Dynamic Games with Noisy Informational Asymmetries*

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May 12, 2025

Abstract

We define and establish the existence of constrained equilibria and trembling hand perfect equilibria in infinite dynamic games with asymmetric and imperfect information. These results rely on the novel notion of sequential absolute continuity, which extends Milgrom and Weber's (1985) absolute continuity condition to dynamic games. We show that sequential absolute continuity holds in a broad class of games with "noisy informational asymmetries," in which players' private information includes some idiosyncratic noise. In Markov settings, our approach allows us to prove the existence of Markov equilibria, where strategies depend only on current payoff-relevant variables.

KEYWORDS: dynamic games, trembling hand perfect equilibrium, Markov perfect equilibrium, absolutely continuous information.

JEL CODES: C72, C73.

*We are thankful to Pierpaolo Battigalli, Sylvain Chassang, Juan Escobar, Hezekiah Grayer, Faruk Gul, Adam Kapor, Alessandro Lizzeri, Benjamin Matta, Stephen Morris, Roger Myerson, Pietro Ortoleva, Wolfgang Pesendorfer, Luciano Pomatto, Giulio Principi, Phil Reny, Luca Rigotti, Larry Samuelson, Fedor Sandomirskiy, Ali Shourideh, Marciano Siniscalchi, Can Urgun, Richard Van Weelden, Mark Whitmeyer, and Leeat Yariv for their insightful comments and discussions. We are grateful to seminar participants at Princeton University, the University of Naples Federico II, the 34th Stony Brook Conference on Game Theory, the Cowles Foundation 2024 Conference on Economic Theory, the 2024 North American Summer Meeting of the Econometric Society, and the 25th ACM Conference of Economics and Computation.

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1 Introduction

Our paper studies the existence of equilibria in infinite dynamic games with asymmetric and imperfect information. We propose a novel approach based on an assumption we call sequential absolute continuity, which extends Milgrom and Weber's (1985) absolute continuity condition to dynamic settings. Under this assumption, we show the existence of constrained equilibria and trembling hand perfect equilibria. A key insight from our analysis is that in many games of economic interest, our assumption is satisfied, and thus equilibrium existence is guaranteed, as long as there is some uncertainty in players' knowledge of the opponent's information—even a bit idiosyncratic noise suffices. Furthermore, our approach leads to a novel method for establishing the existence of equilibria that depend only on current payoff-relevant variables, or Markov equilibria, which can be applied to settings with or without asymmetric information.

We focus on a general class of dynamic games that encompasses but is not limited to multi-stage games. In our framework, a state of the world evolves stochastically based on previous states and actions, while players receive signals about the history of the game, which they may not directly observe. The state and signal spaces are general measure spaces, while players' action sets are countable. Payoffs satisfy two properties: they have a finite upper bound and exhibit a form of continuity in actions.

We introduce equilibrium concepts for our class of games. Building on Selten (1975), we define *constrained equilibrium* as a strategy profile where players are constrained to assign positive probability to all available actions while optimizing within these constraints. A *trembling hand perfect equilibrium*, in turn, is defined as the limit of constrained equilibria as constraints vanish, ensuring that optimal strategies remain robust to small errors—or "trembles"—in players' actions.

The existence of these equilibria (Theorems 1 and 2) hinges on a key assumption we term *sequential absolute continuity* (SAC). SAC imposes two requirements on the transition probability over the history of private signal profiles conditional on the action history. First, it must be *absolutely continuous* with respect to the product of the marginal probability measures of each player's history of private signals. Second, it must be *bounded and continuous* in the action history according to a novel norm on transition probabilities, which implies continuity in the total variation norm. SAC yields continuity of payoffs in strategies for a suitably defined topology, thereby allowing the use of fixed point arguments to establish equilibrium existence.

Next, we study sufficient conditions for SAC and show it holds when players receive "noisy signals" (Propositions 1 and 2). More precisely, we consider settings where each player's signal consists of a fundamental component, potentially common across players, and an idiosyncratic noise term. Assuming appropriate absolute continuity properties between these two (which are satisfied by independence), we show that the absolute continuity part of SAC holds when players' observations are not sufficient to perfectly infer the fundamental components. The second requirement of SAC—boundedness and continuity of the signal distribution—is satisfied when the fundamental signal exhibits continuity in the topology of weak convergence of probability measures (a considerably weaker form of continuity) and satisfies additional absolute continuity conditions. Intuitively, introducing noise to players' observations allows signals to vary more smoothly with changes in the history of actions, preventing the abrupt changes in signal probabilities that occur when actions are perfectly observable.

Taken together, these results have an immediate implication: if we simply add any amount of noise that is absolutely continuous with respect to the Lebesgue measure, we can guarantee existence of a trembling hand perfect equilibrium (Corollary 1). This means that the addition of even an arbitrarily small amount of such noise suffices to restore existence in *any game* with finite actions or that satisfies the weak continuity in actions outlined above, including games in which the literature has demonstrated non-existence. Through two running examples, we illustrate how adding noise simultaneously resolves two issues: *strategic entanglement*, which generates discontinuity of the expected payoffs in players' strategies, precluding classical existence arguments; and in dynamic settings, the *signal discontinuity* with respect to previous actions, as seen in non-existence examples like that of Harris, Reny, and Robson (1995). The conditions of Corollary 1 are met by commonly used noise distributions, including independent uniform and jointly normal, making our results applicable across various economic models.

To illustrate the applicability of our results, consider the following class of games for which the prior literature had not ascertained whether sequentially rational equilibria exist. Our results imply the existence of trembling hand perfect equilibria for this class. In these games, at the beginning of each period, players receive private signals that combine two components: a fundamental component and a noise term. The fundamental component may be correlated with previous actions and signals, while the noise term is independent and absolutely continuous with respect to the Lebesgue measure (e.g., following normal or uniform distributions). After observing the history of action profiles and receiving their signals, players simultaneously choose actions from finite sets. Payoffs are discounted and depend on current signals and actions. Once actions are selected, the game advances to the next period.

The class of games just described represents various economic applications, which include: (i) sequential oligopolistic competition where the actions are quantities or prices and signals are idiosyncratic demand shocks (e.g., Athey et al., 2004; Athey and Bagwell, 2008); (ii) sequential auctions with interdependent values, where the actions are bids and the signals are informative about the value that the auctioned good has for each potential buyer (e.g., Jofre-Bonet and Pesendorfer, 2003); (iii) global games of regime change where players attack a regime after learning about its strength (e.g., Angeletos et al., 2007).

We apply our approach to the analysis of equilibria in Markov strategies. To this end, we introduce *Markov games with informational asymmetries*, where the primitives (state and signal transitions, payoffs, available actions) are determined solely by variables specific to the current period. The set of states decomposes into a *payoff-relevant* dimension, which determines payoffs and state transitions, and a *payoff-irrelevant* dimension. Private signals about the payoff-relevant state are called *payoff-relevant signals*. A strategy is *Markov* if it conditions only on the current payoff-relevant signal and potentially the current period, and *stationary Markov* if it depends solely on the current payoff-relevant signal, regardless of the period.

For Markov strategies to be optimal, *bygones must be bygones*. We formalize this notion through the following conditions. *Markov information* requires that, when players follow Markov strategies, each player's belief about the current payoff-relevant state depends only on their current payoff-relevant signal. The stronger *stationary Markov information* additionally requires that these beliefs are independent of both the period and the strategies used previously. We complement these conditions by assuming *Markov payoffs*, which requires that payoff-relevant signal profiles determine payoffs.

We establish the existence of equilibria in Markov games under a new assumption which we term Markov absolute continuity (MAC) (Theorem 3). The latter replaces SAC in Markov games, modifying it in two ways. First, it applies only to the transition of payoff-relevant signal profiles. Second, it requires that the absolute continuity condition holds with respect to the product of the marginal measures over private signals, taken not only across players but also across periods. Thus, MAC is neither weaker nor stronger than SAC. Under Markov information, Markov payoffs, and MAC, we show the existence of Markov trembling hand perfect equilibria and, when stationary Markov information holds, stationary

Markov trembling hand perfect equilibria.

Some applications for which our analysis yields novel existence results in Markov games are: (1) games with asynchronous moves, including asynchronous revision games (Kamada and Kandori, 2020), and dynamic cheap talk games (Renault et al., 2013); (2) stochastic games in which players receive both a public and a private shock (Balbus et al., 2013), such as in dynamic oligopolies, where the public and private shocks can be interpreted as demand and firm's costs, respectively.

We establish that our equilibrium concepts satisfy some key sequential rationality properties. For constrained equilibria, we show that conditional on sets of private histories occurring with positive probability, they (i) prescribe optimal strategies, and (ii) optimize their payoffs up to $\nu > 0$ utils, with ν vanishing as the size of trembles approaches zero. This establishes the existence of totally mixed ν-equilibria, where strategy profiles are optimal up to ν utils and assign positive probabilities to all actions. For trembling hand perfect equilibria, we prove that a player's strategy is a best response in continuation games starting from histories where that same player has perfect information about past states and actions. This property implies that trembling hand perfect equilibria are subgame perfect. Furthermore, these equilibria are conditional equilibria, maximizing expected payoffs after any set of private histories occurring with positive probability. Finally, in Markov games satisfying our conditions, we show that stationary Markov perfect equilibria optimize against all possible deviations at almost every payoff-relevant signal.

Section 6 shows that our framework comprises games with stochastic move opportunities where players can choose their actions at random times. This illustrates how our results apply to settings that are more general than discounted games, whereas the majority of the literature on Markov equilibria in stochastic games focuses exclusively on the discounted case. In Section 7, we extend our framework to allow for inactive players, accommodating games where players may not know the order of moves or be oblivious to the number of moves that have occurred in the past.

Related literature. Despite the extensive applications of infinite games with asymmetric information in economics, the existence of equilibria in these settings remains an open question. Even in one-period Bayesian games with finite actions, the literature has documented examples where equilibria fail to exist. For instance, Simon (2003) constructs a game without a Bayes-Nash equilibrium, while Hellman (2014) and Simon and Tomkowicz (2018) provide examples where even approximate equilibria do not exist. In a two-period game with almost perfect information, Harris et al. (1995) construct an example featuring compact action spaces and continuous payoffs that lacks a subgame perfect equilibrium.

Our paper builds upon the seminal works of Milgrom and Weber (1985) and Balder (1988). Milgrom and Weber were the first to introduce the absolute continuity condition to prove the existence of equilibrium in distributional strategies for Bayesian games with metric signal spaces. Balder extended their result to general measure spaces and strategies expressed as transition probabilities, also relying on absolute continuity. We generalize their absolute continuity condition to dynamic settings through our sequential absolute continuity and Markov absolute continuity assumptions. An essential part of our arguments relies on showing that under these assumptions, payoffs are continuous in strategies and Markov strategies, respectively, when employing suitable adaptations of Balder's weak topology. Furthermore, we provide sufficient conditions for absolute continuity based on noisy signals, broadening the applicability of this framework even in static cases.

We contribute to the literature studying sequentially rational equilibria in infinite dynamic games. Inspired by Selten's (1975) seminal work on finite games, we focus on trembling hand perfect equilibria, which we define as limits of constrained equilibria. The work that is closest to our paper is Myerson and Reny (2020), which introduces the concept of perfect conditional ε-equilibria, defined as strategy profiles that can be approximated by a net of conditional ε-equilibria.[1] These equilibria eventually assign positive probability to every possible action and almost every move of nature. Their limiting distributions, as ε tends to zero, are termed perfect conditional equilibrium distributions. However, these distributions may not always be induced by a strategy profile.

A vast literature on stochastic games stemming from the seminal work of Shapley (1953) studies the existence of stationary Markov perfect equilibria in settings where players observe the history of play and the current state, and have discounted payoffs. In this environment, existence generally requires continuity assumptions on the transition of the state across periods.[2] In the absence of such restrictions, equilibria may fail to exist as shown by Levy (2013), and subsequently Levy and McLennan (2015), who demonstrate that non-existence of stationary Markov perfect equilibria may arise even in standard stochastic games with finite action sets where the state transition is absolutely continuous with respect to a fixed measure. The literature on Markov equilibria in standard stochastic games

[1] In a conditional ε-equilibrium, players optimize their payoffs up to ε utils, conditional on every positive measure set of private histories.

[2] Duggan (2012) provides an excellent discussion of these assumptions. Nowak and Raghavan (1992), Duffie et al. (1994), and Duggan (2012) establish the existence of stationary Markov perfect equilibria under restrictions on the state transition and other conditions such as payoff-irrelevant and payoff-relevant noise. Parthasarathy and Sinha (1989), and Nowak (2003) assume stronger restrictions than Levy (2013) to prove existence. He and Sun (2017) unifies these results by assuming the "decomposable coarser transition kernel" condition on the state transition.

typically uses value function approaches to proving existence. In contrast, our approach relies on Markov absolute continuity to show existence of equilibria in strategies that depend only on payoff-relevant information. This allows us to derive new results in a class of games where players may or may not observe the history of play and payoffs are not necessarily discounted.

There is a growing literature that studies the existence of a Markov perfect equilibrium in games with private information. Altman et al. (2008) studies the existence of a stationary Nash equilibrium when each player privately observes the realizations of an associated controlled Markov chain. Balbus et al. (2013) establish the existence of a stationary Markov perfect equilibrium in a setting with strategic complementarities in the presence of public and private shocks. To the best of our knowledge, we are the first to provide conditions for the existence of trembling hand and stationary Markov perfect equilibria under general state and signal spaces.

2 Model

We study dynamic games played in countably many periods $t \in \mathbb{N} := \{1, 2, \dots\}$. Any such game is represented by the following tuple:

$$\Gamma = (N, \Omega, \mathcal{M}, (X_i, S_i, A_i, g_i)_{i \in N}, \mu, \gamma),$$

where

- - $N := \{1, \dots, n\}$ is a finite set of n players.
  - Ω is a measurable set of states of the world, and S_i is a measurable set of private signals for each player $i \in N$. $S := \prod_{i \in N} S_i$ denotes the set of signal profiles.
  - X_i is a countable, compact metric space, endowed with its Borel σ-algebra, representing the action set of each player $i \in N$. The set of action profiles is $X := \prod_{i \in N} X_i$, with generic element $a = (a_1, \dots, a_n)$. The set of histories of action profiles up to period $t \in \mathbb{N}$ is $X^t := \prod_{\ell \leq t} X$, with generic element $a^t = (a_1, \dots, a_t)$. For $\ell \leq t$, the element $a^{t,(\ell)} \in X^\ell$ denotes the truncation of history a^t up to and including period ℓ. The action $a^t_{i,\ell}$ corresponds to player i's move in period-ℓ action profile a^t_ℓ. The sets Ω^t, S^t, and their corresponding elements, such as $\omega^{t,(\ell)}$ or $s^t_{i,\ell}$, are defined analogously.[3]
- $\mathcal{M}(\cdot)$ maps each measurable set to its σ-algebra. For instance, $\mathcal{M}(\Omega)$ denotes the σ-algebra of measurable subsets of Ω. We endow product spaces with their product σ-

[3]For $t \geq 1$, we use the notation $Y^t := \prod_{\ell=1}^t Y$ for any set Y; $Y^0 := \{\emptyset\}$.

algebra, subsets of measurable spaces with their relative σ-algebra,[4] and we assume all singleton sets are measurable.

- The correspondence $A_i : \cup_{t\in\mathbb{N}} S_i^t \times X_i^{t-1} \rightrightarrows X_i$ is non-empty closed-valued, weakly measurable,[5] and specifies the *actions available to player* $i \in N$ as a function of i's private signal and action history.
- The function $g_i : \cup_{t\in\mathbb{N}} \Omega^t \times X^t \to \mathbb{R}$ represents the *flow payoff* received by player $i \in N$ as a function of the history of the states of the world and action profiles. It is measurable and bounded.
- The map $\mu : \cup_{t\in\mathbb{N}\cup\{0\}} \Omega^t \times X^t \to \Delta(\Omega)$ is the *state transition probability*[6] which determines the probability of a new state as a function of the history of the states of the world and action profiles. That is, $\mu(Z|\omega^t, a^t)$ is the probability that the period-$t+1$ state belongs to the set $Z \in \mathcal{M}(\Omega)$ given $(\omega^t, a^t) \in \Omega^t \times X^t$.
- The *signal transition function* $\gamma : \cup_{t\in\mathbb{N}} \Omega^t \times X^{t-1} \to S$ is measurable and determines the signal profiles as a function of the history of the states of the world and action profiles. Denote by $\gamma_i : \cup_{t\in\mathbb{N}} \Omega^t \times X^{t-1} \to S_i$ the projection of γ onto i's private signal in S_i.

Following the representation of Myerson and Reny (2020), we assume that the state of the world evolves stochastically, while signals are deterministic functions of the history of the game. As the state space is a general measurable space, this representation is equivalent to an alternative framework in which the signal realization is also stochastic.

For ease of exposition, in the main body of the paper, we focus on games where players are informed about whether their opponents have moved in the past, i.e., they observe the period $t \in \mathbb{N}$ in which the game is being played. However, most of our results extend to a more general setting in which players may not know the order of moves in the game. This extension is discussed in Section 7.

2.1 Histories, Strategies, and Expected Payoffs

Histories. For $t \in \mathbb{N}$, a *period-t history* $h = (\omega^t, a^{t-1})$ is composed of a *history of the states of the world* $\omega^t \in \Omega^t$, and a *history of action profiles* $a^{t-1} \in X^{t-1}$. The set $\mathcal{H}^t$ contains every period-t history, and $\mathcal{H} := \cup_{t\in\mathbb{N}\cup\{0\}} \mathcal{H}^t$ contains every history.

[4] For every $Z \in \mathcal{M}(Y)$, *Z's relative σ-algebra* is $\mathcal{M}(Z) = \{B \cap Z | B \in \mathcal{M}(Y)\}$.

[5] A correspondence $\phi : Z \rightrightarrows Y$ that maps a measurable space Z to a topological space Y is *weakly measurable* if, for every closed subset $B \subseteq Y$, the set $\{z \in Z : \phi(z) \subseteq B\}$ is measurable. See Definition 18.1 in Aliprantis and Border (2006) and the ensuing discussion.

[6] For measurable spaces Z and Y, the function $\xi : Z \to \Delta(Y)$ is a *transition probability* from Z to Y if $\xi(B|z)$ is measurable in z for every $B \in \mathcal{M}(Y)$; $\Delta(Y)$ is the set of probability measures over Y. A *transition measure* is defined analogously when $\xi : Z \to \mathcal{M}(Y)$ and $\xi(\cdot|z)$ is a measure for every $z \in Z$.

Private histories. For $i \in N$, $t \in \mathbb{N}$, $(s_i^t, a_i^{t-1}) \in S_i^t \times X_i^{t-1}$ is a *period-t private history* of player i. Denote by $\mathcal{H}_i^t := S_i^t \times X_i^{t-1}$ the set of period-t private histories of player i, and by $\mathcal{H}_i := \cup_{t\in\mathbb{N}} \mathcal{H}_i^t$ the set of all private histories of player i. The set $\mathcal{H}_{A_i} := \{(s_i^t, a_i^{t-1}) \in \mathcal{H}_i |\, a_{i,\ell}^{t-1} \in A_i(s_i^{t,(\ell)}, a_i^{t-1,(\ell-1)}),\, \ell \leq t-1\}$ denotes player i's private histories that are compatible with the action correspondence A_i. Similarly $\bar{\mathcal{H}}_{A_i} := \{(s_i^t, a_i^t) \in \mathcal{H}_i |\, a_{i,\ell}^t \in A_i(s_i^{t,(\ell)}, a_i^{t,(\ell-1)}),\, \ell \leq t\}$ denotes i's available private histories including period t's action.

Strategies. A *strategy* of player $i \in N$ is a transition probability $\sigma_i : \mathcal{H}_{A_i} \to \Delta(X_i)$ that maps i's available private histories to probability distributions over i's actions satisfying $\operatorname{supp} \sigma_i(h_i) \subseteq A_i(h_i)$ for every $h_i \in \mathcal{H}_{A_i}$. Denote by Σ_i the set of player i strategies,[7] and by $\sigma = (\sigma_j)_{j\in N} \in \prod_j \Sigma_j := \Sigma$ a strategy profile.

Strategy-induced probabilities. For each player $i \in N$, strategy $\sigma_i \in \Sigma_i$, periods $t, \tau \in \mathbb{N} \cup \{0\}$ with $\tau \leq t$, and history of player i's actions $a_i^\tau \in X_i^\tau$, the strategy induces a *transition probability from S_i^t to X_i^t* defined as:

$$p_i(a_i^t | s_i^t, a_i^\tau, \sigma_i) := \prod_{\ell=\tau+1,\ldots,t} \sigma_i\big(a_{i,\ell}^t | s_i^{t,(\ell)}, a_i^{t,(\ell-1)}\big) \tag{1}$$

when $(s_i^t, a_i^{t,(t-1)}) \in \mathcal{H}_{A_i}$ and $a_i^{t,(\tau)} = a_i^\tau$, and zero otherwise.[8] That is, $p_i(a_i^t|s_i^t, a_i^\tau, \sigma_i)$ represents the probability that player i, using strategy σ_i, takes the sequence of actions in a_i^t that follow a_i^τ when signal history s_i^t is realized. The strategy profile $\sigma \in \Sigma$ induces $p(a^t|s^t, a^\tau, \sigma) := \prod_{j\in N} p_j(a_j^t|s_j^t, a_j^\tau, \sigma_j)$ for $(s^t, a^t) \in S^t \times X^t$, $a^\tau \in X^\tau$. When $\tau = 0$, hence $a^\tau = \emptyset$, we write $p_i(a_i^t|s_i^t, \sigma_i) := p_i(a_i^t|s_i^t, \emptyset, \sigma_i)$ and $p(a^t|s^t, \sigma) := p(a^t|s^t, \emptyset, \sigma)$.

State and signal transitions. Conditional on a history of action profiles $a^{t-1} \in X^{t-1}$ and states of the world $\omega^\tau \in \Omega^\tau$, for $t, \tau \in \mathbb{N} \cup \{0\}$ with $\tau \leq t$, the *probability measure over Ω^t induced by the state transition probability μ* is

$$d\mu_\omega^t(\omega^t|\omega^\tau, a^{t-1}) := \prod_{\ell=\tau}^{t-1} d\mu(\omega_{\ell+1}^t|\omega^{t,(\ell)}, a^{t-1,(\ell)})$$

if $\omega^{t,(\tau)} = \omega^\tau$, and zero otherwise. When $\tau = 0$, hence $\omega^\tau = \emptyset$, we write $d\mu_\omega^t(\omega^t|a^{t-1}) := d\mu_\omega^t(\omega^t|\emptyset, a^{t-1})$.

For every $t \in \mathbb{N}$, $(\omega^t, a^{t-1}) \in \mathcal{H}^t$, denote the *history of realized signals up to period t* by $\gamma^t(\omega^t, a^{t-1}) := (\gamma(\omega^{t,(1)}), \gamma(\omega^{t,(2)}, a^{t-1,(1)}), \ldots, \gamma(\omega^t, a^{t-1}))$. To ease notation, we write

[7] Lemma 4 shows that the set of strategies is non-empty. That is, there exists a measurable selector, with support on the available actions set, as a function of $h_i \in \mathcal{H}_i$. This result follows from the weak measurability of A_i.

[8] We use the convention that $\prod_{t=\tau}^{\tau-1} y_t = 1$ for any sequence $(y_t)_{t\in\mathbb{N}\cup\{0\}}$ in $\mathbb{R}$, and $\tau \in \mathbb{N}$.

$p(a^t|\omega^t,\sigma)$ and $p(a^t|\omega^t,a^\tau,\sigma)$, instead of $p(a^t|\gamma^t(\omega^t,a^{t,(t-1)}),\sigma)$ and $p(a^t|\gamma^t(\omega^t,a^{t,(t-1)}),a^\tau,\sigma)$, respectively.

For $a^{t-1} \in X^{t-1}$, the measure $\mu^t_\omega(\cdot|a^{t-1})$ and the function γ induce a *probability measure over private signal profiles*, $\mu^t_s(B|a^{t-1}) := \mu^t_\omega(\{\omega^t : \gamma^t(\omega^t,a^{t-1}) \in B\}|a^{t-1})$, for $B \in \mathcal{M}(S^t)$. Let $\mu^t_{s_i}(\cdot|a^{t-1})$ be the marginal of μ^t_s on player i's signals in S^t_i.

Finally, for every $t \in \mathbb{N}$, we assume throughout that there exists a transition probability $\mu^t_{s_i|s_i} : S_i^{t-1} \times X^{t-1} \to \Delta(S_i)$ satisfying $d\mu^t_{s_i}(s^t_i|a^{t-1}) = d\mu^t_{s_i|s_i}(s^t_{i,t}|s_i^{t,(t-1)},a^{t-1}) \times d\mu^{t-1}_{s_i}(s_i^{t,(t-1)}|a^{t-1,(t-2)})$.

Expected payoffs. For every $t \in \mathbb{N}$, a strategy profile $\sigma \in \Sigma$, the signal history function γ^t, and the state transition probability μ induce a probability $P^t(\cdot|\sigma)$ over $\Omega^t \times X^t$. Player i's *expected payoff* is $U_i(\sigma) := \sum_{t\in\mathbb{N}} U_{i,t}(\sigma)$, where $U_{i,t}(\sigma) := \int_{\Omega^t\times X^t} g_i(\omega^t,a^t)\, dP^t(\omega^t,a^t|\sigma)$.

Similarly, for $\tau \in \mathbb{N}$, we define player i's *continuation expected payoff* from $\sigma \in \Sigma$ after history $(\omega^\tau,a^{\tau-1}) \in \mathcal{H}^\tau$ as $U_i(\sigma|\omega^\tau,a^{\tau-1}) := \sum_{t\geq\tau} U_{i,t}(\sigma|\omega^\tau,a^{\tau-1})$, where $U_{i,t}(\sigma|\omega^\tau,a^{\tau-1}) := \sum\limits_{a^t\in X^t}\int\limits_{\Omega^t} g_i(\omega^t,a^t)\, dP^t(\omega^t,a^t|\sigma,\omega^\tau,a^{\tau-1})$ and $P^t(\cdot|\sigma,\omega^\tau,a^{\tau-1})$ is a probability over $\Omega^t \times X^t$ induced by σ after the realization of $(\omega^\tau,a^{\tau-1})$.

We impose two requirements on players' expected payoffs throughout our analysis.

First, we assume that a bound on the sum of per-period expected payoffs is finite. Formally, for each $i \in N$,

$$\sum_{t\in\mathbb{N}} \sup_{\sigma\in\Sigma} \left|U_{i,t}(\sigma)\right| < \infty. \qquad \text{(boundedness)}$$

Consider the following condition, which is stronger than boundedness: for each $i \in N$,

$$\sum_{t\in\mathbb{N}} \sup_{(\omega^t,a^t)} |g_i(\omega^t,a^t)| < \infty. \tag{2}$$

This condition holds in games with discounted payoffs where $g_i(\omega^t,a^t) = \delta^t \cdot u_i(\omega^t,a^t)$ for some $\delta \in (0,1)$ and bounded $u_i : \cup_{t\in\mathbb{N}}\Omega^t \times X^t \to \mathbb{R}$, as well as in games where payoffs decline more slowly than geometric decay (e.g., $g_i(\omega^t,a^t) = \frac{1}{t^2} \cdot u_i(\omega^t,a^t)$). However, boundedness is weaker than condition (2). For example, boundedness is satisfied in revision games (Kamada and Kandori, 2020), which end in finite time with probability 1 and have bounded expected length (see Lemma 3), while (2) fails in these games.[9]

Second, we require that expected payoffs, when conditioned on previous signals, be continuous with respect to past actions. We assume there is a transition probability $\mu^t_{\omega|s}$:

[9] Inspired by Fudenberg and Levine (1983), one can define *continuity at infinity* in our setting by requiring $\sup_{\sigma,\sigma'} |\sum_{t=\tau}^\infty U_{i,t}(\sigma) - U_{i,t}(\sigma')| \to 0$ as $\tau \to \infty$. Our boundedness condition is stronger as it implies $\sum_{t=\tau}^\infty \sup_{\sigma,\sigma'} |U_{i,t}(\sigma) - U_{i,t}(\sigma')| \to 0$ as $\tau \to \infty$.

$S^t \times X^{t-1} \to \Delta(\Omega^t)$ such that $d\mu^t_\omega(\omega^t|a^{t-1}) = d\mu^t_{\omega|s}(\omega^t|s^t, a^{t-1}) \times d\mu^t_s(s^t|a^{t-1})$, for $t \in \mathbb{N}$. For every $i \in N$, $t \in \mathbb{N}$, the function $\hat{g}_{i,t} : S^t \times X^t \to \mathbb{R}$ defined as

$$\hat{g}_{i,t}(s^t, a^t) := \int_{\Omega^t} g_i(\omega^t, a^t)\, d\mu^t_{\omega|s}(\omega^t|s^t, a^{t,(t-1)}) \qquad \text{(continuity)}$$

is continuous in a^t for every s^t in a μ^t_s-full measure set.[10] This condition holds automatically when the action set, X, is finite. Notice that we require payoff continuity with respect to the history of action profiles, not signals or states.[11]

2.2 An Application

To motivate our findings we formalize an interesting class of games in which players' signals can be written as a fundamental signal plus an (arbitrarily small) noise term. Existence of an equilibrium in this class follows from Corollary 1 in Section 3.3.

APPLICATION 1 (Dynamic games with Lebesgue signals)**.** Each state of the world $\omega \in \Omega$ can be written as $\omega = (\hat{s}, \epsilon)$:

- $\hat{s} = (\hat{s}_i)_{i \in N}$, where $\hat{s}_i \in \mathbb{R}^{\ell_i}$, $\ell_i \in \mathbb{N}$, represents the *fundamental signal component* of player $i \in N$;
- $\epsilon = (\epsilon_i)_{i \in N}$, where $\epsilon_i \in \mathbb{R}^{\ell_i}$ represents the *idiosyncratic noise term* of player $i \in N$.

Each player $i \in N$ observes a private signal $\gamma_i(\hat{s}, \epsilon) = \hat{s}_i + \epsilon_i$, which combines their fundamental component and noise term. Furthermore, each $\hat{s}_i$ and ϵ_i may or may not affect payoffs.

Assume that: (a) for every $t \in \mathbb{N}$, there exists a continuous and bounded density $f^t(\hat{s}^t, \epsilon^t_1, \ldots, \epsilon^t_n, a^{t-1})$ such that the joint distribution of $\hat{s}^t$ and ϵ^t, conditional on the action profile history $a^{t-1} \in X^{t-1}$, can be written as

$$d\mu^t_{\hat{s},\epsilon}(\hat{s}^t, \epsilon^t_1, \ldots, \epsilon^t_n|a^{t-1}) = f^t(\hat{s}^t, \epsilon^t_1, \ldots, \epsilon^t_n, a^{t-1})\, d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) \times d\lambda^t(\epsilon^t)$$

where $\mu^t_{\hat{s}}$ denotes the marginal with respect to $\hat{s}^t$ and λ^t denotes the Lebesgue measure over real vectors, respectively; (b) for every $i \in N$, and almost surely in the private signal $s^t_i \in S^t_i$, the set of fundamental components which, combined with some noise term, can yield s^t_i has positive measure; (c) for every $t \in \mathbb{N}$, $d\mu^t_{\hat{s}}(\cdot|a^{t-1})$ is continuous in the action

[10] For every $t \in \mathbb{N}$, $B \in \mathcal{M}(S^t)$ is a μ^t_s*-full measure set* if $\mu^t_s(S^t \setminus B|a^{t-1}) = 0$ for every $a^{t-1} \in X^{t-1}$.

[11] In some of the literature, state transitions are modeled as actions taken by nature. In this framework, continuity in actions inherently implies continuity in the state. Harris et al. (1995), He and Sun (2020), and Myerson and Reny (2020), among others, assume this form of state continuity.

profile history $a^{t-1} \in X^{t-1}$ in the topology of *weak convergence of probability measures*.[12]

Common noise structures satisfy assumptions (a)-(c). For instance, these conditions are met when the fundamental signal component is composed of the opponents' actions, and the additive noise term is independent across players and periods and follows a normal or uniform distribution. ◂

Application 1 encompasses numerous economic settings. These include dynamic oligopolistic competition, where firms repeatedly interact in markets and set quantities or prices based on idiosyncratic demand signals (Athey et al., 2004; Athey and Bagwell, 2008); sequential auctions with interdependent values, where buyers submit bids informed by signals about their value for the auctioned good (Jofre-Bonet and Pesendorfer, 2003; Aoyagi, 2003; Skrzypacz and Hopenhayn, 2004); and global games such as currency attacks, where strategic decisions are informed by private signals about fundamental values (Morris and Shin, 1998), and regime change games, where players make strategic decisions based on signals about regime strength (Angeletos et al., 2007). In these applications, the noise structures conform to our assumptions, typically featuring additive independent normal or uniform noise terms.

2.3 Discontinuities and Non-existence

Equilibrium existence is not guaranteed without additional requirements on signal transitions. The following two examples relate non-existence to different forms of discontinuities that can arise in infinite games. The first example shows how players' payoffs can be discontinuous in their strategies—a phenomenon known as *strategic entanglement*.[13] This discontinuity prevents the use of standard fixed-point arguments to prove existence. The second adapts Harris, Reny, and Robson's (1995) non-existence example to our framework with countable actions. We later revisit both examples to provide intuition for our results.

Example 1 (Strategic entanglement)**.** Consider a two-player game where both players observe a public signal s, drawn from a uniform in $[0, 1]$, before they choose an action A or B. If both players select the same action, each receives a payoff of 1; otherwise, their payoffs are zero.

The following sequence of strategies generates a payoff discontinuity. For each player

[12] A sequence of probability measures over Y, $(\rho_n)_{n\in\mathbb{N}}$, $\rho_n \in \Delta(Y)$, *converges in the topology of weak convergence of probability measures* to $\rho \in \Delta(Y)$ if $\lim_{n\to+\infty} \int f d\rho_n = \int f d\rho$ for every bounded, continuous function $f : Y \to \mathbb{R}$.

[13] This issue was identified by Simon and Stinchcombe (1989), Börgers (1991), and Harris et al. (1995), among others. The term "strategic entanglement" was coined by Myerson and Reny (2020). Example 1 is analogous to Example 2 in Milgrom and Weber (1985), Example 2.1 in Cotter (1991), and Example 2.1 in Stinchcombe (2011).

$i \in \{1, 2\}$, and $n \in \mathbb{N}$, define $\sigma_i^n(A|s) = 1$ if $s \in [(k-1)/2^n, k/2^n]$ for odd k, and $\sigma_i^n(A|s) = 0$ otherwise. Players choose the same action as a function of the signal, but as n increases, players switch actions over progressively finer intervals. Figure 1 illustrates the first three strategies in this sequence.

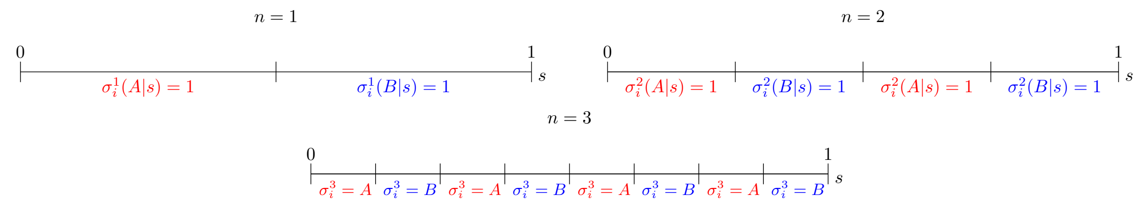

Figure 1: Sequence of strategies generating strategic entanglement.

Throughout the sequence each player earns a payoff of 1, resulting in a limit payoff of 1 as well. However, intuitively, as the intervals over which the players switch actions become progressively finer as n grows, each player's limit strategy must choose A and B with equal probability, *independently* of s, yielding an expected payoff of $1/2$.

More formally, we have:

$$\lim_{n\to\infty} \sigma_1^n(a_1|s) \cdot \sigma_2^n(a_2|s) \neq \lim_{n\to\infty} \sigma_1^n(a_1|s) \cdot \lim_{n\to\infty} \sigma_2^n(a_2|s)$$

for $(a_1, a_2) \in \{A, B\}^2$, where these limits are with respect to the weak convergence of probability measures. Consequently, the players' expected payoffs are discontinuous with respect to their strategies. This discontinuity implies that the best response correspondence fails to have a closed graph, thus invalidating the conditions required for Nash's classical fixed-point argument. ◂

The form of payoff discontinuity illustrated in Example 1 has implications for equilibrium existence. For instance, Simon (2003), Hellman (2014), and Simon and Tomkowicz (2018) construct one-period Bayesian games with finite actions that lack any Bayes-Nash equilibrium.[14]

While strategic entanglement can occur in static settings, the following example illustrates a distinct form of discontinuity that emerges in multi-period games, potentially leading to equilibrium non-existence.

[14]In Example 1, for illustrative purposes, we focus on the weak topology on probability measures, which is natural for the space of strategies and offers desirable properties such as compactness. The assumptions of SAC and MAC, which we introduce later, ensure payoff continuity with respect to this topology. While one might conjecture that alternative topologies could enable fixed-point arguments, examples of non-existence rule out this possibility.

Example 2.1 (Harris, Reny, and Robson (1995)). Consider the following game. In the first period, player A chooses action $a \in \mathcal{A} := \cup_{n\in\mathbb{N}}\{-1/n, 1/n\} \cup \{0\}$ and player B action $b \in \{L, R\}$, while in the second period, after observing the move of player A, players C and D choose $c \in \{L, R\}$ and $d \in \{L, R\}$, respectively. Players C and D's payoff functions are identical and depend only on action a: Playing L yields a payoff of $-a$ and R of a. That is, the second period players strictly prefer to play L if $a < 0$, R if $a > 0$, and are indifferent otherwise. Player B wants to guess the future choice of player C and gets a payoff of 1 when $c = b$ and of -1, otherwise. Player A's payoff is as follows

$$-|a| \cdot \mathbb{1}_{\{b=c\}} + |a| \cdot \mathbb{1}_{\{b\neq c\}} - 10 \cdot \mathbb{1}_{\{c\neq d\}} - \tfrac{1}{2}|a|^2.$$

If B and C make the same choice, A obtains a payoff of $-|a|$, and $|a|$ otherwise; if C and D make different choices, A obtains a negative payoff of -10; A gets $-\frac{1}{2}|a|^2$.[15]

As in Harris et al. (1995), this game lacks a subgame perfect equilibrium. Although our version differs in having a countable rather than uncountable action set for A and in that B's actions are not observed by C and D, their non-existence argument remains essentially valid. A optimally seeks to minimize B's probability of correctly guessing C's action while maintaining perfect coordination between C and D. Such an outcome could be achieved through A's uniform randomization between a positive and negative value, say choosing $a = \delta$ and $a = -\delta$ each with probability $\frac{1}{2}$. Yet, for any non-zero $\delta \in \mathcal{A}$, A faces a cost of $\frac{1}{2}|\delta|^2 > 0$, which vanishes as δ approaches zero. In the limit, A's mixed strategy becomes degenerate, preventing random coordination between C and D. Furthermore, if $a = 0$ in equilibrium, and C and D select identical actions with certainty, then their chosen action must be deterministic, causing B to guess C's action with probability 1. This creates an incentive for A to deviate to a small non-zero a, ensuring $c \neq d$. ◂

3 Equilibrium Existence

We now introduce sequential absolute continuity (SAC), the main assumption of our analysis. This condition restricts the transition of private signal profiles by requiring: (a) absolute continuity with respect to the product of the marginal measures of each player's private signals conditional on the action history; (b) boundedness and continuity in the action profile history according to a novel norm, which implies continuity in the total variation norm. Under SAC, the discontinuities outlined in the previous section do not occur.

[15] Notice that the game falls within our framework by assuming there is no payoff-relevant state, i.e., Ω is a singleton, and second-period players receive a signal equal to A's action.

Let Z be a countable set and Y a measurable space. The *strong total variation norm* of a transition measure $\xi : Z \to \mathcal{M}(Y)$ is

$$\|\xi\|_{SV} \coloneqq \sup\left\{ \sum_{j \in I} |\xi(Y_j|z_j)| \;\middle|\; \{Y_j\}_{j\in I} \in \pi(Y), \{z_j\}_{j\in I} \subseteq Z \right\},$$

where $\pi(Y)$ denotes the set of finite measurable partitions of Y. For a subset $\check{Z} \subseteq Z$, let $\xi|_{\check{Z}}$ be the restriction of ξ to $\check{Z}$, and, for an element $\check{z} \in Z$, let $\xi^{\check{z}}(\cdot|z) := \xi(\cdot|\check{z})$ for every $z \in Z$. We say that ξ is *continuous in Z in strong total variation* if, for every $z^* \in Z$ and $\varepsilon > 0$, there is $\delta > 0$ such that $\left\|(\xi - \xi^{z^*})|_{B(z^*,\delta)}\right\|_{SV} < \varepsilon$, where $B(z^*, \delta)$ denotes the ball with radius δ centered at z^*.

When a transition probability admits a density with respect to some z-independent measure, we show that continuity and boundedness in the strong total variation norm is equivalent to such a density being both continuous in Z and bounded by a z-independent, L^1 function on Y (see Proposition 8 in Online Appendix C.1). A density satisfying these conditions is called a *Carathéodory integrand*.[16]

Notice that, in our notation, ξ is *continuous in the total variation norm* if $\left\|\xi^z - \xi^{z^*}\right\|_{SV} \to 0$ as $z \to z^*$, i.e., as z approaches z^* it cannot vary across partition elements. Therefore, this form of continuity is weaker than continuity in the strong total variation norm.[17]

ASSUMPTION (Sequential absolute continuity)**.** *The following holds:*

(a) *For every $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, $\mu_s^t(\cdot|a^{t-1})$ is absolutely continuous with respect to the product measure $\prod_{i\in N} \mu_{s_i}^t(\cdot|a^{t-1})$;*

(b) *For every $t \in \mathbb{N}$, μ_s^t is bounded, and continuous in X^{t-1} in strong total variation.*[18]

SAC extends the absolute continuity condition of Milgrom and Weber (1985) to dynamic games. In their static setting, signals correspond to types, and the absolute continuity condition requires the joint type distribution to be absolutely continuous with respect to the product of players' marginal type distributions, coinciding with SAC(a). SAC(b) holds trivially in one-period games as $X^0 = \emptyset$.

[16]A measurable function $f : Y \times Z \to \mathbb{R}$ is a *Carathéodory integrand* on $(Y \times Z, \beta)$, where $\beta \in \Delta(Y)$ and Z is a compact metric space endowed with its Borel σ-algebra, if: (i) for every $y \in Y$, $f(y, \cdot)$ is continuous in Z; (ii) there exists $\psi \in L^1(Y, \beta)$ such that $|f(y, \cdot)| \leq \psi(y)$ for every $y \in Y$.

[17]Example 4 in Online Appendix C.1 constructs a transition measure that is continuous in the total variation norm but discontinuous in strong total variation.

[18]By Proposition 8, SAC(b) is equivalent to the following: if a density exists with respect to a measure over S^t that is independent of $a^t \in X^t$, then this density is a Carathéodory integrand. SAC(a) ensures multiple ways to construct such measures and densities, all of which are Carathéodory under SAC(b). We define our condition using continuity in strong total variation due to this multiplicity of possible a^t-independent measures over S^t.

In a dynamic environment, sequential absolute continuity imposes two key requirements. (a) Absolute continuity must hold conditional on every possible history of play. This means that the joint distribution of players' signals, *given any sequence of past actions*, can be expressed in terms of the product of individual players' signal distributions. (b) The distribution of signals must be continuous with respect to past play and bounded in strong total variation. This condition implies continuity in both the total variation norm and set-wise continuity.[19] Consequently, any game violating these weaker forms of continuity necessarily violates continuity in strong total variation. This is the case in Example 2.1, where the distribution of signals fails to be set-wise continuous (see Example 2.2 in Section 3.2).

SAC is always satisfied if the action and signal spaces are not too large. For instance, SAC(a) holds if the set of signals S_i is countable for all but at most one player, while SAC(b) holds if, for every player $i \in N$, the action set X_i is finite.

In general, even with finite action sets, SAC restricts the information structure of the game. For instance, when the state is drawn from a non-atomic distribution on the reals, players cannot commonly observe the state without violating SAC(a). Example 1 illustrates this violation: $\mu_s^1(D) = 1 \neq 0 = \mu_{s_1}^1 \times \mu_{s_2}^1(D)$, where $D = \{(s_1, s_2) \in [0,1]^2 | s_1 = s_2\}$ is the diagonal set, and each μ_{s_i} is uniform on $[0,1]$, $i \in \{1,2\}$.

3.1 Constrained Equilibrium

We define a constrained equilibrium that must (i) put a positive weight on each available action, and (ii) be optimal within the set of constrained strategies given the constrained strategies of the opponents.

For every $\varepsilon > 0$, a measurable function $\tilde{\varepsilon}_i : S_i^t \times X_i^t \to (0,1)$ is an *ε-tremble* of player $i \in N$ if

$$\sum_{a_i \in A_i(s_i^t, a_i^{t-1})} \tilde{\varepsilon}_i(s_i^t, (a_i^{t-1}, a_i)) < \varepsilon \cdot \tilde{\varepsilon}_i(s_i^{t,(t-1)}, a_i^{t-1})$$

for every $(s_i^t, a_i^{t-1}) \in \mathscr{H}_i^t$, and $t \in \mathbb{N}$. For any player $i \in N$, an ε-tremble assigns strictly positive mass to all available action histories. Moreover, at each period, the total mass of trembles assigned to that period's actions cannot exceed ε times the tremble mass of the action history up to the previous period. We denote by $\tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i \in N}$ an ε-tremble profile, by $\mathscr{E}(\varepsilon)$ the set of ε-tremble profiles, and by $\mathscr{E} := \cup_{\varepsilon > 0} \mathscr{E}(\varepsilon)$ the set of ε-tremble profiles for any positive ε.

[19] Formally, set-wise continuity requires that, for each $\tilde{S} \in \mathcal{M}(S^t)$ and sequence $a_m \to a$, we have $\mu_s^t(\tilde{S} \mid a_m) \to \mu_s^t(\tilde{S} \mid a)$.

Given $\tilde{\varepsilon} \in \mathcal{E}$, a strategy profile, σ_i, is *$\tilde{\varepsilon}$-constrained* if, for each player $i \in N$, and private history $(s_i^t, a_i^t) \in \bar{\mathcal{H}}_{A_i}$, $p_i(a_i^t | s_i^t, \sigma_i) \geq \tilde{\varepsilon}_i(s_i^t, a_i^t)$. Hence, σ_i is a totally mixed strategy over the available actions. We denote by $\Sigma_i(\tilde{\varepsilon})$ the set of player i's $\tilde{\varepsilon}$-constrained strategies, and by $\Sigma(\tilde{\varepsilon})$ the set of $\tilde{\varepsilon}$-constrained strategy profiles.

DEFINITION 1. Let $\tilde{\varepsilon} \in \mathcal{E}$. An $\tilde{\varepsilon}$-constrained strategy profile $\sigma \in \Sigma(\tilde{\varepsilon})$ is an *$\tilde{\varepsilon}$-constrained equilibrium* if, for every player $i \in N$,

$$U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i}) \qquad \forall \sigma_i' \in \Sigma_i(\tilde{\varepsilon}).$$

THEOREM 1. *Let Γ be a dynamic game that satisfies sequential absolute continuity. For every $\tilde{\varepsilon} \in \mathcal{E}$, Γ has an $\tilde{\varepsilon}$-constrained equilibrium.*

The proof of our main theorem builds upon Milgrom and Weber (1985) and Balder's (1988) arguments for one-period games. It consists of the following key steps.

First, we identify each player i's constrained strategy, σ_i, with its induced transition probabilities over action histories, $p_i(\cdot|\cdot, \sigma_i)$. For every $i \in N$, we construct measures $\nu_i^t \in \Delta(S_i^t)$, which we refer to as *reference measures*,[20] such that, for each $a^{t-1} \in X^{t-1}$, $\mu_s^t(\cdot|a^{t-1})$ is absolutely continuous with respect to ν_i^t. Using these measures, we endow each player's strategy space, now represented as transition probabilities, with the coarsest topology that makes every expected payoff functional, with Carathéodory flow payoffs, continuous in transition probabilities—a topology Balder (1988) terms the *weak topology*. Equipped with the product topology, the strategy space is both convex and compact, and the best response correspondence is convex-valued.

Next, we demonstrate that each player's expected payoff is continuous in strategies, a result implied by sequential absolute continuity. For every $i \in N$, $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, SAC(a) allows us to write

$$d\mu_s^t(s_1^t, \ldots, s_n^t | a^{t-1}) = f(s_1^t, \ldots, s_n^t, a^{t-1})\, d\nu_1(s_1^t) \times \ldots \times d\nu_n(s_n^t), \tag{3}$$

where f is a density function. Proposition 8 in Online Appendix C.1 establishes that, whenever $\mu_s^t(\cdot|a^{t-1})$ can be written as in equation (3), SAC(b) holds if and only if the density f is a Carathéodory integrand. This implies that, if $\mu_s^t(\cdot|a^{t-1})$ has a density that is a Carathéodory integrand with respect to one product measure, densities with respect to any other product measure will also be Carathéodory integrands.[21] Therefore, equation (3) and

[20] We define these measures formally in Section 3.2; see footnote 24 for a formal construction.

[21] To the best of our knowledge, we are the first to characterize the continuity properties of its supporting densities via continuity with respect to a norm. Example 4 together with Proposition 8 show that continuity with respect to the total variation norm is not sufficient to ensure continuity of the density.

our continuity condition on payoffs imply that players' expected payoffs can be written as the integral of Carathéodory integrands over a product measure over players' signals. Theorem 2.5 in Balder (1988) then implies the continuity of expected payoffs, ensuring the non-emptiness and closed-graph properties of the best-response correspondence.[22]

These properties allow us to apply the Kakutani-Fan-Glicksberg fixed point theorem[23] to establish the existence of a fixed point of the best response correspondence over transition probabilities. From this fixed point, we can recover each player i's equilibrium strategy σ_i by taking the ratio of two probabilities: the probability that player i takes a specific sequence of actions a_i^t up to period t, divided by the probability of player i's action sequence $a_i^{t,(t-1)}$ up to the previous period $t-1$. This ratio is well-defined because the $\tilde{\varepsilon}$-constrained requirement ensures that transition probabilities assign positive weight to every action history.

3.2 Trembling Hand Perfect Equilibrium

Inspired by Selten's (1975) seminal work on finite games, we define trembling hand perfect equilibrium as the limit of constrained strategies as their ε-trembles vanish. Consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ converging to zero, and a corresponding sequence of $\tilde{\varepsilon}_n$-constrained equilibria σ^n, where $\tilde{\varepsilon}_n \in \mathcal{E}(\varepsilon_n)$. Theorem 1 guarantees the existence of such sequences. However, constructing the limit strategy from the limit of such a sequence of transition probabilities presents a challenge. While constrained strategies allow us to recover a strategy by taking the ratio of consecutive transition probabilities, this approach fails when the limit strategy assigns zero probability to certain action histories, leaving the construction undefined along these paths.

To address this challenge, we introduce non-vanishing transition probabilities that preserve information about strategies even after zero-probability histories. These probabilities are analogous to those used in the $\tilde{\varepsilon}$-constrained case but are well-defined in the limit. We then infer the limit strategy by taking the limits of these non-vanishing probabilities, allowing us to characterize the trembling hand perfect equilibrium even after off-path histories.

For every $i \in N$, $t \in \mathbb{N}$, let α_i^t be a probability measure over S_i^t. Consider a sequence of transition probabilities $(\lambda^n)_{n\in\mathbb{N}}$, where $\lambda^n : S_i^t \to \Delta(X_i^t)$. We say that $(\lambda^n)_{n\in\mathbb{N}}$ *converges to* λ^* *in the weak topology of* $(S_i^t \times X_i^t, \alpha_i^t)$ if for every Carathédory integrand ϕ_i on $(S_i^t \times$

[22]Notice that we define the topology over transition probabilities on action histories, rather than period-by-period behavioral strategies, to capture potential strategic entanglement of each player's strategy across periods. If the topology were defined over behavioral strategies, continuity would not follow.

[23]Corollary 17.55 in Aliprantis and Border (2006).

$X_i^t, \alpha_i^t)$, the following convergence holds

$$\int_{S_i^t} \sum_{a_i^t \in X_i^t} \phi_i(s_i^t, a_i^t) \lambda^n(a_i^t|s_i^t)\, d\alpha_i^t(s_i^t) \to \int_{S_i^t} \sum_{a_i^t \in X_i^t} \phi_i(s_i^t, a_i^t) \lambda^*(a_i^t|s_i^t)\, d\alpha_i^t(s_i^t).$$

For every $i \in N$, $t \in \mathbb{N}$, define the function $\mathfrak{a}_i$ over $(s_i^t, a_i^{t,(t-1)}) \in \mathcal{H}_{A_i}$ for any given $\sigma_i \in \Sigma_i$ as follows

$$\mathfrak{a}_i(s_i^t, a_i^t; \sigma_i) := \left\{a_i^{t,(\tau)} | \tau = \min\left\{\hat{\tau} \le t-1 | \Pi_{\ell \ge \hat{\tau}+1}^{t-1} \sigma_i(a_{i,\ell}^t | s_i^{t,(\ell)}, a_i^{t,(\ell-1)}) > 0\right\}\right\}.$$

This function truncates a_i^t up to a period τ, which is the latest period before $t-1$ where player i's action $a_{i,\tau}^t$ has zero probability under strategy σ_i, as a function of i's signals. In particular, if $\sigma_i(a_{i,\tilde{\tau}}^t | s_i^{t,(\tilde{\tau})}, a_i^{t,(\tilde{\tau}-1)}) > 0$, for every $\tilde{\tau} \le t-1$, then $\tau = 0$, and if $\sigma_i(a_{i,\tilde{\tau}}^t | s_i^{t,(\tilde{\tau})}, a_i^{t,(\tilde{\tau}-1)}) = 0$, for every $\tilde{\tau} \le t-1$, then $\tau = t-1$.

For every $i \in N$, $t \in \mathbb{N}$, we define a *reference measure for player i* as $\nu_i^t \in \Delta(S_i^t)$ satisfying:

- For each $a^{t-1} \in X^{t-1}$, $\mu_{s_i}^t(\cdot|a^{t-1})$ is absolutely continuous with respect to ν_i^t;
- There exists a transition probability $\nu_{i,t} : S_i^{t-1} \to \Delta(S_i)$ such that $d\nu_i^t(s_i^t) = d\nu_{i,t}(s_{i,t}^t | s_i^{t,(t-1)}) \times d\nu_i^{t-1}(s_i^{t,(t-1)})$.

A reference measure satisfying these conditions can always be constructed in our environment.[24]

DEFINITION 2. A strategy $\sigma^* \in \Sigma$ is a *trembling hand perfect equilibrium* if:

(i) There exist sequences $(\varepsilon_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$, $(\tilde{\varepsilon}_n)_{n \in \mathbb{N}}$ in $\mathcal{E}$, and $(\sigma^n)_{n \in \mathbb{N}}$ in Σ satisfying:

- $\tilde{\varepsilon}_n \in \mathcal{E}(\varepsilon_n)$ for each $n \in \mathbb{N}$,
- σ^n is a $\tilde{\varepsilon}_n$-constrained equilibrium for each $n \in \mathbb{N}$,
- $\lim_{n \to \infty} \varepsilon_n = 0$;

(ii) For every $i \in N$, and $\tau, t \in \mathbb{N}$ with $\tau \le t$, there exist a transition probability $p_i^*(\cdot|\cdot, a_i^\tau) : S_i^t \to \Delta(X_i^t)$, for $a_i^\tau \in X_i^\tau$, and a reference measure ν_i^t, such that:

- $p_i(\cdot|\cdot, a_i^\tau, \sigma_i^n)$ converges to $p_i^*(\cdot|\cdot, a_i^\tau)$ in the weak topology of $(S_i^t \times X_i^t, \nu_i^t)$,
- For $(s_i^t, a_i^{t,(t-1)}) \in \mathcal{H}_{A_i} \cap \mathcal{H}_i^t$ and $a_i^{t,(\tau)} = \mathfrak{a}_i(s_i^t, a_i^t; \sigma_i)$:

$$\sigma_i^*\left(a_{i,t}^t | s_i^t, a_i^{t,(t-1)}\right) = \frac{p_i^*\left(a_i^t | s_i^t, a_i^{t,(\tau)}\right)}{p_i^*\left(a_i^{t,(t-1)} | s_i^{t,(t-1)}, a_i^{t,(\tau)}\right)}. \tag{4}$$

[24]Define, for instance, $\nu_{i,t} : S_i^{t-1} \to \Delta(S_i)$ as $d\nu_{i,t}(s_{i,t}|s_i^{t-1}) := \sum_{a^{t-1} \in X^{t-1}} \xi_i(a^{t-1}) \cdot d\mu_{s_i|s_i}^t(s_{i,t}|s_i^{t-1}, a^{t-1})$ where ξ_i is an arbitrary collection of strictly positive probability weights, i.e., for every $t \in \mathbb{N}$ and $a^{t-1} \in X^{t-1}$, $\xi_i(a^{t-1}) > 0$ and $\sum_{a^{t-1} \in X^{t-1}} \xi_i(a^{t-1}) = 1$.

This definition characterizes a trembling hand perfect equilibrium (THPE) as the limit of a sequence of $\tilde{\varepsilon}_n$-constrained equilibria $(\sigma^n)_{n\in\mathbb{N}}$ as the trembles ε_n vanish, extending Selten's (1975) notion of *perfect equilibrium* to infinite games. Unlike Selten's approach, which directly restricts behavioral strategies to be strictly positive, our notion of constrained equilibrium restricts the probabilities of action histories. For each player $i \in N$, the limit strategy σ_i^* is constructed from the weak limits p_i^* of the transition probabilities induced by σ_i^n. Specifically, σ_i^* is defined in equation (4) as the ratio of these limiting probabilities, where the denominator is positive as it captures the probability of action histories following the last action with zero probability in the limit strategy.

THEOREM 2. *Let Γ be a dynamic game that satisfies sequential absolute continuity. Then, Γ has a trembling hand perfect equilibrium.*

Theorem 2 ensures that from a sequence of $\tilde{\varepsilon}_n$-constrained equilibria with $\tilde{\varepsilon}_n \in \mathcal{E}(\varepsilon_n)$ and $\varepsilon_n \to 0$, we can extract a convergent subsequence that yields a trembling hand perfect equilibrium.

In Section 5, we establish that trembling hand perfect equilibria exhibit some desirable equilibrium properties. They are conditional equilibria meaning that they are optimal after any positive-probability set of private histories. Moreover, they are optimal at histories where players can perfectly deduce past events, implying subgame perfection. Finally, they are approximated by fully-mixed strategies that satisfy sequential rationality properties. The topology of convergence we employ together with the SAC assumption and our construction of limit strategies, ensures that continuation expected payoffs converge, thereby guaranteeing these properties.

The following example demonstrates that the game introduced by Harris et al. (1995) and described in Example 2.1, which lacks a subgame perfect equilibrium, violates SAC(b), highlighting the importance of this condition in our equilibrium existence result.

EXAMPLE 2.2 (Harris et al. (1995), continued)**.** Consider any sequence of player A's actions $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ where $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a_n \to a_0 := 0$. For every $i \in \{C, D\}$, $n \in \mathbb{N}$,$\gamma_i(a_n, b) = s_{i,2}^2 = a_n$, and $\mu_s^2(\{s_{i,2}^2 = 0\}|a_n) = 0$.
Therefore,

$$\lim_{n\to\infty} \mu_s^2(\{s_{i,2}^2 = 0\}|a_n) = 0 \neq 1 = \mu_s^2(\{s_{i,2}^2 = 0\}|a_0).$$

This set-wise discontinuity in the signal transition implies a violation of SAC(b). ◂

3.3 Noisy Informational Asymmetries

Even in games which lack conditions for existence, adding small amounts of idiosyncratic noise to players' private signals can ensure SAC, provided this noise satisfies certain regularity properties. In this section, we formalize these properties, and show that they hold in many common applications. For instance, SAC often holds when signals are real-valued, and the noise is additive and absolutely continuous with respect to the Lebesgue measure, e.g., additive $i.i.d.$ noise following uniform or normal distributions typically satisfies these regularity properties.

Intuitively, introducing noise to players' observations mitigates discontinuities arising from perfect information about other players' signals or actions. In cases of strategic entanglement, as illustrated in Example 1, noisy observations prevent players from finely tuning their strategies based on their opponents' private signals. This noise effectively smooths out the joint distribution of signals, rendering it absolutely continuous with respect to the product of players' marginal distributions. Similarly, in games like Example 2.1, adding noise to the observation of previous actions allows the signal distribution to vary more smoothly with players' moves. This prevents the abrupt changes in signal probabilities that occur when actions are perfectly observable.

We say that a dynamic game has *decomposable noisy signals* if, for every $i \in N$ and $t \in \mathbb{N}$, each player i's period-t private signal can be represented as

$$s_{i,t} = m_i(\hat{s}_{i,t}, \epsilon_{i,t}),$$

where $\hat{s}_{i,t} \in \hat{S}_i$ denotes a *fundamental signal component* and $\epsilon_{i,t} \in \mathcal{E}_i$ represents a *noisy* random variable. The spaces $\hat{S}_i$ and $\mathcal{E}_i$ are Polish spaces endowed with their Borel σ-algebras.[25]

For every $t \in \mathbb{N}$, denote by $\mu^t_{\hat{s},\epsilon}(\cdot|a^{t-1})$ the joint distribution of $(\hat{s}^t, \epsilon^t)$ conditional on $a^{t-1} \in X^{t-1}$. Let $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ and $\mu^t_{\epsilon}(\cdot|a^{t-1})$ be the marginals with respect to the fundamental signal $\hat{s}^t$ and noise vector ϵ^t, respectively; let $\mu^t_{\hat{s}_i}(\cdot|a^{t-1})$ and $\mu^t_{\epsilon_i}(\cdot|a^{t-1})$ be the marginals of $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ and $\mu^t_{\epsilon}(\cdot|a^{t-1})$ with respect to player i's dimension, for each $i \in N$.

ASSUMPTION (Noisy observability). *For every $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, the following holds:*
(a) $\mu^t_{\hat{s},\epsilon}(\cdot|a^{t-1})$ *is absolutely continuous with respect to* $\mu^t_{\hat{s}}(\cdot|a^{t-1}) \times \prod_{i \in N} \mu^t_{\epsilon_i}(\cdot|a^{t-1})$*;*

[25] We use the notation: $\epsilon_t = (\epsilon_{1,t}, \dots, \epsilon_{n,t})$, $\hat{s}_t = (\hat{s}_{1,t}, \dots, \hat{s}_{n,t})$, $m(\hat{s}_t, \epsilon_t) = (m_1(\hat{s}_{1,t}, \epsilon_{1,t}), \dots, m_n(\hat{s}_{n,t}, \epsilon_{n,t}))$, $m^t(\hat{s}^t, \epsilon^t) = (m(\hat{s}^t_1, \epsilon^t_1), \dots, m(\hat{s}^t_t, \epsilon^t_t))$, $\epsilon^t = (\epsilon_1, \dots, \epsilon_t)$, and $\hat{s}^t = (\hat{s}_1, \dots, \hat{s}_t)$.

(b) *If* $\mu^t_s(\tilde{S}|a^{t-1}) > 0$ *for* $\tilde{S} \in \mathcal{M}(S^t)$*, then*

$$\hat{S}(\tilde{S}) \coloneqq \left\{\hat{s} \in \hat{S}^t \mid \exists \epsilon \in \mathcal{E}^t,\, (\hat{s}, \epsilon) \in (m^t)^{-1}(\tilde{S})\right\}$$

satisfies $(\prod_{i \in N} \mu^t_{\hat{s}_i}(\cdot|a^{t-1}))(\hat{S}(\tilde{S})) > 0$.

Noisy observability imposes two requirements for every period $t \in \mathbb{N}$ and history of actions $a^{t-1} \in X^{t-1}$. (a) The joint measure of the fundamental signal component $\hat{s}^t$ and the noise term ϵ^t is absolutely continuous with respect to the product of the measure of the fundamental signal component and the product of the marginal measures over each player's noise. (b) If a set of histories of signal profiles $\tilde{S}$ has positive measure, then the set of fundamental signals that combined with some noise yield a signal in $\tilde{S}$, denoted by $\hat{S}(\tilde{S})$, has positive measure under the product of the marginal distributions of each player's fundamental signal component, $\prod_{i \in N} \mu^t_{\hat{s}_i}(\cdot|a^{t-1})$.

The next result shows that "adding some noise" to the players' observations, satisfying noisy observability, yields SAC(a).

PROPOSITION 1. *Noisy observability implies SAC*(a) *in any game with decomposable noisy signals.*

The following lemma provides a sufficient condition for noisy observability (b).

LEMMA 1. *Suppose that for every* $i \in N$*,* $t \in \mathbb{N}$*, and* $a^{t-1} \in X^{t-1}$*, there is* $\check{S}^t_i \in \mathcal{M}(S^t_i)$ *with* $\mu^t_{s_i}(\check{S}^t_i|a^{t-1}) = 1$ *such that* $\mu^t_{\hat{s}_i}(\{\hat{s}^t_i \in \hat{S}^t_i | \exists \epsilon^t_i \in \mathcal{E}^t_i,\, m^t_i(\hat{s}^t_i, \epsilon^t_i) = s^t_i\}|a^{t-1}) > 0$ *for every* $s^t_i \in \check{S}^t_i$*. Then noisy observability* (b) *holds.*

Lemma 1 says that noisy observability (b) holds if, for every player, there exists a full-measure set of signals such that the fundamental signals that could generate each observed signal (when combined with some noise) have positive measure. This prevents players from perfectly inferring the fundamental signal from their observations, as a positive measure of distinct fundamental components could have produced the same observed signal.

The following example shows that the noisy observability condition holds in a natural setting where signals are real-valued vectors and noise is additive.

EXAMPLE 3. If signals are real-valued vectors and noise is additive, i.e., for each $i \in N$ there is $\ell_i \in \mathbb{N}$ such that $\hat{S}_i, \mathcal{E}_i \subseteq \mathbb{R}^{\ell_i}$ and $m_i(\hat{s}_i, \epsilon_i) = \hat{s}_i + \epsilon_i$, then noisy observability holds if:

1. ϵ_t is independent of $\hat{s}$ and across periods, and the noise distribution can be written as

$$d\mu^t_{\epsilon,t}(\epsilon_{1,t},\ldots,\epsilon_{n,t}|a^{t-1}) = f^t_\epsilon(\epsilon_{1,t},\epsilon_{2,t},\ldots,\epsilon_{n,t},a^{t-1})\cdot d\lambda^{\ell_1}(\epsilon_{1,t})\times\ldots\times d\lambda^{\ell_n}(\epsilon_{n,t}), \quad (5)$$

where λ^{ℓ_i} is the Lebesgue measure over $\mathbb{R}^{\ell_i}$;

2. For every $a^{t-1} \in X^{t-1}$, $f^t_\epsilon(\epsilon, a^{t-1}) > 0$ implies $f^t_\epsilon(\hat{\epsilon}, a^{t-1}) > 0$ for every $\hat{\epsilon}$ in a neighborhood of ϵ;

3. For each $a^{t-1} \in X^{t-1}$, the measure $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ assigns positive probability to some neighborhood of every $\hat{s}^t \in \hat{S}^t$.

These conditions satisfy noisy observability (a) and the hypotheses of Lemma 1. ◂

Next, we investigate which noise structure implies SAC(b) by characterizing a class of signal profile distributions that are bounded and continuous in strong total variation. For every $t \in \mathbb{N}$, let $\mu^t_{\hat{s},s}(\cdot|a^{t-1})$ be the joint distribution of $(\hat{s}^t, s^t)$ conditional on $a^{t-1} \in X^{t-1}$.

ASSUMPTION (Sequentially continuous noise)**.** *The following holds:*

(a) *For every $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ is continuous in a^{t-1} with respect to the topology of weak convergence of probability measures;*[26]

(b) *For every $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, $d\mu^t_{\hat{s},s}(s^t, \hat{s}^t|a^{t-1}) = f(s^t, \hat{s}^t, a^{t-1})d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1})d\tilde{\mu}^t(s^t)$ where $\tilde{\mu}^t \in \Delta(S^t)$, and f is a bounded density such that, for a $\tilde{\mu}^t$-full-measure set $\check{S} \in \mathcal{M}(S^t)$, $f(s^t, \cdot, \cdot)$ is continuous for every $s^t \in \check{S}$.*

Sequentially continuous noise imposes two conditions. (a) The measure of the fundamental signal component $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ must be continuous in the history of action profiles a^{t-1} with respect to the topology of weak convergence of probability measures. This is a considerably weaker continuity requirement compared to SAC(b). In particular, it holds whenever $\hat{s}^t$ is a deterministic and continuous function of a^{t-1}. (b) The joint measure of the signal and fundamental signal $\mu^t_{\hat{s},s}(\cdot|a^{t-1})$ must be absolutely continuous with respect to the product of the fundamental signal measure $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ and a probability measure over the space of signals $\tilde{\mu}^t(\cdot)$ that does not depend on the history of actions. Furthermore, the corresponding density function $f(s^t, \hat{s}^t, a^{t-1})$ must be bounded and, for s^t in a full-measure set, continuous in $\hat{s}^t$ and a^{t-1}.

PROPOSITION 2. *Sequentially continuous noise implies SAC(b) in any game with decomposable noisy signals.*

[26] See footnote 12 for the definition of this convergence.

When the signals are real vectors and the noise is additive and absolutely continuous with respect to the Lebesgue measure, we provide a simpler sufficient condition for sequentially continuous noise (b).

LEMMA 2. *If, for each $i \in N$, there is $\ell_i \in \mathbb{N}$ such that $\hat{S}_i, \mathcal{E}_i \subseteq \mathbb{R}^{\ell_i}$, and $m_i(\hat{s}_i, \epsilon_i) = \hat{s}_i + \epsilon_i$, then the following conditions imply sequentially continuous noise (b):*

1. *Noisy observability (a) holds;*
2. *For every $t \in \mathbb{N}$ and $a^{t-1} \in X^{t-1}$, the noise distribution $\mu^t_\epsilon(\cdot|a^{t-1})$ is absolutely continuous with respect to the Lebesgue measure λ^t. The density $f^t : \mathcal{E}^t \times \hat{S}^t \times X^{t-1} \to \mathbb{R}$ of $\mu^t_{\hat{s},\varepsilon}(\cdot|a^{t-1})$ with respect to the product of $\mu^t_{\hat{s}}(\cdot|a^{t-1})$ and λ^t is bounded, and there exists a full measure set $\check{S}^t \in \mathcal{M}(S^t)$ such that for all $s^t \in \check{S}^t$, the function $\tilde{f}(s^t, \hat{s}^t, a^{t-1}) := f^t(s^t - \hat{s}^t, \hat{s}^t, a^{t-1})$ is continuous in both $\hat{s}^t$ and a^{t-1}.*

Notice that, by Proposition 2 and Lemma 2, if the density function f^t_ϵ in equation (5) of Example 3.3 is additionally almost surely continuous in ϵ and a^{t-1}, then sequentially continuous noise (b) holds.

Lemmas 1 and 2 enable us to apply Propositions 1 and 2 to the class of games introduced in Application 1, thereby establishing equilibrium existence.

COROLLARY 1. *Let Γ be a dynamic game with Lebesgue signals as in Application 1. Then, Γ satisfies sequential absolute continuity.*

In light of these results, we conclude this section by revisiting the game discussed in Example 2.1. That game features countable signals, ensuring SAC(a) holds. Furthermore, players' signals in period 2 are the previous period's actions. Hence, the signal transition probabilities are continuous in the topology of weak convergence of probability measures, satisfying the sequentially continuous noise (a) requirement for fundamental signals. Proposition 2 then implies that adding independent, uniformly distributed noise to the observation of players A and B's actions yields equilibrium existence. Below, we characterize an SPE for this modified game.

EXAMPLE 2.3 (Harris et al. (1995), continued)**.** Assume that, instead of observing a, each player $i \in \{C, D\}$ observes $s_i = a + u_i$ where $u_i \sim \mathrm{U}[-\delta, \delta]$, for some $\delta \in \mathcal{A}$. Notably, the noise range δ can be arbitrarily close to zero.

The following strategy is an SPE. Player A randomizes uniformly between the actions δ and $-\delta$. The other players' best responses to A's strategy are as follows: each player $i \in \{C, D\}$ plays L if $a + u_i < 0$ and R otherwise, while B sets $\beta = 1/2$.[27] ◂

[27] In Online Appendix B.6, we show that this strategy profile is an SPE.

Harris et al. (1995) restores existence in this example, and more generally in games with almost perfect information, by introducing a *public* correlation device at the start of the second period.[28] Our Corollary 1 demonstrates an alternative approach: existence can be ensured by incorporating idiosyncratic noise into players' *private* signals. This result extends beyond games with almost perfect information to a broad class of dynamic games with informational asymmetries. Notably, as the noise vanishes in Example 2.3, the equilibrium distribution over actions converges to that in the correlated equilibrium in Harris et al. (1995).

4 Markov Games

In this section, we show that our approach can establish the existence of equilibria in which players condition only on a subset of available information. Specifically, we focus on strategies that condition solely on the information that the players deem payoff-relevant. Our framework extends standard stochastic games with almost perfect information to a broader class of dynamic games with informational asymmetries, which we refer to as Markov games.

In *Markov game with informational asymmetries*, the set of states of the world Ω can be written as a product $\Omega = \Omega^R \times \Omega^{\not R}$ of the *payoff-relevant* states Ω^R and *payoff-irrelevant* states $\Omega^{\not R}$. For every $i \in N$, players' payoffs and state transition depend only on the current payoff-relevant state and action profile, $g_i : \Omega^R \times X \to \mathbb{R}$ and $\mu : \Omega^R \times X \to \Delta(\Omega)$. Additionally, in the discounted payoffs case we can allow g_i to depend on the period without including it as part of the payoff-relevant states, i.e., $g_i(t, \omega_t^R, a_t) = \delta^t \cdot u_i(\omega_t^R, a_t)$, $\delta \in (0, 1)$. Each player receives signals about the current payoff-relevant and irrelevant state, which we term payoff-relevant and irrelevant private signals, respectively. That is, $S_i = S_i^R \times S_i^{\not R}$ and $\gamma_i = (\gamma_i^R, \gamma_i^{\not R})$ such that $\gamma_i^R : \Omega^R \to S_i^R$ and $\gamma_i^{\not R} : \cup_{t \in \mathbb{N}} \mathscr{H}^t \to S_i^{\not R}$. In particular, the payoff irrelevant signal may contain information about the past that is no longer payoff relevant. The available actions depend on the current payoff-relevant signals, $A_i : S_i^R \to X_i$. The measurable function $\mathscr{s}_i^R : \mathscr{H}_i \to S_i^R$ projects each private history h_i onto the current payoff-relevant signal in S_i^R.

For every $i \in N$, player i's strategy $\sigma_i \in \Sigma_i$ is *Markov* if it conditions only on the current payoff-relevant private signal and possibly the current period of play. Formally, for each $t \in \mathbb{N}$, $\sigma_i(\cdot|h_i) = \sigma_i(\cdot|\tilde{h}_i)$ for all $h_i, \tilde{h}_i \in \mathscr{H}_i^t \cap \mathscr{H}_{A_i}$ such that $\mathscr{s}_i^R(h_i) = \mathscr{s}_i^R(\tilde{h}_i)$. A

[28] He and Sun (2020) extends this existence result without resorting to a correlation device by requiring the state transition to be atomless. Relatedly, Manelli (1996) restores existence in signalling games by adding cheap talk to the sender's messages.

strategy σ_i is *stationary Markov* if it depends only on the current payoff-relevant signal, regardless of the period: $\sigma_i(\cdot|h_i) = \sigma_i(\cdot|\tilde{h}_i)$ for all $h_i, \tilde{h}_i \in \mathscr{H}_{A_i}$ such that $s_i^R(h_i) = s_i^R(\tilde{h}_i)$. Thus, while Markov strategies may condition on both the current payoff-relevant signal and the period, stationary Markov strategies depend solely on the current payoff-relevant signal. Denote by Σ_i^M and Σ^M the sets of player i's Markov strategies and Markov strategy profiles, respectively. Similarly, let $\Sigma_i^{sM} \subseteq \Sigma_i^M$ and $\Sigma^{sM} \subseteq \Sigma^M$ be the sets of player i's stationary Markov strategies and stationary Markov strategy profiles, respectively.

For ease of exposition, we assume that regular conditional measures exist.[29] In particular, for every $\sigma \in \Sigma$, $t \in \mathbb{N}$, let $\mu_{\omega,t}^R(\cdot|\cdot, \sigma) : \cup_{i \in N} \mathscr{H}_i \to \Delta(\Omega^R)$ be the transition probability from private history $h_i \in \mathscr{H}_i$, $i \in N$, to period-t payoff-relevant states $\omega_t \in \Omega^R$ induced by strategy σ.[30]

Consider the following conditions on information and payoffs.

(i) *Markov information.* For every $i \in N$, $t \in \mathbb{N}$, $\sigma^M \in \Sigma^M$, $\mu_{\omega,t}^R(\cdot|h_i, \sigma^M) = \mu_{\omega,t}^R(\cdot|\tilde{h}_i, \sigma^M)$ for each $h_i, \tilde{h}_i \in \mathscr{H}_i^t$ such that $s_i^R(h_i) = s_i^R(\tilde{h}_i)$.

(ii) *Stationary Markov information.* For every $i \in N$, $t, \tau \in \mathbb{N}$, $\sigma^M, \tilde{\sigma}^M \in \Sigma^{sM}$, $\mu_{\omega,t}^R(\cdot|h_i, \sigma^M) = \mu_{\omega,\tau}^R(\cdot|\tilde{h}_i, \tilde{\sigma}^M)$ for each $h_i \in \mathscr{H}_i^t, \tilde{h}_i \in \mathscr{H}_i^\tau$ such that $s_i^R(h_i) = s_i^R(\tilde{h}_i)$.

(iii) *Markov payoffs.* For every $i \in N$, $t \in \mathbb{N}$, $s^t, \tilde{s}^t \in S^t$, and $a^t \in X^t$, $\hat{g}_{i,t}(s^t, a^t) = \hat{g}_{i,t}(\tilde{s}^t, a^t)$ if $s^t = ((s_\ell^{R,t})_{\ell \leq t}, (s_\ell^{\not{R},t})_{\ell \leq t})$, $\tilde{s}^t = ((s_\ell^{R,t})_{\ell \leq t}, (\tilde{s}_\ell^{\not{R},t})_{\ell \leq t})$.

Markov information requires that if all players follow Markov strategies, then in each period, players' beliefs about the current payoff-relevant state, conditional on their private history, depend only on the current payoff-relevant signal. Stationary Markov information strengthens this condition by requiring that these beliefs are also independent of the period: players with the same payoff-relevant signal form the same beliefs about the current state regardless of when they observed that signal. Additionally, this condition requires that the distribution of the current payoff-relevant state, conditional on the current payoff-relevant signal, is independent of strategies. This requirement is satisfied, for instance, when the history of actions is perfectly observed and incorporated into the payoff-relevant signal, or when past actions do not affect the current payoff relevant state once we condition on the current payoff-relevant signal. Together with Markov payoffs, which ensures

[29] A regular conditional measure is a transition measure which additionally is consistent with the original probability measure when integrating over conditioning events. A sufficient condition for the existence of regular conditional measures is to assume that Ω and S are Souslin spaces. A subset of a Hausdorff space is *Souslin* if it is a continuous image of a Polish space, i.e., a complete, separable metric space. See Definition 10.4.1 and Corollary 10.4.6 in Bogachev (2007).

[30] See Online Appendix B.2 for a formal definition.

that payoff-relevant signal profiles determine payoffs, these conditions formalize the notion that *bygones are bygones* in an environment with informational asymmetries and justify restricting attention to (stationary) Markov strategies.

Stationary Markov information and payoffs are satisfied in various games of interest. In standard stochastic games, these conditions hold naturally: the current state of the world, observed by all players, serves as the payoff-relevant signal, and flow payoffs depend solely on this signal and the current action profile. Notably, these conditions can also be met in certain games with asymmetric information. Examples include stochastic games with public and private shocks (Balbus et al., 2013), dynamic cheap talk games (Renault et al., 2013), and asynchronous revision games with or without observation of previous moves (Kamada and Kandori, 2020). We discuss these Markov games in detail in Applications 2 and 3.

We now define Markov absolute continuity (MAC), which plays an analogous role to SAC in Markov games. For every $t \in \mathbb{N}$, let $S^{R,t} := (S^R)^t$ be the set of histories of payoff-relevant signal profiles up to period t. The function $\gamma^{R,t} : \Omega^t \to S^{R,t}$ projects the signal transition γ^t onto the payoff-relevant dimension. For each action history $a^{t-1} \in X^{t-1}$, this projection induces a transition probability $\mu_s^{R,t}(\cdot|a^{t-1})$ over $S^{R,t}$ through the state transition $\mu_\omega^t(\cdot|a^{t-1})$, $\mu_s^{R,t}(C|a^{t-1}) \coloneqq \mu_\omega^t(\omega^t \in \Omega^t : \gamma^{R,t}(\omega^t) \in C|a^{t-1})$, for $C \in \mathcal{M}(S^{R,t})$. Let $\mu_{s_i,t}^R(\cdot|a^{t-1})$ be the marginal of $\mu_s^{R,t}$ on player i's period-t payoff-relevant signals in S_i^R.

ASSUMPTION (Markov absolute continuity)**.** *The following holds:*

(a) *For every $t \in \mathbb{N}$, $a^{t-1} \in X^{t-1}$, $\mu_s^{R,t}(\cdot|a^{t-1})$ is absolutely continuous with respect to the product measure $\prod_{i\in N}\prod_{\ell\leq t}\mu_{s_i,\ell}^R(\cdot|a^{t,(\ell-1)})$;*

(b) *For every $t \in \mathbb{N}$, $\mu_s^{R,t}$ is bounded, and continuous in X^{t-1} in strong total variation.*

MAC neither weakens nor strengthens SAC, but rather modifies it in two ways. First, it applies exclusively to *payoff-relevant signal profiles*. Second, MAC requires that absolute continuity holds with respect to the product of marginal measures over payoff-relevant private signals, taken not only across players but also across periods.

Next, we adapt of notion of trembles to Markov games. For every $\varepsilon > 0$, a measurable function $\tilde{\varepsilon}_i^M : S_i^R \to (0,1)$ is a *Markov ε-tremble* of player $i \in N$ if $\sum_{a_i\in A_i(s_i^R)}\tilde{\varepsilon}_i^M(s_i^R, a_i) < \varepsilon$ for every $s_i^R \in S_i^R$. Denote by $\tilde{\varepsilon}^M = (\tilde{\varepsilon}_i^M)_{i\in N}$ a Markov ε-tremble profile, by $\mathcal{E}^M(\varepsilon)$ the set of Markov ε-tremble profiles, and by $\mathcal{E}^M := \cup_{\varepsilon>0}\mathcal{E}^M(\varepsilon)$ the set of Markov ε-tremble profiles for any positive ε.

Given $\tilde{\varepsilon}^M \in \mathcal{E}^M$, a strategy profile $\sigma_i \in \Sigma_i$ is $\tilde{\varepsilon}^M$*-constrained* if, for each player $i \in N$

and private history $h_i \in \mathscr{H}_i$, $\sigma_i(a_i|h_i) \geq \tilde{\varepsilon}_i^M(s_i^R(h_i), a_i)$. Thus, unlike the non-Markov case where trembles restrict transition probabilities p_i, Markov trembles directly constrain strategies σ_i period by period. Let $\Sigma_i(\tilde{\varepsilon}^M)$ be the set of $\tilde{\varepsilon}^M$-constrained strategies of player i, and $\Sigma_i^M(\tilde{\varepsilon}^M)$ and $\Sigma_i^{sM}(\tilde{\varepsilon}^M) \subseteq \Sigma_i^M(\tilde{\varepsilon}^M)$ the sets of Markov and stationary Markov $\tilde{\varepsilon}^M$-constrained strategies of player i, respectively. Denote by $\Sigma(\tilde{\varepsilon}^M)$, $\Sigma^M(\tilde{\varepsilon}^M)$ and $\Sigma^{sM}(\tilde{\varepsilon}^M)$ their corresponding sets of strategy profiles.

A strategy profile $\sigma \in \Sigma^M(\tilde{\varepsilon}^M)$ is a *Markov $\tilde{\varepsilon}^M$-constrained equilibrium* if $U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i})$ for every player $i \in N$, and $\sigma_i' \in \Sigma_i(\tilde{\varepsilon}^M)$. Similarly, $\sigma \in \Sigma^{sM}(\tilde{\varepsilon}^M)$ is a *stationary Markov $\tilde{\varepsilon}^M$-constrained equilibrium* if $U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i})$ for every player $i \in N$, and $\sigma_i' \in \Sigma_i(\tilde{\varepsilon}^M)$.

The following definition adapts the notions of Markov and stationary Markov perfect equilibrium to Markov games with informational asymmetries.

DEFINITION 3. A strategy $\sigma^* \in \Sigma^M$ is a *Markov trembling hand perfect equilibrium* if:

(i) There exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}}$ in $\mathscr{E}^M$, and $(\sigma^n)_{n\in\mathbb{N}}$ in Σ^M where, for each $n \in \mathbb{N}$, $\tilde{\varepsilon}_n^M \in \mathscr{E}^M(\varepsilon_n)$, σ^n is a $\tilde{\varepsilon}_n^M$-constrained equilibrium, and $\lim_{n\to\infty} \varepsilon_n = 0$;

(ii) For every $i \in N$, and $t \in \mathbb{N}$, σ_i^n converges to σ_i^* in the weak topology of $(S_i^R \times X_i, \mu_{s_i,t}^M(\cdot|a^{t-1}))$ for each a^{t-1}.

A *stationary Markov trembling hand perfect equilibrium* $\hat{\sigma}^* \in \Sigma^{sM}$ is defined analogously.

In standard stochastic games, where players observe the history of the game before moving, (stationary) Markov perfect equilibria are defined as strategy profiles that maximizes continuation payoffs for each player. However, in settings featuring informational asymmetries, defining continuation payoffs requires specifying beliefs over possible histories that are not fully observed. To circumvent this issue, we define equilibria relying on trembling hand perfection. We refer to them as (stationary) Markov trembling hand perfect equilibria (sMTHPE and MTHPE, respectively).

In Section 5, we show that these notions feature some desirable sequential rationality properties. If a player's private history uniquely determines the sequence of states and actions, a MTHPE prescribes a best response at that history, making it a subgame perfect equilibrium. Additionally, in games satisfying stationary Markov information, an sMTHPE satisfies a condition analogous to the standard Markov perfect requirement: strategies are optimal given players' beliefs about current payoff-relevant states, conditional on their current payoff-relevant signals. In particular, in games with almost perfect information, as in standard stochastic games, our notion of sMTHPE coincides with the classical definition.

THEOREM 3. *Let Γ be a Markov game satisfying Markov information, Markov payoffs, and Markov absolute continuity. Then, Γ has a Markov perfect equilibrium. Moreover, if Γ also satisfies stationary Markov information, then it has a stationary Markov perfect equilibrium.*

The proof of Theorem 3 builds upon the argument establishing Theorem 2, with two main novelties. First, MAC ensures absolute continuity across periods, preventing strategic entanglement across periods and allowing strategies to retain their Markov properties in the limit. This preserves the continuity of payoffs with respect to period-by-period strategies, enabling us to apply a fixed point theorem to best responses over (stationary) Markov strategies. Second, we show that in a Markov environment, if a player's opponents employ Markov strategies, then by Markov information and payoff conditions, the player has a best response within the class of Markov strategies. This guarantees that, at the fixed point, no player can profitably deviate to non-Markov strategies. For stationary Markov equilibria, stationary Markov information ensures that the optimal responses can be chosen to be independent of the period, as players with the same payoff-relevant signal form identical beliefs about the current state regardless of when they observed that signal.

Games in the following class satisfy all the assumptions of Theorem 3.

APPLICATION 2 (Stochastic games with public and private shocks)**.** This class of games enriches standard stochastic games by incorporating payoff-relevant private shocks into the state space. Such a framework is well-suited to model dynamic oligopolies, where firms set prices, while the current public and private shocks represent demand and firm-specific cost characteristics, respectively.

Formally, consider a Markov game with $\Omega^R = \hat{\Omega} \times \prod_{i\in N} \Theta_i$ and $\Omega^{\not R} = X$, where $\hat{\Omega}$ is a countable set of commonly observed, payoff-relevant states (public shocks), each Θ_i is a potentially uncountable set of private shocks observed only by player $i \in N$, and X contains the commonly observed action profile from the previous period. Furthermore, the period-t payoff-relevant state $\hat{\omega}_t \in \hat{\Omega}$ is drawn according to transition probability $\mu(\hat{\omega}_t|\hat{\omega}_{t-1}, a_{t-1})$ that depends only on the previous public state and action profile and, since $\hat{\Omega}$ is countable, can be written as a density $f_{\hat{\omega}}(\hat{\omega}_t|\hat{\omega}_{t-1}, a_{t-1})$. The period-$t$ private shock profile $\theta_t \in \prod_{i\in N} \Theta_i$ is drawn according to a distribution that takes the form $f_\theta(\theta_t, \hat{\omega}_t) \prod_{i\in N} d\mu_{\theta_i}(\theta_{i,t})$ for probability distributions $\mu_{\theta_i} \in \Delta(\Theta_i)$ for each $i \in N$.

For every $i \in N$, MAC(a) holds since $\hat{\Omega}$ is countable and each Θ_i is conditionally independent; MAC(b) holds, by Proposition 8 in Online Appendix C.1, if f_θ and $f_{\hat{\omega}}$ are uniformly bounded and the latter is continuous in X for every $(\hat{\omega}_{t-1}, \hat{\omega}_t)$. Furthermore, stationary Markov information and Markov payoffa are satisfied. Therefore, by Theorem 3, there exists a stationary Markov trembling hand perfect equilibrium where each player's strategy depends only on the current public and private shocks.

The existence of an equilibrium may appear at odds with Example 2.1, since players can observe the history of infinite action profiles. However, existence holds since MAC restricts only the transition of payoff-relevant signal profiles, i.e., public and private shocks, while actions are part of the payoff-irrelevant signals. ◂

5 Properties of Equilibria

This section establishes some key sequential rationality properties of our equilibrium concepts. We begin by examining constrained equilibria, showing that they prescribe optimal strategies for each player conditional on any set of private histories.

For every player $i \in N$, and period $t \in \mathbb{N}$, define $\hat{h}_i^t : \mathcal{H}^t \to \mathcal{H}_i^t$ by $\hat{h}_i^t(\omega^t, a^{t-1}) := (\gamma_i^t(\omega^t, a^{t-1}), a_i^{t-1})$, which maps each t-period history onto player i's private history. For every strategy profile $\sigma \in \Sigma$, and event $Z \in \mathcal{M}(\mathcal{H}_i^t)$, let $P_i(Z|\sigma) = P^t((\hat{h}_i^t)^{-1}(Z)|\sigma)$ be the probability of player i's private history belonging to Z under σ. When $P_i(Z|\sigma) > 0$, player i's period-t *beliefs* about event $C \in \mathcal{M}(\mathcal{H}^t)$ *conditional on* Z are given by $P_i^t(C|Z, \sigma) = P^t(C \cap (\hat{h}_i^t)^{-1}(Z)|\sigma)/P_i(Z|\sigma)$.

For every pair of strategy profiles $\sigma, \hat{\sigma} \in \Sigma$, event $Z \in \mathcal{M}(\mathcal{H}_i^t)$, and period $t \in \mathbb{N}$, player i's *expected payoff from* $\hat{\sigma}$ *conditional on* Z *and* σ is

$$U_i(\hat{\sigma}|Z, \sigma) := \int_{\mathcal{H}^t} U_i(\hat{\sigma}|\omega^t, a^{t-1})\, dP_i^t(\omega^t, a^{t-1}|Z, \sigma).$$

Here, σ determines the conditional beliefs while $\hat{\sigma}$ determines the expected payoffs.[31]

We define two equilibrium refinements both imposing optimality conditional on sets of private histories occurring with positive probability. The first refinement requires each player's strategy to maximize the expected payoffs among $\tilde{\varepsilon}$-constrained strategies, for $\tilde{\varepsilon} \in \mathcal{E}$, while the second requires optimality among all possible strategies.

Definition 4. A strategy profile $\sigma \in \Sigma(\tilde{\varepsilon})$, for $\tilde{\varepsilon} \in \mathcal{E}$, is an *$\tilde{\varepsilon}$-constrained conditional equilibrium*, if, for every $i \in N$, $t \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}_i^t)$ satisfying $P_i(Z|\sigma) > 0$,

$$U_i(\sigma|Z, \sigma) \geq U_i(\sigma_i', \sigma_{-i}|Z, \sigma) \qquad \forall \sigma_i' \in \Sigma_i(\tilde{\varepsilon}).$$

[31] Alternatively, we can define $\hat{\sigma}$ as a continuation strategy of σ after private histories in Z.

A strategy profile $\sigma \in \Sigma$ is a *conditional equilibrium* if, for every $i \in N$, $t \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}_i^t)$ satisfying $P_i(Z|\sigma) > 0$,

$$U_i(\sigma|Z,\sigma) \geq U_i(\sigma_i', \sigma_{-i}|Z,\sigma) \qquad \forall \sigma_i' \in \Sigma_i.$$

PROPOSITION 3. *If $\tilde{\varepsilon} \in \mathcal{E}$, and σ is an $\tilde{\varepsilon}$-constrained equilibrium, then σ is an $\tilde{\varepsilon}$-constrained conditional equilibrium.*

We next relate our notion of constrained equilibrium to other equilibrium concepts where each player's strategy is optimal up to $\nu > 0$ utils. Formally, a strategy profile $\sigma \in \Sigma$ is a *ν-equilibrium* if, for every $i \in N$, and $\sigma_i' \in \Sigma_i$,

$$U_i(\sigma) + \nu \geq U_i(\sigma_i', \sigma_{-i}).$$

A conditional ν-equilibrium additionally requires that each player's strategy is optimal conditional on any positive-probability set of private histories. We adapt Myerson and Reny (2020) definition to our framework.

DEFINITION 5. Let $\nu > 0$. A strategy profile $\sigma \in \Sigma$ is a *conditional ν-equilibrium* if, for every $i \in N$, $\ell \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$ satisfying $P_i(Z|\sigma) > 0$,

$$U_i(\sigma|Z,\sigma) + \nu \geq U_i(\sigma_i', \sigma_{-i}|Z,\sigma), \qquad \forall \sigma_i' \in \Sigma_i.$$

To establish the relationship between constrained and conditional ν-equilibria, we require that players' expected payoffs conditional on any set of private histories are uniformly bounded.

ASSUMPTION (Bounded conditional payoffs)**.** *For every $i \in N$,*

$$\sup\{\bar{U}_i(\hat{\sigma}|Z,\sigma) \mid \hat{\sigma}, \sigma \in \Sigma, Z \in \mathcal{M}(\mathcal{H}_i^\ell), \ell \in \mathbb{N}, P_i(Z|\sigma) > 0\} < \infty,$$

where $\bar{U}_i(\hat{\sigma}|Z,\sigma)$ is calculated as $U_i(\hat{\sigma}|Z,\sigma)$ with g_i replaced by $|g_i|$.

This condition holds naturally in games with finitely many periods or discounted payoffs. Under this assumption, we obtain the following result.

PROPOSITION 4. *Under sequential absolute continuity, if σ^ε is an $\tilde{\varepsilon}$-constrained equilibrium with $\tilde{\varepsilon} \in \mathcal{E}(\varepsilon)$, then σ^ε is also a $\nu(\varepsilon)$-equilibrium, where $\nu(\varepsilon) = O(\varepsilon)$.*[32] *In addition, if bounded conditional payoffs holds, then σ^ε is also a conditional $\nu(\varepsilon)$-equilibrium.*

[32]That is, there exists a constant $K > 0$ such that $\nu(\varepsilon) \leq K\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Together, Propositions 3 and 4 show that trembling hand perfect equilibria are approximated by sequences of strategies that satisfy sequential rationality properties and are fully mixed. The full support property of constrained equilibria ensures that all private histories that can be reached by some strategy have positive probability. This means constrained equilibria test the optimality of players' strategies even after histories that would be off-path in the limit strategy.

We now show that trembling hand perfect equilibria are best responses at identifiable histories—histories where a player can perfectly deduce both the history of states and actions from their private history. As a result, THPE are subgame perfect equilibria and Nash equilibria. Additionally, we show that THPE are conditional equilibria.

A strategy profile $\sigma^* = (\sigma_i^*)_{i \in N}$ is a *Nash equilibrium* (NE) if no player can unilaterally deviate to improve their payoff. Formally, for every $i \in N$, and $\sigma_i' \in \Sigma_i$,

$$U_i(\sigma^*) \geq U_i(\sigma_i', \sigma_{-i}^*).$$

Let $\mathcal{H}_i^{\text{id}}$ denote the set of histories where player $i \in N$ can perfectly deduce both the history of states and actions from her private history. Formally, $\mathcal{H}_i^{\text{id}} := \{(\omega^t, a^{t-1}) \in \mathcal{H}^t : (\hat{h}_i^t)^{-1}(\hat{h}_i^t(\omega^t, a^{t-1})) = \{(\omega^t, a^{t-1})\}\}$. A history (ω^t, a^{t-1}) is the *root of a proper subgame* if it belongs to $\mathcal{H}_i^{\text{id}}$ for all players $i \in N$.[33] A set $H \in \mathcal{M}(\mathcal{H})$ is *negligible* if $P^t(H \cap \mathcal{H}^t | \sigma) = 0$ for every $t \in \mathbb{N}$, $\sigma \in \Sigma$.

A strategy profile σ^* is *optimal at identifiable histories* if there exists a negligible set $H \in \mathcal{M}(\mathcal{H})$ such that, for each player $i \in N$, and history $(\omega^t, a^{t-1}) \in (\mathcal{H}_i^{\text{id}} \cap \mathcal{H}^t) \setminus H$,

$$U_i(\sigma^* | \omega^t, a^{t-1}) \geq U_i(\sigma_i', \sigma_{-i}^* | \omega^t, a^{t-1}). \tag{6}$$

A strategy profile σ^* is a *subgame perfect equilibrium* (SPE) if it induces a Nash equilibrium in every proper subgame. Formally, condition (6) holds for all histories $(\omega^t, a^{t-1}) \in (\bigcap_{i \in N} \mathcal{H}_i^{\text{id}} \cap \mathcal{H}^t) \setminus H$. If σ^* is optimal at identifiable histories then it is also an SPE.

PROPOSITION 5. *Under sequential absolute continuity, if σ^* is a trembling hand perfect equilibrium, then it is a conditional equilibrium and optimal at identifiable histories. As a consequence, σ^* is also a subgame perfect equilibrium and a Nash equilibrium.*

Proposition 5 establishes that a THPE is optimal both after identifiable histories and after private histories that occur with positive probability during play. Notably, sequential absolute continuity does not necessarily preclude the existence of proper subgames or identifiable histories. In games with finite actions, SAC holds if all but one player has countably

[33] This definition adapts Myerson and Reny's (2020) subgame notion to our setting.

infinite signals. In this case, any player can can observe the history of the game in any period, leading to identifiable histories. Moreover, identifiable histories may arise even with uncountable signals. For instance, in a one-period two-player game, SAC is satisfied when player 1 observes the state $\omega \in \Omega$ perfectly while the opponent's signal is drawn according to a density $f(s_2, \omega)$, $s_2 \in S_2$, with respect to some probability measure $\nu \in \Delta(S_2)$.

Next, we examine the properties of stationary Markov trembling hand perfect equilibria. As is shown below, players' continuation payoffs under a stationary Markov strategy profile depend solely on their current payoff-relevant signal. We prove that a sMTHPE maximizes these continuation payoffs at almost every payoff-relevant signal.

Consider a Markov game satisfying stationary Markov information and Markov payoffs. The continuation expected payoff of player $i \in N$ after observing a payoff-relevant signal $s_i^R \in S_i^R$ can be written as

$$U_i(\sigma|s_i^R) = \sum_{\substack{t \geq 1, \\ a^t \in X^t}} \int_{(\Omega^R)^2} g_i(\omega_t, a_t) p(a^t|\omega_t, a^{t,(t-1)}, \sigma) d\mu_{\omega_t|\omega_1}(\omega_t|\omega_1, a^{t,(t-1)}) d\mu_\omega^R(\omega_1|s_i^R),$$

where $\mu_{\omega_t|\omega_1}(\cdot|\omega_1, a^{t-1})$ denotes the transition probability from payoff-relevant state ω_1 to ω_t conditional on action history a^{t-1}, and $\mu_\omega^R(\cdot|s_i^R)$ represents $\mu_{\omega,t}^R(\cdot|h_i, a^{t-1})$ omitting the variables that it does not depend on due to the stationary Markov information assumption. Stationary Markov information and Markov payoffs guarantee that this continuation expected payoff is independent of the strategies used by players before player i observes s_i^R and of the period of play.

For player $i \in N$, a strategy $\sigma_i \in \Sigma_i$ satisfies the *Markov best response condition at the payoff-relevant signal* $s_i^R \in S_i^R$ if $U_i(\sigma^*|s_i^R) \geq U_i(\sigma_i', \sigma_{-i}^*|s_i^R)$ for all $\sigma_i' \in \Sigma_i$.

PROPOSITION 6. *Consider a Markov game satisfying stationary Markov information, Markov payoffs, and Markov absolute continuity. If σ^* is a stationary Markov trembling hand perfect equilibrium, then for each player $i \in N$, there exists $\tilde{H}_i^R \in \mathcal{M}(S_i^R)$ with $\mu_{s_i,t}^R(\tilde{H}_i^R|a^{t-1}) = 1$ for all $a^{t-1} \in X^{t-1}$ and $t \in \mathbb{N}$, such that σ_i^* satisfies the Markov best response condition at every payoff-relevant signal $s_i^R \in \tilde{H}_i^R$.*

This proposition shows that in Markov games, sMTHPE are optimal after every payoff-relevant signal except possibly for a negligible subset of signals. For games with almost perfect information, which is the standard case in the literature, this implies that our notion of stationary Markov perfect equilibrium coincides with the classical definition.

6 Relaxing Discounted Payoffs

Our payoff boundedness condition does not only accommodate but also generalizes the discounted payoffs assumption commonly employed by the literature. This greater generality is substantive: we show that our condition holds for a class of non-discounted games, which we call *games with stochastic move opportunities*, where players choose their actions at randomly drawn times. Revision games (Kamada and Kandori, 2020) exemplify this class of games.

Consider the following class of non-discounted games where players draw opportunities to move at random periods. Opportunities to move are drawn from an interval $[0, T)$, with $T \in \mathbb{R}_+ \cup \{\infty\}$, and states of the world takes the form $\Omega = [0, T) \times \hat{\Omega}$, where $\hat{\Omega}$ is a set of underlying states that contains an absorbing state $\hat{\omega}^{end}$, such that for any $t \in [0, T)$, the game ends if $(t, \hat{\omega}^{end})$ is reached.

For every $\ell \in \mathbb{N}$, period ℓ represents the ℓ'th opportunity to move for any player, and t_ℓ is the corresponding timing. For $\omega_\ell = (t_\ell, \hat{\omega}_\ell)$ and $\omega_{\ell'} = (t_{\ell'}, \hat{\omega}_{\ell'})$, the state transition probability is such that, with probability one, $t_\ell < t_{\ell'}$ whenever $\ell < \ell'$, ensuring that later timings are assigned to later opportunities. Furthermore, for $\omega^t \in \Omega^t$, if $\omega^t_\tau = (\cdot, \hat{\omega}^{end})$ for some $\tau < t$ then $g_i(\omega^t, \cdot) = 0$ for all $i \in N$. That is, after visiting the state $\hat{\omega}^{end}$ for the first time, players do not incur flow payoffs.

Denote by $H^{end,t} := \{(\omega^t, a^t) \in \Omega^t \times X^t | \omega^t_t = (\cdot, \hat{\omega}^{end}), \omega^t_\ell \neq (\cdot, \hat{\omega}^{end}), \ell < t\}$ the subset of $\Omega^t \times X^t$ in which the game ends in period t. The following lemma provides a condition which implies payoff boundedness in games with stochastic move opportunities.

LEMMA 3. *If* $\sum_{t \in \mathbb{N}} t \cdot \sup_{\sigma \in \Sigma} P^t(H^{end,t}|\sigma) < \infty$ *and* $\sum_{t \in \mathbb{N}} P^t(H^{end,t}|\sigma) = 1$*, for every* $\sigma \in \Sigma$*, then the payoff boundedness condition holds.*

In games with stochastic move opportunities, Lemma 3 states that payoff boundedness holds whenever (i) the sequence in which the t-th element is t times an upper bound on the period-t probability that the game ends is summable, and (ii) the game ends in finite time with probability 1.

A consequence of this result is that, when actions do not affect the length of the game, i.e., $P^t(H^{end,t}|\sigma)$ is independent of σ, payoff boundedness is satisfied as long as the expected number of opportunities to move is finite, and (ii) holds. Notably, these conditions hold in revision games, where opportunities to move are drawn at exogenous Poisson rates independently of the previous play.[34]

[34] Revision games can be represented as follows. Move opportunities arrive prior to an exogenous deadline, $T < \infty$, at a constant Poisson rate. The underlying state space $\hat{\Omega}$ contains two elements $\{\omega^0, \hat{\omega}^{end}\}$.

7 Imperfect Observation of Moving Periods

Theorems 1, 2, and 3, and Propositions 3, 4, 5, and 6 hold in a broader class of games that allows players to be *inactive* in certain periods. During these inactive periods, players neither make moves nor record information in their private histories. Due to the additional complexity this generalization entails, we present the complete details in Online Appendix B.1. This section outlines the main elements of this extension and illustrates an application to Markov games.

Whenever *active*, a player receives informative signals about the history of states and action profiles before moving; when *inactive*, a player neither observes signals nor takes non-trivial actions. Private histories exclude information from inactive periods, implying players may be oblivious to the number of moves that have occurred. We assume that at least one player is active in each period, and without loss, that players receive non-zero flow payoffs only when active.

Allowing for active and inactive players extends our analysis to settings where players may not be informed whether moves have taken place in the past, thus introducing uncertainty about the multi-stage structure of the game. Such games have been studied in various contexts: Kreps and Ramey (1987) provide an early example where players lack a sense of calendar time; Matsui (1989) considers an espionage game with private revision opportunities; Kamada and Moroni (2018) examine outcomes in coordination games with private timing; Doval and Ely (2020) investigate the range of equilibrium outcomes that can arise across different information structures and extensive forms for a given base game.

We apply the inactive player framework to the class of Markov games to show the existence of stationary Markov perfect equilibria. See Online Appendix B.7 for a formal description.

APPLICATION 3 (Two classes of asynchronous games)**.** We can model asynchronous games that satisfy MAC by assuming only one player is active at each period.

Asynchronous revision games with finite actions, where moving opportunities are drawn at Poisson rates (Kamada and Kandori, 2020), fall within this application. Players perfectly observe the payoff-relevant state, while previous moving times are treated as part of the payoff-irrelevant state. This ensures MAC holds even though move timings belong to an uncountable set. By Theorem 3 and Proposition 6, there exists a sMTHPE that

The game starts in state ω^0 and switches to $\hat{\omega}^{end}$ as soon as a time larger than the deadline is drawn. There is a finite constant set of available actions, and the players' payoffs realize once the game ends.

satisfies the Markov best response condition at almost every payoff-relevant signal. Since players perfectly observe the current payoff-relevant state, our equilibrium notion coincides with the standard Markov perfect equilibrium.

Furthermore, Theorem 3 implies the existence of a sMTHPE in dynamic cheap talk games (Renault et al., 2013), where an informed sender communicates with a receiver who controls state transitions.[35] With finite message and action spaces, payoff boundedness, continuity, and MAC hold, even with uncountable exogenous states observed only by the sender. The equilibrium has the sender conditioning on the receiver's previous action and current exogenous state, while the receiver conditions on her previous action and current message. ◂

A Appendix

The following lemma shows that each player's expected payoff can be written in a simple way as a function of $p(\cdot|\cdot,\sigma)$.

Lemma 4. *The function $p(\cdot|\cdot,\sigma)$ is measurable, the set Σ is non-empty, and, for each $i \in N$, player i's expected payoff when players follow strategy $\sigma \in \Sigma$ is given by*

$$U_i(\sigma) = \sum_{t\in\mathbb{N}} U_{i,t}(\sigma), \tag{7}$$

where $U_{i,t}(\sigma) := \sum_{a^t \in X^t} \int_{\Omega^t} g_i(\omega^t, a^t) \cdot p(a^t|\omega^t,\sigma)\, d\mu^t_\omega(\omega^t|a^{t,(t-1)})$.

Proof. Define the correspondence $\tilde{A}_i : \mathcal{H}_i \rightrightarrows \Delta(X_i)$ as $\tilde{A}_i(h_i) := \Delta(A_i(h_i))$, where $\Delta(X_i)$ is endowed with the weak topology of probability measures. For each $h_i \in \mathcal{H}_i$, $\tilde{A}_i(h_i)$ is the set of probability measures with support in $A_i(h_i)$. By Himmelberg and Van Vleck (1975), $\tilde{A}_i(h_i)$ is weakly measurable. By Theorems 15.19 and 18.13 in Aliprantis and Border (2006), $\tilde{A}_i$ admits a measurable selector. This shows that the set of strategies is non-empty.

Since g_i is bounded, the Fubini-Tonelli theorem[36] allows us to repeatedly interchange the order of integration between the counting measure over actions and the measure over states. This yields player i's period-t expected payoff $U_{i,t}(\sigma)$ under strategy profile σ. □

[35] To satisfy the Markov information condition, the state transition must be either independent or controlled solely by the receiver's action, without dependence on the current state.

[36] Theorems 11.27 and 11.28 in Aliprantis and Border (2006).

Following Lemma 4, for $t \in \mathbb{N}$, we rewrite player i's period-t expected payoff from $\sigma \in \Sigma$ after history $(\omega^\tau, a^{\tau-1}) \in \mathcal{H}^\tau$, $\tau \leq t$, as $U_{i,t}(\sigma|\omega^\tau, a^{\tau-1}) = \sum_{a^t \in X^t} \int_{\Omega^t} g_i(\omega^t, a^t) \cdot p(a^t|\omega^t, a^{\tau-1}, \sigma)\, d\mu_\omega^t(\omega^t|\omega^\tau, a^{t,(t-1)})$.

A.1 Proof of Theorems 1 and 2

We use the following notation. For any set Y and Z, define $Y^\infty := \cup_{t\in\mathbb{N}} Y^t$ and $(X \times Z)^\infty := \cup_{t\in\mathbb{N}} X^t \times Z^t$. We also write $\mathcal{H}^t_{A_i} = \mathcal{H}_{A_i} \cap \mathcal{H}^t_i$ and $\bar{\mathcal{H}}^t_{A_i} = \bar{\mathcal{H}}_{A_i} \cap (S^t_i \times X^t_i)$. The set of Carathéodory integrands on $(Y \times Z, \beta)$, where Z is a countable metric space endowed with its Borel σ-algebra and $\beta \in \Delta(Y)$ is a probability measure, is denoted by $CI(Y \times Z, \beta)$.

For our arguments we will use nets. A net is a generalization of a sequence indexed by a directed set rather than the natural numbers.[37] A generic net has the form $(x_\lambda)_{\lambda\in\Lambda}$, where Λ is a directed set. In the special case where σ-algebras are countably generated, sequences would suffice in all our arguments (Proposition 2.3 in Balder, 2001).

We start by defining the relevant topology on the set of strategy profiles. For every $i \in N$, we can construct a reference measure $\nu^t_i \in \Delta(S^t_i)$ such that, for each $a^{t-1} \in X^{t-1}$, $\mu^t_{s_i}(\cdot|a^{t-1})$ is absolutely continuous with respect to ν^t_i.[38]

Let $\delta < 1$. For each $i \in N$, we define $\alpha_i \in \Delta(S^\infty_i)$, as $\alpha_i(B) := \frac{1-\delta}{\delta} \sum_{t\in\mathbb{N}} \delta^t \cdot \nu^t_i(B \cap S^t_i)$ for every $B \in \mathcal{M}(S^\infty_i)$. Define $I_{i,\varphi_i} : \Sigma_i \to \mathbb{R}$ as

$$I_{i,\varphi_i}(\sigma_i) := \int_{S^\infty_i} \sum_{\bar{a}_i \in X^\infty_i} \varphi_i(\bar{s}_i, \bar{a}_i) \cdot p_i(\bar{a}_i|\bar{s}_i, \sigma_i), d\alpha_i(\bar{s}_i),$$

The *weak topology* on Σ_i is the coarsest topology such that $I_{i,\varphi}$ is continuous for every $\varphi_i \in CI(S^\infty_i \times X^\infty_i, \alpha_i)$. The *weak topology* on $\Sigma = \times_{i\in N}\Sigma_i$ is the product topology, where each Σ_i is endowed with its weak topology.

Consider the sets of induced transition probabilities $\mathcal{P}_i := \{p_i(\cdot|\cdot, \sigma_i)|\sigma_i \in \Sigma_i\}$ and $\mathcal{P}_i(\tilde{\varepsilon}) := \{p_i(\cdot|\cdot, \sigma_i)|\sigma_i \in \Sigma_i(\tilde{\varepsilon})\}$, for $\tilde{\varepsilon} \in \mathcal{E}$, and endow them with the *weak topology* defined in the same manner as above. Let the *weak topology* on $\mathcal{P} = \times_{i\in N}\mathcal{P}_i$, and $\mathcal{P}(\tilde{\varepsilon}) := \times_{i\in N}\mathcal{P}_i(\tilde{\varepsilon})$ be their product topology.[39] For every $i \in N$, the resulting topological spaces

[37] A *directed set* is a set equipped with a binary relation $\geq$ that is reflexive, transitive, and satisfies: for any $\lambda, \lambda' \in \Lambda$, there exists $\beta \in \Lambda$ such that $\beta \geq \lambda$ and $\beta \geq \lambda'$.

[38] Footnote 24 exemplifies the construction of one such reference measure.

[39] We view $p_i(\cdot|\cdot, \sigma_i)$ as a transition probability from S^∞_i to X^∞_i, where for each $t \in \mathbb{N}$, and $s^t_i \in S^t_i$, $p_i(\cdot|s^t_i, \sigma_i)$ has support in X^t_i.

$\Sigma(\tilde{\varepsilon})$ and $\mathscr{P}(\tilde{\varepsilon})$ are homeormorphic. This holds since, for every $\tilde{p}_i(\cdot|\cdot) \in \mathscr{P}_i(\tilde{\varepsilon})$, there is a strategy, defined recursively as

$$\sigma_i\big(a^t_{i,t}|s^t_i, a^{t,(t-1)}_i\big) = \frac{\tilde{p}_i(a^t_i|s^t_i)}{\tilde{p}_i(a^{t,(t-1)}_i|s^{t,(t-1)}_i\big)}, \tag{8}$$

for $t \in \mathbb{N}$, $(s^t_i, a^{t,(t-1)}_i) \in \mathscr{H}^t_{A_i}$ (where $\tilde{p}_i(a^{t,(0)}_i|s^{t,(0)}_i) := 1$), satisfying $p_i(\cdot|\cdot, \sigma_i) = \tilde{p}_i(\cdot|\cdot)$.

Let us extend the definition of $p_i(a^t_i|s^t_i, a^\tau_i, \sigma_i)$, in equation (1), for $\tau \geq t$ and $(s^t_i, a^t_i) \in S^t_i \times X^t_i$, $a^\tau_i \in X^\tau_i$, setting it equal to 1 if $a^{\tau,(t)}_i = a^t_i$, and zero otherwise.

The following lemma is useful for the proofs of Theorems 1 and 2.

Lemma 5. *Let $\{\sigma^\lambda_i\}_{\lambda\in\Lambda}$ be a net in Σ_i for $i \in N$. Suppose that for each $\tau \in \mathbb{N}$ and $a^\tau_i \in X^\tau_i$, the net $\{p_i(\cdot|\cdot, a^\tau_i, \sigma^\lambda_i)\}_{\lambda\in\Lambda}$ converges to $p^*_i(\cdot|\cdot, a^\tau_i) \in \mathscr{P}_i$ in the weak topology of $\mathscr{P}_i$. Then:*

(i) There exists $\sigma^*_i \in \Sigma_i$ that satisfies equation (4) *for every $t, \tau \in \mathbb{N}$, with $\tau \leq t$.*

(ii) If $\tilde{\varepsilon} \in \mathscr{E}$ and $\{\sigma^\lambda_i\}_{\lambda\in\Lambda} \subseteq \Sigma_i(\tilde{\varepsilon})$, then $\sigma^*_i \in \Sigma_i(\tilde{\varepsilon})$.

Proof. We define $\sigma^*_i : S^t_i \times X^t_i \to [0, 1]$ recursively as follows:

- For $s_i \in S_i$, let $\sigma^*_i(a_i|s_i) = p^*_i(a_i|s_i, \emptyset)$;
- Suppose $\sigma^*_i(a^\ell_{i,\ell}|s^\ell_i, a^{\ell,(\ell-1)}_i)$ has been defined for every $(s^\ell_i, a^\ell_i) \in \bar{\mathscr{H}}^\ell_{A_i}$, with $\ell \leq t-1$. For $(s^t_i, a^{t,(t-1)}_i) \in \mathscr{H}^t_{A_i}$, define

$$\sigma^*_i(a^t_{i,t}|s^t_i, a^{t,(t-1)}_i) := \frac{p^*_i(a^t_i|s^t_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i))}{p_i(a^{t,(t-1)}_i|s^{t,(t-1)}_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i), \sigma^*_i)}.$$

Remark 1. *Consider the following.*

1. *By the definition of $\mathfrak{a}_i$, $p_i(a^{t,(t-1)}_i|s^{t,(t-1)}_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i), \sigma^*_i) > 0$, $\forall (s^t_i, a^t_i) \in S^t_i \times X^t_i$.*
2. *The functions $\mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i)$ and $p_i(a^{t,(t-1)}_i|s^{t,(t-1)}_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i), \sigma^*_i)$ depend only on $\sigma^*_i(a^\ell_{i,\ell}|s^\ell_i, a^{\ell,(\ell-1)}_i)$ for $\ell \leq t-1$. Therefore, $\sigma^*_i(a^t_{i,t}|s^t_i, a^{t,(t-1)}_i)$ is well-defined.*
3. *By the definition of σ^*_i, we have $p^*_i(a^t_i|s^t_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i)) = p_i(a^t_i|s^t_i, \mathfrak{a}_i(s^t_i, a^t_i; \sigma^*_i), \sigma^*_i)$ for every $(s^t_i, a^t_i) \in \mathscr{H}^t_{A_i}$.* $\triangle$

First, we show that $\sigma^*_i(\cdot|s^t_i, a^{t-1}_i) \in \Delta(A_i(s^t_i, a^{t-1}_i))$ for every $(s^t_i, a^{t-1}_i) \in \mathscr{H}^t_{A_i}$. Since $\sum_{a_i\in X_i} \sigma^\lambda_i(a_i|(s^{t-1}_i, s_i), a^{t-1}_i) = 1$ for all $\lambda \in \Lambda$, $((s^{t-1}_i, s_i), a^{t-1}_i) \in \mathscr{H}^t_{A_i}$, by Lemma 15 with $C = \{a^{t,(\tau)}_i\} \times X_i$, we obtain

$$p_i(a^{t-1}_i|s^{t-1}_i, a^{t,(\tau)}_i, \sigma^*_i)\Bigg(\sum_{a_i\in X_i} \sigma^*_i(a_i|(s^{t-1}_i, s_i), a^{t-1}_i) - 1\Bigg) = 0,$$

ν_i^t-almost surely for every $((s_i^{t-1}, s_i), a_i^{t-1}) \in \mathcal{H}^t_{A_i}$ s.t. $a_i^{t,(\tau)} = a_i((s_i^{t-1}, s_i), a_i^{t-1}; \sigma_i^*)$[40]. By Remark 1.1, this shows $\sigma_i^*(\cdot|s_i^t, a_i^{t-1}) \in \Delta(X_i)$ for $(s_i^t, a_i^{t-1}) \in \mathcal{H}^t_{A_i}$.

We now show that, $\operatorname{supp} \sigma_i^*(\cdot|s_i^t, a_i^{t-1}) \subseteq A_i(s_i^t, a_i^{t-1})$ for every $(s_i^t, a_i^{t-1}) \in \mathcal{H}^t_{A_i}$. In fact, define the function $\hat{\varphi}_i : \cup_{t\in\mathbb{N}} S_i^t \times X_i^t \to \mathbb{R}$ as $\hat{\varphi}_i(s_i^t, a_i^t) := \inf\{d(a_i^t, \tilde{a}_i^t)|(s_i^t, \tilde{a}_i^t) \in \bar{\mathcal{H}}_{A_i}\}$, where d is the distance in the metric space X^t. The function $\hat{\varphi}_i$ is continuous in a_i^t, as it is a "distance to a set" function, and it is measurable in s_i^t for each a_i^t.[41] The function $\hat{\varphi}_i(s_i^t, a_i^t)$ is zero for $(s_i^t, a_i^t) \in \bar{\mathcal{H}}_{A_i}$, and strictly positive otherwise, as A_i is closed-valued. For $\tilde{a}_i \in X_i$, by Lemma 15 in Online Appendix C.1 for $C = \{(a_i^{t-1}, \tilde{a}_i)\}$, we obtain

$$\hat{\varphi}_i((s_i^{t-1}, s_i), (a_i^{t-1}, \tilde{a}_i)) \cdot p_i(a_i^{t-1}|s_i^{t-1}, a_i^{t,(\tau)}, \sigma_i^*) \cdot \sigma_i^*(\tilde{a}_i|(s_i^{t-1}, s_i), a_i^{t-1}) = 0,$$

ν_i^t-almost surely for $((s_i^{t-1}, s_i), a_i^{t-1}) \in \mathcal{H}^t_{A_i}$ such that $a_i^{t,(\tau)} = a_i((s_i^{t-1}, s_i), a_i^{t-1}, \sigma_i^*)$. By Remark 1.1, this shows that $\sigma_i^*(\tilde{a}_i|(s_i^{t-1}, s_i), a_i^{t-1}) = 0$ if $((s_i^{t-1}, s_i), a_i^{t-1}) \in \mathcal{H}^t_{A_i}$, but $\tilde{a}_i \notin A_i((s_i^{t-1}, s_i), a_i^{t-1})$. Hence, $\sigma_i^*(\cdot|(s_i^{t-1}, s_i), a_i^{t-1}) \in \Delta(A_i((s_i^{t-1}, s_i), a_i^{t-1}))$.

Finally, suppose $\sigma_i^\lambda \in \Sigma_i(\tilde{\varepsilon})$ for every $\lambda \in \Lambda$, we show that $\sigma_i^* \in \Sigma_i(\tilde{\varepsilon})$. We have $p_i(a_i^t|s_i^t, \sigma_i^\lambda) - \tilde{\varepsilon}_i(s_i^t, a_i^t) \geq 0$, for every $(s_i^t, a_i^t) \in \bar{\mathcal{H}}_{A_i}$. Therefore, by Lemma 15 setting $C = \{a_i^t\}$, we obtain $p_i(a_i^t|s_i^t, \sigma_i^*) \geq \tilde{\varepsilon}_i(s_i^t, a_i^t) > 0$, ν_i^t-almost surely for $(s_i^t, a_i^t) \in \bar{\mathcal{H}}_{A_i}$. $\square$

Proof of Theorem 1. Fix $\tilde{\varepsilon} \in \mathcal{E}$. Let $\sigma \in \Sigma(\tilde{\varepsilon})$ and $p \in \mathcal{P}(\tilde{\varepsilon})$, with $p_i = p_i(\cdot|\cdot, \sigma_i)$ for $i \in N$. We define player i's expected utility evaluated at p as $\tilde{U}_i(p) := U_i(\sigma)$. Player i's *$\tilde{\varepsilon}$-constrained best response correspondence* $r_i : \mathcal{P}_{-i}(\tilde{\varepsilon}) \rightrightarrows \mathcal{P}_i(\tilde{\varepsilon})$ is

$$r_i(p_{-i}) \in \arg\max\{\tilde{U}_i(\tilde{p}_i, p_{-i})|\tilde{p}_i \in \mathcal{P}_i(\tilde{\varepsilon})\},$$

and $r : \mathcal{P}(\tilde{\varepsilon}) \rightrightarrows \mathcal{P}(\tilde{\varepsilon})$ is the Cartesian product of the r_i over $i \in N$.

Due to the homeomorphism between $\Sigma(\tilde{\varepsilon})$ and $\mathcal{P}(\tilde{\varepsilon})$, every fixed point of r is mapped to a strategy profile which, by the definition of r, forms an $\tilde{\varepsilon}$-constrained equilibrium. We establish the existence of such a fixed point by applying the Kakutani-Fan-Glicksberg fixed point theorem. To verify that the hypotheses of the theorem hold we show that: **(1)** $\mathcal{P}(\tilde{\varepsilon})$ is compact, and **(2)** convex; **(3)** r has closed graph, **(4)** is non-empty, and **(5)** convex valued.[42] Point **(5)** follows directly from Lemma 4.

[40] With a slight abuse of notation, we write a_i in terms of a_i^{t-1} as the dependency on a_i^t is immaterial.

[41] This follows by the measurable maximum theorem as A_i is weakly measurable, non-empty, and compact valued. See Theorem 18.19 in Aliprantis and Border (2006).

[42] As discussed by Balder (1988), the requirement that $\mathcal{P}(\tilde{\varepsilon})$ is a subset of a locally convex Hausdorff space is satisfied by considering equivalent classes of transition probabilities that induce the same expected payoff.

(1) $\mathscr{P}(\tilde{\varepsilon})$ is compact. By Theorem 2.3 in Balder (1988), $\mathscr{P}(\tilde{\varepsilon})$ is relatively compact.[43] To see that it is closed, let $(\sigma_i^\lambda)_{\lambda\in\Lambda}$ be a net of player i's strategies, and suppose that $p_i(\cdot|\cdot,\sigma_i^\lambda)$ converges weakly to a transition probability represented by the measurable function $p_i^*(\cdot|\cdot)$. Then $p_i^*(\cdot|\cdot) = p_i(\cdot|\cdot,\sigma_i^*)$, for σ_i^* defined recursively using equation (8). By Lemma 5, $\sigma_i^* \in \Sigma_i(\tilde{\varepsilon})$.

(2) $\mathscr{P}(\tilde{\varepsilon})$ is convex. Let $p', p'' \in \mathscr{P}(\tilde{\varepsilon})$, with associated strategies σ' and σ'', respectively. For every $\beta \in [0,1]$, we show that $\bar{p}_i = \beta\cdot p_i' + (1-\beta)\cdot p_i''$ belongs to $\mathscr{P}_i(\tilde{\varepsilon})$ by showing that σ_i, as defined by equation (8) with $\tilde{p}_i = \bar{p}_i$, belongs to $\Sigma_i(\tilde{\varepsilon})$, and satisfies $p_i(a_i^t|s_i^t,\sigma_i) = (\beta\cdot p_i' + (1-\beta)\cdot p_i'')(a_i^t|s_i^t)$.

First, it is immediate that $p_i(\cdot|\cdot,\sigma_i) \geq \tilde{\varepsilon}_i$. Second, for every $(s_i^t, a_i^{t,(t-1)}) \in \mathscr{H}_{A_i}$, $\sigma_i \in [0,1]$, since $p_i(a_i^{t,(t-1)}|s_i^{t,(t-1)}) \geq p_i(a_i^t|s_i^t)$ for every $p_i \in \mathscr{P}_i(\tilde{\varepsilon})$. Finally, we can write

$$\sigma_i\big(a_{i,t}^t|s_i^t, a_i^{t,(t-1)}\big) = \frac{1}{\bar{p}_i(a_i^{t,(t-1)}|s_i^{t,(t-1)})}\big(\sigma_i'\big(a_{i,t}^t|s_i^t, a_i^{t,(t-1)}\big)\cdot\beta\cdot p_i'(a_i^{t,(t-1)}|s_i^{t,(t-1)}) + \sigma_i''\big(a_{i,t}^t|s_i^t, a_i^{t,(t-1)}\big)\cdot(1-\beta)\cdot p_i''(a_i^{t,(t-1)}|s_i^{t,(t-1)})\big),$$

which implies $\sum_{a_i\in A_i(h_i)}\sigma_i(a_i|h_i) = 1$ for each $h_i \in \mathscr{H}_i$.

(3) and (4) r has closed graph and is non-empty. By Radon-Nikodym theorem, SAC(a) implies the existence of a density function $\tilde{f}^t : S^t \times X^{t-1} \to \mathbb{R}$ such that $d\mu_s^t(s^t|a^{t-1}) = \tilde{f}^t(s^t, a^{t-1}) \cdot d\mu_{s_1}^t(s_1^t|a^{t-1}) \times \ldots \times d\mu_{s_n}^t(s_n^t|a^{t-1})$ for every $t \in \mathbb{N}$. For $t\in\mathbb{N}$, $s^t \in S^t$, let $\nu^t(s^t) = \prod_{i\in N}\nu_i^t(s_i^t)$. Applying again Radon-Nikodym, we can write

$$d\mu_s^t(s^t|a^{t-1}) = f^t(s^t, a^{t-1})\cdot d\nu^t(s^t)$$

where $f^t : S^t\times X^{t-1}\to\mathbb{R}$ denotes another density function. Thus, by combining Lemma 4, payoff continuity and SAC(a), we obtain, for each $t\in\mathbb{N}$

$$U_{i,t}(\sigma) = \int_{S^t}\sum_{a^t\in X^t}\hat{g}_{i,t}(s^t,a^t)\cdot f^t(s^t, a^{t,(t-1)})\cdot p(a^t|s^t,\sigma)\,d\nu^t(s^t).$$

Proposition 8 in Online Appendix C.1 shows that there is a version of $f^t(s^t, a^{t,(t-1)})$ in $CI(S^t\times X^{t-1},\nu^t)$ if and only if SAC(b) holds. Thus, by Theorem 2.5 in Balder (1988), $\tilde{U}_{i,t}(p(\cdot|\cdot,\sigma)) \coloneqq U_{i,t}(\sigma)$ is continuous in $p\in\mathscr{P}(\tilde{\varepsilon})$, since, by payoff continuity, $\hat{g}_{i,t}(s^t,a^t)\cdot f^t(s^t,a^{t,(t-1)})$ is a Carathédory integrand for every $t\in\mathbb{N}$.

Let $\tilde{U}_i(p(\cdot|\cdot,\sigma)) \coloneqq \sum_{t=1}^\infty\tilde{U}_{i,t}(p(\cdot|,\cdot,\sigma))$. By Lemma 4, $|\tilde{U}_i(p(\cdot|\cdot,\sigma))| \leq \sum_{t\in\mathbb{N}}\sup_{\sigma\in\Sigma}|\tilde{U}_{i,t}(p(\cdot|,\cdot,\sigma))| < \infty$, where the last inequality follows by payoff boundedness. Therefore, if $(\sigma^\lambda)_{\lambda\in\Lambda}$ is a net in Σ and $\sigma^*\in\Sigma$, $p(\cdot|\cdot,\sigma^\lambda)\to p(\cdot|\cdot,\sigma^*)$ implies

[43] A set is *relatively compact* if its closure is compact.

$\tilde{U}_i(p(\cdot|\cdot,\sigma^\lambda)) \to \tilde{U}_i(p(\cdot|\cdot,\sigma^*))$ by Proposition 10 in Online Appendix C.2, which establishes a special case of the dominated convergence theorem. This shows that $\tilde{U}_i(p(\cdot|\cdot,\sigma))$ is continuous on $\mathscr{P}$.

The compactness of $\mathscr{P}_i(\tilde{\varepsilon})$ and continuity of $\tilde{U}_i$ imply that r_i is non-empty. To see that r has closed graph, let $(p^\lambda)_{\lambda\in\Lambda}$ and $(\hat{p}^\lambda)_{\lambda\in\Lambda}$ be two nets in $\mathscr{P}(\tilde{\varepsilon})$ such that, $\hat{p}^\lambda \in r(p^\lambda)$ for every $\lambda \in \Lambda$, $\hat{p}^\lambda \to \hat{p}^*$ and $p^\lambda \to p^*$. Let us show that $\hat{p}^* \in r(p^*)$.

The condition $\hat{p}^\lambda \in r(p^\lambda)$ implies that, for every $i \in N$, $p_i \in \mathscr{P}_i(\tilde{\varepsilon})$, $\tilde{U}_i(\hat{p}_i^\lambda, p_{-i}^\lambda) \geq \tilde{U}_i(p_i, p_{-i}^\lambda)$. Also, notice that, by the definition of weak convergence, $(\hat{p}_i^\lambda, p_{-i}^\lambda) \to (\hat{p}_i^*, p_{-i}^*)$ and $(p_i, p_{-i}^\lambda) \to (p_i, p_{-i}^*)$. Thus, by the continuity of $\tilde{U}_i$, we obtain $\tilde{U}_i(\hat{p}_i^*, p_{-i}^*) \geq \tilde{U}_i(p_i, p_{-i}^*)$, for each $i \in N$, which shows that $\hat{p}^* \in r(p^*)$. □

PROOF OF THEOREM 2. By Remark 2.4 in Balder (1988), Σ is weakly sequentially compact. Therefore, for any sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}}$ in $\mathscr{E}$, and $(\sigma^n)_{n\in\mathbb{N}}$ in Σ such that, for each $n \in \mathbb{N}$, $\tilde{\varepsilon}_n \in \mathscr{E}(\varepsilon_n)$, σ_n is $\tilde{\varepsilon}_n$-constrained, and $\lim_{n\to\infty} \varepsilon_n = 0$, we can construct sequences $\{p_i(\cdot|\cdot, a_i^\tau, \sigma_i^n)\}_{n\in\mathbb{N}}$ that converge to $p_i^*(\cdot|\cdot, a_i^\tau)$ in the weak topology of $\mathscr{P}_i$ for each $\tau \in \mathbb{N}$ and $a_i^\tau \in X_i^\tau$. Theorem 2 follows by Lemma 5. □

A.2 Proofs of Section 3.3

For every $a^{t-1} \in X^{t-1}$, and $t \in \mathbb{N}$, define $\mu_s^{P,t}(\cdot|a^{t-1}) := \prod_{i\in N} \mu_{s_i}^t(\cdot|a^{t-1})$, $\mu_{\hat{s}}^{P,t}(\cdot|a^{t-1}) := \prod_{i\in N} \mu_{\hat{s}_i}^t(\cdot|a^{t-1})$, and $\mu_\varepsilon^{P,t}(\cdot|a^{t-1}) := \prod_{i\in N} \mu_{\varepsilon_i}^t(\cdot|a^{t-1})$.

PROOF OF PROPOSITION 1. Suppose that noisy observability holds.

Let $\tilde{S} \in \mathcal{M}(S^t)$ such that $\mu_s^t(\tilde{S}) > 0$, and set $B := (m^t)^{-1}(\tilde{S})$, where $m^t : \hat{S}^t \times \mathcal{E}^t \to S^t$ is defined as $m^t(\hat{s}^t, \epsilon^t) := (m(\hat{s}_1^t, \epsilon_1^t), \ldots, m(\hat{s}_t^t, \epsilon_t^t))$. By Lemma 4.46 in Aliprantis and Border (2006), since a Polish space is second countable, there exist sets $\hat{B} \in \mathcal{M}(\hat{S}^t)$ and $E(\hat{s}^t) \in \mathcal{M}(\mathcal{E}^t)$, for each $\hat{s}^t \in \hat{B}$, such that we can write $B = \{(\hat{s}^t, \epsilon^t) | \hat{s}^t \in \hat{B}, \epsilon^t \in E(\hat{s}^t)\}$. Therefore, by noisy observability (a), we have that, for every $t \in \mathbb{N}$, and $a^{t-1} \in X^{t-1}$,

$$\mu_s^t(\tilde{S}|a^{t-1}) = \mu_{s,\epsilon}^t(B|a^{t-1}) = \int_{\hat{B}} \int_{E(\hat{s}^t)} f(\hat{s}^t, \epsilon^t, a^{t-1}) \, d\mu_\epsilon^{P,t}(\epsilon^t|a^{t-1}) \, d\mu_{\hat{s}}^t(\hat{s}^t|a^{t-1}) > 0,$$

for some measurable density $f : \hat{S}^t \times \mathcal{E}^t \times X^{t-1} \to \mathbb{R}$. Hence, there is a measurable $\hat{B}_0 \subset \hat{B}$ such that $\mu_{\hat{s}}^t(\hat{B}_0|a^{t-1}) > 0$, and, for each $\hat{s}^t \in \hat{B}_0$, $\int_{E(\hat{s}^t)} f(\hat{s}^t, \epsilon^t, a^{t-1}) \, d\mu_\epsilon^{P,t}(\epsilon^t|a^{t-1}) > 0$. Thus, $\mu_s^t(\tilde{S}_0|a^{t-1}) > 0$, for $\tilde{S}_0 := \{m^t(\hat{s}^t, \epsilon^t) | \hat{s}^t \in \hat{B}_0, \epsilon^t \in E(\hat{s}^t)\}$, and we have

$$\mu_s^{P,t}(\tilde{S}_0|a^{t-1}) = \int_{\hat{B}_0} \int_{E(\hat{s}^t)} f(\hat{s}^t, \epsilon^t, a^{t-1}) \, d\mu_\epsilon^{P,t}(\epsilon^t|a^{t-1}) d\mu_{\hat{s}}^{P,t}(\hat{s}^t|a^{t-1}) > 0,$$

since by noisy observability (b), $\mu^t_{\hat{s}}(\hat{B}_0|a^{t-1}) > 0$ implies $\mu^{P,t}_{\hat{s}}(\hat{B}_0|a^{t-1}) > 0$. This shows that $\mu^t_s(\cdot|a^{t-1})$ is absolutely continuous with respect to $\mu^{P,t}_s(\cdot|a^{t-1})$ for every $a^{t-1} \in X^{t-1}$. $\square$

PROOF OF LEMMA 1. Suppose $\mu^t_s(\tilde{S}|a^{t-1}) > 0$ for $\tilde{S} \in \mathcal{M}(S^t)$, $a^{t-1} \in X^{t-1}$, and $t \in \mathbb{N}$. For each $i \in N$, there is $\check{S}^t_i \in \mathcal{M}(S^t_i)$ such that (i) $\mu^t_{\hat{s}_i}(\check{S}^t_i|a^{t-1}) = 1$ and (ii) $\mu^t_{\hat{s}_i}(\{\hat{s}^t_i \in \hat{S}^t_i | \exists \epsilon^t_i \in \mathcal{E}^t_i,\ m^t_i(\hat{s}^t_i, \epsilon^t_i) = s^t_i\}|a^{t-1}) > 0$. By (i), we have $\mu(\tilde{S} \cap \times_{i\in N}\check{S}^t_i|a^{t-1}) > 0$. Let $s^t \in \tilde{S} \cap \times_{i\in N}\check{S}^t_i$. We have $\mu^{P,t}_{\hat{s}}(\hat{S}(\tilde{S})|a^{t-1}) \geq \mu^{P,t}_{\hat{s}}(\hat{S}(s^t)|a^{t-1}) = \prod_{i\in N}\mu^t_{\hat{s}_i}(\{\hat{s}^t_i \in \hat{S}^t_i|\exists \epsilon^t_i \in \mathcal{E}^t_i,\ m^t_i(\hat{s}^t_i, \epsilon^t_i) = s^t_i\}|a^{t-1}) > 0$, where the inequality follows by (ii). $\square$

PROOF OF PROPOSITION 2 AND LEMMA 2. We first show Proposition 2. For every $t \in \mathbb{N}$, $B \in \mathcal{M}(S^t)$, $a^{t-1} \in X^{t-1}$, $\mu^t_s(B|a^{t-1}) = \int_B \int_{\hat{S}^t} f^t(s^t, \hat{s}^t, a^{t-1})\, d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) d\tilde{\mu}^t(s^t)$.

By sequentially continuous noise, $\int_{\hat{S}^t} f^t(s^t, \hat{s}^t, a^{t-1})\, d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) \in CI(S^t \times X^{t-1}, \tilde{\mu}^t)$. Then, by Proposition 8 in Online Appendix C.1, μ^t_s is bounded and continuous in strong total variation. Similarly, to show Lemma 2, notice that for $\hat{B} \in \mathcal{M}(\hat{S}^t)$ we can write

$$\begin{aligned}
\mu^t_{\hat{s},s}(\hat{B} \times B|a^{t-1}) &= \int_{\hat{B}} \int_{B-\hat{s}^t} f^t(\epsilon^t, \hat{s}^t, a^{t-1})\, d\lambda^t(\epsilon^t) d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) \\
&= \int_{\hat{B}} \int_{B} f^t(s^t - \hat{s}^t, \hat{s}^t, a^{t-1})\, d\lambda^t(s^t) d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) \\
&= \int_{B} \int_{\hat{B}} f^t(s^t - \hat{s}^t, \hat{s}^t, a^{t-1})\, d\mu^t_{\hat{s}}(\hat{s}^t|a^{t-1}) d\lambda^t(s^t),
\end{aligned}$$

where the first equality holds by absolute continuity, the second equality holds by translation invariance of the Lebesgue measure λ^t, and the third by Fubini-Tonelli. This shows that sequentially continuous noise (b) is satisfied. $\square$

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B Online Appendix — Omitted Proofs

B.1 Active and Inactive Players

In the remainder of the paper, we enrich the class of games we consider by allowing for active and inactive players. In particular, we assume there is a function $\mathcal{T}_i : \cup_{t\in\mathbb{N}}\Omega^t \times X^{t-1} \to \{0,1\}$ that is measurable and determines if player $i \in N$ is *active* or *inactive* in period $t \in \mathbb{N}$, $\mathcal{T}_i(\cdot) = 1$ and $\mathcal{T}_i(\cdot) = 0$, respectively, as a function of the history of the states of the world and action profiles. We assume that at least one player is active in each period.

Whenever *active*, players move after receiving signals about the history of the state and action profiles. Specifically, for $t \in \mathbb{N}$, after every history $(\omega^t, a^{t-1}) \in \Omega^t \times X^{t-1}$ with $\mathcal{T}_i(\omega^t, a^{t-1}) = 1$, player $i \in N$ observes a signal $s_{i,t} \in S_i$ and chooses an action among the available ones according to A_i. When $\mathcal{T}_i(\omega^t, a^{t-1}) = 0$, player i is *inactive* and does not observe private signals nor chooses any non-trivial action. We assume S_i contains the null signal s_* and X_i the null action a_*, the latter an isolated point of X_i, and, when $\mathcal{T}_i(\omega^t, a^{t-1}) = 0$, we assume that $\gamma_i(\omega^t, a^{t-1}) = s_*$ and $A_i(\hat{\hbar}_i^t(\omega^t, a^{t-1})) = \{a_*\}$.

Without loss, we assume player $i \in N$ receives a non-zero flow payoff only when active, i.e., $g_i(\omega^t, (a^{t-1}, a_t)) = 0$ if $\mathcal{T}_i(\omega^t, a^{t-1}) = 0$ for every $(\omega^t, (a^{t-1}, a_t)) \in \Omega^t \times X^t$.

Private histories. To accommodate active and inactive players, we extend our definition of private history. For every period $t \in \mathbb{N}$, $(s_i^t, a_i^{t-1}) \in S_i^t \times X_i^{t-1}$ is a *signal-action history of player* i if $a_{i,\ell}^{t-1} = a_*$ whenever $s_{i,\ell}^t = s_*$ for $\ell \leq t-1$. The set of player i's signal-action histories contained in $S_i^t \times X_i^{t-1}$ is denoted by $\mathcal{H}_i^t$ and the set of all signal-action histories is $\mathcal{H}_i := \cup_{t\in\mathbb{N}}\mathcal{H}_i^t$. Similarly, $(s_i^t, a_i^t) \in S_i^t \times X_i^t$ is a signal-action history of player i *up to and including period* t*-action* if $a_{i,\ell}^t = a_*$ whenever $s_{i,\ell}^t = s_*$ for $\ell \leq t$. The set of signal-action histories up to and including period t is $\bar{\mathcal{H}}_i^t$, and $\bar{\mathcal{H}}_i := \cup_{t\in\mathbb{N}}\bar{\mathcal{H}}_i^t$.

A *generalized private history* of player i is a function of a signal-action history $(s_i^t, a_i^{t-1}) \in \mathcal{H}_i$ as follows

$$\hbar_i(s_i^t, a_i^{t-1}) := \big((s_{i,\ell}^t)_{\ell\leq t,\, s_{i,\ell}^t \neq s_*}, (a_{i,\ell}^{t-1})_{\ell\leq t-1,\, s_{i,\ell}^t \neq s_*}\big).$$

Generalized private history $\hbar_i(s_i^t, a_i^{t-1})$ collects player i's signals up to t and actions up to $t-1$, describing the information available to player i before playing at period t. Notice that, private histories do not record any information from periods in which players are inactive. For this reason, a player may be oblivious to the number of moves that have occurred, e.g., for any $(s_i^t, a_i^{t-1}) \in \mathcal{H}_i$, we have $\hbar_i(s_i^t, a_i^{t-1}) = \hbar_i((s_i^t, s_*), (a_i^{t-1}, a_*))$. A generalized private history of player i *up to and including period* t*-action* is the function of $(s_i^t, a_i^t) \in \bar{\mathcal{H}}_i$ defined by $\bar{\hbar}_i(s_i^t, a_i^t) := ((s_{i,\ell}^t)_{\ell \le t,\, s_{i,\ell}^t \neq s_*}, (a_{i,\ell}^t)_{\ell \le t,\, s_{i,\ell}^t \neq s_*})$.

We denote by $\mathcal{H}_i := \cup_{h_i \in \mathcal{H}_i} \hbar_i(h_i)$ the set of player i's generalized private histories. The *length* of a generalized private history $h_i = \hbar_i(s_i^t, a_i^{t-1})$ is $|h_i| := |\{\ell \le t |\, s_{i,\ell}^t \neq s_*\}|$; for $\ell \le |h_i|$, $h_i^{(\ell)}$ is the *truncation* of h_i up to and including player i's ℓ'th active period, so that $|h_i^{(\ell)}| = \ell$. Let $\mathcal{H}_i^t := \{h_i \in \mathcal{H}_i || h_i| = t\}$ be the set of i's generalized private histories of length $t \in \mathbb{N}$. Recall that $\hat{\hbar}_i^t$ is the mapping from $\mathcal{H}^t$ onto $\mathcal{H}_i^t$. For any history $(\omega^t, a^{t-1}) \in \mathcal{H}^t$, let $\hbar_i^{h,t}(\omega^t, a^{t-1}) := \hbar_i \circ \hat{\hbar}_i^t(\omega^t, a^{t-1})$ denote the function that maps this history to player i's corresponding generalized private history.

The sub σ-algebra of $\mathcal{M}(\mathcal{H}_i)$ induced by $\hbar_i$ is $\mathcal{M}(\mathcal{H}_i|\hbar_i) := \{\hbar_i^{-1}(A) | A \in \mathcal{M}(\mathcal{H}_i)\}$. The sub σ-algebra $\mathcal{M}(\bar{\mathcal{H}}_i|\bar{\hbar}_i)$ is defined analogously.

For every $i \in N$, we assume the correspondence A_i is measurable with respect to $\mathcal{M}(\mathcal{H}_i|\hbar_i)$.

For every $i \in N$, the sets $\mathcal{H}_{A_i}$, and $\bar{\mathcal{H}}_{A_i}$, and a strategy $\sigma_i : \mathcal{H}_{A_i} \to \Delta(X_i)$ are defined as before with the additional requirement that σ_i is measurable with respect to $\mathcal{M}(\mathcal{H}_i|\hbar_i)$. Thus, we will sometimes write $\sigma_i : \mathcal{H}_{A_i} \to \Delta(X_i)$, where $\mathcal{H}_{A_i} = \hbar_i(\mathcal{H}_{A_i})$.[1] The definitions of the function p_i, expected payoffs, constrained equilibrium, and trembling hand perfect equilibrium apply verbatim to the environment with imperfect observation of moving periods.

We extend the definition of a proper subgame root. A history $(\omega^t, a^{t-1}) \in \mathcal{H}^t$ is called the root of a proper subgame if it uniquely determines players' generalized private histories. More precisely, for all players $i \in N$, and periods $\tau \in \mathbb{N}$, if any history $(\tilde{\omega}^\tau, \tilde{a}^{\tau-1}) \in \mathcal{H}^\tau$ generates the same generalized private history (that is, $\hbar_i^{h,t}(\omega^t, a^{t-1}) = \hbar_i^{h,\tau}(\tilde{\omega}^\tau, \tilde{a}^{\tau-1})$), then it must be the same history (that is, $(\omega^t, a^{t-1}) = (\tilde{\omega}^\tau, \tilde{a}^{\tau-1})$). Let $\mathcal{H}^\varnothing \subseteq \mathcal{H}$ denote the set of all such histories. A negligible subset and an SPE are defined as before.

Markov games. For each player $i \in N$, activity status (active or inactive) is determined by the current payoff-relevant state through the function $\mathcal{T}_i : \Omega^R \to \{0, 1\}$, where 1 indicates active and 0 indicates inactive. The definitions of signal-action histories and all other objects remain unchanged but are now expressed in terms of signal-action histories.

The *Markov information* condition for games with imperfect observation of moving periods requires that beliefs about the current payoff-relevant state depend only on the current payoff-relevant signal: for every player $i \in N$, period t, length ℓ, and Markov strategy σ^M, $\mu_{\omega,t}^R(\cdot|h_i, \sigma^M) = \mu_{\omega,t}^R(\cdot|\tilde{h}_i, \sigma^M)$ whenever $h_i, \tilde{h}_i \in \hbar_i(\mathcal{H}^t) \cap \mathcal{H}_i^\ell$ satisfy $\mathcal{s}_i^R(h_i) = \mathcal{s}_i^R(\tilde{h}_i)$.

[1]For $i \in N$, we can write $\sigma_i : \mathcal{H}_{A_i} \to \Delta(X_i)$ by Theorem 4.41 in Aliprantis and Border (2006).

The *stationary Markov information* condition further requires that these beliefs be independent of both strategy and the timing of player i's active periods: for any periods $t, \tilde{t}$, lengths $\ell, \tilde{\ell}$, and stationary Markov strategies $\sigma^{sM}, \tilde{\sigma}^{sM}$, $\mu^R_{\omega,t}(\cdot|h_i, \sigma^{sM}) = \mu^R_{\omega,\tilde{t}}(\cdot|\tilde{h}_i, \tilde{\sigma}^{sM})$ whenever $h_i \in \hbar_i(\mathcal{H}^t) \cap \mathcal{H}^\ell_i$, $\tilde{h}_i \in \hbar_i(\mathcal{H}^{\tilde{t}}) \cap \mathcal{H}^{\tilde{\ell}}_i$ satisfy $\mathscr{s}^R_i(h_i) = \mathscr{s}^R_i(\tilde{h}_i)$ and $t - \ell = \tilde{t} - \tilde{\ell}$.

Additional notation. Active and inactive players introduce an additional complexity: a single generalized private history may be generated by histories of different lengths. This requires defining some probabilities and conditional payoffs.

Let $\mathcal{H}^{\mathbb{N}} := \Omega^{\mathbb{N}} \times X^{\mathbb{N}}$ be the set of histories with infinite length.[2] Define $\tilde{\hbar}^t : \mathcal{H}^{\mathbb{N}} \to \mathcal{H}^t$ and $\tilde{\hbar}^t_i : \mathcal{H}^{\mathbb{N}} \to \mathcal{H}_i$, for $i \in N$, functions from infinite histories to period-t histories and player's i generalized private histories, respectively, by setting $\tilde{\hbar}^t(\bar{\omega}, \bar{a}) = (\bar{\omega}^{(t)}, \bar{a}^{(t-1)})$ and $\tilde{\hbar}^t_i(\bar{\omega}, \bar{a}) = \hbar_i(\gamma^t_i(\bar{\omega}^{(t)}, \bar{a}^{(t-1)}), \bar{a}^{(t-1)}_i)$, for $(\bar{\omega}, \bar{a}) \in \mathcal{H}^{\mathbb{N}}$.

We define $P(\cdot|\sigma) \in \Delta(\mathcal{H}^{\mathbb{N}})$ as the measure over infinite histories induced by the strategy profile $\sigma \in \Sigma$.[3] For every $i \in N$, $\sigma \in \Sigma$, $\ell \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}^\ell_i)$, the probability that player i's private history belongs to Z under σ is given by $P_i(Z|\sigma) := P(\cup_{t\in\mathbb{N}}(\tilde{\hbar}^t_i)^{-1}(Z)|\sigma)$. If $P_i(Z|\sigma) > 0$, player i's period-t *beliefs* over $C \in \mathcal{M}(\mathcal{H}^t)$ *conditional* on Z can be defined as $P^t_i(C|Z, \sigma) = P\big((\tilde{\hbar}^t)^{-1}(C) \cap (\tilde{\hbar}^t_i)^{-1}(Z)|\sigma\big)/P_i(Z|\sigma)$.

For every pair of strategy profiles $\sigma, \hat{\sigma} \in \Sigma$, and $Z \in \mathcal{M}(\mathcal{H}^\ell_i)$, for $\ell \in \mathbb{N}$, player i's *expected payoff from $\hat{\sigma}$ conditional on Z and σ* becomes

$$U_i(\hat{\sigma}|Z, \sigma) := \sum_{\tau\in\mathbb{N}} \int_{\mathcal{H}^\tau} U_i(\hat{\sigma}|\omega^\tau, a^{\tau-1})\, dP^\tau_i(\omega^\tau, a^{\tau-1}|Z, \sigma),$$

where the sum is over all possible lengths τ of histories for which the conditioning event Z could have occurred.

An $\tilde{\varepsilon}$*-constrained conditional equilibrium* is defined as before.

THEOREMS 1 AND 2 WITH IMPERFECT OBSERVATION OF MOVING PERIODS.

The proofs of Theorems 1 and 2 extend to games with imperfect observation of moving periods, provided that Lemma 5 holds. This lemma is satisfied when the limit strategy is measurable with respect to the sub σ-algebra $\mathcal{M}(\mathcal{H}_i|\hbar_i)$. The latter follows from Lemma 14 in Online Appendix C.1, as it implies that $p^*|_{\bar{\mathcal{H}}_i}$ must be measurable with respect to the sub σ-algebra $\mathcal{M}(\bar{\mathcal{H}}_i|\bar{\hbar}_i)$.[4]

[2] For any set Y, denote $Y^{\mathbb{N}} := \{(y_t)_{t\in\mathbb{N}}|y_t \in Y, \forall t \in \mathbb{N}\}$.

[3] For each $\sigma \in \Sigma$, there is a probability measure $P(\cdot|\sigma) : \mathcal{H}^{\mathbb{N}} \to [0, 1]$ such that for every $t \in \mathbb{N}$, and measurable, bounded functions $r^t : \mathcal{H}^t \to \mathbb{R}$, and $\hat{r} : \mathcal{H}^{\mathbb{N}} \to \mathbb{R}$, defined as $\hat{r}(\bar{\omega}, \bar{a}) := r^t(\hat{\hbar}^t(\bar{\omega}, \bar{a}))$, for $(\bar{\omega}, \bar{a}) \in \mathcal{H}^{\mathbb{N}}$, we have that $\mathbb{E}_{P(\cdot|\sigma)}(\hat{r}) = \mathbb{E}_{P^t(\cdot|\sigma)}(\hat{r})$. See Theorem 4.49 in Pollard (2002).

[4] To apply Lemma 14 we set $Y \times Z = S^\infty_i \times X^\infty_i$, $\tilde{B} = \bar{\mathcal{H}}_i$, and $(T_y, T_z)(\bar{s}_i, \bar{a}_i) = \bar{\hbar}_i(\bar{s}_i, \bar{a}_i)$ if $(\bar{s}_i, \bar{a}_i) \in \bar{\mathcal{H}}_i$, and $T(\bar{s}_i, \bar{a}_i) = (\bar{s}_i, a^*)$, for some $a^* \notin X^\infty_i$, otherwise.

B.2 Proof of Theorem 3

For every $i \in N$, $t \in \mathbb{N}$, a *Markov reference measure* $\nu_{i,t}^M(\cdot) \in \Delta(S_i^R)$ can be constructed by setting $d\nu_{i,t}^M(s_i) := \sum_{a^{t-1} \in X^{t-1}} \xi_i(a^{t-1}) \cdot d\mu_{s_i,t}^M(s_i|a^{t-1})$ for any collection ξ_i satisfying $\xi_i(a^{t-1}) > 0$ for every $a^{t-1} \in X^{t-1}$ and $\sum_{a^{t-1} \in X^{t-1}} \xi_i(a^{t-1}) = 1$. Using these reference measures, we define $\nu^{M,t}(s^t) := \prod_{i \in N} \prod_{\ell \leq t} \nu_{i,\ell}^{R,t}(s_i^t)$.

Define the *weak topology on* Σ_i^M as the coarsest topology such that, for $t \in \mathbb{N}$, the functional $\mathcal{G}_{i,\varphi_i}^t : \Sigma_i^M \to \mathbb{R}$, $\mathcal{G}_{i,\varphi_i}^t(\sigma_i^M) := \int_{S_i^R} \sum_{a_i \in X_i} \varphi_i(s_i, a_i) \cdot \sigma_i^M(a_i|s_i, t)\, d\nu_{i,t}^M(s_i)$, is continuous for every $\varphi_i \in CI(S_i^R \times X_i, \nu_{i,t}^M)$. The *weak topology* on Σ^M is its product topology. The sets $\Sigma^{sM}(\tilde{\varepsilon}^M)$, $\Sigma^M(\tilde{\varepsilon}^M)$, and Σ^{sM}, for $\tilde{\varepsilon}^M \in \mathcal{E}^M$, are endowed with their relative topologies and are closed.[5]

Lemma 6. *Under payoff boundedness, continuity, and MAC, for every $i \in N$, $U_i(\sigma)$ is continuous in the weak topology of Σ^M.*

Proof. Let $(\sigma^\lambda)_{\lambda \in \Lambda}$ be a net in Σ^M that converges to σ^* in the weak topology. We will show that $U_i(\sigma^\lambda) \to U_i(\sigma^*)$, for $i \in N$.

By MAC and Proposition 8 in Online Appendix C.1, for every $t \in \mathbb{N}$, $s^t \in S^{R,t}$, $a^{t-1} \in X^{t-1}$, the payoff-relevant signal profile transition can be written as $d\mu_s^{R,t}(s^t|a^{t-1}) = f^t(s^t, a^{t-1}) \cdot d\nu^{M,t}(s^t)$, where $f^t \in CI(S^{R,t} \times X^{t-1}, \nu^{M,t})$.

For every $\sigma^M \in \Sigma^M$, we can write player i's period-t expected payoff as

$$\begin{aligned} U_{i,t}(\sigma^M) &= \sum_{a^t \in X^t} \int_{S^t} \hat{g}_{i,t}(s^t, a^t) \cdot p(a^t|s^t, \sigma^M)\, d\mu_s^t\left(s^t|a^t\right) \\ &= \sum_{a^t \in X^t} \int_{S^{R,t}} \hat{g}_{i,t}(s^t, a^t) \cdot f^t(s^t, a^{t,(t-1)}) \cdot \prod_{i \in N, \ell \leq t} \sigma_i^M(a_{i,\ell}^t | s_{i,\ell}^t, \ell)\, d\nu^{M,t}(s^t), \end{aligned}$$

where the first equality follows from applying Lemma 4, and Theorems 4.41 and 13.46 in Aliprantis and Border (2006). The second equality holds since by Markov payoffs, $\hat{g}_{i,t}(s^t, a^t)$ depends only on payoff-relevant signals, so we can write it as $\hat{g}_{i,t}(s^{R,t}, a^t)$, while by definition of non-stationary Markov strategies, $p(a^t|s^t, \sigma^M) = p(a^t|s^{R,t}, \sigma^M)$.

By payoff continuity, there is a version of $\hat{g}_{i,t}(s^t, a^t) \cdot f^t(s^t, a^{t,(t-1)}) \in CI(S^{R,t} \times X^t, \nu^{M,t})$. Thus, by Theorem 2.5 in Balder (1988), we obtain $U_{i,t}(\sigma^\lambda) \to U_{i,t}(\sigma^*)$.

By Lemma 4, and payoff boundedness, $|U_i(\sigma)| = \left|\sum_{t=1}^\infty U_{i,t}(\sigma)\right| \leq \sum_{t \in \mathbb{N}} \sup_{\sigma \in \Sigma} |U_{i,t}(\sigma)| < \infty$. Therefore, by Proposition 10 in Online Appendix C.2,

[5] The proof that sets of constrained strategies are closed is exactly analogous to the proof of Lemma 5. Therefore, we omit it.

we obtain $U_i(\sigma^\lambda) \to U_i(\sigma^*)$. $\square$

Let $\tilde{\varepsilon}^M \in \mathcal{E}^M$. Denote by $r^M : \Sigma^M(\tilde{\varepsilon}^M) \to \Sigma^M(\tilde{\varepsilon}^M)$ and $r^{sM} : \Sigma^{sM}(\tilde{\varepsilon}^M) \to \Sigma^{sM}(\tilde{\varepsilon}^M)$ the best response correspondences in Markov and stationary Markov strategies, respectively. Our proof proceeds in two steps. First, we establish that both r^M and r^{sM} have fixed points. Second, we show that, at each of these fixed points, players have no profitable deviations to any $\tilde{\varepsilon}^M$-constrained strategy under their respective assumptions: MAC and Markov information and payoffs for the Markov fixed point, and additionally stationary Markov information for the stationary Markov fixed point.

Part 1: Existence of a fixed point or r^M. We establish that r^M has a fixed point by applying the Kakutani-Fan-Glicksberg fixed point theorem. This requires verifying five conditions: **(1)** $\Sigma^M(\tilde{\varepsilon}^M)$ is compact, **(2)** $\Sigma^M(\tilde{\varepsilon}^M)$ is convex, **(3)** r^M has closed graph, **(4)** r^M is non-empty, and **(5)** r^M is convex valued.[6] Condition **(1)** follows by the closedness of $\Sigma^M(\tilde{\varepsilon}^M)$ and Theorem 2.3 in Balder (1988). Condition **(2)** is straightforward. Condition **(3)** holds by the same argument used in Theorem 1, and by applying Lemma 6. Condition **(4)** follows from **(1)** and the continuity of U_i. All that remains is to establish condition **(5)**, which is implied by the following lemma.

We introduce the following notation. For strategies $\sigma_i, \sigma_i', \sigma_i'' \in \Sigma_i$, we use two types of concatenations: (σ_i^t, σ_i') denotes the strategy that follows σ_i for the first t active periods and switches to σ_i' afterward, while $(\sigma_i^\tau, \sigma_{i,\tau+1}', \sigma_i'')$ denotes the strategy that follows σ_i for the first τ active periods, uses σ_i' in period $\tau+1$, and switches to σ_i'' afterward.

Lemma 7. *If $\hat{\sigma}_i, \tilde{\sigma}_i \in r_i^M(\sigma)$ for $\sigma \in \Sigma^M(\tilde{\varepsilon}^M)$ then $\sigma_i^\beta := \beta \cdot \hat{\sigma}_i + (1-\beta) \cdot \tilde{\sigma}_i \in r_i^M(\sigma)$ for every $\beta \in (0,1)$.*

Proof. We will show, inductively in t, that $U_i((\sigma_i^{\beta,t}, \hat{\sigma}_i), \sigma_{-i}^M) = U_i((\sigma_i^{\beta,t}, \tilde{\sigma}_i), \sigma_{-i}^M) = U_i(\hat{\sigma}_i, \sigma_{-i}^M) = U_i(\tilde{\sigma}_i, \sigma_{-i}^M)$ for every $t \in \mathbb{N} \cup \{0\}$. The statement is immediate for $t = 0$ as $\hat{\sigma}_i$ and $\tilde{\sigma}_i$ are both best responses to the same strategy.

Suppose the induction hypothesis holds for $t \leq \tau$. We show it holds for $t = \tau+1$. We can write $U_i((\sigma_i^{\beta,\tau+1}, \hat{\sigma}_i), \sigma_{-i}^M) = \beta \cdot U_i((\sigma_i^{\beta,\tau}, \hat{\sigma}_i), \sigma_{-i}^M) + (1-\beta) \cdot U_i((\sigma_i^{\beta,\tau}, \tilde{\sigma}_{i,\tau+1}, \hat{\sigma}_i), \sigma_{-i}^M)$.

We can write $U_i(\sigma) = \int_{\mathcal{H}_i^{\tau+2}} U_i(\sigma|h_i, \sigma_{-i}) p_i(\mathbf{a}_i(h_i)|\mathbf{s}_i(h_i), \sigma_i) dP_{-i}(h_i|\sigma_{-i})$, for any $\sigma \in \Sigma$, where the measurable functions $\mathbf{a}_i(h_i)$ and $\mathbf{s}_i(h_i)$ project the generalized private history $h_i \in \mathcal{H}_i$ onto $X_i^{|h_i|}$ and $S_i^{|h_i|}$, respectively, and $U_i(\sigma|h_i, \sigma_{-i})$ is defined as in equation (11), that is,

[6] By Balder (1988), the corresponding quotient space is locally convex Hausdorff.

$$U_i(\sigma|h_i,\sigma_{-i}) \coloneqq \sum_{\tau\in\mathbb{N}} \int_{\mathcal{H}^\tau} U_i(\sigma|\omega^\tau, a^{\tau-1})\, dP_{\omega|h_i}(\omega^\tau, a^{\tau-1}|h_i,\sigma_{-i}).$$

Therefore, condition (13) in Lemma 9 in Online Appendix B.3 implies that $U_i(\hat{\sigma}_i,\sigma_{-i}|h_i,\sigma_{-i}) = U_i(\tilde{\sigma}_i,\sigma_{-i}|h_i,\sigma_{-i})$ for $h_i \in \mathcal{H}_i^{\tau+2}$, and, therefore, $U_i((\sigma_i^{\beta,\tau},\tilde{\sigma}_{i,\tau+1},\hat{\sigma}_i),\sigma_{-i}^M) = U_i((\sigma_i^{\beta,\tau},\tilde{\sigma}_i),\sigma_{-i}^M)$, which concludes our induction argument. The lemma follows by payoff boundedness, as it implies that $U_i((\sigma_i^{\beta,t},\hat{\sigma}_i),\sigma_{-i}^M)$ converges to $U_i(\sigma_i^\beta,\sigma_{-i}^M)$. □

The existence of a fixed point of r^{sM} follows from an identical argument, as $\Sigma^{sM}(\tilde{\varepsilon}^M)$ is a closed subset of $\Sigma^M(\tilde{\varepsilon}^M)$ and conditions (1)-(5) continue to hold when restricted to stationary Markov strategies. In particular, the convexity of r^{sM} follows from the same inductive argument as the above lemma, since concatenations of stationary Markov strategies remain stationary Markov.

Part 2: Ruling out deviations to non-Markov strategies. For each player $i \in N$, strategy profile $\sigma \in \Sigma$, we define the following probability measures.

1. The probability that player i's payoff-relevant signal belongs to $Z \in \mathcal{M}(S_i^R)$ and is generated by a generalized private history of length $\hat{t}$:

$$P_{s_i^R,\hat{t}}(Z|\sigma) \coloneqq P\Big(\bigcup_{t\in\mathbb{N}} (\tilde{\hbar}_i^t)^{-1}((\mathcal{s}_i^R)^{-1}(Z) \cap \mathcal{H}_i^{\hat{t}})|\sigma\Big)$$

2. The transition probability $P_{h_i,\hat{t}|s_i^R}(\cdot|\cdot,\sigma) : S_i^R \to \Delta(\mathcal{H}_i^{\hat{t}})$ from payoff-relevant signals to length-$\hat{t}$ generalized private histories, which satisfies

$$dP_i(h_i|\sigma) = dP_{h_i,\hat{t}|s_i^R}(h_i|s_i^R,\sigma) \times dP_{s_i^R,\hat{t}}(s_i^R|\sigma)$$

3. The transition measure $\mu_{\omega,t}^R(\cdot|\cdot,\sigma) : \cup_{i\in N}\mathcal{H}_i \to \Delta(\Omega^R)$ from private histories to payoff-relevant states induced by σ, which satisfies, for all $\hat{\Omega} \in \mathcal{M}(\Omega^R)$, and $Z \in \mathcal{H}_i^\ell$, where $\ell \in \mathbb{N}$,

$$\int_{(\Omega^R)^{t-1}\times\hat{\Omega}\times X^{t-1}\cap\bar{p}^R\circ(\hbar_i^{h,t})^{-1}(Z)} p(a^{t-1}|\omega^{t-1},\sigma)d\mu_\omega^{R,t}((\omega^{t-1},\omega_t)|a^{t-1}) = \int_Z d\mu_{\omega,t}^R(\hat{\Omega}|h_i,\sigma)dP_i(h_i|\sigma),$$

where $d\mu_\omega^{R,t}(\cdot|a^{t-1})$ represents the transition probability for payoff-relevant states, and $\bar{p}^R$ denotes the mapping from histories in $\mathcal{H}$ to $\cup_{t\in\mathbb{N}}\Omega^{R,t} \times X^{t-1}$. The measure $\mu_{\omega,t}^R(\cdot|h_i,\sigma)$ is the one referenced in the Markov and stationary Markov information conditions.

4. The measure $P_{h_t|h_\tau}(\cdot|\omega_\tau, a_\tau, \sigma)$ over period-t payoff-relevant states and actions in $\Omega^R \times X$, induced by σ, after state-action pair $(\omega_\tau, a_\tau) \in \Omega^R \times X$, with $t, \tau \in \mathbb{N}$, $t > \tau$,

$$P_{h_t|h_\tau}(\hat{\Omega}^R \times \hat{X}|\omega_\tau, a_\tau, \sigma) := \sum_{a^t \in X^{t-1} \times \hat{X}} \int_{\Omega^{R,t-(\tau+1)} \times \hat{\Omega}^R} p(a^t|\omega^t, a_0^\tau, \sigma) \prod_{k=\tau+1}^{t} d\mu_\omega^R(\omega_k^t|\omega_{k-1}^t, a_{k-1}^t),$$

for each measurable set $\hat{\Omega}^R \times \hat{X} \in \mathcal{M}(\Omega^R \times X)$, where $a_0^\tau \in X^\tau$ satisfies $a_{0,\tau}^\tau = a_\tau$, and $\mu_\omega^R : \Omega^R \times X \to \Delta(\Omega^R)$ is the transition probability over payoff relevant states. When σ is a Markov strategy from period τ onward, $P_{h_t|h_\tau}$ is independent of the specific choice of a_0^τ.

Part 2 follows by the next lemma.

LEMMA 8. *The following holds:*

I. If σ^M is a fixed point of r^M, then $U_i(\sigma^M) \geq U_i(\sigma_i, \sigma_{-i}^M)$ for every $i \in N$, $\sigma_i \in \Sigma_i$ that is Markov $\tilde{\varepsilon}^M$-constrained;

II. If σ^{sM} is a fixed point of r^{sM}, then $U_i(\sigma^{sM}) \geq U_i(\sigma_i, \sigma_{-i}^{sM})$ for every $i \in N$, $\sigma_i \in \Sigma_i$ that is Markov $\tilde{\varepsilon}^M$-constrained.

Proof. **Point I.** Suppose that σ^M is a fixed point of r^M and that there is a player $i \in N$, $c > 0$ and a strategy $\sigma_i' \in \Sigma_i(\tilde{\varepsilon}^M)$ such that $U_i(\sigma_i', \sigma_{-i}^M) - U_i(\sigma^M) = c > 0$. By payoff boundedness, there is $\hat{t} \in \mathbb{N}$ such that strategy $\hat{\sigma}_i = (\sigma_i'^{\hat{t}}, \sigma_i^M)$, satisfies $U_i(\hat{\sigma}_i, \sigma_{-i}^M) - U_i(\sigma^M) \geq c/2 > 0$.

We show the existence of a strategy with the following properties: (a) It matches $\hat{\sigma}_i$ in all active periods except period $\hat{t}$; (b) It is Markov in period $\hat{t}$; (c) It yields identical payoffs to $\hat{\sigma}_i$ in period $\hat{t}$. By applying this construction recursively, we establish that there exists a profitable Markov deviation from σ_i^M.

For each payoff-relevant signal $s_i^R \in \mathcal{s}_i^R(\mathcal{H}_i^{\hat{t}})$, we define the $\hat{t}$'th active period strategy as

$$\tilde{\sigma}_{i,\hat{t}}^M(a_i|s_i^R) := \int_{\mathcal{H}_i^{\hat{t}}} \hat{\sigma}_i(a_i|h_i) dP_{h_i,\hat{t}|s_i^R}(h_i|s_i^R, \hat{\sigma}_i, \sigma_{-i}^M). \tag{9}$$

Let $\tilde{\sigma}_i = (\hat{\sigma}_i^{t-1}, \tilde{\sigma}_{i,\hat{t}}^M, \sigma_i^M)$ and $\hat{\sigma} = (\hat{\sigma}_i, \sigma_{-i}^M)$. The sum of player i's expected flow payoffs from period $\hat{t}$ given strategy $\hat{\sigma}$, denoted $U_{i,\hat{t}\to\infty}(\hat{\sigma})$, can be expresed as,

$$\sum_{\tau \geq \hat{t}, t \geq \tau} \int_{S_i^R \times \mathcal{H}_i^{\hat{t}} \times \Omega^R \times X \times \Omega^R \times X} g_i(\omega_t, a_t) \cdot \hat{\sigma}_i(a_{i,\tau}|h_i) \cdot \sigma_{-i}^M(a_{-i,\tau}|\omega_\tau) dP_{h_t|h_\tau}(\omega_t, a_t|\omega_\tau, a_\tau, \hat{\sigma})$$
$$d\mu_{\omega,\tau}^R(\omega_\tau|h_i, \hat{\sigma})\, dP_{h_i,\hat{t}|s_i^R}(h_i|s_i^R, \hat{\sigma})\, dP_{s_i^R,\hat{t}}(s_i^R|\hat{\sigma}) =$$

$$\sum_{\tau\geq\hat{t},t\geq\tau}\int_{S_i^R\times\Omega^R\times X\times\Omega^R\times X} g_i(\omega_t,a_t)\cdot\tilde{\sigma}^M_{i,\hat{t}}(a_{i,\tau}|s_i^R)\cdot\sigma^M_{-i}(a_{-i,\tau}|\omega_\tau)\,dP_{h_t|h_\tau}(\omega_t,a_t|\omega_\tau,a_\tau,\hat{\sigma})$$
$$d\mu^R_{\omega,\tau}(\omega_\tau|s_i^R,\hat{\sigma})\,dP_{s_i^R,\hat{t}}(s_i^R|\hat{\sigma}) = U_{i,\hat{t}\to\infty}(\tilde{\sigma}_i,\sigma^M_{-i}). \quad (10)$$

The first equality follows from the Markov information assumption, as $d\mu^R_{\omega,\tau}(\omega_\tau|h_i,\sigma)$ depends on h_i only through $s_i^R(h_i)$. This allows us to substitute $\tilde{\sigma}^M_{i,\hat{t}}$ from (9) inside the integral. The last equality follows from the definition of $\tilde{\sigma}_i$. Since player i's flow payoffs in periods 1 through $\hat{t}-1$ are identical under both σ_i and $\tilde{\sigma}_i$, we have $U_i(\hat{\sigma}_i,\sigma^M_{-i}) = U_i(\tilde{\sigma}_i,\sigma^M_{-i})$. By applying this construction recursively backward from period $\hat{t}-1$ to period 0, we can construct a strategy $\tilde{\sigma}_i'$ that is Markov in all periods and yields the same payoff as $\hat{\sigma}_i$. This contradicts the optimality of σ_i^M since we have $U_i(\tilde{\sigma}_i',\sigma^M_{-i}) - U_i(\sigma^M) \geq c/2 > 0$.

Point II. Let σ^{sM} be a fixed point of r^{sM}. We now show that, for every $i\in N$, there is a stationary Markov best response to σ^{sM}_{-i} when stationary Markov information holds. While we already know a Markov best response exists (since stationary Markov information implies Markov information), we need to establish that this best response can be chosen to be stationary. Under a stationary Markov strategy σ, the transition probability $dP_{h_t|h_\tau}(\cdot|\omega_\tau,a_\tau,\sigma)$ depends only on the initial state-action pair (ω_τ,a_τ) and the period difference $t-\tau$. Moreover, by the stationary Markov information condition, the transition probability $d\mu^R_{\omega,\tau}(\cdot|h_i,\sigma)$ depends only on the payoff-relevant signal s_i^R and the difference between τ (the length of a history that could have generated a generalized private history h_i) and $|h_i|$. This allows us to write it as $d\mu^R_\omega(\cdot|s_i^R,\tau-|h_i|)$.

Let us first prove that no profitable Markov one-shot deviation exists from σ_i^{sM}. A one-shot deviation is a Markov strategy that differs from σ_i^{sM} in exactly one active period while maintaining identical behavior in all other periods. Specifically, suppose there exists a profitable deviation $\tilde{\sigma}_i^{sM}$ that can be expressed as $\tilde{\sigma}_i = (\sigma_i^{sM,\hat{t}},\tilde{\sigma}_{i,\hat{t}},\sigma_i^{sM})$, where $\tilde{\sigma}_{i,\hat{t}}$ is a stationary Markov strategy used in the deviating period $\hat{t}$. For a strategy σ, define the expected utility conditional on a private signal as

$$U_i(\sigma|s_i^R) \coloneqq \sum_{\tau\geq\hat{t},t\geq\tau}\int_{(\Omega^R\times X)^2} g_i(\omega_t,a_t)\cdot\sigma_i(a_{i,\tau}|s_i^R)\cdot\sigma_{-i}(a_{-i,\tau}|\omega_\tau)\,dP_{h_t|h_\tau}(\omega_t,a_t|\omega_\tau,a_\tau,\sigma)\cdot$$
$$d\mu^R_\omega(\omega_\tau|s_i^R,\tau-\hat{t}).$$

Notice, $U_i(\sigma|s_i^R)$ corresponds to the integrand in equation (10) with respect to $dP_{s_i^R,\hat{t}}(s_i^R|\sigma)$. From our previous discussion, if players follow stationary Markov strategies prior to period $\hat{t}$, this term depends only on s_i^R and not on those strategies or the period $\hat{t}$.

By equation (10), there must exist a set $\hat{S}_i^R\in\mathcal{M}(S_i^R)$ with positive measure under

$P_{s_i^R,\hat{t}}(\cdot|\sigma)$ such that $U_i(\tilde{\sigma}_i, \sigma_{-i}^{sM}|s_i^R) > U_i(\sigma_i^{sM}, \sigma_{-i}^{sM}|s_i^R)$ for $s_i^R \in \hat{S}_i^R$. We can then construct a new strategy $\tilde{\tilde{\sigma}}_i$ as follows:

$$\tilde{\tilde{\sigma}}_i(\cdot|s_i^R) = \begin{cases} \tilde{\sigma}_{i,\hat{t}}(\cdot|s_i^R) & \text{if } s_i^R \in \hat{S}_i^R \\ \sigma_i^{sM}(\cdot|s_i^R) & \text{if } s_i^R \notin \hat{S}_i^R \end{cases}.$$

Since $U_i(\sigma|\cdot)$ is independent of previous play, we can construct a sequence of profitable deviations. Let $\sigma_i^{\hat{t},t'}$ be the strategy that coincides with $\tilde{\tilde{\sigma}}_i$ between periods $\hat{t}$ and t' for $t' > \hat{t}$, $t' \in \mathbb{N}$, and follows σ_i^{sM} otherwise. By payoff boundedness, as $t' \to \infty$, the expected payoff of $\sigma_i^{\hat{t},t'}$ converges to that of $\sigma_i^{\hat{t},\infty}$, where $\sigma_i^{\hat{t},\infty}$ is the strategy that follows σ_i^{sM} up to period $\hat{t}-1$ and switches to $\tilde{\tilde{\sigma}}_i$ thereafter. Therefore, $\sigma_i^{\hat{t},\infty}$ must also be a profitable deviation. However, by stationary Markov information, for any stationary Markov strategy σ_i, we have $U_i((\sigma_i, \sigma_{-i}^{sM})|h_i, \sigma_{-i}^{sM}) = U_i((\sigma_i, \sigma_{-i}^{sM})|s_i^R(h_i))$. This equality, combined with the fact that $U_i(\tilde{\sigma}_i, \sigma_{-i}^{sM}|s_i^R) > U_i(\sigma_i^{sM}, \sigma_{-i}^{sM}|s_i^R)$ for $s_i^R \in \hat{S}_i^R$, contradicts the optimality of $\sigma_i^{sM} \in \tilde{\Sigma}_i = \Sigma_i^{sM}(\tilde{\varepsilon}^M)$ established by condition (13) in Lemma 9.

Let σ_i^{nsM} be a non-stationary Markov best response, and $\sigma_i^{nsM,t}$ a best response among Markov strategies that coincide with σ_i^{sM} starting from the $(t+1)$th active period. These best responses exist by compactness and Lemma 6. By payoff boundedness, for any $\varepsilon > 0$, there exists t such that $|U_i(\sigma_i^{nsM}, \sigma_{-i}^{sM}) - U_i(\sigma_i^{nsM,t}, \sigma_{-i}^{sM})| < \varepsilon$. We will show that for all $t \in \mathbb{N}$, $U_i(\sigma_i^{nsM,t}, \sigma_{-i}^{sM}) = U_i(\sigma^{sM})$, which combined with the previous inequality establishes our claim.

Since σ_i^{sM} has no profitable one-shot deviations, by Lemma 9 (applied to $\tilde{\Sigma}_i$, the set of non-stationary Markov strategies that differ from σ_i^{sM} in at most one active period), the strategy that matches $\sigma_i^{nsM,t}$ for the first $t-1$ active periods and follows σ_i^{sM} thereafter must yield the same expected payoff as $\sigma_i^{nsM,t}$. Applying this argument recursively, we obtain $U_i(\sigma^{sM}) = U_i(\sigma_i^{nsM,t}, \sigma_{-i}^{sM})$. □

Let $(\tilde{\varepsilon}_n^M)_{n\in\mathbb{N}}$ in $\mathcal{E}^M$ with $\tilde{\varepsilon}_n^M \in \mathcal{E}^M(\varepsilon_n)$, and $\varepsilon_n \to 0$. Let $\sigma^n \in \Sigma^M(\tilde{\varepsilon}_n^M)$ and $\sigma^{s,n} \in \Sigma^{sM}(\tilde{\varepsilon}_n^M)$ be Markov and stationary Markov constrained equilibria, respectively. Passing to subsequences, there exist strategies σ^* and σ^{s*} such that σ^n converges to σ^* in the weak topology on Σ^M and σ^n converges to $\sigma^{s,*}$ in the weak topology on Σ^{sM}.

B.3 Proof of Propositions 3, 5, and 6

For $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$, $\ell \in \mathbb{N}$, let $\mathcal{H}_i(Z) := \{h_i \in \mathcal{H}_i | h_i^{(\ell)} \in Z\}$ be the set of private histories that follow private histories in Z. We say that a strategy $\sigma_i' \in \Sigma_i$ is a *continuation of strategy* $\sigma_i \in \Sigma_i$ *after* Z if σ_i coincides with σ_i' in every $h_i \notin \mathcal{H}_i(Z)$. The set of strategies that are a continuation of σ_i after Z is denoted $\Sigma_i(\sigma_i, Z)$. For a pair of strategies $\sigma_i, \sigma_i' \in \Sigma_i$ we write $(\sigma_i'|_Z, \sigma_i)$ for the strategy in $\Sigma_i(\sigma_i, Z)$ that coincides with σ_i' at private histories in $\mathcal{H}_i(Z)$, and is equal to σ_i otherwise.

Let $\ell \in \mathbb{N}$, $a_i^\ell \in (X_i \setminus \{a_*\})^\ell$. Define $\mathcal{H}_i(a_i^\ell) \coloneqq \{h_i \in \mathcal{H}_i^\ell \cap \mathcal{H}_{A_i} | h_i = (s_i^\ell, a_i^\ell),\ s_i^\ell \in S_i^\ell\}$. Let $\sigma_i(a_i^\ell)$ denote a strategy that takes action $a_{i,t}^\ell$ in active period t. Define $P_{-i}(Z|\sigma_{-i}) = \sum_{a_i^\ell \in (X_i \setminus \{a_*\})^\ell} P_i(Z \cap \mathcal{H}_i(a_i^\ell) | (\sigma_i(a_i^\ell), \sigma_{-i}))$ for $Z \in \mathcal{M}(\mathcal{H}_i)$. The expression $P_{-i}(Z|\sigma_{-i})$ sums up the probabilities of each subset of Z that contains a specific sequence of player i's actions, assuming player i selects those actions with certainty, and the opponents play according to σ_{-i}. Since player i's strategy is fixed to follow the actions in their private history, this sum depends only on opponents' strategies σ_{-i}.

Throughout the following analysis, when integrating over action sets, we employ the counting measure unless explicitly specified otherwise by the measure in question.[7]

Let $\sigma_{-i} \in \Sigma_{-i}$, and let the transition probability $P_{\omega|h_i}(\cdot|\cdot, \sigma_{-i}) : \mathcal{H}_i \to \Delta(\mathscr{H})$ be defined by

$$\int_{C \cap (\hbar_i^{h,t})^{-1}(Z)} p_{-i}(a^{t-1} \mid \omega^{t,(t-1)}, \sigma_{-i}) d\mu_\omega^t(\omega^t | a^{t-1}) \quad = \int_Z P_{\omega|h_i}\left(C \mid h_i, \sigma_{-i}\right) dP_{-i}(h_i \mid \sigma_{-i}),$$

for $Z \in \mathcal{M}(\mathcal{H}_i(a_i^\ell))$, $a_i^\ell \in (X_i \setminus \{a_*\})^\ell$, and $C \in \mathcal{M}(\mathscr{H})$. The expression $P_{\omega|h_i}(\cdot|\cdot, \sigma_{-i})$ defines a transition probability mapping player i's private histories to distributions over histories that could have occurred when i observes their private history, given the opponents' strategy profile σ_{-i}. Define

$$U_i(\hat{\sigma}|h_i, \sigma_{-i}) \coloneqq \sum_{\tau \in \mathbb{N}} \int_{\mathscr{H}^\tau} U_i(\hat{\sigma}|\omega^\tau, a^{\tau-1})\, dP_{\omega|h_i}(\omega^\tau, a^{\tau-1}|h_i, \sigma_{-i}). \tag{11}$$

The expression $U_i(\hat{\sigma}|h_i, \sigma_{-i})$ defines player i's expected payoff from strategy profile $\hat{\sigma}$ conditional on their private history h_i, given opponents follow strategy profile σ_{-i}. It sums the expected continuation payoffs over all possible lengths of histories that could have occurred when i observes h_i.

Lemma 9. *Let $(\sigma_i, \sigma_{-i}) \in \Sigma$, $\ell \in \mathbb{N}$, $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$, and $\tilde{\Sigma}_i \subseteq \Sigma_i$ be such that $(\sigma_i'|Z, \sigma_i) \in \tilde{\Sigma}_i$ for every $\sigma_i' \in \tilde{\Sigma}_i$. If*

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \qquad \forall \sigma_i' \in \tilde{\Sigma}_i, \tag{12}$$

then

$$U_i(\sigma_i, \sigma_{-i}|Z, \sigma) \geq U_i(\sigma_i', \sigma_{-i}|Z, \sigma) \qquad \forall \sigma_i' \in \tilde{\Sigma}_i.$$

[7] The counting measure "counts one" for every element of a set. Therefore, if p is a counting measure over a countable set B and $f : B \to \mathbb{R}$, $\int_B f(b)dp(b) = \sum_{b \in B} f(p)$.

Furthermore, if $P_{\omega|h_i}$ exists, then there exists $N \in \mathcal{M}(\mathcal{H}_i)$ with $P_i(N|\sigma) = 0$ such that

$$U_i(\sigma_i, \sigma_{-i}|h_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}|h_i, \sigma_{-i}) \qquad \forall \sigma_i' \in \tilde{\Sigma}_i, \forall h_i \in \mathcal{H}_i \setminus N. \tag{13}$$

Proof. We argue by contradiction. Suppose that σ_i satisfies equation (12), but there are $\ell \in \mathbb{N}$, $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$ with $P_i(Z|\sigma) > 0$, and $\sigma_i' \in \tilde{\Sigma}_i$ such that

$$U_i(\sigma|Z, \sigma) < U_i(\sigma_i', \sigma_{-i}|Z, \sigma). \tag{14}$$

By the definition of P_i^t, for every $C \in \mathcal{M}(\mathcal{H}^t)$, we have that

$$P_i^t(C|Z, \sigma) = \frac{\int_{C \cap (\hbar_i^{h,t})^{-1}(Z)} p(a^{t-1}|\omega^{t,(t-1)}, \sigma) d\mu_\omega^t(\omega^t|a^{t-1})}{P_i(Z|\sigma)}.$$

Therefore, by the definition of $U_i(\cdot|Z, \cdot)$, multiplying both sides of equation (14) by $P_i(Z|\sigma) > 0$, we obtain

$$\begin{aligned}\sum_{\substack{\tau \in \mathbb{N},\\ t \geq \tau}} \int_{(\hbar_i^{h,t})^{-1}(Z) \cap \mathcal{H}^\tau} & U_{i,t}(\sigma|\omega^\tau, a^{\tau-1}) p(a^{\tau-1}|\omega^{\tau,(\tau-1)}, \sigma) \, d\mu_\omega^\tau(\omega^\tau|a^{\tau-1}) \\ < \sum_{\substack{\tau \in \mathbb{N},\\ t \geq \tau}} & \int_{(\hbar_i^{h,t})^{-1}(Z) \cap \mathcal{H}^\tau} U_{i,t}(\sigma_i', \sigma_{-i}|\omega^\tau, a^{\tau-1}) p(a^{\tau-1}|\omega^{\tau,(\tau-1)}, \sigma) \, d\mu_\omega^\tau(\omega^\tau|a^{\tau-1}).\end{aligned} \tag{15}$$

By the definition of $U_{i,t}(\sigma|\omega^\tau, a^{\tau-1})$, the previous expression yields,

$$\begin{aligned}&\sum_{t \in \mathbb{N}} \int_{\mathcal{H}^t} \mathbb{1}\{\hbar_i^{h,t}(\omega^t, a^{t,(t-1)}) \in \mathcal{H}_i(Z)\} \cdot g_i(\omega^t, a^t) \cdot p(a^t|\omega^t, \sigma) \, d\mu_\omega^t(\omega^t|a^{t,(t-1)}) \\ &\quad < \sum_{t \in \mathbb{N}} \int_{\mathcal{H}^t} \mathbb{1}\{\hbar_i^{h,t}(\omega^t, a^{t,(t-1)}) \in \mathcal{H}_i(Z)\} \cdot g_i(\omega^t, a^t) \cdot p(a^t|\omega^t, (\sigma_i'', \sigma_{-i})) \, d\mu_\omega^t(\omega^t|a^{t,(t-1)}),\end{aligned}$$

where $\sigma_i'' = (\sigma_i'|_Z, \sigma_i)$. Since σ_i'' belongs to $\tilde{\Sigma}_i$, and σ_i'' coincides with σ_i at $h_i \notin \mathcal{H}_i(Z)$, this contradicts that σ_i is a best response, as required by condition (12).

Similarly, if there is a set Z with $P_{-i}(Z|\sigma_{-i}) > 0$, and a strategy $\sigma_i' \in \tilde{\Sigma}_i$ such that condition (13) does not hold, then we obtain a contradiction by multiplying equation $U_i(\sigma_i, \sigma_{-i}|h_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}|h_i, \sigma_{-i})$ by the corresponding p_i and integrating by $dP_{-i}(h_i|\sigma_{-i})$ over Z, which yields equation (15). □

PROOF OF PROPOSITION 3. The fact that an $\tilde{\varepsilon}$-constrained equilibrium is an $\tilde{\varepsilon}$-constrained conditional equilibrium follows by Lemma 9 by setting $\tilde{\Sigma}_i = \Sigma_i(\tilde{\varepsilon})$. □

PROOF OF PROPOSITIONS 5 AND 6.

LEMMA 10. *1. If σ is an $\tilde{\varepsilon}$-constrained equilibrium, for $\tilde{\varepsilon} \in \mathcal{E}$, then there exists a negligible set $H \in \mathcal{M}(\mathcal{H})$ such that $U_i(\sigma|\omega^t, a^{t-1}) \geq U_i(\sigma_i', \sigma_{-i}|\omega^t, a^{t-1})$, for every $i \in N$, $\sigma_i' \in \Sigma_i(\tilde{\varepsilon})$, $(\omega^t, a^{t-1}) \in \mathcal{H}_i^{id} \setminus H$.*

2. Let Γ be a Markov game satisfying stationary Markov information and Markov payoffs. If σ^M is a stationary Markov $\tilde{\varepsilon}^M$-constrained equilibrium, for $\tilde{\varepsilon}^M \in \mathcal{E}^M$, then, for each player $i \in N$, there exists $\tilde{H}_i^R \in \mathcal{M}(S_i^R)$ with $\mu_{s_i,t}^R(\tilde{H}_i^R|a^{t-1}) = 1$ for all $a^{t-1} \in X^{t-1}$ and $t \in \mathbb{N}$ such that $U_i(\sigma^M|s_i^R) \geq U_i(\sigma_i', \sigma_{-i}^M|s_i^R)$, for every $\sigma_i' \in \Sigma_i(\tilde{\varepsilon})$ at every payoff-relevant signal $s_i^R \in \tilde{H}_i^R$.

Proof. For each $i \in N$, $t \in \mathbb{N}$, $h \in \mathcal{H}_i^{\text{id}} \cap \mathcal{H}^t$, we have that $(\hbar_i^{h,t})^{-1}(\hbar_i^{h,t}(h)) = \{h\}$. Therefore, $P_{\omega|h_i}(h|h_i, \sigma_{-i}) = \mathbb{1}\{h_i = \hbar_i^{h,t}(h)\}$, since the right hand side is a probability measure. This implies that $U_i(\hat{\sigma}|h_i, \sigma_{-i})$ is independent of σ_{-i} and equal to $U_i(\hat{\sigma}|\omega^t, a^{t-1})$ for $(\omega^t, a^{t-1}) = (\hbar_i^{h,t})^{-1}(h_i)$. Therefore, by condition (13) in Lemma 9 we obtain point 1.

Similarly, point 2 follows by the same Lemma 9 together with Lemma 8.II as $U_i(\sigma|h_i, \sigma_{-i}) = U_i(\sigma|\mathcal{s}_i^R(h_i))$ when σ is a stationary Markov strategy and Γ satisfies the stationary Markov information and Markof payoffs condition. $\square$

The following lemma establishes Propositions 5 and 6. First, it shows that a THPE is optimal at identifiable histories. Second, it shows that a stationary MTHPE satisfies the Markov best response condition. Finally, since every SPE is also a Nash equilibrium, it establishes that a THPE is a conditional equilibrium by Lemma 9 with $\tilde{\Sigma}_i = \Sigma_i$.

LEMMA 11. *(a) Let σ^* be a THPE. There exists a negligible set $H \in \mathcal{M}(\mathcal{H})$ such that $U_i(\sigma^*|\omega^t, a^{t-1}) \geq U_i(\sigma_i', \sigma_{-i}^*|\omega^t, a^{t-1})$, for every $\sigma_i' \in \Sigma_i$, $(\omega^t, a^{t-1}) \in \mathcal{H}_i^{id} \setminus H$.*

(b) Let Γ be a Markov game satisfying stationary Markov information and Markov payoffs and σ^* be a stationary trembling hand Markov perfect equilibrium. Then, for each player $i \in N$, there exists $\tilde{H}_i^R \in \mathcal{M}(S_i^R)$ with $\mu_{s_i,t}^R(\tilde{H}_i^R|a^{t-1}) = 1$ for all $a^{t-1} \in X^{t-1}$ and $t \in \mathbb{N}$ such that $U_i(\sigma^*|s_i^R) \geq U_i(\sigma_i', \sigma_{-i}^*|s_i^R)$, for every $\sigma_i' \in \Sigma_i$, and every payoff-relevant signal $s_i^R \in \tilde{H}_i^R$.

Proof. Let σ^* be a THPE and suppose that there is a set $B \subseteq \mathcal{H}_i^{\text{id}} \cap \mathcal{H}^t$ non-negligible such that $U_i(\sigma^*|\omega^t, a^{t-1}) < U_i(\sigma_i', \sigma_{-i}^*|\omega^t, a^{t-1})$ for $(\omega^t, a^{t-1}) \in B$. We assume, without loss, that for each $i \in N$ there is $\tau_i \in \mathbb{N}$, $a_i(\gamma_i^{t-1}(\omega^{t,(t-1)}, a^{t-1}), a_i^{t-1}; \sigma_i^*) = a_i^{t-1,(\tau_i)}$ for every

$(\omega^t, a^{t-1}) \in B$, and the projection of B onto X^{t-1} is a singleton $\{a^{t-1}\}$. Therefore, we obtain

$$- \int_B U_i(\sigma^*|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, \sigma^*)\, d\mu^t_\omega(\omega^t|a^{t-1})$$
$$+ \int_B U_i(\sigma'_i, \sigma^*_{-i}|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, (\sigma'_i, \sigma^*_{-i}))\, d\mu^t_\omega(\omega^t|a^{t-1}) \coloneqq \beta > 0.$$

By the definition of THPE, Lemma 15 in Online Appendix C.1, and the continuity of U_i in strategies,[8] there exists a sequence $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}}$ with $\tilde{\varepsilon}_n \in \mathcal{E}(\varepsilon_n)$ where $\varepsilon_n \to 0$, a sequence of $\tilde{\varepsilon}_n$-constrained equilibria $(\sigma^{\varepsilon_n})_{n\in\mathbb{N}}$, and an $N_i \in \mathbb{N}$ such that for all $n \geq N_i$,

$$\Bigg| \int_B U_i(\sigma^*|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, \sigma^*)\, d\mu^t_\omega(\omega^t|a^{t-1})$$
$$- \int_B U_i(\sigma^{\varepsilon_n}|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, \sigma^{\varepsilon_n})\, d\mu^t_\omega(\omega^t|a^{t-1})\Bigg| < \beta/3.$$

Further, there exists a sequence $(\sigma_i^{'\varepsilon_n})_{n\in\mathbb{N}}$ of $\tilde{\varepsilon}_n$-constrained strategies for player i such that:

$$\Bigg| \int_B U_i(\sigma'_i, \sigma^*_{-i}|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, (\sigma'_i, \sigma^*_{-i}))\, d\mu^t_\omega(\omega^t|a^{t-1})$$
$$- \int_B U_i(\sigma_i^{'\varepsilon_n}, \sigma^{\varepsilon_n}_{-i}|\omega^t, a^{t-1})\Pi_{j\in N}p_j(a^{t-1}|\omega^{t,(t-1)}, a^{t,(\tau_j)}, (\sigma_i^{'\varepsilon_n}, \sigma^{\varepsilon_n}_{-i}))\, d\mu^t_\omega(\omega^t|a^{t-1})\Bigg| < \beta/3.$$

Combining the last three expressions yields a contradiction with Lemma 10.

Similarly, suppose there exists $a^t \in X^t$, and a set $\tilde{S}^R_i \in \mathcal{M}(S^R_i)$ with $\mu^R_{s_i,t}(\tilde{S}^R_i|a^t) > 0$ such that $U_i(\sigma^*|s^R_i) < U_i(\sigma'_i, \sigma_{-i}|s^R_i)$ for all $s^R_i \in \tilde{S}^R_i$. Then, integrating this inequality with respect to the measure $\mu^R_{s_i,t}(\cdot|a^t)$ over $s^R_i \in \tilde{S}^R_i$ and following steps analogous to those above yields a contradiction, establishing part (b). $\square$

B.4 Conditional ν-equilibrium

In what follows we assume that for every $\hat{\sigma} \in \Sigma$, $\tau \in \mathbb{N}$, and $a^{\tau-1} \in X^{\tau-1}$, there is a transition probability $\mu^\tau_{\omega,a}(\cdot, a^{\tau-1}|\cdot, \hat{\sigma}) : \mathcal{H}_i \to \Delta(\Omega^\tau)$ such that

$$d\mu^\tau_\omega(\omega^\tau|a^{\tau-1}) \cdot p(a^{\tau-1}|\omega^{\tau,(\tau-1)}, \hat{\sigma}) = d\mu^\tau_{\omega,a}(\omega^\tau, a^{\tau-1}|h_i, \hat{\sigma}) \times dP_i(h_i|\hat{\sigma}). \quad (16)$$

We abbreviate bounded conditional payoff as BCP.

[8]Notice that we can view the previous integral as the expected utility of a strategy in which each player $j \in N$ puts weight 1 on actions in a^{t-1} up to τ_j, and in which $g_i = 0$ for $t' < t$.

PROPOSITION 7. *Under SAC and BCP, if σ^ε is an $\tilde{\varepsilon}$-constrained equilibrium with $\tilde{\varepsilon} \in \mathcal{E}(\varepsilon)$, then σ^ε is also a conditional $\nu(\varepsilon)$-equilibrium for some function $\nu : [0,1] \to (0,\infty)$ satisfying* $\lim_{\varepsilon \to 0} \nu(\varepsilon) = 0$.

Proof. The notation used in this proof is introduced in Online Appendix B.3.

The following lemma is useful for our argument.

LEMMA 12. *Under payoff boundedness, payoff continuity, SAC, and BCP, the following statements hold for every $\ell \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$ with $P_i(Z|\sigma) > 0$:*

1. *For any $\hat{\sigma} \in \Sigma$, there exists $\sigma_i' \in \Sigma_i(\hat{\sigma}_i, Z)$ such that*

$$\sigma_i' \in \arg\max\{U_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}|Z, \hat{\sigma}_i)|\tilde{\sigma}_i \in \Sigma_i(\hat{\sigma}_i, Z)\}. \tag{17}$$

2. *If $\sigma_i' \in \Sigma_i(\hat{\sigma}_i, Z)$ satisfies* (17)*, then there is a strategy $\sigma_i'' \in \Sigma_i(\hat{\sigma}_i, Z)$ that is pure at every $h_i \in \mathcal{H}_i(Z)$, and satisfies $U_i(\sigma_i', \hat{\sigma}_{-i}|Z, \hat{\sigma}_i) = U_i(\sigma_i'', \hat{\sigma}_{-i}|Z, \hat{\sigma}_i)$.*

Proof. Let $\ell \in \mathbb{N}$, and $Z \in \mathcal{M}(\mathcal{H}_i^\ell)$ with $P_i(Z|\sigma) > 0$.

Point 1. Define $\check{\Sigma}_i := \{\sigma \in \Sigma | \sigma_i(\cdot|h_i) = \hat{\sigma}_i(\cdot|h_i),\, h_i \in \mathcal{H}_i^r \cap \mathcal{H}_{A_i}, r < \ell\}$ to be the set of strategies that coincide with $\hat{\sigma}_i$ at all private histories of length shorter than ℓ. Since $\check{\Sigma}_i$ is closed in the weak topology, it is compact. Under payoff boundedness and continuity, along with SAC, $U_i(\cdot, \hat{\sigma}_{-i})$ is continuous and therefore attains its maximum over $\check{\Sigma}_i$—call this maximum σ_i''. Notice that, σ_i'' satisfies (12) with $\tilde{\Sigma}_i = \check{\Sigma}_i$ and matches $\hat{\sigma}_i$ for histories shorter than ℓ. This implies that $(\hat{\hat{\sigma}}_i|_Z, \sigma_i'') \in \check{\Sigma}_i$ for all $\hat{\hat{\sigma}}_i \in \check{\Sigma}_i$. By Lemma 9, σ_i'' maximizes $U_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}|Z, \hat{\sigma}_i)$ over $\tilde{\sigma}_i \in \check{\Sigma}_i$. Therefore, $\sigma_i' = (\sigma_i''|_Z, \hat{\sigma}_i) \in \Sigma_i(\hat{\sigma}_i, Z)$ satisfies condition (17).

Point 2. Let $\sigma_i' \in \Sigma_i(\hat{\sigma}_i, Z)$ satisfy (17). For each $h_i \in Z$, define the support correspondence $\xi : Z \rightrightarrows X_i$ by:

$$\xi(h_i) = \{a_i \in X_i \mid \sigma_i'(a_i|h_i) > 0\}.$$

By Lemma 13 in Online Appendix C.1, ξ has a measurable selection $\hat{a}_{i,\ell} : Z \to X_i$. Let $\tilde{\sigma}_i$ be the strategy that equals $\hat{a}_{i,\ell}(h_i)$ for $h_i \in Z$ and equals σ_i' otherwise. We prove that:

$$U_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}|Z, \hat{\sigma}) = U_i(\sigma_i', \hat{\sigma}_{-i}|Z, \hat{\sigma}).$$

We proceed by contradiction, supposing $U_i(\tilde{\sigma}_i, \hat{\sigma}_{-i}|Z, \hat{\sigma}) < U_i(\sigma_i', \hat{\sigma}_{-i}|Z, \hat{\sigma})$.

For any strategy $\sigma_i'' \in \Sigma_i(\hat{\sigma}_i, Z)$, we can write

$$U_i(\sigma_i'', \hat{\sigma}_{-i}|Z, \hat{\sigma}) = \frac{1}{P_i(Z|\hat{\sigma})} \sum_{\substack{\tau \in \mathbb{N}, \\ t \geq \tau}} \int_{(\mathscr{h}_i^{h,\tau})^{-1}(Z)} \int_{\mathscr{H}^{\tau+1}} U_{i,t}(\sigma_i'', \hat{\sigma}_{-i}|\omega^{\tau+1}, a^\tau) \cdot p_i(a_i^\tau|\omega^\tau, \sigma_i'')$$

$$
\cdot\, p_{-i}(a^{\tau}_{-i}|\omega^{\tau},\hat{\sigma}_{-i})\, d\mu^{\tau+1}_{\omega}(\omega^{\tau+1}|\omega^{\tau},a^{\tau})d\mu^{\tau}_{\omega}(\omega^{\tau}|a^{\tau-1})
$$

$$
= \frac{1}{P_i(Z|\hat{\sigma})} \sum_{\substack{\tau\in\mathbb{N},\\ t\geq\tau}} \int_Z \int_{\mathcal{H}^{\tau}} \int_{\mathcal{H}^{\tau+1}} U_{i,t}(\sigma''_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) \cdot \sigma''_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot \hat{\sigma}_{-i}(a^{\tau}_{-i,\tau}|\omega^{\tau},a^{\tau-1})
$$

$$
d\mu^{\tau+1}_{\omega}(\omega^{\tau+1}|\omega^{\tau},a^{\tau})d\mu^{\tau}_{\omega,a}(\omega^{\tau},a^{\tau-1}|h_i,\hat{\sigma})dP_i(h_i|\hat{\sigma}),
$$

where $U_{i,\tau}(\sigma''_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) := g_i(\omega^{\tau+1,(\tau)},a^{\tau})$ and the last equality follows from the definition of P_i and (16). Therefore, $U_i(\tilde{\sigma}_i,\hat{\sigma}_{-i}|Z,\hat{\sigma}) < U_i(\sigma'_i,\hat{\sigma}_{-i}|Z,\hat{\sigma})$ implies that there is $\hat{Z} \in \mathcal{M}(\mathcal{H}^{\ell}_i)$, $\hat{Z} \subseteq Z$ with $P_i(\hat{Z}|\hat{\sigma}) > 0$ such that, for every $h_i \in \hat{Z}$,

$$
\sum_{\substack{\tau\in\mathbb{N},\\ t\geq\tau}} \int_{\mathcal{H}^{\tau+1}} U_{i,t}(\sigma'_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) \cdot \sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot \hat{\sigma}_{-i}(a^{\tau}_{-i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot d\mu^{\tau+1}_{\omega,a}(\omega^{\tau+1},a^{\tau}|h_i,\hat{\sigma})
$$

$$
> \sum_{\substack{\tau\in\mathbb{N},\\ t\geq\tau}} \int_{\mathcal{H}^{\tau+1}} U_{i,t}(\sigma'_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) \cdot \hat{a}_{i,\ell}(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot \hat{\sigma}_{-i}(a^{\tau}_{-i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot d\mu^{\tau+1}_{\omega,a}(\omega^{\tau+1},a^{\tau}|h_i,\hat{\sigma})
$$

where $\hat{a}_{i,\ell}(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) = 1$ if $a^{\tau}_{i,\tau} = \hat{a}_{i,\ell}(\hbar^{h,\tau}_i(\omega^{\tau},a^{\tau-1}))$ and zero otherwise, and $d\mu^{\tau+1}_{\omega,a}(\omega^{\tau+1},a^{\tau}|h_i,\hat{\sigma}) = d\mu^{\tau+1}_{\omega}(\omega^{\tau+1}|\omega^{\tau},a^{\tau})d\mu^{\tau}_{\omega,a}(\omega^{\tau},a^{\tau-1}|h_i,\hat{\sigma})$.

Now, notice that we can write $\sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) = \sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})(1 - \hat{a}_{i,\ell}(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})) + \sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})\hat{a}_{i,\ell}(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})$. Therefore, the previous inequality implies

$$
U_i(\sigma'_i,\hat{\sigma}_{-i}|\hat{Z},\hat{\sigma}) \cdot P_i(\hat{Z}|\hat{\sigma}) < \sum_{\substack{\tau\in\mathbb{N},\\ t\geq\tau}} \int_{\hat{Z}} \int_{\mathcal{H}^{\tau+1}} U_{i,t}(\sigma'_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) \cdot \sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})
$$

$$
\cdot \Big(1 - \hat{a}_{i,\ell}(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1})\Big) \cdot \hat{\sigma}_{-i}(a^{\tau}_{-i,\tau}|\omega^{\tau},a^{\tau-1})\, d\mu^{\tau+1}_{\omega,a}(\omega^{\tau+1},a^{\tau}|h_i,\hat{\sigma})dP_i(h_i|\hat{\sigma})
$$

$$
+ \sum_{\substack{\tau\in\mathbb{N},\\ t\geq\tau}} \int_{\hat{Z}} \int_{\mathcal{H}^{\tau+1}} U_{i,t}(\sigma'_i,\hat{\sigma}_{-i}|\omega^{\tau+1},a^{\tau}) \cdot \sigma'_i(a^{\tau}_{i,\tau}|\omega^{\tau},a^{\tau-1}) \cdot \sigma'_i(\hat{a}_{i,\ell}(\hbar^{h,\tau}_i(\omega^{\tau},a^{\tau-1}))|\omega^{\tau},a^{\tau-1})
$$

$$
\cdot\, \hat{\sigma}_{-i}(a^{\tau}_{-i,\tau}|\omega^{\tau},a^{\tau-1})\, d\mu^{\tau+1}_{\omega,a}(\omega^{\tau+1},a^{\tau}|h_i,\hat{\sigma})dP_i(h_i|\hat{\sigma}).
$$

where we used Tonelli's theorem to write the bound as a sum of two integrals.

However, this last inequality contradicts (17): the payoff on the right hand side of the inequality is the payoff from the strategy that instead of putting weight $\sigma'_i(\hat{a}_{i,\ell}(h_i)|h_i) > 0$ on $\hat{a}_{i,\ell}(h_i)$ randomizes according to σ'_i, for $h_i \in \hat{Z}$. From the inequality, this alternative strategy yields a higher expected payoff conditional on private histories in $\hat{Z}$. Arguing recursively, by defining a new correspondence from $\mathcal{H}_i(Z) \cap \mathcal{H}^{\tau}_i$ to X_i, for each $\tau > \ell$, and finding a measurable selection $a_{i,\tau}$ we can construct a strategy that yields an higher payoff than σ'_i. $\square$

Let $\sigma^* \in \Sigma(\tilde{\varepsilon})$ be an $\tilde{\varepsilon}$-constrained equilibrium. We show that, if $\tilde{\varepsilon}$ is an ε-tremble, then σ^* is a conditional ν-equilibrium for every $\nu > \nu(\varepsilon)$, for some function $\nu : [0,1] \to (0,\infty)$ satisfying $\lim_{\varepsilon\to 0}\nu(\varepsilon) = 0$.

Suppose there is $\nu > 0$, $Z \in \mathcal{M}(\mathcal{H}_i)$, and $\sigma_i' \in \Sigma_i$, such that

$$U_i(\sigma^*|Z,\sigma^*) + \nu < U_i(\sigma_i', \sigma_{-i}^*|Z,\sigma^*). \tag{18}$$

Let $\hat{\sigma}_i' \in$ argmax$\{U_i(\hat{\sigma}_i, \sigma_{-i}^*|Z,\sigma^*)|\hat{\sigma}_i \in \Sigma_i(\sigma_i^*, Z)\}$. The strategy $\hat{\sigma}_i'$ exists by Lemma 12.1, and by Lemma 12.2 it can be chosen to be a pure strategy in $\mathcal{H}_i(Z)$. Let $\hat{a}_i : \mathcal{H}_i(Z) \to X_i$ be the function that yields the action drawn with probability 1 by $\hat{\sigma}_i'(\cdot|h_i)$ for $h_i \in \mathcal{H}_i(Z)$.

For each private history $h_i \in \mathcal{H}_i$, and strategy $\sigma_i \in \Sigma_i$, define

$$prob_i(h_i|\sigma_i) = \prod_{\ell=0}^{|h_i|-1} \sigma_i(\mathbf{a}_{i,\ell+1}(h_i)|h_i^{(\ell)})$$

where $\mathbf{a}_{i,\ell}(h_i)$ denotes player i's ℓth action in history h_i.

Define strategy $\tilde{\sigma}_i$ recursively on the length of h_i as follows:

$$\tilde{\sigma}_i(a_i|h_i) := \begin{cases} 1 - \frac{1}{prob_i(h_i|\tilde{\sigma}_i)} \cdot \sum_{a_i \in X_i\setminus \hat{a}_i(h_i)} \tilde{\varepsilon}(h_i,a_i) & \text{if } a_i = \hat{a}_i(h_i),\ h_i \in \mathcal{H}_i(Z) \\ \tilde{\varepsilon}(h_i,a_i) \cdot \frac{1}{prob_i(h_i|\tilde{\sigma}_i)} & \text{if } a_i \neq \hat{a}_i(h_i),\ h_i \in \mathcal{H}_i(Z) \\ \sigma_i^*(a_i|h_i) & \text{otherwise.} \end{cases}$$

Notice that, $\tilde{\sigma}_i$ is an $\tilde{\varepsilon}$-constrained strategy that assigns the maximum probability to $\hat{a}_i(h_i)$ that still satisfies the $\tilde{\varepsilon}$-constraint requirements. By Lemma 13, the function $\sum_{a_i \in X_i\setminus \hat{a}_i(\cdot)} \tilde{\varepsilon}(\cdot, a_i)$ is $\mathcal{M}(\mathcal{H}_i)$-measurable.

Define $H^t(Z) = \{(\omega^t, a^t) \in \Omega^t \times X^t | \hbar_i^{h,t}(\omega^t, a^{t,(t-1)}) \in \mathcal{H}_i(Z)\}$, and $\alpha(h_i) := \mathbb{1}\{a_{i,\ell}(h_i) = \hat{a}_i(h_i^{(\ell-1)}), \ell \leq |h_i|\} \cdot \sum_{a_i\in X_i} \tilde{\varepsilon}(h_i,a_i)$. We can write

$$U_{i,t}(\tilde{\sigma}_i, \sigma_{-i}^*|Z,\sigma^*)\cdot P_i(Z|\sigma^*) = \int_{H^t(Z)} g_i(\omega^t,a^t)\cdot p_i(a_i^t|\omega^t,\tilde{\sigma}_i)p_{-i}(a_{-i}^t|\omega^t,\sigma_{-i}^*)\, d\mu_\omega^t(\omega^t|a^{t,(t-1)})$$

$$= \int_{H^t(Z)} g_i(\omega^t,a^t)\left(p_i(a_i^t|\omega^t,\hat{\sigma}_i') - \alpha(\hbar_i^{h,t}(\omega^t,a^{t,(t-1)}))\right)\cdot p_{-i}(a_{-i}^t|\omega^t,\sigma_{-i}^*)\cdot\, d\mu_\omega^t(\omega^t|a^{t,(t-1)})$$

$$+ \int_{H^t(Z)} g_i(\omega^t,a^t)\cdot \tilde{\varepsilon}_i(a_{i,t}^t|\hbar_i^{h,t}(\omega^t,a^{t,(t-1)}))\cdot p_{-i}(a_{-i}^t|\omega^t,\sigma_{-i}^*)\, d\mu_\omega^t(\omega^t|a^{t,(t-1)}),$$

Notice that $\alpha(\hbar_i^{h,t}(\omega^t,a^{t,(t-1)})) \leq \varepsilon$, and $U_i(\sigma_i',\sigma_{-i}|Z,\sigma) = \sum_{t\in\mathbb{N}} U_{i,t}(\sigma_i',\sigma_{-i}|Z,\sigma)$.

Therefore, we obtain

$$U_i(\tilde{\sigma}_i, \sigma^*_{-i}|Z, \sigma^*) \geq U_i(\hat{\sigma}'_i, \sigma^*_{-i}|Z, \sigma^*) -$$
$$\frac{1}{P_i(Z|\sigma^*)}\left(\sum_t \int_{H^t(Z)} 2\varepsilon \cdot |g_i(\omega^t, a^t)| \cdot p_i(a^t_i|\omega^t, \hat{\sigma}'_i) \cdot p_{-i}(a^t_{-i}|\omega^t, \sigma^*_{-i})\, d\mu^t_\omega(\omega^t|a^{t,(t-1)})\right)$$
$$\geq U_i(\hat{\sigma}'_i, \sigma^*_{-i}|Z, \sigma^*) - 2\varepsilon \sup_{\hat{\sigma}, \sigma \in \Sigma} \bar{U}_i(\hat{\sigma}|Z, \sigma)$$

Define $\nu(\varepsilon) \coloneqq 2\varepsilon \cdot \sup\left\{\bar{U}_i(\hat{\sigma}|Z, \sigma) \mid \hat{\sigma}, \sigma \in \Sigma, Z \in \mathcal{M}(\mathcal{H}^\ell_i), \ell \in \mathbb{N}, P_i(Z|\sigma) > 0\right\}$. If $\nu > \nu(\varepsilon)$ then

$$U_i(\sigma^*|Z, \sigma^*) + \nu < U_i(\hat{\sigma}'_i, \sigma^*_{-i}|Z, \sigma^*) \leq U_i(\tilde{\sigma}_i, \sigma^*_{-i}|Z, \sigma^*) + \nu(\varepsilon),$$

where the first inequality is implied by equation (18), contradicts that σ^* is an $\tilde{\varepsilon}$-constrained equilibrium.

B.5 Stochastic Move Opportunities

Proof of Lemma 3. Let $M = \sup\{|g_i(\omega^t, a^t)|\,|(\omega^t, a^t) \in \Omega^t \times X^t,\, t \in \mathbb{N}\}$. For each $t \in \mathbb{N}$, define $\bar{g}_i(\omega^t, a^t) \coloneqq M$ if $g_i(\omega^t, a^t) \neq 0$ and $\bar{g}_i(\omega^t, a^t) \coloneqq 0$, otherwise.

For $i \in N$, and $(\omega^t, a^t) \in \Omega^t \times X^t$, $|g_i(\omega^t, a^t)| \leq \bar{g}_i(\omega^t, a^t)$. By Lemma 4, we write

$$\begin{aligned}
|U_i(\sigma)| &\leq \sum_{t=1}^\infty \sup_{\sigma \in \Sigma} \left| \sum_{a^t \in X^t} \int_{\Omega^t} g_i(\omega^t, a^t) \cdot p(a^t|\omega^t, \sigma)\, d\mu^t_\omega(\omega^t|a^{t,(t-1)}) \right| \\
&\leq \sum_{t=1}^\infty \sup_{\sigma \in \Sigma} \sum_{a^t \in X^t} \int_{\Omega^t} \bar{g}_i(\omega^t, a^t) \cdot p(a^t|\omega^t, \sigma)\, d\mu^t_\omega(\omega^t|a^{t,(t-1)}) \\
&= \sum_{t=1}^\infty \sup_{\sigma \in \Sigma} \sum_{a^t \in X^t} \int_{\Omega^t} \mathbb{1}\{(\omega^{t,(\tilde{t})}, a^{t,(\tilde{t})}) \notin H^{end,\tilde{t}}, \tilde{t} \leq t\} \cdot M \cdot p(a^t|\omega^t, \sigma)\, d\mu^t_\omega(\omega^t|a^{t,(t-1)}) \\
&= \sum_{t=1}^\infty \sup_{\sigma \in \Sigma} \left(1 - \sum_{\tilde{t}=1}^t P^{\tilde{t}}(H^{end,\tilde{t}}|\sigma)\right) \cdot M \\
&\leq \sum_{t=1}^\infty \sum_{\tilde{t}=t+1}^\infty \sup_{\sigma \in \Sigma} P^{\tilde{t}}(H^{end,\tilde{t}}|\sigma) \cdot M = M \cdot \sum_{\tilde{t}=1}^\infty \sup_{\sigma \in \Sigma} \left((\tilde{t} - 1) \cdot P^{\tilde{t}}(H^{end,\tilde{t}}|\sigma)\right),
\end{aligned}$$

where the second equality follows from the definition of $P^t(\cdot|\sigma)$, and $\bar{H}^{end,\tau} \cap \bar{H}^{end,\tau'} = \emptyset$ for $\tau \neq \tau'$, the third inequality from the fact that the game has finitely many periods with probability 1,[9] and the last equality is obtained by changing the order of the summation. $\square$

[9]This latter condition implies $(1 - \sum_{\tilde{t}=1}^t P^{\tilde{t}}(H^{end,\tilde{t}}|\sigma)) = \sum_{\tilde{t}=t+1}^\infty P^{\tilde{t}}(H^{end,\tilde{t}}|\sigma)$.

B.6 Examples 2.1, 2.2, and 2.3

Consider the game of Examples 2.1, 2.2, and 2.3 adapted from Harris et al. (1995).

We show that the strategy profile described in Example 2.3 is an SPE. Restrict player A to randomize uniformly between the actions $\delta - \varepsilon$ and $-\delta + \varepsilon$, for $\varepsilon \in [\delta - 1, \delta]$ such that $a_\varepsilon := 1/2 \cdot (\delta - \varepsilon) + 1/2 \cdot (\delta + \varepsilon) \in \Delta(\mathcal{A})$. It is immediate to see that players B, C, and D cannot profitably deviate; we focus on player A's best response. Player A's payoff is

$$-10 \cdot \mathbb{P}(\{c \neq d\}|a_\varepsilon) - \tfrac{1}{2}|\delta - \varepsilon|^2,$$

where $\mathbb{P}(\{c \neq d\}|a_\varepsilon)$ is the probability that C and D play different actions conditional on a_ε, which equals[10]

$$\mathbb{P}(\{c \neq d\}|a_\varepsilon) = \varepsilon \cdot \frac{(2\delta - \varepsilon)}{2\delta^2}$$

for $\varepsilon > 0$ and zero otherwise.

Clearly, $\varepsilon < 0$ cannot be optimal since, compared to $\varepsilon = 0$, it does not affect $\mathbb{P}(\{c \neq d\}|a_\varepsilon)$ and increases $\frac{1}{2}|\delta - \varepsilon|^2$. Furthermore, it is easy to check that A's payoff is decreasing in ε for $\delta \in \mathcal{A}$, making it optimal to set $\varepsilon = 0$ and uniformly randomize between δ and $-\delta$.

To complete the argument, we need to show that player A cannot profitably deviate by playing any other mixed strategy. Consider a mixed strategy $\alpha \in \Delta(\mathcal{A})$ that assigns positive probability to some action $\bar{a} \in (-\delta, \delta)$. Player A would obtain a strictly higher payoff by shifting that probability mass from $\bar{a}$ to δ instead. Therefore, any optimal strategy for player A can only place probability on actions $\pm\delta$. To show $\alpha \in \Delta(\mathcal{A})$ cannot be optimal, we first observe that any strategy with supp$(\alpha) \cap [-\delta, \delta] = \emptyset$ is not optimal. Now consider any $\bar{a} \in$ supp$(\alpha) \cap (-\delta, \delta)$. Using calculations analogous to those in footnote 10, the probability of miscoordination between C and D given $\bar{a}$ is:

$$\mathbb{P}(\{c \neq d\}|\bar{a}) = \frac{\delta^2 - |\bar{a}|^2}{2\delta^2}.$$

Therefore, player A's payoff from playing $\bar{a}$ is:

$$U_A(\bar{a}) = -10 \cdot \frac{\delta^2 - |\bar{a}|^2}{2\delta^2} - \frac{1}{2}|\bar{a}|^2 < -\frac{1}{2}|\delta|^2 = U_A(\delta)$$

where the inequality holds since U_A increases in $|\bar{a}|$ for $\bar{a} \in [-\delta, \delta]$ and $\delta \in \mathcal{A} \subseteq [-1, 1]$.

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[10] Applying conditioning, it follows that

$$\begin{aligned}
\mathbb{P}(\{c \neq d\}|a_\varepsilon) &= 1/2 \cdot \big(\mathbb{P}(\{c \neq d\}|\delta - \varepsilon) + \mathbb{P}(\{c \neq d\}| - \delta + \varepsilon)\big) \\
&= 1/2 \cdot \big(2 \cdot \mathbb{P}(s_i \geq 0|\delta - \varepsilon) \cdot \mathbb{P}(s_{-i} < 0|\delta - \varepsilon) + 2 \cdot \mathbb{P}(s_i \geq 0| - \delta + \varepsilon) \cdot \mathbb{P}(s_{-i} < 0| - \delta + \varepsilon)\big) \\
&= \left(\int_0^{2\delta - \varepsilon} \frac{1}{2\delta} dy\right) \cdot \left(\int_{-\varepsilon}^0 \frac{1}{2\delta} dy\right) + \left(\int_0^{\varepsilon} \frac{1}{2\delta} dy\right) \cdot \left(\int_{-2\delta+\varepsilon}^0 \frac{1}{2\delta} dy\right) = \varepsilon \cdot \frac{(2\delta - \varepsilon)}{2\delta^2}
\end{aligned}$$

for every $i \in \{C, D\}$.

B.7 Applications to Markov Games

Application 3.

Asynchronous revision games. Let $\Omega = \cup_{t\in\mathbb{N}}(X^t \times [0,T))$ represent histories of action profiles and move timings, where X is finite and $T \in \mathbb{R}_+$. Let $\Omega^R = \cup_{t\in\mathbb{N}}(X^t) \times [0,T)$ be the payoff-relevant state space, which contains the history of previous moves and current moving time. For each $\omega^R \in \Omega^R$ and $i \in N$, players observe the payoff-relevant state perfectly through $\gamma_i^R(\omega^R) = \omega^R$. For $\ell \in \mathbb{N}$ and $h^\ell \in \mathscr{H}^\ell$, $\gamma_i^{\not R}(h^\ell) = (t_1, ..., t_{\ell-1})$, that is, the previous moving times are part of the payoff-irrelevant signals. The environment follows Section 6 stochastic move framework, with moves occurring at independent Poisson rates. Payoff boundedness holds since Poisson rates ensure the game has a finite expected length and ends with probability 1. Payoff continuity follows from finite actions. MAC(a) holds due to the Poisson move timing and finite action histories, while MAC(b) follows from finite actions. Stationary Markov information and payoff conditions are satisfied as active players perfectly observe current payoff-relevant states.

Dynamic cheap talk games. The sender observes the whole history of the game, including the current exogenous state $\hat{\omega} \in \hat{\Omega}$, and selects a message $m \in M$; the receiver, observing only the sender's messages, implements an action $a \in A$; the exogenous state transition $\mu : A \to \Delta(\hat{\Omega})$ depends on the current action played by the receiver; payoffs depend on the current state and receiver's action, $(\hat{\omega}, a) \in \hat{\Omega} \times A$.

We can write the set of the states of the world as $\Omega^R = \hat{\Omega} \times M \times A$, incorporating the current message and previous actions played by the receiver. For $i \in \{s, r\}$, representing the sender and the receiver, respectively, the current payoff-relevant signals are: $\mathscr{s}_s^R(h_s) = (\hat{\omega}_t^t, a_{t-1}^{t-1})$ for $h_s = (\hat{\omega}^t, m^{t-1}, a^{t-1}) \in \mathscr{H}_s$; $\mathscr{s}_r^R(h_r) = (a_{t-1}^{t-1}, m_t^t)$ for $h_r = (a^{t-1}, m^t) \in \mathscr{H}_r$. Stationary Markov information is satisfied as, before moving, the sender observes the current exogenous state and the previous receiver's action, which constitute the current payoff-relevant state, and the receiver's inference about the current exogenous state depends only on her previous actions played and the current message received, which constitute her current payoff-relevant signals. Markov payoffs holds as well. We assume the sets M and A are finite to ensure that payoff boundedness, payoff continuity, and MAC hold.

C Online Appendix — Mathematical Results

C.1 Carathéodory integrands and measurability of weak limits

Let $(Y, \mathcal{M}(Y), \beta)$ be a measure space, Z be a countable metric space endowed with its power set σ-algebra and counting measure.

The following result shows that a correspondence from Y to Z, which may not be closed-valued, has a measurable selection under a condition weaker than measurability. It relies on the countability of the set Z.

LEMMA 13. *Let $\phi : Y \rightrightarrows Z$ be a non-empty valued correspondence such that for every $z \in Z$ the set $\{y \in Y | z \in \phi(y)\}$ is $\mathcal{M}(Y)$-measurable. Then ϕ has a $\mathcal{M}(Y)$-measurable selection, and for any $\mathcal{M}(Y) \times \mathcal{M}(Z)$-measurable, real valued function $\hat{g}$, $\sum_{z \in \phi(y)} \hat{g}(y, z)$ is $\mathcal{M}(Y)$-measurable.*

Proof. Let $(z_j)_{j \in \mathbb{N}}$ be an enumeration of the set Z. Define the function

$$m_j(y) := \mathbb{1}\{z_j \notin \phi(y)\} - 1/j.$$

Then, the function $\bar{m}(y) := \inf_{j \in \mathbb{N}} m_j(y)$ is strictly negative (ϕ is non-empty valued), finite for each y, and $\mathcal{M}(Y)$-measurable. It yields $1/j$ for the smallest j such that $z_j \in \phi(y)$.

The selection $z(y) = z_{|1/\bar{m}(y)|}$ is $\mathcal{M}(Y)$-measurable. In fact, for a measurable set $\hat{Z} \in \mathcal{M}(Z)$, we can write $\hat{Z} = (z_{n_j})_{n_j \in \mathbb{N}}$ for some subsequence $(n_j)_{j \in \mathbb{N}}$ of $\mathbb{N}$. Then $z^{-1}(\hat{Z}) = \{y \in Y | \, |1/\bar{m}(y)| \in (n_j)_{j \in \mathbb{N}}\}$ is in $\mathcal{M}(Y)$ since all countable subsets of $\mathbb{R}$ are measurable.

Let $\hat{m}_k(y) = \sum_{j=1}^{k} \hat{g}(y, z_j) \cdot \mathbb{1}\{z_j \in \phi(y)\}$, then $\hat{m}_k(y)$ is measurable and, therefore, $\lim_{k \to \infty} \hat{m}_k(y) = \sum_{z \in \phi(y)} \hat{g}(y, z)$ is $\mathcal{M}(Y)$-measurable. □

The following lemma shows that when measures have densities, those densities are Carathéodory integrands if and only if the measures are continuous and bounded in strong total variation. Let $\mu(\cdot|\cdot) : Z \to \mathcal{M}(Y)$ a transition probability from Z to Y.

PROPOSITION 8. *Suppose that there is a measure β over Y and a function $\varphi : Y \times Z \to \mathbb{R}_+$ that is $\mathcal{M}(Y) \otimes \mathcal{M}(Z)$ measurable and such that for every $B \in \mathcal{M}(Y)$ and $z \in Z$*

$$\mu(B|z) = \int_B \varphi(y, z) \, d\beta(y).$$

Then $\varphi \in CI(Y \times Z, \beta)$ if and only if μ is bounded and continuous in Z in strong total variation.

Proof. **Necessity.** We argue by contradiction. Suppose there is a $B \in \mathcal{M}(Y)$ with $\beta(B) \in (0, \infty)$, such that for each $y \in B$, there is $z^*(y) \in Z$, $\varepsilon(y) > 0$ and a sequence $(z_n(y))_{n\in\mathbb{N}}$ such that $d\left(z_n(y), z^*(y)\right) < \frac{1}{n}$ and $|\varphi(y, z_n(y)) - \varphi(y, z^*(y))| > \varepsilon(y)$. Define the correspondence $\xi : B \rightrightarrows Z$

$$\xi(y) = \left\{z \in Z \middle| \exists\varepsilon \in (0,1),\, \forall n \in \mathbb{N},\, \exists z_n \in Z,\, d(z_n, z) < 1/n,\, |\varphi(y, z_n) - \varphi(y, z)| > \varepsilon\right\}.$$

Since $z^*(y) \in \xi(y)$ for each $y \in B$, ξ is non-empty valued. Furthermore, let $z \in Z$ and let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a countable dense subset of $(0, 1)$. We have

$$\begin{aligned}
&\{y \in B | z \in \xi(y)\} \\
&\quad = \left\{y \in B \middle| \exists\varepsilon \in (0,1),\, \forall n \in \mathbb{N},\, \exists z_n \in Z,\, d(z_n, z) < 1/n,\, |\varphi(y, z_n) - \varphi(y, z)| > \varepsilon\right\} \\
&\quad = \left\{y \in B \middle| \exists j \in \mathbb{N},\, \forall n \in \mathbb{N},\, \exists z_n \in Z,\, d(z_n, z) < 1/n,\, |\varphi(y, z_n) - \varphi(y, z)| > \varepsilon_j\right\} \\
&\quad = \bigcup_{j\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{z_n\in Z} \left\{y \in B \middle|\, d(z_n, z) < 1/n,\, |\varphi(y, z_n) - \varphi(y, z)| > \varepsilon_j\right\},
\end{aligned}$$

where the second equality follows from $\{\varepsilon_j\}_{j\in\mathbb{N}}$ dense in $(0, 1)$. Since φ is $\mathcal{M}(Y) \otimes \mathcal{M}(Z)$ measurable, each set $\{y \in B |\, d(z, z^*) < 1/n,\, |\varphi(y, z) - \varphi(y, z^*)| > \varepsilon_j\}$ is in $\mathcal{M}(Y)$. Therefore, $\{y \in B|z \in \xi(y)\}$ is $\mathcal{M}(Y)$-measurable and, by Lemma 13, ξ has a $\mathcal{M}(Y)$-measurable selection $\hat{z}^*(y)$. Define the correspondence

$$\xi_\varepsilon(y) = \left\{\varepsilon \in \{\varepsilon_j\}_{j\in\mathbb{N}} \middle| \forall n \in \mathbb{N}, \exists z_n \in Z,\, d(z_n, \hat{z}^*(y)) < 1/n,\, |\varphi(y, z_n) - \varphi(y, \hat{z}^*(y))| > \varepsilon\right\}.$$

Since $\hat{z}^*(y) \in \xi(y)$ for each $y \in B$, ξ_ε is non-empty valued. As before, By Lemma 13, ξ_ε has a $\mathcal{M}(Y)$-measurable selection, $\hat{\varepsilon}(y)$. Analogously, define the correspondence

$$\xi_n(y) = \left\{z \in Z \middle| d(z, \hat{z}^*(y)) < 1/n,\, |\varphi(y, z) - \varphi(y, \hat{z}^*(y))| > \hat{\varepsilon}(y)\right\}.$$

Since $\hat{\varepsilon}(y) \in \xi_\varepsilon(y)$, ξ_n is non-empty valued. As before, $\xi_n(y)$ has a $\mathcal{M}(Y)$-measurable selection, $\hat{z}_n(y)$.

Now, for each $z \in Z$, define the set $B_z := \{y \in B|\hat{z}^*(y) = z\}$, which is $\mathcal{M}(Y)$-measurable as $\hat{z}^*$ is measurable. Since $\cup_{z\in Z} B_z = B$ and $\beta(B) > 0$ then there is $\hat{z}$ such that $\beta(B_{\hat{z}}) > 0$. Define the $\mathcal{M}(Y)$-measurable sets $\hat{B}^{n,+}_z := \{y \in B_{\hat{z}}|\hat{z}_n(y) = z,\, \varphi(y, z) - \varphi(y, \hat{z}) > 0\}$, and $\hat{B}^{n,-}_z := \{y \in B_{\hat{z}}|\hat{z}_n(y) = z,\, \varphi(y, z) - \varphi(y, \hat{z}) < 0\}$. Notice that for $y \in \hat{B}^{n,+}_z \cup \hat{B}^{n,-}_z$ we have $z \in B(\hat{z}, 1/n)$. Let $\breve{\varepsilon} := \int_{B_{\hat{z}}} \hat{\varepsilon}(y)\, d\beta(y)$. Since $\hat{\varepsilon}(y) > 0$ for every $y \in B$ and $\beta(B_{\hat{z}}) > 0$, $\breve{\varepsilon} > 0$.

We have, for every $n \in \mathbb{N}$, there is a finite subset of Z, denoted Z_n, such that

$$\left\|(\mu - \mu^{\hat{z}})|_{B(\hat{z},1/n)}\right\|_{SV} \geq \sum_{z\in Z_n} \left(\int_{\hat{B}^{n,+}_z} \left(\varphi(y, z) - \varphi(y, \hat{z})\right)\, d\beta(y) + \int_{\hat{B}^{n,-}_z} (\varphi(y, \hat{z}) - \varphi(y, z))\, d\beta(y) \right)$$

$$\geq \int_{B_{\hat{z}}} \left|\varphi(y,\hat{z}_n(y)) - \varphi(y,\hat{z})\right| \, d\beta(y) - \check{\varepsilon}/2 \geq \int_{B_{\hat{z}}} \hat{\varepsilon}(y)\, d\beta(y) - \check{\varepsilon}/2 = \check{\varepsilon}/2.$$

However, since μ is continuous in z in the strong total variation norm, $\left\|(\mu - \mu^{\hat{z}})|_{B(\hat{z},1/n)}\right\|_{SV}$ ought to converge to zero as $n \to \infty$. This is a contradiction.

We have shown that there is a version of φ that is continous in z for each $y \in Y$. Let $\bar{\varphi}(y) := \max_{z\in Z} \varphi(z,y)$, which is $\mathcal{M}(Y)$-measurable and well-defined because $\varphi(\cdot, y)$ is continuous and Z is compact. If there is no function $\psi \in L^1(Y,\beta)$ such that $\varphi(y,z) \leq \psi(y)$ then $\int_Y \bar{\varphi}(y)\, d\beta(y) = \infty$. Let $\{z_j\}_{j\in\mathbb{N}}$ be an enumeration of Z. For each $j \in \mathbb{N}$, define $\bar{B}_j$ recursively for $j \in \mathbb{N}$ as $\bar{B}_j = \{y \in Y | z_j \in \text{argmax } \varphi(z,y)\} \setminus \cup_{\hat{j}<j} \bar{B}_{\hat{j}}$. Notice, $\{\bar{B}_j\}_{j\in\mathbb{N}}$ is a measurable partition of Y. Hence, for every $M > 0$ there is a finite set $\bar{J} \subseteq \mathbb{N}$ such that

$$M < \sum_{j\in\bar{J}} \int_{\bar{B}_j} \bar{\varphi}(y)\, d\beta(y) = \sum_{j\in\bar{J}} \int_{\bar{B}_j} \varphi(y,z_j)\, d\beta(y) \leq \|\mu\|_{SV},$$

implying $\|\mu\|_{SV} = \infty$.

Sufficiency. Suppose that φ is a Carathéodory integrand, bounded by some function $\psi \in L^1(Y,\beta)$, i.e., $\varphi(y,z) \leq \psi(y)$ for every $y \in Y$. For each $n \in \mathbb{N}$ and $z^* \in Z$, define $\hat{\hat{\varepsilon}}_n(y) := \sup_{z\in B(z^*,1/n)} |\varphi(y,z) - \varphi(y,z^*)|$. This function is measurable and converges to zero almost surely in y. Since $\hat{\hat{\varepsilon}}_n(y) \leq 2\psi(y)$ for each $y \in Y$ and $\left\|(\mu - \mu^{z^*})|_{B(z^*,1/n)}\right\|_{SV} \leq \int_Y \hat{\hat{\varepsilon}}_n(y)\, d\beta(y)$, the continuity of μ in strong total variation follows by the dominated convergence theorem.

Finally, we have $\varphi(z,y) \leq \bar{\varphi}(y) \leq \psi(y)$, which implies $\|\mu\|_{SV} < \infty$. $\square$

The following corollary is a consequence of Proposition 8. Let $\mathcal{M}(Y)_0$ be a sub σ-algebra of $\mathcal{M}(Y)$.

COROLLARY 2. *Let $\varphi \in CI(Y\times Z, \mathcal{M}(Y), \beta)$ and let $\hat{\varphi} = \mathbb{E}(\varphi|\mathcal{M}(Y)_0 \otimes \mathcal{M}(Z))$ denote the conditional expectation of φ with respect to $\mathcal{M}(Y)_0 \otimes \mathcal{M}(Z)$. Then there is a version of $\hat{\varphi} \in CI(Y \times Z, \mathcal{M}(Y)_0, \beta)$.*

The following example shows that continuity in the strong total variation norm is stronger than continuity in the total variation norm.

EXAMPLE 4. Define $Z := \{z_{(n,k)} | z_{(n,k)} = \left(1/n, k/n^2\right), k, n \in \mathbb{N}, k \in [1, n-1]\} \cup (0,0)$. Let $\varphi(y,(0,0)) = 1$ for every $y \in Y$, and

$$\varphi(y, z_{(n,k)}) = \begin{cases} 0 & \text{if } y \in [(k-1)/n, k/n] \\ 1 & \text{otherwise.} \end{cases}$$

Define $\xi(\hat{Y}|z) = \int_{\hat{Y}} \varphi(y,z)\, dy$ for each $\hat{Y} \in \mathcal{M}(Y)$.

Now take a sequence $z_m \to (0,0)$. We can write $z_m = z_{(n_m,k_m)}$ with $k_m/n_m^2 \to 0$. We have

$$\sup_{\pi \in \pi(Y)} \sum_{\tilde{Y} \in \pi} \int_{\tilde{Y}} |\varphi(y, z_m) - \varphi(y,(0,0))|\, dy \leq 1/n_m,$$

and, hence, $\xi(\cdot|z)$ is continuous in z in the total variation norm.

Consider a sequence $\{z_n(y)\}_{n\in\mathbb{N}}$ defined by $z_n(y) = z_{(n,k(y))}$ where $k(y) = \lceil y \cdot n \rceil$. For each n, the function $k(y)$ induces a finite partition of Y. While $|z_n(y) - (0,0)| < 2/n$ for all $y \in Y$, we have $|\varphi(y, z_n(y)) - \varphi(y,(0,0))| = 1$. Consequently:

$$\|(\xi - \xi^{(0,0)})|_{B((0,0),1/n)}\|_{SV} \geq 1.$$

This inequality shows that $\xi(\cdot|z)$ fails to be continuous in z in strong total variation. ◂

In what follows, let $(Z, \mathcal{M}(Z))$ and $(\hat{Z}, \mathcal{M}(\hat{Z}))$ be countable sets endowed with their power sets as σ-algebras and the counting measure, where $\hat{Z}$ contains an isolated point z^*. Consider surjective measurable functions $T_y : Y \to \hat{Y}$ and $T_z : Z \to \hat{Z} \setminus z^*$, where T_z^{-1} is lower hemicontinuous.[11] Let $(Y, \mathcal{M}(Y), \beta)$ be a measure space, and define the measure space $(\hat{Y}, \mathcal{M}(\hat{Y}), \hat{\beta})$ where $\hat{\beta} := \beta \circ T_y^{-1}$ is the pushforward measure of β under T_y.[12] We say a net of measurable functions $(f^\lambda)_{\lambda\in\Lambda} \subseteq \mathbb{R}^{Y\times Z}$ converges to f in the weak topology if for every $\phi \in CI(Y \times Z, \beta)$, the integral $I_\phi(f^\lambda) := \int_Y \sum_{z\in Z} \phi(y,z) f^\lambda(y,z), d\beta(y)$ converges to $I_\phi(f)$. The convergence for functions in $\mathbb{R}^{\hat{Y}\times\hat{Z}}$ is defined analogously.

Let $\tilde{B} \in \mathcal{M}(Y \times Z)$. Define $T : Y \times Z \to \hat{Y} \times \hat{Z}$ by

$$T(y,z) := \begin{cases} (T_y(y), T_z(z)) & \text{if } (y,z) \in \tilde{B} \\ (T_y(y), z^*) & \text{otherwise} \end{cases}$$

LEMMA 14. *Let $\mathcal{M}(Y \times Z|T) = \sigma\{T^{-1}(\hat{B}) | \hat{B} \in \mathcal{M}(\hat{Y} \times \hat{Z})\}$. Suppose that $(f^\lambda)_{\lambda\in\Lambda}$ is a net of functions each of which is measurable with respect to $\mathcal{M}(Y \times Z|T)$ and has support in $\tilde{B}$. If $(f^\lambda)_{\lambda\in\Lambda}$ converges to f, then f is measurable with respect to $\mathcal{M}(Y \times Z|T)$ and has support in $\tilde{B}$.*

Proof. By Theorem 4.41 in Aliprantis and Border (2006) (henceforth A&B), for each $\lambda \in \Lambda$, there exists a $\mathcal{M}(\hat{Y} \times \hat{Z})$-measurable function $\hat{f}^\lambda$ such that $f^\lambda = \hat{f}^\lambda \circ T$. Let $\hat{f}$ be the limit of a subnet of $(\hat{f}^\lambda)_{\lambda\in\Lambda}$; note that $\hat{f}$ also has support in $T(\tilde{B})$. To establish our

[11] A correspondence $\phi : X \rightrightarrows Y$ is lower hemicontinuous if for every $x \in X$, $y \in \phi(x)$, and sequence x_n converging to x, there exists a subsequence x_{n_k} and sequence $y_k \in \phi(x_{n_k})$ such that $y_k \to y$.

[12] For each $\hat{B} \in \mathcal{M}(\hat{Y})$, the pushforward measure $\beta \circ T_y^{-1}$ is defined by $\beta \circ T_y^{-1}(\hat{B}) = \beta(T_y^{-1}(\hat{B}))$.

conclusion, we will show that $f = \hat{f} \circ T$. By Theorem 4.41 in A&B, this will complete the proof.

Suppose not, then there is a test function $\psi \in CI(Y \times Z, \beta)$ such that $I_\psi(f) \neq I_\psi(\hat{f} \circ T)$.

For any measurable set $B \in \mathcal{M}(Y \times Z)$, let $\mathbb{1}_B$ denote its indicator function. Define measures μ_ψ and $\hat{\mu}$ on $\mathcal{M}(Y \times Z|T)$ by $\mu_\psi(B) := I_\psi(\mathbb{1}_B) \quad \text{and} \quad \hat{\mu}(B) := I_{\mathbb{1}_{Y \times Z}}(\mathbb{1}_B)$ for each $B \in \mathcal{M}(Y \times Z|T)$, respectively. Note that $\hat{\mu}$ is σ-finite and μ_ψ is absolutely continuous with respect to $\hat{\mu}$. Therefore, by the Radon-Nikodym theorem, there exists a $\mathcal{M}(Y \times Z|T)$-measurable function $\bar{\psi}$ such that $I_{\bar{\psi}}(\mathbb{1}_B) = I_\psi(\mathbb{1}_B)$ for every $B \in \mathcal{M}(Y \times Z|T)$.

By Theorem 4.41 in A&B, there exists a $\mathcal{M}(\hat{Y} \times \hat{Z})$-measurable function $\hat{\psi} : \hat{Y} \times \hat{Z} \to \mathbb{R}$ such that $\bar{\psi} = \hat{\psi} \circ T$. By Corollary 2 there is a version of $\bar{\psi}$ that is continuous in Z for each $y \in Y$. Therefore, since T_z^{-1} is lower hemicontinuous, there is also a version of $\hat{\psi}(\hat{y}, \cdot)$ that is continuous in $\hat{Z}$. Additionally, since ψ is bounded by a $L^1(Y)$ function, $\hat{\psi}$ must be bounded by a $L^1(\hat{Y})$ function. This shows that $\hat{\psi} \in CI(\hat{Y} \times \hat{Z}, \hat{\beta})$.

Now, by Theorem 13.46 in A&B, for every measurable $g : \hat{Y} \times \hat{Z} \to \mathbb{R}$

$$\hat{I}_{\hat{\psi}}(g) := \int_{\hat{Y}} \sum_{\hat{z} \in \hat{Z}} \hat{\psi}(\hat{y}, \hat{z}) \cdot g(\hat{y}, \hat{z}) \, d\hat{\beta}(\hat{y}) = \int_Y \sum_{z \in Z} \psi(y, z) \cdot g \circ T(y, z) \, d\beta(y). \tag{19}$$

This leads to a contradiction with $\lim_\lambda I_\psi(f^\lambda) \neq I_\psi(\hat{f} \circ T)$ as we have $\lim_\lambda I_\psi(f^\lambda) = \lim_\lambda I_\psi(\hat{f}^\lambda \circ T) = \lim_\lambda \hat{I}_{\hat{\psi}}(\hat{f}^\lambda) = \hat{I}_{\hat{\psi}}(\hat{f}) = I_\psi(\hat{f} \circ T)$, where the third equality follows by the definition of $\hat{f}$, since $\hat{\psi} \in CI(\hat{Y} \times \hat{Z}, \hat{\beta})$, and the second and last equalities follow by (19). $\square$

LEMMA 15. *For each $j \in 1, 2$, let $(\phi_j^\lambda)_{\lambda \in \Lambda}$ be a net of transition probabilities from Y to Z converging to ϕ_j^* in the weak topology. If there exist functions $\varphi_1, \varphi_2 \in CI(X \times Z, \beta)$ such that $\phi_1^\lambda \cdot \varphi_1 + \phi_2^\lambda \cdot \varphi_2 \geq 0$ for every $\lambda \in \Lambda$, then for any closed set $C \in \mathcal{M}(Z)$:*

$$\sum_{z \in C} (\phi_1^*(y, z) \cdot \varphi_1(y, z) + \phi_2^*(y, z) \cdot \varphi_2(y, z)) \geq 0,$$

β-almost surely.

Proof. Let $\hat{\phi} : Y \times Z \to \mathbb{R}$ be of the form $\hat{\phi}(y, z) = \mathbb{1}\{y \in \tilde{Y}\} \cdot \tilde{\phi}(z)$ for some $\tilde{Y} \in \mathcal{M}(Y)$ and some bounded and continuous function $\tilde{\phi} : Z \to \mathbb{R}$. Since $\hat{\phi} \in CI(Y \times Z, \beta)$, we have

$$\int_{\tilde{Y}} \sum_{z \in Z} \tilde{\phi}(z) \cdot \left(\phi_1^\lambda(y, z) \cdot \varphi_1(y, z) + \phi_2^\lambda(y, z) \cdot \varphi_2(y, z)\right) d\beta(y)$$

$$\to \int_Y \sum_{z \in Z} \tilde{\phi}(z) \cdot \left(\phi_1^*(y, z) \cdot \varphi_1(y, z) + \phi_2^*(y, z) \cdot \varphi_2(y, z)\right) d\beta(y) \geq 0.$$

Let $U_n = \cup_{c \in C} B(c, 1/n)$ where $B(c, 1/n)$ is the ball centered at $c \in C$ with radius $1/n$. Since Z is a normal space as it is a maetric space, by the Urysohn's lemma (Theorem 2.46 in A&B), for each $n \in \mathbb{N}$, there is a continuous function $f_n : Z \to [0, 1]$ that is equal to 1 in C and it is equal to zero in the complement of U_n. Since f_n converges to $\mathbb{1}\{z \in C\}$ pointwise, by setting $\tilde{\phi} = f_n$ in the right hand side of the previous expression, taking the limit in n, and applying the dominated convergence theorem, we obtain $\int_{\tilde{Y}} \sum_{z \in C} \left(\phi_1^*(y, z) \cdot \varphi_1(y, z) + \phi_2^*(y, z) \cdot \varphi_2(y, z)\right) d\beta(y) \geq 0$, for every $\tilde{Y} \in \mathcal{M}(Y)$. Finally, since $\tilde{Y}$ is arbitrary, this implies the β-almost surely inequality. $\square$

C.2 A special case of dominated convergence for nets

We extend two fundamental results from measure theory—Fatou's lemma and Lebesgue's dominated convergence theorem—from sequences to nets under the counting measure. The proofs of Lemma 16 and Proposition 10 follow the standard arguments with sequences replaced by nets and use Proposition 9. Note that Proposition 9 is specific to countable sums and does not hold for integrals over uncountable sets. We include the proofs for completeness.

PROPOSITION 9 (Monotone convergence). *Let $(x_{\alpha,k})_{\alpha \in \mathcal{A}, k \in \mathbb{N}} \subseteq \mathbb{R}_+$ be a net, indexed by directed set $(\mathcal{A}, \geq)$ and $\mathbb{N}$, such that (i) $\alpha \geq \alpha'$ implies $x_{\alpha,k} \geq x_{\alpha',k}$ for all $k \in \mathbb{N}$, and (ii) there is M such that $\sum_{k=1}^{\infty} x_{\alpha,k} < M$ for all $\alpha \in \mathcal{A}$. Then $x_k = \lim_\alpha x_{\alpha,k}$ exists for each $k \in \mathbb{N}$ and $\lim_\alpha \sum_{k=1}^{\infty} x_{\alpha,k} = \sum_{k=1}^{\infty} x_k$.*

Proof. The net $x_{\alpha,k}$ has a limit, x_k, for each k, as it is non-decreasing and bounded by (ii).

Let $B = \{(\alpha, n) | \alpha \in \mathcal{A}, n \in \mathbb{N}\}$ be a directed set with $(\alpha, n) \geq (\alpha', n')$ if and only if $\alpha \geq \alpha'$ and $n \geq n'$. And define $y_{\alpha,n} := \sum_{k=1}^{n} x_{\alpha,k}$. The net $(y_b)_{b \in B}$ is non-decreasing and bounded and, therefore, has a limit—its supremum in $\mathbb{R}$—which we denote s. That is, for each $\varepsilon > 0$ there is $\bar{\alpha}$ and $\bar{n}$ such that $\alpha \geq \bar{\alpha}$ and $n \geq \bar{n}$ implies $\left|\sum_{k=1}^{n} x_{\alpha,k} - s\right| < \varepsilon$. For each $\alpha \geq \bar{\alpha}$, the sequence $(y_{\alpha,n})_{n \in \mathbb{N}}$ is monotone non-decreasing and bounded. Therefore, it has a limit $\sum_{k=1}^{\infty} x_{\alpha,k}$. By the continuity of the absolute value we obtain, $\left|\sum_{k=1}^{\infty} x_{\alpha,k} - s\right| < \varepsilon$. This shows that $\lim_\alpha \sum_{k=1}^{\infty} x_{\alpha,k} = s$.

For each fixed $n \geq \bar{n}$, there is $\hat{\alpha}(n) \geq \bar{\alpha}$ such that $\left|\sum_{k=1}^{n} x_{\hat{\alpha}(n),k} - \sum_{k=1}^{n} x_k\right| < \varepsilon$. Therefore, $\left|\sum_{k=1}^{n} x_k - s\right| \leq \left|\sum_{k=1}^{n} x_{\hat{\alpha}(n),k} - \sum_{k=1}^{n} x_k\right| + \left|\sum_{k=1}^{n} x_{\hat{\alpha}(n),k} - s\right| < 2\varepsilon$. This shows that $\sum_{k=1}^{\infty} x_k = s$. $\square$

LEMMA 16 (Fatou's Lemma). *Let $(x_{\alpha,k})_{\alpha \in \mathcal{A}, k \in \mathbb{N}} \subseteq \mathbb{R}_+$ be a net, indexed by directed set $(\mathcal{A}, \geq)$ and $\mathbb{N}$, such that $\sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} \sum_{k=1}^{\infty} x_{\alpha,k} < \infty$. Then $x_k := \sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} x_{\alpha,k}$ exists* *for each* $k \in \mathbb{N}$*, and*

$$\sum_{k=1}^{\infty} \sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} x_{\alpha,k} \leq \sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} \sum_{k=1}^{\infty} x_{\alpha,k}.$$

Proof. Define $y_{k,\alpha} := \inf\{x_{k,\hat{\alpha}} | \hat{\alpha} \geq \alpha\}$. Since $\geq$ is transitive $y_{k,\alpha}$ is non-decreasing in α. Then, $(\sum_{k=1}^{\infty} y_{k,\alpha})_{\alpha \in \mathcal{A}}$ is a non-decreasing and bounded net, and, therefore, has a limit. Furthermore, for $k \in \mathbb{N}, \alpha \in \mathcal{A}$ we have $y_{k,\alpha} \leq x_{k,\alpha}$. Therefore, $\lim_{\alpha} \sum_{k=1}^{\infty} y_{k,\alpha} \leq \sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} \sum_{k=1}^{\infty} x_{k,\alpha} < \infty$. To conclude note that by Proposition 9 the left hand side of the previous expression is equal to $\sum_{k=1}^{\infty} \lim_{\alpha} y_{k,\alpha} = \sum_{k=1}^{\infty} \sup_{\bar{\alpha}} \inf_{\alpha \geq \bar{\alpha}} x_{\alpha,k}$. □

PROPOSITION 10 (Dominated convergence)**.** *Let* $(x_{\alpha,k})_{\alpha \in \mathcal{A}, k \in \mathbb{N}} \subseteq \mathbb{R}$ *be a net indexed by directed set* $(\mathcal{A}, \geq)$ *and* $\mathbb{N}$*. If* $x_k = \lim_{\alpha} x_{\alpha,k}$ *exists for each* $k \in \mathbb{N}$*, and there exists a sequence* $(y_k)_{k \in \mathbb{N}}$ *with* $|\sum_{k=1}^{\infty} y_k| < \infty$ *such that* $|x_{\alpha,k}| \leq y_k$ *for all* α *and* k*, then* $\lim_{\alpha} \sum_{k=1}^{\infty} |x_{\alpha,k} - x_k| = 0$.

Proof. First observe that $|x_{\alpha,k} - x_k| \leq |x_{\alpha,k}| + |x_k| \leq 2y_k$ for all α, k. Applying Lemma 16 to the net $(2y_k - |x_{\alpha,k} - x_k|)_{\alpha \in \mathcal{A}, k \in \mathbb{N}} \subseteq \mathbb{R}_+$, we obtain:

$$\inf_{\bar{\alpha}} \sup_{\alpha \geq \bar{\alpha}} \sum_{k=1}^{\infty} |x_{\alpha,k} - x_k| \leq \sum_{k=1}^{\infty} \inf_{\bar{\alpha}} \sup_{\alpha \geq \bar{\alpha}} |x_{\alpha,k} - x_k| = 0.$$

Therefore, $\lim_{\alpha} \sum_{k=1}^{\infty} |x_{k,\alpha} - x_k| = 0$.

□