

HYDRODYNAMICS

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I Introduction

Interstellar Medium

Components of the interstellar medium (ISM) are

- Cold neutral medium (CNM) – $T \sim 70$ K
- Warm neutral medium (WNM) – $T \sim 8000$ K
- HII regions – photoionized gas
- Warm ionized medium (WIM) – photoionized gas
- Hot intercloud medium (HIM) – collisionally ionized gas
- Molecular gas – mostly in molecular clouds

Property	CNM	WNM	HII reg	WIM	HIM	Mol
Density (cm^{-3})	30	0.3	$10 - 10^4$	0.1	10^{-3}	$> 10^2$
Temperature (K)	30 – 120	8000	$\approx 10^4$		$> 10^6$	~ 10
Thermal speed (km s^{-1})	~ 1	8	~ 10		> 100	~ 0.1
Observed speed (km s^{-1})	few–10	8 – 15	15 – 25		-	~ 1
Total Mass ($10^9 M_{\odot}$)	2	2	0.1	1	< 0.1	1
Scale-height (kpc)	0.1	0.2 – 0.3	-	1	3	0.06
Maximum Radius (kpc)	$\lesssim 25$	25	~ 15	-	-	~ 10
Filling factor	0.01	0.25	0.02	0.25	≈ 0.5	0.001
Main observables	21-cm		recomb/forbid		X-rays	CO

Table 1 / Properties of different interstellar medium phases.

Cold neutral gas is made of clouds, 90% H, 10% He, and trace elements. The photoionized gas is mostly HII regions, and can be seen through emission of **recombination** ($\text{Ly}\alpha$, $\text{H}\alpha$, $\text{H}\beta$, ...), **forbidden lines** ($[\text{OII}]$, $[\text{OIII}]$, $[\text{NII}]$, $[\text{SII}]$, ...), and **bremsstrahlung**. Hot gas is collisionally ionized and emits X-rays. Hot gas extends to hundreds of kiloparsecs making it difficult to observe.

The thermal velocity is given by

$$v_{\text{th}} = \sqrt{\frac{kT}{m}} \quad (2)$$

A magnetic field can be detected by

- Zeeman effect – splitting of spectral lines
- Synchrotron radiation – non-thermal radiation
- Faraday rotation – rotation of polarization angle

- Polarization of stellar light
- Dust polarization

The magnetic field is important for gas dynamics, cloud stability, and shocks.

Equilibriums in Hydrodynamics

The medium can be divided into three phases:

- Cold phase – $T \lesssim 100$ K
- Warm phase – $T \sim 10^4$ K
- Hot phase – $T \gtrsim 10^6$ K

and the pressures are roughly equivalent in the three phases. Thermal equilibrium is only a reasonable approximation locally, not globally for the ISM. This is contrasted by the pressure equilibrium which is often fulfilled globally in the ISM.

Hydrostatic equilibrium is the balance between the pressure and gravitational force. It is important for molecular clouds, galaxy discs, hot halos, stars, ...

Collisional Processes

In ideal fluids there are two types of collisions, elastic (change in \mathbf{v}_p , and energy is conserved), and inelastic collisions (with the transfer of energy). The mean free path is the average distance between collisions,

Note (Mean free path).

$$\ell = \frac{1}{n \sigma}, \quad (3)$$

Number density Cross-section

and the average time between collisions is

$$\tau = \frac{\ell}{v_{\text{th}}} = \frac{1}{n \sigma v_{\text{th}}} \sim 10^4 \left(\frac{n}{\text{cm}^{-3}} \right)^{-1} \text{ s} \quad (\text{WIM}). \quad (4)$$

The shortest inelastic collision in WIM is the ionization of O^+ to OII , and the excitation time is,

$$\tau_{\text{exc}}(\text{O}^+) \sim 10^{11} \left(\frac{n_{\text{O}^+}/n_e}{10^{-3}} \right)^{-1} \text{ s}. \quad (5)$$

In CNM, the average collision time is $\sim 10^8$ s, mostly due to the C^+ . The velocities are distributed as a **Maxwellian distribution**,

Definition .1 (Maxwellian distribution).

$$f(v) dv = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv. \quad (6)$$

Using the distribution, the following characterizing speeds can be derived,

Note (Peak velocity).

$$v_{\text{peak}} = \sqrt{\frac{2kT}{m}}, \quad (7)$$

Note (RMS velocity).

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}. \quad (8)$$

Note (Mean velocity).

$$\langle v \rangle = \int_0^\infty v f(v) dv = \sqrt{\frac{8kT}{\pi m}}, \quad (9)$$

Fluids as continuums

1.1 Fluid element

A fluid element is a portion of the fluid of size l that is large compared to the mean free path ℓ , but small compared to the characteristic length scale of the system L ,

$$\ell \ll l \ll L. \quad (1.1)$$

The length scale is defined as

$$L \sim \frac{q}{\nabla q} \quad (1.2)$$

where q is a macroscopic physical quantity. In a CNM, the “worst case scenario”, $A = A(\text{H}, \text{H}) \approx 10^{-15} \text{ cm}^2$, and $n \approx 30 \text{ cm}^{-3}$, so $\ell \approx 10^{13} \text{ cm}$. This means that any structure in the ISM would have an interaction that is smaller than this.

1.2 Advection

The advection of a fluid element is the transport of the element by the flow. Consider a fluid element moving with velocity \mathbf{u} and has a generic quantity q which is a function of position and time, $q(\mathbf{r}, t)$. The change in q is given by the Lagrangian derivative,

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q. \quad (1.3)$$

It is explicitly defined as,

$$\begin{aligned} \frac{Dq}{Dt} &= \lim_{\delta t \rightarrow 0} \frac{q(\mathbf{r} + \delta \mathbf{r}, t + \delta t) - q(\mathbf{r}, t)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{q(\mathbf{r} + \delta \mathbf{r}, t + \delta t) - q(\mathbf{r}, t) + q(\mathbf{r}, t + \delta t) - q(\mathbf{r}, t + \delta t)}{\delta t} \\ &= \left(\frac{\partial q}{\partial t} \right)_{(\mathbf{r}, t)} + \lim_{\delta t \rightarrow 0} \frac{q(\mathbf{r} + \delta \mathbf{r}, t + \delta t) - q(\mathbf{r}, t + \delta t)}{\delta t} \\ &= \left(\frac{\partial q}{\partial t} \right)_{(\mathbf{r}, t)} + \lim_{\delta t \rightarrow 0} \frac{\delta \mathbf{r} \cdot (\nabla q)_{(\mathbf{r}, t)}}{\delta t} \\ &= \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q. \end{aligned} \quad (1.4)$$

Taking q to be a generic quantity,

Definition 1.1 (Lagrangian derivative).

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\text{Eulerian derivative}} + \underbrace{\mathbf{u} \cdot \nabla}_{\text{Advection operator}}. \quad (1.5)$$

1.3 Continuity equation

Consider a portion of fluid of density ρ contained in a volume V . The mass inside the volume is,

$$M = \int_V \rho dV. \quad (1.6)$$

The change in mass is given by the rate of change of mass inside the volume,

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int_V \rho dV = - \oint_S \rho \mathbf{u} \cdot d\mathbf{S}, \quad (1.7)$$

where S is the surface of the volume V . Using the divergence theorem,

Theorem 1.1 (Divergence theorem).

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV, \quad (1.8)$$

the equation becomes,

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho dV &= - \oint_S \rho \mathbf{u} \cdot d\mathbf{S} \\ &= - \int_V \nabla \cdot (\rho \mathbf{u}) dV \\ &\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \end{aligned} \quad (1.9)$$

which is the continuity equation or the mass conservation equation. It can be rewritten as,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \quad (1.10)$$

For an incompressible fluid, $\nabla \cdot \mathbf{u} = 0 \Leftrightarrow D\rho/Dt = 0$.