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Enhancing the ACS Heuristic: Parameter Tuning and Computational Experiments for Solving MIP Problems

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Abstract

This thesis focuses on enhancing the Alternating Criteria Search (ACS) heuristic for solving large and complex Mixed Integer Programming (MIP) problems. While the Parallel ACS (PACS) approach has proven effective in generating high-quality solutions through large-scale parallelization, it remains limited by reliance on specialized hardware and by the need for manual parameter calibration. To overcome these issues, this work develops a refined PACS framework that adapts automatically to different problem instances and computing environments. The improvements center on a dynamic mechanism for adjusting the variable fixing strategy, a lightweight initialization method that avoids parameter tuning, and a reformulation of subproblems that leverages slack-based constraints for greater efficiency. Extensive computational experiments on hard instances from MIPLIB 2017 confirm that the refined PACS consistently achieves better solution quality than the baseline approach, while remaining robust, architecture-agnostic, and easy to integrate into modern MIP solvers.

Abstract

Questa tesi si concentra sul perfezionamento dell'euristica Alternating Criteria Search (ACS) per la risoluzione di problemi di Programmazione Mista Intera (MIP) di grande dimensione e complessità. Sebbene l'approccio Parallel ACS (PACS) abbia dimostrato di essere efficace nel generare soluzioni di alta qualità attraverso la parallelizzazione su larga scala, esso presenta ancora limiti legati alla dipendenza da hardware specializzato e alla necessità di una calibrazione manuale dei parametri. Per superare tali criticità, questo lavoro propone un framework PACS perfezionato, capace di adattarsi automaticamente a diversi tipi di problemi e a differenti ambienti di calcolo. Le migliorie principali riguardano un meccanismo dinamico per l'adattamento della strategia di fissaggio delle variabili, un metodo di inizializzazione leggero che elimina la necessità di tarature manuali, e una riformulazione dei sottoproblemi basata su vincoli slack per garantire maggiore efficienza. Ampi test computazionali su istanze complesse tratte da MIPLIB 2017 confermano come il PACS perfezionato raggiunga sistematicamente risultati di qualità superiore rispetto all'approccio di riferimento, mantenendosi al contempo robusto, indipendente dall'architettura e facilmente integrabile nei moderni solver MIP.

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Chapter 1

Introduction

1.1 Mixed Integer Programming (MIP)

Mixed Integer Programming (MIP) is a powerful optimization technique used to model complex real-world problems where discrete decisions are essential—such as resource allocation, scheduling, and logistics. MIP models aim to find the best solution according to a given objective function, which is either maximized or minimized. A generic mixed-integer program (MIP) will be defined as

$$\left\{ \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & l \leq x \leq u \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \end{array} \right. \quad (1.1)$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $\mathcal{I} \subseteq \{1 \dots n\}$ is the subset of integer variables indices. The solution vector x is bounded by $l \in \bar{\mathbb{R}}^n$ and $u \in \bar{\mathbb{R}}^n$ where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Here, some decision variables are constrained to take integer values, which increases the model's complexity.

1.1.1 Heuristic in MIP solving

The presence of integer variables significantly increases the computational complexity of MIP problems, especially as the number of variables and constraints grows. Solvers such as IBM ILOG CPLEX^[1] and GUROBI^[2] incorporate a variety of built-in strategies to improve performance. Among these strategies are heuristics, which are specialized algorithms designed to quickly find feasible, but not necessarily optimal, solutions. Although heuristics do not guarantee optimality, they are computationally efficient and can significantly accelerate the solution process by providing good initial solutions or guiding the search within the solution space.

1.2 Alternating Criteria Search (ACS)

Finding high-quality feasible solutions is a key aspect of the discrete optimization process, and constructing them has been one of the main focuses of research in the last few decades. Primal heuristics differ in whether they require a starting feasible solution or not. In the first case, the heuristics are called starting heuristics, and state-of-the-art techniques rely on Large Neighborhood Search (LNS)^[3] to find high-quality feasible solutions. The effectiveness of such strategies relies on the power of the MIP solver to solve the subproblems generated during the LNS heuristic. Improvement heuristics, in contrast, require a feasible starting solution and aim to improve its quality with respect to the objective. The simplest improvements are the 1-opt and 2-opt^[4] methods, but—as with starting heuristics—there are many improvement heuristics based on LNS ideas. The neighborhoods explored are defined based on branch-and-bound information—such as the best incumbent and the LP relaxation. This is essentially a limitation in terms of diversification: the strategies may reach good solutions, but it is difficult to find them in the early stages of the search. Therefore, to address this issue, the Alternating Criteria Search (ACS)^[5]—or Parallel ACS, its straightforward parallel implementation—can be utilized. The search neighborhoods are defined based on randomization instead of branch-and-bound information, allowing for a wider exploration of the search space. The aim of this strategy is to find high-quality feasible solutions at the beginning of the search, increasing the heuristic’s effectiveness.

1.2.1 Parallelization of ACS

Parallelism can be exploited in MIP optimization to accelerate computation and enhance solver performance, in terms of execution time and scalability. The exploration of the branch-and-bound tree in parallel is now incorporated in most state-of-the-art MIP solvers—such as CPLEX or GUROBI—and it essentially consists of solving multiple subproblems of the original MIP simultaneously. However, this strategy may not scale well to a large number of cores^[6]. To address this limitation, the proposed heuristic—Parallel ACS (PACS)—is a parallel algorithm that integrates elements of both starting solution and improvement heuristics, offering the capability of generating starting solutions and improving them with respect to the original objective. To leverage parallelism, Parallel ACS performs a large number of LNS runs simultaneously over a diversified set of neighborhoods, with the aim of increasing the chances of finding higher-quality solutions. After each LNS exploration, the strategy consolidates the local improvements through a recombination phase. PACS combines parallelism and diversified LNS in order to address large instances of MIPs arising from various application domains.

1.2.2 Limitations of the PACS Approach

Although the experiments about the PACS strategy bring evidence about the effectiveness on hard MIP instances, there are several considerations that must be addressed. First of all, the PACS algorithm was executed on an 8-node computing cluster, each equipped with two Intel Xeon X5650 6-core processors and 24 GB of memory, totaling 96 cores and 192 GB of RAM. Such computing resources are not typically available in consumer-grade or general-purpose environments. Furthermore, as discussed in the original study, the PACS strategy could assist the MIP solver in accelerating convergence. This suggests that integrating PACS heuristics within a MIP solver could provide a meaningful contribution to solver performance. Second, before the PACS execution, an initial calibration phase is required to tune the parameters prior to execution. This undermines the objective of designing a self-contained, efficient heuristic. With these considerations in mind, the goal of this thesis is to refine the PACS strategy to make it more compatible with MIP solvers. In particular, the aim is to establish a fixed, robust parameter configuration that remains effective regardless of the input instance.

Chapter 2

The ACS Framework

The Alternating Criteria Search (ACS) heuristic is designed to pursue a twofold objective: to identify a feasible solution and to subsequently enhance its quality with respect to the objective function $c^T x$. To this end, ACS employs a Large Neighborhood Search (LNS) strategy, which iteratively solves two auxiliary mixed-integer programming (MIP) subproblems in order to address both objectives. The heuristic requires an initial vector, which is not mandated to be a feasible solution for the original MIP. As illustrated in Figure 2.1, this vector is iteratively refined by solving sub-MIPs, where a subset of variables is fixed to the values defined in the initial input.

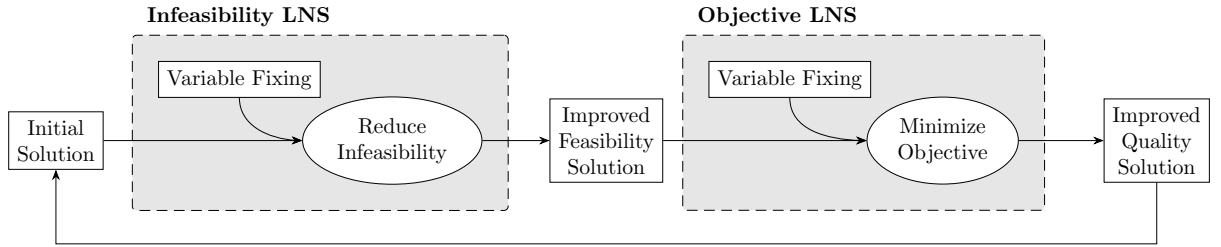


Figure 2.1: ACS Heuristic Workflow Diagram

2.1 Feasibility-MIP (FMIP)

In linear programming theory, it is well established that the feasibility problem can be addressed using the two-phase Simplex method, in which an auxiliary optimization problem is solved to obtain a feasible starting basis, and hence a feasible solution. In a similar manner, the following auxiliary MIP problem, denoted as the Feasibility-MIP

(FMIP), is formulated to identify a feasible starting solution:

$$\left\{ \begin{array}{ll} \min & \sum_{i=0}^m \Delta_i^+ + \Delta_i^- \\ \text{s.t.} & Ax + I_m \Delta^+ - I_m \Delta^- = b \\ & x_i = \hat{x}_i, \forall i \in F \\ & l \leq x \leq u \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \\ & \Delta^+ \geq 0, \Delta^- \geq 0 \end{array} \right. \quad (2.1)$$

Here, I_m denotes an $m \times m$ identity matrix, while Δ^+ and Δ^- are vectors in R^m corresponding to the m constraints. These are introduced as slack variables, and the objective is to minimize their sum. Analogous to the two-phase Simplex method, a vector x is feasible for the original MIP if and only if it can be extended to a solution of value 0 for the associated FMIP. Since solving an FMIP is as computationally demanding as solving the original problem, the neighborhoods are restricted by fixing a given subset F of the integer variables to the values of an input vector $[\hat{x}, \Delta^+, \Delta^-]$. Because of the introduction of slack variables, the vector \hat{x} is not required to be a feasible solution itself, but it must be integral and within the variable bounds to preserve the feasibility of the model. In this way, the FMIP guarantees that feasibility is preserved under any arbitrary variable-fixing scheme. Once a feasible solution is obtained, any LNS-based improving heuristic—such as RINS^[7], DINS^[8], or local branching^[9]—can then be applied to refine the solution.

2.2 Optimality-MIP (OMIP)

Rather than executing the FMIP until convergence to a feasible solution vector, the following auxiliary MIP problem, denoted as Optimality-MIP (OMIP), is designed to improve a partially feasible solution $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$, which satisfies $\sum_{i=1}^m (\hat{\Delta}_i^+ + \hat{\Delta}_i^-) \neq 0$, with respect to the original objective $c^T x$:

$$\left\{ \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + I_m \Delta^+ - I_m \Delta^- = b \\ & \sum_{i=0}^m \Delta_i^+ + \Delta_i^- \leq \sum_{i=0}^m \hat{\Delta}_i^+ + \hat{\Delta}_i^- \\ & x_i = \hat{x}_i, \forall i \in F \\ & l \leq x \leq u \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \\ & \Delta^+ \geq 0, \Delta^- \geq 0 \end{array} \right. \quad (2.2)$$

Analogous to the FMIP, the OMIP represents a reformulation of the original MIP model in which auxiliary slack variables are introduced for each constraint. This formulation

enables the OMIP to enhance a solution even if it is not feasible for the original MIP. Moreover, to ensure that the optimal solution of the OMIP does not exceed the infeasibility of the input solution \hat{x} , an additional budget constraint is imposed, limiting the total slack to $\sum_{i=0}^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-$.

By iteratively solving subproblems of both auxiliary MIPs, the ACS heuristic is designed to converge—although convergence is not formally guaranteed—to a high-quality feasible solution. By construction, infeasibility decreases monotonically after each iteration. However, the quality of the solution with respect to the original objective function may fluctuate.

2.3 Parallelization of ACS (PACS)

The parallelization of ACS exploits parallelism by generating a diversified set of large neighborhood searches, which are solved simultaneously. Exploring multiple search neighborhoods in parallel is expected to increase the likelihood of identifying high-quality solutions. Following this parallelization step, the improvements obtained in parallel must be combined efficiently. To this end, an additional search subproblem is generated in which variables with identical values across different solutions are fixed. Consequently, the recombination phase constitutes a crucial step in PACS: by merging the improvements achieved during the parallel phase, the subsequent phase can explore a newly diversified set of large neighborhood searches based on the recombined solution, thereby enhancing the probability of further improvements. The Algorithm 1 illustrates the overall workflow. In this process, each processor generates a set of randomized variable fixings and then solves the corresponding sub-MIP—either FMIP or OMIP—until the allotted time limit is reached. Subsequently, the solutions are exchanged, and the set U , containing the indices of variables common across solutions, is constructed. The recombination MIP then consists of a subproblem, again associated with either FMIP or OMIP, in which the variables in U are fixed. The best solution obtained will be either the most feasible or the most optimal, depending on whether a recombination FMIP or OMIP is employed. Moreover, every solution used as input can be incorporated as a MIP start, as it remains feasible under any variable-fixing strategy. Processor synchronization and memory communication are handled via the Message Passing Interface (MPI)^[10], owing to its efficient all-to-all collective communication primitives in distributed large-scale architectures. To better illustrate the framework, Figure 2.2 presents a schematic overview of the PACS heuristic, with an emphasis on its parallelization.

2.4 Initialization of ACS

As introduced earlier in this chapter, ACS only requires a starting vector that is integer feasible and within the variable bounds. However, a stronger starting point is

Algorithm 1 Parallel Alternating Criteria Search (PACS)

function FMIP_LNS(MIP, F, \hat{x})
 return $\min\{\sum_i \Delta_i^+ + \Delta_i^- \mid Ax + I_m \Delta^+ - I_m \Delta^- = b, x_j = \hat{x}_j \forall j \in F, x_j \in \mathbb{Z} \forall j \in \mathcal{I}\}$
end function

function OMIP_LNS($MIP, F, \hat{x}, \hat{\Delta}$)
 return $\min\{c^T x \mid Ax + I_m \Delta^+ - I_m \Delta^- = b, \sum_i^m \Delta_i^+ + \Delta_i^- \leq \hat{\Delta}, x_j = \hat{x}_j \forall j \in F, x_j \in \mathbb{Z} \forall j \in \mathcal{I}\}$
end function

Require: Original MIP formulation; PACS Execution time limit TL_{PACS} ; Number of threads T ; Initial solution parameter $\theta \in (0, 100]$; Variable fixing parameter $\rho \in (0, 1)$

Ensure: Feasible solution \hat{x} if found

```

1: function PACS( $MIP, TL_{PACS}, T, \theta, \rho$ )
2:    $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-] \leftarrow \text{INITTHETASOLUTION}(MIP, \theta, 10^6)$ 
3:   while TIMEELAPSED()  $\leq TL_{PACS}$  do
4:     if  $\sum_i \Delta_i^+ + \Delta_i^- > 0$  then
5:       for all threads  $t_i : i \in \{0, T-1\}$  in parallel do
6:          $F_{t_i} \leftarrow \text{RANDOMFIXINGS}(MIP, \rho)$ 
7:          $[x^{t_i}, \Delta^{+t_i}, \Delta^{-t_i}] \leftarrow \text{FMIP\_LNS}(MIP, F_{t_i}, \hat{x})$ 
8:       end for
9:        $U \leftarrow \{j \in \mathcal{I} \mid x_j^{t_i} = x_j^{t_k}, 0 \leq i < k < T\}$ 
10:       $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-] \leftarrow \text{FMIP\_LNS}(MIP, U, x^{t_0})$ 
11:     end if
12:      $\Delta^{UB} \leftarrow \sum_i^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-$ 
13:     for all threads  $t_i : i \in \{0, T-1\}$  in parallel do
14:        $F_{t_i} \leftarrow \text{RANDOMFIXINGS}(MIP, \rho)$ 
15:        $[x^{t_i}, \Delta^{+t_i}, \Delta^{-t_i}] \leftarrow \text{OMIP\_LNS}(MIP, F_{t_i}, \hat{x}, \Delta^{UB})$ 
16:     end for
17:      $U \leftarrow \{j \in \mathcal{I} \mid x_j^{t_i} = x_j^{t_k}, 0 \leq i < k < T\}$ 
18:      $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-] \leftarrow \text{OMIP\_LNS}(MIP, U, x^{t_0}, \Delta^{UB})$ 
19:   end while
20:   return  $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$ 
21: end function

```

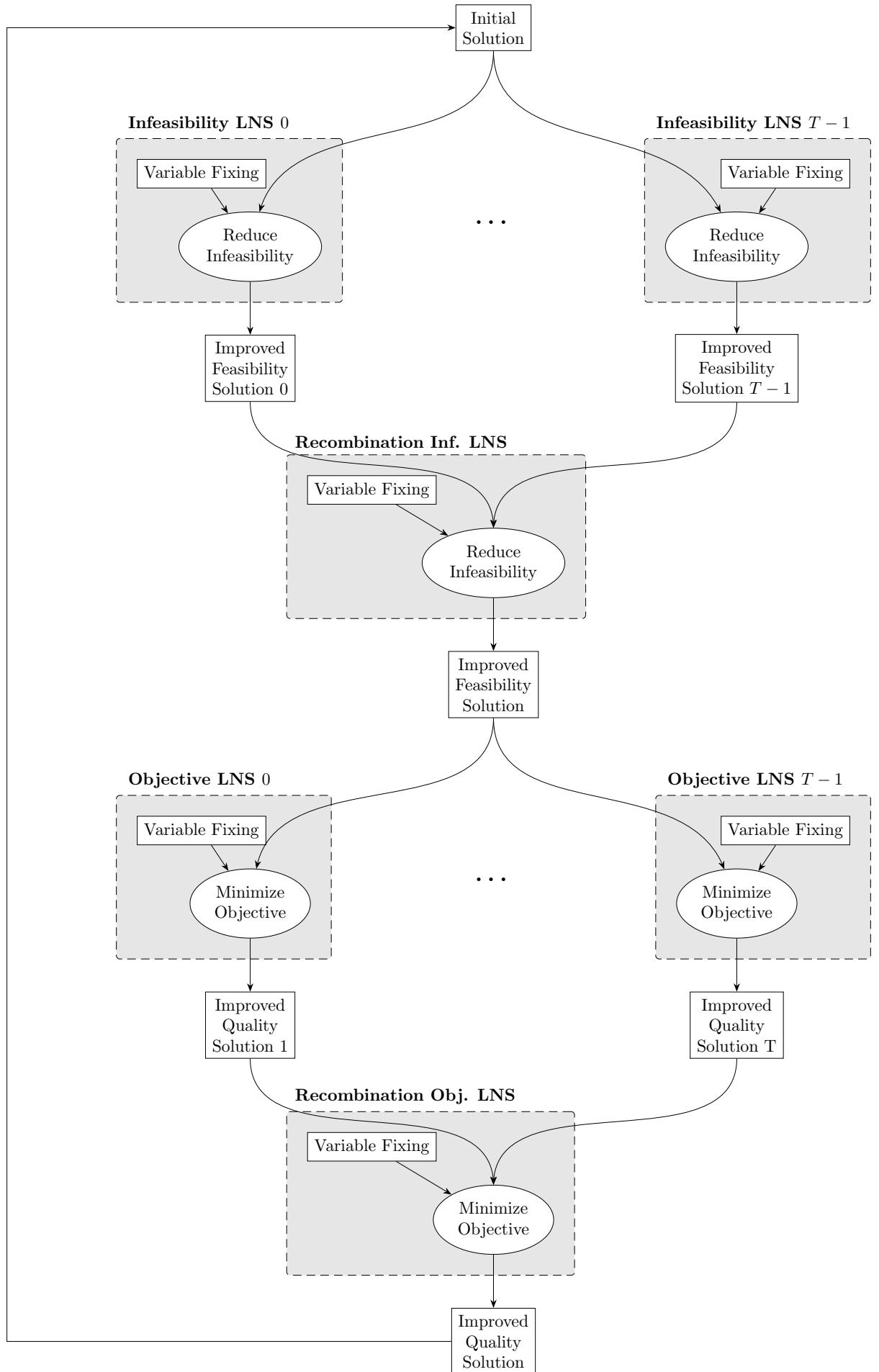


Figure 2.2: PACS Heuristic Workflow Diagram

a solution that is as feasible as possible with respect to the objective function of the FMIP. The proposed algorithm provides a lightweight heuristic that seeks to minimize the infeasibility of the initial solution. The Algorithm 2 first sorts the list of integer

Algorithm 2 Starting vector heuristic

```

function RELAXED_MIP( $MIP, F, \hat{x}$ )
  return  $\min\{\sum_i \Delta_i^+ + \Delta_i^- \mid Ax + I_m \Delta^+ - I_m \Delta^- = b, x_j = \hat{x}_j \ \forall j \in F\}$ 
end function

```

Require: Original MIP formulation; Percentage of variables to fix $\theta \in (0, 100]$; Fixed bound constant c_b

Ensure: Starting integer-feasible vector \hat{x}

```

1: function INITTHETASOLUTION( $MIP, \theta, c_b$ )
2:    $V \leftarrow \text{sort}_{\uparrow}(\{x \in \mathcal{I}\}, (u_x - l_x))$ 
3:    $F \leftarrow \emptyset$ 
4:   while  $\neg \text{ISINTEGERFEASIBLE}(\hat{x}) \wedge F \neq \mathcal{I}$  do
5:      $\mathcal{K} \leftarrow \text{top } \theta\% \text{ of unfixed variables from } V$ 
6:     for  $k \in \mathcal{K}$  do
7:        $\hat{x}_k \leftarrow \text{RANDOMINTEGER}(\max(l_k, -c_b), \min(u_k, c_b))$ 
8:     end for
9:      $F \leftarrow F \cup \mathcal{K}$ 
10:     $[x, \Delta^+, \Delta^-] \leftarrow \text{RELAXED\_MIP}(MIP, F, \hat{x})$ 
11:     $Q \leftarrow \{i \mid x_i \in x, x_i \in \mathbb{Z}, x_i \in \mathcal{I}\}$ 
12:     $\hat{x}_q \leftarrow x_q, \forall q \in Q$ 
13:     $F \leftarrow F \cup Q$ 
14:  end while
15:  return  $\hat{x}$ 
16: end function

```

variables in order of increasing bound range. It then fixes the top $\theta\%$ of variables to random integer values within their respective bounds. The input parameter θ controls the trade-off between the difficulty of the LP relaxation and the quality of the resulting starting solution. In cases where the bounds are infinite, a constant value $c_b = 10^6$ is used to clamp the bounds. The rationale behind the sorting step is to prioritize binary variables first, followed by the remaining integer variables. Until all integer variables are fixed, the LP relaxation of the FMIP is solved to optimize the unfixed variables. Any variables that attain integer values in this process are then fixed. Since at least $\theta\%$ of the variables are fixed at each iteration, the algorithm is guaranteed to terminate after at most $\lceil 100/\theta \rceil$ iterations.

2.5 Variable Fixing Strategy

Selecting an appropriate variable fixing scheme is a challenging task: an overly restrictive strategy may fail to yield improvements, whereas an excessively loose strategy can lead to a search space that is too large to explore efficiently within a reasonable timespan.

The proposed algorithm constitutes a simple yet intuitive variable fixing method: it incorporates randomness to promote diversification and allows for controlling the number of variables to be fixed through an adjustable parameter. In the Algorithm 3, the fixings are

Algorithm 3 Variable Fixing Selection Algorithm

Require: Original *MIP* formulation; Fraction of variables to fix $\rho \in (0, 1)$

Ensure: Set of integer indices F

```

1: function RANDOMFIXINGS( $MIP, \rho$ )
2:    $i \leftarrow \text{UNIFORMFROM}(\mathcal{I})$ 
3:    $F \leftarrow [(i, i + 1, \dots, i + \rho \cdot |\mathcal{I}| - 1) \bmod |\mathcal{I}|]$ 
4:   return  $F$ 
5: end function

```

determined by selecting a random integer variable x_i and fixing a consecutive sequence of integer variables starting from x_i up to a cap determined by ρ , an input parameter that specifies the number of variables to be fixed. The fixing is performed in a circular fashion: if the end of the set \mathcal{I} is reached before the required number of variables are fixed, the algorithm continues from the beginning of \mathcal{I} . The effectiveness of this strategy relies on the fact that, for many problems—such as network flow and routing—the variables are arranged consecutively, often defining a cohesive substructure within the problem.

Chapter 3

The PACS Tuning

According to the original study, although the baseline PACS algorithm demonstrates strong performance on many MIP instances, its exclusive focus on integer variables and dependence on fixed parameters can restrict diversification and increase the risk of premature convergence.

This section presents enhancements aimed at overcoming these limitations:

1. Generalizing starting vector construction and variable fixing to include both integer and continuous variables.
2. Implementing architecture-agnostic parallelization for consistent behavior across hardware.
3. Removing manual parameter tuning via adaptive sub-MIP time limits and dynamic adjustment of ρ .
4. Introducing a parameter-free initial solution strategy.
5. Adding optimization procedures, including slack upper bound enforcement and WalkMIP-based variable fixing, to improve feasibility and solution quality.

These improvements increase PACS’s robustness, efficiency, and applicability while preserving its strengths in diversification and parallelism.

3.1 PACS with Generalized Fixing

The baseline PACS algorithm enforces both the starting vector construction and the fixing scheme to operate exclusively on the set \mathcal{I} of integer variables. While this restriction may appear efficient and straightforward, it can, in fact, be limiting. In particular, if the MIP instance contains only a small fraction of integer variables relative to the total, focusing solely on them may reduce diversification and cause the algorithm to converge prematurely to a local minimum, which is undesirable in an optimization process. To overcome this issue, Algorithm 2 and Algorithm 3 are generalized into Algorithm 4 and

Algorithm 5, respectively, thereby allowing both integer and continuous variables to be considered.

Algorithm 4 Generalized Starting vector heuristic

Require: Original *MIP* formulation; Percentage of variables to fix $\theta \in (0, 100]$; Fixed bound constant c_b

Ensure: Starting integer-feasible vector \hat{x}

```

1: function INITTHETASOLUTION(MIP,  $\theta$ ,  $c_b$ )
2:    $V \leftarrow \text{sort}_{\uparrow}(\{x \in \{x_1, \dots, x_n\}\}, (u_x - l_x))$ 
3:    $F \leftarrow \emptyset$ 
4:   while  $\neg \text{ISINTEGERFEASIBLE}(\hat{x}) \wedge |F| \neq |V|$  do
5:      $\mathcal{K} \leftarrow \text{top } \theta\% \text{ of unfixed variables from } V$ 
6:     for  $k \in \mathcal{K}$  do
7:        $\hat{x}_k \leftarrow \text{RANDOMINTEGER}(\max(l_k, -c_b), \min(u_k, c_b))$ 
8:     end for
9:      $F \leftarrow F \cup \mathcal{K}$ 
10:     $[x, \Delta^+, \Delta^-] \leftarrow \text{RELAXED\_MIP}(\textit{MIP}, F, \hat{x})$ 
11:     $Q \leftarrow \{i \mid x_i \in x, x_i \in \mathbb{Z}, x_i \notin \mathcal{I}\}$ 
12:     $\hat{x}_q \leftarrow x_q, \forall q \in Q$ 
13:     $F \leftarrow F \cup Q$ 
14:  end while
15:  return  $\hat{x}$ 
16: end function

```

Algorithm 5 Generalized Variable Fixing Selection Algorithm

Require: Original *MIP* formulation; Fraction of variables to fix $\rho \in (0, 1)$

Ensure: Set of integer indices F

```

1: function RANDOMFIXINGS(MIP,  $\rho$ )
2:    $i \leftarrow \text{UNIFORMFROM}(\{1, \dots, n\}) \quad \triangleright n$ : number of variables in the original MIP
3:    $F \leftarrow [(i, i + 1, \dots, i + \rho \cdot n - 1) \bmod |n|]$ 
4:   return  $F$ 
5: end function

```

Although these algorithms operate on the auxiliary MIP problems—FMIP or OMIP—the variables subject to fixing are the same as those in the original MIP formulation. By extending the fixing strategy to include both integer and continuous variables, diversification is improved, reducing the likelihood of stagnation in local minima and enhancing exploration of the solution space.

3.2 Architecture-Agnostic Parallelization

In the original study, the Message Passing Interface (MPI) was employed to synchronize processors at each recombination phase. While this approach is well suited to high-performance computing environments, it may be unnecessarily complex in general-purpose scenarios, where a simpler multi-threading implementation is often preferable. In this thesis, communication is instead managed through a set of logical threads, which may differ from the number of available hardware threads. This abstraction ensures that, even on machines with fewer physical cores, the algorithm can reproduce the same behavior across different architectures, provided sufficient computational time is allowed. More specifically, during the coordination phase, either the most feasible or the most optimal solution—depending on whether a recombination FMIP or OMIP is performed—is shared among the logical processors. Each processor then continues working independently on its own copy, with updates to the incumbent solution handled exclusively through a thread-safe update function, formally described in Algorithm 6.

Algorithm 6 Parallel ACS Incumbent Update Procedure

Require: Original *MIP* formulation; Candidate solution $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$; Incumbent x_{ACS} with slack sum Δ_{ACS} and objective value $C_{ACS} = c^T x_{ACS}$; Zero-tolerance ϵ

Ensure: Updated incumbent x_{ACS} in a thread-safe manner

```

1: function UPDATEINCUMBENT(MIP,  $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$ )
2:   acquire lock
3:   if  $(|\sum_i^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-| < |\Delta_{ACS}|) \vee (|\sum_i^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-| < \epsilon \wedge c^T \hat{x} < C_{ACS})$  then
4:      $[x_{ACS}, \Delta_{ACS}, C_{ACS}] \leftarrow [\hat{x}, \sum_i^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-, c^T \hat{x}]$ 
5:   end if
6:   release lock
7: end function

```

This mechanism is crucial to ensure that the algorithm consistently improves and converges within the given time limit. The incumbent solution is updated whenever a better solution is identified: either one with a smaller slack sum, indicating improved feasibility, or one with a lower objective value $c^T x$ provided that the slack sum is less than the tolerance parameter ϵ , the zero-feasibility threshold.

3.3 Eliminating Calibration in PACS Parameter Selection

After adapting the PACS algorithm to a more general-purpose environment, another important challenge arises: parameter selection. Since PACS must be applicable to a wide range of hard MIP instances, it is necessary to identify parameter settings that generally perform well, both in terms of computational efficiency and heuristic solution quality. The

goal is to remove the need for an explicit calibration phase, while still ensuring reliable performance. The tuning process specifically concerns the following parameters:

1. The time span assigned to each sub-MIP, either FMIP_LNS or OMIP_LNS
2. The parameter ρ governing the fixing strategy
3. The parameter θ governing the starting vector

3.3.1 Adaptive Determination of Sub-MIP Time Span

To guarantee determinism in the implementation, each sub-MIP is assigned both a time limit equal to the remaining computation time and a deterministic time limit. The latter is defined as the maximum number of instructions that the solver may execute before termination. The deterministic time limit is computed according to the following formula^[11]:

$$TL_{DET} = \max \left(x, \min \left(\frac{nz}{y}, X \right) \right) \quad (3.1)$$

which provides a dynamic mechanism for adapting the computational effort of each sub-MIP. In this formulation, x and X denote the minimum and maximum allowable deterministic time limits, respectively, nz represents the number of nonzeros in the constraint matrix A of the MIP problem, and y acts as a scaling factor. The chosen parameter values are:

1. Minimum deterministic time limit: $x = 10^3$
2. Scaling factor: $y = 10^2$
3. Maximum deterministic time limit: $X = 10^7$

This dynamic adjustment removes the need to explicitly set a fixed time limit for each subproblem, thereby eliminating the necessity of parameter tuning in this respect.

3.3.2 Adaptive Variable Fixing through ρ Adjustment

Fixed ρ Initialization

As discussed in Section 2.5, selecting an appropriate variable fixing scheme is a non-trivial task. In particular, the random fixings scheme presents additional challenges: if the fraction ρ of variables to be fixed is set too high, the procedure may fail to yield meaningful improvements; conversely, if ρ is set too low, the resulting search space may become excessively large, making it computationally intractable within a reasonable time frame.

For this reason, in the first instance, the parameter ρ is selected from a set of predetermined candidate values. Anticipating the results presented in Section 4.2.1, it can be observed that certain values of ρ are more effective in terms of solution quality, as they grant the solver greater flexibility during the optimization process.

Dynamic Adjustment of Parameter ρ

Since the LNS heuristics in the PACS algorithm restrict the search space by fixing a number of variables according to the value of ρ , it is natural to design a mechanism for dynamically adapting this parameter in order to increase the likelihood of discovering high-quality solutions.

Algorithm 7 Parallel ACS Rho Update (Parallel Phases)

Require: Status code MIP_{code} returned by the solver; Adjustment step δ_ρ for ρ ; The variable fixing parameter ρ ; Number of parallel sub-MIPs num_{MIP}

Ensure: Updated value of ρ , synchronized across parallel sub-MIPs

```

1: function DYNRHOUPDATE( $MIP_{code}$ ,  $\delta_\rho$ ,  $\rho$ ,  $num_{MIP}$ )
2:    $\hat{\delta}_\rho \leftarrow \frac{\delta_\rho}{num_{MIP}}$ 
3:   acquire lock
4:   if  $MIP_{code} = OPT \vee MIP_{code} = OPT_{TOL}$  then
5:      $\rho \leftarrow \rho - \hat{\delta}_\rho$ 
6:   end if
7:   if  $MIP_{code} = FEAS_{TL} \vee MIP_{code} = FEAS_{DET.TL}$  then
8:      $\rho \leftarrow \rho + \hat{\delta}_\rho$ 
9:   end if
10:   $\rho \leftarrow \text{CLAMP}(\rho, 0.01, 0.99)$ 
11:  if *Tie Case detected* then
12:     $\rho \leftarrow \rho - \delta_\rho$ 
13:  end if
14:  release lock
15: end function

```

The procedure, described in Algorithm 7, is applied after each sub-MIP optimization phase in the parallel step. Based on the status returned by the solver, the value of ρ is updated as follows:

- If the solver hits the time limit—either the deterministic limit or the global remaining time—while still producing a feasible solution, ρ is increased by $\frac{\delta_\rho}{num_{MIP}}$. This adjustment suggests fixing more variables in subsequent phases, thereby simplifying the subproblem to be solved.
- Conversely, if the solver converges to an optimal solution within the tolerance, this indicates that the corresponding region of the search space has already been sufficiently explored. In this case, ρ is decreased by $\frac{\delta_\rho}{num_{MIP}}$, enlarging the search space and granting the solver greater freedom in the following optimization steps.

Since each sub-MIP independently attempts to modify the value of ρ , synchronization through locking is required to prevent inconsistencies. The final value of the parameter

ρ is determined as the average of the adjustments proposed by the parallel optimization phases. In case of a tie, a deterministic rule is applied: ρ is decreased by δ_ρ .

Algorithm 8 Parallel ACS Rho Update (Recombination Phases)

Require: Status code MIP_{code} returned by the solver; Adjustment step δ_ρ for ρ ; The variable fixing parameter ρ ; Number of parallel sub-MIPs num_{MIP}

Ensure: Updated value of ρ after recombination adjustment

```

1: function DYNRHOUPDATE( $MIP_{code}$ ,  $\delta_\rho$ ,  $\rho$ ,  $num_{MIP}$ )
2:    $\hat{\delta}_\rho \leftarrow \frac{2\delta_\rho}{num_{MIP}}$ 
3:   if  $MIP_{code} = OPT \vee MIP_{code} = OPT_{TOL}$  then
4:      $\rho \leftarrow \rho - \hat{\delta}_\rho$ 
5:   end if
6:   if  $MIP_{code} = FEAS_{TL} \vee MIP_{code} = FEAS_{DET.TL}$  then
7:      $\rho \leftarrow \rho + \hat{\delta}_\rho$ 
8:   end if
9:    $\rho \leftarrow \text{CLAMP}(\rho, 0.01, 0.99)$ 
10: end function

```

For the recombination phase, the procedure is slightly modified into Algorithm 8, where the adjustment step is doubled, i.e. $2\delta_\rho/num_{MIP}$, in order to resolve potential tie cases.

Finally, although an initial value of ρ must be provided to start the fixing process, experimental results in Section 4.2.2 show that performance is only marginally affected by this initialization. Consequently, the effectiveness of the method does not critically depend on the initial choice of ρ .

3.3.3 Parameter-Free Strategy for Initial Solution Construction

Another non-trivial challenge concerns the selection of the initial solution vector. The parameter θ , which regulates the trade-off between solution quality and execution time, must be tuned to suit the most general cases. Similar to the fixing strategy, the analysis considers a set of predetermined values for θ , while also introducing a novel initialization strategy for the starting solution vector. The motivation is that using a small value of θ —which corresponds to generating a high-quality initial solution—may waste computation time, since the true strength of PACS lies in its diversification and parallelization capabilities.

Therefore, Algorithm 9 is designed with the following goal: to provide a heuristic starting solution that is deterministic and computationally lightweight, while remaining reasonably feasible for the original MIP problem.

Algorithm 9 Heuristic Initialization of Starting Solution

Require: Original *MIP* formulation with n variables, objective coefficients c_i , bounds $[l_i, u_i]$; Zero-tolerance ϵ

Ensure: Starting solution vector $\hat{x} \in \bar{\mathbb{R}}^n$

```
1: function STARTSOLMAXFEAS(MIP)
2:    $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-] \leftarrow (+\infty, \dots, +\infty) \in \bar{\mathbb{R}}^{n+2m}$ 
3:   for  $i = 1 \rightarrow n$  do
4:      $(l_i, u_i) \leftarrow$  variable bounds of variable  $i$ 
5:     if  $l_i = -\infty \wedge u_i = +\infty$  then
6:        $\hat{x}_i \leftarrow 0$ 
7:     else if  $l_i > -\infty \wedge u_i = +\infty$  then
8:        $\hat{x}_i \leftarrow l_i$ 
9:     else if  $l_i = -\infty \wedge u_i < +\infty$  then
10:       $\hat{x}_i \leftarrow u_i$ 
11:    else
12:      if  $c_i \leq -\epsilon$  then
13:         $\hat{x}_i \leftarrow l_i$ 
14:      else if  $c_i \geq \epsilon$  then
15:         $\hat{x}_i \leftarrow u_i$ 
16:      else
17:         $\hat{x}_i \leftarrow \text{RANDOMINTEGER}(l_i, u_i)$ 
18:      end if
19:    end if
20:  end for
21:  return  $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$ 
22: end function
```

The initialization procedure in Algorithm 9 operates as follows:

- If a variable has no finite bounds, it is set to 0.
- If a variable has only one finite bound, it is set to that bound.
- If both bounds are finite:
 - set the variable to the lower bound if $c_i \leq -\epsilon$, i.e. the coefficient is negative,
 - set it to the upper bound if $c_i \geq \epsilon$, i.e. the coefficient is positive,
 - choose a random integer within $[l_i, u_i]$ if $-\epsilon \leq c_i \leq \epsilon$, i.e. the coefficient is close to zero.

In this way, the solution initialization is almost entirely deterministic and computationally lightweight, although feasibility with respect to the original MIP problem may be partially

compromised. Furthermore, the proposed strategy is parameter-free, which makes it particularly well-suited for the purposes of this thesis.

3.4 Additional PACS Optimizations

While the previous modifications focused on tuning or slightly adjusting specific components of the PACS workflow, the following procedures—analyzed in Section 4.2.5, 4.2.6, 4.2.7—aim to further enhance the quality of the final solution.

3.4.1 Slack Upper Bound Enforcement and Budget Constraint Removal

The first idea is to exploit the values of the slack variables Δ^+ and Δ^- obtained in a previous iteration to enforce these values as upper bounds. In the PACS algorithm, this mechanism is already embedded in the budget constraint $\sum_{i=0}^m \Delta_i^+ + \Delta_i^- \leq \sum_{i=0}^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-$, which provides only a global upper bound. The proposed strategy, instead, aims at fixing the upper bounds of individual slack variables, thereby assisting subsequent FMIP optimizations and improving feasibility. The procedure is implemented as shown in Algorithm 10.

Algorithm 10 Fixing Slack Variables to Upper Bound

Require: Original *MIP* formulation; Solution $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$; Zero-tolerance ϵ

Ensure: Set S_{UB} of variable–upper bound pairs to be fixed

```

1: function SLACKUPPERBOUNDFIXINGS(MIP,  $[\hat{x}, \hat{\Delta}^+, \hat{\Delta}^-]$ )
2:    $S_{UB}^+ \leftarrow \emptyset$  ▷ Set of variable-upper bound pairs for  $\Delta^+$ 
3:    $S_{UB}^- \leftarrow \emptyset$  ▷ Set of variable-upper bound pairs for  $\Delta^-$ 
4:   for  $i = 1 \rightarrow m$  do
5:      $u_i^+ \leftarrow$  upper bound of  $\Delta_i^+$ 
6:     if  $u_i^+ - \hat{\Delta}_i^+ > \epsilon$  then
7:        $S_{UB}^+ \leftarrow S_{UB}^+ \cup \{(i, \Delta_i^+)\}$ 
8:     end if
9:      $u_i^- \leftarrow$  upper bound of  $\Delta_i^-$ 
10:    if  $u_i^- - \hat{\Delta}_i^- > \epsilon$  then
11:       $S_{UB}^- \leftarrow S_{UB}^- \cup \{(i, \Delta_i^-)\}$ 
12:    end if
13:  end for
14:  return  $[S_{UB}^+, S_{UB}^-]$ 
15: end function

```

Here, the solution vector \hat{x} corresponds to the current PACS incumbent—valid in both the parallel and recombination phases—and each slack variable is compared with its

respective upper bound. Whenever the value of a slack variable is strictly smaller than its upper bound by more than the tolerance ϵ , the variable index together with its solution value is recorded in either S_{UB}^+ or S_{UB}^- , depending on the sign of the slack, as a pair $(i, \hat{\Delta}_i)$. The resulting sets S_{UB}^+ and S_{UB}^- are subsequently employed to update the upper bounds of the slack variables, thereby modifying the auxiliary FMIP and OMIP formulations as follows.

FMIP Reformulation

$$\left\{ \begin{array}{ll} \min & \sum_{i=0}^m \Delta_i^+ + \Delta_i^- \\ \text{s.t.} & Ax + I_m \Delta^+ - I_m \Delta^- = b \\ & x_i = \hat{x}_i, \forall i \in F \\ & \Delta_i^+ \leq \hat{\Delta}_i, \forall (i, \hat{\Delta}_i) \in S_{UB}^+ \\ & \Delta_i^- \leq \hat{\Delta}_i, \forall (i, \hat{\Delta}_i) \in S_{UB}^- \\ & l \leq x \leq u \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \\ & \Delta^+ \geq 0, \Delta^- \geq 0 \end{array} \right. \quad (3.2)$$

OMIP Reformulation

$$\left\{ \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + I_m \Delta^+ - I_m \Delta^- = b \\ & \cancel{\sum_{i=0}^m \Delta_i^+ + \Delta_i^- \leq \sum_{i=0}^m \hat{\Delta}_i^+ + \hat{\Delta}_i^-} \\ & x_i = \hat{x}_i, \forall i \in F \\ & \Delta_i^+ \leq \hat{\Delta}_i, \forall (i, \hat{\Delta}_i) \in S_{UB}^+ \\ & \Delta_i^- \leq \hat{\Delta}_i, \forall (i, \hat{\Delta}_i) \in S_{UB}^- \\ & l \leq x \leq u \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I} \\ & \Delta^+ \geq 0, \Delta^- \geq 0 \end{array} \right. \quad (3.3)$$

It is worth noting that the budget constraint in OMIP can now be removed, since the slack upper bound fixing provides a stronger restriction, as will be further confirmed by the results in Section 4.2.5.

3.4.2 WalkMIP: Violated-Constraints-Based Variable Fixing

It is worthwhile to investigate a more sophisticated and potentially efficient variable fixing strategy, aiming to enrich diversification and consequently increase the likelihood of

finding high-quality solutions within a relatively short time span. The WalkMIP strategy adapts ideas from the WalkSAT^[12] algorithm, targeting specifically the variables that cause infeasibility in a partially feasible solution.

Algorithm 11 presents the WalkMIP procedure adapted for MIP constraint satisfaction problems.

Algorithm 11 Walk-based Repair Heuristic for MIP

Require: Auxiliary formulation \overline{MIP} (FMIP or OMIP); Initial solution \hat{x} ; Fixing parameter ρ ; Random walk probability $p \in (0, 1)$; Solution kick probability p_{HUGE_KICK}

Ensure: Updated solution \hat{x} and set F of integer indices

```

1: function WALKMIP( $\overline{MIP}$ ,  $\hat{x}$ ,  $\rho$ ,  $p$ ,  $p_{HUGE\_KICK}$ )
2:    $violConstr \leftarrow$  Set of indices of violated constraints, under  $\hat{x}$ 
3:    $F \leftarrow \emptyset$ 
4:   if *First Execution* then  $numInitialConstr \leftarrow |violConstr|$ 
5:   end if
6:   if  $violConstr = \emptyset$  then return
7:   end if
8:   if with  $\mathbb{P} = p_{HUGE\_KICK}$  then
9:     return RANDOMFIXINGS( $\overline{MIP}$ ,  $\rho$ )
10:  end if
11:   $k \leftarrow \text{NUMCONSTRTOFIX}(numInitialConstr, |violConstr|)$ 
12:  repeat
13:     $c \leftarrow \text{UNIFORMFROM}(violConstr)$ 
14:     $[i^*, x^*, \delta^*] \leftarrow \text{MINDAMAGEMOVE}(\overline{MIP}, c, \hat{x})$ 
15:    if  $minDMG \leq -\epsilon$  then
16:       $F \leftarrow F \cup i^*$ 
17:       $\hat{x} \leftarrow \text{UPDATESOL}(\hat{x}, i^*, x^*)$  ▷ Apply best move to  $\hat{x}$ 
18:    else if with  $\mathbb{P} = 1 - p \wedge i^* \neq -1$  then
19:       $F \leftarrow F \cup i^*$ 
20:       $\hat{x} \leftarrow \text{UPDATESOL}(\hat{x}, i^*, x^*)$  ▷ Apply min-damage move to  $\hat{x}$ 
21:    else
22:       $i_{rnd} \leftarrow \text{RANDOMINTEGER}(0, n - 1)$ 
23:       $F \leftarrow F \cup i_{rnd}$ 
24:       $\hat{x} \leftarrow \text{UPDATESOL}(\hat{x}, i_{rnd}, \hat{x}_{i_{rnd}})$  ▷ Randomly fixing a var in  $\hat{x}$ 
25:    end if
26:     $violConstr \leftarrow$  Set of indices of violated constraints, under  $\hat{x}$ 
27:    if  $violConstr = \emptyset$  then break
28:    end if
29:  until  $k$  times
30:  if  $k < \rho \cdot n$  then
31:     $F \leftarrow \text{RANDOMFIXINGS}(\overline{MIP}, \rho - \frac{k}{n})$  ▷ Fix remaining vars (up to  $\rho \cdot n$ ) from  $\hat{x}$ 
32:  end if
33:  return  $F$ 
34: end function

```

The WalkMIP algorithm is executed independently by each processor, replacing the standard random fixing procedure. The strategy can be decomposed into three main phases:

1. **Initialization and Huge-Kick Phase (lines 1-10 in Algorithm 11):** If this is the first execution of the routine, the algorithm records the number of violated constraints under the current solution x , which is later used to determine the number of iterations for the main loop. To enhance diversification and escape local minima, a probability p_{HUGE_KICK} is introduced to randomly fix variables according to ρ . In the implementation, this parameter is set to $1/16$.
2. **Constraint Selection and Optimization Loop (lines 11-29 in Algorithm 11):** The number of constraints to target is determined by

$$k(v, v_0) = \begin{cases} \lceil k_{\min} + (k_{\max} - k_{\min}) \cdot \hat{r}^\beta \rceil & \text{if } v > k_{\min}, \\ v & \text{otherwise,} \end{cases}$$

where

$$\hat{r} = \min\left(1, \max\left(0, \frac{v}{\max(1, v_0)}\right)\right), \quad k_{\min} = 32, \quad k_{\max} = 1024, \quad \beta = \frac{\sqrt{5}-1}{2}.$$

Here, v_0 is the initial number of violated constraints, and v is the current number of violated constraints. This rule guarantees that the number of touched constraints never exceeds k_{\max} , while adapting to the reduction of violations over time. At each iteration, the loop samples k violated constraints and, for each, selects one variable to modify, in analogy to the WalkSAT strategy. The function `MINDAMAGEMOVE` (Algorithm 12) evaluates candidate moves and returns the one that minimizes constraint violation. The update proceeds as follows:

- If a fixing reduces infeasibility, the best move is applied.
- Otherwise, with probability $1 - p$, the minimal-damage move is applied.
- Otherwise, with probability p , a random variable is fixed using its current solution value $x_{i_{\text{rnd}}}$.

3. **Fixing Remaining Variables (lines 30-33 in Algorithm 11):** To maintain consistency with the original variable fixing strategy, the total number of fixed variables is set to $\rho \cdot n$. If the previous phases fixed fewer variables, the remaining variables are fixed according to the standard strategy.

Algorithm 12 Minimum Damage Move Selection for a Violated Constraint in WalkMIP

Require: Auxiliary formulation \overline{MIP} (FMIP or OMIP); Violated constraint c ; Current solution vector \hat{x}

Ensure: $[i^*, x^*, \delta^*]$ with i^* the selected index, x^* its new value, and δ^* the minimum violation change

```
1: function MINDAMAGEMOVE( $\overline{MIP}$ ,  $c$ ,  $\hat{x}$ )
2:    $[i^*, x^*, \delta^*] \leftarrow [-1, +\infty, +\infty]$ 
3:   for each variable  $i$  in  $c$  do
4:     if  $i.type = \text{BIN}$  then  $\bar{x}_i \leftarrow \neg \hat{x}_i$ 
5:     else if  $i.type = \text{INT}$  then  $\bar{x}_i \leftarrow \hat{x}_i + \text{RANDOMINTEGER}(-1, 1)$ 
6:     else if  $i.type = \text{DOUBLE}$  then  $\bar{x}_i \leftarrow \hat{x}_i + \text{RANDOMDOUBLE}(-0.5, 0.5)$ 
7:     end if
8:      $\bar{x}_i \leftarrow \text{CLAMP}(\bar{x}_i, l_i, u_i)$ 
9:      $\delta \leftarrow \text{COMPUTEVIOLATION}(\hat{x}, \bar{x}_i)$ 
10:    if  $\delta < \delta^*$  then
11:       $[i^*, x^*, \delta^*] \leftarrow [i, \bar{x}_i, \delta]$ 
12:    end if
13:  end for
14:  return  $[i^*, x^*, \delta^*]$ 
15: end function
```

Any modifications of ρ introduced through the dynamic adjustment routine are subsequently propagated to later WalkMIP iterations. Nevertheless, it is important to emphasize that WalkMIP can only be applied when the slack variables are nonzero. Consequently, the original random fixing strategy, described in Section 3.3.2, remains necessary during OMIP optimization phases where the slack sum is below the zero-tolerance threshold ϵ . Although the results in Section 4.2.7 indicate that this strategy is less effective than the simpler, more straightforward variable fixing approach, it offers valuable insights and suggests a potential direction for future research.

Chapter 4

Experimental Results

4.1 Experimental Setup

The experiments are designed to compare the PACS framework with the standalone IBM ILOG CPLEX Optimization Studio (version 22.1). The PACS framework is implemented in C++ for performance reasons, interfacing with the IBM ILOG CPLEX MIP solver through the C API. The source code is available under a non-commercial MIT license at https://github.com/francesco-biscaccia-carrara/PACS_Tuning.

All experiments were conducted on a cluster within the UniPD DEI Blade infrastructure, running Rocky Linux 8.10. Each node is equipped with an Intel(R) Xeon(R) E5-2623 v3 processor (4 cores, 3.00 GHz) and 16 GB of RAM. Although the infrastructure is cluster-based, the computational power of a single node is comparable to that of a general-purpose laptop, or even lower. For fairness in comparison, PACS executions use 4 logical threads, implemented as C++ standard threads, each running a CPLEX instance restricted to a single core. The baseline counterpart is a standalone CPLEX execution restricted to 4 cores.

4.1.1 Dataset

As anticipated in the previous sections, the benchmark for this comparison must consist of hard MIP instances. To this end, the set of hard instances from MIPLIB 2017^[13] has been used as the test bed.¹ While CPLEX directly processes each MIP instance, the PACS framework additionally requires a random seed in order to reproduce the same randomized choices across different runs. For this purpose, PACS was evaluated on each instance using the seeds {38472910, 56473829, 27384910, 91827364, 8374659}. This setup increases statistical reliability and can be considered representative of the algorithm’s standard behavior. Consequently, for each instance, five independent PACS runs and one CPLEX run were executed.

¹The instance *tpl-tub-ss16* was excluded, as its execution was terminated due to insufficient computational resources.

4.1.2 Metrics

Both PACS and plain CPLEX terminate as soon as an a feasible solution is identified, subject to a global time limit of 5 minutes. Since the PACS framework was executed five times for each instance, the reported solution quality and computation time are given as the arithmetic mean across all runs. Moreover, if the majority of PACS executions for a given instance—at least three out of five seeds—fail to produce a solution, the instance is recorded as unsolved. Instead of directly comparing the objective values of the solutions returned by the two approaches, the MIP gap metric^[14] has been adopted. The MIP gap is defined as follows: given a solution \hat{x} for a MIP with an optimal solution x , the primal gap $\gamma(\hat{x}) \in [0, 1]$ is

$$\gamma(\hat{x}) = \begin{cases} 0 & \text{if } |c^T x| = |c^T \hat{x}| = 0, \\ 1 & \text{if } c^T x \cdot c^T \hat{x} < 0, \\ \frac{|c^T x - c^T \hat{x}|}{\max\{|c^T \hat{x}|, |c^T x|\}} & \text{otherwise.} \end{cases} \quad (4.1)$$

This metric captures the relative gap between an incumbent and the best-known solution, normalized to the interval $[0, 1]$. Since the algorithms operate under a strict time limit, the metric must be extended accordingly. Given a time limit t_{\max} , the primal function $p : [0, t_{\max}] \rightarrow [0, 1]$ is defined as:

$$p(\hat{x}) = \begin{cases} 1 & \text{if no solution is found up to time } t, \\ \gamma(\hat{x}(t)) & \text{if } \hat{x}(t) \text{ is the incumbent at time } t. \end{cases} \quad (4.2)$$

For each test, two types of plots are produced to summarize and compare performance:

1. **Success Rate vs. Computation Time:** The x -axis represents elapsed time, and the y -axis reports the cumulative number of instances for which an incumbent was found. The closer a curve is to the top-left corner, the more efficient the algorithm.
2. **Success Rate vs. MIP Gap:** The x -axis represents the MIP gap, and the y -axis reports the cumulative number of instances solved with a gap less than or equal to the given value. Consistently, curves closer to the top-left corner indicate better performance

To provide a more compact quantitative comparison, the integral of each curve has been computed and reported in a corresponding bar chart, one for each plot.

4.1.3 Tolerance Parameters

Because both algorithms involve floating-point computations and comparisons, PACS employs a set of tolerance parameters. In these experiments, the parameters were selected to be reasonable relative to the defaults used by CPLEX:

- **Zero-tolerance parameter ϵ :** values smaller than ϵ are treated as zero. In all tests, ϵ was set to 10^{-5} .
- **Absolute maximum constraint violation:** the maximum permissible violation of any constraint under a candidate solution \hat{x} . In these tests, it was set equal to ϵ , i.e., 10^{-5} .
- **Absolute maximum integrality violation:** the maximum deviation allowed for variables constrained to take integer values. This tolerance was likewise set to $\epsilon = 10^{-5}$.
- **Relative objective error:** the maximum admissible relative error between the recomputed objective value $c^T \hat{x}$ and the value stored internally by PACS. This threshold was again set to $\epsilon = 10^{-5}$.

In the latter three cases, if the corresponding tolerance is exceeded, the algorithm terminates with an error and the run is recorded as unsolved.

4.2 Results

This section presents the experimental evaluation and details the progressive tuning of PACS parameters. Each test builds upon the outcomes of the previous one, providing a stepwise refinement of the algorithm.

4.2.1 Fixed ρ Initialization Test

The first experiment focuses on the choice of the parameter ρ in the variable fixing strategy. In this experiment, PACS was employed with generalized fixing strategies under an architecture-agnostic parallelization scheme. The predefined values of ρ considered are $\{.1, .25, .5, .75, .9\}$. It is worth noting that smaller values of ρ leave greater freedom to the MIP solver.

Since the PACS algorithm also requires the parameter θ for the initial solution construction strategy, it was fixed to $\theta = .25$ for this test.

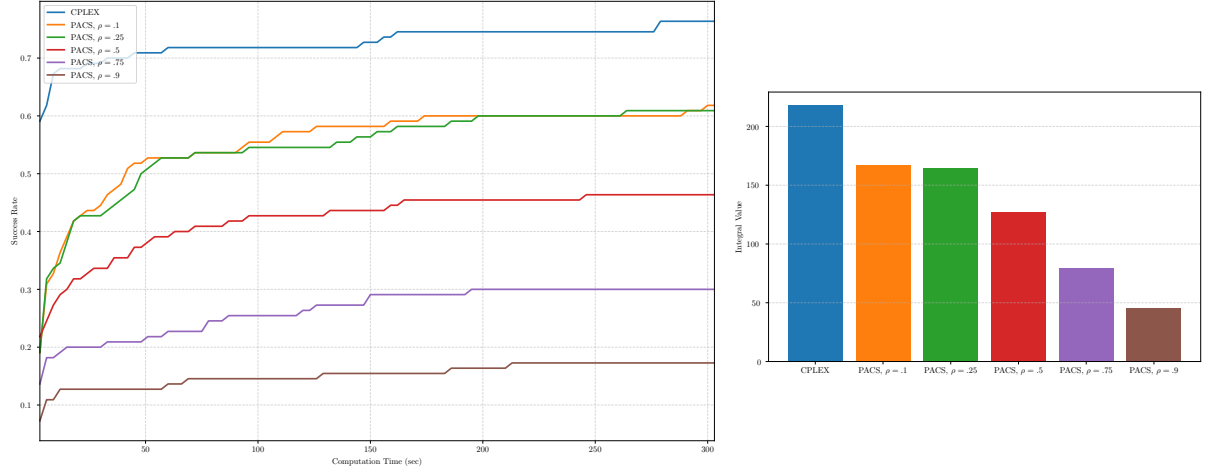


Figure 4.1: Success Rate vs. Computation Time for Fixed ρ Initialization Test

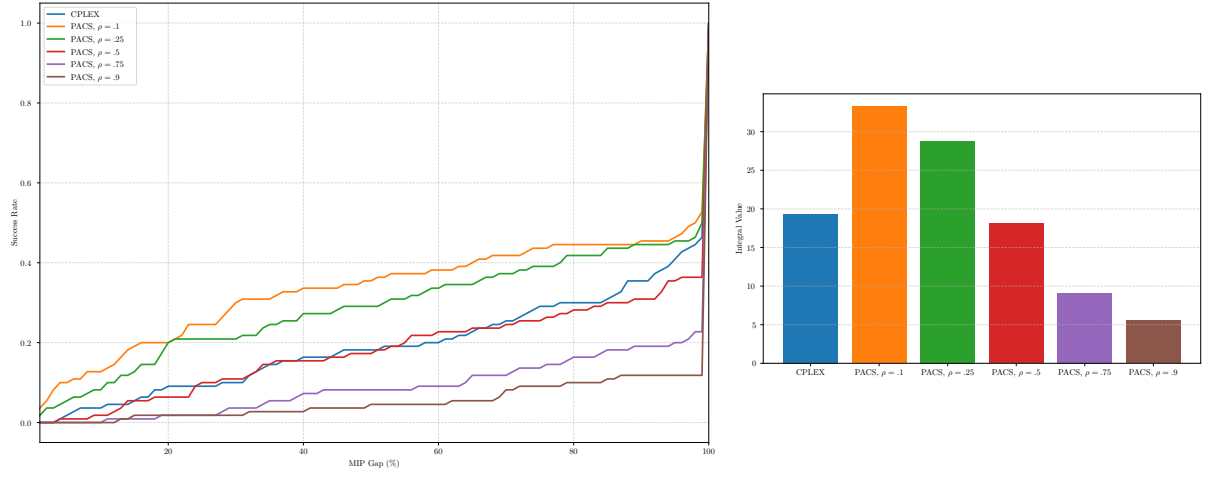


Figure 4.2: Success Rate vs. MIP Gap for Fixed ρ Initialization Test

From Figure 4.1, it is evident that standalone CPLEX consistently outperforms PACS in terms of success rate across all values of ρ . However, Figure 4.2 reveals a different picture: PACS with $\rho = .1$ outperforms all other settings of ρ , and even surpasses CPLEX in terms of solution quality, providing higher-quality solutions approximately $1.72\times$ more frequently than the solver alone. This result may be attributed to the fact that greater freedom allows CPLEX to optimize more effectively, whereas PACS primarily benefits from enhanced diversification through its parallelization strategy.

Based on these observations, the baseline PACS configuration under the static fixing scenario is set to $\theta = .25$ and $\rho = .1$.

4.2.2 Dynamic ρ Adjustment Test

This experiment investigates the dynamic adjustment strategy² for the variable fixing parameter ρ , as introduced in Section 3.3.2. Although PACS requires an initial ρ value,

²In the plots, the PACS algorithm with dynamically adjusted ρ is denoted as DACS for brevity.

it is automatically updated during execution. In this test, the candidate values for ρ are .1, .25, .5, while the starting solution parameter θ is set to .25, consistent with the previous experiment.

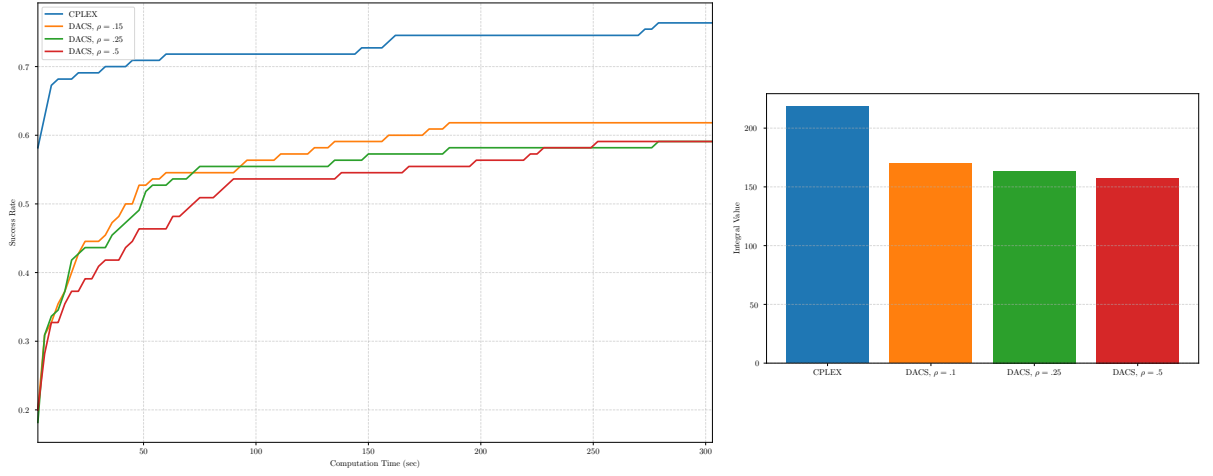


Figure 4.3: Success Rate vs. Computation Time for Dynamic ρ Adjustment Test

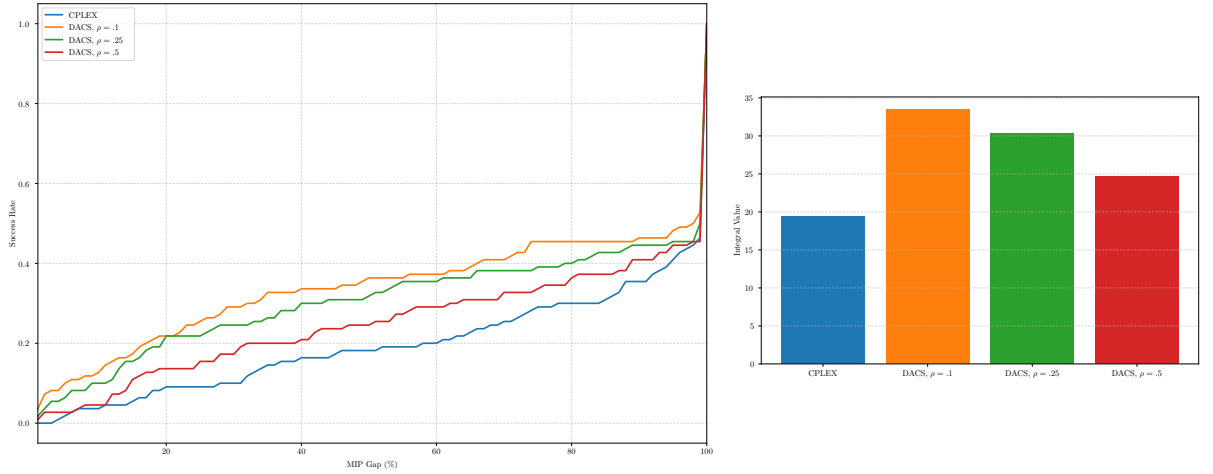


Figure 4.4: Success Rate vs. MIP Gap Plot for Dynamic ρ Adjustment Test

As in the static case, Figure 4.3 shows that CPLEX consistently outperforms PACS in terms of success rate. However, Figure 4.4 demonstrates that dynamic adjustment of ρ improves solution quality: selecting $\rho = .1$ produces higher-quality solutions roughly $1.73\times$ more frequently than CPLEX alone.

The key observation from this experiment is that, unlike the fixed ρ strategy, where the choice of ρ significantly affects PACS performance, the dynamic adjustment greatly reduces sensitivity to the initial parameter. Regardless of the starting ρ , PACS achieves comparable results in both success rate and solution quality, providing consistently reliable outcomes.

Based on these results, the baseline PACS configuration under the dynamic fixing scenario is set to $\theta = .25$ and initial $\rho = .1$.

4.2.3 Fixed vs. Dynamically Adjusted ρ Test

This experiment compares the fixed and dynamically adjusted strategies for selecting the variable fixing parameter ρ , aiming to identify the most effective approach for the final PACS algorithm. Since θ is consistent across the previous experiments, a direct comparison is possible.

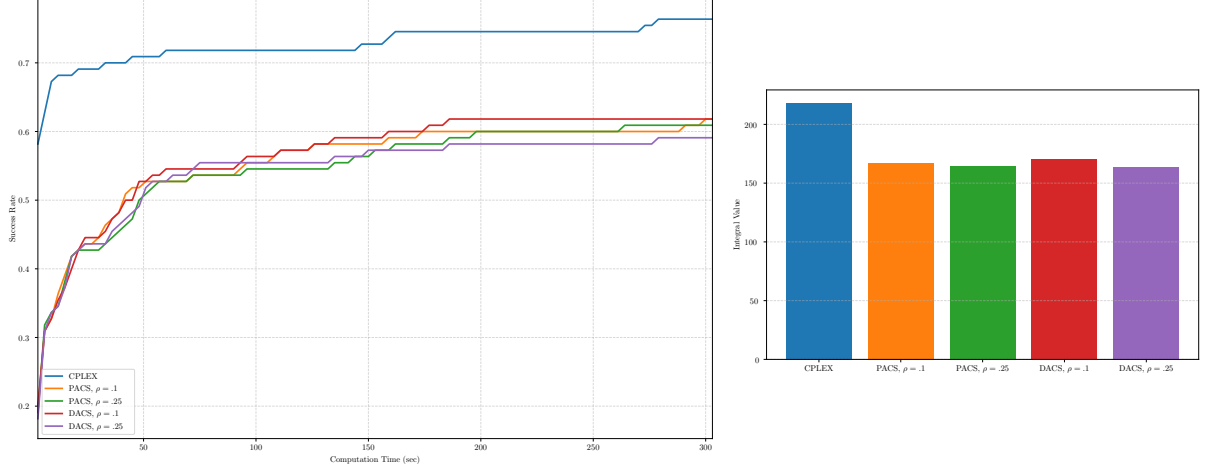


Figure 4.5: Success Rate vs. Computation Time for Fixed vs. Dynamically Adjusted ρ Test

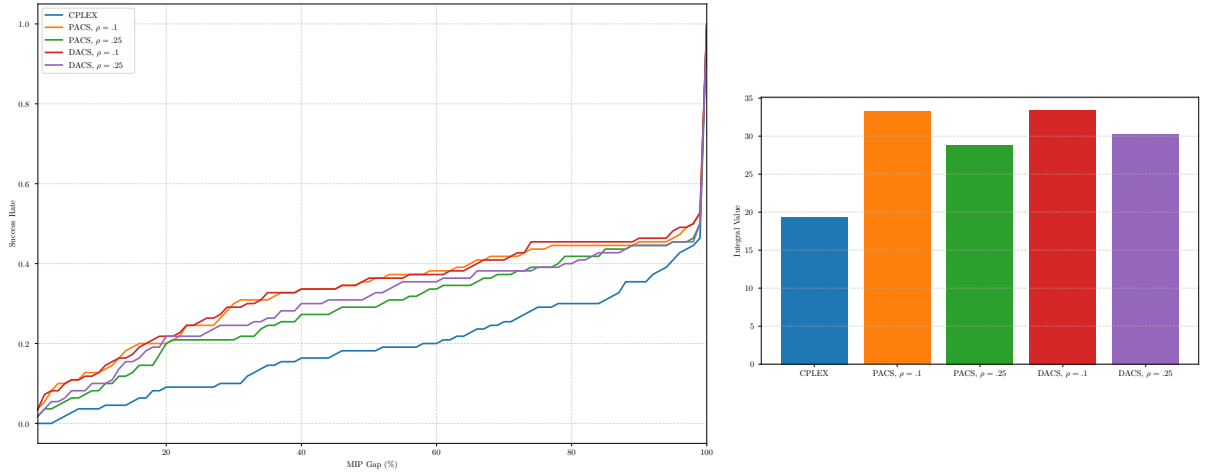


Figure 4.6: Success Rate vs. MIP Gap Plot for Fixed vs. Dynamically Adjusted ρ Test

As evident from Figures 4.5 and 4.6, there is no significant difference in performance between the two strategies, both in terms of success rate and solution quality.

Considering the results from the previous experiment, the dynamic adjustment strategy for ρ is preferred due to its reduced sensitivity to the initial parameter choice. Consequently, the final PACS framework adopts dynamic variable fixing with an initial $\rho = .1$ and $\theta = .25$.

4.2.4 Initial Solution Construction Test

This experiment evaluates different strategies for constructing the initial solution, with the dual objectives of identifying the most effective parameter settings and, if possible, reducing parameter dependence. Specifically, the initialization strategy described in Section 3.1 is analyzed with $\theta \in \{.25, 1\}$. The choice $\theta = .25$ encourages the algorithm to produce solutions of higher quality with respect to the original MIP objective, whereas $\theta = 1$ corresponds to constructing a solution by randomly fixing variables within their bounds. These two strategies are compared with the parameter-free approach introduced in Section 3.3.3.

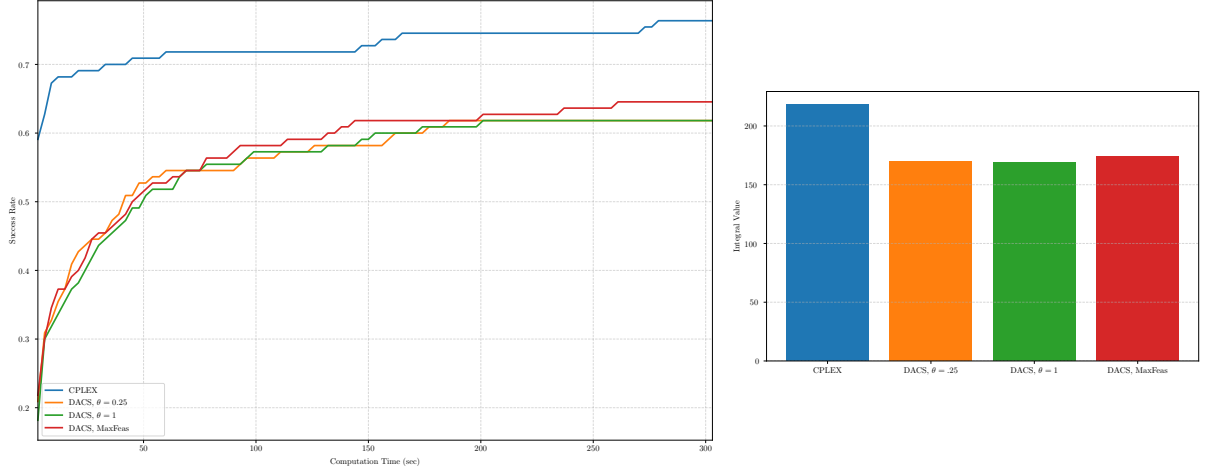


Figure 4.7: Success Rate vs. Computation Time for Initial Solution Construction Test

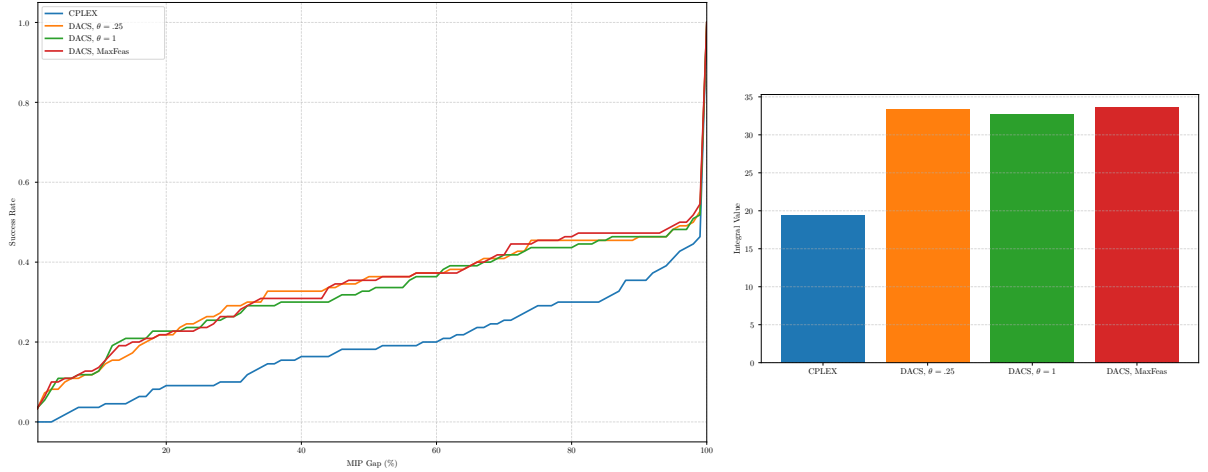


Figure 4.8: Success Rate vs. MIP Gap Plot for Initial Solution Construction Test

Figures 4.7 and 4.8 indicate that overall performance is broadly comparable across all strategies. However, a closer inspection of the integral values reveals that the parameter-free strategy achieves the best results: 173.66 vs. 169.98 in terms of success rate, and 33.63 vs. 33.40 in terms of MIP gap integrals (MaxFeas vs. DACS, $\theta = .25$). Moreover, since the parameter-free approach eliminates the need to tune θ , it represents a more robust and practical choice.

Consequently, based on this test, the best configuration of PACS employs a variable fixing strategy with dynamic adjustment of ρ initialized at $\rho = .1$, combined with a parameter-free initial solution construction strategy that maximizes feasibility.

4.2.5 Slack Upper Bound Enforcement Test

In this experiment, the PACS algorithm—tuned with the best strategies and parameters identified in the previous tests—was evaluated to assess the effectiveness of the slack upper bound enforcement described in Section 3.4.1.

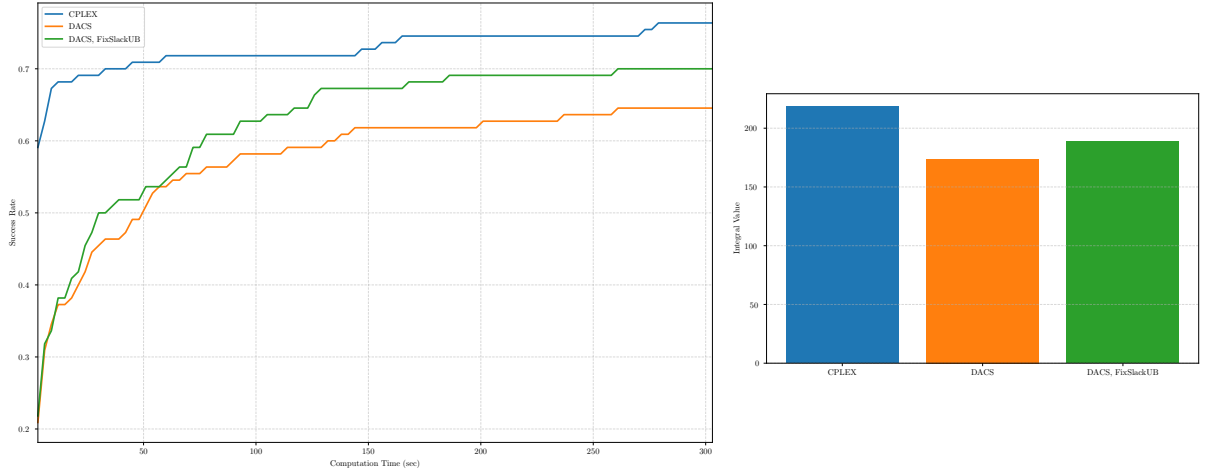


Figure 4.9: Success Rate vs. Computation Time for Slack Upper Bound Enforcement Test

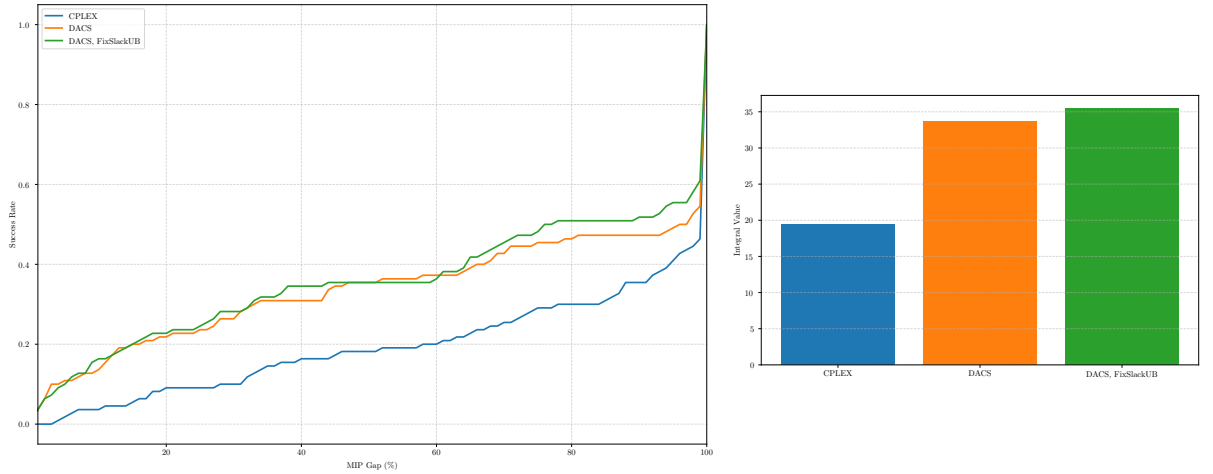


Figure 4.10: Success Rate vs. MIP Gap for Slack Upper Bound Enforcement Test

As shown in Figure 4.9, this strategy narrows the gap between PACS and standalone CPLEX in terms of success rate. It is worth noting, however, that CPLEX solves around 60% of the instances within the very first seconds, whereas PACS requires between 75 and 125 seconds to reach the same level of solved instances.

On the other hand, Figure 4.10 indicates that this modification leads to a noticeable improvement in solution quality. Compared with the previously tuned PACS configuration, the enforced slack upper bound yields consistently better-quality solutions.

Therefore, the final PACS configuration integrates slack upper bound enforcement, as it enhances both efficiency and solution quality.

4.2.6 Budget Constraint Removal Test

This experiment evaluates the impact of removing the budget constraint from the OMIP formulation. Intuitively, since the slack upper bound enforcement already fulfills the role of the budget constraint, its explicit inclusion should be unnecessary and potentially redundant.

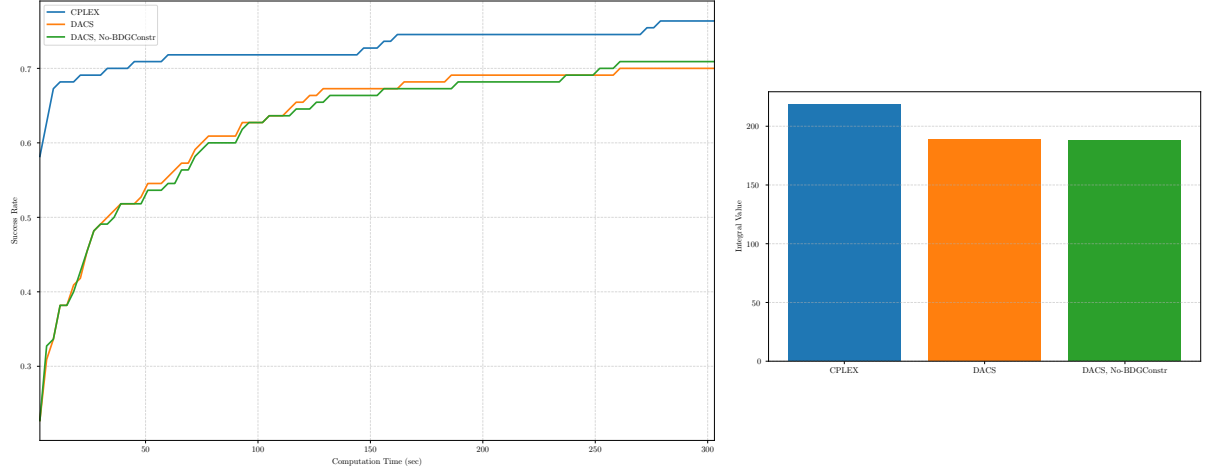


Figure 4.11: Success Rate vs. Computation Time for Budget Constraint Removal Test

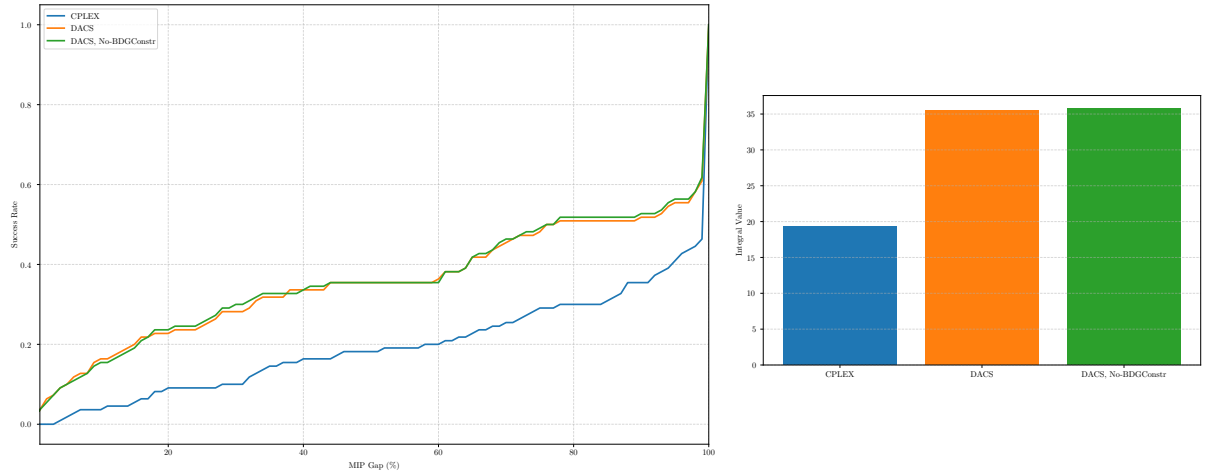


Figure 4.12: Success Rate vs. MIP Gap for Budget Constraint Removal Test

As shown in Figures 4.11 and 4.12, removing the budget constraint slightly improves both success rate and solution quality. Beyond this performance gain, the simplification also reduces model complexity.

Therefore, the budget constraint is omitted in the final PACS framework, as its enforcement through slack upper bounds proves both sufficient and more efficient.

4.2.7 WalkMIP Strategy Test

The final experiment evaluates the performance of the WalkMIP strategy, a more sophisticated variable fixing approach compared to the original PACS method. The key parameter here is the probability p , which governs the likelihood of selecting a minimum-damage move rather than a random move. In practice, higher values of p increase the chance of random moves being chosen. To investigate this effect, p was tested with values $\{.6, .7, .8\}$.

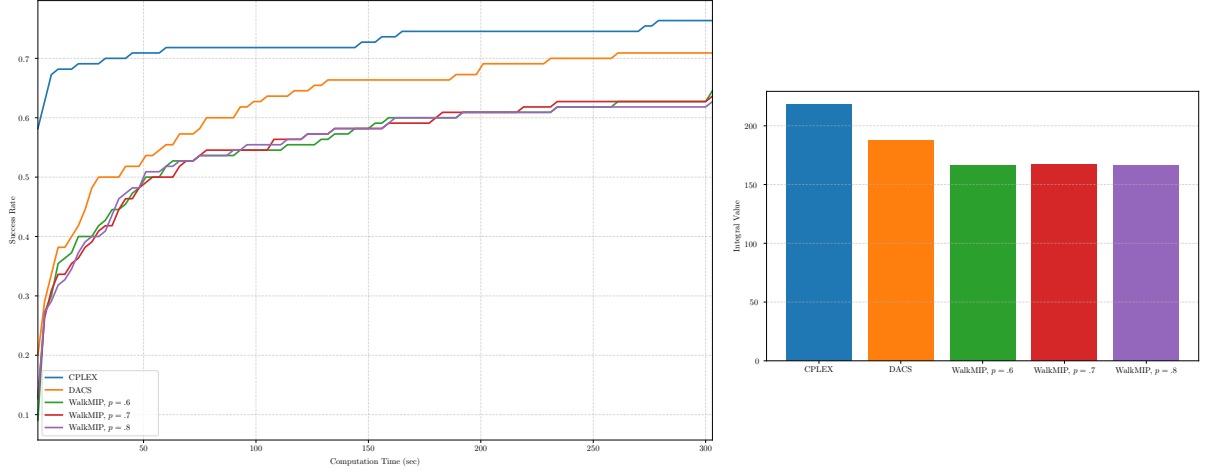


Figure 4.13: Success Rate vs. Computation Time for WalkMIP Strategy Test

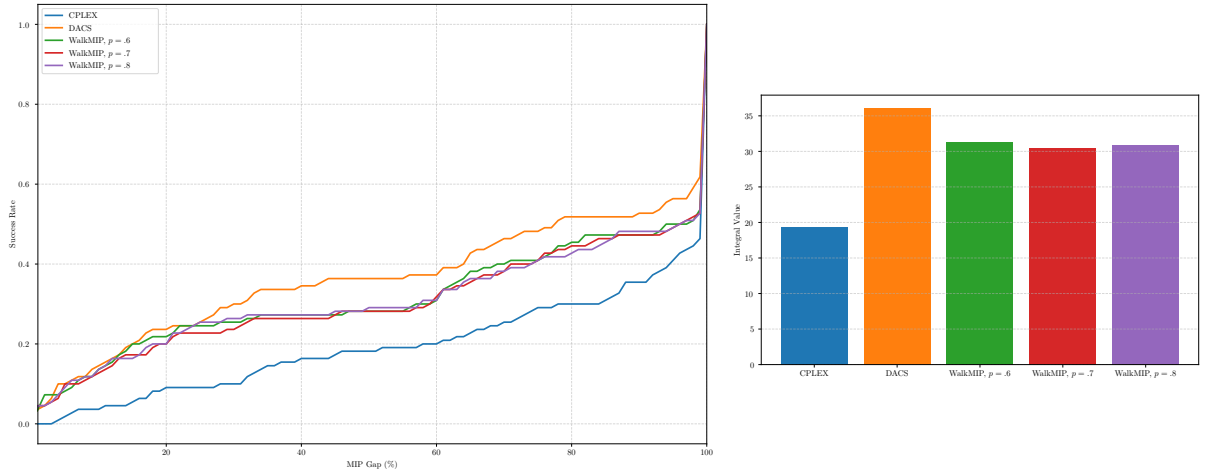


Figure 4.14: Success Rate vs. MIP Gap for WalkMIP Strategy Test

As shown in Figures 4.13 and 4.14, the simpler and more direct variable fixing strategy consistently outperforms WalkMIP in both success rate and solution quality.

For this reason, WalkMIP was not included in the final PACS framework. The final configuration of PACS therefore incorporates all modifications validated up to the budget constraint removal test, with parameters set as follows: dynamic adjustment of ρ , initialized at $\rho = .1$, parameter-free initial solution construction, slack upper bound enforcement, and removal of the budget constraint.

Chapter 5

Conclusion and Future Work

In this thesis, several techniques have been explored to refine and tune the original PACS algorithm into a more general-purpose and largely parameter-free strategy suitable for integration into modern MIP solvers. The experimental evaluation presented in Section 4.2 demonstrated that the tuned algorithm consistently improves both success rate and solution quality over the baseline PACS.

Key contributions include:

1. the introduction of a dynamic adjustment mechanism for the fixing parameter ρ , which reduces sensitivity to the initial parameter choice;
2. the adoption of a parameter-free initialization strategy that maximizes feasibility while eliminating the need to tune θ ;
3. the incorporation of slack enforcement and budget constraint removal, which simplify the formulation while further improving performance.

Together, these refinements yield a lightweight and standalone algorithm that can be deployed in the early stages of MIP optimization, helping to speed up computation and reduce dependence on manual parameter tuning. In this sense, PACS provides both robustness and efficiency, making it a valuable addition to the toolbox of modern MIP solvers.

5.1 Future Work

Building on these results, several promising directions remain open for further research:

- **Refined fixing strategies:** While WalkMIP did not outperform the simpler PACS fixing strategy in this study, exploring refined or hybrid versions could lead to more effective diversification without sacrificing efficiency.
- **Solver generalization:** The present experiments focused on IBM ILOG CPLEX, but evaluating PACS within other state-of-the-art solvers such as GUROBI would provide a broader assessment of its robustness and portability.

- **Lightweight optimization of partial solutions:** The current approach relies on the OMIP formulation to refine partial feasible solutions, which may introduce computational overhead. Investigating more efficient or lightweight refinement mechanisms could further accelerate the overall framework.

Pursuing these directions would not only deepen the understanding of PACS but also help to position it as a general-purpose, solver-agnostic, and practical strategy for modern mixed-integer programming.

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