RELATIVISTIC QM

FRAN ŠTIMAC

2025

university of groningen

faculty of science and engineering

Contents

Introduction

Throughout the course **natural units** will be used

Definition .1 (Natural units).

$$c = \hbar = 1, \tag{1}$$

making [L] = [T] and [Mass] = [Energy]. This means that the speed of light c and the reduced Planck constant $\hbar$ are set to unity. This is a common practice in high energy physics and quantum field theory. The units of length, time and mass are then expressed in terms of energy. The consequence of this is when short scales are probed, high energies are needed.

1 Classical Field Theory

1.1 Dynamics of fields

Fields and their properties

A scalar field, named a has a formula,

$$\phi_a(\mathbf{x}, t), \tag{1.1}$$

like the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$. As the space and time are now combined, the manifold has to be described using Minkowski spacetime.

Definition 1.1 (Metric).

$$ds^2 = -(dx^0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2, \tag{1.2}$$

or can be rewritten as a four-vector,

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}), \tag{1.3}$$

where $x^0 = ct$ and $x^i = x^i$ for $i = 1, 2, 3$. The metric tensor is then,

Definition 1.2 (Metric tensor for Minkowski space).

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1.4}$$

giving the metric as,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta^{\mu\nu} dx_\mu dx_\nu. \tag{1.5}$$

Note (Properties of the metric tensor).

$$\eta^{-1}\eta = \mathbb{1}, \quad \eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho. \tag{1.6}$$

The covariant and contravariant vectors are related by the metric tensor,

$$x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu, \tag{1.7}$$

and another tensor can be defined as,

$$B^\mu = \eta^{\mu\nu} A_\nu, \quad A_\mu = \eta_{\mu\nu} B^\nu. \tag{1.8}$$

Definition 1.3 (Contraction of tensors).

$$\eta_{\mu\alpha}T^{\alpha\nu\rho} = T_{\mu}{}^{\nu\rho} \tag{1.9}$$

Returning back to the definitions of the electric and magnetic fields, a field can be written as

$$A^{\mu}(\mathbf{x}, t) = (\phi, \mathbf{A}), \tag{1.10}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The electric and magnetic fields are connected to it by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1.11}$$

Action and Lagrangian

The action for a system is defined as,

Definition 1.4 (Action).

$$S = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.12}$$

Note (Partial derivatives).

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right). \tag{1.13}$$

A field is defined with no variation at the endpoints, $\delta\phi_a = 0$, and the action is stationary. Varying the action gives,

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu\phi_a)\right). \tag{1.14}$$

Using the property,

Lemma 1.1.

$$\delta(AB) = A\,\delta B + B\,\delta A, \tag{1.15}$$

the variation of the action can be written as,

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right). \tag{1.16}$$

As the last term is a total derivative, it becomes 0 as it vanishes at the endpoints. The Euler-Lagrange equation is then,

Definition 1.5 (Euler-Lagrange equation).

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0. \tag{1.17}$$

Maxwell's equations

For example, taking the Lagrangian of the electromagnetic field,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{1.18}$$

the Euler-Lagrange equation is,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \tag{1.19}$$

Question. Derive the Maxwell's equations for the electromagnetic field, given the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial_\alpha A_\beta)\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{1.20}$$

Lemma 1.2 (Kronecker delta).

$$\frac{\partial A_\mu}{\partial A_\nu} = \delta^\nu_\mu \quad \partial_\mu A^\nu = \delta^\nu_\mu \tag{1.21}$$

Knowing that there is no A_ν dependence as all A_ν terms are in the derivative, the Euler-Lagrange equation simplifies to,

$$\partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\tau)} \right) = 0, \tag{1.22}$$

and the Maxwell's equations are derived as,

Derivation 1.1.

$$\begin{aligned}
-\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_\mu A_\nu \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\sigma A_\tau)} + \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\sigma A_\tau)}\partial_\alpha A_\beta \right) + \partial_\sigma \left(\partial_\mu A^\mu \eta^{\mu\nu} \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\sigma A_\tau)} \right) &= 0 \\
-\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_\mu A_\nu \delta^\sigma_\alpha \delta^\tau_\beta + \delta^\sigma_\mu \delta^\tau_\nu \partial_\alpha A_\beta \right) + \partial_\sigma \eta^{\mu\nu}(\partial_\mu A^\mu \delta^\sigma_\mu \delta^\tau_\nu) &= 0 \\
-\frac{1}{2}\partial_\sigma \left(\eta^{\mu\sigma}\eta^{\nu\tau}\partial_\mu A_\nu + \eta^{\sigma\alpha}\eta^{\tau\beta}\partial_\alpha A_\beta \right) + \partial_\sigma(\eta^{\sigma\tau}\partial_\mu A^\mu) &= 0 \\
-\partial_\sigma \partial^\sigma A^\tau + \partial^\tau \partial_\mu A^\mu &= 0 \\
\text{reindexing } \sigma \to \mu, \tau \to \nu \quad -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= 0 \\
-\partial_\mu F^{\mu\nu} &= 0,
\end{aligned} \tag{1.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the **field strength tensor**.

Klein-Gordon equation

Another example is the Klein-Gordon equation,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}m^2\phi^2 \\ &= \underbrace{\frac{1}{2}\dot{\phi}^2}_{T=\int d^3x \frac{1}{2}\dot{\phi}^2} - \underbrace{\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2}_{V=\int d^3x\left(\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2\right)},\end{aligned} \tag{1.24}$$

which is a Lagrangian for a scalar field. To find the equations of motion, the Euler-Lagrange equation is used,

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \tag{1.25}$$

Derivation 1.2.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} &= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi \frac{\partial(\partial_\nu \phi)}{\partial(\partial_\sigma \phi)} + \frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)}\partial_\nu \phi\right) \\ &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi \delta^\sigma_\nu + \delta^\sigma_\mu \partial_\nu \phi) \\ &= \frac{1}{2}(\eta^{\mu\sigma}\partial_\mu\phi + \eta^{\sigma\nu}\partial_\nu\phi) \\ &= \partial^\sigma \phi,\end{aligned} \tag{1.26}$$

and the Klein-Gordon equation is derived as,

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0. \tag{1.27}$$

First Order Lagrangians

Consider a Lagrangian that is linear in time derivatives,

$$\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \nabla\psi^*\cdot\nabla\psi - m\psi^*\psi, \tag{1.28}$$

treating ψ and ψ^* as independent variables. The Euler-Lagrange equations are then,

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = \frac{i}{2}\dot{\psi} - m\psi \quad \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = -\frac{i}{2}\psi \quad \frac{\partial \mathcal{L}}{\partial \nabla\psi^*} = -\nabla\psi, \tag{1.29}$$

giving the equation of motion,

$$i\dot{\psi} = -\nabla^2\psi + m\psi. \tag{1.30}$$

Compared to the Klein-Gordon equation, only ψ and ψ^* are needed to determine the future evolution compared to needing ϕ and $\dot{\phi}$.

Note on locality

All examples above, as well as all thories of nature are local. This means that there are no terms in the Lagrangian that look like,

$$\mathcal{L} = \int d^3x\, d^3y \phi(\mathbf{x})\phi(\mathbf{y}). \tag{1.31}$$

One of the main reasons for introducing field theories in classical physics is to account for locality.

1.2 Lorenz invariance

Since relativity has to be accounted for, the field theory has to be invariant under Lorentz transformations, i.e.,

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.32}$$

where,

Lemma 1.3 (Lorentz tranformation).

$$\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}. \tag{1.33}$$

The field has to transform as well, a simple example is a scalar field which transforms as,

Note (Scalar field transformation).

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x), \tag{1.34}$$

and the derivative of a scalar field transforms as,

Note (Derivative of scalar field transformation).

$$\partial_\mu \phi(x) \to (\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(\Lambda^{-1}x). \tag{1.35}$$

The inverse transformation is used as coordinates are not relabeled after the transformation. From the Lorentz invariant theory, if $\phi(x)$ is a solution to the equations of motion, then $\phi(\Lambda^{-1}x)$ is also a solution.

Maxwell's equations

Using the Lorentz transformation $A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$,

Derivation 1.3.

$$\begin{aligned}(\partial_\mu A_\nu(x))(\partial_\beta A^\nu(x))\eta^{\mu\beta} &\to (\partial_\mu \eta_{\nu\alpha}\Lambda^\alpha{}_\gamma \eta^{\gamma\varepsilon} A_\varepsilon(\Lambda^{-1}x))(\partial_\beta \Lambda^\nu{}_\delta \eta^{\delta\zeta} A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= \eta_{\nu\alpha}\Lambda^\alpha{}_\gamma \eta^{\gamma\varepsilon}\Lambda^\nu{}_\delta \eta^{\delta\zeta}(\partial_\mu A_\varepsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= \Lambda_\nu{}^\varepsilon \Lambda^{\nu\zeta}(\partial_\mu A_\varepsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= (\partial_\mu A_\varepsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta}\eta^{\varepsilon\zeta} \\ &= (\partial_\mu A_\varepsilon(\Lambda^{-1}x))(\partial_\beta A^\varepsilon(\Lambda^{-1}x))\eta^{\mu\beta},\end{aligned} \tag{1.36}$$

where the derivation used,

Derivation 1.4.

$$\begin{aligned}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} &= \eta_{\alpha\beta} \\ \eta^{\alpha\varepsilon}\eta^{\beta\zeta}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} &= \eta^{\alpha\varepsilon}\eta^{\beta\zeta}\eta_{\alpha\beta} \\ \Rightarrow \quad \Lambda_\nu{}^\varepsilon \Lambda^{\nu\zeta} &= \eta^{\varepsilon\zeta}.\end{aligned} \tag{1.37}$$

For the second part of the Lagrangian,

Derivation 1.5.

$$\begin{aligned}\partial_\mu A^\mu(x) &\to (\Lambda^{-1})^\alpha{}_\mu \partial_\alpha \Lambda^\mu{}_\beta A^\beta(\Lambda^{-1}x)\\ &= \delta^\alpha_\beta \partial_\alpha A^\beta(\Lambda^{-1}x)\\ &= \partial_\alpha A^\alpha(\Lambda^{-1}x),\end{aligned} \tag{1.38}$$

showing that Maxwell's equations are Lorentz invariant.

Klein-Gordon equation

The Klein-Gordon equation is Lorentz invariant, as the derivatives transform as,

Derivation 1.6.

$$\begin{aligned}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} &\to (\Lambda^{-1})^\rho{}_\mu \partial_\rho\phi(\Lambda^{-1}x)(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma\phi(\Lambda^{-1}x)\eta^{\mu\nu}\\ &= \partial_\rho\phi(\Lambda^{-1}x)\partial_\sigma\phi(\Lambda^{-1}x)\Lambda^\rho{}_\mu\eta^{\mu\nu}\Lambda^\sigma{}_\nu\\ &= \partial_\rho\phi(\Lambda^{-1}x)\partial_\sigma\phi(\Lambda^{-1}x)\eta^{\rho\sigma}\end{aligned} \tag{1.39}$$

The action is then Lorentz invariant, however there is no Jacobian factor in the integral after change of variables due to $\det\Lambda = 1$.

First Order Lagrangians

Since the Lagrangian is linear in time derivative, but quadratic in spatial derivatives, the theory is not Lorentz invariant.

1.3 Symmetries

Noeter's theorem

For every continous symmetry of the Lagrangian, there is a conserved current, such that,

$$\partial_\mu j^\mu = 0. \tag{1.40}$$

For example, a transformation $\delta\phi_a(x) = X_a(\phi(x))$ is a symmetry if the Lagrangian changes by a total derivative, $\delta\mathcal{L} = \partial_\mu F^\mu$.

$$\begin{aligned}\delta\mathcal{L} &= \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\\ \Rightarrow \quad j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a - F^\mu.\end{aligned} \tag{1.41}$$

Translation invariance

In classical particle mechanics, invariance in spatial translations leads to momentum conservation, while invariance in time translations leads to energy conservation. Consider the infinitesimal translation,

$$x^\mu \to x^\mu + \epsilon^\mu \quad \Rightarrow \quad \phi(x)_a \to \phi(x)_a + \epsilon^\mu \partial_\mu \phi(x)_a, \tag{1.42}$$

making the Lagrangian transform as,

$$\mathcal{L} \to \mathcal{L} + \epsilon^\mu \partial_\mu \mathcal{L} \tag{1.43}$$

and the conserved current is,

Energy momentum tensor, $\partial_\mu T^\mu{}_\nu = 0$

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \overset{\text{def}}{=} T^\mu{}_\nu . \tag{1.44}$$

The **energy momentum tensor** satisfies,

$$E = \int d^3x\, T^{00}, \qquad P^i = \int d^3x\, T^{0i}. \tag{1.45}$$

Total energy Total momentum

Klein-Gordon equation

The energy momentum tensor for the Klein-Gordon equation is,

Derivation 1.7.

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \\ &= \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \\ \cdot/\ \eta^{\sigma\nu} \quad T^{\mu\nu} &= \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \end{aligned} \tag{1.46}$$

Lorenty transformations and angular momentum

The infinitesimal form of the Lorentz transformation is,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu , \tag{1.47}$$

Infinitesimal Lorentz transformation

From equation (1.33), the infinitesimal Lorentz transformation satisfies,

$$\begin{aligned} (\delta^\mu{}_\sigma + \omega^\mu{}_\sigma)\eta^{\sigma\rho}(\delta^\nu{}_\rho + \omega^\nu{}_\rho) &= \eta^{\mu\nu} \\ \Rightarrow \quad \omega^{\mu\nu} + \omega^{\nu\mu} &= 0, \end{aligned} \tag{1.48}$$

showing that $\omega^{\mu\nu}$ is antisymmetric. The scalar field transformation is then,

$$\phi(x) \to \phi(\Lambda^{-1}x) = \phi(x^\mu - \omega^\mu{}_\nu x^\nu) = \phi(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \phi(x^\mu) + \delta\phi(x). \tag{1.49}$$

The Lagrangian tranforms as,

$$\delta \mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L}), \tag{1.50}$$

and applying Noether's theorem, the conserved current is,

$$j^\mu = -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu . \tag{1.51}$$

For each of the six choices for $\omega^\mu{}_\nu$, the conserved current can be rewritten as,

Definition 1.6 (Lorentz transformation conserved current)**.**

$$(\mathcal{J}^{\mu})^{\rho\sigma} = x^{\rho}T^{\mu\sigma} - x^{\sigma}T^{\mu\rho}. \tag{1.52}$$

1.4 Hamiltonian formalism

The momentum conjugate to the field is defined as,

Definition 1.7.

$$\pi^{a}(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{a}}. \tag{1.53}$$

The Hamiltonian is then,

Definition 1.8.

$$\mathcal{H} = \pi^{a}\dot{\phi}_{a} - \mathcal{L} \quad H = \int d^{3}x\, \mathcal{H}. \tag{1.54}$$

Unlinke the Lagrangian formalism, the Hamiltonian is not manifestly Lorentz invariant.

2 Free Fields

2.1 Canonical quantization

Take the generalized coordinates q_a and their conjugate momenta p_a and promote them to operators $\hat{q}_a$ and $\hat{p}_a$ that satisfy the canonical commutation relations

$$[\hat{q}_a, \hat{p}^b] = i\delta_{ab} \quad \text{and} \quad [\hat{q}_a, \hat{q}_b] = [\hat{p}^a, \hat{p}^b] = 0. \tag{2.1}$$

In field theories this transforms into the commutation relations

Definition 2.1.

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_a^b \quad \text{and} \quad [\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] = 0. \tag{2.2}$$

Free field theories typically have Lagrangians which are quadratic in the fields, so that the equations of motion are linear. To decouple the degrees of freedom, a Fourier tranform is done,

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t) \quad \left(\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2)\right)\phi(\mathbf{p}, t) = 0, \tag{2.3}$$

so for each value of $\mathbf{p}$ we have a harmonic oscillator vibrating at a frequency $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. To quantize the field, an infinite number of harmonic oscillators need to be quantized.

Simple harmonic oscillator

The Hamiltonian for a simple harmonic oscillator is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \tag{2.4}$$

and the creation and annihilation operators are defined as

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2}}q + i\sqrt{\frac{1}{2\omega}}p, & a^\dagger &= \sqrt{\frac{\omega}{2}}q - i\sqrt{\frac{1}{2\omega}}p \\ q &= \sqrt{\frac{1}{2\omega}}(a + a^\dagger), & p &= -i\sqrt{\frac{\omega}{2}}(a - a^\dagger). \end{aligned} \tag{2.5}$$

The commutation relations are

$$[a, a^\dagger] = 1 \quad \text{and} \quad [q, p] = i, \tag{2.6}$$

allowing us to write the Hamiltonian as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{2.7}$$

The commutation relations between the Hamiltonian and the creation and annihilation operators are

$$[H, a] = -\omega a \quad \text{and} \quad [H, a^\dagger] = \omega a^\dagger. \tag{2.8}$$

2.2 Free scalar field

We now apply the quantization of the harmonic oscillator to the free scalar field. We write ϕ and π as a linear sum of an infinite number of creation and annihilation operators,

$$
\begin{aligned}
\phi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \\
\pi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right).
\end{aligned}
\tag{2.9}
$$