The Origins of l'Hospital's Rule

Francesca Drăguț, Alexandru Fanu, Alexandru Petreuș,

Department of Computer Science,
West University,
Timişoara, Romania,

Email: francesca.dragut01@e-uvt.ro
alexandru.fanu02@e-uvt.ro
alexandru.petreus02@e-uvt.ro

May 2021

1 History

The area of Mathematics re-flourished in Europe beginning with the Renaissance and continued to evolve and develop ever since. Centuries 17 and 18 have seen some of the brightest minds in this domain. Sir Isaac Newton, Gottfried Wilhelm Leibniz, René Descartes, Blaise Pascal, the Bernoulli family - Johann, Jakob, Daniel -, Marquis de l'Hospital, Brook Taylor, Pierre de Fermat, Christian Huygens were all contemporary and most of them kept correspondance with other scholars.

This correspondence created a competitive academic envirnoment. For example, contests were announced, like the one that Jakob Bernoulli proposed, regarding the equation of the catenary, which transformed Johann Bernouli into a renowned Mathematician (see [1], [2]). Johann Bernoulli himself kept correspondence with over 100 scholars [3].

This global movement in pursuit of knowledge contributed a great deal to the formulas, lemmas, definitions and models that we now learn in school and base out Mathematical knowledge upon.

One of the most interesting encounters in the recent history of Mathematics is the one between Johann Bernoulli and Marquis de l'Hospital. Their interaction produced the famous "L'Hospital Rule", which is a one of the most fundamental and used formulas in Calculus, due to its wide applicability.

Our goal is to present the origin of this rule, in a historical and mathematical perspective and to show how a simple rule that we use so often, became to be what it is. We will see it in the context of its discoverers, Johann Bernoulli and Marquis de l'Hospital, and their own discoveries. We will show how Bernoulli's and l'Hospital's rules are basically the same and how can that be deduced.

2 Johann Bernoulli and Marquis de l'Hospital

Johann Bernoulli I was born in Basel, Switzerland, in 1667 and died in the same place in 1748. Since he was not suited to become a business man, his family granted him permission to study in the university where his older brother, Jakob Bernoulli, lectured. Johann started the study of medicine, but even his medical thesis was a mathematical work.

Later, Johann's brother, Jakob, started teaching him Mathematics. They were among the first ones to understand the new discoveries of Leibniz in the area of differential calculus. As a diligent follower of Leibniz' ideas, Johann traveled to Paris, where he was a representative of Leibniz' findings in calculus.

During his stay in Paris, he met Marquis de l'Hospital, who was then an amateur mathematician. Guillaume François Antoine, Marquis de l'Hospital, was born in 1661 in Paris, France, where he died in 1704. Needless to say, based upon his noble title, he came from a wealthy, noble family. From a young age, he stood out with his remarkable mathematical skills. He managed to solve a very difficult cycloid problem, proposed by Pascal. Later, he solved another difficult problem, raised by Johann Bernoulli, the problem of the brachistochrone. Both cycloids and the brachistochrone are special types of curves. Unsurprisingly, he later published his most famous book, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes (Analysis of the Infinitely Small to Understand Curved Lines)*, in 1696. But in order to see how he managed to publish such a powerful book in Mathematics, we will go back to Johann Bernoulli's story in Paris.

When Johann Bernoulli and Marquis de l'Hospital met, Bernoulli was only a young, poor mathematician, descending from a family in which he was the tenth child. Meanwhile, l'Hospital was an older, wealthier man, whose attempts in the army were unsuccessful, due to eyesight issues and who focused all of his ambitions on his passion, Mathematics.

When they met, l'Hospital asked Bernoulli to tutor him and guide him through the new findings of Leibniz. Of course, this service was not free of charge, but generously paid by the Marquis. Bernoulli tutored him between 1691 and 1692, when he retured home, in Switzerland. In 1965, Bernoulli received a chair at the University of Groeningen, but throughout this period, he kept corresponding with l'Hospital [1]. In fact, Bernoulli kept many correspondences and throughout his life, he had had such correspondence with more than 100 scholars and academic figures.

In 1696, l'Hospital published the book that made him famous, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes (Analysis of the Infinitely Small to Understand Curved Lines)*. At the end of the book, he acknowledged that his findings were based upon Leibniz' and Bernoulli's work and if they found their own work in the book, they should claim it. While Bernoulli praised l'Hospital's book, in his private letters, he claimed that much of the content of the book was his own, especially the rule of 0/0, which was later named "L'Hospital's Rule" [1].

The correspondence between the two mathematicians later proved to be more than informal and relaxed. In 1922, it was proved what Bernoulli stated in 1704, after l'Hospital's death, that much of the book, including the rule, was, in fact, his. In 1922, Bernoulli's manuscript, which he had written between 1691 and 1692, was published and a lot of similarities were found with l'Hospital's book. Later, in 1955, Bernoulli's correspondence was published. There, correspondence with Marquis de l'Hospital was found. L'Hospital asked Bernoulli not only to continue tutoring him from distance, but he asked Bernoulli to work on problems sent by him and to make all his discoveries known only to l'Hospital and nobody else. Because of financial problems, Bernoulli accepted and it seems that what we know today to be l'Hospital's rule, might more probably than not, be actually Bernoulli's rule [1].

(For this section, the main reference was [3], but certain paragraphs had other sources. Where other sources are referenced, they refer to the whole idea or paragraph.)

3 The Rule

L'Hospital's Rule, as we know it today, states that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. These conditions refer to both the numerator and the denominator approaching to either 0 or $\frac{\pm}{\infty}$, thus having an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Suppose f and g are differentiable and $g'(x)0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \; and \; \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \infty \; and \; \lim_{x \to a} g(x) = \infty$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right side exists (or is $\pm\infty$) [4].

Interestingly enough, L'Hospital and Bernoulli didn't even use the concept of limits when describing the rule. Instead, they formulated it geometrically and gave the answer in terms of differentials, which L'Hospital fundamentally defined as "The infinitely small part by which a variable quantity increases or decreases continually is called the differential of that quantity" in his famous calculus textbook. *Analyse des Infiniment Petits* also contains two important postulates on which L'Hospital's Rule is based:

1. "Grant that two quantities, whose difference is an infinitely small quantity, may be taken (or used) indifferently for each other; or (which is the same thing) that a quantity which is increased or decreased only by an infinitely small quantity may be considered as remaining the same." [5]
2. "Grant that a curve may be considered as the assemblage of an infinite number of infinitely small straight lines; or (which is the same thing) as a polygon of an infinite number of sides, each infinitely small, which determine the curvature of the curve by the angles they make with each other." [5]

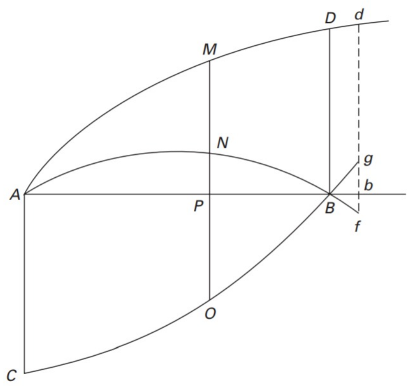

Figure 1: Graphical representation of l'Hospital's Rule, from [6]

In Figure 1, we see an illustration of L'Hospital's diagram that perfectly shows how L'Hospital's Rule works. Let AMD, ANB and COB and P a point on AB, such that $PM = \frac{(AB \cdot PN)}{PO}$. When the point P falls in B, PN and PO would get the value 0. If an ordinate bd is imagined infinitely near to BD, cutting the curves ANB and COB in the points f and g respectively, then $bd = \frac{(AB \cdot bg)}{bf}$. Because we expressed bd as a line that has the abscissa approximately the same as BD, then the value of bd would approach to the value of BD. When $B = P$, $PN = PO = 0$ and when $P = b$, PN and PO become bg and bf. bf is the differential of the ordinate B with regard to the curve ANB and bg is the differential of the ordinate b with regard to the curve. Therefore, when ANB and COB transform from curves to lines with the ordinate 0 (the value of the two functions is 0), we can still compute the value of BD (which would be $\frac{0}{0}$) with the help of bg and bf, the derivatives of PN and PO in the points f and g.

In [4], we can see a visual representation of the L'Hospital's Rule and why it might be true.

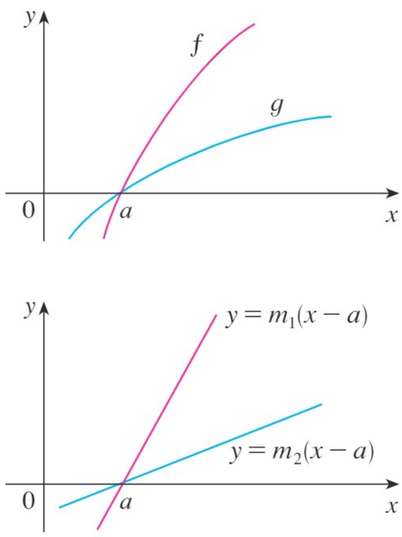

Figure 2: Visualization of l'Hospital's Rule, from [4]

In 2, we can see that the first graph shows two differentiable functions f and g, both approaching 0 as $x \to a$. If we looked closer towards the point $(a, 0)$, the graphs would start to look almost linear. Suppose the functions f and g were actually linear (as the second graph shows), then the ratio would be:

$$\frac{m_1 \cdot (x-a)}{m_2 \cdot (x-a)} = \frac{m_1}{m_2},$$

which is the quotient of their derivatives. This, of course, suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{(see [4])}$$

We can observe how L'Hospital's and Bernoulli's statement is essentially describing the rule in depth, demonstrating through geometrical representation that what their arguments are valid, while the course book explains us in a more relaxed way how the rule was created and its applicability in geometry and function graphs.

d x $\overline{\mathrm{d}y{=}x{-}2y;y(0){=}1{\Rightarrow}x_0{=}0,y_0{=}1;{\Leftrightarrow}y'{=}x{-}2y{\Rightarrow}y'(x_0){=}x_0{-}2y(x_0){\Rightarrow}y''(x_0){=}1{-}2y'(x_0){\Rightarrow}y'''(x_0){=}{-}2y''(x_0){\Rightarrow}y^{IV}{=}{-}2y'''(x_0)}$

4 Conclusions

In this report, we presented a historical background of the period when the rule was discovered. We also presented the encounters between some of the most remarkable figures in science. Among these encounters, one of the most interesting ones was the one between Marquis de l'Hospital and Johann Bernoulli. This encounter resulted in one of the most well known rules in Calculus.

We showed why it is considered that l'Hospital took Bernoulli's ideas in order to formulate his rule. It was all based upon what we would call today a non-disclosure agreement. Johann Bernoulli accepted a significant sum of money at exchange with his fame and public exposure. Only after l'Hospital's death, he tried to claim the rule as his own.

When claiming the rule, nobody believed Bernoulli. It was only later proved, in the 20th century, that there is an undeniable similarity between some of his unpublished work and l'Hospital's book. Later, some of Bernoulli's private correspondence was found, where he confessed about their agreement.

Nevertheless, time brought light over this issue. But what we know for sure is the importance of this rule in Calculus and in Mathematics, overall. The applicability of this rule in calculating limits made it famous and widely-used.

The rule can be proved by geometry, which we did in this report, using several resources.

References

[1] Howard Eves. *In Mathematical Circles (Volume 2: Quadrants III and IV)*. Boston: Prindle, Weber and Schmidt, 1969.

[2] Sepideh Alassi. Jacob Bernoulli and the Problem of Catenary. 2018.

[3] C. C. Gillispie. *Dictionary of Scientific Biography*, volume II and VIII (See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII). New York: Scribner's, 1974.

[4] James Stewart. *Calculus - Early Transcendentals (Eighth Edition)*. Cengage Learning, 2015.

[5] Victor Katz. *A History of Mathematics: An Introduction (3rd Edition)*. Pearson Education, 2009.

[6] Dirk Jan Struik. *A Sourcebook in Mathematics, 1200–1800*. Princeton, NJ: Princeton University, Press, 1969.