The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

Propositional Logic

Section 1.1

Chapter Summary

Propositional Logic
- The Language of Propositions
- Applications
- Logical Equivalences

Predicate Logic
- The Language of Quantifiers
- Logical Equivalences
- Nested Quantifiers

Proofs
- Rules of Inference
- Proof Methods
- Proof Strategy

Propositions

A ***proposition*** is a declarative sentence that is **either true or false (not both)**.

Examples of propositions:

a) The Moon is made of green cheese.
b) Trenton is the capital of New Jersey.
c) Toronto is the capital of Canada.
d) $1 + 0 = 1$
e) $0 + 0 = 2$

Examples that are not propositions.

a) Sit down!
b) What time is it?
c) $x + 1 = 2$
d) $x + y = z$

Propositional Logic

Constructing Propositions

- Propositional Variables: *p, q, r, s, …*
- The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
- Compound Propositions; constructed from logical connectives and other propositions
  - Negation ¬
  - Conjunction ∧
  - Disjunction ∨
  - Implication →
  - Biconditional ↔

Conjunction

The *conjunction* of propositions *p* and *q* is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Compound Propositions: Negation

The *negation* of a proposition *p* is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Example: If *p* denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Disjunction

The *disjunction* of propositions *p* and *q* is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

In English “or” has two distinct meanings.

- “**Inclusive Or**” - In the sentence “Students who have taken CS202 or Math120 may take this class,” we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \lor q$ to be true, either one or both of *p* and *q* must be true.
- “**Exclusive Or**” - When reading the sentence “Soup or salad comes with this entrée,” we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of *p* and *q* must be true, but not both. The truth table for $\oplus$ is:

p	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Understanding Implication₁

In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The “meaning” of $p \rightarrow q$ depends only on the truth values of *p* and *q*.

These implications are perfectly fine, but would not be used in ordinary English.

- “If the moon is made of green cheese, then I have more money than Bill Gates. ”
- “If the moon is made of green cheese then I’m on welfare.”
- “If 1 + 1 = 3, then your grandma wears combat boots.”

Implication

If *p* and *q* are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as “if *p*, then *q*” and has this truth table:

p	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If *p* denotes “I am at home.” and *q* denotes “It is raining.” then $p \rightarrow q$ denotes “If I am at home then it is raining.”

In $p \rightarrow q$, *p* is the *hypothesis* (*antecedent* or *premise*) and *q* is the *conclusion* (or *consequence*).

Understanding Implication₂

One way to view the logical conditional is to think of an <u>obligation or contract</u>.

- “If I am elected, then I will lower taxes.”
- “If you get 100% on the final, then you will get an A.”

If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where *p* is true and *q* is false.

Different Ways of Expressing $p \to q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

Biconditional

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Converse, Contrapositive, and Inverse

From $p \to q$ we can form new conditional statements .

- $q \to p$ is the **converse** of $p \to q$
- $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
- $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Expressing the Biconditional

Some alternative ways "p if and only if q" is expressed in English:

- p **is necessary and sufficient for** q
- **if** p **then** q**, and conversely**
- p **iff** q
- p **exactly when** q

Truth Tables For Compound Propositions

Construction of a truth table:

Rows

- Need a row for every possible combination of values for the atomic propositions.

Columns

- Need a column for the compound proposition (usually at far right)
- Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
  - This includes the atomic propositions

Equivalent Propositions

Two propositions are *equivalent* if they always have the same truth value.

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example Truth Table

Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Translating English Sentences

Steps to convert an English sentence to a statement in propositional logic

- Identify atomic propositions and represent using propositional variables.
- Determine appropriate logical connectives

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's
- q: I go to the country.
- r: I will go shopping.

If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Applications of Propositional Logic

Section 1.2

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let a, c, and f represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \rightarrow (c \vee \neg f)$$

System Specifications

System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

$$q \rightarrow \neg p$$

Logic Puzzles

Raymond Smullyan (Born 1919)

An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.

You go to the island and meet A and B.

- A says "B is a knight."
- B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \land \neg q) \lor (\neg p \land q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \lor q$, $\neg p$, $p \rightarrow q$. When p is false and q is true all three statements are true. So the specification is consistent.

- What if "The diagnostic message is not retransmitted" is added.

  Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Propositional Equivalences

Section 1.3

Tautologies, Contradictions, and Contingencies

A *tautology* is a proposition which is always true.

- Example: $p \lor \neg p$

A *contradiction* is a proposition which is always false.

- Example: $p \land \neg p$

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	**¬p**	**p ∨ ¬p**	**p ∧ ¬p**
T	F	T	F
F	T	T	F

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	**q**	**¬p**	**¬q**	**(p ∨ q)**	**¬(p ∨ q)**	**¬p ∧ ¬q**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Logically Equivalent

Two compound propositions p and q are **logically equivalent if *p↔q* is a tautology.**

We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table shows that $\neg p \lor q$ is equivalent to $p \rightarrow q$.

p	**q**	**¬p**	**¬p ∨ q**	**p → q**
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Key Logical Equivalences [1]

Identity Laws: $p \land T \equiv p, \quad p \lor F \equiv p$

Domination Laws: $p \lor T \equiv T, \quad p \land F \equiv F$

Idempotent laws: $p \lor p \equiv p, \quad p \land p \equiv p$

Double Negation Law: $\neg(\neg p) \equiv p$

Negation Laws: $p \lor \neg p \equiv T, \quad p \land \neg p \equiv F$

Key Logical Equivalences[2]

Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$

Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$

$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$

Constructing New Logical Equivalences

We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Proofs[1]

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
$\equiv (\neg p \wedge \neg q)$	By the identity law for **F**

Equivalence Proofs₂

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative laws laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Disjunctive Normal Form (optional)₂

Example: Find the Disjunctive Normal Form (DNF) of

$$(p \vee q) \rightarrow \neg r$$

Solution: This proposition is true when *r* is false or when both *p* and *q* are false.

$$(\neg p \wedge \neg q) \vee \neg r$$

Disjunctive Normal Form (*optional*)₁

A propositional formula is in *disjunctive normal form* if it consists of a disjunction of (1, … ,*n*) disjuncts where each disjunct consists of a conjunction of (1, …, *m*) atomic formulas or the negation of an atomic formula.

- Yes $(p \wedge \neg q) \vee (\neg p \vee q)$
- No $p \wedge (p \vee q)$

Disjunctive Normal Form (optional)₃

Example: Show that every compound proposition can be put in disjunctive normal form.

Solution: Construct the truth table for the proposition. Then an equivalent proposition is the disjunction with *n* disjuncts (where *n* is the number of rows for which the formula evaluates to **T)**. Each disjunct has m conjuncts where *m* is the number of distinct propositional variables. Each conjunct includes the positive form of the propositional variable if the variable is assigned **T** in that row and the negated form if the variable is assigned **F** in that row. This proposition is in disjunctive normal from.

Conjunctive Normal Form (optional) 1

A compound proposition is in *Conjunctive Normal Form* (CNF) if it is a conjunction of disjunctions.

Every proposition can be put in an equivalent CNF.

- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Propositional Satisfiability

A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is *unsatisfiable*.

A compound proposition is unsatisfiable if and only if its negation is a tautology.

Conjunctive Normal Form (optional) 2

Example: Put the following into CNF:

$$\neg(p \to q) \vee (r \to p)$$

Solution:

1. Eliminate implication signs:

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

2. Move negation inwards; eliminate double negation:

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

3. Convert to CNF using associative/distributive laws

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Notation

$\bigvee_{j=1}^{n} p_j$ is used for $p_1 \vee p_2 \vee \ldots \vee p_n$

$\bigwedge_{j=1}^{n} p_j$ is used for $p_1 \wedge p_2 \wedge \ldots \wedge p_n$

Sudoku

A **Sudoku puzzle** is represented by a 9×9 grid made up of nine 3×3 subgrids, known as **blocks**. Some of the 81 cells of the puzzle are assigned one of the numbers 1,2, …, 9.

The puzzle is solved by assigning numbers to each blank cell so that every row, column and block contains each of the nine possible numbers.

Example

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding as a Satisfiability Problem 1

Let $p(i, j, n)$ = **true** when the number *n* is in the cell in the *i*th row and the *j*th column.

There are 9 × 9 × 9 = 729 such propositions.

In the previous sample puzzle:

- $p(5,1,6)$ is true
- $p(5, j, 6)$ is false for $j = 2,3, \ldots 9$

Note: In this case, the notation $p(i, j, n)$ is used as a variable, we can also look at it as $p_{i,j,n}$

Encoding as a Satisfiability Problem 2

For each cell with a given value, assert $p(i, j, n)$, when the cell in row *i* and column *j* has the given value.

Assert that **every row contains every number**.

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

Assert that **every column contains every number**.

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

Encoding as a Satisfiability Problem 3

Assert that **each of the 3 × 3 blocks contain every number**.

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigwedge_{i=1}^{3}\bigvee_{j=1}^{3} p\left(3r+i,3s+j,n\right)$$

Assert that **no cell contains more than one number**. Take the conjunction over all values of *n*, *n′*, *i*, and j, where each variable ranges from 1 to 9 and n ≠ *n′* , of

$$p\left(i,j,n\right)\rightarrow\neg p\left(i,j,n'\right)$$

Solving Satisfiability Problems

To solve a Sudoku puzzle, we need to find an assignment of truth values to the 729 variables of the form $p(i,j,n)$ that makes the conjunction of the assertions true. Those variables that are assigned T yield a solution to the puzzle.

A truth table can always be used to determine the satisfiability of a compound proposition. But this is too complex even for modern computers for large problems.

There has been much work on developing efficient methods for solving satisfiability problems as many practical problems can be translated into satisfiability problems.