Calculus(II) Exam 1, April 08, 2020

Please show <u>all work (80%)</u> and <u>simplify solutions (20%).</u>

1. (20 points) Evaluate each of the following integrals.

   (a) $\int \frac{1}{\sqrt{9x^2+6x-8}}\,dx$; (b) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$;

   (c) $\int \sin(\ln x)\,dx$; (d) $\int \sin^{-1} x\,dx$.

2. (10 points) Determine each of the following integrals converges or diverges? If the integral converges, find its value.

   (a) $\int_0^4 \frac{x}{x^2-9}\,dx$; (b) $\int_0^3 \frac{dx}{\sqrt[3]{(x-1)^2}}$.

3. (6 points) Find the area of the region enclosed by $y = \cos x$ and $y = \sin 2x$ for $0 \le x \le \frac{\pi}{2}$.

4. (6 points) Find the number a such that the line $x = a$ bisects the area under the curve $y = \frac{1}{x^2}$ for $1 \le x \le 4$.

5. (8 points) Find the number b such that the line $y = b$ bisects the area under the curve $y = \frac{1}{x^2}$ for $1 \le x \le 4$.

6. (12 points) Find the volume of the solid obtained from rotating the region bounded by the given curves about the specified line.

   (a) $y = \frac{1}{4}x^2, x = 2, y = 0$, about the y-axis;

   (b) $y = 0, y = x^2, x = 0, x = 1$, about the x-axis.

7. (14 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

   (a) $x = y^2 + 1, x = 2$, about $y = -2$;

   (b) $y = e^{-x^2}, y = 0, x = 0, x = 1$, about the y-axis.

8. (8 points) Use the cylindrical shells to find the volume of solid: a right circular cone with height h and base radius r.

9. (16 points) Find the exact length of each of the following curves.

   (a) $y = \ln(\sec(x)), 0 \le x \le \frac{\pi}{4}$; (b) $x = \frac{y^4}{8} + \frac{1}{4y^2}, 1 \le y \le 2$.

10. (20 points) Find the area of the surface obtained from rotating each of the following curves about the specified axis.

    (a) $x^2 + y^2 = r^2$, about the line $y = r$;

    (b) $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}, 1 \le y \le 2$, about the x-axis.

Calculus(II) Exam 1 Answer, April 08, 2020

1. (20 points) Evaluate each of the following integrals.

(a) $\int \frac{1}{\sqrt{9x^2+6x-8}} dx = \int \frac{1}{\sqrt{9x^2+6x+1-9}} dx = \int \frac{1}{\sqrt{(3x+1)^2-9}} dx$

Let $u = 3x+1, du = 3dx \Rightarrow \frac{1}{3}\int \frac{1}{\sqrt{u^2-3^2}} dx$

Use: $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + c;$

So: $\frac{1}{3}\int \frac{1}{\sqrt{u^2-3^2}} dx = \frac{1}{3}\ln\left|3x+1+\sqrt{u^2-a^2}\right| + \mathrm{c} = \frac{1}{3}\ln\left|3x+1+\sqrt{(3x+1)^2-9}\right| + \mathrm{c}.$

(b) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2} dx \Rightarrow \frac{x^3+6x^2+3x+6}{x^3+2x^2} = \frac{x^3+2x^2}{x^3+2x^2} + \frac{4x^2+3x+6}{x^3+2x^2} = 1 + \frac{4x^2+3x+6}{x^2(x+2)} = 1 + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)}.$

$\Rightarrow Ax(x+2) + B(x+2) + Cx^2 = 4x^2+3x+6$

$\Rightarrow A(x^2+2x) + B(x+2) + Cx^2 = 4x^2+3x+6$

$\begin{cases} A+C=4 \\ 2A+B=3 \\ 2B=6 \end{cases} \Longrightarrow \begin{cases} A=0 \\ B=3 \\ C=4 \end{cases}$

$\therefore \int \frac{x^3+6x^2+3x+6}{x^3+2x^2} dx = \int \left(1+\frac{3}{x^2}+\frac{4}{x+2}\right) dx = x + 3(-1)\frac{1}{x} + 4\ln|x+2| + c$

$= x - \frac{3}{x} + 4\ln|x+2| + c.$

(c) $\int \sin(\ln x)\, dx$

Let $u = \sin(\ln x), du = \cos(\ln x)\frac{1}{x}dx, dv = dx, v = x$

$= x\sin(\ln x) - \int x\frac{1}{x}\cos(\ln x)\, dx = x\sin(\ln x) - \int \underline{\cos(\ln x)\, dx}$

$u = \cos(\ln x), du = -\sin(\ln x)\frac{1}{x}\, dx, dv = dx, v = x$

$\cos(\ln x)x + \int x\frac{1}{x}\sin(\ln x)\, dx = \cos(\ln x)x + \int \sin(\ln x)\, dx.$

$\Rightarrow \int \sin(\ln x)\, dx = x\sin(\ln x) - \left(\cos(\ln x)x + \int \sin(\ln x)\, dx\right)$

$\Rightarrow 2\int \sin(\ln x)\, dx = x\sin(\ln x) - x\cos(\ln x)$

$\Rightarrow \int \sin(\ln x)\, dx = \frac{1}{2}[x\sin(\ln x) - x\cos(\ln x)] + \mathrm{c}.$

(d) $\int \sin^{-1} x\, dx$

Let $u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}}dx, dv = dx, v = x$

$= \sin^{-1} x\, x - \int \underline{x\frac{1}{\sqrt{1-x^2}} dx}$

Let $\mathrm{t} = 1-x^2, dt = -2xdx$ (change all x by t)

$\int x\frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{1}{\sqrt{t}} dt = -\frac{1}{2}\int t^{\frac{-1}{2}} dt = -\frac{1}{2} \times \frac{1}{\frac{1}{2}} \times t^{\frac{1}{2}} + c = -t^{\frac{1}{2}} + c = -(1-x^2)^{\frac{1}{2}} + c.$

$= x\sin^{-1} x + \sqrt{1-x^2} + c.$

2. (10 points) Determine each of the following integrals converges or diverges? If the integral converges, find its value

(a) $\int_0^4 \frac{x}{x^2-9}dx$

Singularity: $x = 3$

$$= \int_0^3 \frac{x}{x^2-9}dx + \int_3^4 \frac{x}{x^2-9}dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}dx + \lim_{t\to 3^+}\int_0^4 \frac{x}{x^2-9}dx$$

$$\because \int \frac{x}{x^2-9}dx = \frac{1}{2}\int \frac{1}{x^2-9}d(x^2-9) = \frac{1}{2}\ln|x^2-9| + c$$

$$\therefore \int_0^4 \frac{x}{x^2-9}dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln|x^2-9|\right)|_0^t + \lim_{s\to 3^+}\left(\frac{1}{2}\ln|x^2-9|\right)|_s^4$$

$$= \lim_{t\to 3^-}\left(\frac{1}{2}\ln|t^2-9| - \frac{1}{2}\ln 9\right) + \lim_{s\to 3^+}\left(\frac{1}{2}\ln 7 - \frac{1}{2}\ln|s^2-9|\right)$$

$$= \left[-\infty - \tfrac{1}{2}\ln 9\right] + \left[\tfrac{1}{2}\ln 7 + \infty\right].$$ Answer: Diverges.

(b) $\int_0^3 \frac{dx}{\sqrt[3]{(x-1)^2}}$

Singularity: $x = 1$

$$= \lim_{t\to 1^-}\int_0^t (x-1)^{\frac{-2}{3}}dx + \lim_{t\to 1^+}\int_s^3 (x-1)^{\frac{-2}{3}}dx$$

$$= \lim_{t\to 1^-} 3(x-1)^{\frac{1}{3}}|_0^t + \lim_{s\to 1^+} 3(x-1)^{\frac{1}{3}}|_s^3$$

$$= 3[0-(-1)] + 3\left[2^{\frac{1}{3}} - 0\right] = 3 + 3\cdot\sqrt[3]{2}.$$

3. (6 points) Find the area of the region enclosed by $y = \cos x$ and $y = \sin 2x$ for $0 \le x \le \frac{\pi}{2}$.

$\Rightarrow \cos x = \sin 2x = 2\sin x\cos x \quad \cos x\,(2\sin x - 1) = 0$ or $\cos x = 0, when\ x = \frac{\pi}{2}$; $\sin x = \frac{1}{2}, when\ x = \frac{\pi}{6}$

$$\int_0^{\frac{\pi}{6}} \cos x - \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x\,dx = \sin x + \frac{1}{2}\cos 2x]_0^{\frac{\pi}{6}} + \left(\frac{-1}{2}\cos 2x - \sin x\right)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{4} - 0 - \tfrac{1}{2}\right) + \left(\tfrac{-1}{2}(-1) - 1 + \tfrac{1}{4} + \tfrac{1}{2}\right) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

4. (6 points) Find the number a such that the line $x = a$ bisects the area under the curve $y = \frac{1}{x^2}$ for $1 \le x \le 4$.

$y = \frac{1}{x^2}$, Let $x = k$. Line bisects the area under the curve $y = \frac{1}{x^2}$

$$\Rightarrow so \int_1^k \frac{1}{x^2} dx = \int_k^4 \frac{1}{x^2} dx \Rightarrow \left[\frac{1}{-x}\right]_1^k = \left[\frac{1}{-x}\right]_k^4$$

$\Rightarrow \frac{1}{-k} + 1 = \frac{1}{-4} + \frac{1}{k} \Rightarrow k = \frac{8}{5}$.

5. (8 points) Find the number b such that the line $y = b$ bisects the area under the curve $y = \frac{1}{x^2}$ for $1 \le x \le 4$.

$\Rightarrow$ Bisects area $= \int_1^{\frac{8}{5}} \frac{1}{x^2} dx = \frac{3}{8}$, half area $= \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}}$, $y = \frac{1}{x^2}$ when $x = 4, y = \frac{1}{16}$

So secondary bisecting line b, lies above $y = \frac{1}{16} \Rightarrow \int_{\frac{1}{16}}^{b} \left(\frac{1}{\sqrt{y}} - 1\right) dy = \frac{3}{16}$

$$\Rightarrow 2\sqrt{b} - b - 2\sqrt{\frac{1}{16}} + \frac{1}{16} = \frac{3}{16} \Rightarrow 64b^2 - 176b + 25 = 0$$

$$\Rightarrow b = \frac{176 \pm \sqrt{176^2 - 4 \times 64 \times 25}}{2 \times 64} = \frac{176 \pm \sqrt{24576}}{128} = \frac{176 \pm 64\sqrt{6}}{128} = \frac{11 \pm 4\sqrt{6}}{8}$$

$\because \cancel{\frac{11+4\sqrt{6}}{8}}, \therefore b = \frac{11-4\sqrt{6}}{8}$.

6. (12 points) Find the volume of the solid obtained from rotating the region bounded by the given curves about the specified line.

(a) $y = \frac{1}{4}x^2, x = 2, y = 0$, about the y-axis

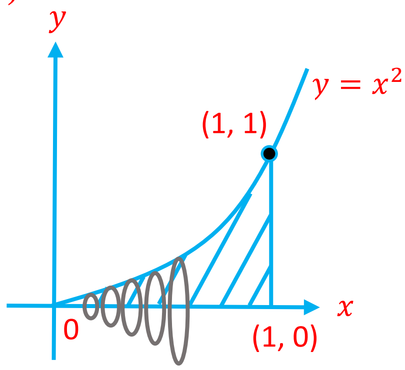
$y = \frac{1}{4}x^2 \Rightarrow x = 2\sqrt{y}$ $\quad \Rightarrow \int_0^1 \pi\left[2^2 - \left(2\sqrt{y}\right)^2\right] dy = \pi \int_0^1 4 - 4y\, dy$

$r_1 \; x = 2$ $\quad = 4\pi\left[1 - \frac{1}{2} - 0\right] = \frac{4\pi}{2} = 2\pi.$

$r_2 \; x = 2\sqrt{y}$

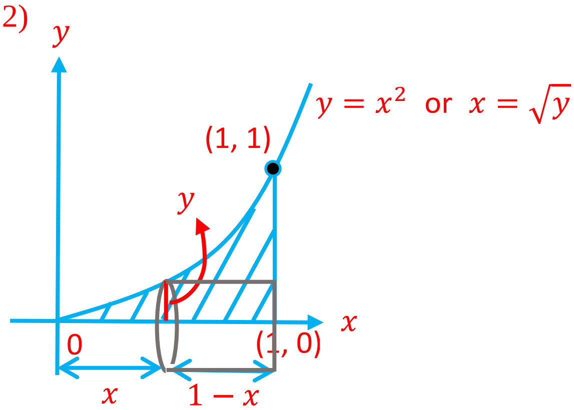
(b) $y = 0, y = x^2, x = 0, x = 1$, about the x-axis

(Sol 1)



$$V = \int_0^1 \pi y^2\, dx = \int_0^1 \pi x^4\, dx = \pi \frac{1}{5} x^5 |_0^1 = \frac{\pi}{5}(1-0) = \frac{\pi}{5}.$$

(Sol 2)



$$V = \int_0^1 2\pi y(1-x)\, dy = \int_0^1 2\pi y\left(1-\sqrt{y}\right) dy = 2\pi \int_0^1 \left(y - y^{\frac{3}{2}}\right) dy$$

$$= 2\pi \left(\frac{1}{2}y^2 - \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^1 = \frac{\pi}{5}.$$

7. (14 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

(a) $x = y^2 + 1, x = 2$, about $y = -2$

Rotating about $y = -2$, using cylinder method

So $x = 2 \Rightarrow 2 = y^2 + 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

y from -1 to 1, so from y to $y = -2$ radius is $r \longrightarrow r = y - (-2) = y + 2$

Height of cylinder is $h \longrightarrow h = 2 - (y^2 + 1) = 1 - y^2$

That $V = \int_{-1}^{1} 2\pi r h = \int_{-1}^{1} 2\pi\, (y + 2)(\,1 - y^2)dy = 2\pi \int_{-1}^{1} (y - y^3 + 2 - 2y^2)\, dy$

$$= 2\pi\left[\frac{1}{2}y^2 - \frac{1}{4}y^4 + 2y - \frac{2}{3}y^3\right]_{-1}^{1} = 2\pi\left[\frac{1}{2} - \frac{1}{4} + 2 - \frac{2}{3} - \left(\frac{1}{2} - \frac{1}{4} - 2 + \frac{2}{3}\right)\right] = 2\pi\left[\frac{8}{3}\right] = \frac{16}{3}\pi.$$

(b) $y = e^{-x^2}, y = 0, x = 0,\ x = 1,$ about the y-axis

$$V = \int_0^1 2\pi x e^{-x^2}\,dx = \frac{2\pi x e^{-x^2}}{-2x}\Big]_0^1 = -\pi e^{-x^2}\Big]_0^1 = -\pi(e^{-1} - 1) = \pi\left(1 - \frac{1}{e}\right).$$

8. (8 points) Use the cylindrical shells to find the volume of solid: a right circular cone with height h and base radius r.

$$\Rightarrow V = 2\pi \int_0^r x\left(h - \frac{h}{r}x\right)dx = 2\pi h \int_0^r \left(x - \frac{x^2}{r}\right)dx = 2\pi h\left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r = 2\pi h\left[\frac{r^2}{6}\right] = \frac{1}{3}\pi r^2 h.$$

9. (16 points) Find the exact length of each of the following curves.

(a) $y = \ln(\sec(x)), 0 \le x \le \frac{\pi}{4}$

$$\Rightarrow y = \ln(\sec x) \text{ then } y' = -\frac{1}{\cos x} \times -\sin x = \tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|]_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1).$$

(b) $x = \frac{y^4}{8} + \frac{1}{4y^2}, 1 \le y \le 2$

$$\Rightarrow \frac{dx}{dy} = \frac{y^3}{2} - \frac{y^{-3}}{2}, \left(\frac{dx}{dy}\right)^2 = \frac{y^6}{4} - \frac{1}{2} + \frac{y^{-6}}{4}$$

$$\Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{y^6}{4} + \frac{1}{2} + \frac{y^{-6}}{4} = \left(\frac{y^3}{2} + \frac{y^{-3}}{2}\right)^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{y^3}{2} + \frac{y^{-3}}{2}, h = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^2 \frac{y^3}{2} + \frac{y^{-3}}{2}dy$$

$$\Rightarrow h = \frac{y^4}{8} - \frac{y^{-2}}{4}\Big]_1^2 = \left(\frac{16}{8} - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) = \frac{33}{16}.$$

10. (20 points) Find the area of the surface obtained from rotating each of the following curves about the specified axis.

(a) $x^2 + y^2 = r^2$, about the line $y = r$

$$\Rightarrow f(x) = \sqrt{r^2 - x^2} = y, f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\Rightarrow S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right)dx$$

$$f(x) = -\sqrt{r^2 - x^2} = y, f'(x) = \frac{x}{\sqrt{r^2 - x^2}}, \text{so } S_2 = 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right)dx$$

$$\Rightarrow \text{ Total area } S = S_1 + S_2 = 8\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 + x^2}}\right)dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2 \frac{\pi}{2} = 4\pi^2 r^2.$$

(b) $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}, 1 \le y \le 2,$ about the x-axis

$$\Rightarrow S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}(y^2 + 2)^{\frac{1}{2}} \times 2y = y(y^2 + 2)^{\frac{1}{2}}$$

$$\left(\frac{dx}{dy}\right)^2 = y^2(y^2 + 2) = y^4 + 2y^2$$

$$S = \int_1^2 2\pi y\sqrt{1 + y^4 + 2y^2}\,dy = 2\pi\int_1^2 y\sqrt{(y^2 + 1)^2}\,dy = 2\pi\int_1^2 y^3 + y\,dy$$

$$= 2\pi\left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \frac{1}{4} - \frac{1}{2}\right) = \frac{21\pi}{2}.$$