[redacted] - Exercise class [redacted]

1. A plant has to fill an order of 1000 tons of a product requiring manganese (at least 1%), chrome (at least 18%) and molybdenum (at least 2%). The suppliers sell these three metals in three different kinds of packagings. The first packaging contains 2 Kilos of manganese, 2 Kilos of chrome and 1 Kilo of molybdenum and costs 20 EUROS. The second packaging contains 2 Kilos of manganese, 3 Kilos of chrome and 1 Kilo of molybdenum and costs 30 EUROS. The third packaging contains 1 Kilo of manganese, 2 Kilos of chrome and 5 Kilos of molybdenum and costs 40 EUROS. The goal is to minimize the total cost.
   Formulate the corresponding LP model.

2. A refinery produces a fuel, which is made by blending five raw stocks of differing octane rating, ethanol, additive and cost.

Stock	Octane rating	Ethanol	Additive (ppM)	Price (E/l)
1	70	15	500	9
2	80	20	400	12
3	85	25	400	13
4	90	35	350	27
5	99	40	300	29

   The fuel must have a minimum level of ethanol of 30 and at least 450 parts per million of additive. The octane rating must be on average at least 88. It is requested to produce at the minimum cost.
   Formulate the corresponding LP model.

3. A call center employs 10 workers to be divided into 3 teams. For each worker the speed of completing phone calls has been measured as the average number of phone calls processed in a minute.

Operator	1	2	3	4	5	6	7	8	9	10
Speed	3.5	6	7	2.8	1.5	1.5	3	4	6.5	4

   The operators must be assigned to 3 teams of at least 2 and at most 4 workers. Each team must process at least 8 calls/minute. Operators 1,3 and 5 must be assigned to distinct teams. The objective is to maximize the speed of the slowest team.
   Formulate the corresponding LP model.

4. A firm producing bookshelves has received an order for 200 shelves. Each shelf requires four 4m boards, three 2m boards, four 1m boards and nine 0.4m boards. The boards can be purchased from the suppliers with the following fixed lengths only: 6m (12 EUROS), 3m (8 EUROS) and 1.5m (5 EUROS).
   It is possible to cut the 6m boards as follows: (4m/2m) or (1m/1m/1m/1m/1m/1m) or (4m/1m/1m) or (2m/2m/2m) or (2m/2m/1m/1m) or (4m/.4m/.4m/.4m/.4m/.4m) or (4m/1m/0.4m/0.4m)
   It is possible to cut the 3m boards as follows: (2m/1m) or (1m/1m/.4m/.4m) or (1m/1m/1m) or (.4m/.4m/.4m/.4m/.4m/.4m/.4m)
   It is possible to cut the 1.5m boards as follows: (1m/.4m) or (.4m/.4m/.4m).
   The objective is to minimize the total cost.
   Formulate the corresponding LP model.

5. A horse-breeder can use four different types of foods to determine the most appropriate diet for his horses. Each kg. of a given food has the following characteristics depicted in the following table (cost in EUROS, calories in cal., proteins in g., vitamins in g.)

Food	Cost	Calories	Proteins	Vitamins
1	1300	4500	0.4	0.6
2	900	3600	0.7	0.5
3	700	3000	0.4	0.4
4	400	1800	0.6	0.3

   A balanced diet is required for each horse that guarantees a quantity of proteins P between 2 and 6 g., at least 5 g. of vitamins V and a quantity of calories C between 15000 and 22000. It is also required that the blend includes at least 20% of food 2 and non more than 50% of food 4. Foods 1 and 3 are not compatible. The horse-breeder wants to determine the blend maximizing expression

$$100V - 12L - 20|C - 14000| - 7500|P - 4|$$

   where L is the total blending cost.
   Formulate the corresponding LP model.

SOLUTIONS

1. See [redacted] the Course website.

2. Let denote by x_i the quantity in liters of each stock. The model is as follows.

$$\min 9x_1 + 12x_2 + 13x_3 + 27x_4 + 29x_5$$

$$\begin{array}{rcl}
70x_1 + 80x_2 + 85x_3 + 90x_4 + 99x_5 & \geq & 88(x_1 + x_2 + x_3 + x_4 + x_5) \\
15x_1 + 20x_2 + 25x_3 + 35x_4 + 40x_5 & \geq & 30(x_1 + x_2 + x_3 + x_4 + x_5) \\
500x_1 + 400x_2 + 400x_3 + 350x_4 + 300x_5 & \geq & 450(x_1 + x_2 + x_3 + x_4 + x_5) \\
x_1 + x_2 + x_3 + x_4 + x_5 & = & 1 \\
x_i & \geq & 0 \quad \forall i = 1, ..., 5
\end{array}$$

where the first three constraints relate to the requirements on octane, ethanol and additive, respectively. The fourth constraint indicates that the blending refers to a liter of product and the fifth constraint is related to the positivity of the variables.

3. Let $x_{i,j}$ be a binary variable denoting whether operator j is assigned or not to team i. Let $y \geq 0$ be a positive real variable denoting the speed of the slowest team. The model is as follows.

$$\max y$$

$$\begin{array}{rcl}
3.5x_{i,1} + 6x_{i,2} + 7x_{i,3} + 2.8x_{i,4} + 1.5x_{i,5} + 1.5x_{i,6} + & & \\
+3x_{i,7} + 4x_{i,8} + 6.5x_{i,9} + 4x_{i,10} & \geq & y \quad \forall i = 1, ...3 \\
3.5x_{i,1} + 6x_{i,2} + 7x_{i,3} + 2.8x_{i,4} + 1.5x_{i,5} + 1.5x_{i,6} + & & \\
+3x_{i,7} + 4x_{i,8} + 6.5x_{i,9} + 4x_{i,10} & \geq & 8 \quad \forall i = 1, ...3 \\
x_{i,1} + x_{i,2} + x_{i,3} + x_{i,4} + x_{i,5} + x_{i,6} + x_{i,7} + x_{i,8} + x_{i,9} + x_{i,10} & \leq & 4 \quad \forall i = 1, ...3 \\
x_{i,1} + x_{i,2} + x_{i,3} + x_{i,4} + x_{i,5} + x_{i,6} + x_{i,7} + x_{i,8} + x_{i,9} + x_{i,10} & \geq & 2 \quad \forall i = 1, ...3 \\
x_{i,1} + x_{i,3} + x_{i,5} & \leq & 1 \quad \forall i = 1, ...3 \\
x_{1,j} + x_{2,j} + x_{3,j} & = & 1 \quad \forall j = 1, ...10 \\
x_{i,j} & \in & \{0, 1\} \ \forall i = 1, ..., 3, \ j = 1, ..., 10
\end{array}$$

where the max-min constraint on the slowest team is handled by the introduction of variable y plus the first type of constraints. The second set of constraints indicates that every team must have speed ≥ 8, The third and fourth sets of constraints indicate that every team must be composed by at most 4 operators and at least 2. The fifth set of constraints indicates that operators 1, 3 and 5 must be assigned to different teams. The sixth set of constraints indicates that each operator must be assigned to exactly 1 team. The last set of constraints indicates that the $x_{i,j}$ variables are binary.

4. Let $x_{i,j}$ be a positive integer variable denoting how many boards of type i ($i = 1 \rightarrow 6m$, $i = 2 \rightarrow 3m$, $i = 3 \rightarrow 1.5m$) are cut according to a specific way. There are seven ways of cutting for boards of size 6m, four ways for boards of size 3m and 2 ways for boards of size 1.5m. The model is as follows.

$$\min \sum_{j=1}^{7} 12x_{1,j} + \sum_{j=1}^{4} 8x_{2,j} + \sum_{j=1}^{2} 5x_{3,j}$$

$$\begin{array}{rcc}
x_{1,1} + x_{1,3} + x_{1,6} + x_{1,7} & \geq & 800 \\
x_{1,1} + 3x_{1,3} + 2x_{1,5} + x_{2,1} & \geq & 600 \\
6x_{1,2} + 2x_{1,3} + 2x_{1,5} + x_{1,7} + x_{2,1} + 2x_{2,2} + 3x_{2,3} + x_{3,1} & \geq & 800 \\
5x_{1,6} + 2x_{1,7} + 2x_{2,2} + 7x_{2,4} + x_{3,1} + 3x_{3,2} & \geq & 1800 \\
x_{i,j} & \mathit{integer} \geq 0 & \forall i \ \forall j
\end{array}$$

where the first four constraints indicate that the 200 shelves require at least 800 boards of size 4m, 600 boards of size 2m, 800 boards of size 1m and 1800 boards of size .4m. The last set of constraints indicates that the $x_{i,j}$ variables are positive integer.

5. Let denote by x_i the quantity in kilos of each food. Let denote by y_i a 0/1 variable indicating if food i is present or not in the diet. Let z be a positive variable representing $|P - 4|$. Let V, L, C and P correspond to what is indicated in the text of the exercise. The model is as follows.

$$\max 100V - 12L - 20(C - 14000) - 7500z$$

$$
\begin{array}{rcl}
V & = & 0.6x_1 + 0.5x_2 + 0.4x_3 + 0.3x_4 \\
L & = & 1300x_1 + 900x_2 + 700x_3 + 400x_4 \\
C & = & 4500x_1 + 3600x_2 + 3000x_3 + 1800x_4 \\
P & = & 0.4x_1 + 0.7x_2 + 0.4x_3 + 0.6x_4 \\
P & \geq & 2 \\
P & \leq & 6 \\
z & \geq & P - 4 \\
z & \geq & 4 - P \\
V & \geq & 5 \\
C & \geq & 15000 \\
C & \leq & 22000 \\
x_2 & \geq & 0.2(x_1 + x_2 + x_3 + x_4) \\
x_4 & \leq & 0.5(x_1 + x_2 + x_3 + x_4) \\
x_1 & \leq & 15y_1 \\
x_3 & \leq & 15y_3 \\
y_1 + y_3 & \leq & 1 \\
V, L, C, P, z, x_i & \geq & 0 \quad \forall i = 1, ..., 4 \\
y_1, y_3 & \in & \{0, 1\}
\end{array}
$$

where the first four constraints link V, L, C and P to the x_i variables. Constraints from fifth to eighth indicate that P must be between 2 and 6 and that z is $\geq P - 4$ and $\geq 4 - P$ (minabs constraint on $|P - 4|$; no minabs constraint is necessary for $|C - 14000|$ as $C \geq 15000$). The ninth constraint states that $V \geq 5$. The tenth and eleventh constraints indicate that C must be between 15000 and 22000. The last constraints indicate that the blend includes at least 20% of food 2 and non more than 50% of food 4 and that foods 1 and 3 are not compatible plus the nonnegativity of the variables and the presence of binary variables y_1 and y_3.