Local Volatility Surface Calibration: An Arbitrage-Free Implementation

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Abstract

This paper presents a comprehensive implementation of the Dupire (1994) local volatility framework for equity option pricing. We develop a production-quality pipeline that transforms market-observed implied volatilities into arbitrage-free local volatility surfaces. The methodology employs Stochastic Volatility Inspired (SVI) parametrization for smile interpolation, coupled with explicit convexity enforcement to guarantee no-arbitrage conditions. Through systematic optimization, we achieve zero post-enforcement arbitrage violations while maintaining excellent calibration quality (RMSE $< 0.5\%$). The implementation is validated on U.S. equity options data from WRDS OptionMetrics, demonstrating robustness across different underlyings (AAPL, TSLA, NVDA) and market conditions. This work contributes both a rigorous theoretical framework and a practical tool suitable for academic research and industry applications.

Contents

1 Introduction

1.1 Motivation

The accurate modeling of volatility surfaces is fundamental to modern derivatives pricing and risk management. While the Black-Scholes (1973) model assumes constant volatility, market-observed option prices exhibit the well-documented "volatility smile" phenomenon, where implied volatility varies systematically with strike and maturity [8].

Local volatility models, pioneered by Dupire (1994) [2] and Derman & Kani (1994) [3], provide a deterministic volatility function $\sigma_{\text{local}}(S, t)$ that perfectly calibrates to all vanilla option prices while maintaining theoretical consistency. This approach has become the industry standard for pricing exotic options and managing volatility risk.

1.2 Research Objectives

This research addresses three fundamental questions:

1. **Calibration**: How can we efficiently extract a smooth, arbitrage-free implied volatility surface from discrete market option prices?

2. **Transformation**: What is the optimal methodology for converting the implied volatility surface to local volatility via Dupire's formula?

3. **Validation**: How do we ensure the resulting surface satisfies no-arbitrage conditions and produces stable derivatives?

1.3 Key Contributions

Our implementation makes several contributions to the existing literature:

- **SVI Calibration with Tighter Constraints**: We demonstrate that restricting the SVI parameter space (Section 4.4.1) reduces arbitrage violations from 18% to approximately 10% in pre-enforcement measurements.

- **Explicit Convexity Enforcement**: Unlike standard approaches that rely solely on smoothing, we implement direct enforcement of monotonicity and convexity conditions (Section 3.3.3), achieving 0% post-enforcement violations.

- **Savitzky-Golay Derivatives**: We employ polynomial least-squares filtering for numerical differentiation (Section 3.3), which preserves surface features better than traditional finite differences.

- **Comprehensive Validation**: The framework is tested across multiple equity underlyings with rigorous quality metrics (RMSE, arbitrage violations, numerical stability).

1.4 Document Structure

The remainder of this paper is organized as follows. Section 2 reviews the theoretical foundations of local volatility and the SVI parametrization. Section 3 describes our implementation methodology, including data processing, calibration, and validation procedures. Section 4 documents the systematic optimization process that led to our final arbitrage-free implementation. Section 5 presents empirical results and validation on real market data. Section 6 concludes with implications for research and practice.

2 Theoretical Framework

2.1 Dupire's Local Volatility Model

The local volatility model posits that the underlying asset price S_t follows a diffusion process with deterministic, state-dependent volatility:

$$dS_t = \mu S_t dt + \sigma_{\text{local}}(S_t, t) S_t dW_t \tag{1}$$

where μ is the drift rate and W_t is a standard Brownian motion.

Theorem 1 (Dupire's Formula, 1994)**.** *Given a continuum of European call option prices $C(K, T)$ for all strikes K and maturities T, the local volatility function is uniquely determined by:*

$$\sigma^2_{local}(K, T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} \tag{2}$$

where r is the risk-free interest rate.

Sketch. The result follows from the Fokker-Planck (forward Kolmogorov) equation for the risk-neutral probability density of the terminal asset price, combined with the fundamental pricing equation. See Dupire (1994) for the complete derivation. □

2.2 No-Arbitrage Conditions

For a call price surface to be arbitrage-free, it must satisfy:

Proposition 2 (Call Price Monotonicity)**.** *The call price is monotonically decreasing in strike:*

$$\frac{\partial C}{\partial K} \leq -e^{-rT} \tag{3}$$

This ensures non-negative prices for call spread strategies.

Proposition 3 (Call Price Convexity)**.** *The call price is convex in strike:*

$$\frac{\partial^2 C}{\partial K^2} \geq 0 \tag{4}$$

This ensures non-negative prices for butterfly spread strategies.

Violations of these conditions indicate arbitrage opportunities and must be eliminated for a valid local volatility surface.

2.3 SVI Parametrization

The Stochastic Volatility Inspired (SVI) model [4] provides a parsimonious parametric form for the total implied variance:

Definition 1 (SVI Raw Parametrization)**.** *For a given maturity T, the total implied variance as a function of log-moneyness $k = \ln(K/F)$ is:*

$$w(k;\theta) = a + b\left[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right] \tag{5}$$

where $\theta = (a, b, \rho, m, \sigma)$ are parameters with economic interpretations:

- *a: overall level of variance (vertical translation)*
- *$b > 0$: slope of the wings*
- *$-1 < \rho < 1$: correlation (skew parameter)*
- *m: horizontal translation (where smile is centered)*
- *$\sigma > 0$: ATM curvature*

The SVI model has several advantages for practical implementation:

1. **Analytic Form**: Closed-form expressions for implied volatility
2. **Flexibility**: Can capture various smile shapes (skew, smile, smirk)
3. **No-Arbitrage**: Satisfies butterfly arbitrage conditions under parameter restrictions
4. **Interpolation**: Provides smooth extrapolation beyond observed strikes

3 Methodology

3.1 Data Pipeline

3.1.1 Data Source

We employ WRDS OptionMetrics, an institutional-grade database providing:

- End-of-day option prices with pre-computed implied volatilities
- Coverage: 1996–present for U.S. equity options
- Quality controls: filtered for bid-ask errors, early exercise, corporate actions
- Greeks: delta, gamma, vega computed using proprietary models

3.1.2 Data Filters

To ensure calibration quality, we apply the following filters:

Table 1: Data Quality Filters

Filter	Threshold	Rationale
Bid-Ask Spread	$< 10\%$ of mid	Exclude illiquid/stale quotes
Moneyness (K/S)	$[0.80, 1.20]$	Focus on liquid near-money options
Time to Maturity	[7 days, 1 year]	Avoid short-term gamma, long-term model error
Volume	≥ 10 contracts	Minimum liquidity threshold
Implied Volatility	$(0, 200\%)$	Sanity check for errors

3.2 SVI Calibration Procedure

3.2.1 Slice-by-Slice Fitting

For each maturity T_i, we calibrate SVI parameters by minimizing:

$$\min_{\theta_i} \sum_{j=1}^{N_i} \left[w(k_j; \theta_i) - w_j^{\text{market}}\right]^2 + \lambda \cdot P(\theta_i) \tag{6}$$

where $w_j^{\text{market}} = (\sigma_j^{\text{BS}})^2 \cdot T_i$ is the market total variance and $P(\theta_i)$ is an arbitrage penalty function.

3.2.2 Parameter Constraints

Based on theoretical requirements and empirical testing (Section 4.4.1), we impose:

Table 2: SVI Parameter Bounds

Parameter	Original Bounds	Optimized Bounds
a	$[0, 1.0]$	$[0.001, 0.6]$
b	$[0.001, 2.0]$	$[0.02, 0.8]$
ρ	$[-0.99, 0.99]$	$[-0.90, 0.90]$
m	$[-1.0, 1.0]$	$[-0.3, 0.3]$
σ	$[0.001, 1.0]$	$[0.08, 0.4]$

Rationale: Tighter bounds prevent pathological smile shapes that lead to arbitrage violations when interpolated across maturities. The optimized bounds were determined through systematic grid search (documented in Section 4).

3.2.3 Maturity Interpolation

SVI parameters are interpolated across maturities using:

- **Cubic splines** when ≥ 4 maturity slices available
- **Linear interpolation** otherwise
- **Bound enforcement**: $\rho \in [-0.95, 0.95]$ post-interpolation

3.3 Dupire Formula Implementation

3.3.1 Call Price Computation

From the calibrated SVI surface, call prices are computed using Black-Scholes:

$$C(K,T) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{7}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{8}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{9}$$

where $\sigma = \sqrt{w(k;T)/T}$ is the implied volatility from SVI.

3.3.2 Numerical Differentiation

We employ **Savitzky-Golay filters** [9] for computing derivatives:

Algorithm 1 Local Volatility Computation

Input: Call price grid $C(K_i, T_j)$, grid spacing ΔK, ΔT
Apply light Gaussian smoothing: $C_{\text{smooth}} = G_{\sigma=1.0}(C)$
Enforce convexity (Algorithm 2)
Compute strike derivatives using Savitzky-Golay filter (window=11, order=3):
  $\frac{\partial C}{\partial K} = \text{SavGol}(C_{\text{smooth}}, \text{deriv} = 1)$
  $\frac{\partial^2 C}{\partial K^2} = \text{SavGol}(C_{\text{smooth}}, \text{deriv} = 2)$
Compute time derivative using Savitzky-Golay filter (window=7, order=3):
  $\frac{\partial C}{\partial T} = \text{SavGol}(C_{\text{smooth}}, \text{deriv} = 1)$
Apply Dupire formula: $\sigma^2_{\text{local}} = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$
Clip to realistic range: $\sigma_{\text{local}} \in [0.05, 0.80]$[1]
Apply final smoothing: $\sigma_{\text{local}} = G_{\sigma=1.5}(\sigma_{\text{local}})$
Output: Local volatility surface $\sigma_{\text{local}}(K,T)$

3.3.3 Explicit Convexity Enforcement

Justification: This explicit enforcement (Algorithm 2) guarantees that Dupire's formula denominator remains positive, preventing numerical instabilities and ensuring the local volatility is well-defined everywhere.

4 Systematic Optimization

4.1 Initial Problem Statement

The baseline implementation exhibited approximately 18% convexity violations (measured using coarse metrics). Academic standards typically require violations below 5% for publication-quality research.

[1] The clipping bounds [5%, 80%] were chosen based on historical volatility ranges for U.S. equities. The lower bound prevents numerical instabilities from near-zero denominators, while the upper bound excludes unrealistic volatility levels that arise from edge effects near grid boundaries.

Algorithm 2 Convexity Enforcement

Input: Smoothed call price grid $C(K_i, T_j)$
for each maturity j **do**
  // Enforce monotonicity: $C(K)$ decreasing in K
  $C(:, j) \leftarrow \text{CumulativeMinimum}(C(:, j))$
  for iteration = 1 to 3 **do**
    for each interior point i **do**
      Compute second difference: $\Delta^2 C_i = C_{i+1} - 2C_i + C_{i-1}$
      if $\Delta^2 C_i < 0$ **then**
        // Violates convexity, adjust to midpoint
        $C_i \leftarrow \frac{C_{i-1}+C_{i+1}}{2}$
      end if
    end for
  end for
end for
Apply light smoothing to remove discontinuities: $C \leftarrow G_{\sigma=0.5}(C)$
Output: Arbitrage-free call price grid

4.2 Hypothesis Testing

4.2.1 Smoothing Parameter Grid Search

We first hypothesized that increasing Gaussian smoothing would reduce violations. A systematic grid search over $\sigma_{\text{IV}} \in [2.5, 5.0]$ and $\sigma_{\text{call}} \in [2.5, 5.0]$ yielded:

Table 3: Smoothing Parameter Grid Search Results[2]

Configuration	$\boldsymbol{\sigma}_{\text{IV}}$	$\boldsymbol{\sigma}_{\text{call}}$	Pre-Violations	Result
Original	2.5	3.0	10.66%	Baseline
Moderate	3.5	3.5	10.72%	Worse
Balanced	4.0	4.0	10.76%	Worse
Maximum	5.0	5.0	10.90%	Worse

Key Finding: Increasing smoothing *worsens* violations, contrary to intuition. The small magnitude of differences ($< 0.3\%$) indicates that smoothing is not the root cause of violations—the problem is structural rather than numerical noise.

4.3 Root Cause Analysis

Investigation revealed three fundamental issues:

1. **SVI Parameter Freedom**: Unconstrained parameters allowed extreme smile shapes that violate convexity when interpolated across maturities.

2. **Post-Calibration Smoothing**: Applying heavy Gaussian smoothing after SVI calibration introduced new violations by distorting the carefully calibrated smile.

[2] Grid search conducted on AAPL options dated 2025-08-29 with $N = 198$ contracts across 12 maturity slices. Violations measured after SVI calibration with tightened parameter bounds (Table 2).

3. **Lack of Explicit Enforcement**: No mechanism to guarantee convexity before computing Dupire derivatives.

4.4 Implemented Solutions

4.4.1 SVI Parameter Tightening

Through empirical testing, we determined optimal parameter bounds (Table 2). The tightening process:

- Reduced parameter a maximum from 1.0 to 0.6 (prevents excessive variance levels)
- Increased parameter b minimum from 0.001 to 0.02 (ensures non-trivial smile slope)
- Tightened ρ to $[-0.90, 0.90]$ (prevents extreme skew)
- Significantly tightened m to $[-0.3, 0.3]$ (keeps smile centered)
- Tightened σ to $[0.08, 0.4]$ (realistic ATM curvature)

Impact: RMSE improved from 0.82% to 0.32%, demonstrating that tighter constraints do not sacrifice fit quality.

4.4.2 Optimized Smoothing Strategy

- **IV surface**: Reduced from $\sigma = 2.5$ to $\sigma = 1.5$ (lighter, preserves smile features)
- **Call prices**: $\sigma = 1.0$ before enforcement, then $\sigma = 0.5$ after
- **Final local vol**: $\sigma = 1.5$ to remove any remaining discontinuities

4.4.3 Explicit Enforcement

Implementation of Algorithm 2 ensures:

- Monotonicity: enforced via cumulative minimum
- Convexity: enforced via iterative midpoint adjustment
- Smoothness: light post-enforcement smoothing removes discontinuities

4.5 Final Results

Table 4: Quality Metrics: Before and After Optimization

Metric	Target	Before	After
Pre-enforcement violations	$< 15\%$	18%	10%
Post-enforcement violations	0%	N/A	**0%**
SVI fit RMSE	$< 1\%$	0.82%	**0.32%**
Local vol range	Realistic	5-80%	5-80%
NaN/Inf values	0	0	0

Note: The 18% baseline reflects the original implementation with wide SVI parameter bounds ($\rho \in [-0.99, 0.99]$, $m \in [-1.0, 1.0]$). After tightening constraints (Table 2), pre-enforcement violations reduced to ∼10%, demonstrating the importance of parameter restrictions. The 10.66% in Table 3 represents violations *after* parameter tightening.

5 Empirical Results

5.1 Calibration Quality

Table 5: SVI Calibration Results by Ticker

Ticker	Date	Options	Slices	RMSE
AAPL	2025-08-29	198	12	0.13%
TSLA	2025-08-29	245	14	0.28%
NVDA	2025-08-29	312	16	0.19%

All tickers achieve RMSE well below the 1% threshold, demonstrating excellent calibration quality while maintaining arbitrage-free constraints.

5.2 Arbitrage Validation

Table 6: Arbitrage Violations by Ticker

	Pre-Enforcement		Post-Enforcement	
Ticker	**Monoton.**	**Convex.**	**Monoton.**	**Convex.**
AAPL	0.00%	10.16%	0.00%	**0.00%**
TSLA	0.00%	9.84%	0.00%	**0.00%**
NVDA	0.00%	11.23%	0.00%	**0.00%**

Interpretation: Pre-enforcement violations ($\sim 10\%$) are numerical artifacts inherent to discrete grid approximations. The enforcement mechanism successfully eliminates all violations, yielding a rigorously arbitrage-free surface used for local volatility computation.

Note on Monotonicity: Monotonicity violations are absent even pre-enforcement due to the SVI model's inherent structure, which guarantees decreasing call prices in strike. Convexity violations ($\sim 10\%$) arise from numerical differentiation and maturity interpolation artifacts.

5.3 Local Volatility Characteristics

The local volatility ranges are economically reasonable, with higher-volatility stocks (TSLA) exhibiting higher mean local volatility, as expected.

Table 7: Local Volatility Surface Statistics

Ticker	Mean	Std Dev	Min	Max
AAPL	29.08%	17.56%	5.00%	80.00%
TSLA	42.15%	21.33%	5.00%	80.00%
NVDA	35.72%	19.44%	5.00%	80.00%

6 Conclusions and Future Work

6.1 Summary of Contributions

This research presents a production-quality implementation of the local volatility framework with the following key achievements:

1. **Arbitrage-Free Surfaces**: Zero post-enforcement violations across all tested equities
2. **Excellent Calibration**: SVI RMSE consistently below 0.5%
3. **Numerical Stability**: No NaN/Inf values, robust derivatives
4. **Systematic Optimization**: Documented improvement from 18% to 0% violations
5. **Academic Rigor**: Literature-backed methodology suitable for publication

6.2 Academic Implications

The explicit enforcement mechanism (Algorithm 2) addresses a gap in existing literature. While Andersen & Brotherton-Ratcliffe (2005) and others acknowledge the need for enforcement in discrete implementations, few published works provide detailed algorithms. Our contribution makes the process transparent and reproducible.

6.3 Practical Applications

This framework is suitable for:

- **Exotic Options Pricing**: Risk-neutral path simulation under local vol
- **Model Risk Assessment**: Comparison with stochastic vol models (Heston, SABR)
- **Hedging Strategies**: Delta/vega computation using local vol Greeks
- **Academic Research**: Time-series analysis of local vol dynamics

6.4 Future Research Directions

Several extensions merit investigation:

1. **Extended SVI (eSSVI)**: Gatheral & Jacquier (2014) provide an extended parametrization with additional flexibility for extreme strikes.

2. **Stochastic Local Volatility**: Combine local vol with stochastic volatility to capture both smile and path-dependency.

3. **Machine Learning Approaches**: Neural networks for arbitrage-free interpolation (Hernandez 2017).

4. **High-Frequency Dynamics**: Intraday local vol surface evolution and microstructure effects.

5. **Multi-Asset Extensions**: Local correlation surfaces for basket options.

6.5 Code Availability

The complete implementation, including:

- Python source code (`svi_calibration.py`, `local_vol_builder.py`)
- Jupyter notebook with detailed explanations (`local_vol_notebook.ipynb`)
- Validation scripts and test cases
- Documentation (technical report, optimization summary)

is available at: https://github.com/francesco-degirolamo/local-vol-surface

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