HYDRODYNAMICS

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Contents

Introduction

Interstellar Medium

Components of the interstellar medium (ISM) are

- Cold neutral medium (CNM) – $T \sim 70$ K
- Warm neutral medium (WNM) – $T \sim 8000$ K
- HII regions – photoionized gas
- Warm ionized medium (WIM) – photoionized gas
- Hot intercloud medium (HIM) – collisionally ionized gas
- Molecular gas – mostly in molecular clouds

Property	CNM	WNM	HII reg	WIM	HIM	Mol
Density (cm^{-3})	30	0.3	$10 - 10^4$	0.1	10^{-3}	$> 10^2$
Temperature (K)	$30 - 120$	8000	$\approx 10^4$		$> 10^6$	~ 10
Thermal speed (km s^{-1})	~ 1	8	~ 10		> 100	~ 0.1
Observed speed (km s^{-1})	few$-$10	$8 - 15$	$15 - 25$		-	~ 1
Total Mass ($10^9 M_\odot$)	2	2	0.1	1	< 0.1	1
Scale-height (kpc)	0.1	$0.2 - 0.3$	-	1	3	0.06
Maximum Radius (kpc)	$\lesssim 25$	25	~ 15	-	-	~ 10
Filling factor	0.01	0.25	0.02	0.25	≈ 0.5	0.001
Main observables	21-cm		recomb/forbid		X-rays	CO

Table 1 / Properties of different interstellar medium phases.

Cold neutral gas is made of clouds, 90% H, 10% He, and trace elements. The photoionized gas is mostly HII regions, and can be seen through emission of recombination (Lyα, Hα, Hβ, ...), forbidden lines ([OII], [OIII], [NII], [SII], ...), and bremsstrahlung. Hot gas is collisionally ionized and emits X-rays. Hot gas extends to hunderds of kiloparsecs making it difficult to observe.

The thermal velocity is given by

$$v_{\text{th}} = \sqrt{\frac{kT}{m}} \tag{2}$$

A magnetic field can be detected by

- Zeeman effect – splitting of spectral lines
- Synchrotron radiation – non-thermal radiation
- Faraday rotation – rotation of polarization angle

- Polarization of stellar light
- Dust polarization

The magnetic field is important for gas dynamics, cloud stability, and shocks.

Equilibriums in Hydrodynamics

The medium can be divided into three phases:

- Cold phase – $T \lesssim 100$ K
- Warm phase – $T \sim 10^4$ K
- Hot phase – $T \gtrsim 10^6$ K

and the pressures are roughly equivalent in the three phases. Thermal equilibrium is only a reasonable approximation locally, not globally for the ISM. This is contrasted by the pressure equilibrium which is often fulfilled globally in the ISM.
Hydrostatic equilibrium is the balance between the pressure and gravitational force. It is important for molecular clouds, galaxy discs, hot halos, stars, ...

Collisional Processes

In ideal fluids there are two types of collisions, elastic (change in $\mathbf{v}_p$, and energy is conserved), and inelastic collisions (with the transfer of energy). The mean free path is the average distance between collisions,

Note (Mean free path).

$$\ell = \frac{1}{\underbrace{n}_{\text{Number density}}\ \underbrace{\sigma}_{\text{Cross-section}}}, \tag{3}$$

and the average time between collisions is

$$\tau = \frac{\ell}{v_{\text{th}}} = \frac{1}{n\sigma v_{\text{th}}} \sim 10^4 \left(\frac{n}{\text{cm}^{-3}}\right)^{-1} \text{s} \quad \text{(WIM)}. \tag{4}$$

The shortest inelastic collision in WIM is the ionization of O^+ to OII, and the excitation time is,

$$\tau_{\text{exc}}(\text{O}^+) \sim 10^{11} \left(\frac{n_{\text{O}^+}/n_e}{10^{-3}}\right)^{-1} \text{s}. \tag{5}$$

In CNM, the average collision time is $\sim 10^8$ s, mostly due to the C^+. The velocities as distributed as a Maxwellian distribution,

Definition .1 (Maxwellian distribution).

$$f(v)\,dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv. \tag{6}$$

Using the distribution, the following characteristing speeds can be derived,

Note (Peak velocity).

$$v_{\text{peak}} = \sqrt{\frac{2kT}{m}}, \tag{7}$$

Note (RMS velocity).

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}. \tag{8}$$

Note (Mean velocity).

$$\langle v\rangle = \int_0^\infty v f(v)\,dv = \sqrt{\frac{8kT}{\pi m}}, \tag{9}$$

1 Fluids as continuums

1.1 Fluid element

A **fluid element** is a portion of the fluid of size l that is large compared to the mean free path ℓ, but small compared to the characteristic length scale of the system L,

$$\ell \ll l \ll L. \tag{1.1}$$

The length scale is defined as

$$L \sim \frac{q}{\nabla q} \tag{1.2}$$

where q is a macroscopic physical quantity. In a CNM, the "worst case scenario", $A = A(\mathrm{H},\mathrm{H}) \approx 10^{-15}\,\mathrm{cm}^2$, and $n \approx 30\,\mathrm{cm}^{-3}$, so $\ell \approx 10^{13}\,\mathrm{cm}$. This means that any structure in the ISM would have an interaction that is smaller than this.

1.2 Advection

The **advection** of a fluid element is the transport of the element by the flow. Consider a fluid element moving with velocity $\mathbf{u}$ and has a generic quantity q which is a function of position and time, $q(\mathbf{r}, t)$. The change in q is given by the **Lagrangian derivative**,

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q. \tag{1.3}$$

It is explicitly defined as,

$$\begin{aligned}
\frac{\mathrm{D}q}{\mathrm{D}t} &= \lim_{\delta t\to 0} \frac{q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - q(\mathbf{r}, t)}{\delta t} \\
&= \lim_{\delta t\to 0} \frac{q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - q(\mathbf{r}, t) + q(\mathbf{r}, t+\delta t) - q(\mathbf{r}, t+\delta t)}{\delta t} \\
&= \left(\frac{\partial q}{\partial t}\right)_{(\mathbf{r},t)} + \lim_{\delta t\to 0} \frac{q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - q(\mathbf{r}, t+\delta t)}{\delta t} \\
&= \left(\frac{\partial q}{\partial t}\right)_{(\mathbf{r},t)} + \lim_{\delta t\to 0} \frac{\delta\mathbf{r}\cdot(\nabla q)_{(\mathbf{r},t)}}{\delta t} \\
&= \frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q.
\end{aligned} \tag{1.4}$$

Taking q to be a generic quantity,

Definition 1.1 (Lagrangian derivative).

$$\frac{\mathrm{D}}{\mathrm{D}t} = \underbrace{\frac{\partial}{\partial t}}_{\text{Eulerian derivative}} + \overbrace{\mathbf{u}\cdot\nabla}^{\text{Advection operator}}. \tag{1.5}$$

1.3 Continuity equation

Consider a portion of fluid of density ρ constained in a volume V. The mass inside the volume is,

$$M = \int_V \rho \, dV. \tag{1.6}$$

The change in mass is given by the rate of change of mass inside the volume,

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int_V \rho \, dV = -\oint_S \rho \mathbf{u} \cdot d\mathbf{S}, \tag{1.7}$$

where S is the surface of the volume V. Using the divergence theorem,

Theorem 1.1 (Divergence theorem).

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, dV, \tag{1.8}$$

the equation becomes,

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho \, dV &= -\oint_S \rho \mathbf{u} \cdot d\mathbf{S} \\ &= -\int_V \nabla \cdot (\rho \mathbf{u}) \, dV \\ &\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \end{aligned} \tag{1.9}$$

which is the **continuity equation** or the **mass conservation equation**. It can be rewritten as,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} = 0. \tag{1.10}$$

For an **incompressible fluid**, $\nabla \cdot \mathbf{u} = 0 \Leftrightarrow \mathrm{D}\rho/\mathrm{D}t = 0$.

1.4 Euler equation

Consider an infinitesimal fluid element, its mass would be,

$$m = \rho \delta V \tag{1.11}$$

and the acceleration is,

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}. \tag{1.12}$$

The two forces acting on the fluid element are the pressure force and the gravitational force. The pressure force is,

$$\mathbf{F}_P = -\int_S P \, d\mathbf{S} = -\int_V \nabla P \, dV, \tag{1.13}$$

which then for an infinitesimal volume becomes,

$$\mathbf{F}_P = -\nabla P \delta V. \tag{1.14}$$

The gravitational force for an infinitesimal volume is,

$$m\mathbf{g} = \rho \delta V \mathbf{g}. \tag{1.15}$$

$g = -\nabla\Phi$

Bringing everything together,

Definition 1.2 (Euler equation).

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla P - \rho \nabla \Phi \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi. \tag{1.16}$$

Currently we have 6 unknowns, $\rho, \mathbf{u}, P, \Phi$, and 3 equations, the continuity equation, the Euler equation, and the Poisson equation,

Definition 1.3 (Poisson equation).

$$\nabla^2 \Phi = 4\pi G(\rho + \rho_{\text{ext}}). \tag{1.17}$$

1.5 Equation of state

The equation of state is a relation between the pressure, density, and temperature of a fluid. For an ideal gas,

$$P = \frac{\rho k T}{\underbrace{\mu}_{\text{Atomic weight}} \underbrace{m_p}_{\text{Proton mass}}}, \tag{1.18}$$

A gas can be assumed to be **baroclinic**, $P = P(\rho, T)$, or **barotropic**, $P = P(\rho)$. Typical situations are **isotropic fluids**, $P \propto \rho$, and **adiabatic fluids**, $P \propto \rho^\gamma$.

1.6 Energy equation

Firstly, the total energy density of a system is,

$$\epsilon = \rho\left(\underbrace{\frac{1}{2}\mathbf{u}^2}_{\text{Kinetic energy density}} + \underbrace{\mathcal{U}}_{\text{Specific internal energy, } \mathcal{U} = U/M} + \overbrace{\Phi}^{\text{Gravitational potential energy density}}\right), \tag{1.19}$$

Taking the derivative,

$$\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = \frac{\epsilon}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\mathbf{u}^2 + \mathcal{U} + \Phi\right). \tag{1.20}$$

The derivative of the spefic kinetic energy is,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\mathbf{u}^2\right) = \mathbf{u} \cdot \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{\mathbf{u}}{\rho} \cdot \nabla P - \mathbf{u} \cdot \nabla \Phi. \tag{1.21}$$

Using the first law of thermodynamics,

$$d\mathcal{U} + P\, d\mathcal{V} = \delta Q, \tag{1.22}$$

and taking the derivative,

$$\frac{\mathrm{D}\mathcal{U}}{\mathrm{D}t} + P\frac{\mathrm{D}\mathcal{V}}{\mathrm{D}t} = \dot{Q}. \tag{1.23}$$

The specific volume is $\mathcal{V} = 1/\rho$, so the derivative is,

$$\frac{\mathrm{D}\mathcal{V}}{\mathrm{D}t} = -\frac{1}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{\rho}\nabla \cdot \mathbf{u}. \tag{1.24}$$

The derivative of the gravitational potential energy is,

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \mathbf{u} \cdot \nabla\Phi + \frac{\partial \Phi}{\partial t}. \tag{1.25}$$

Putting everything together,

Definition 1.4 (Conservation of energy density)**.**

$$\begin{aligned} \frac{\mathrm{D}\epsilon}{\mathrm{D}t} &= -\epsilon\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla P - P\nabla \cdot \mathbf{u} + \rho\dot{Q} + \rho\underbrace{\frac{\partial \Phi}{\partial t}}_{\text{Excluded as being negligible}}, \\ \frac{\partial \epsilon}{\partial t} &+ \nabla \cdot ((\epsilon + P)\mathbf{u}) = \rho\dot{Q}. \end{aligned} \tag{1.26}$$

The physical meaning of this equation can be unveiled by integrating the equation, using the divergence theorem,

$$\frac{\partial}{\partial t}\int_V \epsilon\, dV + \underbrace{\oint_S \epsilon\mathbf{u} \cdot d\mathbf{S}}_{\text{Energy flux}} + \underbrace{\oint_S P\mathbf{u} \cdot d\mathbf{S}}_{\text{Work done by the fluid } dW/dt} = \int_V \rho\dot{Q}\, dV. \tag{1.27}$$

1.7 Heat exchange

Conduction

Thermal conduction is the transfer of heat from a region of high temperature to a region of low temperature. Mathematically it can be assumed that the flux of thermal conduction is linearly dependent on the temperature gradient,

$$\begin{aligned} \mathbf{q} &= -\kappa\nabla T, \\ &= -\kappa_{\mathrm{Sp}}T^{5/2}\nabla T. \end{aligned} \tag{1.28}$$

If there is a magnetic field present, the supression factor f has to be taken into account, and is inserted in the beginning of the formula. It is usually between $0.01 - 0,2$, making it a non-negligible factor.
Putting everything into the energy equation,

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot ((\epsilon + P)\mathbf{u}) = \rho\dot{Q} - \nabla \cdot \mathbf{q}. \tag{1.29}$$

Convection