École Polytechnique Fédérale de Lausanne

Primal and mixed formulation of the linear elasticity problem

MATH-468: Numerics for Fluids, Structures and Electromagnetics

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15th January 2023

1 THE LINEAR ELASTICITY PROBLEM

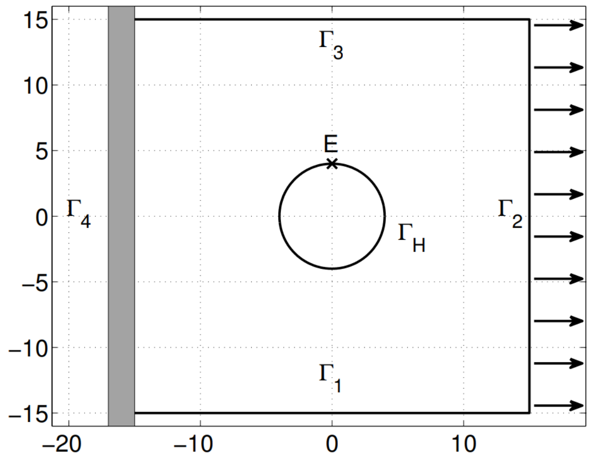

FIGURE 1
Plate with a hole

We consider a 30×30 plate with a central hole of radius 4 as in Figure 1. The plate is assumed elastic and weightless, its left end is clamped to a rigid wall, its right end is pulled by a constant force $\mathbf{f}$ and no force is present on the rest of the boundary. Our goal is to compute the displacement of the plate from its rest position; to this end, we solve the so-called Lamé equation:

$$\begin{cases} -\operatorname{div}\sigma(\mathbf{u}) = \mathbf{0} & \text{in } \Omega = (-15,15)\times(-15,15), \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_4, \\ \sigma(\mathbf{u})\cdot\mathbf{n} = \mathbf{0} & \text{on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_H, \\ \sigma(\mathbf{u})\cdot\mathbf{n} = \mathbf{f} & \text{on } \Gamma_2, \end{cases}$$

where $\mathbf{u} = (u_1, u_2)$ is the displacement vector, $\sigma(\mathbf{u})$ is the Cauchy stress tensor

$$\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda\,\mathrm{Tr}[\epsilon(\mathbf{u})]\mathrm{I},$$

I is the identity tensor, $\epsilon(\mathbf{u})$ is the strain tensor

$$\epsilon(\mathbf{u}) = sym(\nabla\mathbf{u}) = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2},$$

and $\mathrm{Tr}[A]$ denotes the trace of the tensor A, e.g. $\mathrm{Tr}[A] = A_{1,1} + A_{2,2}$ for $A \in \mathbb{R}^{2\times 2}$. Note that $\mathrm{Tr}[\epsilon(\mathbf{u})] = \operatorname{div}(\mathbf{u})$.

2 PRIMAL FORMULATION

2.1 THE WEAK PROBLEM

Fist of all it is worth noticing that, thanks to the definition of the Cauchy stresses tensor $\sigma(\mathbf{u})$, the left hand side of the Lamé equation can be written as follows:

$$\operatorname{div}(\sigma(\mathbf{u})) = 2\mu\operatorname{div}(\epsilon(\mathbf{u})) + \lambda\nabla(\operatorname{div}(\mathbf{u})) \tag{1}$$

The weak formulation of the problem is thus obtained by multiplying the Lamé equation by a generic test function $\mathbf{v} \in \mathbf{V}$, where $\mathbf{V}$ is the 2D Sobolev space where the problem is set:

$$-\int_\Omega [2\mu\operatorname{div}(\epsilon(\mathbf{u}) + \lambda\nabla(\operatorname{div}(\mathbf{u}))]\cdot\mathbf{v} = 0 \quad \forall\mathbf{v}\in\mathbf{V} \tag{2}$$

then proceeding with integration by parts:

$$\int_\Omega 2\mu\epsilon(\mathbf{u}) : \nabla\mathbf{v} + \lambda\operatorname{div}(\mathbf{u})\operatorname{div}(\mathbf{v}) - \int_{\partial\Omega}[2\mu(\epsilon(\mathbf{u})\cdot\mathbf{n})\cdot\mathbf{v} + \lambda\operatorname{div}(\mathbf{u})\mathbf{n}\cdot\mathbf{v}] = 0 \quad \forall\mathbf{v}\in\mathbf{V} \tag{3}$$

Moreover, one notices that:

$$\epsilon(\mathbf{u}) = sym(\nabla\mathbf{u}) \implies \epsilon(\mathbf{u}) : \nabla\mathbf{v} = \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \tag{4}$$

In addition, we can consider the boundary conditions, so that the right hand side of the equation is uniquely defined on the non-homogeneous Neumann boundary Γ_2, indeed:

$$2\mu(\epsilon(\mathbf{u})\cdot\mathbf{n})\cdot\mathbf{v}+\lambda\,\mathrm{div}(\mathbf{u})\mathbf{n}\cdot\mathbf{v}=(\sigma(\mathbf{u})\cdot\mathbf{n})\cdot\mathbf{v}=\begin{cases}\mathbf{0}, & \text{on } \Gamma_1,\Gamma_3,\Gamma_4,\Gamma_H\\ \mathbf{f}, & \text{on } \Gamma_2\end{cases} \tag{5}$$

Finally, the primal weak formulation of the problem becomes, in its most general form:

$$\text{Find } \mathbf{u}\in\mathbf{V} \quad : \quad a(\mathbf{u},\mathbf{v})=F(\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{V}, \tag{6}$$

$$\text{where} \quad a(\mathbf{u},\mathbf{v})=\int_\Omega 2\mu\epsilon(\mathbf{u}):\nabla\mathbf{v}+\lambda\,\mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v}) \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V} \quad \text{and} \quad F(\mathbf{v})=\int_{\Gamma_2} f\mathbf{v} \quad \forall\mathbf{v}\in\mathbf{V}$$

The choice of the space $\mathbf{v}$ is straightforward, due to the homogeneous Dirichlet boundary condition on Γ_4: $\mathbf{v}=[H^1_{\Gamma_4}(\Omega)]^2$.

2.2 Well Posedness and Stability Estimate

The analysis of the well posedness of (6) completely relies on the Lax-Milgram Lemma, whose statement is:

Lemma 2.1 (Lax-Milgram) *Let $\mathbf{V}$ be a Hibert space, $F:\mathbf{V}\to\mathbb{R}$ a linear and continuous operator on $\mathbf{V}$ (i.e. $F\in\mathbf{V}'$) and $a:\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ a bilinear form.*
If:
- *$a(\cdot,\cdot)$ is continuous on $\mathbf{V}\times\mathbf{V}$, i.e. $\exists M>0:|a(\mathbf{u},\mathbf{v})|\le M\|\mathbf{u}\|_V\|\mathbf{v}\|_V \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V}$*
- *$a(\cdot,\cdot)$ is coercive over $\mathbf{V}\times\mathbf{V}$, i.e. $\exists\alpha>0:a(\mathbf{v},\mathbf{v})\ge\alpha\|\mathbf{v}\|_V^2 \quad \forall\mathbf{v}\in\mathbf{V}$*

Then (6) is well-posed, i.e. $\exists\,!\ \mathbf{u}\in\mathbf{V}$ solution. Moreover: $\|\mathbf{u}\|\le C\|\mathbf{f}\|$.

To ensure the well-posedness of (6), we have to check that the hypothesis of Lax-Milgram Lemma are fulfilled:

- $\mathbf{v}$ is a closed subspace of $[H^1(\Omega)]^2$, hence it is a Hilbert space. For this reason $\|\cdot\|_{\mathbf{V}}=\|\cdot\|_1$
- $a(\mathbf{u},\mathbf{v}):\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is a bilinear form (trivial)
- $a(\mathbf{u},\mathbf{v}):\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is continuous over $\mathbf{V}\times\mathbf{V}$, indeed

$$\begin{aligned}|a(\mathbf{u},\mathbf{v})|&=\left|\int_\Omega 2\mu\epsilon(\mathbf{u}):\nabla\mathbf{v}+\lambda\int_\Omega \mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v})\right|\le 2\mu\|\mathbf{u}\|_1\|\mathbf{v}\|_1+\lambda\|\mathbf{u}\|_1\|\mathbf{v}\|_1\\&\le 2\max\{2\mu,\lambda\}\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V}\end{aligned}$$

  As a consequence of the above computations, we deduce that the continuity constant is $M=2\max\{2\mu,\lambda\}$.
- $a(\mathbf{u},\mathbf{v}):\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is coercive over $\mathbf{V}\times\mathbf{V}$, indeed

$$\begin{aligned}a(\mathbf{v},\mathbf{v})&=\int_\Omega 2\mu\,|\epsilon(\mathbf{v})|^2+\lambda\int_\Omega|\mathrm{div}(\mathbf{v})|^2\ge 2\mu\|\epsilon(\mathbf{v})\|_0^2\ge\\&\ge\frac{2\mu}{1+C_k^2}\left(\|\mathbf{v}\|_0^2+\|\epsilon(\mathbf{v})\|_0^2\right)=\frac{2\mu}{1+C_k^2}\|\mathbf{v}\|_1^2 \quad \forall\mathbf{v}\in\mathbf{V}\end{aligned}$$

  By exploiting the Korn inequality[1].Therefore, the coercivity constant is $\alpha=\frac{2\mu}{1+C_k^2}$.
- $F(\mathbf{v}):\mathbf{V}\to\mathbb{R}$ is linear and continuous over $\mathbf{V}$, indeed

$$|F(\mathbf{v})|=\left|\int_{\Gamma_2}\mathbf{f}\cdot\mathbf{v}\right|\le\|\mathbf{f}\|_{L^2(\Gamma_2)}\cdot\|\mathbf{v}\|_{L^2(\Gamma_2)}\le\|\mathbf{f}\|_{L^2(\Gamma_2)}\cdot C_{tr}\|\mathbf{v}\|_1 \quad \forall\mathbf{v}\in\mathbf{V}$$

  Exploiting Holder's inequality and the Trace inequality[2].

Since all the hypothesis of Lax-Milgram lemma are fulfilled, the thesis follows.
In particular, it is worth studying the aforementioned stability inequality with respect to the $[H^1(\Omega)]^2$ norm. The stability estimate is obtained exploiting the coercivity of the bilinear form, the continuity of the forcing term and the weak formulation of the problem itself, indeed:

$$\begin{aligned}&\alpha\|\mathbf{u}\|_1^2\le a(\mathbf{u},\mathbf{u})=F(\mathbf{u})\le\|\mathbf{f}\|_{L^2(\Gamma_2)}C_{tr}\|\mathbf{u}\|_1\\ \Longrightarrow\quad&\|\mathbf{u}\|_1\le\frac{C_{tr}}{\alpha}\|\mathbf{f}\|_{L^2(\Gamma_2)}=C_{tr}\frac{1+C_k^2}{2\mu}\|\mathbf{f}\|_{L^2(\Gamma_2)}\end{aligned} \tag{7}$$

From this formula, it emerges that the stability constant is $C=C_{tr}\frac{1+C_k^2}{2\mu}$ only depends on μ, while it does not on λ. For this reason we can conclude that, once the incompressibility limit is applied, i.e. $\lim_{\lambda\to\infty}C$, the stability of the problem is not affected.

[1] Korn inequality: if $\mathbf{u}\in[H^1(\Omega)]^d$ and $\gamma_D(\mathbf{u})=0$ on $\Gamma_D\in\partial\Omega \implies \exists C_k>0:\|\mathbf{u}\|_0\le C_k\|\epsilon(\mathbf{u})\|_0 \quad \forall\mathbf{u}\in[H^1(\Omega)]^d$

[2] Trace inequality (continuity of the trace operator): $\|\mathbf{u}\|_{L^2(\Gamma_2)}\le C_{tr}\|\mathbf{u}\|_1 \quad \forall\mathbf{u}\in[H^1(\Omega)]^d$

2.3 Convergence Estimates

Now, the problem needs to be discretized in order to be solved numerically. In particular, we choose piecewise quadratic finite elements to approximate the Sobolev space v:

$$\mathbf{v}_h = \{\mathbf{v}_h \in \mathbf{V} \cap C^0(\bar{\Omega}) : \mathbf{v}_h\big|_K \in \mathbb{P}^2(K) \quad \forall K \in \tau_h\}, \quad \mathbf{v}_h \subset \mathbf{v}, \quad \dim \mathbf{v}_h < \infty$$

Where the notation is the following:

- $C^0(\bar{\Omega})$: continuous functions over the closure of Ω
- τ_h: conforming triangulation of the domain Ω, K is an element of the triangulation
- $\mathbb{P}^2(K)$: space of piecewise quadriatic polynomials

Therefore, one can state the discretized weak formulation of the problem:

$$\text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ such that } a(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{8}$$

Once discretization happens, analyzing the orders of convergence is fundamental. Indeed convergence estimates allow to manage the trade-off between the accuracy of the numerical solution and the computational cost to generate it. As a preliminary step, we recall two fundamental notions:

- *Galerkin Orthogonality*:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}_h) = F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h \\ a(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h \end{cases} \implies a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{9}$$

- *Interpolation Error Estimate for piecewise polynomial finite elements of degree r*:

$$\|\varphi - \Pi^r \varphi\|_{H^m} \leq C h^{r+1-m} |\varphi|_{H^{r+1}} \quad \text{provided } \varphi \in \mathbf{V} \cap H^{r+1}(\Omega) \tag{10}$$

As a consequence, we now derive the convergence estimates in two different norms, highlighting the dependency on the mesh size h:

- $[H^1(\Omega)]^2$: To obtain the following convergence estimate, we first state the *Ceà Lemma*:

$$\begin{aligned} \alpha \|\mathbf{u} - \mathbf{u}_h\|_1^2 &\leq a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h + \mathbf{v}_h - \mathbf{u}_h) = \\ &= a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \leq \\ &\leq M \|\mathbf{u} - \mathbf{u}_h\|_1 \|\mathbf{u} - \mathbf{v}_h\|_1 \quad \forall \mathbf{v}_h \in \mathbf{V}_h \end{aligned}$$

$$\implies \|\mathbf{u} - \mathbf{u}_h\|_1 \leq \frac{M}{\alpha} \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1 \quad (\textit{Ceà Lemma}) \tag{11}$$

Secondly, we recall that the best approximation error (i.e. $\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1$) is by definition smaller than the interpolation error (whose polynomial degree r is 2). In addition, we exploit (11) and get:

$$\begin{aligned} &\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1 \leq \|\mathbf{u} - \Pi_h^2 \mathbf{u}\| \leq C h^2 |\mathbf{u}|_{H^3(\Omega)} \\ &\implies \|\mathbf{u} - \mathbf{u}_h\|_1 \leq \tilde{C} h^2 |\mathbf{u}|_{H^3(\Omega)} \\ &\text{where} \quad \tilde{C} = \frac{M}{\alpha} C = 2C \max\{2\mu, \lambda\} \frac{1 + C_k^2}{2\mu} \end{aligned} \tag{12}$$

Now, we can immediately observe that $\lim_{\lambda \to \infty} \tilde{C} = \infty$, which means that continuity fails (i.e. $\lim_{\lambda \to \infty} M = \infty$) and the problem falls out of Lax-Milgram theory.

- $[L^2(\Omega)]^2$: According to what normally happens in numerical analysis, one expects the L^2 rate of convergence to be one degree higher than the H^1 one: this is what we derive hereafter[3].
  To proceed with the convergence estimate it is necessary to take advantage of the Aubin-Nitsche Lemma, that we apply to the linear elasticity problem.

  Lemma 2.2 (Aubin-Nitsche) *Let* H *be a Hilbert space, with norm* $\|\cdot\|_H$ *and inner product* $(\cdot,\cdot)_H$*, such that* $\bar{\mathbf{V}} = H$ *and* $\mathbf{V} \hookrightarrow H$*. Then one has*

$$\|\mathbf{u} - \mathbf{u}_h\|_H \leq M \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} \left(\sup_{g \in H} \left\{ \frac{1}{\|g\|_H} \inf_{\varphi_h \in \mathbf{V}_h} \|\varphi_g - \varphi_h\|_{\mathbf{V}} \right\} \right) \tag{13}$$

where, $\forall g \in H$, $\varphi_{\mathrm{g}} \in V$ is the unique solution of the variational problem:

$$\forall v \in V, \quad a(v, \varphi_g) = (g, v). \tag{14}$$

[3]The following discussion is taken from Ciarlet, Philippe G (2002). The finite element method for elliptic problems. SIAM., Chapter3, pp 136-139.

In our case of study, $H = L^2$ and $\mathbf{V} = [H^1_{\Gamma_4}(\Omega)]^2$. Furthermore, (14) is a special case of the following variational problem:

$$\forall g \in \mathbf{V}' \quad \text{find } \varphi \in \mathbf{V} \quad : \quad a(v, \varphi) = g(v) \quad \forall v \in \mathbf{V} \tag{15}$$

Such a problem is called the *adjoint problem* of (6) and it clearly coincides with the original problem if the bilinear form is symmetric.

Proposition 2.3 *If the adjoint problem is sufficiently regular, that is:*

$$\forall g \in L^2(\Omega)\ \exists C > 0\ st\ \|\phi_g\|_0 \leq C|g|_0$$

Then there exists a constant C independent of the mesh size h such that:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq Ch^{r+1}|\mathbf{u}|_{H^{r+1}(\Omega)} \tag{16}$$

Proof. Consider the result of the Aubin-Nitsche lemma (15) and bound the term referring to the best approximation error of the adjoint problem, namely:

$$\inf_{\varphi_h \in \mathbf{V}_h} \|\varphi_g - \varphi_h\|_{H^1} \leq \|\varphi_g - \Pi^r \varphi_g\|_{H^1} \leq Ch|\varphi_g|_{H^2} \leq Ch\|\varphi_g\|_{H^2} \leq \tilde{C}h\|g\|_{L^2}$$

So, from the Aubin-Nitsche lemma, it turns out that:

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{L^2} &\leq M\|\mathbf{u} - \mathbf{u}_h\|_{H^1} \left(\sup_{g \in L^2} \left\{ \frac{1}{\|g\|_{L^2}} \inf_{\varphi_h \in \mathbf{V}_h} \|\varphi_g - \varphi_h\|_{H^1} \right\} \right) \leq \\
&\leq M\|\mathbf{u} - \mathbf{u}_h\|_{H^1} \left(\sup_{g \in L^2} \left\{ \frac{1}{\|g\|_{L^2}} \tilde{C}h\|g\|_{L^2} \right\} \right) = \\
&= M\tilde{C}h\|\mathbf{u} - \mathbf{u}_h\|_{H^1} \leq M\tilde{C}h\frac{M}{\alpha}Ch^r|u|_{H^{r+1}}
\end{aligned}$$

$$\implies \|\mathbf{u} - \mathbf{u}_h\|_{L^2} = \frac{M^2}{\alpha}Ch^{r+1}|u|_{H^{r+1}} \tag{17}$$

One immediately observes that the L^2 norm gain one order of convergence with respect to the H^1 norm, indeed:

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq \bar{C}h^3|\mathbf{u}|_{H^3(\Omega)} \quad \text{where} \quad \bar{C} = \frac{M^2}{\alpha}C = 4C\left(\max\{2\mu, \lambda\}\right)^2 \frac{1 + C_k^2}{2\mu} \tag{18}$$

From this estimate, we deduce that as one approaches the incompressibility limit, the problem escapes the Lax-Milgram theory, hence well-posedness fails. This is not surprising since the same trend occurs with the H^1 norm.

2.4 Numerical Simulation

To obtain a discretized solution of the problem, first of all we define the data of the problem (dimensions, Lamè parameters, traction, polynomial degree of the discretization). Each time the mesh is initialized, the geometry of the problem (Figure 1) is reproduced by constructing a rectangle from which a disk is cut out. We point out the following choices for the code.

- The space V is the space of vector valued Lagrange elements defined on the mesh, obtained as `V = fem.VectorFunctionSpace(msh,("CG", degree))`.
- Dirichlet boundary conditions on Γ_4 are enforced by locating the left edge of the domain and the correspondent degrees of freedom, and imposing zero displacement for them.
- The other boundary conditions are obtained by defining a constant vector function on the whole domain equal to **f** (representing the traction) which is null in the y direction. Thus, when computing the term `dot(T,v)*ds`, representing $\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}\, ds$, thanks to symmetry and the traction being parallel to Γ_3 and Γ_1, what we actually compute is $F(\mathbf{v}_h)$ as defined for the primal weak formulation (6).
- The bilinear form is computed as `a=inner(sigma(u),epsilon(v))*dx`.

Hence, iterating over several values of the parameter λ and discretizing the problem for different refinements, the simulation is carried out according to Algorithm 1.

Notice that since the exact solution is not available, we evaluate errors with respect to a reference solution computed on a finer mesh.

We are interested in studying the convergence of the following quantities

$$Q_1(\mathbf{u}) = \frac{1}{\Gamma_2} \int_{\Gamma_2} u_1 \qquad \text{and} \qquad Q_2(\mathbf{u}) = \mathbf{u}(x_E, y_E)$$

Algorithm 1 Numerical Simulations for the Primal Formulation

```
for all values of λ do
    Define functions to compute ϵ(u) and σ(u)
    Compute reference solution on fine mesh (done as below)
    Initialize mesh
    for number of refinements of the mesh do
        Define vector function space V on current mesh
        Define dirichlet boundary conditions and constant forcing term
        Initialize trial function u and test function v in V
        Define the bilinear form a(·,·) and right hand side F(·)
        Solve the linear problem using a direct solver, based on LU factorization, as a preconditioner
        Refine mesh by halving mesh size
    end for
end for
```

where u_1 is the horizontal displacement and $(x_E, y_E) = (0, 4)$. Q_1 is obtained at each iteration by extrating 500 interpolated values of u_1 on Γ_2 and computing the integral through the trapezoidal method. Q_2 is obtained by interpolating the solution at the given point. Both are evaluated for the reference solution, against which each value for the simulations of interest is compared, thus obtaining the convergence estimates.
From Figure 2 we can observe that for λ not too large, $|Q_1 - Q_{1,ref}|$ and $\|Q_2 - Q_{2,ref}\|_{L^2}$ converge with order between 2 and 3. Both quantities are evaluated on the boundary of the domain where the trace of the displacement $\gamma_D(\mathbf{u}_h) \in [H^{1/2}(\partial\Omega)]^2$. Hence we expect to lose half an order with respect to the order of convergence of $\mathbf{u}_h$ in $[L^2(\Omega)]^2$, which is 3 (see the theoretical estimate (18)). Unfortunately, we might observe slightly different orders since we only have a reference solution on a finer mesh instead of the exact solution to evaluate the errors against. Most importantly, however, for $\lambda \to \infty$ we observe the deterioration of the convergences. Indeed, this follows from the conclusions drawn for the constant in the convergence estimates and its dependence on λ.

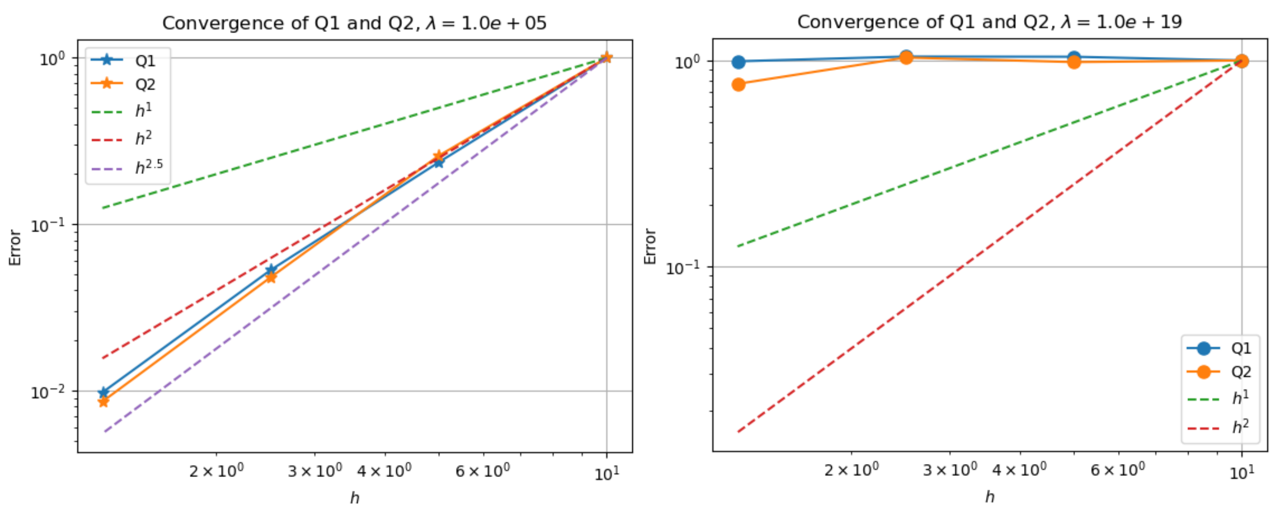

FIGURE 2
Convergence of Q_1 and Q_2 for $\lambda = 1e5, 1e19$

3 MIXED FORMULATION

We introduce the new variable $p = -\lambda \operatorname{div} \mathbf{u} \in L^2(\Omega)$. The main consequence of adding this new variable p is the appearance of a second equation within the weak formulation. In fact, to obtain the mixed problem it is also necessary to express in weak form the constraint introduced above, i.e. $p = -\lambda \operatorname{div} \mathbf{u}$ that relates the two variables $\mathbf{u}$ and p. Multiplying the equation by a generic test function $q \in L^2(\Omega)$ leads to:

$$\int_\Omega pq = \int_\Omega -\lambda \operatorname{div}(\mathbf{u})q \quad \forall q \in L^2(\Omega) \implies -\int_\Omega \operatorname{div}(\mathbf{u})q - \int_\Omega \frac{1}{\lambda} pq = 0 \quad \forall q \in L^2(\Omega) \tag{19}$$

As a consequence, the *mixed formulation* for the problem becomes:

$$\text{Find } (\mathbf{u}, p) \in (\mathbf{V}, Q) \text{ such that } \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = F(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) - c(p, q) = 0 & \forall q \in Q \end{cases} \tag{20}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega 2\mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \qquad b(\mathbf{v}, q) = -\int_\Omega q \operatorname{div} \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}, q \in Q$$

$$c(p, q) = \int_\Omega \frac{1}{\lambda} pq \quad \forall p, q \in Q, \qquad \mathbf{V} = [H^1_{\Gamma_4}(\Omega)]^2, \qquad Q = L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$$

Alternatively, we can introduce the global variable

$$U = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \in W = V \times Q$$

and obtain the corresponding global weak formulation (analogous to 20)

$$\text{Find } U \in W \text{ such that } \quad B(U, V) = F(V) \quad \forall V \in W \tag{21}$$

where

$$B(U, V) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) - c(p, q), \quad F(V) = F(\mathbf{v})$$

3.1 Well Posedness and Stability Estimate [4]

Let B be the linear and continuous operator associated with the bilinear form $b(\cdot, \cdot)$ and identify the following spaces

$$K := Ker(B),\ K : \mathbf{V} \to Q' \qquad K_T := Ker(B^T),\ K_T : Q \to \mathbf{V}'$$

So, to be explicit, we recall that $K = \{\mathbf{v} \in \mathbf{V} : b(\mathbf{v}, q) = 0 \quad \forall q \in Q\}$ and $K_T = \{q \in Q : b(\mathbf{v}, q) = 0 \quad \forall \mathbf{v} \in \mathbf{V}\}$. The well posedness of problem (20) is therefore stated by the following theorem:

Theorem 3.1 *Under the following assumptions*

- $\mathbf{V}$ *and* Q *Hilbert spaces*
- $a(\mathbf{u}, \mathbf{v}) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ *continuous on* $\mathbf{V}$*, coercive on* K*, symmetric and positive semi-definite*
- $b(\mathbf{v}, q) : \mathbf{V} \times Q \to \mathbb{R}$ *continuous on* $\mathbf{V} \times Q$ *and it satisfies the inf-sup condition*

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1 \|q\|_0} \geq \beta > 0$$

- $c(p, q) : Q \times Q \to \mathbb{R}$ *continuous on* Q*, coercive on* K_T*, symmetric and positive semi-definite*
- $F : \mathbf{V} \to \mathbb{R}$ *linear and bounded*

there exists a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *to problem (20), that moreover satisfies*

$$\|\mathbf{u}\|_1 + \|p\|_0 \leq C \|\boldsymbol{f}\|_{\mathbf{V}'}$$

In our case, the hypotheses are fulfilled. Indeed:

- Bilinear form $a(\cdot, \cdot)$
  - Continuity: $a(\cdot, \cdot)$ is continuous since $\exists M > 0 : |a(\mathbf{u}, \mathbf{v})| \leq M \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}$, indeed:

$$|a(\mathbf{u}, \mathbf{v})| = \left| \int_\Omega 2\mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \right| \leq 2\mu \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

  - Coercivity: $a(\cdot, \cdot)$ is coercive on K since $\exists \alpha > 0 : a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_1 \quad \forall \mathbf{v} \in \mathbf{V}$, indeed:

$$a(\mathbf{v}, \mathbf{v}) = \int_\Omega 2\mu \, |\epsilon(\mathbf{v})|^2 = 2\mu \|\epsilon(\mathbf{v})\|^2_0 \geq$$
$$\geq \frac{2\mu}{1 + C_k^2} (\|\mathbf{v}\|^2_0 + \|\epsilon(\mathbf{v})\|^2_0) = \frac{2\mu}{1 + C_k^2} \|\mathbf{v}\|^2_1 \quad \forall \mathbf{v} \in K$$

    by exploiting the Korn inequality
  - Symmetry: $a(\cdot, \cdot)$ is clearly symmetric

[4]This section follows the discussion by Boffi, Daniele, Franco Brezzi, Michel Fortin et al. (2013). Mixed finite element methods and applications. Vol. 44. Springer., Chapter 4.3

– Positive Semi-Definiteness: $a(\cdot,\cdot)$ is positive semi-definite since

$$a(\mathbf{v},\mathbf{v}) = \int_\Omega 2\mu\,|\epsilon(\mathbf{v})|^2 = 2\mu\|\epsilon(\mathbf{v})\|_0^2 \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}$$

- Bilinear form $b(\cdot,\cdot)$
  – Continuity: $b(\cdot,\cdot)$ is continuous since $\exists N > 0 : |b(\mathbf{v},q)| \leq N\|\mathbf{v}\|_1\|q\|_0 \quad \forall \mathbf{v} \in \mathbf{V}, \forall q \in Q$, indeed:

$$|b(\mathbf{v},q)| = \left|\int_\Omega q \operatorname{div}\mathbf{v}\right| \leq \|q\|_0\|\operatorname{div}\mathbf{v}\|_0 \leq \|q\|_0\|\mathbf{v}\|_1 \quad \forall \mathbf{v} \in \mathbf{V}, \forall q \in Q$$

  – Inf-Sup Condition: Let $B^T : Q \to V'$ s.t. $< B^T q, \mathbf{v} >= b(q,\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}$. To prove the inf-sup condition, we proceed in two steps:
    * step 1: We prove that B^T is injective, namely $q = 0 \iff < B^T q, \mathbf{v} >= 0 \quad \forall \mathbf{v}$
      Proof $\Rightarrow$: $< B^T q, \mathbf{v} >= b(q,\mathbf{v}) = -\int_\Omega q \operatorname{div}\mathbf{v} = 0 \quad \forall \mathbf{v} \implies q = 0$
      Proof $\Leftarrow$: $q = 0 \implies b(q,\mathbf{v}) = -\int_\Omega q \operatorname{div}\mathbf{v} =< B^T q, \mathbf{v} >= 0 \quad \forall \mathbf{v}$
    * step 2: We prove that B^T injective $\implies$ inf-sup
      Proof : Take $q \in Q$ s.t. $\|q\|_Q = 1$ and notice that by the definition of operator norm

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{< B^T q, \mathbf{v} >}{\|\mathbf{v}\|_1} =: \|B^T q\|_{\mathbf{V}'}$$

      Moreover, since $\|q\|_0 = 1$ defines a compact set and B^T is linear and continuous

$$\inf_{\|q\|_0=1} \|B^T q\|_{\mathbf{V}'} = \min_{\|q\_0|=1} \|B^T q\|_{\mathbf{V}'} =: \beta > 0$$

$$\implies \inf_{\|q\|_0=1} \|B^T q\|_{\mathbf{V}'} = \inf_{\|q\|_0=1} \sup_{\mathbf{v}\in\mathbf{V}} \frac{< B^T q, \mathbf{v} >}{\|\mathbf{v}\|_1} = \beta > 0$$

$$\implies \inf_{q\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{< B^T q, \mathbf{v} >}{\|\mathbf{v}\|_1\|q\|_0} = \beta > 0$$

- Bilinear form $c(\cdot,\cdot)$
  – Continuity: $c(\cdot,\cdot)$ is continuous since $\exists P > 0 : |c(p,q)| \leq P\|p\|_0\|q\|_0 \quad \forall p,q \in Q$, indeed:

$$|c(p,q)| = \left|\int_\Omega \frac{1}{\lambda} pq\right| \leq \frac{1}{\lambda}\|p\|_0\|q\|_0 \quad \forall p,q \in Q$$

  – Coercivity: $c(\cdot,\cdot)$ is coercive on K^T since $\exists \gamma > 0 : c(q,q) \geq \gamma\|q\|_0^2 \quad \forall q \in K_T$, indeed:

$$c(q,q) = \int_\Omega \frac{1}{\lambda}|q|^2 = \frac{1}{\lambda}\|q\|_0^2 \geq \frac{1}{\lambda+1}\|q\|_0^2 \quad \forall q \in K_T$$

  – Simmetry: $c(\cdot,\cdot)$ is clearly symmetric
  – Positive Semi-Definiteness: $c(\cdot,\cdot)$ is positive semi-definite since

$$c(q,q) = \int_\Omega \frac{1}{\lambda}|q|^2 = \frac{1}{\lambda}\|q\|_0^2 \geq 0 \quad \forall q \in Q$$

- $F(\cdot)$ is linear and bounded as already seen

As to the stability estimate we state the following lemma:
Lemma 3.2 *Under the assumptions of the previous theorem, it holds* $\|\mathbf{u}\|_1 + \|p\|_0 \leq C\|\boldsymbol{f}\|_{\mathbf{V}'}$*, where*

$$C = \frac{1}{\alpha} + \frac{1}{\delta}\left(1 + \frac{M}{\alpha}\right)\left[\frac{1}{\lambda\beta}\left(1 + \frac{M}{\alpha}\right) + 1\right] \text{ and } \delta \text{ s.t. } 0 < \delta \leq \beta - \frac{M^2}{\alpha\beta\lambda} - \frac{M}{\beta\lambda}$$

Proof. Let $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}^\perp$, with $\mathbf{u}_0 \in K$ and $\mathbf{u}^\perp \in K^\perp$ and estimate the two terms in the decomposition separately. For the first one, note that

$$\alpha\|\mathbf{u}_0\|_1^2 \leq a(\mathbf{u}_0,\mathbf{u}_0) = \langle f, \mathbf{u}_0\rangle - a(\mathbf{u}^\perp,\mathbf{u}_0) \leq \|f\|_{V'}\|\mathbf{u}_0\|_1 + M\|\mathbf{u}^\perp\|_1\|\mathbf{u}_0\|_1$$

$$\implies \|\mathbf{u}_0\|_1 \leq \frac{1}{\alpha}\left(\|f\|_{V'} + M\|\mathbf{u}^\perp\|_1\right) \tag{22}$$

For the second one, we get:

$$b(\mathbf{u}, q) = b\left(\mathbf{u}^{\perp}, q\right) = \frac{1}{\lambda}(p, q)_0 \leq \frac{1}{\lambda}\|p\|_0\|q\|_0 \quad \forall q \in Q.$$

$$\implies \beta\left\|\mathbf{u}^{\perp}\right\|_1 \leq \sup_{q\in Q}\frac{b\left(\mathbf{u}^{\perp}, q\right)}{\|q\|_0} \leq \frac{1}{\lambda}\|p\|_0 \implies \left\|\mathbf{u}^{\perp}\right\|_1 \leq \frac{1}{\lambda\beta}\|p\|_0 \tag{23}$$

At this point, we are only left to estimate the Q-norm of p. To this end, from the first equation, we then get

$$b(v,p) \leq \langle f, v\rangle + |a(\mathbf{u}, v)| \quad \forall v \in V \implies \sup_{v\in V}\frac{b(v,p)}{\|\mathbf{v}\|_1} \leq \sup_{v\in V}\left[\frac{1}{\|\mathbf{v}\|_1}\langle f, v\rangle + \frac{1}{\|\mathbf{v}\|_1}|a(\mathbf{u}, v)|\right]$$

Then, using the inf-sup condition, the continuity of f and the continuity of $a(\cdot,\cdot)$, we write

$$\beta\|p\|_0 \leq \sup_{v\in V}\frac{b(v,p)}{\|\mathbf{v}\|_1} \leq \|f\|_{V'} + M\|\mathbf{u}\|_1 \tag{24}$$

Combining (22), (23), (24) we have

$$\beta\|p\|_0 \leq \left(1 + \frac{1}{\alpha}\right)\|f\|_{V'} + \left(\frac{M^2\frac{1}{\lambda}}{\alpha\beta} + \frac{M\frac{1}{\lambda}}{\beta}\right)\|p\|_0,$$

and thus

$$\left(\beta - \frac{M^2\frac{1}{\lambda}}{\alpha\beta} - \frac{M}{\beta}\frac{1}{\lambda}\right)\|p\|_0 \leq \left(1 + \frac{M}{\alpha}\right)\|f\|_{V'}.$$

By noticing that $\frac{M}{\alpha} \geq 1$ and recalling that $\beta > 0$ strictly, we can always find a $\delta > 0$ such that $\beta - \frac{M^2}{\alpha\beta\lambda} - \frac{M}{\beta\lambda} \geq \delta$ for λ sufficiently large. Hence we have

$$\delta\|p\|_0 \leq \left(1 + \frac{M}{\alpha}\right)\|f\|_{V'} \tag{25}$$

Finally, combining (22), (23), (25) we can prove the given inequality.

From the proof, we then obtain that $\lim_{\lambda\to\infty} C = \frac{1}{\alpha} + \frac{1}{\delta}\left(1 + \frac{M}{\alpha}\right)$. This means that the mixed formulation of the problem has a stabilizing effect when approaching the incompressibility limit.

3.2 Convergence Estimates

Once again we discretize the problem, now in the mixed formulation, to solve it numerically. We consider a finite elements approximation using $\mathbb{P}_2^C$ elements for the displacement and $\mathbb{P}_1^C$ elements for the pressure, defining:

$$\mathbf{v}_h = \{\mathbf{v}_h \in \mathbf{V} \cap C^0(\bar{\Omega}) : \mathbf{v}_h\big|_K \in \mathbb{P}^2(K) \quad \forall K \in \tau_h\}, \quad \mathbf{v}_h \subset \mathbf{v}, \quad \dim \mathbf{v}_h < \infty$$
$$Q_h = \{q_h \in Q \cap C^0(\bar{\Omega}) : q_h\big|_K \in \mathbb{P}^1(K) \quad \forall K \in \tau_h\}, \quad Q_h \subset Q, \quad \dim Q_h < \infty$$

We can state the discretized weak formulation of the problem as

$$\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ such that}$$
$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) - c(p_h, q_h) = 0 & \forall q_h \in Q_h \end{cases} \tag{26}$$

with the bilinear forms $a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)$ defined as for (20).

To obtain convergence estimates for the solutions, first of all we notice that for every $(\mathbf{u}_k, p_k) \in \mathbf{V}_h \times Q_h$ it holds that $(\mathbf{u}_h - \mathbf{u}_k, p_h - p_k)$ solves

$$\begin{cases} a(\mathbf{u}_h - \mathbf{u}_k, \mathbf{v}_h) + b(\mathbf{v}_h, p_h - p_k) = \langle F, \mathbf{v}_h\rangle & \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h - \mathbf{u}_k, q_h) - c(p_h - p_k, q_h) = \langle G, q_h\rangle & \forall q_h \in Q_h \end{cases} \tag{27}$$

where

$$\langle F, \mathbf{v}_h\rangle = a(\mathbf{u} - \mathbf{u}_k, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_k) \quad \text{and} \quad \langle G, q_h\rangle = b(\mathbf{u} - \mathbf{u}_k, q_h) - c(p - p_k, q_h)$$

Theorem 3.3 *Let $\mathbf{v}_h \subset \mathbf{V}$, $Q_h \subset Q$ be finite dimensional subspaces that satisfy the inf-sup condition and let $K_h := ker(B_h) \subset K$. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be solution to (26). Then, for $\frac{1}{\lambda}$ sufficiently small, the following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C_u h^2 |u|_{H^3} + C_p h^2 |p|_{H^2} + C(\|F\|_{\mathbf{V}'} + \|G\|_{Q'})$$

Proof.

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0 \leq \|\mathbf{u}-\Pi^2\mathbf{u}+\Pi^2\mathbf{u}-\mathbf{u}_h\|_1+\|p-\Pi^1 p+\Pi^1 p-p_h\|_0 \leq$$
$$\leq \|\mathbf{u}-\Pi^2\mathbf{u}\|+\|\Pi^2\mathbf{u}-\mathbf{u}_h\|_1+\|p-\Pi^1 p\|+\|\Pi^1 p-p_h\|_0$$

We already stated that

$$\|\mathbf{u}-\Pi^2\mathbf{u}\|_1 \leq C_u h^2 |u|_{H^3} \quad \text{and} \quad \|p-\Pi^1 p\|_0 \leq C_p h^2 |p|_{H^2}$$

where C_u and C_p are independent of λ[5].
Set $\mathbf{w}_h=\mathbf{u}_h-\mathbf{u}_k$ and $r_h=p_h-p_k$. Again $\mathbf{w}_h=\mathbf{w}_h^0+\mathbf{w}_h^\perp$, with $\mathbf{w}_h^0 \in K_h$ and $\mathbf{w}_h^\perp \in K_h^\perp$.
By exploiting the coercivity of $a(\cdot,\cdot)$ we get

$$\alpha\|\mathbf{w}_h^0\|_{\mathbf{V}}^2 \leq a(\mathbf{w}_h^0,\mathbf{w}_h^0)=\langle F,\mathbf{w}_h^0\rangle-a(\mathbf{w}_h^\perp,\mathbf{w}_h^0) \implies \|\mathbf{w}_h^0\|_{\mathbf{V}} \leq \frac{1}{\alpha}\|F\|_{\mathbf{V}'}+\frac{M}{\alpha}\|\mathbf{w}_h^\perp\|_{\mathbf{V}} \qquad (a)$$

From the second equation of (27)

$$b(\mathbf{w}_h,q_h)=b(\mathbf{w}_h^\perp,q_h)=\langle G,q_h\rangle+c(r_h,q_h) \quad \forall q_h \in Q_h$$

and exploiting the reversed inf-sup condition

$$\beta\|\mathbf{w}_h^\perp\|_{\mathbf{V}} \leq \sup_{q_h\in Q_h}\frac{b(\mathbf{w}_h^\perp,q_h)}{\|q_h\|_Q}=\|G\|_{Q'}+\|r_h\|_Q \qquad (b)$$

From the first equation of (27)

$$b(\mathbf{v}_h,r_h) \leq \langle F,\mathbf{v}_h\rangle+|a(\mathbf{w}_h,\mathbf{v}_h)| \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$

so, using the inf-sup condition

$$\beta\|r_h\|_Q \leq \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{b(\mathbf{v}_h,r_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \leq \|F\|_{\mathbf{V}'}+M\|\mathbf{w}_h\|_{\mathbf{V}} \leq \|F\|_{\mathbf{V}'}+M\|\mathbf{w}_h^0\|_{\mathbf{V}}+M\|\mathbf{w}_h^\perp\|_{\mathbf{V}} \qquad (c)$$

By combining (a), (b) and (c) we obtain that

$$\|\mathbf{u}_h-\mathbf{u}_k\|_{\mathbf{V}}+\|p_h-p_k\|_Q \leq C(\|F\|_{\mathbf{V}'}+\|G\|_{Q'})$$

This last estimate, together with the estimates for the interpolation errors, yields the conclusion.

The constant C only depends on M,α,β which are independent from λ. Thus, all constants in the estimate (C, C_u, C_p) are not affected for $\lambda\to\infty$, and the convergence is ensured also at the incompressible limit.

3.3 Numerical Simulation

To solve the mixed formulation numerically we proceed analogously to what was done for the first formulation, with some key differences:

- We now define a *mixed space* for the solutions, which is made up of the two finite element spaces already defined for the problem.
- We exploit the global formulation to solve the problem, by defining the global bilinear form
- Once we solve the linear system, the solution has to be "split" to obtain $(\mathbf{u}_h, p_h)$

First of all we check the convergence of $\mathbf{u}_h$ and p_h. To evaluate the convergence of the errors, we interpolate each solution on the mesh of the reference solution to compare the two directly. From the estimates deduced above, we expect convergence with order 2 for $\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$ and $\|p-p_h\|_{L^2}$, and order 3 for $\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$. Yet, the orders which emerge from our numerical simulation are worse than the theoretical estimates, generally displaying one order less. However, as the mesh is refined, the convergence of the variables seems to improve after the fourth refinement (see Figure 3). Due to computational limitations we are not able to compute solutions on finer meshes, but we expect that, for further refinements, the convergence orders would adjust towards the aforementioned values.
Next we turn our attention to the convergence of $Q_1(\mathbf{u}_h)$ and $Q_2(\mathbf{u}_h)$, obtaining similar results to what we found for the primal formulation. Nevertheless, we are able to observe that in this case the estimates don't deteriorate when $\lambda\to\infty$ (a part from slight differences due to the different parameter λ and not too refined meshes), proving the advantages of the mixed formulation. Indeed, according to Theorem 3.3, the convergence constant does not explode as one approaches the incompressibility limit, thus retaining the orders.

[5]The interpolation constants don't depend on the physics of the problem, see (10)

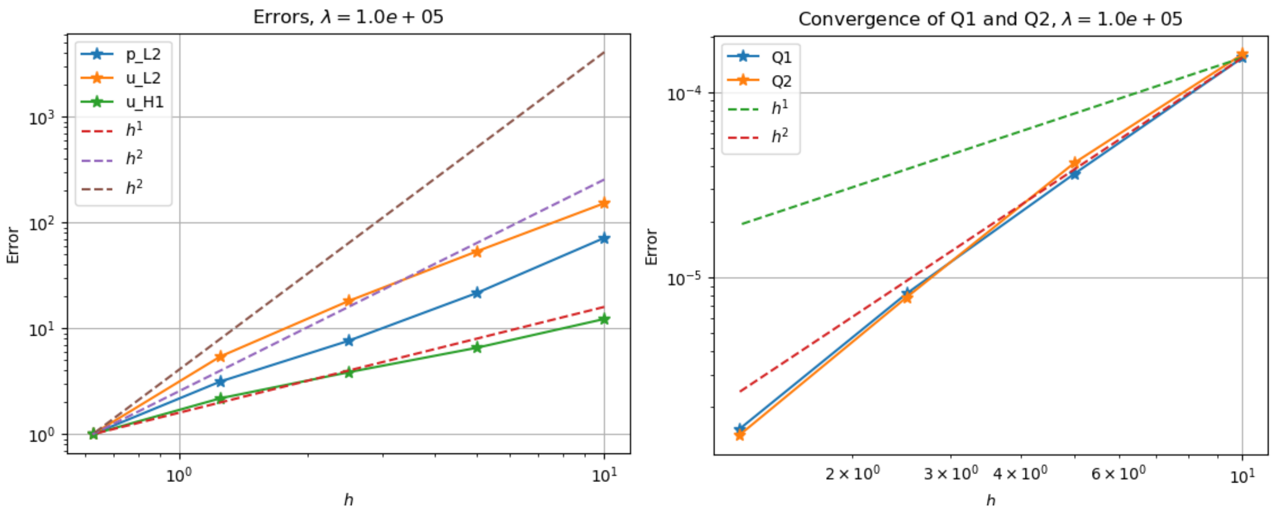

FIGURE 3
Convergence estimates for $\lambda = 1e5$

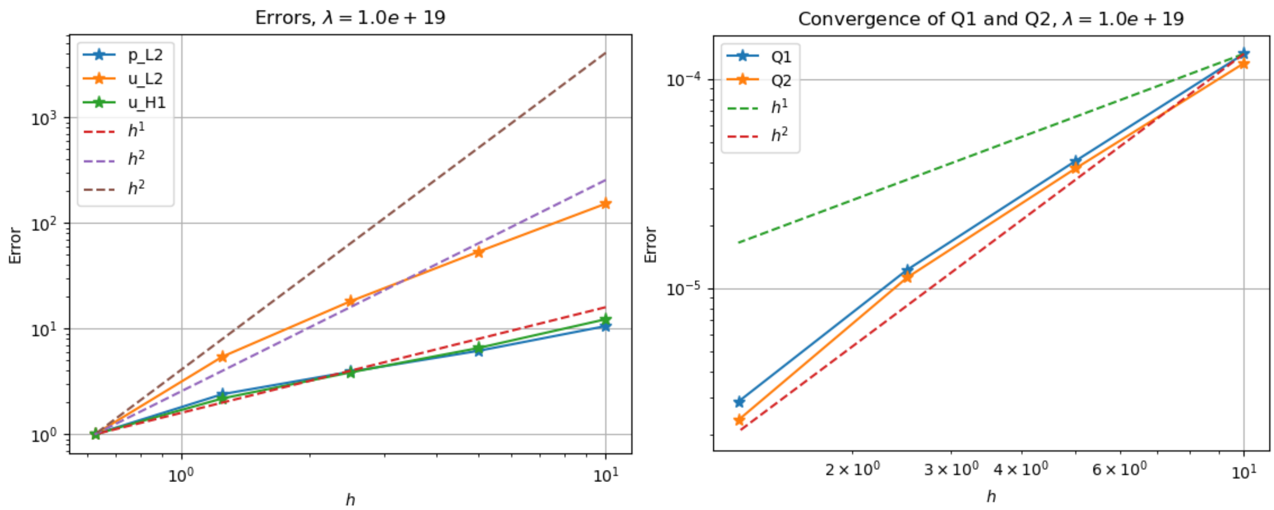

FIGURE 4
Convergence estimates for $\lambda = 1e19$

Finally, we perform an "inf-sup test" on our discrete formulation. The inf-sup condition can be written as

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\int_\Omega q_h \operatorname{div} \mathbf{v}_h}{\|\mathbf{v}_h\|_1 \|q_h\|_0} = \beta_h \geq \beta > 0$$

Where β is a constant independent of the mesh size h. For $\mathbf{v}_h$, Q_h given, we define

$$\int_\Omega q_h \operatorname{div} \mathbf{v}_h = Q_h^T B V_h, \quad \|q_h\|_0^2 = Q_h^T S Q_h, \quad \|\mathbf{v}_h\|_1^2 = V_h^T T V_h$$

To perform the test we compute λ_k the smallest non-zero eigenvalue of $(BT^{-1}B)\phi = \lambda S \phi$, so that we obtain $\beta_h = \sqrt{\lambda_k}$. These last calculations are done in `Matlab`. By running the computation for four different mesh sizes, we obtain values of at least $\beta_h = 0.248 > 0$, hence the constant is bounded away from zero as desired to prove the inf-sup condition holds.

4 Code extract

4.1 Mesh creation

```
meshsize = 1.0

gmsh.initialize()
gmsh.option.setNumber("Mesh.Algorithm", 6)
gmsh.option.setNumber("Mesh.MeshSizeMin", meshsize)
gmsh.option.setNumber("Mesh.MeshSizeMax", meshsize)
gmsh.model.occ.addRectangle(-L_half, -L_half, 0, 2*L_half, 2*L_half, tag=1)
gmsh.model.occ.addDisk(0, 0, 0, R, R, tag=2)
gmsh.model.occ.cut([(2, 1)], [(2, 2)],tag=3) # boolean operation
gmsh.model.occ.synchronize()

gmsh.model.addPhysicalGroup(2, [3], tag=4)
gmsh.model.mesh.generate(2)
```

4.2 Primal formulation and solution

```

msh, cell_markers, facet_markers = gmshio.model_to_mesh(gmsh.model,\
                                                        MPI.COMM_SELF, 0, gdim=2)

V = fem.VectorFunctionSpace(msh, ("CG", degree)) #Vector valued Lagrange elements
x = SpatialCoordinate(msh)

################ Definition of Traction b.c. functions
T=fem.Constant(msh,ScalarType((100,0)))

################ Assembling the weak formulation

def left_edge(x):
    return np.isclose(x[0], -L_half)

bound_face_left = mesh.locate_entities_boundary(msh, 1, left_edge)
bound_dofs_left = fem.locate_dofs_topological(V, 1, bound_face_left)

bc = [fem.dirichletbc(ScalarType((0,0)), bound_dofs_left, V)]

u = TrialFunction(V)
v = TestFunction(V)

a = inner(sigma(u), epsilon(v)) * dx
L = dot(T, v) * ds

problem = fem.petsc.LinearProblem(a, L, bcs=bc,\
    petsc_options={"ksp_type": "preonly", "pc_type": "lu"})
u_h = problem.solve()
```

4.3 Mixed formulation and solution

```
msh, cell_markers, facet_markers = gmshio.model_to_mesh(gmsh.model,\
            MPI.COMM_SELF, 0, gdim=2)

x = SpatialCoordinate(msh)

################ Definition of functional spaces and r.h.s.
P2 = VectorElement("CG", msh.ufl_cell(), 2)
P1 = FiniteElement("CG",msh.ufl_cell(), 1)

MIXED_SPACE = fem.FunctionSpace(msh, P2*P1)

V, _ = MIXED_SPACE.sub(0).collapse()
Q, _ = MIXED_SPACE.sub(1).collapse()

print("Setup spaces on refined mesh number ",mesh_iterator)

u_zero = fem.Function(V)
T = fem.Constant(msh,ScalarType((100,0)))
 ############################################################

```

```python
################ Assembling the weak formulation
def left_edge(x):
    return np.isclose(x[0], -L_half)

bound_face_left = mesh.locate_entities_boundary(msh, 1, left_edge)
bound_dofs_left = fem.locate_dofs_topological((MIXED_SPACE.sub(0),V),\
            entity_dim=1,entities=bound_face_left)
bc = [fem.dirichletbc(u_zero,bound_dofs_left, MIXED_SPACE.sub(0))]


# Define variational problem
(u, p) = TrialFunctions(MIXED_SPACE)
(v, q) = TestFunctions(MIXED_SPACE)

# All terms of the mixed formulation go in one form!
A_form = fem.form(inner(2.0*mu*epsilon(u), epsilon(v))*dx   - \
                  inner(p,nabla_div(v))*dx  -\
                  inner(nabla_div(u), q)*dx - \
                  inner((1/mylambda)*p,q)*dx  )
L_form = fem.form( dot(T, v) *ds)

# Assemble LHS matrix and RHS vectorS
A = fem.petsc.assemble_matrix(A_form, bcs=bc)
A.assemble()
b = fem.petsc.assemble_vector(L_form)

print("Setup system on refined mesh number ",mesh_iterator)

# Create and configure solver
ksp = PETSc.KSP().create(msh.comm)
ksp.setOperators(A)
ksp.setType("preonly")
ksp.getPC().setType("lu")
ksp.getPC().setFactorSolverType("superlu_dist")

# Compute the solution
Sol = fem.Function(MIXED_SPACE)
ksp.solve(b, Sol.vector)

# Split the mixed solution and collapse
(u_h, p_h) = Sol.split()
```

4.4 Evaluation of Q1 and Q2

```python
################ COMPUTING Q2
tol = 0.001 # Avoid hitting the outside of the msh
n_samples = 101
y = np.linspace(R + tol, L_half - tol, n_samples)
points = np.zeros((3, n_samples))
points[1] = y
u1_values = []

bb_tree = geometry.BoundingBoxTree(msh, msh.topology.dim)

cells = []
points_on_proc = []
# Find cells whose bounding-box collide with the the points
cell_candidates = geometry.compute_collisions(bb_tree, points.T)
# Choose one of the cells that contains the point
colliding_cells = geometry.compute_colliding_cells(msh, cell_candidates, points.T)
for i, point in enumerate(points.T):
    if len(colliding_cells.links(i))>0:
        points_on_proc.append(point)
        cells.append(colliding_cells.links(i)[0])

points_on_proc = np.array(points_on_proc, dtype=np.float64)
u1_values = u_h.eval(points_on_proc, cells)
Q2_x[iterator,mesh_iterator] = u1_values[0,0]
Q2_y[iterator,mesh_iterator] = u1_values[0,1]


################ COMPUTING Q1
tol = 0.001 # Avoid hitting the outside of the msh
n_samples = 301
y = np.linspace(-L_half + tol, L_half - tol, n_samples)
```

```python
points = np.zeros((3, n_samples))
points[1] = y
points[0] = np.ones((1, n_samples))*(L_half-tol)
u1_values = []

bb_tree = geometry.BoundingBoxTree(msh, msh.topology.dim)

cells = []
points_on_proc = []
# Find cells whose bounding-box collide with the the points
cell_candidates = geometry.compute_collisions(bb_tree, points.T)
# Choose one of the cells that contains the point
colliding_cells = geometry.compute_colliding_cells(msh, cell_candidates, points.T)
for i, point in enumerate(points.T):
    if len(colliding_cells.links(i))>0:
        points_on_proc.append(point)
        cells.append(colliding_cells.links(i)[0])

points_on_proc = np.array(points_on_proc, dtype=np.float64)
u1_values = u_h.eval(points_on_proc, cells)

# Compute integral
dy = 2*L_half/(n_samples-1)
Q1[iterator,mesh_iterator] = 1/(2*L_half) * dy/2 * 2 * np.sum(u1_values[:,0]) \
            - u1_values[0,0]/2 - u1_values[n_samples-1,0]/2
```

4.5 Matrix assembling for inf-sup test

```python
################ Definition of functional spaces and forms
v_cg2 = VectorElement("CG", msh.ufl_cell(), 2)
s_cg1 = FiniteElement("CG", msh.ufl_cell(), 1)
V = fem.FunctionSpace(msh, v_cg2)
Q = fem.FunctionSpace(msh, s_cg1)
u = TrialFunction(V)
v = TestFunction(V)
p = TrialFunction(Q)
q = TestFunction(Q)

T = fem.petsc.assemble_matrix(fem.form( inner(u,v)*dx))
S = fem.petsc.assemble_matrix(fem.form( inner(p,q)*dx))
B = fem.petsc.assemble_matrix(fem.form( q*div(u)*dx))

T.assemble()
T = petsc2array(T)
S.assemble()
S = petsc2array(S)
B.assemble()
B = petsc2array(B)
```