Homework #3

Problem 1: The Power of the Polynomial

What is the coefficient of x^3 in the expansion of $(2-3x)^5$?

Solution. By Newton's binomial formula, we know that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Hence, for $(2-3x)^5$, we know that that for x^3, the term is as such:

$$\binom{5}{2} \cdot (-3x)^{(5-2)} \cdot 2^2 = -\frac{5!}{2! \cdot 3!} \cdot 27 \cdot 4 \cdot x^3 = -1080x^3.$$

The coffiecient, thus, is $\boxed{-1080}$.

Problem 2: Coefficients Count

What is the coefficient of $a^3b^2c^4$ in the expansion of $(a+b+c)^9$?

Solution. We solve for a formula for the trinomial expansion:

$$(a + \underbrace{b+c}_{=d})^p = \sum_{k=0}^{p} \binom{p}{k} a^{p-k} d^k = \sum_{k=0}^{p} \binom{p}{k} a^{p-k} (b+c)^k = \sum_{k=0}^{p} \binom{p}{k} a^{p-k} \left(\sum_{i=0}^{k} \binom{k}{i} b^{k-i} c^i \right).$$

Hence, for $(a+b+c)^9$, we can calculate the coeffcient of $a^3b^2c^4$ as follows:

$$Ca^3b^2c^4 = \binom{9}{6} a^{9-6} \left[\binom{6}{4} b^{6-4} c^4 \right] \implies C = \binom{9}{6}\binom{6}{4} = \frac{9!}{6! \cdot 3!} \cdot \frac{6!}{4! \cdot 2!} = 20 \cdot 63 = \boxed{1260}$$

Problem 3: Digit Drama - Odds Edition

An integer is selected randomly between 100 and 999. What is the probability that the number has distinct digits?

Solution. To pick the first digit, there are 9 options (excluding 0), for the second also 9 (including 0, excluding the first picked digit), and for the third there are 8 options (one excluding the first and second digits). We use the fundamental theorem of probability to calculate the total number of ways we can pick: $9 \cdot 9 \cdot 8 = 648$. Thus, the total probability is: $\frac{648}{900} = \boxed{\frac{18}{25}}$.

Problem 4: Twin Rivalry

Three students are selected randomly from a group of twelve, including twins Anna and Brian. What is the probability that Anna is selected, but Brian is not?

Solution.

$$P(\text{Anna selected but not Brian}) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{\binom{10}{2}}{\binom{12}{3}} = \boxed{\frac{9}{44}}$$

We know that there are $\binom{10}{2}$ favorable outcomes because we assume that Anna is already given a spot, and we choose 2 more spots out of 10 people, which excludes Brian.

Problem 5: Marble Mayhem

Three marbles are selected randomly from a bag containing 6 yellow marbles and 4 green marbles. What is the probability that there will be more green marbles than yellow marbles selected?

Solution. For there to be green marbles than yellow marbles selected, we can either have 3 green and 0 yellow or 2 green and 1 yellow. We calculate the number of favorable outcomes over unfavorable:

$$P = \frac{\binom{4}{2} \cdot \binom{6}{1} + \binom{4}{3}}{\binom{10}{3}} = \frac{6 \cdot 6 + 4}{120} = \boxed{\frac{1}{3}}$$

Problem 6: Selecting Several Socks

We select randomly two socks from a drawer containing 4 white socks, 6 blue socks, and 2 red socks. What is the probability that the socks will match?

Solution.

$$P(\text{socks will match}) = \frac{\binom{4}{2} + \binom{6}{2} + \binom{2}{2}}{\binom{12}{2}} = \frac{22}{66} = \boxed{\frac{1}{3}}$$

Problem 7: Pair-ly Lucky

Five cards are dealt from a regular deck of 52 cards. What is the probability of being dealt *one pair*?

Solution. The total amount of ways five cards can be picked is $\binom{52}{5}$. There are 13 ranks, and four identical cards of each rank. Therefore, we can select two cards from each quartet, and select out of the remaining 12 ranks, 3 ranks and one of four cards from each, to avoid any extra pairs. Therefore, we have the following probability:

$$P(\text{one pair from a deck}) = \frac{\binom{4}{2} \cdot 13 \cdot \binom{12}{3} \cdot 64}{\binom{52}{5}} = \frac{13 \cdot 6 \cdot 220 \cdot 64}{2598960} = \frac{22 \cdot 13 \cdot 16}{10829} = \boxed{0.422569}$$

Problem 8: One and Done

a. What is the probability of having at least one 1 in four rolls of a die?

b. What is the probability of having at least one double 1 in 24 rolls of two dice?

Solution.

a. Having at least one 1 in four rolls is the complement of not having any, so the probability is:

$$P(\text{at least one 1}) = 1 - \left(\frac{5}{6}\right)^4 = \boxed{0.51775}$$

b. Having at least one double 1 in twenty-four rolls is the complement of not having any, so the probability is:

$$P(\text{at least one double 1}) = 1 - \left(\frac{35}{36}\right)^4 = \boxed{0.491404}$$