Earthquake Shaking of Multistory Buildings

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Introduction.

Buildings are generally severely damaged by large earthquakes. For instance, much of San Francisco was devastated during the infamous earthquake of 1906.

More recently, the Loma Prieta earthquake struck the region, which many people in the US and other countries witnessed firsthand while watching on television the 1989 Major League Baseball World Series game that was taking place in San Francisco.

In this project, we try to model how an earthquake would affect a multi-story building, to solve and interpret the mathematics behind this natural phenomenon.

The Loma Prieta Earthquake.

Figure: Building damaged by the Loma Prieta Earthquake.

Exercise 1.

1. Consider a three-story building with the same m and k values as in the first example. Write down the corresponding system of differential equations. What are the matrices **M**, **K**, and **A**? Find the eigenvalues for **A**. What range of frequencies of an earthquake would place the building in danger of destruction?

Figure: Requirement for the first exercise.

Exercise 1.

Given: $m = 5000kg; \quad k_i = 10000kg/s^2$ 3-story building

$$\frac{d^2x_1}{dt^2} = -4x_1 + 2x_2$$

$$5\cancel{000}\frac{d^2x_2}{dt^2} = -10\cancel{000}(x_2 - x_1) + 10\cancel{000}(x_3 - x_2) \quad \Rightarrow$$

$$\Rightarrow \frac{d^2x_2}{dt^2} \quad = \quad -2x_2 + 2x_1 + 2x_3 - 2x_2 \quad = \quad 2x_1 - 4x_2 + 2x_3$$

$$m_3\frac{d^2x_3}{dt^2} = -k(x_3 - x_2)$$

$$\Rightarrow 5\cancel{000}\frac{d^2x_3}{dt^2} = -10\cancel{000}x_3 + 10\cancel{000}x_2 \quad = \quad 2x_2 - 2x_3$$

Exercise 1.

$$\Rightarrow \begin{cases} \frac{d^2x_1}{dt^2} = -4x_1 + 2x_2 \\ \frac{d^2x_2}{dt^2} = 2x_1 - 4x_2 + 2x_3 \\ \frac{d^2x_3}{dt^2} = 2x_2 - 2x_3 \end{cases}$$

$$M = \begin{pmatrix} 5000 & 0 & 0 \\ 0 & 5000 & 0 \\ 1 & 0 & 5000 \end{pmatrix} \rightarrow \textbf{mass matrix}$$

$$\mathsf{K} = \begin{pmatrix} -20000 & 10000 & 0 \\ 10000 & -20000 & 10000 \\ 0 & 10000 & -10000 \end{pmatrix} \rightarrow \textbf{stiffness matrix}$$

Exercise 1.

$$M^{-1} = \begin{pmatrix} \frac{1}{5000} & 0 & 0 \\ 0 & \frac{1}{5000} & 0 \\ 0 & 0 & \frac{1}{5000} \end{pmatrix}$$

$$A = M^{-1} \cdot K = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \text{determine eigenvalues}$$

$$det(A - rI_3) = -r^3 - 10r^2 - 24r - 8 = 0 \quad \Rightarrow$$

Exercise 1.

$$\begin{cases} r_1 = -6.494 \Rightarrow \omega_1 = \sqrt{(6.494)} = 2.548; \quad \text{osc} = \frac{2\Pi}{\omega_1} = 2.431 \\ r_2 = -3.110 \Rightarrow \omega_2 = \sqrt{3.110} = 1.764; \quad \text{osc} = \frac{2\Pi}{\omega_2} = 3.561 \\ r_3 = -0.396 \Rightarrow \omega_3 = \sqrt{(0.396)} = 0.6295; \quad \text{osc} = \frac{2\Pi}{\omega_3} = 9.989 \end{cases}$$

Exercise 1.

The range of frequencies of an earthquake that would place a building in danger of destruction is determined by the natural frequency of vibration of the building. The closer the natural frequencies of the building are to the frequency of the earthquake, the most likely it is for the building to be damaged by the tectonic motion. Therefore, the range of frequency of an earthquake that would place the building in danger of destruction is between 0.629 and 2.548 Hz.

Exercise 2.

2. Consider a three-story building with the same m and k values as in the second example. Write down the corresponding system of differential equations. What are the matrices $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{A}$? Find the eigenvalues for $\mathbf{A}$. What range of frequencies of an earthquake would place the building in danger of destruction?

Figure: Requirement for the second exercise.

Exercise 2.

Given: $m = 10000kg; \quad k_i = 5000kg/s^2$ 3-story building

$$m_1 \frac{d^2x_1}{dt^2} = -k_0 x_1 + k_1(x_2 - x_1) \;=\; -\frac{\cancelto{1}{5000}}{\cancelto{2}{10000}}x_1 + \frac{\cancelto{1}{5000}}{\cancelto{2}{10000}}(x_2 - x_1) =$$

$$= \; -\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1 \;\Rightarrow\; \frac{d^2x_1}{dt^2} = -x_1 + \tfrac{1}{2}x_2$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_1(x_2 - x_1) + k_2(x_3 - x_2) \;=\; -\tfrac{1}{2}x_2 + \tfrac{1}{2}x_1 + \tfrac{1}{2}x_3 - \tfrac{1}{2}x_2 \Rightarrow$$

$$\Rightarrow \frac{d^2x_2}{dt^2} \;=\; \tfrac{1}{2}x_1 - x_2 + \tfrac{1}{2}x_3$$

$$m_3 \frac{d^2x_3}{dt^2} = -k_2(x_3 - x_2) \;\Rightarrow\; \frac{d^2x_3}{dt^2} \;=\; \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3$$

Exercise 2.

$$M = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 1 & 0 & 10000 \end{pmatrix} \rightarrow \textbf{mass matrix}$$

$$\mathsf{K} = \begin{pmatrix} -10000 & 5000 & 0 \\ 5000 & -10000 & 5000 \\ 0 & 5000 & -5000 \end{pmatrix} \rightarrow \textbf{stiffness matrix}$$

Exercise 2.

$$M^{-1} = \begin{pmatrix} \frac{1}{10000} & 0 & 0 \\ 0 & \frac{1}{10000} & 0 \\ 0 & 0 & \frac{1}{10000} \end{pmatrix}$$

$$A = M^{-1} \cdot K = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \rightarrow \text{determine eigenvalues}$$

$$det(A - rI_3) = \mathrm{r}^3 - \frac{5}{2}r^2 - \frac{3}{2}r - \frac{1}{8} = 0 \quad \Rightarrow$$

Exercise 2.

$$\begin{cases} r_1 = -1.623 \Rightarrow \omega_1 = \overset{\Rightarrow}{\sqrt{(1.623)}} = 1.274; \quad \text{osc} = \frac{2\Pi}{\omega_1} = 4.932 \\ r_2 = -0.777 \Rightarrow \omega_2 = \sqrt{0.777} = 0.821; \quad \text{osc} = \frac{2\Pi}{\omega_2} = 7.129 \\ r_3 = -0.099 \Rightarrow \omega_3 = \sqrt{(0.099)} = 0.315; \quad \text{osc} = \frac{2\Pi}{\omega_3} = 19.970 \end{cases}$$

The range of frequency of an earthquake that would place the building in danger of destruction is between 0.315 and 1.274 Hz.

Exercise 3.

3. Consider the tallest building on your campus. Assume reasonable values for the mass of each floor and for the proportionality constants between floors. If you have trouble coming up with such values, use the ones in the example problems. Find the matrices **M**, **K**, and **A**, and find the eigenvalues of **A** and the frequencies and periods of oscillation. Is your building safe from a modest-sized period-2 earthquake? What if you multiplied the matrix **K** by 10 (that is, made the building stiffer)? What would you have to multiply the matrix **K** by in order to put your building in the danger zone?

Figure: Requirement for the third exercise.

Exercise 3.

The tallest building on our campus has 7 floors, with each floor weighing about $10000kg$. Each restoring force is constant, having a value of $k = 5000kg/s^2$. Then, the differential equations are:

Exercise 3.

- $\frac{d^2x_1}{dt^2} = -x_1 + \frac{1}{2}x_2$
- $\frac{d^2x_2}{dt^2} = -\frac{1}{2}x_2 + \frac{1}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_2 \quad = \quad \frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3$
- $\frac{d^2x_3}{dt^2} = -\frac{1}{2}(x_3 - x_2) + \frac{1}{2}(x_4 - x_3) \quad = \quad \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4$
- $\frac{d^2x_4}{dt^2} = \frac{1}{2}x_3 - x_4 + \frac{1}{2}x_5$
- $\frac{d^2x_5}{dt^2} = \frac{1}{2}x_4 - x_5 + \frac{1}{2}x_6$
- $\frac{d^2x_6}{dt^2} = \frac{1}{2}x_5 - x_6 + \frac{1}{2}x_7$
- $\frac{d^2x_7}{dt^2} = -\frac{1}{2}(x_7 - x_6) \quad = \quad \frac{1}{2}x_6 - \frac{1}{2}x_7$

Exercise 3.

$$M = \begin{pmatrix} 10000 & 0 & 0 & \dots & 0 \\ 0 & 10000 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 10000 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} \frac{1}{10000} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{10000} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{10000} \end{pmatrix}$$

Exercise 3.

$$K = \begin{pmatrix} -10000 & 5000 & 0 & \dots & 0 & 0 & 0 \\ 5000 & -10000 & 5000 & \dots & 0 & 0 & 0 \\ 0 & 5000 & -10000 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 5000 & -10000 & 5000 \\ 0 & 0 & 0 & \dots & 0 & 5000 & -5000 \end{pmatrix}$$

$$A = M^{-1} \cdot K = \begin{pmatrix} -1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}$$

Exercise 3.

Using the calculator we obtain the following eigenvalues:

- $r_1 = -1 \implies \omega_1 = 1$ with $p = 6.283$
- $r_2 = -1.5 \implies \omega_2 = 1.22$ with $p = 5.15$
- $r_3 = -0.5 \implies \omega_3 = 0.707$ with $p = 8.887$
- $r_4 = -1.866 \implies \omega_4 = 1.366$ with $p = 4.6$
- $r_5 = -0.134 \implies \omega_5 = 0.366$ with $p = 17.167$
- $r_6 = -1.309 \implies \omega_6 = 1.144$ with $p = 5.492$
- $r_7 = -0.191 \implies \omega_7 = 0.437$ with $p = 14.378$

Exercise 3.

Our building seems to be safe from a modest-sized period 2 earthquake because none of the periods of oscillations of the floors are close to this value. If we make the building stiffer by multiplying the matrix K by 10, the 6th period would be 2.058 which would place the building in danger of destruction.

Conclusions.

In this project we have got to observe the influence of earthquakes on the urban world that surrrounds us. It was interesting to see how the number of floors, their mass and the materials which have been used play a significant role in the buildings' resistance to different types of earthquakes with various frequency ranges.

In the following video, we can see a summed-up conclusion: `https://www.youtube.com/watch?v=bIqV95gIf9o`

Any questions?

Thank You!