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Note on the Bondi–Metzner–Sachs Group*

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It is shown that, in space–times which are asymptotically flat, there are reasonable physical restrictions that allow one to impose coordinate conditions (in addition to the usual Bondi-type conditions) which restrict the allowed coordinate group to a subgroup of the Bondi–Metzner–Sachs group. This subgroup is isomorphic to the improper orthochronous inhomogeneous Lorentz group.

I. INTRODUCTION

In recent years the subject of gravitational radiation has received a great deal of attention.[1–5] One of the unusual results of this study was that, even in space-times which are asymptotically Minkowskian, one apparently cannot extract the inhomogeneous Lorentz group as an asymptotic symmetry group if finite (retarded) time intervals, only, are considered. Instead, one obtains what is known as the Bondi–Metzner–Sachs group (BMS group)—an infinite parameter group. The purpose of this note is to show that if one imposes certain apparently reasonable physical restrictions at retarded time $u = -\infty$ (or, alternatively, at $u = +\infty$), it is possible to introduce further coordinate conditions such that a well-defined (non-normal) subgroup of the BMS group—isomorphic to the improper orthochronous inhomogeneous Lorentz group—is geometrically singled out by preserving these coordinate conditions.

The physical situations we allow would appear to include a general type of scattering problem in which the sources, and perhaps some waves, can come in from infinity and can again escape to infinity after interacting. This generalizes certain situations considered by Sachs. He observed[6] that the inhomogeneous Lorentz group could be singled out at, say, $u = +\infty$ if, for example, all the matter were radiated away as zero rest-mass energy, leaving Minkowski space for sufficiently large u; alternatively, the system might ultimately simply become static. The situations we consider here appear to be much more general than this, although whether or not they are quite as general as they seem to be, depends, to some extent, on the validity of some heuristic arguments we give, which have, as their basis, some exact results in the linear theory of gravitation.

As an additional purpose of this note, we introduce an invariant differential operation on the sphere (which we denote by ð) and use it to define a type of "spin s spherical harmonic." This is done in slightly greater generality than is absolutely necessary for the present work, since it is also felt that these ideas should find applications elsewhere[7] and that they could be used to simplify earlier work.[2,4,6,8]

II. THE BMS GROUP

Let u, θ, ϕ, r be standard (Bondi-type) coordinates[1,2,4,6] for asymptotically flat space–time. Thus u is a *retarded time* parameter (so that u = const are null hypersurfaces opening into the future); θ, ϕ are spherical polar coordinates for the sphere at infinity on each hypersurface u = const (θ, ϕ, u = const giving the null geodesic generators of these hypersurfaces); r is suitably defined radial coordinate such as an affine or luminosity parameter on each of the generators of the hypersurfaces. The BMS group is defined by the following transformations[2,4,6] on the θ, ϕ, u coordinates:

$$\begin{aligned} \theta' &= \theta'(\theta, \phi), \\ \phi' &= \phi'(\theta, \phi), \\ u' &= K(\theta, \phi)\{u - \alpha(\theta, \phi)\}, \end{aligned} \tag{2.1}$$

where $(\theta, \phi) \rightarrow (\theta', \phi')$ is a conformal transformation

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of the (θ, ϕ)-sphere into itself, K is the corresponding conformal factor, given by

$$d\theta'^2 + \sin^2 \theta' \, d\phi'^2 = K^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \qquad (2.2)$$

and where α is an arbitrary (suitably smooth) real function on the sphere. (The r coordinate may also be transformed, if desired, but the transformation is somewhat arbitrary since it depends on the precise type of radial coordinate used and it is not relevant to the structure of the group.)

The particular BMS transformations for which $\theta' = \theta, \phi' = \phi$ are called, in general, *supertranslations*. Under a supertranslation, the system of null hypersurfaces $u =$ const. is transformed into a different system of null hypersurfaces ($u' =$ const) but no (Lorentz) rotation is involved. We may write α in terms of spherical harmonics:

$$\alpha = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{m,l} Y_{l,m}(\theta, \phi),$$

where the $a_{m,l}$ are constants. The *infinite set* of parameters $a_{m,l}$ [subject to $a_{-m,l} = (-1)^m \bar{a}_{m,l}$, so that α is real] then define the supertranslation. If $a_{m,l} = 0$ for $l > 2$, so that α takes the form:

$$\alpha = \epsilon_0 + \epsilon_1 \sin \theta \cos \phi + \epsilon_2 \sin \theta \sin \phi + \epsilon_3 \cos \theta, \qquad (2.3)$$

then the supertranslations reduce to a special case, called the *translations*, with just *four* parameters $\epsilon_0, \cdots, \epsilon_3$.

This terminology is, in fact, consistent with that for the (Lorentz) translations in Minkowski space. We may call a hypersurface $u =$ const a "good" cone in Minkowski space if it is the null cone of some point, and a "bad" cone if, on the other hand, the generators of $u =$ const do not all meet in a point. Thus, an actual translation in Minkowski space must send "good" cones into "good" cones; under a general supertranslation, the "good" cones will be warped into "bad" cones. It is precisely the condition that α be given by (2.3), which is required to preserve the "goodness" of the Minkowski null cones.

In curved asymptotically flat space–times the difficulty is to find an appropriate analog of the Minkowskian concept of "good" and "bad" cones. It is not, in fact, necessary to do this in order to single out the translations from the remaining supertranslations, since the translations are already determined by (2.3).[9] But if we wish, in addition, to isolate the "pure" Lorentz *rotations* from Lorentz rotations which have a "supertranslation component," then some concept of a distinction between "good" and "bad" cones is necessary. For we might try to define Lorentz rotations (homogeneous Lorentz transformation) as given by (2.1) with $\alpha = 0$. The hypersurface $u = 0$ is then transformed into itself. In Minkowski space, if $u = 0$ is a "good" cone, the resulting transformation indeed represents a Lorentz rotation and it sends other "good" cones into "good" cones. If, on the other hand $u = 0$ is a "bad" cone, then we do *not* get a Lorentz rotation in general. Thus, for asymptotically flat spaces, in order to know which of the BMS transformations are to be regarded as "supertranslation-free Lorentz rotations", we must have some definition of "goodness" of $u =$ const hypersurfaces.

In Minkowski space, the "good" cones can be characterized locally by the fact that the null rays generating them possess no shear. In asymptotically Minkowskian spaces, it will only be the *asymptotic* behavior of the (complex) shear σ of these null rays that will concern us. With null vector l^μ tangent to the null rays and complex null vector m^μ orthogonal to l^μ satisfying $m^\mu \bar{m}_\mu = -1$, l^μ and m^μ being parallelly propagated along each ray, we have[3]

$$l_{\mu;\nu} m^\mu m^\nu = \sigma(u, \theta, \phi, r) = \sigma^0(u, \theta, \phi)/r^2 + O(r^{-4}), \qquad (2.4)$$

where r is scaled so that $r,_\mu l^\mu \to 1$ at infinity. Thus σ^0 defines the asymptotic shear of the hypersurface given, say, by $u =$ const. The complex quantity σ^0 is of special interest in gravitational radiation theory. It forms part of the initial data on $u = 0$ used to determine the space-time asymptotically.[2,4] Furthermore, $\partial\sigma^0/\partial u$ and $\partial^2\sigma^0/\partial u^2$ both have physical significance. We may call $\partial^2\sigma^0/\partial u^2$ the *gravitational radiation field* since it represents the r^{-1} part of the Riemann curvature field.[1,2,4] Bondi *et al*[1] and Sachs[2] call $\partial\sigma^0/\partial u$ the "news function" since it can be used as asymptotic initial data for the gravitational radiation field and $|\partial\sigma^0/\partial u|^2$ represents the flux of energy of the gravitational radiation in their analysis.

We cannot, however, attempt to define "good" cones, in general, simply by requiring $\sigma^0 = 0$. In many cases it is simply not possible to arrange $\sigma^0 = 0$ for all values of θ, ϕ, on one hypersurface, but even in cases where it is possible (e.g. in the axially symmetric, reflection symmetric cases), it is clear from the above remarks that, in the presence of gravitational radiation, if $\sigma^0 = 0$ for one value of u, we will generally have $\sigma^0 \neq 0$ for a

[9] In fact, Sachs has pointed out (Ref. 6) that the translation subgroup of the BMS group is uniquely singled out by group theoretic considerations, namely as the only four-parameter normal subgroup of the BMS group.

later value of u (i.e., "goodness" would not be invariant under time translation). The idea of the present paper is that, if we make apparently reasonable physical assumptions as to how the gravitational radiation falls off at $u = -\infty$ (or alternatively at $u = +\infty$), then we can effectively minimize σ^0 at $u = -\infty$ (or alternatively at $u = +\infty$). This will restrict our coordinates to such an extent that only a subgroup of the BMS group remains—which is isomorphic to the improper orthochronous inhomogeneous Lorentz group.

III. SPIN-WEIGHTED FUNCTIONS ON A SPHERE

In order to analyze the structure of σ^0 as a function of the angular coordinates θ, ϕ, it is important first to realize that, although it is given as a scalar, it is really a tensorlike quantity. The directions of minimum and maximum shear are determined by arg σ^0. Under rotation of the spacelike vectors Re (m^μ), Im (m^μ) in their plane given by

$$(m^\mu)' = e^{i\psi} m^\mu \tag{3.1}$$

[with $(l^\mu)' = l^\mu$, ψ real], we then have

$$(\sigma^0)' = e^{2i\psi}\sigma^0. \tag{3.2}$$

We shall say that σ^0 has spin weight 2. Generally, a quantity η will be said to have *spin weight s* if it transforms as

$$\eta' = e^{si\psi}\eta \tag{3.3}$$

under (3.1). (Here, s is in general integral, but half integral values can also occur.) The quantity σ^0 has, in addition, a conformal weight of -1. That is, under (2.1) and (2.2), if we choose $\alpha = 0$ and examine the σ^0 for $u = 0$ (keeping the m^μ vectors fixed), we find[2]

$$(\sigma^0)' = K^{-1}\sigma^0. \tag{3.4}$$

Generally, a quantity η defined on the (θ, ϕ)-sphere has *conformal weight w* if under conformal transformation of the sphere with conformal factor K as in (2.2) (and with fixed m^μ vectors) we have

$$\eta' = K^w\eta. \tag{3.5}$$

[K is, in effect, the relativistic Doppler factor[2] $(c + v)^{\frac{1}{2}}(c - v)^{-\frac{1}{2}}$.] For consistency with the coordinate conditions,[1,2,4] this conformal transformation should be accompanied by

$$(l^\mu)' = Kl^\mu, \qquad r' = K^{-1}r, \tag{3.6}$$

with $(m^\mu)' = m^\mu$, whence (3.4) [cf. (2.4)].

Effectively, the concepts of spin-weight and conformal weight refer to the behavior of functions on the (θ, ϕ)-sphere at infinity only, and do not refer to the remainder of the space–time. Indeed, the concepts will apply to *any* two-dimensional abstract surface, with a Riemannian or conformal structure. Quantities with spin weights correspond to irreducible tensor quantities on the surface. The vectors Re (m^μ), Im (m^μ) may be regarded as orthogonal tangent vectors (of length $2^{-\frac{1}{2}}$) at each point of the surface. But we shall be concerned, here, only with quantities defined on a *sphere*. If spherical polar coordinates are used, a natural choice[2,4,7] for m^μ is to make Re (m^μ) and Im (m^μ) tangential, respectively, to the curves ϕ = const and θ = const. Another convenient coordinate system for the sphere is $(\zeta, \bar{\zeta})$ where the complex parameter ζ is related to (θ, ϕ) by

$$\zeta = e^{i\phi}\cot\tfrac{1}{2}\theta. \tag{3.7}$$

In this case, the natural choice for m^μ is to make Re (m^μ) and Im (m^μ) tangential, respectively, to the curves Im (ζ) = const and Re (ζ) = const.

Let η be a quantity defined on the sphere of spin weight s. Define the operator $\eth$, in a particular (θ, ϕ) coordinate system, by

$$\eth\eta = -(\sin\theta)^s\left\{\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right\}\{(\sin\theta)^{-s}\,\eta\}. \tag{3.8}$$

(The operator $\eth$ is effectively a covariant differentiation operator in the surface.) Under a rotation of m^μ (but, for the moment, keeping the coordinates fixed) we demand that $\eth\eta$ behave as a quantity of spin weight $s + 1$. From (3.7) and the fact that the m^μ vectors are rotated through an angle $-\phi$ in the passage from (θ, ϕ) to $(\zeta, \bar{\zeta})$, we obtain

$$\eth\eta = 2P^{1-s}\,\partial(P^s\eta)/\partial\zeta \tag{3.9}$$

as the definition of $\eth$ in the $(\zeta, \bar{\zeta})$ system, where

$$P = \tfrac{1}{2}(1 + \zeta\bar{\zeta}). \tag{3.10}$$

[The coordinate ζ and the P used here are related to the ζ and P (now called ζ' and P') of Ref. 8 by $\zeta = -\frac{1}{2}\bar{\zeta}'$, $P = \frac{1}{2}\sqrt{2}P'$.]

Now, the first important property of the operator $\eth$ is that it is invariant (with spin weight unity) under change of coordinate system which preserves the sphere metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = P^{-2}\,d\zeta\,d\bar{\zeta}. \tag{3.11}$$

This is most easily seen in the $(\zeta, \bar{\zeta})$ system and the result for the (θ, ϕ) system will then follow. With ψ as in (3.1), so that $\eta' = e^{si\psi}\eta$, we have

$$e^{-2i\psi} = \frac{\partial\zeta'/\partial\zeta}{\partial\bar{\zeta}'/\partial\bar{\zeta}}, \tag{3.12}$$

where $(\zeta', \bar{\zeta}')$ is the new coordinate system, with new complex tangent vector $m^{\mu'}$. Since $ds' = ds$, we have

$$\partial\zeta'/\partial\zeta = e^{-i\psi} P'/P \tag{3.13}$$

with $\partial\zeta'/\partial\bar{\zeta} = 0$. From this it follows that $\eth'\eta' \equiv 2P'^{1-s}\partial(P'^{s}\eta')/\partial\zeta' = e^{(s+1)i\psi}\,\eth\eta$, as required. [In fact, this argument applies equally to any surface, with metric written $ds^2 = P^{-2}\,d\zeta\,d\bar{\zeta}$, where P need not have the form (3.10).]

The second important property of $\eth$ concerns its behavior under a conformal transformation. Let η have conformal weight w and spin weight s (both integers or both half odd integers) where $w \geq s$. Then,

$$\eth^{w-s+1}\eta \ \textit{is a quantity of conformal weight}\ s-1, \tag{3.14}$$

and, of course, spin weight $w + 1$. To prove this, we need only consider conformal transformations of the form

$$\zeta' = a\zeta, \tag{3.15}$$

a being a constant, with $(P')^{-2}\,d\zeta'\,d\bar{\zeta}' = K^2P^{-2}\,d\zeta\,d\bar{\zeta}$ so that

$$K = aP/P' = (1 + \zeta\bar{\zeta})/(a^{-1} + a\zeta\bar{\zeta}). \tag{3.16}$$

[The general conformal self-transformation of the sphere, excluding reflections can be built up from transformations of the type (3.15)—corresponding to a Lorentz velocity transformation in the direction of the z axis—and rotations which preserve the metric. These rotations have been already dealt with, since they correspond to metric preserving coordinate transformations.] It is required to prove that, under (3.15), $(\eth^{w-s+1}\eta)' = K^{s-1}\,\eth^{w-s+1}\eta$. [The m^μ vectors are unaltered under (3.15).] This can be done by a kind of inductive argument which is somewhat tedious for general values of $w - s$. For small value of $w - s$ it is not hard to verify directly that

$$(P')^{1-w}\frac{\partial}{\partial\zeta'}\left((P')^2\frac{\partial}{\partial\zeta'}\left((P')^2\cdots\frac{\partial}{\partial\zeta'}((P')^sK^w\eta)\cdots\right)\right) = K^{s-1}P^{1-w}\frac{\partial}{\partial\zeta}(\cdots(P^s\eta)\cdots),$$

where from (3.16) we use the fact that $P^2\,\partial K^{-1}/\partial\zeta$ is a constant. It is a curious fact that, given η with $w \geq s$, there is just the *one* power of $\eth$ given in (3.14) which yields a quantity correctly transforming (i.e., with a "weight") under a conformal transformation. If $w \geq -s$, then $\bar{\eth}^{w+s+1}\eta$ also correctly transforms with conformal weight $-s - 1$. Here $\bar{\eth}$ is defined [in the $(\zeta, \bar{\zeta})$ system] by

$$\bar{\eth}\eta = 2P^{1+s}\,\partial(P^{-s}\eta)/\partial\bar{\zeta}. \tag{3.17}$$

If $w > |s|$, then we can apply both operators to get a quantity with conformal weight $-w - 2$:

$$\bar{\eth}^{w+s+1}\,\eth^{w-s+1}\eta = \eth^{w-s+1}\,\bar{\eth}^{w+s+1}\eta. \tag{3.18}$$

That these two quantities are equal is not immediately obvious since the operators $\eth$ and $\bar{\eth}$ do not in general commute. In fact, we have

$$(\bar{\eth}\,\eth - \eth\,\bar{\eth})\eta = 2s\eta. \tag{3.19}$$

However, it is a consequence of the following considerations that $\eth^p\bar{\eth}^q\eta = \bar{\eth}^q\eth^p\eta$ whenever $q - p = 2s$.

The third important property of $\eth$ concerns its effect on spherical harmonics. Let $Y_{l,m}(l = 0, 1, \cdots;$ $m = -l, \cdots, +l)$ be the usual scalar spherical harmonics. Then we can define the *spin s spherical harmonics*[7,8] (for integral spin) as follows:

$${}_sY_{l,m} = \begin{cases} \left[\dfrac{(l-s)!}{(l+s)!}\right]^{\frac{1}{2}}\eth^sY_{l,m} & (0 \leq s \leq l), \\ (-1)^s\left[\dfrac{(l+s)!}{(l-s)!}\right]^{\frac{1}{2}}\bar{\eth}^{-s}Y_{l,m} & (-l \leq s \leq 0). \end{cases} \tag{3.20}$$

The ${}_sY_{l,m}$ are not defined for $|s| > l$. In the $(\zeta, \bar{\zeta})$ system, the spin s harmonics take the form

$${}_sY_{l,m} = \frac{a_{l,m}}{[(l-s)!\,(l+s)!]^{\frac{1}{2}}}(1 + \zeta\bar{\zeta})^{-l} \times \sum_p \zeta^p(-\bar{\zeta})^{p+s-m}\binom{l-s}{p}\binom{l+s}{p+s-m}, \tag{3.21}$$

summed over integral values of p, the $a_{l,m}$ being numerical constants whose exact values are inessential for our purposes. [In fact, (3.21) applies also to "spinor harmonics" for which l, m, and s are all half odd integers.] We have ${}_s\bar{Y}_{l,m} = (-1)^{m+s}{}_{-s}Y_{l,-m}$ and, for *all* s with $|s| \leq l$,

$$\begin{aligned} \eth({}_sY_{l,m}) &= [(l-s)(l+s+1)]^{\frac{1}{2}}{}_{s+1}Y_{l,m}, \\ \bar{\eth}({}_sY_{l,m}) &= -[(l+s)(l-s+1)]^{\frac{1}{2}}{}_{s-1}Y_{l,m}. \end{aligned} \tag{3.22}$$

In particular, $\eth$ annihilates ${}_sY_{l,m}$ whenever $l = s$ and $\bar{\eth}$ annihilates ${}_sY_{l,m}$ if $l = -s$. We see, furthermore, that $\eth^p$ annihilates any quantity of spin weight s which is composed only of harmonics with $l < s + p$.

The ${}_sY_{l,m}$ are eigenfunctions of the operator $\bar{\eth}\eth$ for each spin weight s:

$$\bar{\eth}\eth({}_sY_{l,m}) = -(l-s)(l+s+1)\,{}_sY_{l,m}. \tag{3.23}$$

(If $s = 0$, $\bar{\eth}\eth$ is essentially the total angular momentum operator.) More generally,

$$\bar{\eth}^p \eth^p({}_sY_{l,m}) = (-1)^p \frac{(l-s)!}{(l-s-p)!}\frac{(l+s+p)!}{(l+s)!}\,{}_sY_{l,m}. \tag{3.24}$$

For each given s, the ${}_sY_{l,m}$ form a set of orthonormal functions of spin weight s on the sphere:

$$\int {}_sY_{l,m}\,{}_s\bar{Y}_{l',m'}\,dS = \delta_{ll'}\,\delta_{mm'}, \tag{3.25}$$

dS being the surface area element on the sphere. This can be proved (s being an integer) by induction on s using (3.22), the orthonormality of $Y_{l,m}$, and the fact that

$$\int (\eth A)B\,dS = -\int A(\eth B)\,dS \tag{3.26}$$

with (spin weight of A) + (spin weight of B) = -1. [It is immediate from (3.8) that $\int \eth C\,dS = 0$ (where C has spin weight -1) and (3.26) follows.] The orthonormal functions ${}_sY_{l,m}$ are also *complete*, for spin weight s quantities on the sphere. That is, if η has spin weight s and is suitably regular[10] on the sphere, there exist constants $\eta_{l,m}$ such that

$$\eta = \sum_{l,m} \eta_{l,m}\,{}_sY_{l,m}. \tag{3.27}$$

To see this, without loss of generality choose $s > 0$ and consider $\bar{\eth}^s\eta$. This has spin weight zero, so from the completeness of the $Y_{l,m}$, we have, for some constants $\phi_{l,m}$

$$\bar{\eth}^s\eta = \sum_{l,m} \phi_{l,m}Y_{l,m}. \tag{3.28}$$

By (3.24), we can write this

$$\bar{\eth}^s\{\eta - \sum_{\substack{l,m\\ l\geq s}} \chi_{l,m}\,\eth^s Y_{l,m}\} = \sum_{\substack{l,m\\ l<s}} \phi_{l,m}Y_{l,m}, \tag{3.29}$$

where

$$\chi_{l,m} = \frac{(l-s)!}{(l+s)!}(-1)^s\phi_{l,m}(l \geq s).$$

Moreover, the right-hand side of (3.29) must vanish since, if we multiply (3.29) by $\bar{Y}_{l',m}(l' < s)$ and integrate over the sphere, we get $\phi_{l',m}$ on the right and $\int \bar{Y}_{l'm}\,\bar{\eth}^s\{\cdots\}\,dS$ on the left. Applying (3.26) and $\eth^s\,Y_{l'm} = 0(l' < s)$, we see that this vanishes. Thus, all the ϕ's (with $l' < s$) vanish, so the left-hand side of (3.29) also vanishes. Now, if μ is any suitably regular quantity on the sphere, of spin weight s' where $s' > 0$, then

$$\bar{\eth}\mu = 0 \quad implies \quad \mu = 0. \tag{3.30}$$

For, by (3.17), $\bar{\eth}\mu = 0$ implies that $P^{-s'}\mu$ is an analytic function of ζ. Hence $\lim_{|\zeta|\to\infty} |P^{-s'}\mu| > 0$ unless $\mu = 0$. But $s' > 0$, so that $P^{-s'} \to 0$ and μ must be bounded for all ζ. Therefore $\mu = 0$. Repeated application of (3.30) to (3.29) now gives (3.27) as required, with

$$\eta_{l,m} = \{(l+s)!/(l-s)!\}^{\frac{1}{2}}\chi_{l,m}.$$

Another way of expressing this result is: given any suitably regular η on the sphere,[10] of (integral) spin weight $s > 0$, there exists ξ of spin weight zero for which

$$\eta = \eth^s\xi. \tag{3.31}$$

The relevance to this paper of the foregoing analysis lies essentially in this result.[11] (In the above, $\xi = \sum \chi_{l,m}Y_{l,m}$.) One reason this will be of interest to us here is that it enables us to define the "electric" and "magnetic" parts of η, denoted, respectively, by η_e and η_m, where

$$\eta_e = \eth^s\,\mathrm{Re}\,(\xi), \qquad \eta_m = \eth^s\,\mathrm{Im}\,(\xi). \tag{3.32}$$

[If, instead, $s < 0$, then we would use $\bar{\eth}$ in place of $\eth$ and $-s$ in place of s in (3.31), (3.32).] We would get the same η_e and η_m if a different choice of ξ were made in (3.31), since the arbitrariness in ξ lies only in harmonics $Y_{l,m}$ with $l < s$. We can characterize η_e and η_m by the fact that

$$\eta = \eta_e + \eta_m \tag{3.33}$$

with

$$\bar{\eth}^s\eta_e = \eth^s\bar{\eta}_e, \qquad \bar{\eth}^s\eta_m = -\eth^s\bar{\eta}_m. \tag{3.34}$$

The invariance properties of $\eth$ imply that the concepts of "magnetic" and "electric" parts of η are suitably invariant under rotations of the sphere or of the m^μ vectors. Furthermore if (and *only* if) η has conformal weight -1, the splitting of η into its "electric" and "magnetic" parts is also invariant under *conformal* transformation of the sphere [cf. (3.14)].

Here we shall apply these concepts to σ^0. The spin weight is 2 and the conformal weight is -1, so on each hypersurface u = const, we have a splitting $\sigma^0 = \sigma_e^0 + \sigma_m^0$, which is invariant under conformal transformations of the (θ, ϕ)-sphere, i.e., under the "Lorentz" transformations (2.1) which leave the hypersurface invariant. The σ^0 and its u-derivatives describe properties of the gravitational

[10] We use the phrase "suitably regular" in the sense that η is a smooth function of ζ and $\bar{\zeta}$ in the $(\zeta, \bar{\zeta})$ system and that if we transform coordinates by, say, $\zeta' = \zeta^{-1}$, then η' is a smooth function of ζ' and $\bar{\zeta}'$. This deals with the point $\zeta = \infty$ in the $(\zeta, \bar{\zeta})$ system.

[11] This result, in the case $\eta = \sigma^0$, is considered by D. Lamb, J. Math. Phys. 7, 458 (1966).

radiation field. By analogy with the linear theory, we may suppose μ_e^0 to be associated with "electric"-type radiation (e.g., arising from changes in the mass quadrupole) and σ_m^0 to be associated with "magnetic"-type radiation (e.g., arising from changes in the angular momentum quadrupole). The analogy with electrodynamics arising here is the reason for the use of the terms "electric" and "magnetic" in (3.32). In the next section, we consider these matters more explicitly.

IV. PHYSICAL ASSUMPTIONS

We wish to impose physical restrictions on the systems under consideration, which have sufficient generality to include scattering problems in which sources can come in from and go out to spatial infinity with initial and final velocities less than c. In particular, the sources might all be confined to a bounded region of space. We shall require that the angular momentum, with respect to some origin, of each component part of the source remains bounded for all time (where it is assumed that there are a finite number of "parts" to the source). Generally, the physical restrictions that we impose are intended to be such that, $u \to -\infty$ (or alternatively as $u \to +\infty$), σ^0 behaves as

$$\sigma^0(u, \theta, \phi) \to S(\theta, \phi), \tag{4.1}$$

with S independent of u. If the analogy with the linear theory is to be trusted, we would normally expect S to have no "magnetic" part. This would add some force to the arguments but it is not actually essential. On the basis of (4.1) alone, we shall be able to extract the inhomogeneous Lorentz group.

In order to investigate what is entailed physically by the requirement that (4.1) hold, we must resort to analogies with the linear theory. The full theory is, unfortunately, insufficiently developed as yet to enable us to infer the exact relation between the motion of the sources and asymptotic quantities such as σ^0. In the linear theory, we may take $u =$ const to be "good" cones in Minkowski space. Since the shear of these cones now vanishes, the linearized σ^0, denoted σ^0_{lin}, describes a first-order deviation from "goodness" of these cones. We can then obtain σ^0_{lin} from the multipole moments $C_{l,m}(u)$ of the source distribution by[6,7]

$$\sigma^0_{\text{lin}} = \eth^2\left\{\sum_{l,m} \beta_{l,m} \frac{d^l}{du^l}(C_{l,m}Y_{l,m}) + f\right\}, \tag{4.2}$$

where $f = f(\theta, \phi)$ is real, independent of u, and corresponds to the gauge freedom in σ^0, and where the $\beta_{l,m}$ are some inessential numerics. (Both $\beta_{l,m}$ and $C_{l,m}$ have spin weight zero.)

The type of behavior for the multipole moments that we may assume as reasonable is

$$C^{e}_{l,m} \sim u^l, \tag{4.3}$$

$$C^{m}_{l,m} \sim u^{l-1}, \tag{4.4}$$

where $C^e_{l,m}$ and $C^m_{l,m}$ correspond, respectively, to the "electric" and "magnetic" parts of the multipole moments, so that $C^e_{l,m}Y_{l,m}$ and $C^m_{l,m}Y_{l,m}$ are the real and imaginary parts of $C_{l,m}Y_{l,m}$. For example, for $l = 2$, the $C^e_{2,m}$ and $C^m_{2,m}$ are, respectively, the five components of the mass quadrupole moment and spin (or angular momentum) quadrupole moment, defined so that $C^e_{2,m} \sim \sum MR^2$ and $C^m_{2,m} \sim \sum LR$, where M, L, and R are the mass, angular momentum, and distance from the origin of each component of the source. If each component is moving with uniform limiting velocity and each angular momentum is bounded, we have $R \sim u$, $L \sim 1$, whence $C^e_{2,m} \sim u^2$ and $C^m_{2,m} \sim u$. The corresponding argument for higher moments leads to (4.3) and (4.4).

By substituting (4.3) and (4.4) into (4.2), a reasonable guess for σ^0_{lin} results:

$$\sigma^0_{\text{lin}} = \eth^2\{A + iB + f\}, \tag{4.5}$$

where

$$A = \sum_{l,m} \beta_{l,m} \frac{d^l(C^e_{l,m}Y_{l,m})}{du^l}$$

is real and tends to a finite limit as $u \to -\infty$ (or as $u \to +\infty$), and where

$$B = \sum_{l,m} \beta_{l,m} \frac{d^l(C^m_{l,m}Y_{l,m})}{du^l}$$

is real and tends to zero as $u \to -\infty$ (or $u \to +\infty$). Of course, (4.5) cannot be inferred rigorously from (4.3) and (4.4) since it is not permissible simply to differentiate order of magnitudes in this way. But if the behavior (4.3) and (4.4) is sufficiently "smooth" asymptotically, then the deduction is valid. This would rule out oscillatory behavior for the sources[12] in the distant past (or future), since such behavior would certainly contradict (4.5). We assume, here, that the components of the system have no accelerations in the limit $u \to -\infty$ (or $u \to +\infty$) which would lead to a violation of (4.5).

[12] An interesting case that would be excluded by our assumptions is that of two masses revolving about one another in ever decreasing circles as they lose energy by gravitational radiation. Here the angular momentum is unbounded as $u \to -\infty$ and apparently $\sigma^0 \sim u^{3/8}$.

We can rewrite (4.5) as

$$\sigma^0_{1\mathrm{in}} \rightarrow S_e(\theta, \phi) \qquad (4.6)$$

when $u \rightarrow -\infty$ (or $u \rightarrow +\infty$), S_e being purely "electric" and independent of u. Thus, it seems not unreasonable to infer that, under similar physical assumptions in the full theory, (4.1) will hold [with $S(\theta, \phi)$ purely electric].

It may be felt that the analogy with the linear theory would be a little more trustworthy, however, if given in terms of gauge independent quantities such as the radiation field $\partial^2\sigma^0/\partial u^2$, rather than σ^0. If we assume $\partial^2\sigma^0/\partial u^2$ behaves, in the full theory, in the same way as a $\partial^2\sigma^0_{1\mathrm{in}}/\partial u^2$ consistent with (4.6), then the only difference in the behavior of σ^0 that could arise would lie in the constants of integration. Moreover, the constant of integration involved in the passage from $\partial^2\sigma^0/\partial u^2$ to $\partial\sigma^0/\partial u$ must be zero. This follows from a requirement that the total energy radiated away, as measured according to the Bondi–Sachs formula[1,2]

$$\iiint \left|\frac{\partial\sigma^0}{\partial u}\right|^2 du\, d\theta \sin\theta\, d\phi,$$

be finite. In fact, this finiteness requirement for the gravitationally radiated energy greatly restricts the behavior of $\partial\sigma^0/\partial u$ as $u \rightarrow \pm\infty$ in any case in the full theory, in a *rigorous* way (e.g., any simple oscillatory behavior would be ruled out). However, this is not sufficient for our purposes, since the radiated energy is finite if $\sigma^0 \sim u^{\frac{1}{2}-\epsilon}(0 < \epsilon < \frac{1}{2})$ but this would violate (4.1). The remaining constant of integration arises in the passage from $\partial\sigma^0/\partial u$ to σ^0, and this would be the S of (4.1). In the linear theory the $S = S_e$ is purely "electric" and this may be reasonable in the full theory also. But, in any case, the possibility of S having a "magnetic" part can also be treated here.

We now show that, under the assumptions stated above, it is possible to introduce coordinate conditions such that $S_e(\theta, \phi) = 0$. This will be irrespective of whether the "magnetic" part S_m, of $S(= S_e + S_m)$, vanishes. Sachs[2] has shown that, under a BMS transformation (2.1), σ^0 transforms [cf. also (3.2), (3.4)] as

$$\sigma^{0\prime}(u, \theta, \phi) = K^{-1} e^{2i\psi}(\sigma^0(u, \theta, \phi) - \tfrac{1}{2}\eth^2\alpha) \qquad (4.7)$$

(α having spin weight zero). It should be pointed out that $\sigma'^0(u, \theta, \phi)$ refers to the asymptotic shear of the hypersurfaces $u' = $ const of the *transformed* coordinate system, although evaluated at the (u, θ, ϕ) of the *original* coordinate system. The complete transformation to $\sigma^{0\prime}(u', \theta', \phi')$ is, of course, more complicated. Applied to (4.1), (4.7) gives

$$S_e'(\theta, \phi) = K^{-1} e^{2i\psi}(S_e(\theta, \phi) - \tfrac{1}{2}\eth^2\alpha), \qquad (4.8)$$

$$S_m'(\theta, \phi) = K^{-1} e^{2i\psi} S_m(\theta, \phi), \qquad (4.9)$$

since α is real. Since by (3.32) we can write $S_e(\theta, \phi) = \eth^2 G$ for some real G and since α can be chosen arbitrarily on the (θ, ϕ)-sphere, it follows that a BMS transformation exists (e.g., $\alpha = 2G$) for which $S_e'(\theta, \phi) = 0$, as required.

We thus may adopt as our coordinate condition the requirement that $S_e(\theta, \phi) = 0$. This can be done for the S_e defined *either* at $u = -\infty$ *or* at $u = +\infty$. For definiteness we could choose the condition defined at $u = -\infty$. (If $S_m = 0$, this means that in the limit $u \rightarrow -\infty$ the hypersurfaces $u = $ const become nearer and nearer to being asymptotically like the "good" cones of Minkowski space.) The BMS transformations which preserve the coordinate condition $S_e = 0$ are now those for which $\eth^2\alpha = 0$ [cf. (4.8)]. Thus α has the form (2.3) and the allowed supertranslations are simply the translations. The "Lorentz rotations" given by $\alpha = 0$ clearly do not destroy the coordinate condition $S_e = 0$. [The fact that S has conformal weight -1 is essential here, since the splitting $S = S_e + S_m$ must be invariant under conformal transformation of the (θ, ϕ)-sphere, $\eth^2$ being also suitably conformally invariant here.] The transformations (2.1) with α as in (2.3) have the same form as inhomogeneous Lorentz transformations in Minkowski space. Thus, the group of coordinate transformations which preserve all the usual coordinate conditions in asymptotic gravitational radiation theory *as well* as keeping $S_e = 0$ at $u = -\infty$ (or else at $u = +\infty$) is isomorphic to the improper orthochronous inhomogeneous Lorentz group.

V. DISCUSSION

There are several inconclusive aspects of this work (even apart from the heuristic nature of the argument concerning the physical assumptions) which should be mentioned. In particular, the ambiguity as to where the coordinate condition $S_e = 0$ is to be imposed is disturbing. If the condition is imposed at $u = -\infty$, there seems no reason at all to believe that the condition would then automatically hold also at $u = +\infty$ (although it is just conceivable that "coherence" relations between the radiation-field data at infinity, of the type considered by Friedlander,[13] might link the con-

[13] F. G. Friedlander, Proc. Roy. Soc. (London) **A279**, 386 (1964).

dition $S_e = 0$ at $u = -\infty$ with that at $u = +\infty$). In fact, the situation is worse than this since all the above arguments could be repeated using an advanced time parameter v in place of the retarded time u. We would then have four different alternative ways of choosing coordinate conditions, namely $S_e = 0$ at $u = \pm\infty$, or the corresponding $S_e = 0$ at $v = \pm\infty$, any one of which would lead to the inhomogeneous Lorentz group, but the different choices would *apparently* be quite unrelated.

It would be interesting to know whether the conditions at $u = -\infty$ are in any way related to those at $v = +\infty$. In the conformal approach to asymptotic analysis,[5] $v = -\infty$, $u = -\infty$, and $u = +\infty$ appear as distinct "points" I^-, I^0, and I^+, respectively, but $v = +\infty$ represents the *same* "points" I^0 as $u = -\infty$. Another question of interest is the relation of the present work to that of the Bergmann,[14] who concludes that if one examines asymptotically flat space–times by proceeding to infinity in *spacelike* directions, then one cannot extract the inhomogeneous Lorentz group as an asymptotic symmetry group. From the conformal point of view, this is again concerned with the "point" I^0, but, since I^0 is approached quite differently in the two analyses, there appears to be no essential conflict here.

A further unanswered question concerns the relevance of the present work to some recent suggestions[6,15,16] that possibly the BMS group can play a role in elementary-particle physics. It is not at all clear that considerations of this paper affect this possibility. The arguments here require consideration of infinite times, while the BMS group emerges if only finite (retarded) time intervals are considered. For such finite time intervals, the BMS group must still be regarded as the relevant asymptotic symmetry group for situations involving gravitational radiation in asymptotically flat space–time.

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