1 Syntactic Definitions

1.1 Axiomatic System

We also define an axiomatization of *ReLo* based on other dynamic logics tailored to reason about programs. We discuss and define *ReLo*'s axioms as follows. Let φ and ψ be formulas.

Definition 1. *Axiomatic System*

(PL) *Enough Propositional Logic tautologies*

(K) $[t, \pi](\varphi \to \psi) \to ([t, \pi]\varphi \to [t, \pi]\psi)$

(And) $[t, \pi](\varphi \wedge \psi) \leftrightarrow [t, \pi]\varphi \wedge [t, \pi]\varphi$

(Du) $[t, \pi]\varphi \leftrightarrow \neg\langle t, \pi\rangle\neg\varphi$

(R) $\langle t, \pi\rangle\varphi \leftrightarrow \varphi$ *if* $f(t, \pi) = \epsilon$

(It) $\varphi \wedge [t, \pi][t, \pi^\star]\varphi \leftrightarrow [t, \pi^\star]\varphi$

(In) $\varphi \wedge [t, \pi^\star](\varphi \to [t, \pi]\varphi) \to [t, \pi^\star]\varphi$

and the following rules.

$$\textbf{\textit{(MP)}} \quad \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

$$\textbf{\textit{(Gen)}} \quad \frac{\varphi}{[t, \pi]\varphi}$$

Axioms **(PL)**, **(K)**, **(And)** and **(Du)** are standard in Modal Logic literature (along with rules **(MP)** and **(Gen)**). Axiom **(It)** respectively denotes the reasoning over nondeterministic iteration denoted by the operator *, following a similar idea portrayed by Axiom (vii) as in Propositional Dynamic Logic (PDL), denoting an intuitive meaning regarding the notion of program iteration: if φ holds in the current state and after a single execution of π with t, any finite nondeterministic number of iterations of π with t preserves φ's truth value, then φ must hold after any (nondeterministic finite) number of iterations of π with t.

Axiom **(In)** is an axiom which carries the same ideia for PDL. It enables the inductive reasoning on programs by carrying the following intuitive meaning: "considering that φ holds in the current state, if after any (nondeterministic finite) number of iterations of π with its respective input t φ holds, then it will hold after any number of iterations of π taking t as its input.

We discuss the proofs of validity (w.r.t. *ReLo*'s model) of **(R)**, **(It)** and **(In)** as follows.

Definition 2. *Validity of ReLo's axiomatic system*

1. ***(R)***

Proof.
Suppose by contradiction that exists a state s from a model $\mathcal{M} = \langle S, R_\pi, \delta, \lambda, V \rangle$ where **(R)** does not hold. There are two possible cases.
$(\rightarrow)$
Suppose by contradiction $\mathcal{M}, s \Vdash \langle t, \pi \rangle \varphi$ and $\mathcal{M}, s \nVdash \varphi$. $\mathcal{M}, s \Vdash \langle t, \pi \rangle \varphi$ iff there is a state $v \in S$ such that $sR_\pi v$. Because $f(t, \pi) = \epsilon, s = v$ (i.e., in this execution no other state is reached from s). Therefore, $\mathcal{M}, s \Vdash \varphi$, contradicting $\mathcal{M}, s \nVdash \varphi$.
$(\leftarrow)$
Suppose by contradiction $\mathcal{M}, s \Vdash \varphi$ and $\mathcal{M}, s \nVdash \langle t, \pi \rangle \varphi$. In order to $\mathcal{M}, s \nVdash \langle t, \pi \rangle \varphi$, for every state $v \in S$ such that $sR_\pi v$, $\mathcal{M}, v \nVdash \varphi$. Because $f(t, \pi) = \epsilon, s = v$ (i.e., in this execution no other state is reached from s). Therefore, $\mathcal{M}, v \Vdash \varphi$, contradicting $\mathcal{M}, v \nVdash \varphi$.

□

2. ***(It)***

Proof.
Suppose by contradiction that exists a state s from a model $\mathcal{M} = \langle S, R_\pi, \delta, \lambda, V \rangle$ where **(It)** does not hold. There are two possible cases.
$(\rightarrow)$
Suppose by contradiction $\mathcal{M}, s \Vdash \varphi \wedge [t, \pi][t, \pi^\star]\varphi$ and $\mathcal{M}, s \nVdash [t, \pi^\star]\varphi$. Therefore, $\mathcal{M}, s \Vdash \varphi$ and $\mathcal{M}, s \Vdash [t, \pi][t, \pi^\star]\varphi$. For $\mathcal{M}, s \Vdash [t, \pi][t, \pi^\star]\varphi$, every state $w \in S$ such that $sR_\pi w$, $\mathcal{M}, w \Vdash [t, \pi^\star]\varphi$. Then, for $\mathcal{M}, w \Vdash [t, \pi^\star]\varphi$, every state $v \in S$ such that $wR_{\pi^\star} v$, $\mathcal{M}, v \Vdash \varphi$. From $\mathcal{M}, s \nVdash [t, \pi^\star]\varphi$, there is a state $u \in S$ such that $sR_{\pi^\star} u$ and $\mathcal{M}, u \nVdash \varphi$. Because $sR_\pi w$ and $wR_{\pi^\star} v$ from $R_{\pi^\star}$ we have $sR_{\pi^\star} v$. Then, for all u such that $sR_{\pi^\star} u$, $\mathcal{M}, u \Vdash \varphi$ which contradicts the existence of a state u such that $sR_{\pi^\star} u$ and $\mathcal{M}, u \nVdash \varphi$.
$(\leftarrow)$
Suppose by contradiction $\mathcal{M}, s \Vdash [t, \pi^\star]\varphi$ and $\mathcal{M}, s \nVdash \varphi \wedge [t, \pi][t, \pi^\star]\varphi$. Then, $\mathcal{M}, s \Vdash \neg(\varphi \wedge [t, \pi][t, \pi^\star]\varphi)$ which is $\mathcal{M}, s \Vdash \neg\varphi$ or $\mathcal{M}, s \Vdash \neg[t, \pi][t, \pi^\star]\varphi$. In order to $\mathcal{M}, s \Vdash [t, \pi^\star]\varphi$, for each state $w \in S$ such that $sR^\star_\pi w$, $\mathcal{M}, w \Vdash \varphi$. By the definition of $R^\star_\pi$, $sR^\star_\pi s$. Then, $\mathcal{M}, s \Vdash \varphi$ which contradicts $\mathcal{M}, s \Vdash \neg\varphi$. Now, considering $\mathcal{M}, s \Vdash \neg[t, \pi][t, \pi^\star]\varphi$, for each state $v \in S$ such that $sR_\pi v$, $\mathcal{M}, v \nVdash [t, \pi^\star]\varphi$. Then, to $\mathcal{M}, v \nVdash [t, \pi^\star]\varphi$, there exists a state $u \in S$ such that $vR^\star_\pi u$ and $\mathcal{M}, u \nVdash \varphi$. Because $sR_\pi v$ and $vR^\star_\pi u$, $sR^\star_\pi u$. From $\mathcal{M}, w \Vdash \varphi$, there is no state u such that $\mathcal{M}, u \nVdash \varphi$, which is a contradiction.

□

3. ***(Ind)***

Proof.
Suppose by contradiction that exists a state s from a model $\mathcal{M} = \langle S, R_\pi, \delta, \lambda, V \rangle$ where **(Ind)** does not hold. Therefore, suppose $\mathcal{M}, s \Vdash \varphi \wedge [t, \pi^\star](\varphi \rightarrow [t, \pi]\varphi)$ and $\mathcal{M}, s \nVdash [t, \pi^\star]\varphi$. Then, $\mathcal{M}, s \Vdash \varphi$ and $\mathcal{M}, s \Vdash [t, \pi^\star](\varphi \rightarrow [t, \pi]\varphi)$. Then, for all states $w \in S$ such that $sR^\star_\pi w$, $\mathcal{M}, w \Vdash \varphi \rightarrow [t, \pi]\varphi$ which is $\mathcal{M}, w \Vdash \neg\varphi \vee [t, \pi]\varphi$. Then, $\mathcal{M}, w \Vdash \neg\varphi$ or $\mathcal{M}, w \Vdash [t, \pi]\varphi$. Considering $\mathcal{M}, w \Vdash \neg\varphi$, by $R^\star_\pi$ we have $sR^\star_\pi s$. Because $\mathcal{M}, s \Vdash \varphi$, there is a state w (namely, $s = w$) where $\mathcal{M}, w \Vdash \neg\varphi$ does not hold, resulting in a contradiction. Now, considering $\mathcal{M}, w \Vdash [t, \pi]\varphi$, for each state $v \in S$ such that $wR_\pi v$, $\mathcal{M}, v \Vdash \varphi$. Because $sR^\star_\pi w$ and $wR_\pi v$, by the definition of $R^\star_\pi$ $sR^\star_\pi v$. Then, if $\mathcal{M}, s \nVdash [t, \pi^\star]\varphi$ then there is a state $u \in S$ such that $sR_\pi u$ and $\mathcal{M}, u \nVdash \varphi$. Because for each state $v \in S$ such that $wR_\pi v$, $\mathcal{M}, v \Vdash \varphi$, such state u cannot exist as it is a contradiction. $\square$