

Right mixed case

In case $i <_1 j <_2 k$, we can write $\Gamma = Y_1|X_1iX_2|X_3jX_4kX_5|Y_3$, where Y_i 's are clouds of $<_1$ and X_j 's are clouds of $<_2$. We denote by ∂Y_i the summands of $D_0 Y_i$ in which we split a cloud of $<_2$ belonging to Y_i .

When we apply the differential D_3 to a monomial, the derived variables cannot be simply doubled, as Fox monomials can contain only one derived variable. This issue can be addressed by imposing a "splitting rule" for monomials containing more than one derived variable. One can verify that the following equation makes the two sides equal for any evaluation, providing a way to resolve double derivations:

$$M_1|\partial Y_i|M_2|\partial Y_i|M_3 = M_1|\partial Y_i|M_2|Y_i|M_3 + M_1|Y_i|M_2|\partial Y_i|M_3$$

This is a consequence of the fact that $D_0(M||N) = D_0(M)||N + M||D_0(N)$.

Putting all together, we have the following formulas:

$$\begin{aligned} D_1^i D_2^{jk} (Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3) = \\ &= Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3 \\ &+ Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3 \\ &+ Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\ &+ Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \end{aligned}$$

$$\begin{aligned} D_1^j D_2^{ik} (Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3) = \\ &= Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3 \\ &+ Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\ &+ Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\ &+ Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \end{aligned}$$

$$\begin{aligned}
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|X_1X_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|X_3X_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|Y_2|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1X_2|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4X_5|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1X_2|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|X_3X_4X_5|Y_3 \\
& + Y_1|X_1X_2|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|X_1X_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4X_5|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4X_5|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|Y_3||Y_1|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|Y_2|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|Y_3 \\
& + Y_1|Y_2|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3||Y_1|X_1iX_2|Y_2|X_3X_4kX_5|X_3jX_4X_5|Y_3 \\
& + Y_1|X_1iX_2|Y_2|X_3jX_4kX_5|Y_3||Y_1|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|Y_3 \\
& + Y_1|X_1iX_2|Y_2|Y_3||Y_1|Y_2|X_3jX_4kX_5|Y_3||Y_1|X_1iX_2|Y_2|X_3jX_4X_5|X_3X_4kX_5|Y_3
\end{aligned}$$

