Numerical Methods for Partial Differential Equations
A.Y. 2025/2026

Laboratory 05
Finite Element method for non linear equations

Exercise 1.

Let $\Omega = (0,1)^3$ be the unit cube and let us consider the following non linear problem:

$$\begin{cases} -\boldsymbol{\nabla} \cdot \left(\left(\mu_0 + \mu_1 u^2 \right) \boldsymbol{\nabla} u \right) = f & \text{in } \Omega, \qquad (1a) \\ u = 0 & \text{on } \partial\Omega, \qquad (1b) \end{cases}$$

where $\mathbf{x} = (x, y, z)^T$, $\mu_0 = 1$, $\mu_1 = 10$ and $f(\mathbf{x}) = 1$.

1.1. Write the weak formulation of problem (1), expressing it in the residual form $R(u, v) = 0$.

1.2. Compute the Fréchet derivative $a(u)(\delta, v)$ of the residual $R(u)(v)$, then write Newton's method for the solution of problem (1).

1.3. Using Newton's method, implement a solver for problem (1). Then, solve the problem on the mesh `mesh/mesh-cube-20.msh`, with polynomial degree $r = 1$, and using a tolerance of 10^{-6} on the norm of the residual for the Newton's method.

Possibilities for extension

Automatic differentiation. Computing the Fréchet derivative might be difficult and error prone. There exist tools based on *automatic differentiation* (AD) that allow to only implement the computation of the residual (where no derivative is required), and then obtain the matrix through AD. `deal.II` offers wrappers to AD libraries from Trilinos that can be used to this purpose. Based on the `deal.II` tutorials, modify the code for Exercise 1 to use AD and compare the computational costs of the two strategies.