

# Ontological Logs, Higher Categories and a path to programmatic representation

Noah Chrein

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# Part 1: Intuitions and Definitions

- Ologs
- $\mathcal{Cat} \implies (\mathcal{Cat} \downarrow \mathcal{C}) \implies sm(\mathcal{C})$
- Colimits / Coverings
- Anisotropy / monoidal \$
- $\mathcal{O} / \oplus$
- Site / Ontological Generation

## Part 2: $sm : \mathcal{C}at \rightarrow \mathcal{C}at$ , 2-categories

- $\mathcal{C}at$  as a category,  $sm, Fun : \mathcal{C}at \rightarrow \mathcal{C}at$
- $\Delta : Id_{\mathcal{C}at} \rightarrow Fun$
- $colim : Fun(\mathcal{C}) \rightarrow \mathcal{C}$
- $\eta : Id \rightarrow \Delta \circ colim$  as initial in  $\mathcal{L}_{fun, \mathcal{C}}$

## Part 3: Higher Categories: Simplicial sets, Python Ologs

- $\mathcal{Cat}$  as a 2-category
- Simplicial Sets
- Functoriality as Naturality
- Pythonic Simplicial Sets
- Face and Degeneracy as ontological expansions

## Ontological Log

An **Ontological Log** is a labeled category.

- I.e. labeled objects and labeled morphisms
- An ontology represents concepts and their relations via category theory

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This is formalized by an **ontological expansion**

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## Submorphism

A **submorphism**  $\mathcal{F} : J \rightarrow J'$  is a set of maps between the images of  $J$  and  $J'$

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in this case, for an ontological expansion  $O : \mathfrak{C} \rightarrow sm(\mathfrak{D})$  we then have a covering  $\{\eta_s : O(c)(s) \rightarrow colim(O(c))\}$

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If we make the further assumption that:

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This assertion is the first condition of an **Ontological Generator**

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## Goal

To organize different expansions and how they relate

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- We also want to relate two ontological expansions

So our proto-definition of an ontological generator is now a functor

$$OG : \$ \rightarrow sm(\mathcal{C})^{\mathcal{C}}$$



# Operations on Ontological Expansions: 9

- recall that  $sm : \mathcal{C}at \rightarrow \mathcal{C}at$  is a functor
- given  $O : \mathcal{C} \rightarrow sm(\mathcal{C})$ ,  $sm(O) : sm(\mathcal{C}) \rightarrow sm(sm(\mathcal{C}))$

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This composition yields the **spiral product**

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- Finally collect ontologies into one large on that "looks like"

$$\bigcup_{c' \in O(c)} O'(c')$$

(careful, in general the spiral product isn't just an ordinary union)

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## intuition

We want to actually measure data about the objects of  $\mathcal{C}$ . The site assumption allows us to formalize data in terms of sheaves

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Moreover given an ontological expansion  $OG_s(c)$ , we can use the sheaf condition to "glue together" data from  $c' \in OG_s(c)$  to  $c$ -data

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## Part 2: $sm : \mathcal{C}at \rightarrow \mathcal{C}at$ and colim

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- $\eta : Id \rightarrow \Delta \circ colim$  is initial in  $\mathcal{L}_{fun, \mathcal{C}}$
- Conjecture:  $\eta : Id \rightarrow \Delta \circ colim$  is initial in  $\mathcal{L}_{sm, \mathcal{C}}$

$$(\mathcal{C}at \downarrow \mathcal{C})$$

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$P$  is a functor and  $\phi$  a natural transformation as in the diagram:

$$\begin{array}{ccc} S & \xrightarrow{P} & S' \\ & \searrow \scriptstyle J \quad \swarrow \scriptstyle J' & \\ & \mathcal{C} & \end{array} \quad \begin{array}{c} \phi \\ \Rightarrow \end{array}$$

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That is,  $fun(F)$  is a functor:

- for  $J : S \rightarrow \mathcal{C}$ ,  $fun(F)(J) = F \circ J : S \rightarrow \mathcal{D}$
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$$fun(F)\left( \begin{array}{ccc} S & \xrightarrow{P} & S' \\ & \searrow \phi \nearrow & \\ J & & J' \end{array} \right) = \begin{array}{ccc} S & \xrightarrow{P} & S' \\ & \searrow F(\phi) \nearrow & \\ F \circ J & & F \circ J' \end{array}$$

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Consider the identity functor  $Id_{\mathcal{C}at}$ . Both  $fun$  and  $Id$  are endofunctors of the category  $\mathcal{C}at$

- $\Delta$  is a natural transformation  $\Delta : Id_{\mathcal{C}at} \rightarrow fun$ .

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If  $\mathfrak{C}$  is cocomplete, colimit becomes a functor  $\text{colim} : \text{fun}(\mathfrak{C}) \rightarrow \mathfrak{C}$

The canonical morphisms give us a natural  $\eta : \text{Id} \rightarrow \Delta \circ \text{colim}$

let  $\mathcal{L}_{fun, \mathfrak{C}}$  be the category such that:

- objects are pairs  $(L, l : Id_{fun(\mathfrak{C})} \rightarrow \Delta_C \circ L)$
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$$\begin{array}{ccc} & Id_{fun(\mathfrak{C})} & \\ I \swarrow & & \searrow I' \\ \Delta(L) & \xrightarrow{\Delta(\phi)} & \Delta(L') \end{array}$$

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Let's step back: for a small functor  $J : S \rightarrow \mathfrak{C}$ ,  
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If  $\mathfrak{C}$  is cocomplete, the universal property of the colimit is exactly the statement that, for all  $L, J$ , there is a unique  $\phi_J : colim(J) \rightarrow L(J)$  such that

$$\begin{array}{ccc} & J(s) & \\ \eta_{J,s} \swarrow & & \searrow l_{J,s} \\ colim(J) & \xrightarrow{\Delta(\phi)} & L(J) \end{array}$$

that is, colim is initial in the category  $\mathcal{L}_{fun, \mathfrak{C}}$

Want: initial object in  $\mathcal{L}_{sm, \mathcal{C}}$

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that is,  $i$  forgets that  $\phi$  is a natural transformation, and instead regards it just as a collection of maps  $\phi_s : J(s) \rightarrow J'(P(s))$ , i.e. a submorphism.

Conjecture:  $i^*$  is faithful and essentially surjective

Lift  $i$  to a functor  $i^* : \mathcal{C}^{sm}(\mathcal{C}) \rightarrow \mathcal{C}^{fun}(\mathcal{C})$ :

$$i^*(L)(J) = L \circ i(J)$$

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if this is true then  $\mathcal{L}_{sm,\mathfrak{C}}$  has an initial object, which will be the colimit  $sm(\mathfrak{C}) \rightarrow \mathfrak{C}$  we are looking for, but this is still left to prove.

## Part 3: Towards higher ontologies; programmatic representations

- $\mathcal{C}at$  and the 2-categorical secret
- simplicial sets
- higher ontologies
- faces and degeneracies as ontological expansions
- programming higher ontologies

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2-morphisms are commutative diagrams:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{P} & \mathcal{D} \\ \downarrow F & \nearrow \eta & \downarrow F' \\ \mathcal{C} & \xrightarrow{Q} & \mathcal{D}' \end{array}$$

where  $\eta : Q \circ F \rightarrow F' \circ P$  is a natural transformation

# Simplexes in $\mathcal{C}at$

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Can actually be described entirely by a "2-simplex":

$$\begin{array}{ccc} \mathcal{C} & & \\ F \downarrow & \nearrow \eta & \searrow F' \circ P \\ \mathcal{C}' & \xrightarrow{Q} & \mathcal{D}' \end{array}$$

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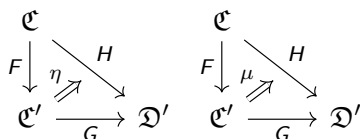
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- functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  as 1-simplicies
- commutative triangles up to natural transformation as 2-simplicies

Just as two 0-simplicies(categories) can have multiple 1-simplicies(functors) between them  
3 functors (1-simplicies) can have multiple natural transformations (2-simplicies) between them





# face and degeneracy

given a 2-simplex  $\sigma^2 =$

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The diagram shows a 2-simplex  $\sigma^2$  with vertices  $\mathfrak{C}$ ,  $\mathfrak{C}'$ , and  $\mathfrak{D}'$ . The edges are labeled  $F$  (from  $\mathfrak{C}$  to  $\mathfrak{C}'$ ),  $H$  (from  $\mathfrak{C}$  to  $\mathfrak{D}'$ ), and  $G$  (from  $\mathfrak{C}'$  to  $\mathfrak{D}'$ ). A double arrow labeled  $\eta$  points from  $\mathfrak{C}$  to  $\mathfrak{C}'$ .

we can extract 3 bits of information, namely the faces:  
 $face_0(\sigma^2) = H$ ,  $face_1(\sigma^2) = F$ ,  $face_2(\sigma^2) = G$

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if  $\Sigma_n$  is the set of  $n$ -simplexes, the faces and degeneracies are, in general, functions:

$$\Sigma_0 \begin{array}{c} \xrightarrow{\{d_0^0\}} \\ \xleftarrow{\{f_i^1\}_{i=0,1}} \end{array} \Sigma_1 \begin{array}{c} \xrightarrow{\{d_i^1\}_{i=1,2}} \\ \xleftarrow{\{f_i^2\}_{i=0,1,2}} \end{array} \Sigma_2$$

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of course, in more general situations, this chain continues as:

$$\Sigma_{n-1} \begin{array}{c} \xleftarrow{\{f_i^n\}_{0 \leq i \leq n}} \\ \xrightarrow{\{d_i^n\}_{0 \leq i \leq n}} \end{array} \Sigma_n \xrightarrow{\{d_i^n\}_{0 \leq i \leq n}} \Sigma_{n+1}$$

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A morphism of simplicial sets, is then just a natural transformation  $\eta : \Sigma \rightarrow \Sigma'$

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i.e. a natural transformation between 1-categories (as simplicial sets) is actually just a functor.



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- the naive definition of a higher ontology, then, should be a simplicial set

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- Ontologies represent concepts and relations between them
- Simplicial Sets allow us to stage morphisms as concepts and use 2-simplexes to find relations between morphisms, (or relations between relations).
- the naive definition of a higher ontology, then, should be a simplicial set

The theory of Simplicial Sets is pretty well developed (almost as well as category theory) and so  $sSet$  gives a great starting point for capturing the intuition behind what a higher ontology should be

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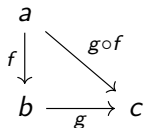
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A simplicial set is then just a collection of simplex objects containing its degeneracies and faces



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you can view my working code at  
<https://github.com/nopouch/golog>

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The functoriality here isn't apparent, and may necessitate working with something close to but not exactly simplicial sets.

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