Taylor Series Method

Adelina Anescu, Alexandru Petreuș, Francesca Drăguț

Faculty of Mathematics and Informatics
West University of Timișoara

November 16, 2022

Introduction.

General form of Taylor Polynomials:

$$P_n(x) := y(x_0)+y'(x_0)(x-x_0)+\frac{y^n(x_0)}{2!}(x-x_0)^2+\ldots+\frac{y^n(x_0)}{n!}(x-x_0)^n$$

$$\sum_{n=0}^{\infty}\frac{y^k(x_0)}{k!}(x-x_0)^k$$

Introduction.

To determine the Taylow series for the solution $\varphi(x)$ to the initial value problem

$$dy/dx = f(x, y), \quad y(x_0) = y_0,$$

we need only determine the values of the derivatives of φ (assuming they exist) at x_0; that is, $\varphi(x_0), \varphi'(x_0), \ldots$

The initial condition gives the first value $\varphi(\mathbf{x_0}) = \mathbf{y_0}$. Using the equation $y' = f(x, y)$, we find $\varphi'(x_0) = f(x_0, y_0)$.

Introduction.

To determine $\varphi''(x_0)$, we differentiate the equation $y' = f(x, y)$ implicitly with respect to x to obtain

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

First Problem.

Problem: Compute the Taylor polynomials of degree 4 for the solutions to the given initial value problems.
Use these Taylor polynomials to approximate the solution at x = 1.

$$(i)\quad \frac{dy}{dx} = x - 2y;\quad y(0) = 1$$

$$(ii)\quad \frac{dy}{dx} = y(2 - y);\quad y(0) = 4$$

First Problem. Part I.

Problem 1(i): $\frac{dy}{dx} = x - 2y; \;\; y(0) = 1$

$$\frac{dy}{dx} = x - 2y; \;\; y(0) = 1 \implies x_0 = 0, y_0 = 1 \Leftrightarrow$$

$$\Leftrightarrow y' = x - 2y \implies y'(x_0) = x_0 - 2y(x_0) \;\; \Big| \cdot \frac{d}{dx}$$

$$\implies y''(x_0) = 1 - 2y'(x_0)$$

$$\implies y'''(x_0) = -2y''(x_0)$$

$$\implies y^{IV}(x_0) = -2y'''(x_0)$$

First Problem. Part I.

$$y'(x_0) = y'(0) = 0 - 2y(0) = 0 - 2 \cdot 1 = 0 - 2 = -2$$

$$y''(x_0) = y''(0) = 1 - 2y'(0) = 1 - 2 \cdot (-2) = 5$$

$$y'''(x_0) = y'''(0) = -2y''(0) = -2 \cdot 5 = -10$$

$$y^{IV}(x_0) = y^{IV}(0) = -2y'''(0) = -2 \cdot (-10) = 20$$

First Problem. Part I.

$$P_4(x) :=$$
$$y(x_0)+y'(x_0)(x-x_0)+\frac{y''(x_0)}{2!}(x-x_0)^2+\frac{y'''(x_0)}{3!}(x-x_0)^3+\frac{y^{IV}(x_0)}{4!}(x-x_0)^4$$

$$\Longrightarrow\ P_4(x) := y(0)+y'(0)\cdot x+\frac{y''(0)}{2}\cdot x^2+\frac{y'''(0)}{6}\cdot x^3+\frac{y^{IV}(0)}{24}\cdot x^4 =$$

$$= 1-2x+\frac{5}{2}\cdot x^2-\frac{5}{3}\cdot x^3+\frac{5}{6}\cdot x^4$$

$$x=1 \implies P_4(1) = {}^{6)}\frac{1}{1}-{}^{6)}\frac{2}{1}+{}^{3)}\frac{5}{2}-{}^{2)}\frac{5}{3}+\frac{5}{6} =$$

$$= \frac{6-12+15-10+5}{6} = \frac{4}{6} \simeq 0.6666...7$$

First Problem. Part II.

Problem 1(ii): $\frac{dy}{dx} = y(2-y);\ \ y(0) = 4$

$$\frac{dy}{dx} = y(2-y);\ \ y(0) = 4 \implies x_0 = 0,\ \ y_0 = 4 \Leftrightarrow$$

$$\Leftrightarrow y' = 2y - y^2 \implies y'(x_0) = 2y(x_0) - y^2(x_0)$$

$$\implies y''(x_0) = 2y'(x_0) - 2y(x_0) \cdot y'(x_0)$$

$$\implies y'''(x_0) = 2y''(x_0) - 2 \cdot (y'(x_0))^2 - 2y(x_0) \cdot y'(x_0)$$

$$\implies y^{IV}(x_0) = 2y'''(x_0) - 4y'(x_0) \cdot y''(x_0) - 2y'(x_0) \cdot y''(x_0) - 2y(x_0) \cdot y'''(x_0)$$

$$= 2y'''(x_0) - 6y'(x_0) \cdot y''(x_0) - 2y(x_0) \cdot y'''(x_0)$$

First Problem. Part II.

$$y'(x_0) = 2y(0) - y^2(0) = 2 \cdot 4 - 16 = 9 - 16 = -8$$

$$y''(x_0) = 2y'(0) - 2y(0) \cdot y'(0) = -16 - 2 \cdot 4 \cdot (-8) = 48$$

$$y'''(x_0) = 2y''(0) - 2 \cdot (y'(x_0))^2 - 2y(0) \cdot y''(0) = 2 \cdot 48 - 2 \cdot 64 - 8 \cdot 48 = -416$$

$$y^{IV}(x_0) = 2y'''(0) - 6y'(0) \cdot y''(0) - 2y(0) \cdot y'''(0) =$$
$$-2 \cdot 416 + 6 \cdot 8 \cdot 48 - 2 \cdot 4 \cdot (-416) = 4800$$

First Problem. Part II.

$$P_4(x) := y(0) + y'(0)\cdot x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y^{IV}(0)}{24}x^4 =$$

$$= 4 - 8x + \frac{48}{2}x^2 - \frac{416}{6}x^3 + \frac{4800}{24}x^4 =$$

$$= 4 - 8x + 24x^2 - \frac{208}{3}x^3 + 200x^4 =$$

$$x = 1 \implies P_4(1) = 4 - 8 + 24 - \frac{208}{3} + 200 =$$

$$= 220 - \frac{208}{3} \simeq 150.666...7$$

Second Problem.

Problem: Compare the use of Euler's method with Taylor Series Method to approximate solution $\varphi(x)$ to the problem:
$\frac{dy}{dx} + y = cosx - sinx, \;\; y(0) = 2.$

Indications: Give approximations for $\varphi(1)$ and $\varphi(3)$ to the nearest thousandth. Verify that $\varphi(x) = cos(x) + e^{-x}$. Decide which method yields the closest approximation to $\varphi(10)$.

Second Problem. Solve by Euler's Method[1].

Find $y(0)$ for $y' = -y = sin(x) + cos(x)$, **when** $y(0) = 2, \quad h = \frac{1}{10}$.

Solution: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$, where $x_{n+1} = x_n + h$

We have: $h = \frac{1}{10},\ x_0 = 0,\ y_0 = 2,$ and $f(x, y) = -y - sin(x) + cos(x)$.

Step 1: $x_1 = x_0 + h = 0 + \frac{1}{10} = \frac{1}{10}$

$y_1 = y(x_1) = y(\frac{1}{10}) = y_0 + h \cdot f(x_0, y_0) = 2 + h \cdot f(0, 2) = 2 + \frac{1}{10}(-1) = 1.9$

$\vdots$

Step 10: $x_{10} = x_9 + h = \frac{9}{10} + \frac{1}{10} = 1$

$y_{10} = y(x_{10}) = y(1) = y_0 + h \cdot f(x_0, y_0) = 1.035 + h \cdot f(\frac{9}{10}, 1.035) = 1.035 + \frac{1}{10}(-1.196) = 0.915$

Second Problem. Solve by Taylor's Method.

$$P_2(x) = y(0) + y'(x) \cdot x + \frac{y''(0)}{2}x^2$$

$$y' + y = cos(x) - sin(x) \implies y' = cos(x) - sin(x) - y \implies y'(0) = -1$$

$$y' = cos(x) - sin(x) - y \implies y'' = -sin(x) - cos(x) - y' \implies y''(0) = 0$$

$$\implies P_2(x) = 2 - x + 0 = 2 - x \implies P_2(1) = 1,\ P_2(3) = -1$$

$$y''' = -cos(x) + sin(x) - y'' \implies y'''(0) = -1$$
$$y^{IV} = sin(x) + cos(x) - y''' \implies y^{IV}(0) = 2$$
$$y^{V} = cos(x) - sin(x) - y^{IV} \implies y^{V}(0) = -1$$

Second Problem. Solve by Taylor's Method.

$$P_5(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{4!}x^4 + \frac{y^{V}(0)}{5!}x^5$$

$$\implies P_5(x) = 2 - x - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 - \tfrac{1}{120}x^5$$

$$\implies P_5(1) = 2 - 1 - \tfrac{1}{6} + \tfrac{1}{12} - \tfrac{1}{120} = 0.908$$

$$\implies P_5(3) = 2 - 3 - \tfrac{1}{6} \cdot 27 + \tfrac{1}{120} \cdot 27 \cdot 9 =$$

$$= -1 - \tfrac{9}{2} + \tfrac{27}{4} - \tfrac{81}{40} = -1 - 4.5 + 6.75 - 2.025 = -0.075$$

Second Problem. Comparison table.

Method	Approximation of $\varphi(1)$	Approximation of $\varphi(3)$
Euler's method using step of size 0.1	0.915	-0.971
Euler's method using step of size 0.01	0.909	-0.942
Taylor polynomial of degree 2	1	-1
Taylor polynomial of degree 5	**0.908**	-0.775
Exact value of $\varphi(x)$ to nearest thousandth	**0.908**	-0.940

Second Problem. Conclusions.

Conclusion: Taylor Series Method yields a much closer approximation to the exact solution than Euler's Method. For $\varphi(10)$ we should use the fourth method, i.e. compute it with a higher degree Taylor polynomial, which in our case was 5.

Third Problem.

Problem: Compute the Taylor polynomial of degree 6 for the solution to the Airy equation

$$\frac{d^2y}{dx^2} = xy$$

with the initial conditions y(0) = 1, y'(0) = 0.

Question: Do you see how, in general, the Taylor series method for an nth-order DE will employ each of the n initial conditions mentioned in Definition 3, Section 1.2?

Third problem. Initial Value Problem.

Definition 3. By an **initial value problem** for an nth-order equation

$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0$, we mean: find a solution to the DE on an interval

I that satisfies at x_0 the n initial conditions

$y(x_0) - y_0,$

$\frac{dy}{dx}(x_0) = y_1$

$\vdots$

$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$

where $x_0 \in I$ and $y_0, y_1, ..., y_{n-1}$ are given constants.

Third Problem.

$$\frac{d^2y}{dx^2} = xy,\ y(0) = 1,\ y'(0) = 0 \implies x_0 = 0,\ y_0 = 1$$

$$\Leftrightarrow y''(x) = xy \implies y''(x_0) = \mathbf{x_0} \cdot y(x_0)$$

$$\implies y'''(x_0) = y(x_0) + \mathbf{x_0} \cdot y'(x_0)$$

$$\implies y^{IV}(x_0) = 2y'(x_0) + \mathbf{x_0} \cdot y''(x_0)$$

$$\implies y^{V}(x_0) = 3y''(x_0) + \mathbf{x_0} \cdot y'''(x_0)$$

$$\implies y^{VI}(x_0) = 4y'''(x_0) + \mathbf{x_0} \cdot y^{IV}(x_0)$$

$$y''(x_0) = 0,\ y'''(x_0) = 1,\ y^{IV}(x_0) = 0,\ y^{v}(x_0) = 0,\ y^{VI}(x_0) = 0$$

$$P_6(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{4!}x^4 + \frac{y^{V}(0)}{5!}x^5 + \frac{y^{VI}(0)}{6!}x^6$$

$$P_6(x) = 1 + 0 + 0 + \tfrac{1}{6}x^3 + 0 + 0 + 0$$

$$\boxed{\mathbf{P_6(x) = 1 + \tfrac{1}{6}x^3}}$$

Third Problem.

We can observe that the computation of every derivative of y becomes a recursive process, with the formula defined as follows:

$$y^{(n)}(x_0) = (n-2) + x_0 \cdot y^{(n-2)}(x_0)$$

Based on Definition 3 of an initial value problem and on the stated exercise, we can agree that in order to make use of the **Taylor Series Method** for an nth-order DE, we compute each nth-order derivative of the function y.

Third Problem.

Hence, we observe that the computation of each nth-order derivative demands the use of the initial condition $x_0 = 0$, but also lower degree derivatives based on the initial conditions of $y_0 = 1$ and $y'(0) = 0$.

We can conclude that the **Taylor Series Method** for an nth-order DE will employ each of the n initial conditions mentioned in *Definition 3, Section 1.2*.

References

[1] Software used for computations: **eMathHelp**

1.1. Euler for step size 0.1 and $x = 1$: Link here

1.2 Euler for step size 0.1 and $x = 3$: Link here

1.3 Taylor for step size 0.01 and $x = 1$: Link here

1.4 Taylor for step size 0.01 and $x = 3$: Link here

Any questions?

Thank You!