Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

Section Summary [1]

Definition of sets

Describing Sets

- Roster Method
- Set-Builder Notation

Some Important Sets in Mathematics

Empty Set and Universal Set

Subsets and Set Equality

Cardinality of Sets

Tuples

Cartesian Product

Sets

Section 2.1

Introduction

Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.

- Important for counting.
- Programming languages have set operations.

Set theory is an important branch of mathematics.

- Many different systems of axioms have been used to develop set theory.
- Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

Set of all positive integers less than 100:

$S = \{1,2,3,\ldots\ldots,99\}$

Set of all integers less than 0:

$S = \{\ldots, -3,-2,-1\}$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$S = \{a,b,c,d\} = \{b,c,a,d\}$

Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

Ellipsis (…) may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a,b,c,d, \ldots\ldots,z\}$

Some Important Sets

N = *natural numbers* = $\{0,1,2,3\ldots\}$

Z = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^+$ = *positive integers* = $\{1,2,3,\ldots\}$

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used: $S = \{x \mid P(x)\}$

$S = \{x \mid \text{Prime}(x)\}$

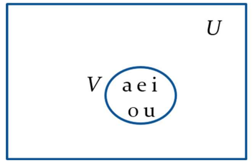
$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

Venn Diagram



The empty set is the set with no elements. Symbolized ∅, but {} also used.

John Venn (1834-1923)
Cambridge, UK

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Some things to remember

Sets can be elements of sets.

$\{\{1,2,3\}, a, \{b,c\}\}$

$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets.

  $\{1,3,5\} = \{3, 5, 1\}$

  $\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Showing a Set is or is not a Subset of Another Set

Showing that A is a subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Subsets

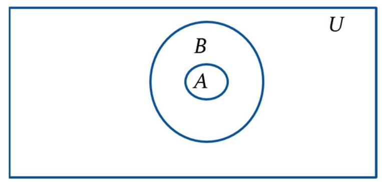
Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- Note:
  1. $\emptyset \subseteq S$, for every set S.
  2. $S \subseteq S$, for every set S.

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Tuples

The *ordered n-tuple* $(a_1,a_2,\ldots,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Power Sets

Definition: The set of all subsets of a set *A*, denoted *P*(*A*) (sometimes 2^S), is called the *power set* of *A*.

Example: If $A = \{a,b\}$ then

$$P(A)=2^A = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n.

Cartesian Product

René Descartes (1596-1650)

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset *R* of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The Cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Section Summary[2]

Set Operations

- Union
- Intersection
- Complementation
- Difference

More on Set Cardinality

Set Identities

Proving Identities

Membership Tables

Set Operations

Section 2.2

Union

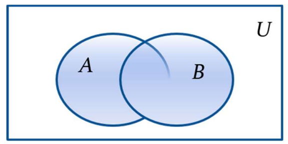
Definition: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \text{ or } x \in B\}$$

Example: What is {1,2,3} ∪ {3, 4, 5}?

Solution: {1,2,3,4,5}

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x \mid x \in A \text{ and } x \in B\}$$

Note if the intersection is empty, then A and B are said to be *disjoint*.

Example: What is? {1,2,3} ∩ {3,4,5} ?

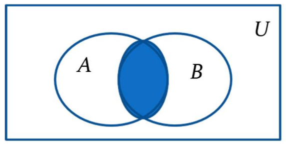
Solution: {3}

Example: What is?

{1,2,3} ∩ {4,5,6} ?

Solution: ∅

Venn Diagram for A ∩B



Complement

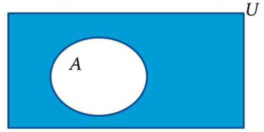
Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set

$$U - A = \{x \mid x \in U \text{ and } x \notin A\}$$

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

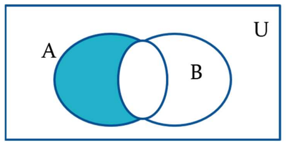


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

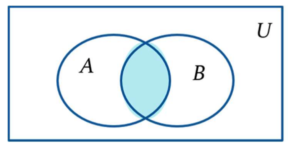
Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors.

- To count the number of students who are either math majors or CS majors ⟹ add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Venn Diagram for A, B, A ∩ B, A ∪ B



Set Identities

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Set Identities

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1. $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and
2. $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	by defn. of union
$(x \notin A) \vee (x \in \overline{B})$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge \neg(x \in B))$	1st De Morgan law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	by defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B}$	$= x \in \overline{A \cap B}$	by defn. of complement
	$= \{x \mid \neg(x \in (A \cap B))\}$	by defn. of does not belong symbol
	$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by defn. of intersection
	$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by 1st De Morgan law for Prop Logic
	$= \{x \mid x \notin A \vee x \notin B\}$	by defn. of not belong symbol
	$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by defn. of complement
	$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by defn. of union
	$= \overline{A} \cup \overline{B}$	by meaning of notation

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	*B*	*C*	*B∩C*	*A∪ (B∩C)*	*A∪B*	*A∪C*	*(A∪B) ∩ (A∪C)*
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

These are well defined, since union and intersection are associative.

Example: For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$

Functions

Section 2.3

Section Summary

Definition of a Function.

- Domain, Codomain
- Image, Preimage

Injection, Surjection, Bijection

Inverse Function

Function Composition

Graphing Functions

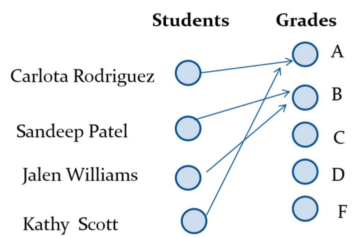
Floor, Ceiling, Factorial

Partial Functions (optional)

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to **exactly one** element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Representing Functions

Functions may be specified in different ways:

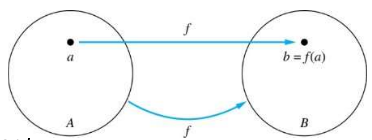
- An explicit statement of the assignment.
  - Eg. Students and grades in this class.
- A formula, for example

  $f(x) = x + 1$
- A computer program.
  - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Chapter 5).

Functions

Given a function $f: A \to B$:

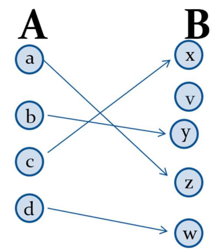
- We say f *maps* A to B or f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The *range* of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Jump to long description

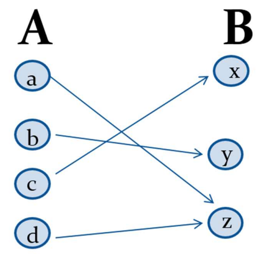
Injections

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



Surjections

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
A function f is called a *surjection* if it is *onto*.



Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

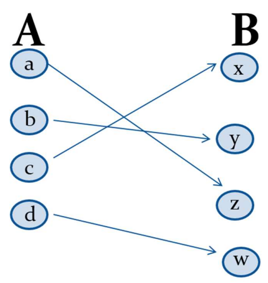
Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

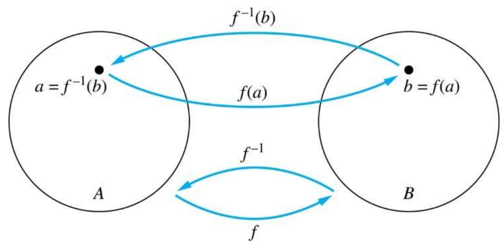
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Functions

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

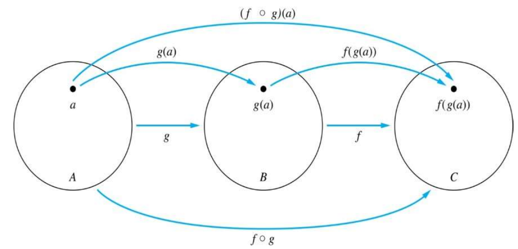
No inverse exists unless *f* is a bijection. Why?



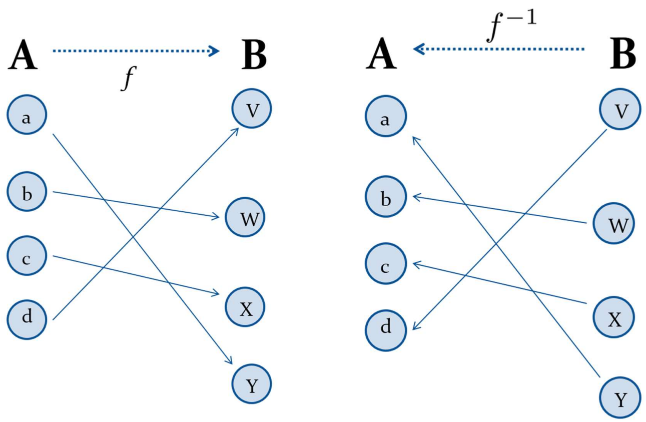
Jump to long description

Composition

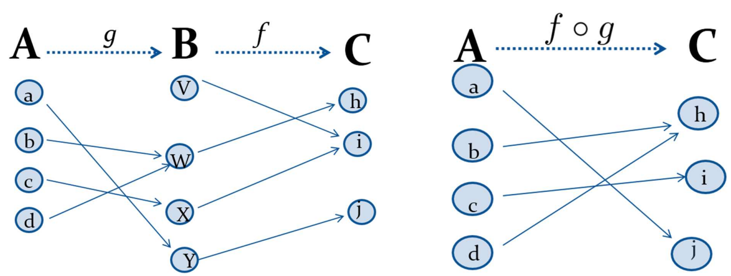
Definition: Let *f*: *B*→*C*, *g*: *A*→*B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f(g(x))$



Inverse Functions



Composition



Composition $f \circ g \neq g \circ f$

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Some Important Functions

The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

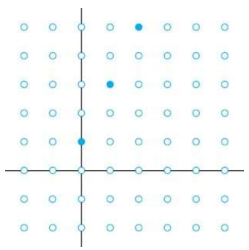
is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

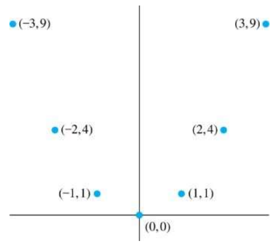
$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Graphs of Functions

Let *f* be a function from the set *A* to the set *B*. The *graph* of the function *f* is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



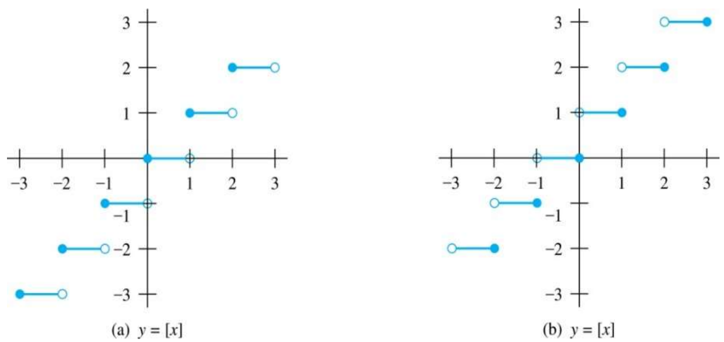
Graph of *f*(*n*) = 2*n* + 1 from Z to Z



Graph of *f*(*x*) = x^2 from Z to Z

Jump to long description

Floor and Ceiling Functions



Graph of (a) Floor and (b) Ceiling Functions

Jump to long description

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(*n* is an integer, *x* is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n = x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x = n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n = x$
(1d)	$\lceil x \rceil = n$ if and only if $x = n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Factorial Function

Definition: f: **N** → **Z**$^+$ **,** denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer

$$f(n) = 1 \cdot 2 \ldots (n-1) \cdot n \quad \text{and} \quad f(0) = 0! = 1$$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \ge ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \le 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \le \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Partial Functions

Definition: A *partial function* f from a set A to a set B is an assignment to each element a in a subset of A**,** called the *domain of definition* of f, of a unique element b in B.

- The sets A and B are called the *domain* and *codomain* of f, respectively.
- We day that f is *undefined* for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a *total function*.

Example: f: **N** → **R** where $f(n) = \sqrt{n}$ is a partial function from **Z** to **R** where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

Sequences and Summations

Section 2.4

Introduction

Sequences are ordered lists of elements.

- 1, 2, 3, 5, 8
- 1, 3, 9, 27, 81, …….

Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.

Section Summary

Sequences.

- Examples: Geometric Progression, Arithmetic Progression

Recurrence Relations

- Example: Fibonacci Sequence

Summations

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.

The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,…..} to S. We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3 \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

Arithmetic Progression

Definition: A arithmetic progression is a sequence of the form: $a,\ a+d,\ a+2d, \ldots,\ a+nd, \ldots$
where the *initial term a* and the *common difference d* are real numbers.

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar^2, \ldots, ar^n, \ldots$
where the *initial term a* and the *common ratio r* are real numbers.

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

Sequences of characters or bits are important in computer science.

The *empty string* is represented by λ.

The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots$. and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solving Recurrence Relations

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

Such a formula is called a *closed formula*.

Iterative Solution Example

Method 1: Working upward, forward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$a_2 = 2 + 3$

$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$

$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$

.

.

.

$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$

Summations

Sum of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable j is called the *index of summation*. It runs through all the integers starting with its *lower limit* m and ending with its *upper limit* n.

Iterative Solution Example

Method 2: Working downward, backward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$a_n = a_{n-1} + 3$

$= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$

$= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$

.

.

.

$= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1)$

Summations

More generally for a set S: $\sum_{j \in s} a_j$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{j=0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

If $S = \{2,5,7,10\}$ then $\sum_{j \in s} a_j = a_2 + a_5 + a_7 + a_{10}$

Product Notation

Product of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod\nolimits_{j=m}^{n} a_j \qquad \prod\nolimits_{m \le j \le n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & r \ne 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

Geometric Series

$= \sum_{j=0}^{n} ar^{j+1}$ From previous slide.

$= \sum_{k=1}^{n+1} ar^k$ Shifting the index of summation with $k = j + 1$.

$= \left(\sum_{k=0}^{n} ar^k\right) + \left(ar^{n+1} - a\right)$ Removing $k = n + 1$ term and adding $k = 0$ term.

$= S_n + \left(ar^{n+1} - a\right)$ Substituting S for summation formula

$\therefore \quad rS_n = S_n + \left(ar^{n+1} - a\right)$

$S_n = \dfrac{ar^{n+1} - a}{r-1}$ if r ≠1

$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$ if r = 1

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	***Closed From***
$\sum_{k=0}^{n} ar^k \ (r \ne 0)$	$\dfrac{ar^{n+1} - a}{r-1}, \ r \ne 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)