

## Left mixed case

In case  $i <_2 j <_1 k$ , we can write  $\Gamma = Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3$ , where  $Y_i$ 's are clouds of  $<_1$  and  $X_j$ 's are clouds of  $<_2$ . We denote by  $\partial Y_i$  the summands of  $D_0 Y_i$  in which we split a cloud of  $<_2$  belonging to  $Y_i$ .

When we apply the differential  $D_3$  to a monomial, the derived variables cannot be simply doubled, as Fox monomials can contain only one derived variable. This issue can be addressed by imposing a "splitting rule" for monomials containing more than one derived variable. One can verify that the following equation makes the two sides equal for any evaluation, providing a way to resolve double derivations:

$$M_1|\partial Y_i|M_2|\partial Y_i|M_3 = M_1|\partial Y_i|M_2|Y_i|M_3 + M_1|Y_i|M_2|\partial Y_i|M_3$$

This is a consequence of the fact that  $D_0(M||N) = D_0(M)||N + M||D_0(N)$ .

Putting all together, we have the following formulas:

$$\begin{aligned} D_1^i D_2^{jk} (Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3) = & Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3 \end{aligned}$$

$$\begin{aligned} D_1^j D_2^{ik} (Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3) = & Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3 \end{aligned}$$

$$\begin{aligned} D_1^k D_2^{ij} (Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3) = & Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|X_1X_2jX_3|Y_2|Y_3 \\ & + Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|X_1X_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1iX_2jX_3|Y_2|Y_3||Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|X_1X_2jX_3|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1X_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3 \\ & + Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1X_2jX_3|X_1iX_2X_3|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|Y_3 \\ & + Y_1|X_1X_2jX_3|X_1iX_2X_3|Y_2|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3||Y_1|Y_2|X_4kX_5|Y_3 \\ & + Y_1|X_1X_2jX_3|X_1iX_2X_3|Y_2|Y_3||Y_1|Y_2|X_4kX_5|Y_3||Y_1|X_1iX_2jX_3|Y_2|X_4kX_5|Y_3 \end{aligned}$$

$$\begin{aligned}
D_2^{ij} D_1^k (Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3) = & Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | X_1 X_2 j X_3 | Y_2 | Y_3 || Y_1 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | X_1 i X_2 j X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 X_3 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | X_1 X_2 j X_3 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | Y_3 || Y_1 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | X_1 X_2 j X_3 | X_1 i X_2 X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | X_1 X_2 j X_3 | Y_2 | Y_3 \\
& + Y_1 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 X_3 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | Y_3 \\
& + Y_1 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | X_1 i X_2 X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3
\end{aligned}$$

$$\begin{aligned}
D_2^{ik} D_1^j (Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3) = & Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | Y_2 | Y_3 \\
& + Y_1 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | Y_2 | Y_3 \\
& + Y_1 | X_1 X_2 j X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | X_1 X_2 j X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3
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D_2^{jk} D_1^i (Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3) = & Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | Y_2 | Y_3 \\
& + Y_1 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 X_3 | Y_2 | Y_3 \\
& + Y_1 | X_1 i X_2 X_3 | Y_2 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 \\
& + Y_1 | X_1 i X_2 X_3 | Y_2 | Y_3 || Y_1 | X_1 X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3 || Y_1 | X_1 i X_2 j X_3 | Y_2 | X_4 k X_5 | Y_3
\end{aligned}$$





