Supplementary Material to: The Sensitivity of the Center of Pressure: a further Criterion to assess Contact Stability and Balancing Controllers

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This document contains the supplementary material for the paper The Sensitivity of the Center of Pressure: a further Criterion to assess Contact Stability and Balancing Controllers, currently under review for publication on IEEE Robotics and Automation Letters (RA-L).

It contains additional remarks and experiments that did not fit into the page length required by the Journal. It also contains the proofs of the Lemmas introduced in the paper. All the equations and figures introduced in this document are prefixed with S, i.e. a reference to an equation or figure not prefixed by S refers to the original paper.

Appendix

A. Remarks on the sensitivity in minimal coordinates (Sect III-C)

Section III-C defines the sensitivity of the center of pressure in a minimal coordinate representation, see Definition 2 in the paper. We defined ζ such that $\dot{q} = Z\zeta$, where Z is a base onto the null space of the constraint Jacobian J as in Eq. (4). It is important to notice that in general ζ is not the velocity of the minimal coordinates of the system. To clarify this, let us make the following assumption.

Assumption 1: The holonomic constraints due to contacts restrain the configuration space $\mathbb{Q}$ into the set $\mathbb{R}^{n+6-k}$, i.e. there exists a differentiable mapping $\chi : \mathbb{R}^{n+6-k} \to \mathbb{Q}$ such that

$$\forall q \in \mathbb{Q} \quad \exists\, \xi \in \mathbb{R}^{n+6-k} \quad : \quad q = \chi(\xi). \tag{S.1}$$

Under the assumption above, $\zeta = \dot{\xi}$ only if $Z = \frac{\partial \chi}{\partial \xi}$. In this case, however, one has to know the mapping ξ identifying the system minimal coordinates.

Note that, by its definition in Eq. (11), the actual value of the sensitivity depends on the choice of the null space Z. Different choices of parametrisation may lead to different values of the sensitivity. We suggest to use the minimal coordinate representation so as $\zeta = \dot{\xi}$, and to choose the minimal coordinates ξ with clear physical meaning. In this way, the sensitivity clearly represents the rate-of-change w.r.t. physical meaningful values.

B. Four-bar linkage model (Sect. IV-A)

Consider the four-bar linkage shown in Figure S.2, where l and d denote the lengths of the leg and of the upper rod, respectively. Each leg possesses a mass of m_l while the upper rod has mass m_b. The inertial frame $\mathcal{I}$ is chosen so as the z axis opposes gravity, the x axis exits from the plane, and the y axis completes the right-hand base. Being a planar model, only translations in the $y-z$ plane and rotations about the x axis are allowed. For notation clarity, we define a new set of canonical vectors $\tilde{e}_i \in \mathbb{R}^2$, where $\tilde{e}_1$ identifies the y axis and $\tilde{e}_2$ identifies the z axis.

The vectors

$$\lambda = \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}^\top \in \mathbb{R}^2, \quad \rho = \begin{bmatrix}\rho_1 & \rho_2\end{bmatrix}^\top \in \mathbb{R}^2$$

connect the center of mass to the left and right foot frames respectively. Let ${}^{\mathcal{I}}p_{\mathcal{B}} \in \mathbb{R}^2$ denote the position of the origin of the frame $\mathcal{B}$ – which is located at the center of the upper link – expressed in the inertial frame. The configuration space is then parametrised by the following coordinates

$$q = ({}^{\mathcal{I}}p_{\mathcal{B}}, \theta, q_1, q_2, q_3, q_4) \in \mathbb{R}^7.$$

Being a vector space, we use the Lagrange formalism to derive the equations of motion [1, Ch. 4], i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = \begin{bmatrix}0_3 \\ \tau\end{bmatrix}, \tag{S.2}$$

with $L := T - V$ the Lagrangian, and T and V respectively the kinetic and potential energy of the multi-body system.

Assumption 2: For the four-bar linkage model, it holds that:

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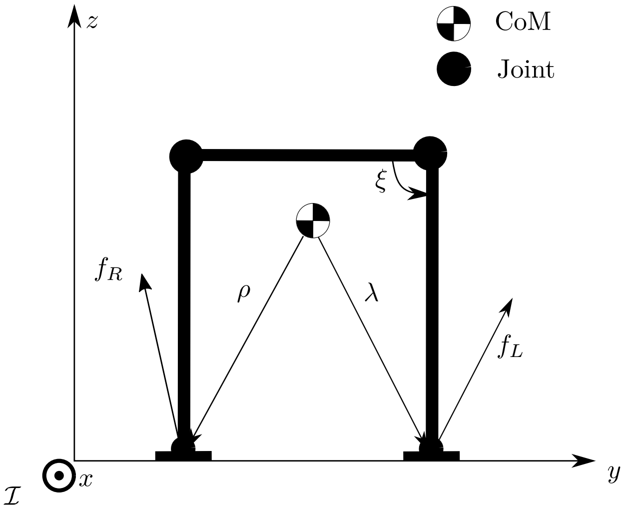

Fig. S.1: Planar four-bar linkage exploiting two rigid contact to stand

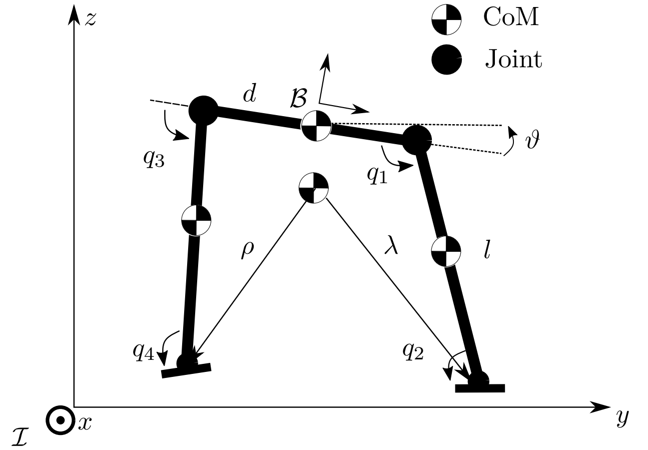

Fig. S.2: Free floating planar four-bar linkage. The figure also shows the joints and the base frame.

i) Each link is approximated as a point mass at its center of mass.
ii) The mass of the left leg equals the mass of the right leg.
iii) The upper body mass, m_b, is twice the leg mass, i.e. $m_b = 2m_l$.

It is worth noting that hypothesis ii) and iii) are in general satisfied in humanoid robots, e.g. in the iCub robot used for the experiments.

For the sake of simplicity we present only the terms of the dynamic and kinematic model that are necessary for the comprehension of the remainder of the section. In particular, partition the mass matrix $\overline{M}$ as follows

$$\overline{M} = \begin{bmatrix} M_b & M_{bj} \\ M_{bj}^\top & M_j \end{bmatrix}.$$

Let $\sin(q_i) = s_i$ and $\cos(q_i) = c_i$. One can verify that

$$M_{bj} = \frac{m_l}{2} \begin{bmatrix} ls_1 & 0 & ls_3 & 0 \\ -lc_1 & 0 & -lc_3 & 0 \\ \frac{l^2 s_1^2 - lc_1(d - lc1)}{2} & 0 & \frac{l^2 s_3^2 + lc_3(d + lc_3)}{2} & 0 \end{bmatrix}. \tag{S.3}$$

The feet Jacobians are given by the following expression

$$J_{\mathcal{C}_L} = \begin{bmatrix} 1_2 & R(\theta)\begin{bmatrix} ls_1 \\ \frac{d}{2} - lc_1 \end{bmatrix} & R(\theta)\begin{bmatrix} ls_1 \\ -lc_1 \end{bmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ 0 \quad 0 & 1 & 1 & 1 \quad 0 \quad 0 \end{bmatrix},$$

$$J_{\mathcal{C}_R} = \begin{bmatrix} 1_2 & R(\theta)\begin{bmatrix} ls_3 \\ -\frac{d}{2} + lc_3 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & R(\theta)\begin{bmatrix} ls_3 \\ -lc_3 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 & 0 \quad 0 & 1 & 1 \end{bmatrix}.$$

To represent a classical balancing context of humanoid robots, we assume that the mechanism makes contact with the environment at the two extremities. Also, we assume that the distance between the contacts equals d, i.e. the length of the upper rod – see Figure S.1. Then, the rigid constraint acting on the system takes the form (4), with

$$J = \begin{bmatrix} J_{\mathcal{C}_L}^\top & J_{\mathcal{C}_R}^\top \end{bmatrix}^\top \in \mathbb{R}^{6\times 7},$$

and $\nu = \dot{q}$. The associated contact wrenches are

$$f = \begin{bmatrix} f_L \\ f_R \end{bmatrix} \in \mathbb{R}^6.$$

As a consequence of these constraints, the mechanism possesses one degree of freedom when standing on the extremities.

With the aim of evaluating the sensitivity of the static center of pressure we define the minimal coordinate ξ as the angle between the upper link and one of the legs (see Figure S.1). Observe that the mapping $q = \chi(\xi)$ in Eq. (S.1) is given by:

$$\begin{bmatrix} {}^{\mathcal{I}}p_b \\ \theta \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2}d + l\cos(\xi))\tilde{e}_1 + l\sin(\xi)\tilde{e}_2 \\ 0 \\ \xi \\ \pi - \xi \\ \xi \\ \pi - \xi \end{bmatrix}. \tag{S.4}$$

C. Proof of Lemma 1

Lemma 1: Assume that Assumption 2 holds. If the contact wrenches redundancy is chosen to minimise the norm of the contact wrenches, i.e. criterion 1), then the induced contact wrenches are given by the following expression:

$$\begin{aligned} {}^{L}f_L &= mg \begin{bmatrix} 0 \\ \frac{1}{2\left(1+\frac{d^2}{4}\right)} + d\frac{5d-6l\cos(\xi)}{10(d^2+4)} \\ \frac{6l\cos(\xi)}{5(d^2+4)} \end{bmatrix}, \\ {}^{R}f_R &= mg \begin{bmatrix} 0 \\ \frac{1}{2\left(1+\frac{d^2}{4}\right)} + d\frac{5d+6l\cos(\xi)}{10(d^2+4)} \\ \frac{6l\cos(\xi)}{5(d^2+4)} \end{bmatrix}. \end{aligned} \tag{S.5}$$

Furthermore, the static center of pressure sensitivity is given by:

$$\begin{aligned} \eta^s_L &= -60l\sin(\xi)\frac{d^2+4}{(5d^2+6ld\cos(\xi)+20)^2}, \\ \eta^s_R &= -60l\sin(\xi)\frac{d^2+4}{(5d^2-6ld\cos(\xi)+20)^2}. \end{aligned} \tag{S.6}$$

Proof: Consider the equilibrium condition of the centroidal dynamics ([2]) of the mechanism, i.e. the balance of the external wrenches acting on the mechanism when written w.r.t. a frame centered in the center of mass and with the orientation of the inertial frame.

$$0 = X_L\,{}^{L}f_L + X_R\,{}^{R}f_R - mge_2 = Xf - mge_2 \tag{S.7}$$

where mge_2 is the wrench due to gravity and $X := \begin{bmatrix} X_L & X_R \end{bmatrix}$,

$$X_L = \begin{bmatrix} 1_2 & 0_{2\times 1} \\ (\tilde{S}\lambda)^\top & 1 \end{bmatrix}, \quad X_R = \begin{bmatrix} 1_2 & 0_{2\times 1} \\ (\tilde{S}\rho)^\top & 1 \end{bmatrix},$$

$\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}^\top \in \mathbb{R}^2$ and $\rho = \begin{bmatrix} \rho_1 & \rho_2 \end{bmatrix}^\top \in \mathbb{R}^2$ the vectors connecting the center of mass to the left and right foot frames respectively, see Figure S.1. The matrix $\tilde{S}$ is the $2D$ equivalent of the $3D$ skew-symmetric matrix $S(x)$ introduced in Sect. II-A in the paper. It is defined as $\tilde{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Since $X \in \mathbb{R}^{3\times 6}$ by construction is always full row rank, (S.7) admits infinite solutions, i.e.

$$f = X^\dagger mge_2 + N_f f_0 \tag{S.8}$$

where $X^\dagger \in \mathbb{R}^{6\times 3}$ is the Moore-Penrose pseudoinverse of X, $N_f \in \mathbb{R}^{6\times 6}$ is the projector onto the nullspace of X and $f_0 \in \mathbb{R}^6$ is a free variable. The unique minimum norm solution corresponds to the choice $f_0 = 0$. The actual expression of the contact wrenches are thus the ones given in Eq. (S.5).

The center of pressure in the planar case possesses only one coordinate, which is given by the following expressions

$${}^{L}\mathrm{CoP}_L = \frac{e_3^\top f}{e_2^\top f}, \quad {}^{R}\mathrm{CoP}_R = \frac{e_6^\top f}{e_5^\top f}$$

which evaluated with the expression of the wrenches in Eq. (S.5) yield

$$\begin{aligned} {}^{L}\mathrm{CoP}_L &= \frac{12l\cos(\xi)}{5d^2+6ld\cos(\xi)+20}, \\ {}^{R}\mathrm{CoP}_R &= \frac{12l\cos(\xi)}{5d^2-6ld\cos(\xi)+20}. \end{aligned}$$

To evaluate the static center of pressure sensitivity, we can differentiate the above expression w.r.t ξ, that is

$$
\begin{aligned}
\eta_L^s &= \frac{\partial\ \mathrm{CoP}_R}{\partial \xi} = -60l\sin(\xi)\frac{d^2+4}{(5d^2+6ld\cos(\xi)+20)^2}, \\
\eta_R^s &= \frac{\partial\ \mathrm{CoP}_R}{\partial \xi} = -60l\sin(\xi)\frac{d^2+4}{(5d^2-6ld\cos(\xi)+20)^2}.
\end{aligned}
$$

■

D. Proof of Lemma 2

Lemma 2: Assume that Assumption 2 holds. If the contact wrenches redundancy is chosen to minimise the norm of the internal joints torques, i.e. criterion 2), the induced wrenches are given by the following expressions.

$$
{}^{L}f_L = mg\left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\lambda_1}{2} \end{bmatrix} + \begin{bmatrix} \frac{3l\cos(\xi)^2}{8d\sin(\xi)} \\ \frac{3l\cos(\xi)}{8d} \\ \frac{3l\cos(\xi)}{8d}\left(\lambda_2\frac{\cos(\xi)}{\sin(\xi)} - \lambda_1\right) + \frac{d}{4} \end{bmatrix}\right), \tag{S.9a}
$$

$$
{}^{R}f_R = mg\left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\rho_1}{2} \end{bmatrix} - \begin{bmatrix} \frac{3l\cos(\xi)^2}{8d\sin(\xi)} \\ \frac{3l\cos(\xi)}{8d} \\ \frac{3l\cos(\xi)}{8d}\left(\rho_2\frac{\cos(\xi)}{\sin(\xi)} - \rho_1\right) + \frac{d}{4} \end{bmatrix}\right). \tag{S.9b}
$$

Furthermore, the static center of pressure sensitivity is given by:

$$
\begin{aligned}
\eta_L^s &= -\frac{45d^2l\sin(\xi)}{(10d+3l\cos(\xi))^2}, \\
\eta_R^s &= -\frac{45d^2l\sin(\xi)}{(10d-3l\cos(\xi))^2}.
\end{aligned} \tag{S.10}
$$

Proof: Consider the balance of the external wrenches acting on an articulated rigid body in the general 3D case:

$$
X_L\,{}^{L}f_L + X_R\,{}^{R}f_R - mge_3 = 0 \tag{S.11}
$$

where $X_L, X_R \in \mathbb{R}^{6\times 6}$ are used to express the wrenches respectively in L and R at the frame located at center of mass with the orientation of the inertial frame. Impose the following expression for the contact wrenches:

$$
\begin{aligned}
{}^{L}f_L &= X_L^{-1}\frac{mg}{2}e_3 + X_L^{-1}\Delta \\
{}^{R}f_R &= X_R^{-1}\frac{mg}{2}e_3 - X_R^{-1}\Delta
\end{aligned} \tag{S.12}
$$

where $\Delta \in \mathbb{R}^6$ is a free variable. We can notice that with the above choice, (S.11) is always satisfied independently of the choice of Δ. In what follows, the redundancy Δ is used to minimize the joint torques. To simplify the calculations we perform a state transformation as described in ([3]). In particular, the dynamics in the new state variables is decoupled in the acceleration, i.e. the mass matrix is block diagonal, with the two blocks being the terms relative to the base and the joints. We refer the reader to ([3]) for additional details, and we report here only the relevant transformations. To avoid confusion, all the dynamic quantities in the new state variables are denoted with the overline.

The new base frame is chosen to correspond to a frame with the same orientation of the inertial frame $\mathcal{I}$ and the origin instantaneously located at the center of mass. We denote with ${}^{c}X_{\mathcal{B}}$ the corresponding 6D transformation. Of particular interest is the form of the Jacobian of the left (right) foot frame L (R) in the new state variables:

$$
\begin{aligned}
\overline{J}_L &= \begin{bmatrix} \overline{J}_{L,b} & \overline{J}_{L,j} \end{bmatrix} = \begin{bmatrix} X_L^\top & J_{L,j} - \frac{1}{m}X_L^\top \Lambda M_{bj} \end{bmatrix}, \\
\overline{J}_R &= \begin{bmatrix} \overline{J}_{R,b} & \overline{J}_{R,j} \end{bmatrix} = \begin{bmatrix} X_R^\top & J_{R,j} - \frac{1}{m}X_R^\top \Lambda M_{bj} \end{bmatrix}, \\
\Lambda &:= 1_6 - P(1_6 + {}^{c}X_B^{-1}P)^{-1}\,{}^{c}X_B^{-1}.
\end{aligned}
$$

P has the following expression:

$$
P := \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ S(p_{\mathrm{CoM}} - p_{\mathcal{B}}) & \frac{I}{m} - 1_3 \end{bmatrix}
$$

where I is the 3D inertia of the robot expressed w.r.t. the orientation of $\mathcal{I}$ and the center of mass as origin and $p_{\mathrm{CoM}} - p_{\mathcal{B}}$ is the distance between the center of mass and the base frame before the state transformation.

The relation between the joint torques and the contact wrenches is given by Eq. (5). So, by inverting this relation, we evaluate the effects of the contact wrenches redundancy Δ on the joint torques at the equilibrium configuration, i.e.

$$\tau = (\bar{J}_j \overline{M}_j^{-1})^{\dagger} \bar{J} \overline{M}^{-1} (mge_3 - \bar{J}^{\top} f). \tag{S.13}$$

We can now substitute the definition of f as in (S.12) into the obtained expression of $\tau = \tau(f)$ in (S.13), i.e.

$$\tau = (\bar{J}_j \overline{M}_j^{-1})^{\dagger} \bar{J} \overline{M}^{-1} [(2 - \bar{J}_L^{\top} X_L^{-1} - \bar{J}_R^{\top} X_R^{-1}) \frac{mg}{2} e_3 - (\bar{J}_L^{\top} X_L^{-1} - \bar{J}_R^{\top} X_R^{-1}) \Delta]. \tag{S.14}$$

Now, it is easy to verify that

$$\bar{J}_L^{\top} X_L^{-1} + \bar{J}_R^{\top} X_R^{-1} = \begin{bmatrix} 2\ 1_6 \\ J_{L,j}^{\top} X_L^{-1} + J_{R,j}^{\top} X_R^{-1} - 2 \frac{M_{bj}^{\top}}{m} \Lambda^{\top} \end{bmatrix}$$

$$\bar{J}_L^{\top} X_L^{-1} - \bar{J}_R^{\top} X_R^{-1} = \begin{bmatrix} 0_{6\times 6} \\ J_{L,j}^{\top} X_L^{-1} - J_{R,j}^{\top} X_R^{-1} \end{bmatrix}.$$

Observe also that

$$P^{\top} e_3 = \begin{bmatrix} 0_{3\times 3} & -S(r) \\ 0_{3\times 3} & \frac{I}{m} - 1_3 \end{bmatrix} e_3 = 0_6 \Rightarrow \Lambda^{\top} e_3 = 1_6.$$

Then (S.14) becomes

$$\tau = -\left(\bar{J}_j \overline{M}_j^{-1}\right)^{\dagger} \bar{J}_j \overline{M}_j^{-1} \left[\left(J_{L,j}^{\top} X_L^{-1} + J_{R,j}^{\top} X_R^{-1} - 2 \frac{M_{bj}^{\top}}{m} \right) \frac{mg}{2} e_3 + \left(J_{L,j}^{\top} X_L^{-1} - J_{R,j}^{\top} X_R^{-1} \right) \Delta \right]. \tag{S.15}$$

If we consider the four-bar linkage planar model, we can notice that the number of rows of J_j are greater than the DoFs of the system and thus the product of the first two terms simplifies into the identity matrix. As a consequence, the vector Δ that minimizes the joint torques is given by:

$$\Delta = -\frac{mg}{2} \left(J_{L,j}^{\top} X_L^{-1} - J_{R,j}^{\top} X_R^{-1} \right)^{\dagger} \left(J_{L,j}^{\top} X_L^{-1} + J_{R,j}^{\top} X_R^{-1} - 2 \frac{M_{bj}^{\top}}{m} \right) e_2 \tag{S.16}$$

which, by using the actual expression for the Jacobians, boils down to

$$\Delta = \frac{mg}{2} \begin{bmatrix} \frac{m_l + m_b}{m} \frac{l \cos^2(\xi)}{d \sin(\xi)} \\ \frac{m_l + m_b}{m} \frac{l \cos(\xi)}{d} \\ \frac{d}{2} \end{bmatrix}. \tag{S.17}$$

The complete expression of the contact wrenches can be obtained by substituting the expression of Δ in Eq. (S.17) into the following equation

$$^{L}f_L = X_L^{-1} \frac{mg}{2} e_2 + X_L^{-1} \Delta,$$

$$^{R}f_R = X_R^{-1} \frac{mg}{2} e_2 - X_R^{-1} \Delta,$$

yielding the following final expression for the contact wrenches:

$$^{L}f_L = mg \left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\lambda_1}{2} \end{bmatrix} + \begin{bmatrix} \frac{m_b + m_l}{m} \frac{l \cos(\xi)^2}{2d \sin(\xi)} \\ \frac{m_b + m_l}{m} \frac{l \cos(\xi)}{2d} \\ \frac{m_b + m_l}{m} \frac{l \cos(\xi)}{2d} \left(\lambda_2 \frac{\cos(\xi)}{\sin(\xi)} - \lambda_1 \right) + \frac{d}{4} \end{bmatrix} \right), \tag{S.19a}$$

$$^{R}f_R = mg \left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\rho_1}{2} \end{bmatrix} - \begin{bmatrix} \frac{m_b + m_l}{m} \frac{l \cos(\xi)^2}{2d \sin(\xi)} \\ \frac{m_b + m_l}{m} \frac{l \cos(\xi)}{2d} \\ \frac{m_b + m_l}{m} \frac{l \cos(\xi)}{2d} \left(\rho_2 \frac{\cos(\xi)}{\sin(\xi)} - \rho_1 \right) + \frac{d}{4} \end{bmatrix} \right). \tag{S.19b}$$

Note that by assuming also Assumption 2-iii) one obtains the expressions in Eq. (S.9).

We can now compute the center of pressures CoP_L and CoP_R. Taking the forces expression in (S.12), the static CoPs are thus:

$$\begin{aligned}\text{CoP}_L &= \frac{-\frac{1}{2}\rho_1 + \rho_2\Delta_1 - \rho_1\Delta_2 + \Delta_3}{\frac{1}{2} + \Delta_2}, \\ \text{CoP}_R &= \frac{-\frac{1}{2}\rho_1 - \rho_2\Delta_1 + \rho_1\Delta_2 - \Delta_3}{\frac{1}{2} - \Delta_2}.\end{aligned} \tag{S.20}$$

By plugging the expression of Δ in Eq. (S.17) we obtain the expression of the static center of pressure as a function of the free variable ξ, which is

$$\begin{aligned}\text{CoP}_L &= \frac{9ld\cos(\xi)}{20d + 6l\cos(\xi)}, \\ \text{CoP}_R &= \frac{9ld\cos(\xi)}{20d - 6l\cos(\xi)}.\end{aligned} \tag{S.21}$$

To obtain the expression of the sensitivity of the static center of pressure we can differentiate Eq. (S.21) w.r.t ξ, finally obtaining

$$\begin{aligned}\eta^s_L &= \frac{\partial\ \text{CoP}_L}{\partial\xi} = -\frac{45d^2l\sin(\xi)}{(10d + 3l\cos(\xi))^2} \\ \eta^s_R &= \frac{\partial\ \text{CoP}_R}{\partial\xi} = -\frac{45d^2l\sin(\xi)}{(10d - 3l\cos(\xi))^2}.\end{aligned}$$

■

E. Additional remarks on the choice of contact wrenches minimisation criteria

A criterion similar to the minimisation of the norm of the contact wrenches that is sometimes found in literature is the minimisation of the tangential components of the forces [4]. The following lemma states the equivalence of the two solutions when planar contacts are considered.

Lemma 3: The solution exploiting the force redundancy to minimise the tangential component of the contact wrenches is equivalent to the minimum contact wrench norm solution (Lemma 1).

Proof:

Consider Eq. (S.8). We want to use f_0 to minimize the tangential component of the contact wrenches, i.e. we want find the solution to

$$\begin{bmatrix} e_1^\top \\ e_4^\top \end{bmatrix} f = 0 \iff \begin{bmatrix} e_1^\top \\ e_4^\top \end{bmatrix} N_f f_0 + \begin{bmatrix} e_1^\top \\ e_4^\top \end{bmatrix} X^\dagger mge_2 = 0.$$

Because of the expression of the nullspace projector and of the pseudoinverse, the above equation reduces to

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} f_0 = 0$$

which has the minimum norm solution in $f_0 = 0$. ■

Remark 1: Most of the balancing controllers using the criterion 1, i.e. the minimisation of the norm of the contact wrenches, specify a desired contact wrench to be tracked as a low priority objective. This usually requires that the contact forces must be planned in advance throughout the whole robot motion. It can be noticed from Figure S.3 that controllers satisfying criterion 2, i.e. minimisation of the joint torques, naturally lead to weight shifting between contacts.

F. Balancing controller for the iCub Humanoid Robot

The balancing controller implemented on the iCub humanoid robot is a dynamic controller composed of a strict hierarchy of two control objectives. The first, and with highest priority, is responsible of tracking a desired robot momentum through the use of the contact wrenches. Note that the desired robot momentum contains feedback terms on the center of mass position and velocity. The second one, instead, is responsible of stabilising the resulting zero dynamics. Discussion about the stability of the zero dynamics is out of the scope of the present paper and we refer the interested reader to [5] for further details on the zero dynamics stability problem.

Assuming only two contacts located at the feet, the dynamics of the robot momentum is given by the following equation:

$$\dot{H} = X_L f_L + X_R f_R + m\bar{g} \tag{S.22}$$

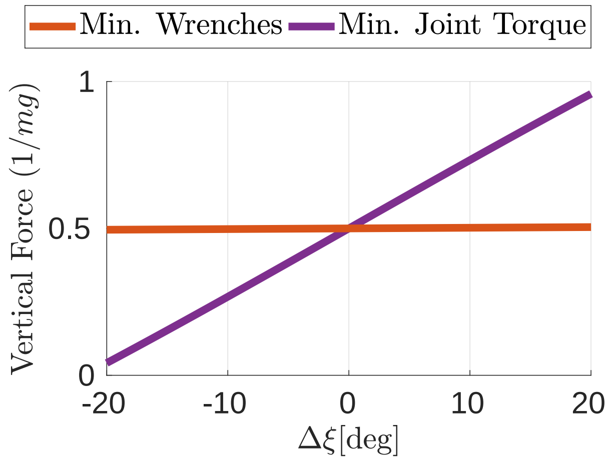

Fig. S.3: Comparison of the vertical force component of the right foot for the two contact forces choice. The vertical force is normalised w.r.t. the total mass of the mechanism, i.e. divided by $1/mg$

where $H \in \mathbb{R}^6$ is the robot linear and angular momentum,

$$X_L = \begin{bmatrix} 1_3 & 0_{3\times 3} \\ S(x_L - x_{\mathrm{CoM}}) & 1_3 \end{bmatrix}, \; X_R = \begin{bmatrix} 1_3 & 0_{3\times 3} \\ S(x_R - x_{\mathrm{CoM}}) & 1_3 \end{bmatrix}$$

are transformation matrices, x_{CoM} is the position of the center of mass and x_L, x_R are the positions of the left and right foot respectively. Finally $m\bar{g}$ is the wrench due to gravity.

Solutions to Eq. (S.22) yield the desired f_L and f_R to be applied by the robot, which are in turn generated by the actuation torques so as to satisfy the first control objective, i.e. tracking a desired robot momentum. To obtain the torques to be applied, we can invert Eq. (5). If we denote with $f^* = [f_L^{*\top}, f_R^{*\top}]^\top$ a possible solution to Eq. (S.22), we obtain the following expression for the control variable τ:

$$\tau = (JM^{-1}B)^\dagger [JM^{-1}(h - J^\top f^*) - \dot{J}\nu], \tag{S.23}$$

where $(JM^{-1}B)^\dagger$ denotes the Moore-Penrose pseudoinverse of the matrix $JM^{-1}B$. The choice of f^*, and as a consequence the torques applied by the robot, depends on the criterion chosen to obtain a solution to Eq. (S.22).

G. Comparison of the center of pressure in the simplified model and on the real robot.

In this section we report an additional experimental validation: we compare how the two different criteria behave with respect to the minimal coordinate variables. Given the kind of movements performed by the robot, we assume as minimum coordinate variable ξ the left hip roll joint, which corresponds to the joint q_1 in the four-bar linkage model shown in Figure S.2. Having performed this choice, we can now properly compare the CoPs obtained with the two force criteria. Fig. S.4 shows the aforementioned comparison considering only the left foot, where Fig S.4a shows the analytical results on the four-bar linkage model and Fig. S.4b shows the results collected on the iCub humanoid robot. It is worth noting the similarity between the two results, thus proving the validity of the reduced model we adopted for the theoretical analysis. The main difference that can be observed, is the range of the minimal coordinate ξ between the simulated mechanism and the real robot. This difference is mainly due to the presence of joint limits and on the small support polygon, i.e. feet, of the real platform.

References

[1] L. Sciavicco, B. Siciliano, and B. Sciavicco, Modelling and Control of Robot Manipulators, 2nd ed. Secaucus, NJ, USA: Springer-Verlag New York, Inc., 2000.

[2] D. Orin, A. Goswami, and S.-H. Lee, "Centroidal dynamics of a humanoid robot," Autonomous Robots, vol. 35, no. 2-3, pp. 161–176, 2013.

[3] S. Traversaro, D. Pucci, and F. Nori, "A Unified View of the Equations of Motion used for Control Design of Humanoid Robots." [Online]. Available: http://bit.ly/2jaSndc

[4] L. Righetti, J. Buchli, M. Mistry, M. Kalakrishnan, and S. Schaal, "Optimal distribution of contact forces with inverse-dynamics control," The International Journal of Robotics Research, vol. 32, no. 3, pp. 280–298, 2013.

[5] G. Nava, F. Romano, F. Nori, and D. Pucci, "Stability analysis and design of momentum-based controllers for humanoid robots," in 2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), Oct 2016, pp. 680–687.

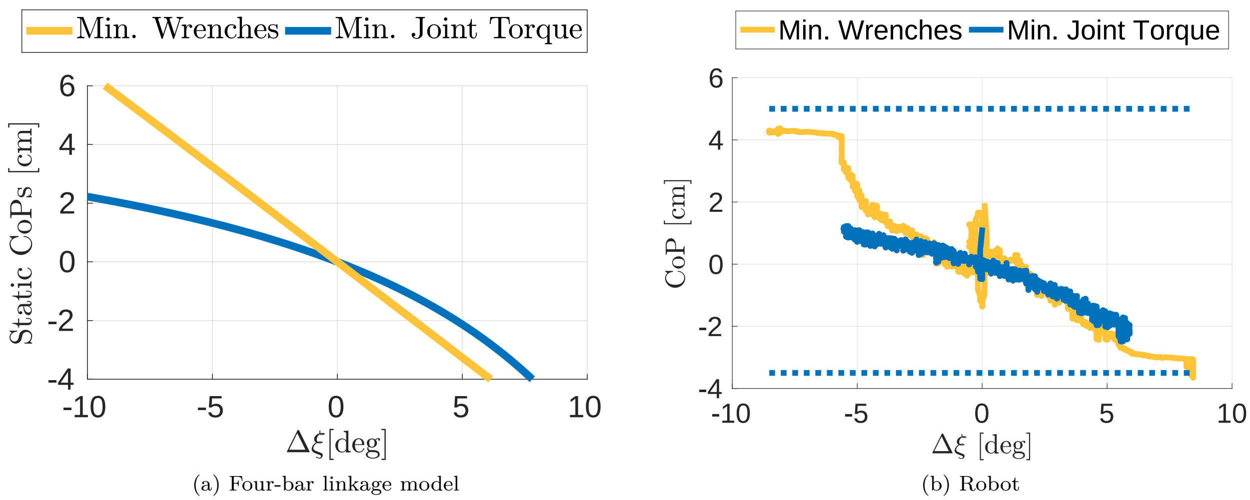

Fig. S.4: Comparison of the center of pressure of the left foot for the two criteria on the simplified model (a) and on the real robot (b). The dotted blue lines in Figure (b) denote the convex hull of the left foot.