CONTROL OF LINEAR MULTI-VARIABLE SYSTEM

Basic control for the Mars Climate Orbiter (MCO)

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Contents

Chapter 1

Problem statement

The Mars Climate Orbiter was a 638 Kg robotic space probe [1] launched by NASA on December 11/1998 to:

1. study the Martian climate
2. study Martian atmosphere
3. study surface changes
4. act as the communications relay in the Mars Surveyor '98 program for Mars Polar Lander

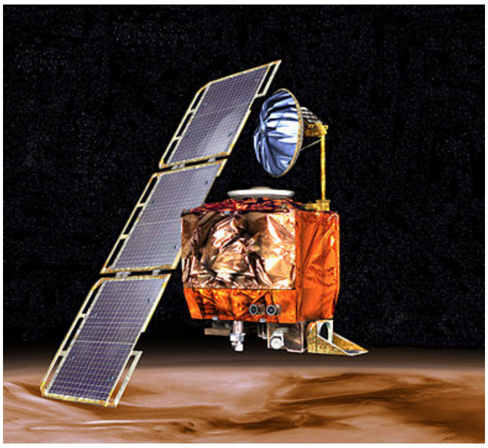

Figure 1.1: Mars Climate Orbiter (MCO)

[1] A space probe is a robotic spacecraft is an uncrewed spacecraft designed to make scientific research measurements.

However, on September 23/1999, communication with the spacecraft was lost as the robot went into orbital insertion, due to a mismatch of units of measurement between the output of ground-based computer software and the ones expected by the robot. In fact, the output of the ground-based computer software was in non-SI units of pound-force seconds (lbf·s) instead of the SI units of newton-seconds (N·s), as was instead specified in the contract between NASA and Lockheed.

For that mistake the spacecraft encountered Mars on a trajectory that brought it too close to the planet, and it was either destroyed in the atmosphere or re-entered heliocentric space after leaving Mars' atmosphere.

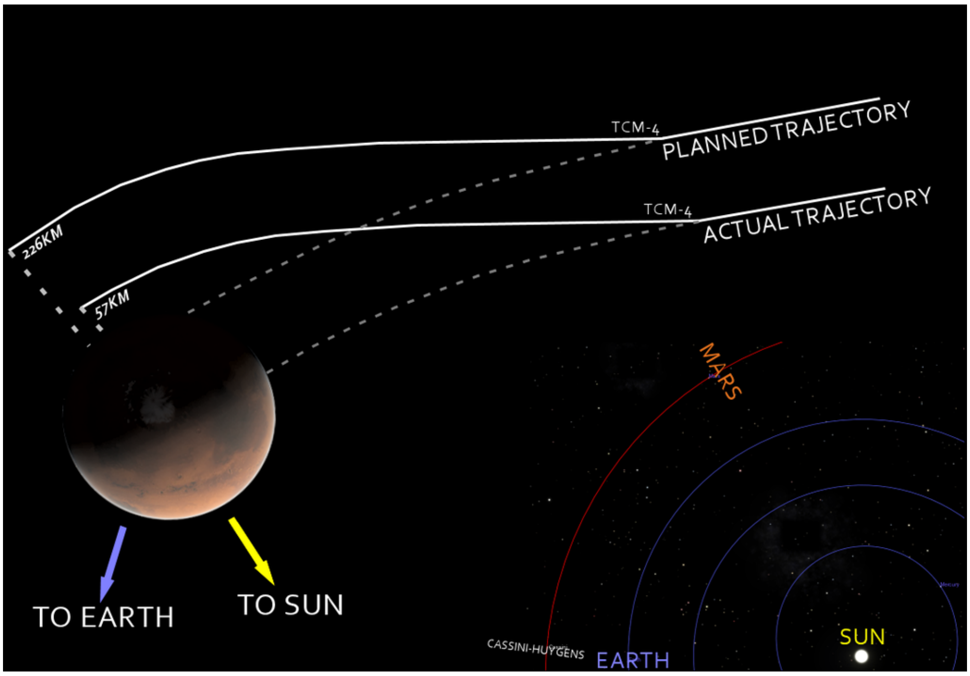

Figure 1.2: Trajectory of the MCO

For more information on MCO:
`https://en.wikipedia.org/wiki/Mars_Climate_Orbiter`

The spacecraft was 3-axis stabilized and included eight hydrazine monopropellant thrusters (four 22 N thrusters to perform *trajectory corrections*; four 0.9 N thrusters to *control attitude*).

Orientation of the spacecraft was determined by:

- a star tracker
- two Sun sensors
- two inertial measurement units (gyroscopes)

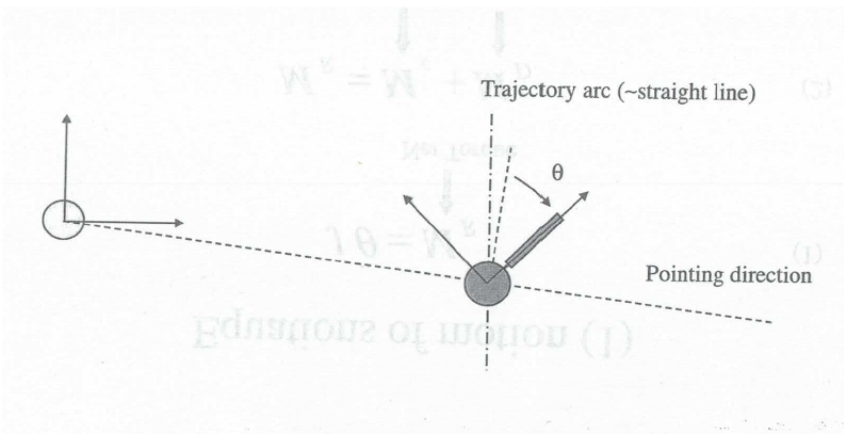

Figure 1.3: Attitude control for MCO

For more information on attitude control:
`https://en.wikipedia.org/wiki/Attitude_control`

Chapter 2

Model of the system

The simplified linear model of the MCO robot is the following one:

$$S : \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + D\underline{d}(t) \quad \longrightarrow \quad \text{state equation} \\ \\ \underline{y}(t) = C\underline{x}(t) \quad \longrightarrow \quad \text{output equation} \end{cases} \tag{2.1}$$

with:

$\underline{x} \in \mathbb{R}^n$	$A \in \mathbb{R}^{n \times n}$	A describes how the internal states are all connected to each other (underlying the dynamic of the system)
$\underline{u} \in \mathbb{R}^m$	$B \in \mathbb{R}^{n \times m}$	B describes how the inputs enter into the system so which states are they affecting
$\underline{d} \in \mathbb{R}^d$	$D \in \mathbb{R}^{n \times d}$	D describes how the disturbances enter into the system so which states are they affecting
$\underline{y} \in \mathbb{R}^l$	$C \in \mathbb{R}^{l \times n}$	C describes how the states are combined to get the output

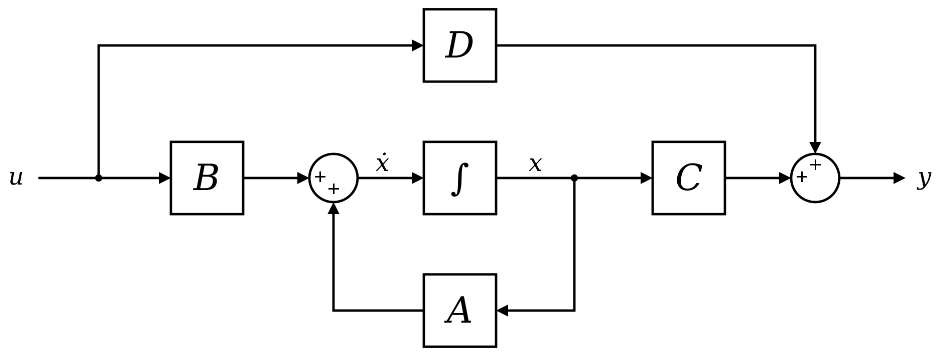

Figure 2.1: Block diagram of a linear state-space equation

In particular in MCO case we have:

- 2 main sensors[a] $\implies$ $n = 2$
- 1 control input $\implies$ $m = 1$
- 1 output $\implies$ y = scalar = θ

and the system becomes:

- in case of **both measurements** and null disturbance:

$$S : \begin{cases} \underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d(t) \\ \\ \underline{y}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t) \end{cases} \tag{2.2}$$

- in case of **gyroscope failure** and null disturbance:

$$S : \begin{cases} \underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d(t) \\ \\ \underline{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t) \end{cases} \tag{2.3}$$

[a] The two sensors are: sun sensor and gyroscope.

- in case of **sun sensor failure** and null disturbance:

$$S : \begin{cases} \underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d(t) \\ \\ \underline{y}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t) \end{cases} \tag{2.4}$$

Each of these systems is:

- **LSI:** Linear system time invariant

$$\alpha \text{ y(t)} \longleftarrow \alpha \text{ u(t)}$$
$$\text{y(t - }\Delta) \longleftarrow \text{u(t - }\Delta)$$

- **strictly causal:** system where the output depends only on past and current states but not on future inputs

2.1 Main goal

The aims of this project is to generate a control signal

$$\underline{\text{u}}\text{(t) such that } \underline{\text{y}} \approx \underline{y}_R$$

despite external disturbances.

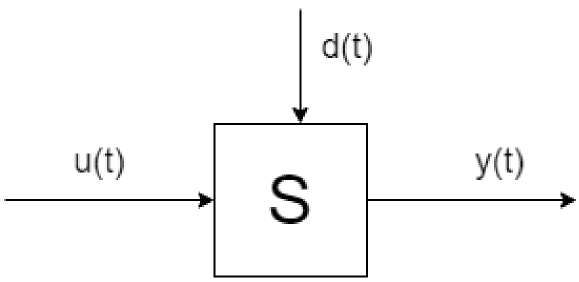

Figure 2.2: Control problem

The main specifications that we are going to analyse are:

- **STABILITY:** A system is said to be stable, if its output is under control so if it produces a bounded output for a given bounded input. Otherwise, it is said to be unstable.

- **ACCURACY:** A system is as much accurate as much $\underline{y}(t)$ is similar to $\underline{y}_R(t)$ when transient expires

- **DISTURBANCE REJECTION:** A system is as much disturbance rejection as much is able to drive its output $\underline{y}(t)$ closed to desired output $\underline{y}_R(t)$ even under external disturbances.

2.2 Overview of the main phases for design a control scheme

During this report we are going to follow these main steps to construct an appropriate control scheme for the MCO:

1. **Offline Analysis** of the system in order to get the main properties such as: stability, controllability, observability

2. Design of a **State Space Observer** to estimate the missing state in case of failure of one sensor

3. Design of a **Reduced State Space Observer** to estimate only the missed state due to the failure of the sensor

4. Design of a **control/regulation scheme** with pre-filters and/or dynamic compensator in order to drives the output toward a desired reference value

5. Design of a **Disturbance Observer** to reduce the effect of the disturbances in system's output

NOTE: The first four steps are going to be analyzed without considering any disturbance.

2.3 Design of the control scheme

2.3.1 Offline Analysis

As the start of the analysis it is important to analyse the stability of system S.

$$S: \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) \quad \longrightarrow \quad \text{state equation} \\ \\ y(t) = C\underline{x}(t) \quad \longrightarrow \quad \text{output equation} \end{cases} \tag{2.5}$$

From theory it's known that the solution of state equation is:

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}Bu(t)d\tau \tag{2.6}$$

where the first part represents the *free response* while the second part represents the *forced response*.
By considering only the free response (u(t) = 0) we have the following result:

$$\phi = e^{At} = \begin{bmatrix} e^{\lambda_1}t & 0 \\ 0 & e^{\lambda_2}t \end{bmatrix} \tag{2.7}$$

From that result we can make this distinction:

1. **Asymptotically stable system**

$$Re(\lambda_i) < 0 \quad \forall i = 1,..,n \tag{2.8}$$

An asymptotically stable system satisfy the *bounded-input, bounded-output (BIBO)* stability criteria so for every bounded input to the system the output will be bounded.

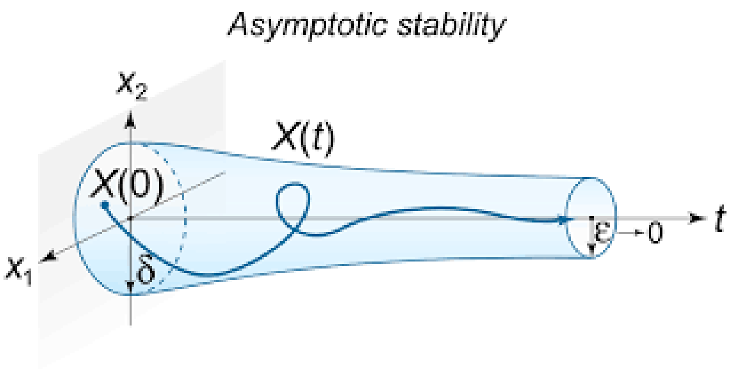

Figure 2.3: Asymptotically stable behaviour

2. **Simply stable system**

$$Re(\lambda_i) \leq 0 \quad \forall i = 1, .., n \quad \text{and the pole in zero has multiplicity 1} \tag{2.9}$$

3. **Unstable system**

$$Re(\lambda_i) > 0 \quad \forall i = 1, .., n \tag{2.10}$$

In our case the system is **UNSTABLE** since we have two poles in zero. This property simply comes from the way the states, and its derivatives, are connected to each other.

2.3.2 Control system design requirements at high level

Controller design comes down to figuring out how to use the sensor data, along with the reference signal, to generate the correct actuator commands by which the system S can become stable.

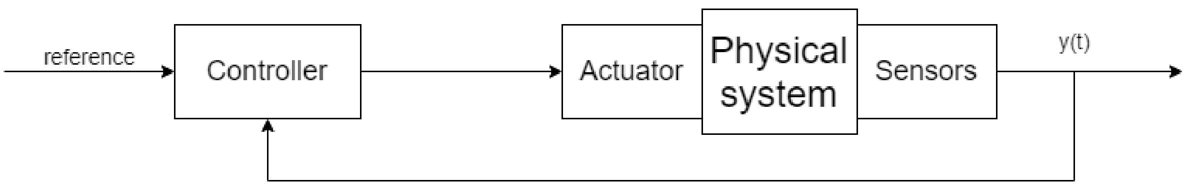

Figure 2.4: Control system design at high level

It is important to highlight that the controller design is about to failing if the system doesn't have the appropriate actuators that can affect the right part of the system, or if you don't have the appropriate sensors in place that can measure the right states.

Definitions:

- **CONTROLLABILITY:** A system is controllable if is possible, by using an appropriate control signal u(t), drives a generic initial condition $\underline{x}(t_0)$ to $\underline{0}$ in a finite amount of time
- **STABILIZABILITY:** A system is stabilizable if the uncontrollable states are stable and the controllable ones are unstable
- **OBSERVABILITY:** A system is observable if is possible to determine its initial condition $\underline{x}(0)$, by starting from y(t), in a finite amount of time
- **DETECTABILITY:** A system is detectable if the non observable states are stable and the observable ones are unstable

NOTE: The detectability concept is the dual of the stabilizability one.

2.3.3 Controllability check

To check the controllability we have to analyze the **Kalman controllability matrix**:

$$K_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{2.11}$$

Necessary and sufficient condition for the system to be **fully controllable** is that Kalman controllability matrix is full rank.

fully controllable $\Longleftrightarrow$ rank(K_c) = n

where Span(K_c) represents the space of all the controllable and reachable states. If the systems S is fully controllable then there exist a **feedback control law** u(t)

$$u(t) = -K\underline{x}(t) + \underline{v}(t) \tag{2.12}$$

such that

$$\underline{\dot{x}}(t) = (A - BK)\underline{x}(t) + B\underline{v}(t) \tag{2.13}$$

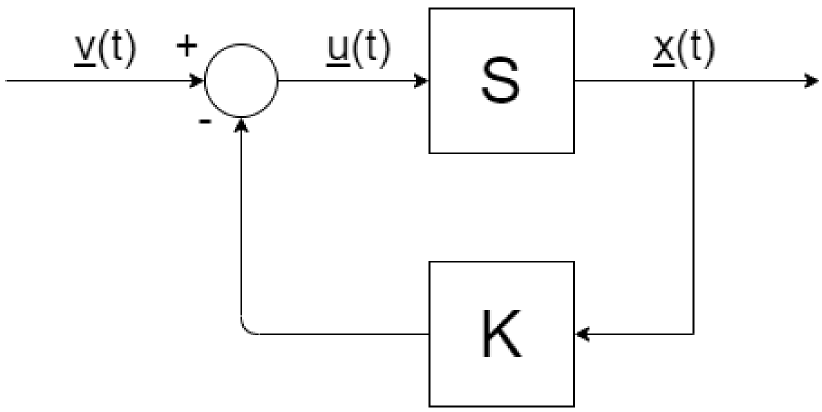

Figure 2.5: Closed-loop system

The system in closed-loop therefore becomes:

$$S_{OL} \text{ :(A,B)} \quad \Longrightarrow \quad S_{CL} \text{ :(A-BK,B)}$$

and the problem now comes down to stabilize (A-BK) instead of A. This result is really powerful since, by acting on K, is possible to arbitrarily place the eigenvalues (poles) of (A- BK) and stabilize system S.

In case of not full rank of K_c there could be two possibilities:

$$\text{all uncontrallable states are stable} \iff S \text{ is } \textbf{stabilizable}$$

$$\text{not all uncontrallable states are stable} \iff S \text{ is } \textbf{non controllable}$$

In our case S is **FULLY CONTROLLABLE** because:

$$K_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad rank(K_c) = 2 \tag{2.14}$$

We can proceed on computing an appropriate K matrix only in case of system with both measurements (2.2). In the other two cases system (2.3) and system (2.4), even if they are still both controllable, we have first to estimate the missing state.

2.3.4 Observability check

To check the observability we have to analyze the **Kalman observability matrix**:

$$K_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix} \tag{2.15}$$

Necessary and sufficient condition for the system to be **fully observable** is that Kalman observability matrix is full rank.

$$\text{fully observable} \iff \text{rank}(K_o) = \text{n}$$

where Span(K_o) represents the space of all the observable states.

In case of not full rank of K_o there could be two possibilities:

all unobservable states are stable $\iff$ S is **detectable**

not all unobservable states are stable $\iff$ S is **non observable**

Both system (2.2) (bith sensors available) and system (2.3) (gyroscope failure) they are **FULLY OBSERVABLE** because:

$$K_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad rank(K_o) = 2 \tag{2.16}$$

and this means that will be possible to implement a state observer to estimate the missing state instead, system (2.4) (sun sensor failure) the system is no more observable and this is a big problem.

2.3.5 State space observer

The state space observe is a subsystem that allow to reconstruct the state vector $\underline{x}$(t) starting from the input vector u(t) and the measured variable y_m(t).

We are going to have different case based on sensor failure or not.

- both sensors available: $\underline{y_m}$(t) = C $\underline{x}$(t)
- gyroscope failure: y_m(t) = $C_{11}x_1$(t)
- sun sensor failure: y_m(t) = $C_{22}x_2$(t)

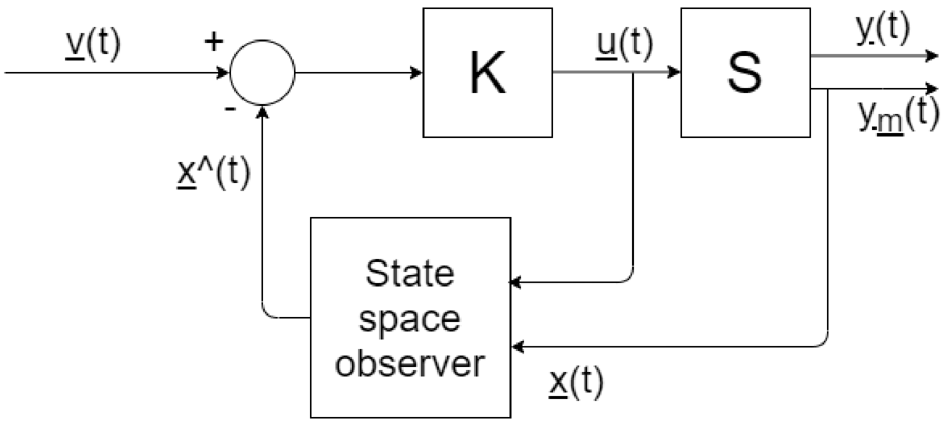

Figure 2.6: Closed-loop scheme with state space estimator

By considering the following system:

$$S : \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) \\ \underline{y}_m(t) = C_m\underline{x}(t) \end{cases} \tag{2.17}$$

I want to design a State Space Observer $\hat{S}$ such that:

$$\underline{e}(t) := \underline{x}(t) - \hat{\underline{x}}(t) \quad \longrightarrow \quad 0 \tag{2.18}$$

Behavior without feedback Imagine to consider an estimate system as follows:

$$\hat{S} : \begin{cases} \dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + Bu(t) \\ \hat{\underline{y}}(t) = C\hat{\underline{x}}(t) \end{cases} \tag{2.19}$$

The corresponding dynamic of the error would be:

$$\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t) \tag{2.20}$$

$$\underline{\dot{e}}(t) = \underline{\dot{x}}(t) - \underline{\dot{\hat{x}}}(t) = A\underline{e}(t) \tag{2.21}$$

So this dynamic is still related to the eigenvalues of A matrix because this quantity tends to zero iff all the eigenvalues of A have Re(λ_i)<0 $\forall$ i=1,..,n.

However, we have already proved that this condition is not verified for the system since S in open loop is not stable.

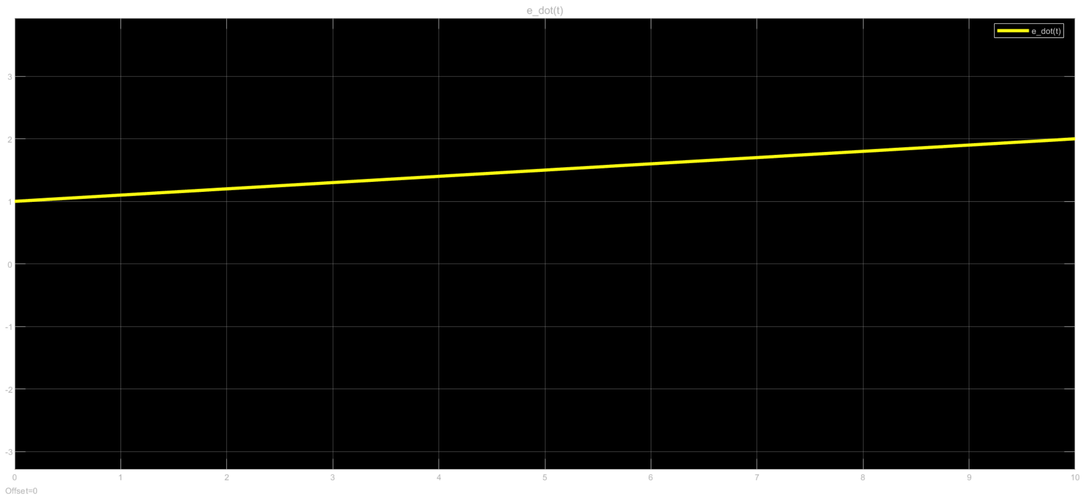

Figure 2.7: Signal $\dot{e}$ not convergent

For that reason we have to add to system (2.19) a term that connects the two outputs value $\underline{y}$(t) and $\underline{y_m}$(t) together.

Behavior with feedback

The new state space observe $\hat{S}$ becomes:

$$\hat{S}: \begin{cases} \underline{\dot{\hat{x}}}(t) = A\underline{\hat{x}}(t) + Bu(t) + H[C\underline{x}(t) - C\underline{\hat{x}}(t)] \\ \underline{\hat{y}}(t) = C\underline{\hat{x}}(t) \end{cases} \tag{2.22}$$

and, therefore, the dynamic of the error becomes:

$$\underline{\dot{e}}(t) = \underline{\dot{x}}(t) - \underline{\dot{\hat{x}}}(t) = (A - HC)\underline{e}(t) \tag{2.23}$$

We can notice that now the dynamic of the error $\underline{e}(t)$ is no more strictly related to A matrix but to (A-HC). For that reason, by acting on H matrix, we can obtain that S becomes asymptotically stable in such a way that:

$$\lim_{t\to\infty} \underline{e}(t) = 0 \tag{2.24}$$

because

$$\lim_{t\to\infty} \underline{\hat{x}}(t) = \underline{x}(t) \tag{2.25}$$

This problem has a similar structure as the problem of pole assignment, and this property is called **DUALITY**:

$$\text{S: (A,B,}\bullet) \longrightarrow S_{CL}\text{: (A-BK,B, }\bullet)$$
$$\text{S: (A, }\bullet\text{,C)} \longrightarrow \hat{S}\text{: (A-HC, }\bullet\text{, C)}$$

Consider this problem with the primal and dual system we can highlight some important properties:

$$S_{primal}\text{: (A,B,C)} \qquad S_{dual}\text{: } (A^\top, B^\top, C^\top)$$

$$K_o^{dual} = \begin{bmatrix} B \\ B^\top A^\top \ldots \\ B^\top (A^\top)^{n-1} \end{bmatrix} = K_{c,primal}^\top \tag{2.26}$$

$$K_c^{dual} = \begin{bmatrix} C^\top & A^\top C^\top & \ldots & (A^\top)^{n-1} C^\top \end{bmatrix} = K_{o,primal}^\top \tag{2.27}$$

So, by writing the dual system:

$$S_{dual}: \begin{cases} \underline{\dot{z}}(t) = A^\top \underline{z}(t) + C^\top w(t) \\ \underline{v}(t) = B^\top \underline{z}(t) \end{cases} \tag{2.28}$$

if the system S_{dual} is fully controllable, we can use a feedback control law

$$w(t) = -H^\top \underline{z}(t) \tag{2.29}$$

that allow us to arbitrarily place the poles of $A^\top$ in the complex plane.

From theory we know that:

$$(A^\top - C^\top H^\top) = (A - HC)^\top \tag{2.30}$$

This leads to this important result:

$$S_{primal} \text{ is fully observable} \Longleftrightarrow S_{dual} \text{ is fully controllable}$$

So, to ensure relation (2.24), we need that the system S_{primal}, so system S, is fully observable. In conclusion, we have shown that, for an adequate value of H we are ensuring:

$$\lim_{t\to\infty} \underline{\hat{x}}(t) = \underline{x}(t) \tag{2.31}$$

By rewriting the previous equations in matrix form we have:

$$\begin{bmatrix} \underline{\dot{x}}(t) \\ \underline{\dot{e}}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{e}(t) \end{bmatrix} \tag{2.32}$$

The diagonal of the central matrix shows that matrix K and H can be chosen independently one from each other and they represent the eigenvalues of this matrix. This is called **SEPARATION PRINCIPLE**.

It is important also to highlight that BK is the quantity related to $\underline{e}$(t) signal so, the more K is high the more the error is amplified.

PRACTICAL ASPECT:
It is important to guarantee a dynamic of the state S slower than the state space observer $\hat{S}$ in such a way to have that the latter can generate a good state estimation $\underline{\hat{x}}(t)$.

In case of gyroscope failure K and H are computed by using the **Linear Quadratic Regulator (LQR)** in order to obtain the optima control. LQR is a type of optimal control based on state space representation.

For a continuous time system the quadratic cost function is a weighted sum between the performance of the system (Q) and the actuator effort (R)

$$J(u) = \int_0^\infty (x^\top Qx + u^\top Ru)dt \tag{2.33}$$

where:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.34}$$

and it is associated to the **performance** (it must be strictly positive and diagonal in such a way to target the state where we want to have a really low error. The higher will be the corresponding value on the Q matrix the lower will be the error associated to this state)

and

$$R = 1 \tag{2.35}$$

and it is associated to the **actuator performance** (it must be strictly positive and diagonal in such a way that if one of the actuator is really expensive (like thrusters), we can penalize it by making very large the corresponding value on the R matrix).

So, by doing this instruction

[K,S,e] = lqr(A,B,Q,R);

we obtain:

- $\boldsymbol{K}^{1\times 2}$: optimal gain $\longrightarrow$ this vector contains the optimal gain values for the closed-loop system
- $\boldsymbol{E}^{2\times 1}$: closed-loop eigenvalues of (A-BK)

By recalling that B and H are independent (for separation principle) and that we want the state space observer faster than the system, we can obtain H by following these steps:

1. take the eigenvalues of (A-BK) and increase the values

   eig_values = e*2;

2. place the eigenvalues in the dual system with place command

   Ht = place(A',C',eig_values);

3. transpose Ht matrix in such a way to obtain H matrix

   $H^{2\times1}$ = Ht' $^{1\times2}$;

2.3.6 Control/regulation scheme

Up to know we have only considered how to control the states but not how to control the output. It is now the time to design a control scheme that ensures the output y(t) to follows a desired output signal $y^*(t)$.

To do so we want to ensure:

$$\lim_{t\to\infty} \delta y(t) = 0 \tag{2.36}$$

with

$$\delta y(t) = y(t) - y^*(t) \tag{2.37}$$

So the system based on the desired values becomes:

$$S^* : \begin{cases} \dot{\underline{x}}^*(t) = A\underline{x}^*(t) + B\underline{u}^*(t) \\ \underline{y}^*(t) = C\underline{x}^*(t) \end{cases} \tag{2.38}$$

NOTE: in principle we could have that, by starting at the right initial condition $\underline{x}(0) = \underline{x}^*$ and by applying the right u(t) = u^*, the output of S becomes y(t) = y^* but this is physically impossible (i.e. impossible to follow an ideal unitary step).

Considering now the dynamic of the error we have:

$$\begin{cases} \delta \underline{x}(t) = \underline{x}(t) - \underline{x}^*(t) \\ \delta u(t) = u(t) - u^*(t) \\ \delta y(t) = y(t) - y^*(t) \end{cases} \tag{2.39}$$

from which we have:

$$\delta \underline{\dot{x}}(t) = A\delta \underline{x}(t) + B\delta u(t) \tag{2.40}$$

Since we want that:

$$\lim_{t\to\infty} \delta \underline{x}(t) = 0 \tag{2.41}$$

to guarantee eq. (2.36).
If (A,B) is controllable, we can design a feedback control law

$$\delta \underline{u}(t) = -K\delta \underline{x}(t) \tag{2.42}$$

which can ensure:

$$\delta \underline{\dot{x}}(t) = (A - BK)\delta \underline{x}(t) \tag{2.43}$$

By rearranging eq. (2.42) we have:

$$\delta \underline{u}(t) = -K\underline{x}(t) + \underline{u}^* + K\underline{x}^*(t) = -K\underline{x}(t) + \underline{v}^*(t) \tag{2.44}$$

If we consider also that $\underline{x}$(t) is not fully measurable, the scheme becomes:

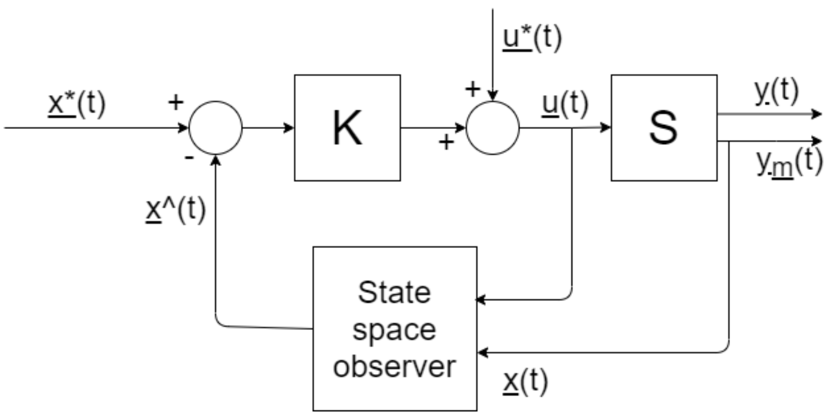

Figure 2.8: Control/regulation scheme

All the previous analysis is still valid, even with the state space observer, because:

$$\underline{\hat{x}}(t) \longrightarrow \underline{x}(t) \longrightarrow \underline{x}^*(t) \tag{2.45}$$

PRACTICAL PROBLEMS:

- the reference signals $\underline{x}^*$ and u^* are computed offline based on the knowledge of y^* this leads to the problem that, in case of dynamic reference signal, it would not follow anymore by $y(t)$
- u^* is a open-loop signal
- most of the time signal $y_n^*(t)$ cannot be physically reached so, in order to not stress too much the actuators, the signal is smoothed with a filter in such a way to obtain a signal which is easier to follow.

$$Y^*(s) = F(s)Y_n^*(s) \tag{2.46}$$

2.3.6.1 Prefilters N_u and N_x

Supposing that y^* is known, so also $Y^*(s)$, the problem comes down into design the appropriate block P that allows to obtain u^* and u^*.

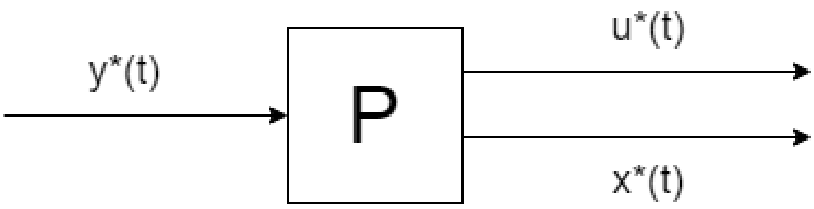

Figure 2.9: Black box

In case of constant reference value, we are going to have the following system:

$$S^* \begin{cases} \underline{0} = A\underline{x}^* + B\underline{u}^* \\ y^* = C\underline{x}^* \end{cases} \tag{2.47}$$

and this leads to the following equations:

$$\underline{x}^* = N_x(s)y^* \tag{2.48}$$

$$u^* = N_u(s)\,y^* \tag{2.49}$$

where:

$$N_x = A^{-1}B\left[CA^{-1}B\right]^{-1} = \frac{\underline{x}^*}{y^*} \tag{2.50}$$

$$N_u = -\left[CA^{-1}B\right]^{-1} = \frac{u^*}{y^*} \tag{2.51}$$

with the assumptions that: A is invertible, $y^* \in$ Span(C), B and C are full rank matrices.
All the previous relations can be represented with the following scheme:

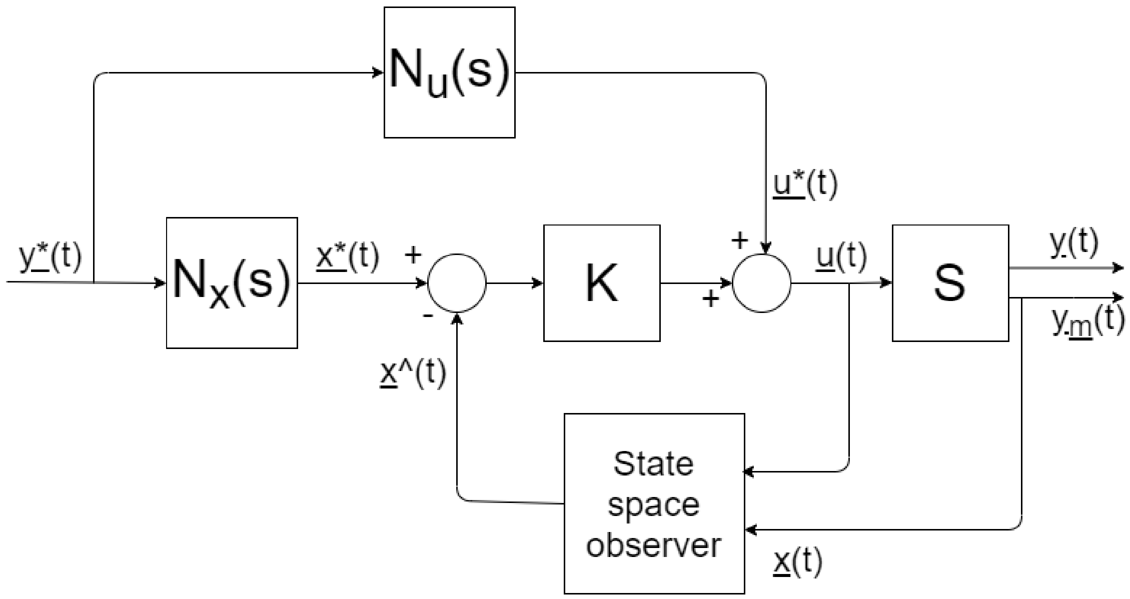

Figure 2.10: Control/regulator scheme

In our case, to compute N_x and N_u we cannot use relations (2.50) and (2.51) because A in not invertible since det(A)=0. For that reason we have that the problem becomes a numerical problem.

So, by calling:

$$A_s = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad X = \begin{bmatrix} \underline{0} \\ u^* \end{bmatrix} \quad B_s = \begin{bmatrix} \underline{0} \\ y^* \end{bmatrix} \tag{2.52}$$

and by using

X=lsqminnorm(A_s, B_s);

we can obtain the best X matrix minimizes the value of norm(As*X-Bs) in such a way to have the best approximation of eq. (2.47).
Finally we can obtain N_x and N_u matrix as follows:

$$N_x^{2\times 1} = \frac{X(1:2,1)}{y^*}; \tag{2.53}$$

$$N_u^{1\times 1} = \frac{X(3,1)}{y^*}; \tag{2.54}$$

By considering $y^* = 5$ we have the following plot.

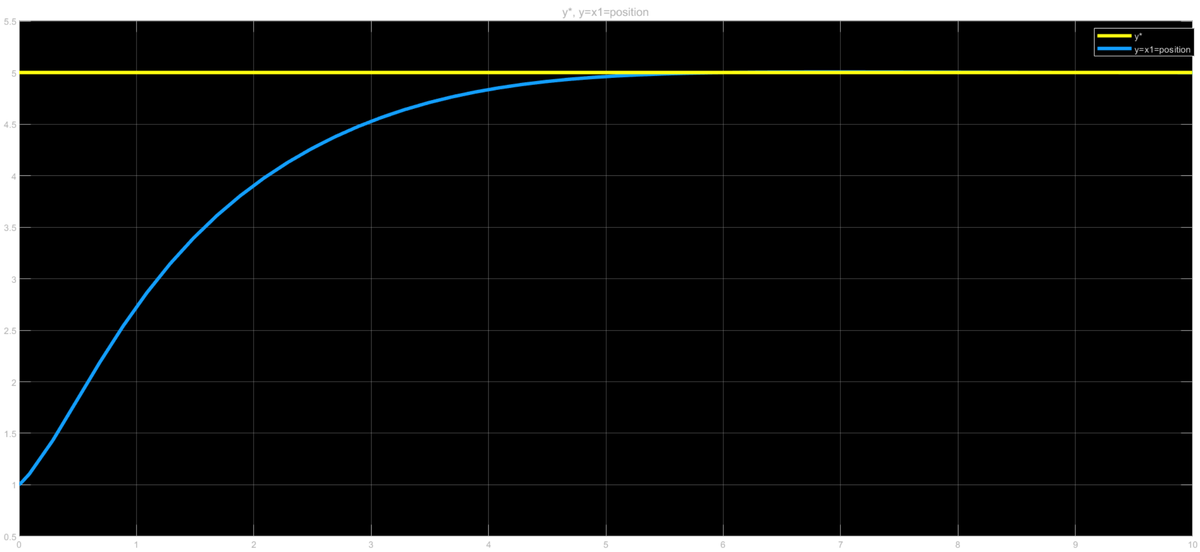

Figure 2.11: Comparison between y(t) (blue) and y*(t) (yellow)

2.3.6.2 Dynamic compensator

The control scheme design up to now cannot correct eventual disturbances. A first approach to solve this problem is to implement a dynamic compensator, with the form $-\frac{K_I}{s}$, instead of prefilter N_u(s). The new scheme is the following one:

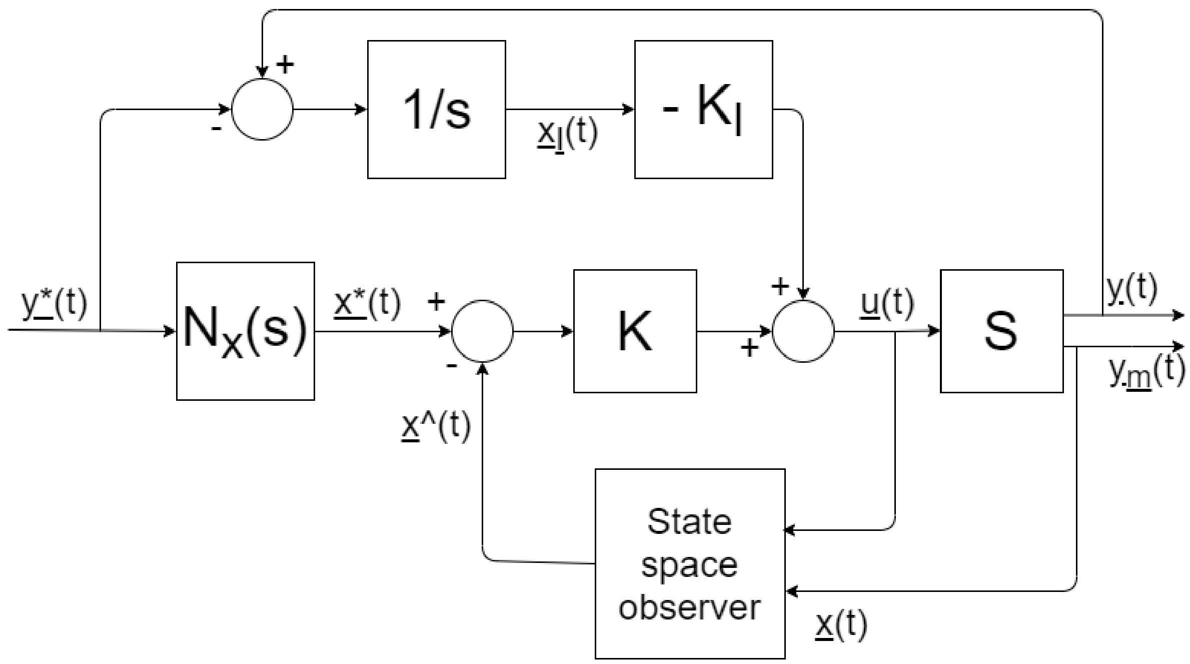

Figure 2.12: Control/regulation scheme with dynamic compensator

From the scheme above we can notice how, with this approach, there is a direct connection between the actual y(t) and the desired y^*(t) therefore, an eventual output variations can be corrected thanks to the dynamic compensator.

To better analyse the dynamic of the system we suppose to open the feedback loop as follows:

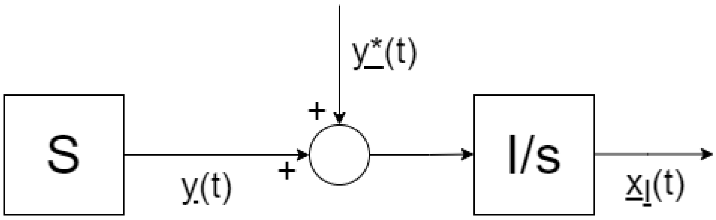

Figure 2.13: Open loop for analysis of dynamic compensator

and to consider the augmented state vector:

$$\underline{x_A}(t) = \begin{bmatrix} \underline{x}(t) \\ x_I(t) \end{bmatrix} \tag{2.55}$$

From that we can define the following augmented state system

$$S_A : \begin{cases} \underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t) \\ \underline{y}(t) = C\underline{x}(t) \\ \dot{x}_I(t) = y(t) - y^*(t) \end{cases} \tag{2.56}$$

If S_A is controllable then we can find a feedback control law such that:

$$u(t) = -k_I x_I(t) + kx(t) + k\underline{x}^*(t) \tag{2.57}$$

$$u(t) = -K\underline{x_A}(t) + k\underline{x}^*(t) \tag{2.58}$$

where:

$$K = \begin{bmatrix} k & k_I \end{bmatrix} \tag{2.59}$$

In conclusion, the problem comes out to find k_I and k which get a better performance of system S. Same analysis can be applied even in presence of a state space observer for relation (2.3.6).

OBSERVATION: even if $\underline{x}^*$ and u^* don't play a big role in output precision, we have that they can play a big role in speed during output transient phase.

In case of gyroscope failure, by rewriting eq. (2.56) in matrix form, we obtain:

$$\dot{\underline{x}}_A(t) = \begin{bmatrix} A & \underline{0} \\ C & \underline{0} \end{bmatrix} \underline{x_A}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} \underline{0} \\ I \end{bmatrix} y^*(t) \tag{2.60}$$

$$u(t) = -\begin{bmatrix} k & k_I \end{bmatrix} \begin{bmatrix} \underline{\hat{x}}(t) \\ x_I(t) \end{bmatrix} \tag{2.61}$$

By calling:

$$A_K = \begin{bmatrix} A & \underline{0}^{2\times1} \\ C & 0 \end{bmatrix} B_K = \begin{bmatrix} B \\ 0^{1\times1} \end{bmatrix} K = \begin{bmatrix} k^{1\times2} & k_I^{1\times1} \end{bmatrix} \tag{2.62}$$

and by using LQR, we can find the optimal K and H matrices:

$$[K_{k\_big}, S_k, e_k] = lqr(A_k, B_k, Q_k, R_k);$$

$$K_k^{1\times2} = K_{k\_big}(:, 1:2); \qquad K_I^{1\times2} = K_{k\_big};$$

From that we can obtain H_k matrix by following these steps:

1. take the first two eigenvalues of $(A_k - B_k K_{k\_big})$ and increase the values

   eig_values = e_k(1:2,:)*2;

2. place the eigenvalues in the dual system with place command

   H_{kt} = place(A_k',C_k',eig_values);

3. transpose Ht matrix in such a way to obtain H matrix

   $H_k^{2\times1}$ = $H_{kt}'^{\ 1\times2}$;

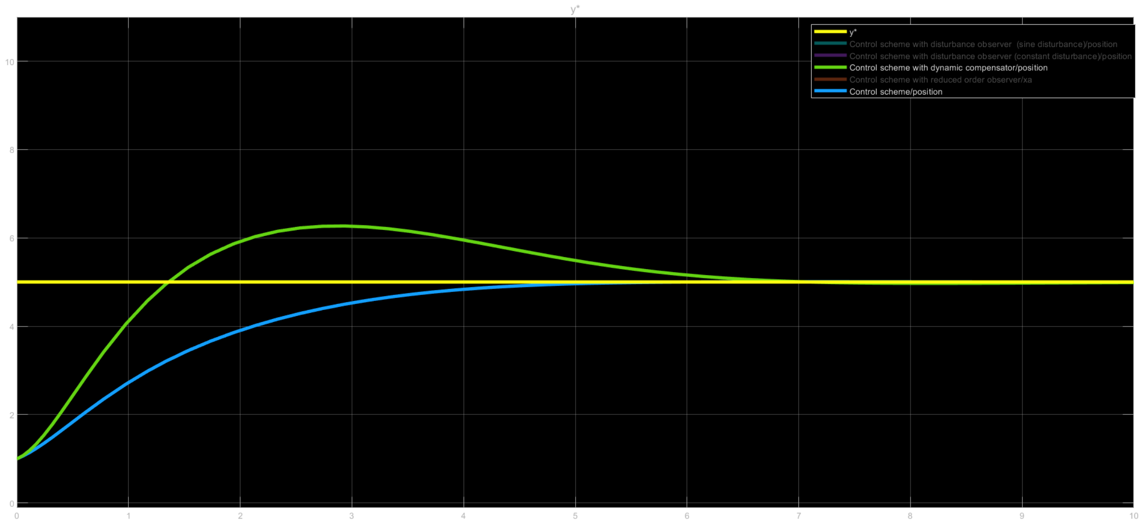

Figure 2.14: Comparison of y(t) for a system without dynamic compensator (blue) and for another one with dynamic compensator (green)

> From the plot above we can notice that the the two response has the same accuracy but the one with the dynamic compensator, green line, is faster.

2.3.7 Reduced state space observer

We can now proceed to analyse the response that implement a reduced state space observer instead of a State space observe. With this approach we are going to estimates only the missing state instead of the whole state space vector.

To do so consider the following system:

$$S: \begin{cases} \dot{\underline{z}}(t) = \bar{A}\underline{z}(t) + \bar{B}\underline{u}(t) \\ \underline{y}(t) = \bar{C}\underline{z}(t) \end{cases} \tag{2.63}$$

and, if $\bar{C}$ is full rank, then its columns are linear independent from each other. For that reason it is possible to define a square, non-singular submatrix.

Assuming that S is observable, we can define the following new matrix:

$$T = \begin{bmatrix} \bar{C} \\ N \end{bmatrix} \tag{2.64}$$

where N arbitrary matrix such that T is square and invertible.

By using matrix T we can make a change of coordinates and split $\underline{z}(t)$ into two subsystems:

$$\underline{x}(t) = T\underline{z}(t) \tag{2.65}$$

$$\begin{bmatrix} \underline{x}_a(t) \\ x_b(t) \end{bmatrix} = \begin{bmatrix} \bar{C}\underline{z}(t) \\ N\underline{z}(t) \end{bmatrix} \tag{2.66}$$

By recalling that $\underline{x}_a(t)$ is associated to the measurable quantity, we have:

$$\underline{x}_a(t) = \underline{y}(t)$$

therefor we can rewrite the system in the following way:

$$S: \begin{cases} \begin{bmatrix} \underline{\dot{x}_a}(t) \\ \dot{x}_b(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \underline{x_a}(t) \\ x_b(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ \underline{y}(t) = \underline{x_a}(t) \end{cases} \tag{2.67}$$

From the above relation we can rewrite the following equations:

$$\underline{\dot{x}_a}(t) = A_{11}\underline{x_a}(t) + B_1 u(t) + A_{12}x_b(t) \tag{2.68}$$

$$\dot{x}_b(t) = A_{22}x_b(t) + u_b(t) \tag{2.69}$$

where:

$$u_b(t) = B_2 u(t) + A_{21}\underline{x_a}(t) \tag{2.70}$$

The idea now is to design a state space observer that estimates the values only for x_b(t) in the following way:

$$\dot{\hat{x}}_b(t) = A_{22}x_b(t) + u_b(t) + H_b[\underline{y_b}(t) - C_b\hat{x}_b(t)] \tag{2.71}$$

where:

$$C_b = A_{12} \tag{2.72}$$

and y_b(t) can be obtain by rewriting eq. (2.68)

$$y_b(t) = \underline{\dot{x}_a}(t) - u_a(t) \tag{2.73}$$

where

$$u_a(t) = A_{11}\underline{x_a}(t) + B_1 u(t) \tag{2.74}$$

$$y_b(t) = A_{12}\underline{x_b}(t) \tag{2.75}$$

We can rewrite eq. (2.71) in the following way:

$$\dot{\hat{x}}_b(t) = A_{22}x_b(t) + \underline{u_b}(t) + H_b[\underline{\dot{x}_a}(t) - u_a(t) - A_{12}\hat{x}_b(t)] \tag{2.76}$$

The only missing problem now is to figure out how to compute $\underline{\dot{x}_a}(t)$ because in principle it's a known quantity but in practice is not computable.

To solve this problem we have to introduce a change of coordinate:

$$\eta(t) = x_b(t) - H_b\underline{x_a}(t) \tag{2.77}$$

From that we have all these relations:

$$\hat{\eta}(t) = \hat{x}_b(t) - H_b\underline{\hat{x}_a}(t) \tag{2.78}$$

$$\dot{\eta}(t) = \dot{x}_b(t) - H_b\underline{\dot{x}_a}(t) \tag{2.79}$$

$$\dot{\hat{\eta}}(t) = \dot{\hat{x}}_b(t) - H_b\underline{\dot{x}_a}(t) \tag{2.80}$$

and, from eq. (2.78) we can obtain:

$$\hat{x}_b(t) = \hat{\eta}(t) + H_b\underline{\hat{x}_a}(t) \tag{2.81}$$

From all the above equations we can write:

$$\hat{s}_b : \begin{cases} \dot{\hat{\eta}}(t) = (A_{22} - H_bA_{12})\hat{\eta}(t) + (A_{22} - H_bA_{12})H_b\underline{x_a}(t) + w(t) \\ \hat{x}_b(t) = \hat{\eta}(t) + H_b\underline{x_a}(t) \end{cases} \tag{2.82}$$

where:

$$w(t) = u_b(t) - H_bA_{11}\underline{x_a}(t) - H_bB_1u(t) \tag{2.83}$$

and

$$u_b(t) = A_{21}\underline{x_a}(t) + B_2 u(t) \tag{2.84}$$

The input equation of system (2.82) has the same structure of a state space observer in which we have to choose an appropriate H_b matrix in such a way to have:

$$\hat{x}_b(t) \quad \longrightarrow \quad x_b(t)$$

To do so the desired H_b matrix has to guarantee that each eigenvalue of $(A_{22} - H_b A_{12})$ has the real part strictly negative.

It is also important to highlight the effect of a reduced observer on the feedback control law:

$$u(t) = -\begin{bmatrix} k_a & k_b \end{bmatrix} \begin{bmatrix} \underline{x_a}(t) \\ \hat{x}_b(t) \end{bmatrix} + \underline{v}(t) \tag{2.85}$$

from which the closed-loop dynamic of the system becomes:

$$\underline{\dot{x}}(t) = (A - BK)\underline{x}(t) + BK \begin{bmatrix} \underline{0} \\ e_b(t) \end{bmatrix} \tag{2.86}$$

where $\underline{e}_a(t) = \underline{0}$ for definition.

With this approach we have that

$$||BK\underline{e}(t)|| \tag{2.87}$$

is lower with respect to the one obtained with the full state space observer since $\underline{e}$(t) is smaller due to the fact that $\underline{e}_a$(t)= $\underline{0}$.

In case of gyroscope failure we have:

$$x_a(\mathrm{t}) = x_1(\mathrm{t}) = y_m(\mathrm{t})$$
$$x_b(\mathrm{t}) = x_2(\mathrm{t})$$

So, by using the structure of eq. (2.67), we have

$$\begin{bmatrix} \dot{x}_a(t)^{1\times 1} \\ \dot{x}_b(t)^{1\times 1} \end{bmatrix} = \begin{bmatrix} A_{11}^{1\times 1} & A_{12}^{1\times 1} \\ A_{21}^{1\times 1} & A_{22}^{1\times 1} \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix} + \begin{bmatrix} B_1^{1\times 1} \\ B_2^{1\times 1} \end{bmatrix} u(t) \tag{2.88}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.89}$$

We have now to place the eigenvalues in the dual system in order to get the desired scalar value H_b. We can do that by following these steps:

1. take the most relevant eigenvalues of (A-BK), so the first element e(1,1) since we are looking for a scalar H_b value, and double the value

   eig_value = e*2;

   in order to ensure a reduce observer faster than the system

2. place the eigenvalue in the dual system with place command

   Ht = place(A(2,2),A(1,2),eig_value);

   noticing that the transpose of a scalar is the scalar itself

3. transpose Ht matrix in such a way to obtain H matrix

   $H^{2\times 1}$ = Ht' $^{1\times 2}$;

From the plot below we can observe that the blue line, so the dynamic of $\dot{e}$(t) given from the system with a reduced observer, goes faster to zero with respect the yellow one, so the $\dot{e}$(t) given from a system which implements the full state space observer.

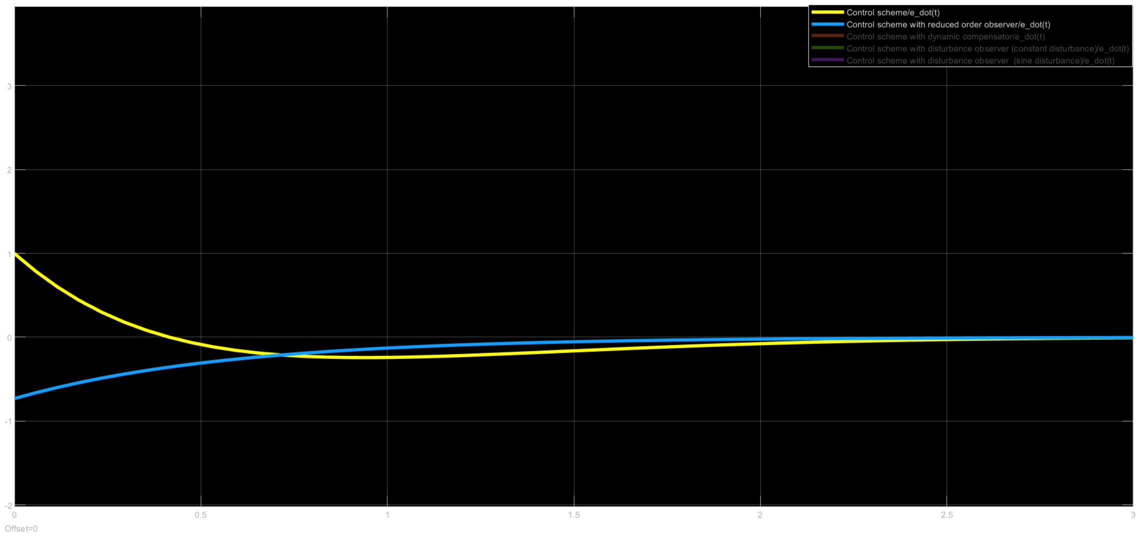

Figure 2.15: Comparison of $\dot{e}$(t) produced by a system which implements a full state observer (yellow) and another one with reduce observer (blue)

2.3.8 Disturbance observer

Up to now we have design a control regulator which does not keep into account the case where an external disturbance occurs and, even with a small one, the system output can completely change.

2.3.8.1 Constant disturbance

By considering a constant disturbance

$$d(t) = d_0 \; 1(t) \quad \text{with } d_0 = 0.5$$

the model of the system becomes:

$$S: \begin{cases} \dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 0.5 \; 1(t) \\ \\ \underline{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t) \end{cases} \tag{2.90}$$

and, without any disturbance observer, we have the following output transient:

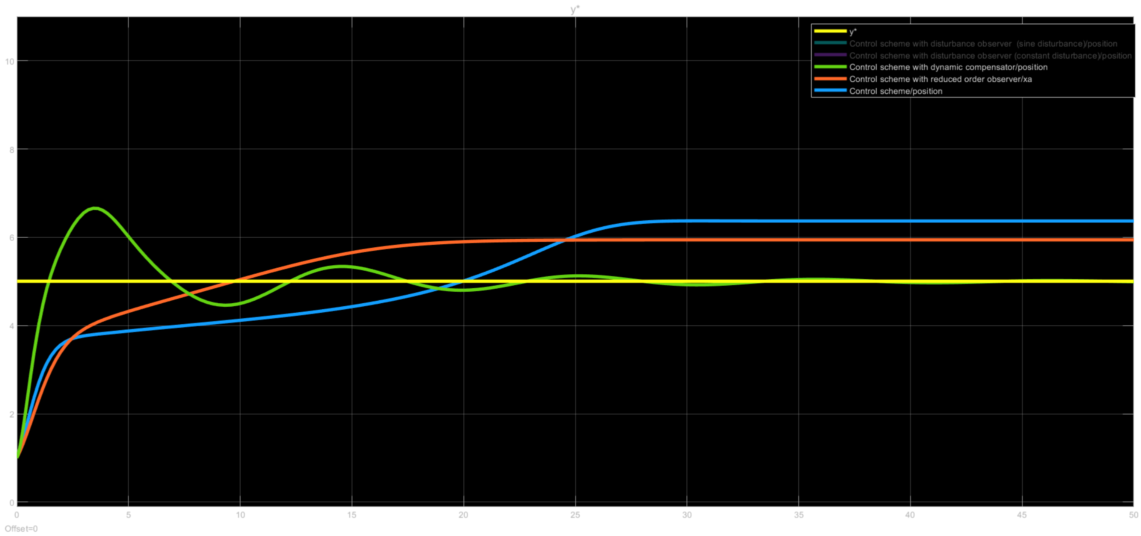

Figure 2.16: Comparison of y(t) generated by a system with a full state observer (blue), another one with reduce observer (red) and the last one with dynamic compensator in presence of a constant disturbance

From the image we can observe that the only system that is still able to stay closed to the desired output value y^*(t) = 5 is only the one that implements the dynamic compensator. This is reasonable since it is the only one that has a direct comparison between y^*(t) and the actual y(t) as shown in Fig. 2.12 and therefore can compensate a variation of y(t) due to a disturbance.

The idea is to consider the effect of the disturbance as the response of fictitious dynamic system. In case of constant constant disturbance the dynamic system is:

$$S_d : \begin{cases} \dot{x}_d(t) = 0 \\ y_d(t) = x_d(t) = d(t) = d_0 \;\; 1(t) \end{cases} \tag{2.91}$$

where d_0 can be considered as the initial condition of the system.

In general S_d can be view as a system with its own dynamic but without any input signal to simulate a non controllable system.

$$S_d : \begin{cases} \dot{x}_d(t) = A_d x_d(t) \\ y_d(t) = C_d x_d(t) = d(t) \end{cases} \tag{2.92}$$

A general view of the current problem can be represented by the following scheme:

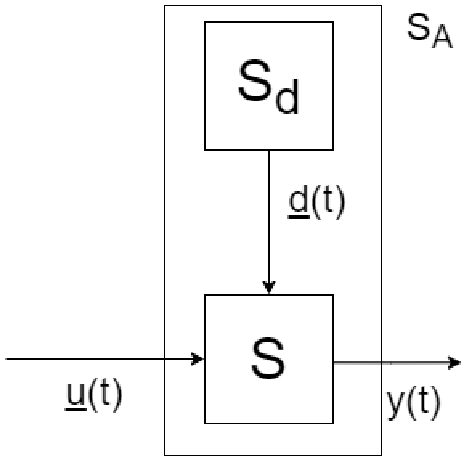

Figure 2.17: General scheme of a system in case of an external disturbance

By considering the augmented state

$$\underline{x_A}(t) = \begin{bmatrix} \underline{x}(t) \\ x_d(t) \end{bmatrix} \tag{2.93}$$

we can rewrite S_A as follows:

$$S_A : \begin{cases} \underline{\dot{x}}(t) = \begin{bmatrix} A & DC_d \\ 0 & A_d \end{bmatrix} \underline{x_A}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \underline{x_A}(t) \end{cases} \tag{2.94}$$

which is equal to:

$$S_A : \begin{cases} \underline{\dot{x}}(t) = A_A \underline{x_A}(t) + B_A u(t) \\ \\ \underline{y}(t) = C_A \underline{x_A}(t) \end{cases} \tag{2.95}$$

Now, by analysing S_A, we can obtain some information:

- **CASE 1:** if S was fully observable (or at least detectable) and S_A is no more observable means that the loss of observability is due to the disturbance. From this we can say that d(t) has none impact on the output and so we don't need to design a disturbance observer.

- **CASE 2:** if S_A remains observable then it is possible to design a state observer

$$\hat{S}_A : \ \underline{\dot{\hat{x}}_A}(t) = A_A \underline{\hat{x}_A}(t) + B_A u(t) + H_A[y(t) - C_A \underline{\hat{x}_A}(t)] \tag{2.96}$$

which, with an appropriate H_A, guarantees to have:

$$\lim_{t\to\infty} \underline{\hat{x}_A}(t) = \underline{x_A}(t) \tag{2.97}$$

therefore also

$$\lim_{t\to\infty} \hat{x}_d(t) = x_d(t) \tag{2.98}$$

This result is really useful since from the estimation of $x_d(t)$ we can reconstruct the disturbance value as follows:

$$\lim_{t\to\infty} \hat{d}(t) = d(t) \tag{2.99}$$

where

$$\hat{d}(t) = C_d \hat{x}_d(t) \tag{2.100}$$

In conclusion we want to use a control signal u(t) capable of guarantee the stability and accuracy of the output but also of compensating the effect of the disturbance. For that reason we can consider the control signal as the sum of three components:

$$u(t) = -k \ \left(\underline{x}(t) - \underline{x}^*(t)\right) + u^*(t) + u_d(t) \tag{2.101}$$

where

- FIRST TERM aims to correct the error
- SECOND TERM aims to guarantee the accuracy
- THIRD TERM aims to compensate the effect of the disturbance

By applying control law (2.101) to the system, the component that has to compensate the effect of the disturbance has to be such that:

$$Bu_d(t) + Dd(t) = 0 \tag{2.102}$$

from which, by supposing that B is full rank matrix, we have:

$$u_d(t) = -B^{\#} Dd(t) \tag{2.103}$$

where $B^{\#}$ is the pseudo-inverse of B and can be computed:

$$B^{\#} = (B^{\top} B)^{-1} B^{\top} \tag{2.104}$$

In case of gyroscope failure, by considering the dynamic of the disturbance expressed in eq. (2.92), we have:

A_d=0 C_d=1 D=D

Based on that we can rewrite eq. (2.95) as follows:

$$S_A : \begin{cases} \underline{\dot{x}}(t) = \begin{bmatrix} A & DC_d \\ 0 & A_d \end{bmatrix} \underline{x_A}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \underline{x_A}(t) \end{cases} \tag{2.105}$$

Now, since S_A is **FULLY OBSERVABLE**, we have to design a disturbance observer. In particular, we have to find the appropriate H_A and $u_d(t)$ that ensure that the system is capable of rejecting constant external disturbance.

To compute H_A I have to follow these steps:

1. Since S_A is fully observable, for the duality principle, I can arbitrarily place the poles in its dual system in such a way to obtain that the eigenvalues of

   $$(A_A - H_A C_A)$$

   have all real part strictly negative. In this way we can ensure:

   $$\lim_{t\to\infty} \underline{\hat{x}_A}(t) = \underline{x_A}(t)$$

   Same result can be obtain with the following values:

   egn_val_Ha=[-1 -3 -5];

2. After that we have to place the pole in the dual system Hat = place(Aa',Ca',egn_val_Ha);

3. Finally we can transpose Hat matrix in order to have H matrix

   $$Ha^{3\times 1}=Hat'^{1\times 3};$$

To compute $u_d(t)$ I have to implement eq. (2.104) by recalling that the pseudo-inverse in MATLAB is done with pinv command

G = pinv(B)*D;

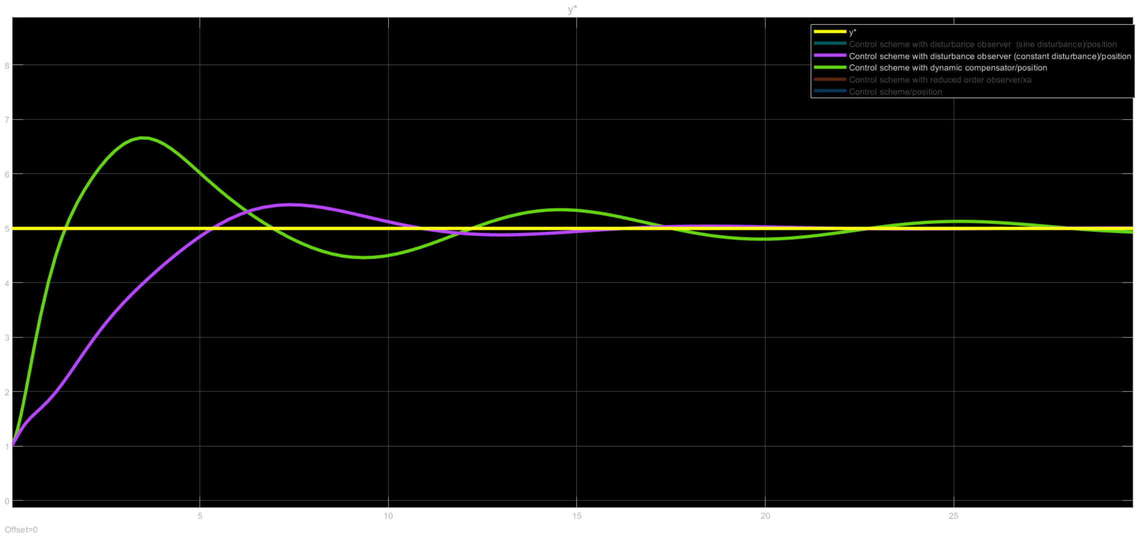

Figure 2.18: Comparison of disturbance rejection between a system with dynamic compensator (green) and the one with a disturbance observer (purple)

From the figure above we can conclude that the **control scheme for the State space system with disturbance observer** is the one that better reach all the specifications of:

- **STABILITY:** since the system is BIBO
- **PRECISION:** since it can generate an output signal $\underline{y}(t)$ similar to $\underline{y}_R$ with the shortest transient
- **DISTURBANCE REJECTION:** since it is the one that better reject the effects of the external disturbance