} = e^{\ln x^{x^a}} \times \frac{d}{dx}\left(\ln x^{x^a}\right) = x^{x^a} \times \frac{d}{dx}(x^a \ln x)$$

$$= x^{x^a} \times (ax^{a-1} \times \ln a + x^a \times \tfrac{1}{x}).$$

$$\therefore \frac{d}{dx}\left(a^{a^x} + a^{x^a} + x^{x^a}\right)$$
$$= a^{a^x} \times a^x \times (\ln a)^2 + aa^{x^a} \times x^{a-1} \times \ln a + x^{x^a}(ax^{a-1} \times \ln a + x^{a-1}).$$

(b) Sol:

$$= \frac{d}{dx}\left(\frac{\ln(\sec x + \tan x)}{\ln 10}\right) = \frac{1}{\ln 10}\frac{d}{dx}\ln(\sec x + \tan x) = \frac{1}{\ln 10} \times \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{1}{\ln 10} \times \frac{\sec x(\sec x \tan x)}{\sec x + \tan x} = \frac{\sec x}{\ln 10}.$$

6. (15 points) Evaluate the following integrals:

(a) Sol:

Let $f(x) = \frac{\sin x}{x^2+2}$, then $f(-x) = \frac{\sin(-x)}{(-x)^2+2} = \frac{-\sin x}{x^2+2} = -\left(\frac{\sin x}{x^2+2}\right) = -f(x)$

$\because f$ is odd function.

$\therefore$ By ch 4-5 $\boxed{6}$, If f is odd function, then $\int_{-a}^{a} f(x)\,dx = 0 \Rightarrow \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sin x}{x^2+2}dx = 0.$

(b) Sol:

Let $u = \sin^{-1} x\,, du = \frac{1}{\sqrt{1-x^2}}dx$

$$\Rightarrow \int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{6}} u du = \frac{1}{2}u^2|_0^{\frac{\pi}{6}} = \frac{1}{2} \times \frac{\pi^2}{36} = \frac{\pi^2}{72}.$$

(c) Sol:

$$\int_0^1 10^{3x}\,dx = \int_0^1 e^{\ln 10^{3x}}\,dx = \int_0^1 e^{3x \ln 10}\,dx$$

Let $u = 3x \ln 10\,, du = 3 \ln 10\,dx$

$$\Rightarrow \int_0^1 e^{3x \ln 10}\,dx = \int_0^1 e^{3x \ln 10} \times \frac{3 \ln 10}{3 \ln 10}dx = \frac{1}{3 \ln 10}\int_0^{3 \ln 10} e^u\,du = \frac{1}{3 \ln 10} \times e^u|_0^{3 \ln 10}$$

$$= \frac{1}{3 \ln 10} \times \left(e^{3 \ln 10} - 1\right) = \frac{1}{3 \ln 10} \times (1000 - 1) = \frac{333}{\ln 10}.$$

7. (10 points) Suppose I have $\$1,000$ in savings.

(a) How much will I have in 10 years if the interest rate is 5%?

(Note that for (ii)&(iii) the answer is written in this format: $1000 \times \square^{\square}$.)

(i) Sol:

$$A = 1000 \times (1 + 0.05 \times 10) = 1000 \times (1.5) = 1500.$$

(ii) Sol:

By the formula.

$$A(t) = A_0(1 + \frac{r}{n})^{nt} \Rightarrow A(10) = 1000 \times (1 + \frac{0.05}{1})^{1\times 10} = 1000 \times (1.05)^{10}.$$

(iii) Sol:

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt} \Rightarrow A(10) = 1000 \times \left(1 + \frac{0.05}{12}\right)^{12\times 10} \fallingdotseq 1000 \times (1.004167)^{120}.$$

(b) How long should I take to double my savings under:

(i) Sol:

Double to $\$2,000$.

$2000 = 1000 \times (1 + 0.05t)$

$$\Rightarrow 2 = 1 + 0.05t \Rightarrow t = \frac{1}{0.05} = 20 \text{ years.}$$

(ii) Sol:

Double to $\$2,000$.

Annual: $A(t) = 1000 \times (1 + 0.05)^t$

$2000 = 1000 \times (1.05)^t$

$\Rightarrow 2 = 1.05^t \Rightarrow \ln 2 = \ln 1.05^t \Rightarrow \ln 2 = t \ln 1.05$

$$\Rightarrow t = \frac{\ln 2}{\ln 1.05} = \frac{0.693}{0.049} \fallingdotseq 14 \text{ years.}$$

(Hint: $\ln 0.95 = -0.051, \ln 1.05 \fallingdotseq 0.049, \ln 1.95 \fallingdotseq 0.668$, $\ln 2 \fallingdotseq 0.693, \ln 2.05 \fallingdotseq 0.718$)

(Note that the answer is rounded to the nearest integer.)