

Planar case

A general term with $i <_1 j <_1 k$ has the form $\Gamma = Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4$, where Y_i 's are clouds of $<_1$ and X_j 's are clouds of $<_2$. We denote by ∂Y_i the summands of $D_0 Y_i$ in which we split a cloud of $<_2$ belonging to Y_i .

When we apply the differential D_3 to a monomial, the derived variables cannot be simply doubled, as Fox monomials can contain only one derived variable. This issue can be addressed by imposing a "splitting rule" for monomials containing more than one derived variable. One can verify that the following equation makes the two sides equal for any evaluation, providing a way to resolve double derivations:

$$M_1|\partial Y_i|M_2|\partial Y_i|M_3 = M_1|\partial Y_i|M_2|Y_i|M_3 + M_1|Y_i|M_2|\partial Y_i|M_3$$

This is a consequence of the fact that $D_0(M||N) = D_0(M)||N + M||D_0(N)$.

Putting all together, we have the following formulas:

$$\begin{aligned} D_1^i D_2^{jk} (Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4) = \\ &= Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|Y_4 \\ &+ Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \end{aligned}$$

$$\begin{aligned} D_1^j D_2^{ik} (Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4) = \\ &= Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \end{aligned}$$

$$\begin{aligned} D_1^k D_2^{ij} (Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4) = \\ &= Y_1|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|Y_3|X_5kX_6|Y_4||Y_1|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|Y_3|X_5kX_6|Y_4 \end{aligned}$$

$$\begin{aligned}
D_2^{ij} D_1^k (Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4) = \\
= Y_1 | Y_2 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | Y_4 \\
+ Y_1 | Y_2 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | Y_4 | | Y_1 | Y_2 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | Y_2 | Y_3 | X_5 k X_6 | Y_4
\end{aligned}$$

$$\begin{aligned}
D_2^{ik} D_1^j (Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4) = \\
= Y_1 | Y_2 | X_3 j X_4 | Y_3 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | Y_2 | X_3 j X_4 | Y_3 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | Y_2 | X_3 j X_4 | Y_3 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | Y_2 | X_3 j X_4 | Y_3 | Y_4
\end{aligned}$$

$$\begin{aligned}
D_2^{jk} D_1^i (Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4) = \\
= Y_1 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | Y_3 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | Y_3 | Y_4 | | Y_1 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | Y_3 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 \\
+ Y_1 | X_1 i X_2 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | Y_2 | X_3 j X_4 | Y_3 | X_5 k X_6 | Y_4 | | Y_1 | X_1 i X_2 | Y_2 | Y_3 | Y_4
\end{aligned}$$

[illegible]

$$\begin{aligned}
& D_3^{ijk} D_0(Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4) = \\
& \partial Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|\partial Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|\partial Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|\partial Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||\partial Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|\partial Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|\partial Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|\partial Y_4 \\
& + Y_1|Y_2|Y_3|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5X_6|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|X_5X_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5X_6|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|X_5X_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|Y_2|Y_3|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|X_3X_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3X_4|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_1X_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1X_2|Y_2|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4|X_3X_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1iX_2|Y_2|X_1X_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4 \\
& + Y_1|X_1X_2|Y_2|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4||Y_1|X_1iX_2|Y_2|X_3jX_4|Y_3|X_5kX_6|Y_4
\end{aligned}$$