

# RELATIVISTIC QM

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# I Introduction

Throughout the course natural units will be used

**Definition .1** (Natural units).

$$c = \hbar = 1, \tag{1}$$

making  $[L] = [T]$  and  $[\text{Mass}] = [\text{Energy}]$ . This means that the speed of light  $c$  and the reduced Planck constant  $\hbar$  are set to unity. This is a common practice in high energy physics and quantum field theory. The units of length, time and mass are then expressed in terms of energy. The consequence of this is when short scales are probed, high energies are needed.

# Classical Field Theory

## 1.1 Dynamics of fields

### Fields and their properties

A scalar field, named  $a$  has a formula,

$$\phi_a(\mathbf{x}, t), \quad (1.1)$$

like the electric and magnetic fields,  $\mathbf{E}(\mathbf{x}, t)$ ,  $\mathbf{B}(\mathbf{x}, t)$ . As the space and time are now combined, the manifold has to be described using Minkowski spacetime.

**Definition 1.1 (Metric).**

$$ds^2 = -(dx^0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2, \quad (1.2)$$

or can be rewritten as a four-vector,

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}), \quad (1.3)$$

where  $x^0 = ct$  and  $x^i = x^i$  for  $i = 1, 2, 3$ . The metric tensor is then,

**Definition 1.2 (Metric tensor for Minkowski space).**

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (1.4)$$

giving the metric as,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta^{\mu\nu} dx_\mu dx_\nu. \quad (1.5)$$

**Note (Properties of the metric tensor).**

$$\eta^{-1}\eta = 1, \quad \eta^{\mu\nu}\eta_{\nu\rho} = \delta_\rho^\mu. \quad (1.6)$$

The covariant and contravariant vectors are related by the metric tensor,

$$x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu, \quad (1.7)$$

and another tensor can be defined as,

$$B^\mu = \eta^{\mu\nu} A_\nu, \quad A_\mu = \eta_{\mu\nu} B^\nu. \quad (1.8)$$

**Definition 1.3** (Contraction of tensors).

$$\eta_{\mu\alpha} T^{\alpha\nu\rho} = T_{\mu}^{\nu\rho} \quad (1.9)$$

Returning back to the definitions of the electric and magnetic fields, a field can be written as

$$A^\mu(\mathbf{x}, t) = (\phi, \mathbf{A}), \quad (1.10)$$

where  $\phi$  is the scalar potential and  $\mathbf{A}$  is the vector potential. The electric and magnetic fields are connected to it by

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1.11)$$

## Action and Lagrangian

The action for a system is defined as,

**Definition 1.4** (Action).

$$S = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \quad (1.12)$$

**Note** (Partial derivatives).

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right). \quad (1.13)$$

A field is defined with no variation at the endpoints,  $\delta\phi_a = 0$ , and the action is stationary. Varying the action gives,

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right). \quad (1.14)$$

Using the property,

**Lemma 1.1.**

$$\delta(AB) = A \delta B + B \delta A, \quad (1.15)$$

the variation of the action can be written as,

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \right) \delta\phi_a + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta\phi_a \right). \quad (1.16)$$

As the last term is a total derivative, it becomes 0 as it vanishes at the endpoints. The Euler-Lagrange equation is then,

**Definition 1.5** (Euler-Lagrange equation).

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) = 0. \quad (1.17)$$

### Maxwell's equations

For example, taking the Lagrangian of the electromagnetic field,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2, \quad (1.18)$$

the Euler-Lagrange equation is,

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \quad (1.19)$$

**Question.** Derive the Maxwell's equations for the electromagnetic field, given the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial_\alpha A_\beta)\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}(\partial_\mu A^\mu)^2. \quad (1.20)$$

**Lemma 1.2** (Kronecker delta).

$$\frac{\partial A_\mu}{\partial A_\nu} = \delta_\mu^\nu \quad \partial_\mu A^\nu = \delta_\mu^\nu \quad (1.21)$$

Knowing that there is no  $A_\nu$  dependence as all  $A_\nu$  terms are in the derivative, the Euler-Lagrange equation simplifies to,

$$\partial_\sigma \left( \frac{\partial \mathcal{L}}{\partial (\partial_\sigma A_\tau)} \right) = 0, \quad (1.22)$$

and the Maxwell's equations are derived as,

**Derivation 1.1.**

$$\begin{aligned} -\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta} \left( \partial_\mu A_\nu \frac{\partial (\partial_\alpha A_\beta)}{\partial (\partial_\sigma A_\tau)} + \frac{\partial (\partial_\mu A_\nu)}{\partial (\partial_\sigma A_\tau)} \partial_\alpha A_\beta \right) + \partial_\sigma \left( \partial_\mu A^\mu \eta^{\mu\nu} \frac{\partial (\partial_\mu A_\nu)}{\partial (\partial_\sigma A_\tau)} \right) &= 0 \\ -\frac{1}{2}\partial_\sigma \eta^{\mu\alpha}\eta^{\nu\beta} \left( \partial_\mu A_\nu \delta_\alpha^\sigma \delta_\beta^\tau + \delta_\mu^\sigma \delta_\nu^\tau \partial_\alpha A_\beta \right) + \partial_\sigma \eta^{\mu\nu} (\partial_\mu A^\mu \delta_\nu^\sigma \delta_\tau^\tau) &= 0 \\ -\frac{1}{2}\partial_\sigma \left( \eta^{\mu\sigma} \eta^{\nu\tau} \partial_\mu A_\nu + \eta^{\sigma\alpha} \eta^{\tau\beta} \partial_\alpha A_\beta \right) + \partial_\sigma (\eta^{\sigma\tau} \partial_\mu A^\mu) &= 0 \\ -\partial_\sigma \partial^\sigma A^\tau + \partial^\tau \partial_\mu A^\mu &= 0 \\ \text{reindexing } \sigma \rightarrow \mu, \tau \rightarrow \nu &\quad -\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \\ &\quad -\partial_\mu F^{\mu\nu} = 0, \end{aligned} \quad (1.23)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength tensor.

### Klein-Gordon equation

Another example is the Klein-Gordon equation,

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^2 \\
 &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}m^2 \phi^2 \\
 &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}m^2 \phi^2,
 \end{aligned} \tag{1.24}$$

$\xrightarrow{\text{red}} T = \int d^3x \frac{1}{2}\dot{\phi}^2$ 
 $\xrightarrow{\text{blue}} V = \int d^3x \left( \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2 \right)$

which is a Lagrangian for a scalar field. To find the equations of motion, the Euler-Lagrange equation is used,

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \tag{1.25}$$

#### Derivation 1.2.

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} &= \frac{1}{2}\eta^{\mu\nu} \left( \partial_\mu \phi \frac{\partial(\partial_\nu \phi)}{\partial(\partial_\sigma \phi)} + \frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)} \partial_\nu \phi \right) \\
 &= \frac{1}{2}\eta^{\mu\nu} (\partial_\mu \phi \delta_\nu^\sigma + \delta_\mu^\sigma \partial_\nu \phi) \\
 &= \frac{1}{2}(\eta^{\mu\sigma} \partial_\mu \phi + \eta^{\sigma\nu} \partial_\nu \phi) \\
 &= \partial^\sigma \phi,
 \end{aligned} \tag{1.26}$$

and the Klein-Gordon equation is derived as,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{1.27}$$

### First Order Lagrangians

Consider a Lagrangian that is linear in time derivatives,

$$\mathcal{L} = \frac{i}{2}(\psi^* \dot{\psi} - \dot{\psi}^* \psi) - \nabla \psi^* \cdot \nabla \psi - m \psi^* \psi, \tag{1.28}$$

treating  $\psi$  and  $\psi^*$  as independent variables. The Euler-Lagrange equations are then,

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = \frac{i}{2}\dot{\psi} - m\psi \quad \frac{\partial \mathcal{L}}{\partial \psi} = -\frac{i}{2}\dot{\psi}^* - m\psi^* \quad \frac{\partial \mathcal{L}}{\partial \nabla \psi^*} = -\nabla \psi, \tag{1.29}$$

giving the equation of motion,

$$i\dot{\psi} = -\nabla^2 \psi + m\psi. \tag{1.30}$$

Compared to the Klein-Gordon equation, only  $\psi$  and  $\psi^*$  are needed to determine the future evolution compared to needing  $\phi$  and  $\dot{\phi}$ .

### Note on locality

All examples above, as well as all theories of nature are local. This means that there are no terms in the Lagrangian that look like,

$$\mathcal{L} = \int d^3x d^3y \phi(\mathbf{x}) \phi(\mathbf{y}). \tag{1.31}$$

One of the main reasons for introducing field theories in classical physics is to account for locality.

## 1.2 Lorenz invariance

Since relativity has to be accounted for, the field theory has to be invariant under Lorentz transformations, i.e.,

$$x^\mu \rightarrow (x')^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.32)$$

where,

**Lemma 1.3** (Lorentz transformation).

$$\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}. \quad (1.33)$$

The field has to transform as well, a simple example is a scalar field which transforms as,

**Note** (Scalar field transformation).

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x), \quad (1.34)$$

and the derivative of a scalar field transforms as,

**Note** (Derivative of scalar field transformation).

$$\partial_\mu \phi(x) \rightarrow (\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(\Lambda^{-1}x). \quad (1.35)$$

The inverse transformation is used as coordinates are not relabeled after the transformation. From the Lorentz invariant theory, if  $\phi(x)$  is a solution to the equations of motion, then  $\phi(\Lambda^{-1}x)$  is also a solution.

### Maxwell's equations

Using the Lorentz transformation  $A^\mu(x) \rightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$ ,

**Derivation 1.3.**

$$\begin{aligned} (\partial_\mu A_\nu(x))(\partial_\beta A^\nu(x))\eta^{\mu\beta} &\rightarrow (\partial_\mu \eta_{\nu\alpha} \Lambda^\alpha{}_\gamma \eta^{\gamma\epsilon} A_\epsilon(\Lambda^{-1}x))(\partial_\beta \Lambda^\nu{}_\delta \eta^{\delta\zeta} A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= \eta_{\nu\alpha} \Lambda^\alpha{}_\gamma \eta^{\gamma\epsilon} \Lambda^\nu{}_\delta \eta^{\delta\zeta} (\partial_\mu A_\epsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= \Lambda_\nu{}^\epsilon \Lambda^{\nu\zeta} (\partial_\mu A_\epsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \\ &= (\partial_\mu A_\epsilon(\Lambda^{-1}x))(\partial_\beta A_\zeta(\Lambda^{-1}x))\eta^{\mu\beta} \eta^{\epsilon\zeta} \\ &= (\partial_\mu A_\epsilon(\Lambda^{-1}x))(\partial_\beta A^\epsilon(\Lambda^{-1}x))\eta^{\mu\beta}, \end{aligned} \quad (1.36)$$

where the derivation used,

**Derivation 1.4.**

$$\begin{aligned} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} &= \eta_{\alpha\beta} \\ \eta^{\alpha\epsilon} \eta^{\beta\zeta} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} &= \eta^{\alpha\epsilon} \eta^{\beta\zeta} \eta_{\alpha\beta} \\ \Rightarrow \Lambda_\nu{}^\epsilon \Lambda^{\nu\zeta} &= \eta^{\epsilon\zeta}. \end{aligned} \quad (1.37)$$



For the second part of the Lagrangian,

**Derivation 1.5.**

$$\begin{aligned}\partial_\mu A^\mu(x) &\rightarrow (\Lambda^{-1})^\alpha{}_\mu \partial_\alpha \Lambda^\mu{}_\beta A^\beta(\Lambda^{-1}x) \\ &= \delta^\alpha_\beta \partial_\alpha A^\beta(\Lambda^{-1}x) \\ &= \partial_\alpha A^\alpha(\Lambda^{-1}x),\end{aligned}\tag{1.38}$$

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showing that Maxwell's equations are Lorentz invariant.

### Klein-Gordon equation

The Klein-Gordon equation is Lorentz invariant, as the derivatives transform as,

**Derivation 1.6.**

$$\begin{aligned}\partial_\mu \phi(x) \partial_\nu \phi(x) \eta^{\mu\nu} &\rightarrow (\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi(\Lambda^{-1}x) (\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi(\Lambda^{-1}x) \eta^{\mu\nu} \\ &= \partial_\rho \phi(\Lambda^{-1}x) \partial_\sigma \phi(\Lambda^{-1}x) \Lambda^\rho{}_\mu \eta^{\mu\nu} \Lambda^\sigma{}_\nu \\ &= \partial_\rho \phi(\Lambda^{-1}x) \partial_\sigma \phi(\Lambda^{-1}x) \eta^{\rho\sigma}\end{aligned}\tag{1.39}$$

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The action is then Lorentz invariant, however there is no Jacobian factor in the integral after change of variables due to  $\det \Lambda = 1$ .

### First Order Lagrangians

Since the Lagrangian is linear in time derivative, but quadratic in spatial derivatives, the theory is not Lorentz invariant.

## 1.3 Symmetries