Investigating the motion of a rocket in orbit

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INTRODUCTION AND THEORY

A satellite in orbit around a body follows an elliptical path, with the barycenter at one of the foci of the ellipse, as described by Kepler's First Law [1]. This path is defined by the changing velocity of the satellite as it orbits, being lowest when it is at the apoapsis, which is derived from Newton's laws of motion and gravitation. The force on two bodies with mass is given by the latter, being [2]

$$\boldsymbol{F_{12}} = -\frac{Gm_1m_2}{\mid \boldsymbol{r_{12}} \mid^2}\hat{\boldsymbol{r}}, \tag{1}$$

where $\boldsymbol{F_{12}}$ is the force between objects 1 and 2, G is the gravitational constant, $\boldsymbol{r_{12}}$ is the distance between the center of the objects (see figure 1), and m_1 and m_2 are the masses of the objects.

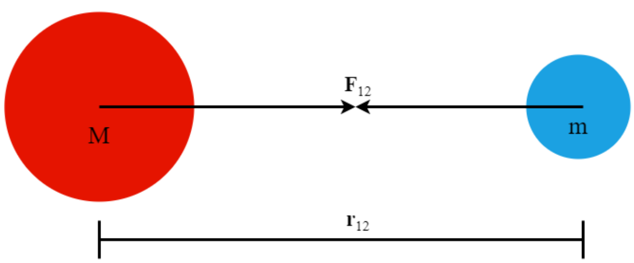

FIG. 1. Newton's Law of Universal Gravitation

From this, the equation of motion for an orbiting satellite can be derived using Newton's second law, giving [2]

$$m\ddot{\boldsymbol{r}} = -\frac{mMG}{\mid \boldsymbol{r} \mid^3}\boldsymbol{r}, \tag{2}$$

where m is the mass of the satellite, M is the mass of the large body, and r is the position of the satellite relative to the centre of the large body. This is an ordinary differential equation, which can be solved using numerical analysis. One solution of this is using the Runge-Kutta family of iterative methods.

Runge-Kutta Methods

The Runge-Kutta methods are used to provide approximate solutions to ordinary differential equations. Such methods discretise a continuous function in both space and time, allowing for integration over discrete time intervals, a technique easily done using computer simulations [3]. For this simulation, the 4th-order Runge-Kutta (RK4) method was used. This method evaluates the slope of a function with known initial values at four different points in a given interval, shown in figure 2, given by the following [4, 5]

$$k_1 = f(x_n, y_n) \tag{3}$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}) \tag{4}$$
$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}) \tag{5}$$
$$k_4 = f(x_n + h, y_n + hk_3) \tag{6}$$

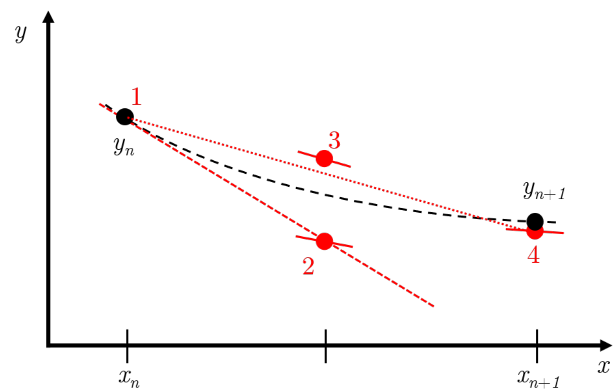

FIG. 2. Slopes used by the 4th-order Runge-Kutta method.

In the context of solving equation 2, h is the time step, and x and y are the positional coordinates of the rocket. k_1 is the slope at the beginning of the interval, calculated using y. k_2 is the slope at the midpoint of the interval, calculated using y and k_1. k_2 is also the slope at the midpoint of the interval, but is instead calculated using k_2 instead of k_1. Finally, k_4 is the slope at the end of the interval, calculated using y and k_4.

The solution to this is then given by the weighted average of the increments [4]

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{7}$$
$$t_{n+1} = t_n + h. \tag{8}$$

As this method is of 4th-order, the global truncation error is of 4th-order, so this method is much more accurate than comparable methods, such as Euler's method, which is 2nd-order [5, 6]. Also of note is that if f is independent of y, then RK4 simplifies to Simpson's rule [6].

A two-dimensional orbit with the larger body at the origin requires several variables that need to be evaluated in four functions, as RK4 couples the variables together:

$$f_1(t, xx, y, v_x, v_y) = \frac{dx}{dt} = v_x \tag{9}$$

$$f_2(t, xx, y, v_x, v_y) = \frac{dy}{dt} = v_y \tag{10}$$

$$f_3(t, xx, y, v_x, v_y) = \frac{dv_x}{dt} = \frac{-GMx}{(x^2 + y^2)^{3/2}} \tag{11}$$

$$f_4(t, xx, y, v_x, v_y) = \frac{dv_y}{dt} = \frac{-GMy}{(x^2 + y^2)^{3/2}} \tag{12}$$

Ignoring the arguments that are not needed, we can apply these functions to each of equations 3-6 to obtain values for x, y, v_x and v_y at each increment in the interval. Using this, and applying it to equation 7, a series of time-stepping equations is obtained, where y is replaced by each of the variables listed. These can easily be calculated using a computer, by iterating through a loop.

RESULTS

DISCUSSION

CONCLUSIONS

APPENDIX

REFERENCES

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