# Homework #4

**Problem 1: Dies by Calculation**

Suppose a fair die is tossed three times.

a. Let $X$ be the largest of the faces that appear. Write with justification the probability density function of $X$.

b. Let $Y$ be the number of different faces that appear. Write with justification the probability density function and the cumulative distribution function $F_Y$ of Y. Plot the graph of $F_Y$ .

**Solution.**

a. Probability mass function of $X$:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{216}$ | $\frac{7}{216}$ | $\frac{19}{216}$ | $\frac{37}{216}$ | $\frac{61}{216}$ | $\frac{91}{216}$ | $\frac{127}{216}$ |

To find the probabilities for each value of $X$, we use the formula $\frac{k^3}{6^3} - \frac{(k-1)^3}{6^3}$, since $P(x \leq k) - P(x \leq k-1) = P(x = k)$

b. Probability mass function of $Y$:

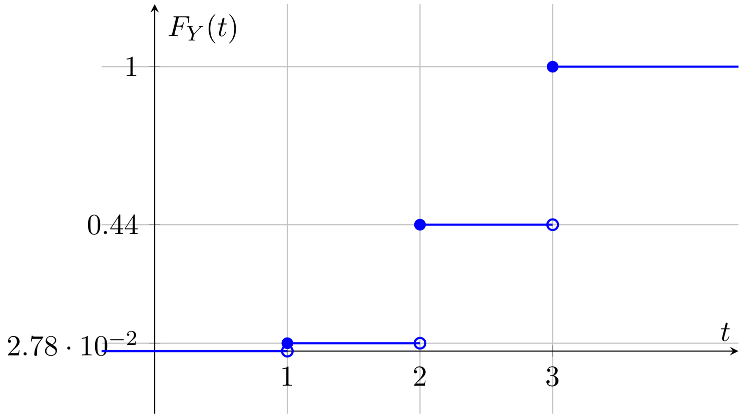
| $y$ | 1 | 2 | 3 |
|---|---|---|---|
| $P(Y = y)$ | $\frac{6}{216}$ | $\frac{90}{216}$ | $\frac{120}{216}$ |

$P(x = 1) = \dfrac{6 \cdot 1 \cdot 1}{216} = \dfrac{6}{216}$, since there must be only 1 distinct number.
$P(x = 2) = \dfrac{6 \cdot 5 \cdot 1}{216} = \dfrac{30}{216}$, since there must be 2 distinct numbers, and one repeated.
$P(x = 3) = \dfrac{6 \cdot 5 \cdot 4}{216} = \dfrac{120}{216}$, since there must be 3 distinct numbers.

Cumulative distribution function of $Y$:

$$F_Y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{36} & 1 \leq t < 2 \\ \frac{4}{9} & 2 \leq t < 3 \\ 1 & 3 \leq t \end{cases}$$

Plot of $F_Y$:




## Problem 2: Triple Flip Tally

A fair coin is flipped three times. Let $Y$ be the number of heads minus the number of tails. Write with justification the probability density function and the cumulative distribution function $F_Y$ of $Y$. Plot the graph of $F_Y$.

**Solution.** Probability mass function of $Y$:

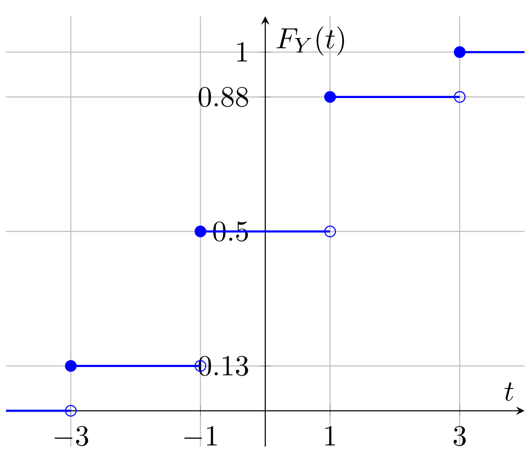
| $y$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $P(Y=y)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

To find the probability for each value of $Y$, we use the formula $\binom{3}{|x|} \cdot \frac{1}{2^3}$.

Cumulative distribution function of $Y$:

$$F_Y(t) = \begin{cases} 0 & t < -3 \\ \frac{1}{8} & -3 \le t < -1 \\ \frac{1}{2} & -1 \le t < 1 \\ \frac{7}{8} & 1 \le t < 3 \\ 1 & 3 \le t \end{cases}$$

Plot of $F_Y$:



## Problem 3: Triple Flip Tally

Let $X$ be a discrete random variable which can take only the value $x = 0, 1, 2, 3, 4, 5, 6$ such that the cumulative distribution function is defined by $F_X(x) = \dfrac{x^2 + x}{42}$ for the above values. Find the probability density function of $X$.

**Solution.** Probability mass function of $X$:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $P(X = x)$ | 0 | $\frac{2}{42}$ | $\frac{4}{42}$ | $\frac{6}{42}$ | $\frac{8}{42}$ | $\frac{10}{42}$ | $\frac{12}{42}$ |

To find the probabilities for each value of $X$, we use the formula $P(x = k) = F_X(k) - F_X(k-1)$.

## Problem 4

Suppose $X$ is a random variable with binomial distribution $B\left(4, \dfrac{2}{3}\right)$. Find the probability density function of $2X + 1$.

**Solution.** Probability mass function of $X$:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{3^4}$ | $\binom{4}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^3$ | $\binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$ | $\binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1$ | $\binom{4}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0$ |

Probability mass function of $2X + 1$:

| $x$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $P(2X + 1 = x)$ | $\frac{1}{81}$ | $\frac{8}{81}$ | $\frac{8}{27}$ | $\frac{32}{81}$ | $\frac{16}{81}$ |

## Problem 5

Suppose that only 30% of all drivers come to a complete stop at an intersection. Let $X$ be the number of drivers which come to a complete stop, among 10 randomly chosen drivers coming to an intersection.

a. What is the probability distribution of $X$?

b. Determine $P(X = 2)$.

c. Determine the probability that at least 2 out of the 10 drivers will come to a complete stop.

d. What is the most likely number of drivers to come to a full stop among 10 drivers?

**Solution.**

a. $X \sim B(10, 0.3)$

b. $P(X = 2) = \dbinom{10}{2}(0.3)^2(0.7)^8 = \boxed{0.2335}$

c. $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1) = 1 - (0.7)^{10} - \binom{10}{1}(0.3)(0.7)^9 = \boxed{0.8507}$.

d. The highest probability in the PMF is that of $P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7 = 0.2668$, which means that $\boxed{3}$ is the most likely number of drivers to come to a full stop.

**Problem 6**

Of the items manufactured by a certain process, 20% are defective. Of the defective items 60% can be repaired.

a. Find the probability that a randomly chosen item is defective and cannot be repaired.

b. Find the probability that exactly 2 of 20 randomly chosen items are defective and cannot be repaired.

**Solution.**

a. We know $P(R \mid D) = 0.6$ and $P(D) = 0.2$, hence we are looking for $P(D \cap R^{\mathsf{c}}) = P(D)P(R^{\mathsf{c}} \mid D) = P(D)(1 - P(R \mid D)) = 0.2(1 - 0.6) = \boxed{0.08}$.

b. $X \sim B(20, 0.08)$, where $X$ is the number of defective items that cannot be repaired.
$P(X = 2) = \binom{20}{2}(0.08)^2(0.92)^{18} = \boxed{0.2712}$.

**Problem 7**

A distributor receives a large shipment of components. The distributor would like to accept the shipment if 10 or fewer of the components are defective and to return it otherwise. She decides to sample 10 components and to return the shipment if more than 1 component is defective.

a. If the proportion of defectives in the batch is in fact 10%, what is the probability that she will return the shipment?

b. If the proportion of defectives in the batch is in fact 20%, what is the probability that she will return the shipment?

c. The distributor decides to accept the shipment only if none of the components in the batch is defective. What is the minimum number of items she should sample if she wants a probability no greater than 1% of accepting the shipment if 20% of the items are defective?

**Solution.**

a. $X \sim B(10, 0.1)$, where $X$ is the number of defective components in the sample.
$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - (0.9)^{10} - \binom{10}{1}(0.1)(0.9)^9 = \boxed{0.2639}$.

b. $X \sim B(10, 0.2)$, where $X$ is the number of defective components in the sample.
$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - (0.8)^{10} - \binom{10}{1}(0.2)(0.8)^9 = \boxed{0.7316}$.

c. Now, $X \sim B(y, 0.2)$, where $y$ is the number of items she samples. We want to find the smallest integer $y$ such that $P(X = 0) < 0.01$. This means we solve $(0.8)^y < 0.01 \iff y > \log_{0.8} 0.01 \implies y > 20.6377$. Hence, we need to sample at least $\boxed{21}$ items.