On the Turing Completeness of Recurrent Neural Networks

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Motivations

Why neural networks

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- In general, all perceptual tasks

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From a theory-of-computation standpoint:

- It would be nice to know that neural networks are as powerful as Turing machines (TMs)
- So as to ensure that any effectively calculable function is computable by a neural network (Church–Turing thesis)

The result

As it turns out, given a TM, we can build a recurrent neural network (RNN) with rational weights and biases that computes the same function as the TM

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- "Neural" computation (RNNs):

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This result bridges two worlds:

- Symbolic computation (TMs):
  - Finite alphabets of symbols
  - Explicit separation of control (state) and memory (tape)
  - If–then conditionals used for updates
- "Neural" computation (RNNs):
  - Continuous values
  - No intrinsic separation of state vs memory
  - No intrinsic Boolean logic

Our contribution

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- The article dates back to 1992

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- Adding an example in which we apply the construction showed in the proof
  - And providing a Python implementation of such example
- Most importantly, giving a more detailed and (hopefully) pleasant structure to the proof itself
  - Which, in our experience, was pretty frustrating to make sense of

Preliminaries

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- And, actually, instead of considering a TM, we will start from a double-stack pushdown automaton (double-stack PDA)
- This is not a problem, since double-stack PDAs can be proven to be computationally equivalent to single-tape TMs
  - That is, a function is computable by a double-stack PDA if and only if it is computable by a single-tape TM

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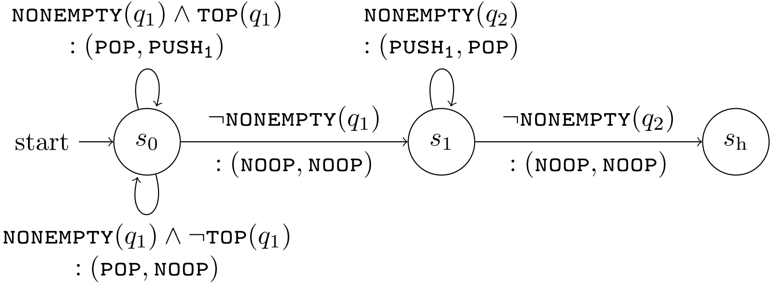
- $q \in [0, 1)$
  - So, it might be a good candidate for a neuron activation ...

PDA $\rightarrow$ RNN

Double-stack PDA computing unary addition



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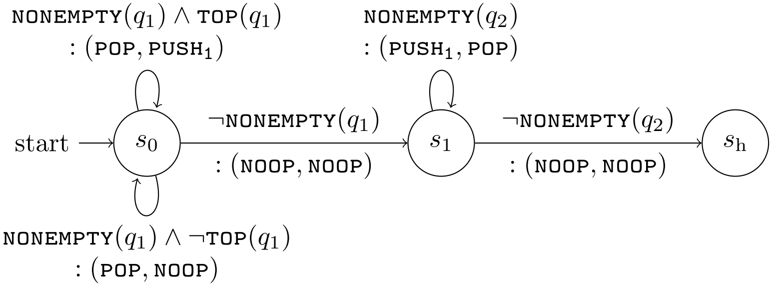

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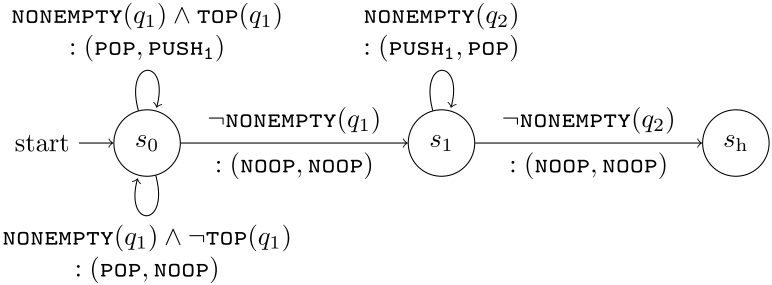

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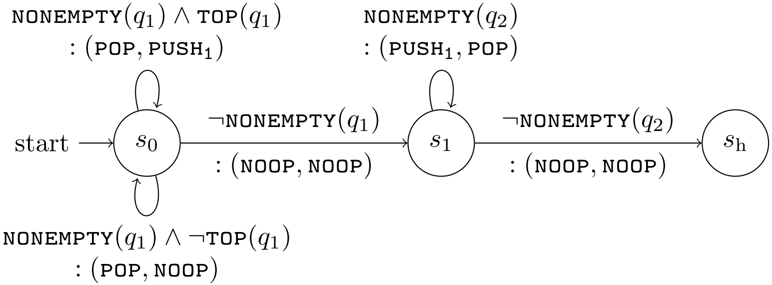

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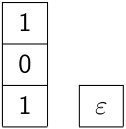
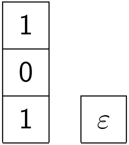
  - E.g. $1 + 1 \longrightarrow$ 

Double-stack PDA computing unary addition

Start



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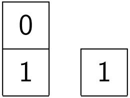

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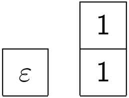

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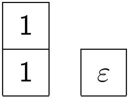
Halt



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  - $q_{1,2} \in [0,1)$ is the rational encoding of the content of stack $1, 2$
- Therefore, $(x_0, x_1, x_h, q_1, q_2)$ will be the input layer of the neural network we are building

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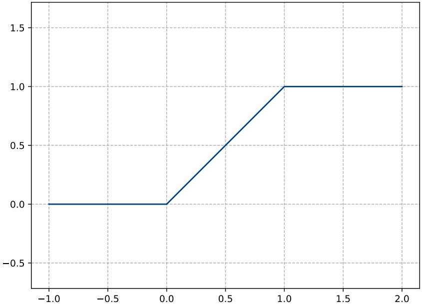
$$x_0^+ = \sigma(x_0^+)$$

where σ is

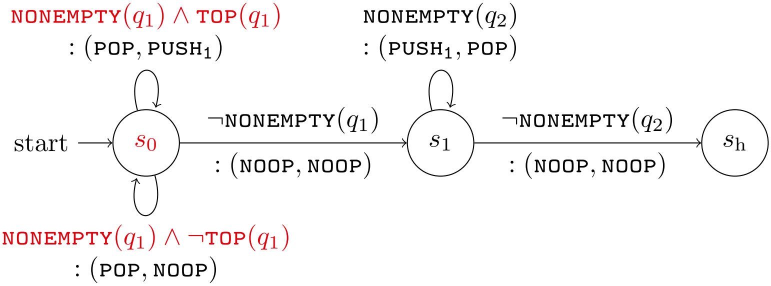


(remember that $x_0^+ \in \{0, 1\}$)

- Yes, it is!
- Let us start from x_0^+:

$$
\begin{aligned}
x_0^+ &= \sigma(x_0^+) \\
&= \sigma(x_0 \cdot \texttt{NONEMPTY}(q_1) \cdot \texttt{TOP}(q_1) + \\
&\qquad x_0 \cdot \texttt{NONEMPTY}(q_1) \cdot \neg\texttt{TOP}(q_1))
\end{aligned}
$$

since



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&= \sigma(\sigma(x_0 + \texttt{NONEMPTY}(q_1) + \texttt{TOP}(q_1) - 2) + \\
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because of the following lemma:

Lemma

Let $a_1, a_2, \ldots, a_k \in \{0, 1\}$. Then

$$a_1 a_2 \ldots a_k = \sigma(a_1 + a_2 + \cdots + a_k - k + 1)$$

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&= \sigma(\sigma(x_0 + \sigma(4q_1) + \sigma(4q_1 - 2) - 2) + \\
&\qquad \sigma(x_0 + \sigma(4q_1) + 1 - \sigma(q_1 - 2) - 2))
\end{aligned}
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because of the following theorem:

Theorem

Let q be the rational encoding of a stack content. Then

$$\textsc{top}(q) = \sigma(4q - 2), \quad \textsc{nonempty}(q) = \sigma(4q)$$

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- From the expression we got, though, we can see that we will not be able to go from $(x_0, x_1, x_h, q_1, q_2)$ to x_0^+ in just one step
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$$x_0^+ = \underbrace{\sigma(\underbrace{\sigma(\overset{\overset{\text{input}}{\uparrow}}{x_0} + \underbrace{\sigma(4\overset{\overset{\text{input}}{\uparrow}}{q_1})}_{\ell_1} + \underbrace{\sigma(4\overset{\overset{\text{input}}{\uparrow}}{q_1} - 2)}_{\ell_1} - 2)}_{\ell_2} + \underbrace{\sigma(\overset{\overset{\text{input}}{\uparrow}}{x_0} + \underbrace{\sigma(4\overset{\overset{\text{input}}{\uparrow}}{q_1})}_{\ell_1} - \underbrace{\sigma(4\overset{\overset{\text{input}}{\uparrow}}{q_1} - 2)}_{\ell_1} - 1)}_{\ell_2})}_{\text{output}}$$

Input layer $\rightarrow$ output layer

- We can do the same for:

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Input layer $\to$ output layer

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$$(x_0, x_1, x_h, q_1, q_2) \longrightarrow x_h^+$$
$$(x_0, x_1, x_h, q_1, q_2) \longrightarrow q_1^+$$

Input layer $\rightarrow$ output layer

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$$
\begin{aligned}
(x_0, x_1, x_h, q_1, q_2) &\longrightarrow x_1^+ \\
(x_0, x_1, x_h, q_1, q_2) &\longrightarrow x_h^+ \\
(x_0, x_1, x_h, q_1, q_2) &\longrightarrow q_1^+ \\
(x_0, x_1, x_h, q_1, q_2) &\longrightarrow q_2^+
\end{aligned}
$$

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$$(x_0, x_1, x_h, q_1, q_2) \longrightarrow q_1^+$$
$$(x_0, x_1, x_h, q_1, q_2) \longrightarrow q_2^+$$

- Let us skip the calculations (which are similar to those we have already seen anyway) and just show the final results:

Input layer $\rightarrow$ output layer

$$x_1^+ = \cdots = \sigma(\sigma(x_0 - \sigma(4q_1)) + \sigma(x_1 + \sigma(4q_2) - 1))$$

Input layer $\to$ output layer

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Input layer $\rightarrow$ output layer

$$
\begin{aligned}
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q_1^+ &= \cdots = \sigma\big(\sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + 4q_1 - 4) + \\
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&\qquad\qquad \sigma(x_0 - \sigma(4q_1) + q_1 - 1) + \\
&\qquad\qquad \sigma\big(x_1 + \sigma(4q_2) + \tfrac{1}{4}q_1 - \tfrac{5}{4}\big) + \\
&\qquad\qquad \sigma(x_1 - \sigma(4q_2) + q_1 - 1)\big)
\end{aligned}
$$

Input layer $\to$ output layer

$$
\begin{aligned}
x_1^+ &= \cdots = \sigma(\sigma(x_0 - \sigma(4q_1)) + \sigma(x_1 + \sigma(4q_2) - 1)) \\
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q_2^+ &= \cdots = \sigma\big(\sigma\big(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + \tfrac{1}{4}q_2 - \tfrac{9}{4}\big) + \\
&\qquad\quad \sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + q_2 - 2) + \\
&\qquad\quad \sigma(x_0 - \sigma(4q_1) + q_2 - 1) + \\
&\qquad\quad \sigma(x_1 + \sigma(4q_2) - 2\sigma(4q_2 - 2) + 4q_2 - 3) + \\
&\qquad\quad \sigma(x_1 - \sigma(4q_2) + q_2 - 1)\big)
\end{aligned}
$$

The network

- We are done: we built a neural network whose input layer is the description of the PDA at time t, namely,

$$(x_0, x_1, x_{\mathrm{h}}, q_1, q_2)$$

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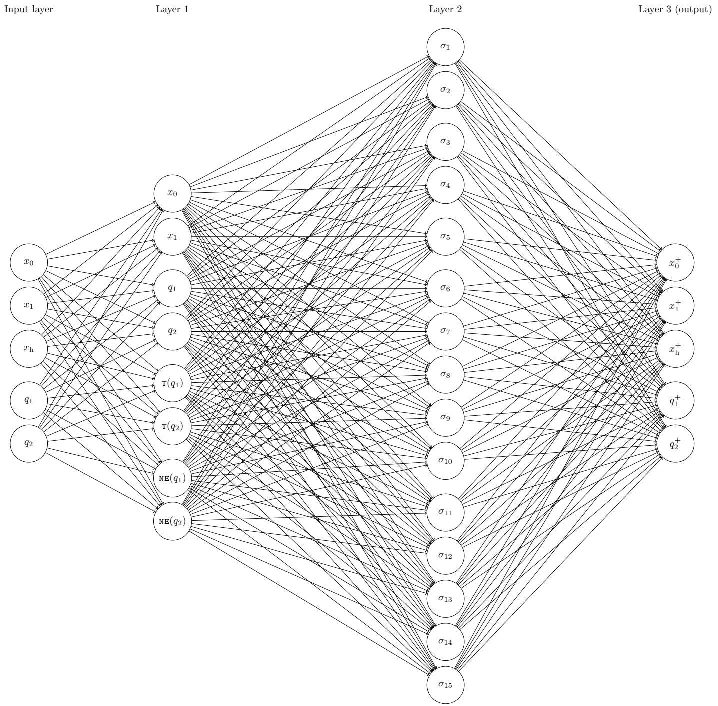
- Let us look at its graphical depiction:



Input layer
Layer 1
Layer 2
Layer 3 (output)
x_0
x_1
x_h
q_1
q_2
x_0
x_1
q_1
q_2
$T(q_1)$
$T(q_2)$
$NE(q_1)$
$NE(q_2)$
σ_1
σ_2
σ_3
σ_4
σ_5
σ_6
σ_7
σ_8
σ_9
σ_{10}
σ_{11}
σ_{12}
σ_{13}
σ_{14}
σ_{15}
x_0^+
x_1^+
x_h^+
q_1^+
q_2^+

Simulation

Now that we have built the network, how do we simulate a PDA execution with it?

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Simulation algorithm

1. Initialize the network with input layer

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2. Compute the output layer $(x_0^+, x_1^+, x_h^+, q_1^+, q_2^+)$
3. If $x_h^+ = 0$
   - Copy the output layer into the input layer
   - Go back to point 2

   If $x_h^+ = 1$, then $q_{1,2}^+$ is the rational encoding of the content of stack $1, 2$ at the end of the PDA execution

Implementation (Python 3 + NumPy)

Fragment of class `Network`

```
def __init__(self):
    self.weights = [
        numpy.array(...),  # matrix W1
        numpy.array(...),  # matrix W2
        numpy.array(...)   # matrix W3
    ]
    self.biases = [
        numpy.array(...),  # vector b1
        numpy.array(...),  # vector b2
        numpy.array(...)   # vector b3
    ]
```

Implementation (Python 3 + NumPy)

Fragment of class `Network`

```
def execute(self, stack1):

    def feedforward(a):
        if a[2] == 1:
            return a[3], a[4]
        for w, b in zip(self.weights, self.biases):
            a = sigmoid(numpy.dot(w, a) + b)
        return feedforward(a)

    a = numpy.array(
        [1, 0, 0, Stack(stack1).encoding, Stack([]).encoding]
    )
    return feedforward(a)
```

Test

- What to expect:

Test

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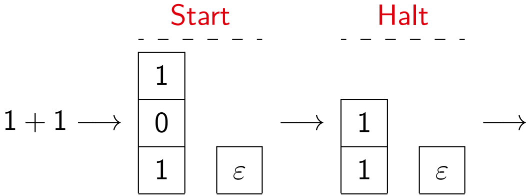

$$q_1 = \frac{2 \cdot 1}{4} + \frac{2 \cdot 1}{4^2} = 0.9375$$

$$q_2 = 0$$

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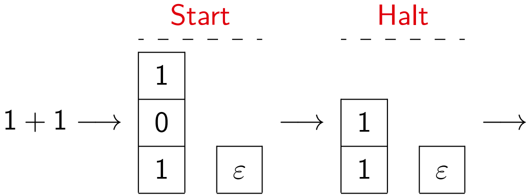

$$q_1 = \frac{2\cdot 1}{4} + \frac{2\cdot 1}{4^2} = 0.9375$$
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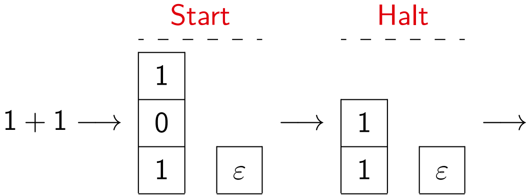

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```
>>> Network().execute([1, 0, 1])
```

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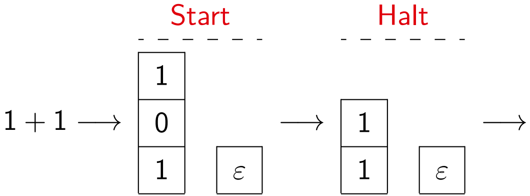

$$q_1 = \frac{2 \cdot 1}{4} + \frac{2 \cdot 1}{4^2} = 0.9375$$
$$q_2 = 0$$

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Test

```
>>> Network().execute([1, 0, 1])
(0.9375, 0.0)
```

Conclusions

Critical observations and limits of our construction

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- In order to be be able to represent any rational number, no matter with how many digits, we would need arbitrary-precision arithmetic and an infinite amount of memory
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Let us consider some implementation issues:

- In order to be be able to represent any rational number, no matter with how many digits, we would need arbitrary-precision arithmetic and an infinite amount of memory
  - But this is just the Turing machine infinite-tape requirement in a different guise
  - And it comes up because we encoded stack contents as rational numbers but we didn't put a limit on stack size
- In the example we saw, the derivation of the network was carried on "by hand", and the implementation needed precomputed weight matrices and bias vectors
  - One might think of automating the construction by devising a program that derives the weight matrices and bias vectors from any given double-stack PDA, and then runs a simulation

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Lemma (Properties of q)

Let $a_1 a_2 \dots a_n \in \{0,1\}^*$ *and* $q = \sum_{i=1}^{n} \frac{2a_i + 1}{4^i} \in \mathbb{Q}$. *Then*

1. $q \in [0, 1)$
2. $q = 0 \iff \omega = \varepsilon$ *(ε is the empty string)*
3. $q \in \left[\frac{1}{4}, \frac{1}{2}\right) \iff a_1 = 0$
4. $q \in \left[\frac{3}{4}, 1\right) \iff a_1 = 1$

Theorem (stack-operation encodings)

1. $\textsc{top}(q) = \sigma(4q - 2)$
2. $\textsc{nonempty}(q) = \sigma(4q)$
3. $\textsc{push}_0(q) = \frac{q}{4} + \frac{1}{4}$
4. $\textsc{push}_1(q) = \frac{q}{4} + \frac{3}{4}$
5. $\textsc{pop}(q) = 4q - 2\sigma(4q - 2) - 1$
6. $\textsc{noop}(q) = q$

Proof.

$$
\begin{aligned}
\textsc{top}(q) = 0 &\implies q \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right) \\
&\implies 4q - 2 \in [-1, 0) \\
&\implies \sigma(4q - 2) = 0 \\
\textsc{top}(q) = 1 &\implies q \in \left[\tfrac{3}{4}, 1\right) \\
&\implies 4q - 2 \in [1, 2) \\
&\implies \sigma(4q - 2) = 1 \\
\textsc{nonempty}(q) = 0 &\implies q = 0 \\
&\implies 4q = 0 \\
&\implies \sigma(4q) = 0 \\
\textsc{nonempty}(q) = 1 &\implies q \neq 0 \\
&\implies q \in \left[\tfrac{1}{4}, \tfrac{1}{2}\right) \cup \left[\tfrac{3}{4}, 1\right) \\
&\implies 4q \in [1, 2) \cup [3, 4) \\
&\implies \sigma(4q) = 1
\end{aligned}
$$

$$
\begin{aligned}
\textsc{push}_0(q) &= \frac{q}{4} + \frac{2 \cdot 0 + 1}{4} \\
&= \frac{q}{4} + \frac{1}{4} \\
\textsc{push}_1(q) &= \frac{q}{4} + \frac{2 \cdot 1 + 1}{4} \\
&= \frac{q}{4} + \frac{3}{4} \\
\textsc{pop}(q) &= 4q - (2\textsc{top}(q) + 1) \\
&= 4q - 2\sigma(4q - 2) - 1
\end{aligned}
$$

$\square$

Lemma (technical)

Let $a_1, a_2, \ldots, a_k \in \{0, 1\}$ *and* $q \in [0, 1]$. *Then*

$$a_1 a_2 \ldots a_k q = \sigma(a_1 + a_2 + \cdots + a_k - k + q)$$

Proof.

Case 1: $\exists\, i : a_i = 0$

$$\begin{aligned}
&\implies a_1 + a_2 + \cdots + a_k \leq k - 1 \\
&\implies a_1 + a_2 + \cdots + a_k - k + q \leq k - 1 - k + q = q - 1 \in [-1, 0] \\
&\implies \sigma(a_1 + a_2 + \cdots + a_k - k + q) = 0 = a_1 a_2 \ldots a_k q
\end{aligned}$$

Case 2: $a_i = 1\ \forall\, i$

$$\begin{aligned}
&\implies a_1 + a_2 + \cdots + a_k = k \\
&\implies a_1 + a_2 + \cdots + a_k - k + q = k - k + q = q \in [0, 1] \\
&\implies \sigma(a_1 + a_2 + \cdots + a_k - k + q) = q = a_1 a_2 \ldots a_k q
\end{aligned}$$

$\square$

$$
\begin{aligned}
x_1^+ &= \sigma\left(x_1^+\right) \\
&= \sigma(x_0 \cdot \neg\texttt{NONEMPTY}(q_1) + x_1 \cdot \texttt{NONEMPTY}(q_2)) \\
&= \sigma(\sigma(x_0 + \neg\texttt{NONEMPTY}(q_1) - 1) + \sigma(x_1 + \texttt{NONEMPTY}(q_2) - 1)) \\
&= \sigma(\sigma(x_0 + 1 - \sigma(4q_1) - 1) + \sigma(x_1 + \sigma(4q_2) - 1)) \\
&= \sigma(\sigma(x_0 - \sigma(4q_1)) + \sigma(x_1 + \sigma(4q_2) - 1))
\end{aligned}
$$

$$
\begin{aligned}
x_{\mathsf{h}}^+ &= \sigma\left(x_{\mathsf{h}}^+\right) \\
&= \sigma(x_1 \cdot \neg\texttt{NONEMPTY}(q_2)) \\
&= \sigma(\sigma(x_1 + \neg\texttt{NONEMPTY}(q_2) - 1)) \\
&= \sigma(\sigma(x_1 + 1 - \sigma(4q_2) - 1)) \\
&= \sigma(\sigma(x_1 - \sigma(4q_2)))
\end{aligned}
$$

$$
\begin{aligned}
q_1^+ &= \sigma(q_1^+) \\
&= \sigma(x_0 \cdot \texttt{NONEMPTY}(q_1) \cdot \texttt{TOP}(q_1) \cdot \texttt{POP}(q_1) + \\
&\qquad x_0 \cdot \texttt{NONEMPTY}(q_1) \cdot \neg\texttt{TOP}(q_1) \cdot \texttt{POP}(q_1) + \\
&\qquad x_0 \cdot \neg\texttt{NONEMPTY}(q_1) \cdot \texttt{NOOP}(q_1) + \\
&\qquad x_1 \cdot \texttt{NONEMPTY}(q_2) \cdot \texttt{PUSH}_1(q_1) + \\
&\qquad x_1 \cdot \neg\texttt{NONEMPTY}(q_2) \cdot \texttt{NOOP}(q_1)) \\
&= \sigma(\sigma(x_0 + \texttt{NONEMPTY}(q_1) + \texttt{TOP}(q_1) - 3 + \texttt{POP}(q_1)) + \\
&\qquad \sigma(x_0 + \texttt{NONEMPTY}(q_1) + \neg\texttt{TOP}(q_1) - 3 + \texttt{POP}(q_1)) + \\
&\qquad \sigma(x_0 + \neg\texttt{NONEMPTY}(q_1) - 2 + \texttt{NOOP}(q_1)) + \\
&\qquad \sigma(x_1 + \texttt{NONEMPTY}(q_2) - 2 + \texttt{PUSH}_1(q_1)) + \\
&\qquad \sigma(x_1 + \neg\texttt{NONEMPTY}(q_2) - 2 + \texttt{NOOP}(q_1))) \\
&= \sigma(\sigma(x_0 + \sigma(4q_1) + \sigma(4q_1 - 2) - 3 + 4q_1 - 2\sigma(4q_1 - 2) - 1) + \\
&\qquad \sigma(x_0 + \sigma(4q_1) + 1 - \sigma(4q_1 - 2) - 3 + 4q_1 - 2\sigma(4q_1 - 2) - 1) + \\
&\qquad \sigma(x_0 + 1 - \sigma(4q_1) - 2 + q_1) + \\
&\qquad \sigma(x_1 + \sigma(4q_2) - 2 + \tfrac{1}{4}q_1 + \tfrac{3}{4}) + \\
&\qquad \sigma(x_1 + 1 - \sigma(4q_2) - 2 + q_1)) \\
&= \sigma(\sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + 4q_1 - 4) + \\
&\qquad \sigma(x_0 + \sigma(4q_1) - 3\sigma(4q_1 - 2) + 4q_1 - 3) + \\
&\qquad \sigma(x_0 - \sigma(4q_1) + q_1 - 1) + \\
&\qquad \sigma(x_1 + \sigma(4q_2) + \tfrac{1}{4}q_1 - \tfrac{5}{4}) + \\
&\qquad \sigma(x_1 - \sigma(4q_2) + q_1 - 1))
\end{aligned}
$$

$$
\begin{aligned}
q_2^+ &= \sigma(q_2^+) \\
&= \sigma(x_0 \cdot \text{NONEMPTY}(q_1) \cdot \text{TOP}(q_1) \cdot \text{PUSH}_1(q_2) + \\
&\qquad x_0 \cdot \text{NONEMPTY}(q_1) \cdot \neg\text{TOP}(q_1) \cdot \text{NOOP}(q_2) + \\
&\qquad x_0 \cdot \neg\text{NONEMPTY}(q_1) \cdot \text{NOOP}(q_2) + \\
&\qquad x_1 \cdot \text{NONEMPTY}(q_2) \cdot \text{POP}(q_2) + \\
&\qquad x_1 \cdot \neg\text{NONEMPTY}(q_2) \cdot \text{NOOP}(q_2)) \\
&= \sigma(\sigma(x_0 + \text{NONEMPTY}(q_1) + \text{TOP}(q_1) - 3 + \text{PUSH}_1(q_2)) + \\
&\qquad \sigma(x_0 + \text{NONEMPTY}(q_1) + \neg\text{TOP}(q_1) - 3 + \text{NOOP}(q_2)) + \\
&\qquad \sigma(x_0 + \neg\text{NONEMPTY}(q_1) - 2 + \text{NOOP}(q_2)) + \\
&\qquad \sigma(x_1 + \text{NONEMPTY}(q_2) - 2 + \text{POP}(q_2)) + \\
&\qquad \sigma(x_1 + \neg\text{NONEMPTY}(q_2) - 2 + \text{NOOP}(q_2))) \\
&= \sigma(\sigma(x_0 + \sigma(4q_1) + \sigma(4q_1 - 2) - 3 + \tfrac{1}{4}q_2 + \tfrac{3}{4}) + \\
&\qquad \sigma(x_0 + \sigma(4q_1) + 1 - \sigma(4q_1 - 2) - 3 + q_2) + \\
&\qquad \sigma(x_0 + 1 - \sigma(4q_1) - 2 + q_2) + \\
&\qquad \sigma(x_1 + \sigma(4q_2) - 2 + 4q_2 - 2\sigma(4q_2 - 2) - 1) + \\
&\qquad \sigma(x_1 + 1 - \sigma(4q_2) - 2 + q_2)) \\
&= \sigma(\sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + \tfrac{1}{4}q_2 - \tfrac{9}{4}) + \\
&\qquad \sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + q_2 - 2) + \\
&\qquad \sigma(x_0 - \sigma(4q_1) + q_2 - 1) + \\
&\qquad \sigma(x_1 + \sigma(4q_2) - 2\sigma(4q_2 - 2) + 4q_2 - 3) + \\
&\qquad \sigma(x_1 - \sigma(4q_2) + q_2 - 1))
\end{aligned}
$$

$$
\begin{aligned}
x_0^+ &= \sigma\big(\underbrace{\sigma(x_0 + \sigma(4q_1) + \sigma(4q_1 - 2) - 2)}_{=:\,\sigma_1} + \\
&\qquad \underbrace{\sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) - 1)}_{=:\,\sigma_2}\big) \\
x_1^+ &= \sigma\big(\underbrace{\sigma(x_0 - \sigma(4q_1))}_{=:\,\sigma_3} + \underbrace{\sigma(x_1 + \sigma(4q_2) - 1)}_{=:\,\sigma_4}\big) \\
x_{\mathsf{h}}^+ &= \sigma\big(\underbrace{\sigma(x_1 - \sigma(4q_2))}_{=:\,\sigma_5}\big)
\end{aligned}
$$

$$
\begin{aligned}
q_1^+ = \sigma\Big(& \underbrace{\sigma(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + 4q_1 - 4)}_{=:\,\sigma_6} + \\
& \underbrace{\sigma(x_0 + \sigma(4q_1) - 3\sigma(4q_1 - 2) + 4q_1 - 3)}_{=:\,\sigma_7} + \\
& \underbrace{\sigma(x_0 - \sigma(4q_1) + q_1 - 1)}_{=:\,\sigma_8} + \\
& \underbrace{\sigma\big(x_1 + \sigma(4q_2) + \tfrac{1}{4} q_1 - \tfrac{5}{4}\big)}_{=:\,\sigma_9} + \\
& \underbrace{\sigma(x_1 - \sigma(4q_2) + q_1 - 1)}_{=:\,\sigma_{10}} \Big)
\end{aligned}
$$

$$
\begin{aligned}
q_2^+ = \sigma\Big(&\underbrace{\sigma\big(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + \tfrac{1}{4}q_2 - \tfrac{9}{4}\big)}_{=:\,\sigma_{11}} + \\
&\underbrace{\sigma\big(x_0 + \sigma(4q_1) - \sigma(4q_1 - 2) + q_2 - 2\big)}_{=:\,\sigma_{12}} + \\
&\underbrace{\sigma\big(x_0 - \sigma(4q_1) + q_2 - 1\big)}_{=:\,\sigma_{13}} + \\
&\underbrace{\sigma\big(x_1 + \sigma(4q_2) - 2\sigma(4q_2 - 2) + 4q_2 - 3\big)}_{=:\,\sigma_{14}} + \\
&\underbrace{\sigma\big(x_1 - \sigma(4q_2) + q_2 - 1\big)}_{=:\,\sigma_{15}}\Big)
\end{aligned}
$$

W^1	x_0	x_1	x_h	q_1	q_2	b^1
x_0	1	0	0	0	0	0
x_1	0	1	0	0	0	0
q_1	0	0	0	1	0	0
q_2	0	0	0	0	1	0
$\texttt{T}(q_1)$	0	0	0	4	0	-2
$\texttt{T}(q_2)$	0	0	0	0	4	-2
$\texttt{NE}(q_1)$	0	0	0	4	0	0
$\texttt{NE}(q_2)$	0	0	0	0	4	0

Table: weights from input layer to layer 1 and biases of layer 1

W^2	x_0	x_1	q_1	q_2	$\texttt{T}(q_1)$	$\texttt{T}(q_2)$	$\texttt{NE}(q_1)$	$\texttt{NE}(q_2)$	b^2
σ_1	1	0	0	0	1	0	1	0	-2
σ_2	1	0	0	0	-1	0	1	0	-1
σ_3	1	0	0	0	0	0	-1	0	0
σ_4	0	1	0	0	0	0	0	1	-1
σ_5	0	1	0	0	0	0	0	-1	0
σ_6	1	0	4	0	-1	0	1	0	-4
σ_7	1	0	4	0	-3	0	1	0	-3
σ_8	1	0	1	0	0	0	-1	0	-1
σ_9	0	1	$\frac{1}{4}$	0	0	0	0	1	$-\frac{5}{4}$
σ_{10}	0	1	1	0	0	0	0	-1	-1
σ_{11}	1	0	0	$\frac{1}{4}$	1	0	1	0	$-\frac{9}{4}$
σ_{12}	1	0	0	1	-1	0	1	0	-2
σ_{13}	1	0	0	1	0	0	-1	0	-1
σ_{14}	0	1	0	4	0	-2	0	1	-3
σ_{15}	0	1	0	1	0	0	0	-1	-1

Table: weights from layer 1 to layer 2 and biases of layer 2

W^3	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{15}	b^3
x_0^+	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^+	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
x_h^+	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
q_1^+	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0
q_2^+	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0

Table: weights from layer 2 to layer 3 and biases of layer 3