RELATIVISTIC QM

Fran Štimac

2025

university of groningen

faculty of science and engineering

Contents

Contents ii

Introduction iii

1 Classical Field Theory 1

1.1 Dynamics of fields 1

Fields and their properties 1 • Action and Lagrangian 2

Introduction

Throughout the course **natural units** will be used

Definition .1 (Natural units).

$$c = \hbar = 1, \tag{1}$$

making [L] = [T] and [Mass] = [Energy]. This means that the speed of light c and the reduced Planck constant $\hbar$ are set to unity. This is a common practice in high energy physics and quantum field theory. The units of length, time and mass are then expressed in terms of energy. The consequence of this is when short scales are probed, high energies are needed.

1 Classical Field Theory

1.1 Dynamics of fields

Fields and their properties

A scalar field, named a has a formula,

$$\phi_a(\mathbf{x}, t), \tag{1.1}$$

like the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$. As the space and time are now combined, the manifold has to be described using Minkowski spacetime.

Definition 1.1 (Metric).

$$ds^2 = -(dx^0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2, \tag{1.2}$$

or can be rewritten as a four-vector,

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}), \tag{1.3}$$

where $x^0 = ct$ and $x^i = x^i$ for $i = 1, 2, 3$. The metric tensor is then,

Definition 1.2 (Metric tensor for Minkowski space).

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1.4}$$

giving the metric as,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta^{\mu\nu} dx_\mu dx_\nu. \tag{1.5}$$

Note (Properties of the metric tensor).

$$\eta^{-1}\eta = \mathbb{1}, \quad \eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho. \tag{1.6}$$

The covariant and contravariant vectors are related by the metric tensor,

$$x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu, \tag{1.7}$$

and another tensor can be defined as,

$$B^\mu = \eta^{\mu\nu} A_\nu, \quad A_\mu = \eta_{\mu\nu} B^\nu. \tag{1.8}$$

Definition 1.3 (Contraction of tensors).

$$\eta_{\mu\alpha}T^{\alpha\nu\rho} = T_{\mu}^{\nu\rho} \tag{1.9}$$

Returning back to the definitions of the electric and magnetic fields, a field can be written as

$$A^{\mu}(\mathbf{x}, t) = (\phi, \mathbf{A}), \tag{1.10}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The electric and magnetic fields are connected to it by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1.11}$$

Action and Lagrangian

The action for a system is defined as,

Definition 1.4 (Action).

$$S = \int d^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.12}$$

Note (Partial derivatives).

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right). \tag{1.13}$$

A field is defined with no variation at the endpoints, $\delta\phi_a = 0$, and the action is stationary. Varying the action gives,

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a)\right). \tag{1.14}$$

Using the property,

Lemma 1.1.

$$\delta(AB) = A\,\delta B + B\,\delta A, \tag{1.15}$$

the variation of the action can be written as,

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right). \tag{1.16}$$

As the last term is a total derivative, it becomes 0 as it vanishes at the endpoints. The Euler-Lagrange equation is then,

Definition 1.5 (Euler-Lagrange equation).

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0. \tag{1.17}$$

Maxwell's equations

For example, taking the Lagrangian of the electromagnetic field,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{1.18}$$

the Euler-Lagrange equation is,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \tag{1.19}$$

Question. Derive the Maxwell's equations for the electromagnetic field, given the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial_\alpha A_\beta)\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{1.20}$$

Lemma 1.2 (Dirac delta).

$$\frac{\partial A_\mu}{\partial A_\nu} = \delta^\nu_\mu \tag{1.21}$$

Knowing that there is no A_ν dependence as all A_ν terms are in the derivative, the Euler-Lagrange equation simplifies to,

$$\partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\tau)} \right) = 0, \tag{1.22}$$

and the Maxwell's equations are derived as,

$$\begin{aligned} -\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_\mu A_\nu \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\sigma A_\tau)} + \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\sigma A_\tau)}\partial_\alpha A_\beta \right) + \partial_\sigma \left(\partial_\mu A^\mu \eta^{\mu\nu} \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\sigma A_\tau)} \right) &= 0 \\ -\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_\mu A_\nu \delta^\sigma_\alpha \delta^\tau_\beta + \delta^\sigma_\mu \delta^\tau_\nu \partial_\alpha A_\beta \right) + \partial_\sigma \eta^{\mu\nu}(\partial_\mu A^\mu \delta^\sigma_\mu \delta^\tau_\nu) &= 0 \\ -\frac{1}{2}\partial_\sigma \left(\eta^{\mu\sigma}\eta^{\nu\tau}\partial_\mu A_\nu + \eta^{\sigma\alpha}\eta^{\tau\beta}\partial_\alpha A_\beta \right) + \partial_\sigma(\eta^{\sigma\tau}\partial_\mu A^\mu) &= 0 \\ -\partial_\sigma \partial^\sigma A^\tau + \partial^\tau \partial_\mu A^\mu &= 0 \\ \text{reindexing } \sigma \to \mu, \tau \to \nu \qquad -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= 0 \\ -\partial_\mu F^{\mu\nu} &= 0, \end{aligned} \tag{1.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

Klein-Gordon equation

Another example is the Klein-Gordon equation,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2,\end{aligned} \tag{1.24}$$

which is a Lagrangian for a scalar field.