Optimal Redundancy Control of Robot Manipulators

Department of Computer, Control, and Management Engineering

Optimal Control Project

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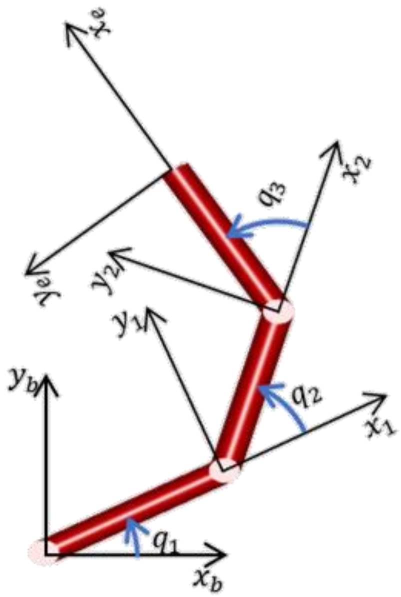
Aim

With respect to a given task, a robot manipulator is said to be redundant if the dimension of the joint space is bigger than the dimension of the operational space.

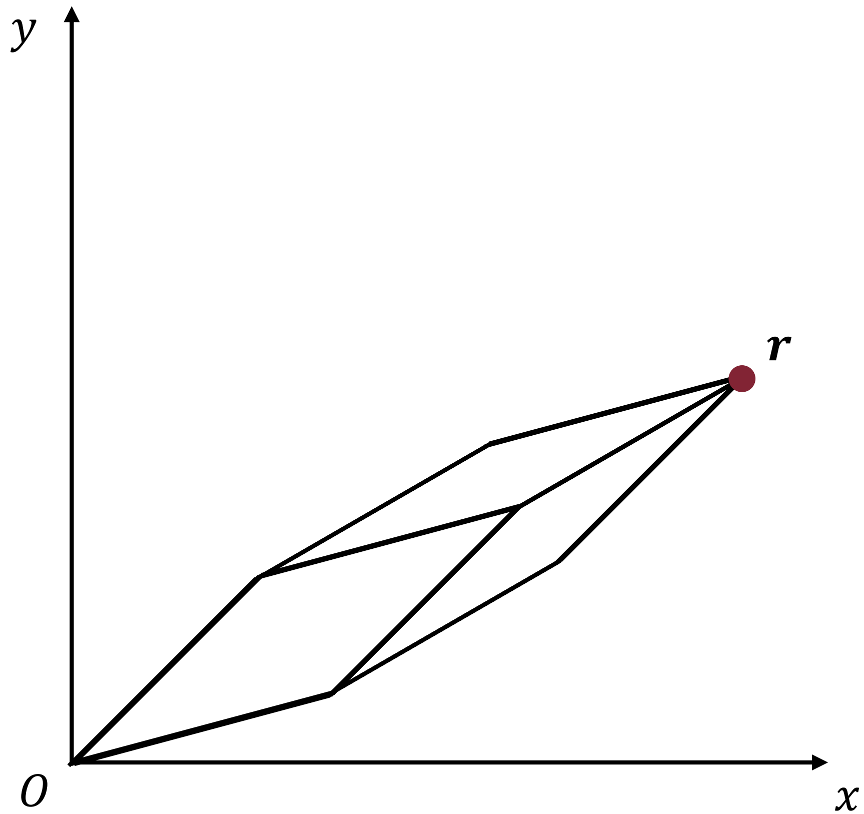
The problem is to find the optimal joint trajectory that maximizes a cost index, for a given end effector velocity and an initial joint configuration. It has been handled with the Pontryagin maximum principle.

Some simulations have been performed on a planar 3R manipulator.



Aim

Which of the following joint configurations that realize a given end effector position is “good” with respect to a certain criterion?



Manipulator model

The relation between the end effector position $\boldsymbol{r} \in \mathbb{R}^m$ and the joint variables $\boldsymbol{q} \in \mathbb{R}^n$, $n > m$, is

$$\boldsymbol{r}(t) = \boldsymbol{dkf}\big(\boldsymbol{q}(t)\big)$$

The considered cost index (to be maximized) is

$$\mathcal{J} = \int_{t_i}^{t_f} \mathcal{L}\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)\big) dt\,,$$

where t_i, t_f and $\boldsymbol{q}(t_i) = \boldsymbol{q}_i$ are given.

The dimension of the redundant space is $n - m$.

Manipulator model

Direct kinematics function

$$\boldsymbol{r}(t) = \boldsymbol{dkf}\left(\boldsymbol{q}(t)\right)$$

Differential kinematics function

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Inverse differential kinematics function

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{r}}(t)$$

Manipulator model

Call $\dot{\boldsymbol{q}}^{\dagger}$ a solution of inverse differential kinematics function.
Suppose $\boldsymbol{\mathcal{P}} \in \mathbb{R}^{n \times n}$ to be such that

$$\mathcal{R}(\boldsymbol{\mathcal{P}}) \equiv \mathcal{N}(\boldsymbol{J})$$

Then

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^{\dagger} + \boldsymbol{\mathcal{P}}\boldsymbol{u}$$

is solution too, with $\boldsymbol{u} \in \mathbb{R}^{n}$ as an arbitrary vector that can generate internal motions.

The inverse differential kinematics function can be replaced by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}(t) + \boldsymbol{\mathcal{P}}\boldsymbol{u}(t)$$

Manipulator model

A possible choice of $\boldsymbol{\mathcal{P}}$ is the matrix that allows the projection of the vector $\boldsymbol{u}$ onto the null space of $\boldsymbol{J}$, so as not to violate the direct kinematics function

$$\boldsymbol{\mathcal{P}} = \boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}$$

Inverse differential kinematics function

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}(t) + \left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\boldsymbol{u}(t) = \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{u}, t)$$

Manipulator model

Fix $\dot{\boldsymbol{r}}(t), \forall t$. A manipulator can be seen as an affine with the respect to the input, time varying system where $\boldsymbol{q}$ is the state, $\boldsymbol{u}$ is the input and $\boldsymbol{r}$ is the output

$$\begin{cases} \dot{\boldsymbol{q}}(t) = \boldsymbol{J}^{\#}(\boldsymbol{q}(t))\dot{\boldsymbol{r}}(t) + \left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}(\boldsymbol{q}(t))\boldsymbol{J}(\boldsymbol{q}(t))\right)\boldsymbol{u}(t) \\ \boldsymbol{r}(t) = \boldsymbol{dkf}(\boldsymbol{q}(t)) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{\boldsymbol{q}} = \boldsymbol{\varphi}(\boldsymbol{q}, t) + \boldsymbol{\Gamma}(\boldsymbol{q})\boldsymbol{u} \\ \boldsymbol{r} = \boldsymbol{\psi}(\boldsymbol{q}) \end{cases}$$

We are looking for the $\boldsymbol{u}$ affecting those $\boldsymbol{q}$ that do not change $\boldsymbol{r}$: i.e. we are analyzing the unobservable distribution of the system.

Pontryagin maximum principle

Find the optimal arbitrary vector $\boldsymbol{u}^o$ and the optimal joint variable vector $\boldsymbol{q}^o$ such that:

- the dynamical constraint $\dot{\boldsymbol{q}}^o = \boldsymbol{f}(\boldsymbol{q}^o, \boldsymbol{u}^o, t)$ is satisfied;
- the final position condition $\boldsymbol{dkf}\left(\boldsymbol{q}(t_f)\right) - \boldsymbol{r}(t_f) = \boldsymbol{\chi}\left(\boldsymbol{q}(t_f)\right) = \mathbf{0}$ is satisfied;
- the performance index $\mathcal{J} = \int_{t_i}^{t_f} \mathcal{L}\left(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)\right) dt$ evaluated in the optimum is maximized,

given

- the initial and final time instants t_i and t_f
- the initial joint variable $\boldsymbol{q}_i$.

It is an optimal control problem of a time-variant, nonlinear dynamical system whereas $\boldsymbol{q}$ is the state vector and $\boldsymbol{u}$ is the input vector

Pontryagin maximum principle

The Hamiltonian is

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{u}, \lambda_0, \boldsymbol{\lambda}, t) = \lambda_0 \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\lambda}^T(t) \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{u}, t)$$

with $\lambda_0 < 0$.

To ensure the absence of singular solutions, the only joint variables considered are the ones that avoid singular configurations of the manipulator, i.e. the set of $\boldsymbol{q}(t)$ such that the Jacobian $\boldsymbol{J}\big(\boldsymbol{q}(t)\big)$ has full rank in the whole time interval. Then

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = -\mathcal{L} + \boldsymbol{\lambda}^T \boldsymbol{f}$$

Pontryagin maximum principle

Necessary conditions for an admissible point $(\boldsymbol{q}^*, \boldsymbol{u}^*)$ to be a maximum are

$$\dot{\boldsymbol{\lambda}}^* = -\left.\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}\right|^{*T}$$

$$\mathcal{H}(\boldsymbol{q}^*, \boldsymbol{\omega}, \boldsymbol{\lambda}^*, t) \leq \mathcal{H}(\boldsymbol{q}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*, t), \forall \boldsymbol{\omega} \text{ admissible}$$

$$\left.\mathcal{H}\right|^{*T} = c \in \mathbb{R}, \forall t \in [t_i, t_f]$$

Since the Hamiltonian is a continuous function of $\boldsymbol{u}$, the second condition can be replaced by

$$\left.\frac{\partial \mathcal{H}}{\partial \boldsymbol{u}}\right|^{*T} = \boldsymbol{0}$$

Boundary conditions

$2n$ boundary conditions are needed:

- n come by fixing the initial joint variable $\boldsymbol{q}_i$
- n are obtained from the transversality conditions at the initial time instant t_i: $\boldsymbol{\lambda}(t_i)$ should intersect orthogonally with the manifold $\boldsymbol{r}(t_i) = \boldsymbol{dkf}(\boldsymbol{q}_i)$

$$\exists\, \boldsymbol{\zeta} \in \mathbb{R}^n : \boldsymbol{J}^T(\boldsymbol{q}_i)\boldsymbol{\zeta} = \boldsymbol{\lambda}(t_i)$$

$$\Updownarrow$$

$$[\boldsymbol{I}_n - \boldsymbol{J}^{\#}(\boldsymbol{q}_i)\boldsymbol{J}(\boldsymbol{q}_i)]\,\boldsymbol{\lambda}(t_i) = \boldsymbol{0}$$

i.e. $\boldsymbol{\lambda}(t_i) \in \mathcal{N}(\boldsymbol{\mathcal{P}})$.

Performance indices

The cost index is

$$\mathcal{J} = \int_{t_i}^{t_f} \{k_0 w(\boldsymbol{q}) + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}}\} dt$$

where $k_0 \geq 0$ weights the secondary cost index.

Manipulability Measure	Distance from Joint Limits
$\sqrt{det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)}$	$-\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}}\right)$
Redundancy is exploited to move away from singularities	Redundancy is exploited to keep the joint variables as close as possible to the center of their ranges

Application of Pontryagin principle

$$\begin{aligned}\mathcal{H}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\lambda},t) &= -k_0 w(\boldsymbol{q}) - \dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T\dot{\boldsymbol{q}} \\ &= -\left(\boldsymbol{f} - \frac{1}{2}\boldsymbol{\lambda}\right)^T\left(\boldsymbol{f} - \frac{1}{2}\boldsymbol{\lambda}\right) + \frac{1}{4}\boldsymbol{\lambda}^T\boldsymbol{\lambda} - k_0 w(\boldsymbol{q})\end{aligned}$$

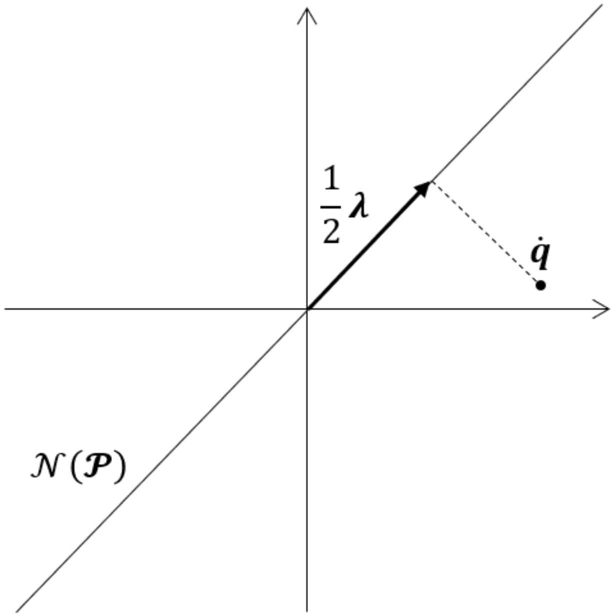
$$\dot{\boldsymbol{\lambda}}^* = -\left.\frac{\partial\mathcal{H}}{\partial\boldsymbol{q}}\right|^{*T} \Rightarrow \dot{\boldsymbol{\lambda}}^* = k_0\left.\frac{\partial w}{\partial\boldsymbol{q}}\right|^{*T} + \left.\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{q}}\right|^{*T}(2\dot{\boldsymbol{q}}^* - \boldsymbol{\lambda}^*)$$

Application of Pontryagin principle

$$\begin{aligned}\mathcal{H}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\lambda},t) &= -k_0 w(\boldsymbol{q}) - \dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T\dot{\boldsymbol{q}} \\ &= -\left(\boldsymbol{f} - \frac{1}{2}\boldsymbol{\lambda}\right)^T\left(\boldsymbol{f} - \frac{1}{2}\boldsymbol{\lambda}\right) + \frac{1}{4}\boldsymbol{\lambda}^T\boldsymbol{\lambda} - k_0 w(\boldsymbol{q})\end{aligned}$$

$$\boldsymbol{0} = \left.\frac{\partial\mathcal{H}}{\partial\boldsymbol{u}}\right|^{*T} \Rightarrow \max_{\boldsymbol{u}} \mathcal{H} = \min_{\boldsymbol{u}} \left\|\boldsymbol{f} - \frac{1}{2}\boldsymbol{\lambda}\right\|$$

Since $\boldsymbol{\lambda} \in \mathcal{N}(\boldsymbol{\mathcal{P}})$, then $\boldsymbol{\lambda}^* = 2proj_{\mathcal{N}(\boldsymbol{\mathcal{P}})}(\dot{\boldsymbol{q}}^*)$



Application of Pontryagin principle

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}} + \left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\boldsymbol{u}$$

$$\boldsymbol{\lambda}^* = 2proj_{\mathcal{N}(\boldsymbol{\mathcal{P}})}(\dot{\boldsymbol{q}}^*)$$

$$\boldsymbol{u}^* = \left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)^{\#}\left(\frac{1}{2}\boldsymbol{\lambda}^* - \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}\right)$$

$$\boldsymbol{u}^* = \frac{1}{2}\left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\boldsymbol{\lambda}^*$$

$$\dot{\boldsymbol{q}}^* = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}} + \frac{1}{2}\left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\boldsymbol{\lambda}^*$$

Application of Pontryagin principle

The problem is described by

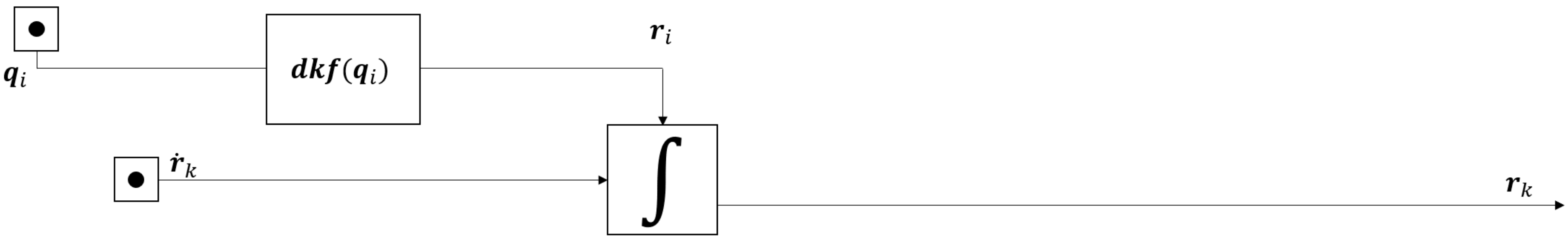
$$\dot{\boldsymbol{q}}^* = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}} + \frac{1}{2}\left(\boldsymbol{I}_n - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\boldsymbol{\lambda}^*$$

$$\dot{\boldsymbol{\lambda}}^* = k_0 \left.\frac{\partial w}{\partial \boldsymbol{q}}\right|^{*T} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right|^{*T} (2\dot{\boldsymbol{q}}^* - \boldsymbol{\lambda}^*)$$

$$\boldsymbol{q}^*(t_i) = \boldsymbol{q}_i$$

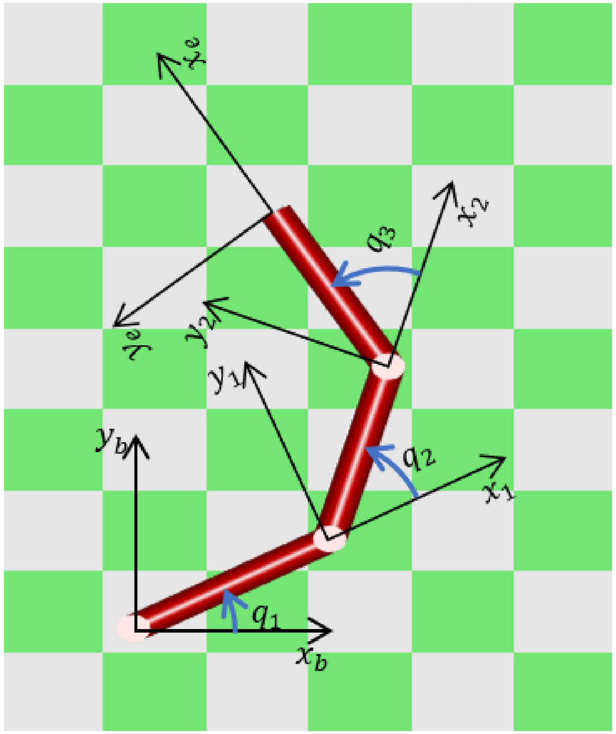
$$\boldsymbol{\lambda}^*(t_i) = 2proj_{\mathcal{N}(\boldsymbol{\mathcal{P}})}\left(\dot{\boldsymbol{q}}^*(t_i)\right)$$

Discretization of the problem



Discretization introduces error, since there is no feedback control action. The smaller the sampling time is, the better the performances of the algorithm are.

Simulations



Joint space dimension: $n = 3$
Operational space dimension: $m = 2$
Degree of redundancy: $n - m = 1$

$\boldsymbol{i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$	$\boldsymbol{d_i}$	$\boldsymbol{\vartheta_i}$
1	$1\,m$	$0\,rad$	$0\,m$	q_1
2	$1\,m$	$0\,rad$	$0\,m$	q_2
3	$1\,m$	$0\,rad$	$0\,m$	q_3

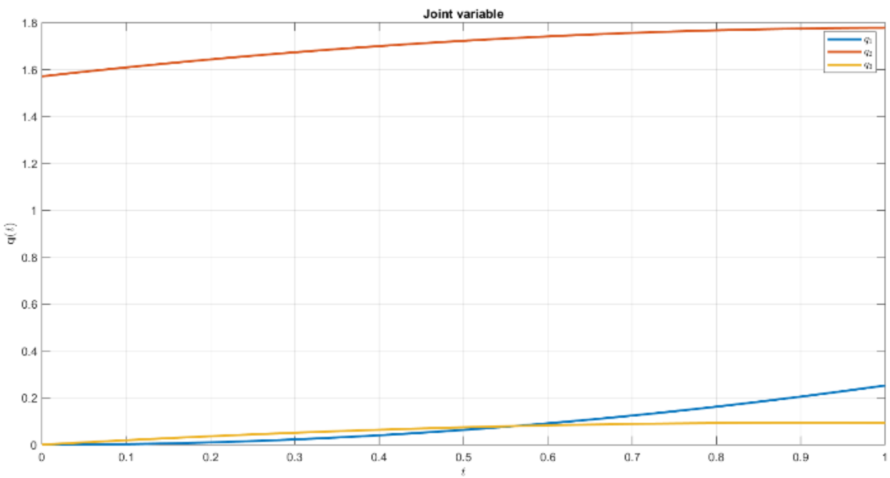
Simulations: common data

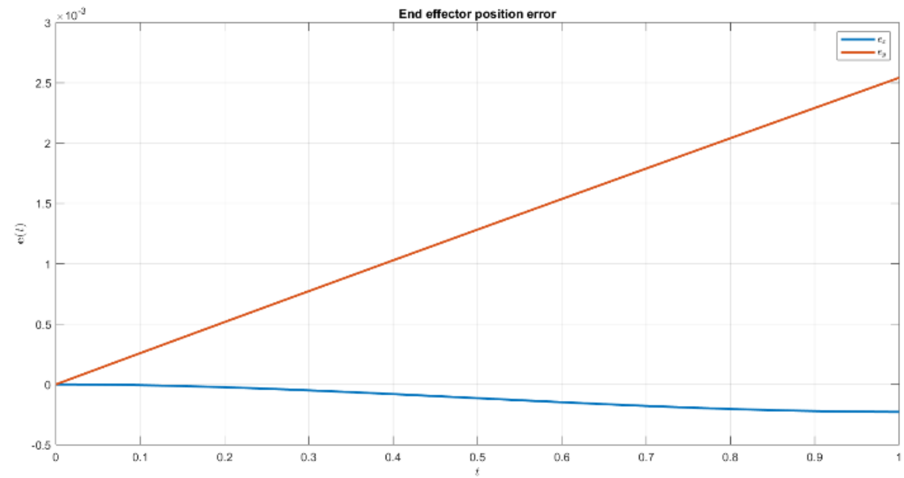
Initial instant [s]	$t_i = 0$
Final instant [s]	$t_f = 1$
Initial configuration [rad]	$\boldsymbol{q}_i = \begin{bmatrix} 0 & \frac{\pi}{2} & 0 \end{bmatrix}^T$
End effector velocity [m/s]	$\boldsymbol{v}_e = [-1 \quad 0]^T$
Sampling time [s]	$\Delta t = 0.01$

$$\boldsymbol{dkf}(\boldsymbol{q}) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{123} + c_{12} + c_1 \\ s_{123} + s_{12} + s_1 \end{bmatrix} \qquad \boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} -s_{123} - s_{12} - s_1 & -s_{123} - s_{12} & -s_{123} \\ c_{123} + c_{12} + c_1 & c_{123} + c_{12} & c_{123} \end{bmatrix}$$

Case 1: no secondary cost index

$$\mathcal{J} = \int_{t_i}^{t_f} \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \, dt$$



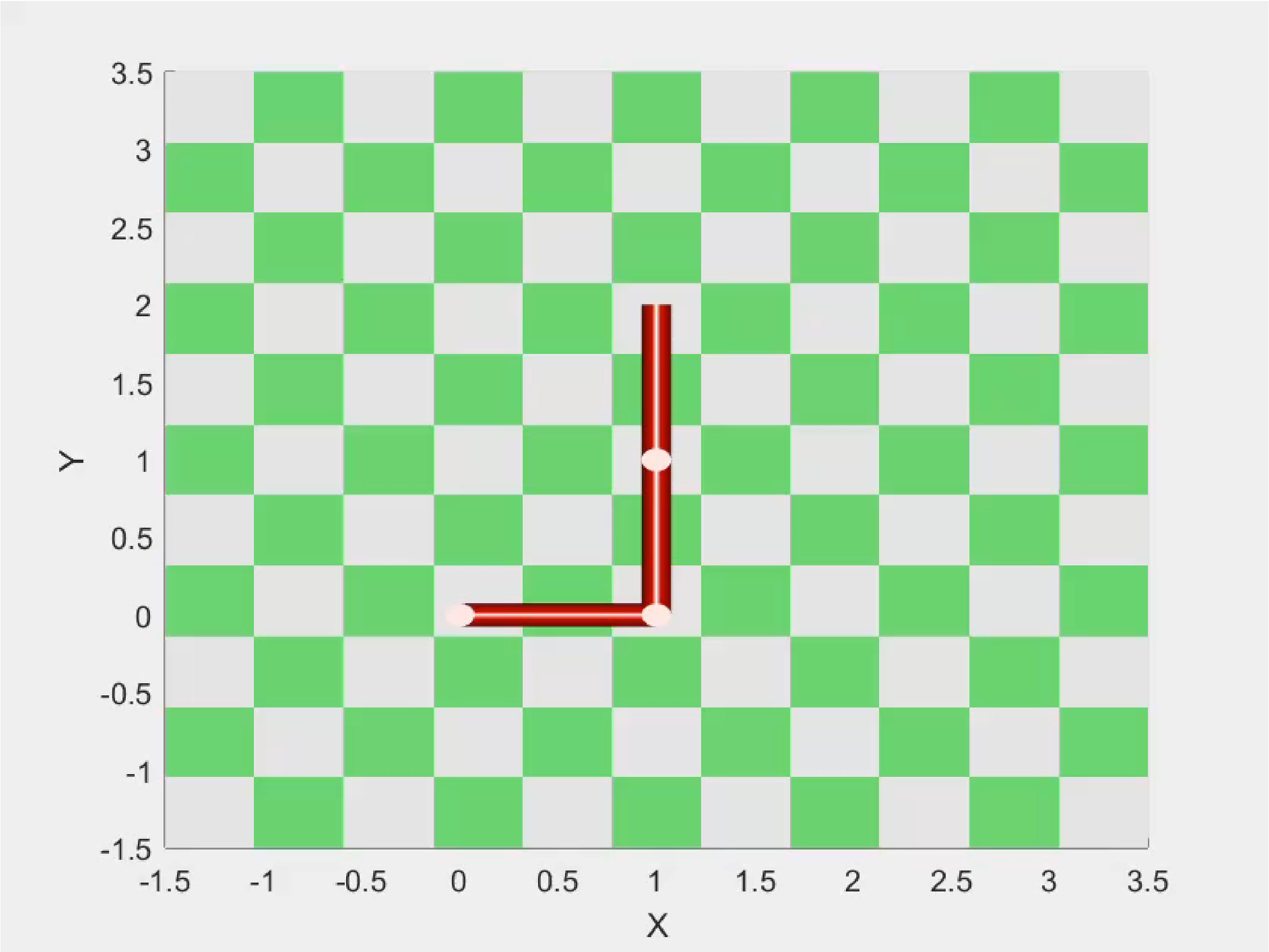


Redundancy is not exploited.

Joint velocities are minimized by means of the pseudoinverse of the Jacobian.

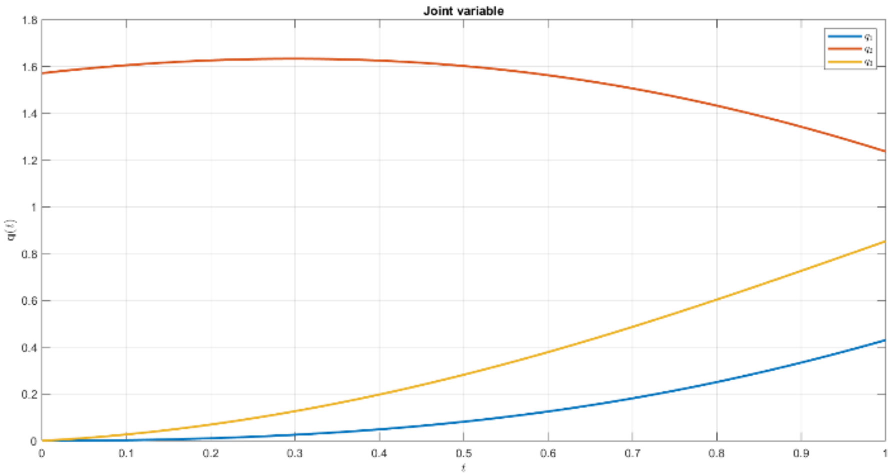
The magnitude of the error is about $0.0026 \, m$.

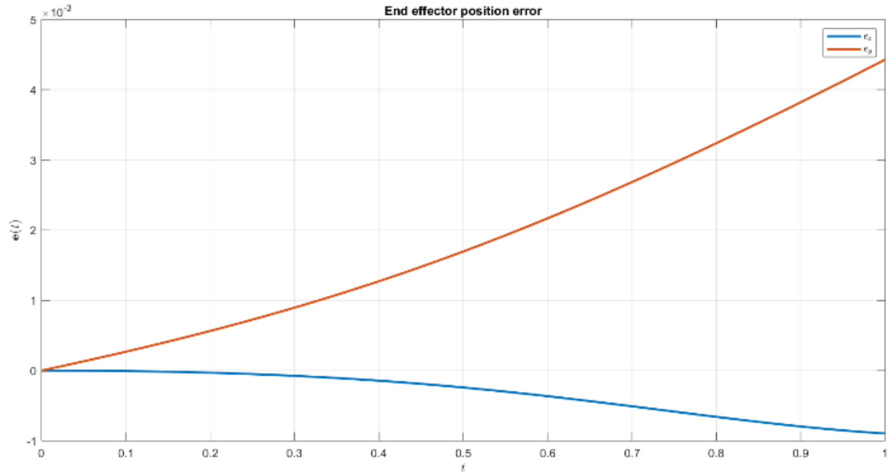
Case 1: no secondary cost index



Case 2: manipulability measure

$$\mathcal{J} = \int_{t_i}^{t_f} \left\{ k_0 \sqrt{det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)} + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \right\} dt$$



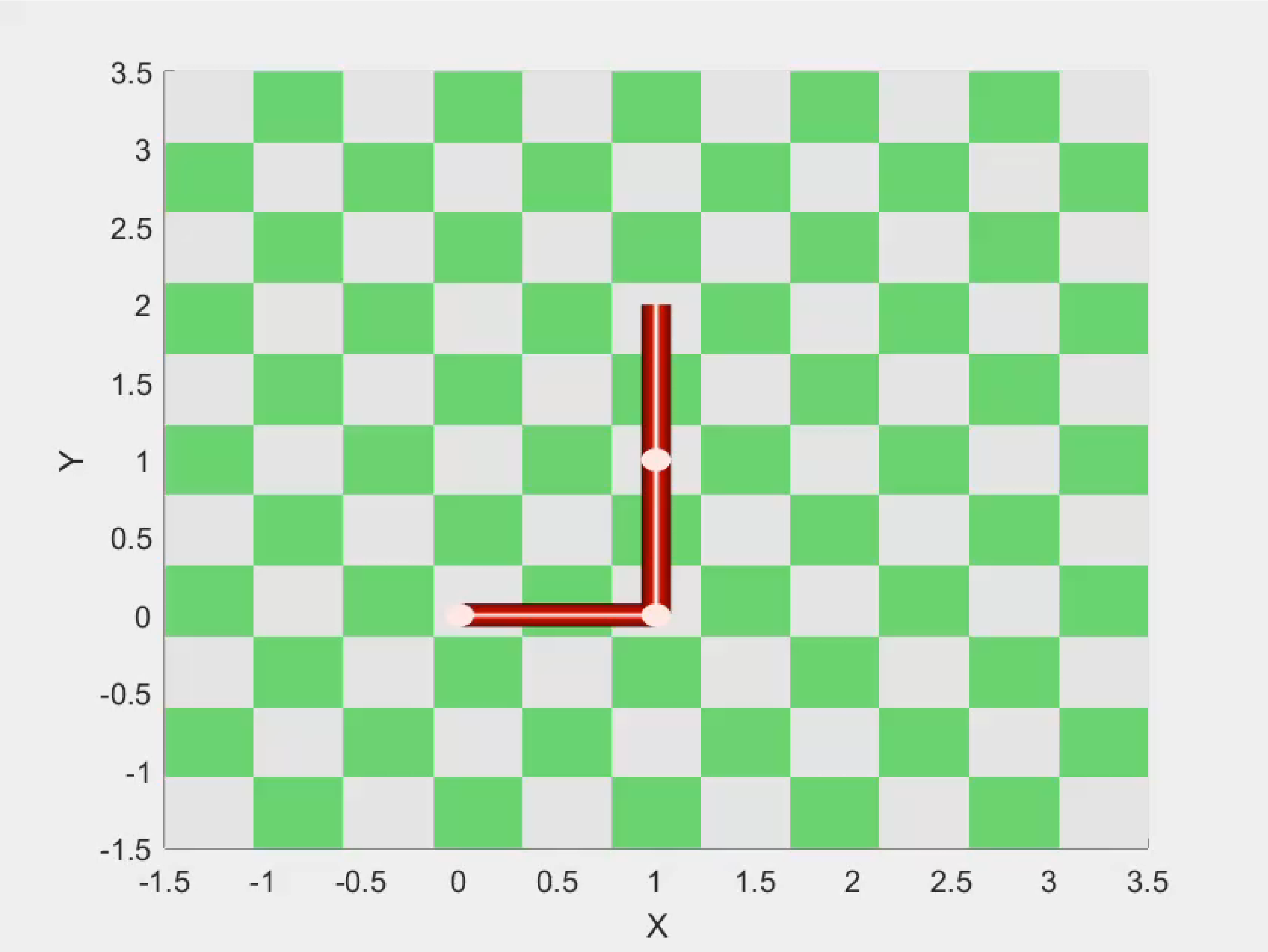


Redundancy is exploited to move away from singularities.

Weight of the manipulability measure: $k_0 = 10$.

Small increase of the magnitude of the error.

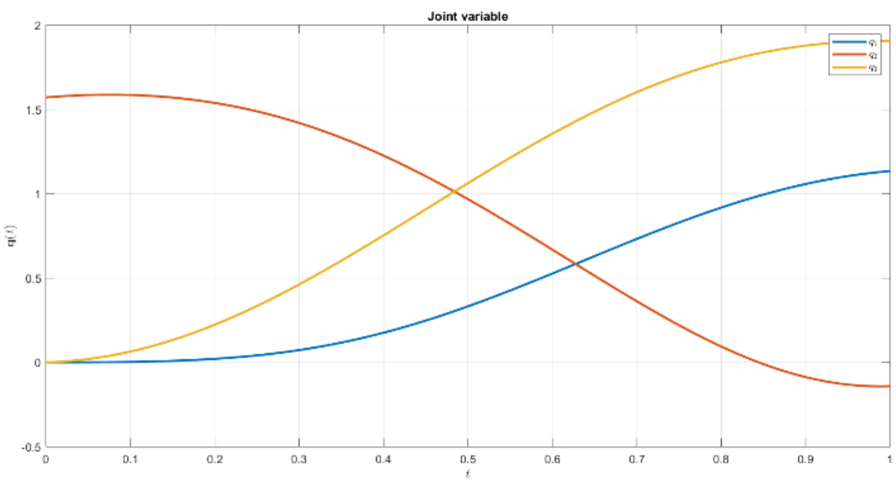
Case 2: manipulability measure

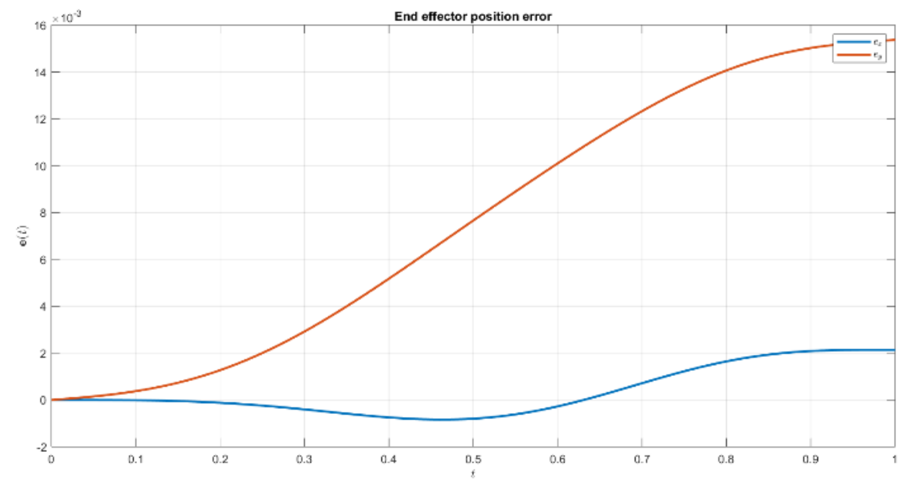


Case 3: distance from joint limits

$$-180° \leq q_1 \leq 180°$$
$$-95° \leq q_2 \leq 95°$$
$$-180° \leq q_3 \leq 180°$$

$$\mathcal{J} = \int_{t_i}^{t_f} \left\{ k_0 \left[-\frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right) \right] + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \right\} dt$$



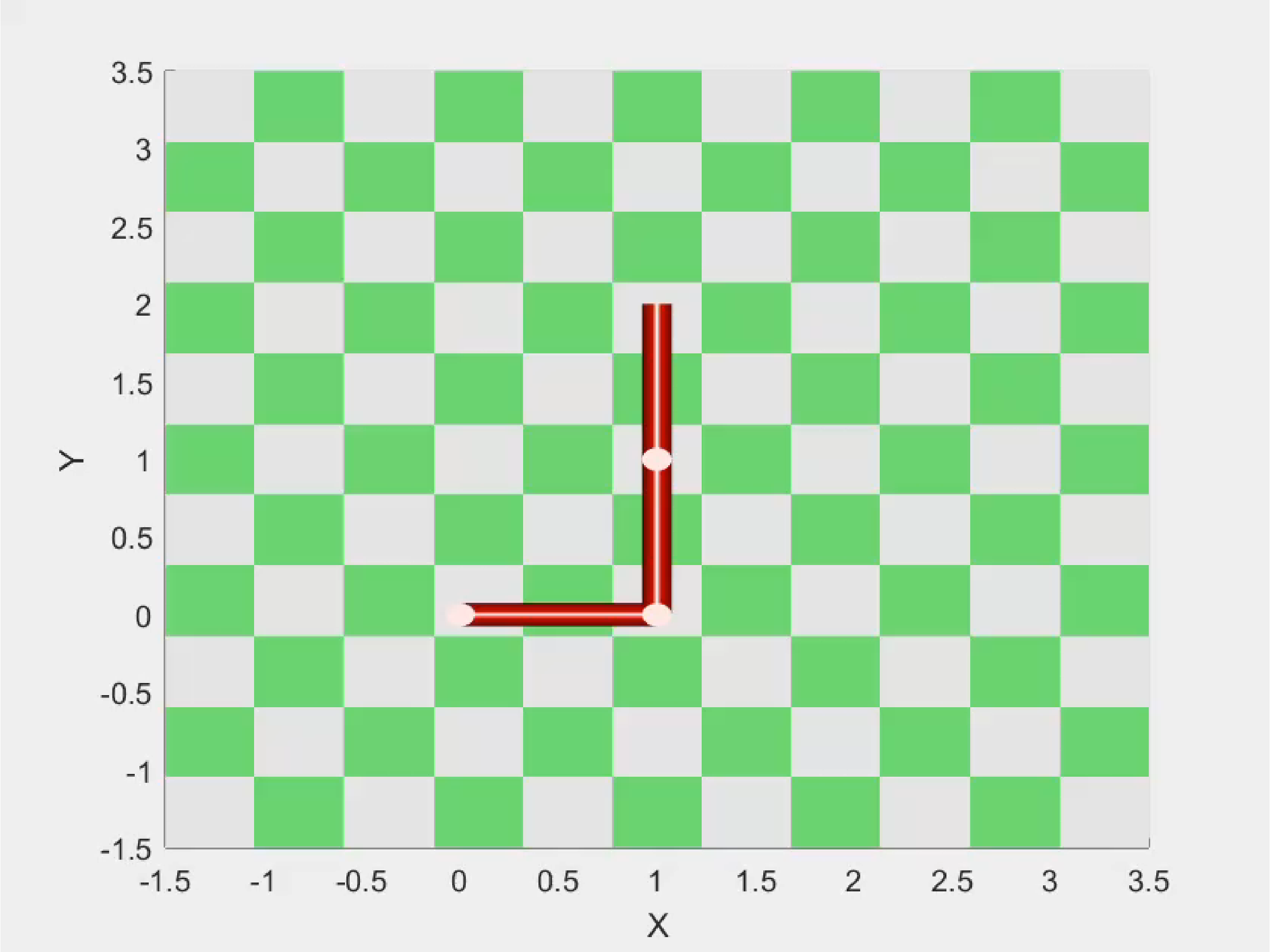


Redundancy is exploited to keep the joint variables close to the mean value of their ranges.

Weight of the distance from joint limits: $k_0 = 1000$.

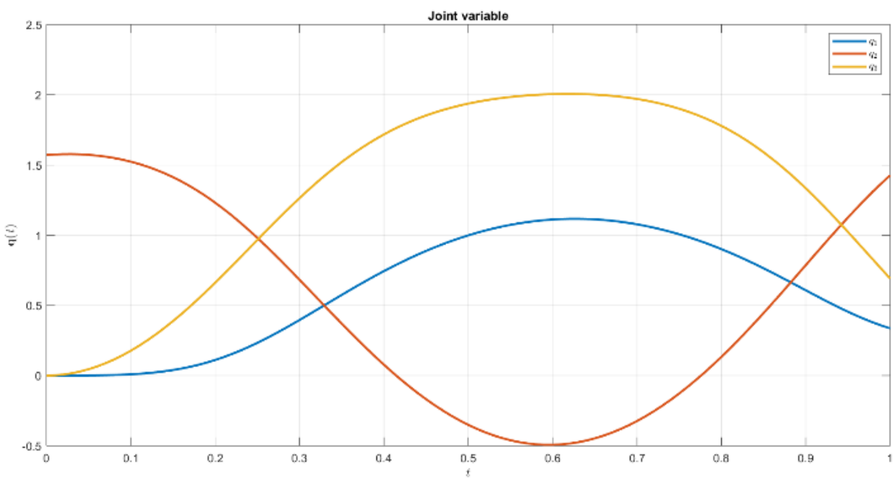
The magnitude of the error is of the order of $10^{-2}\ m$.

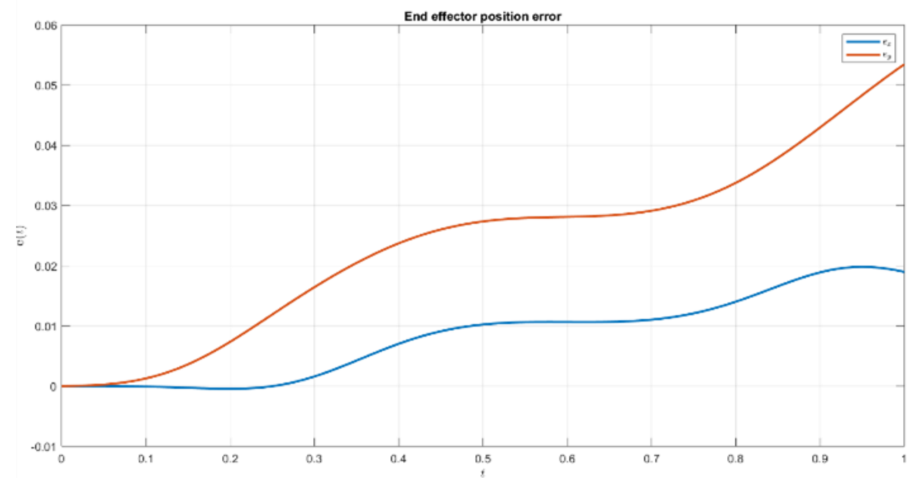
Case 3: distance from joint limits



Case 4a: hybrid secondary cost index

$$\mathcal{J} = \int_{t_i}^{t_f} \left\{ k_0 \left[\sqrt{det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)} - \frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right) \right] + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \right\} dt$$





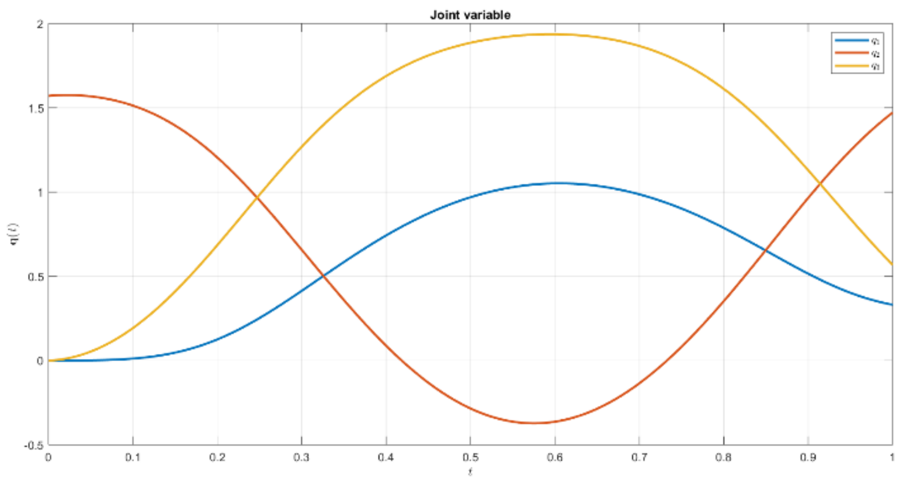
Redundancy is exploited to maximize the sum of the manipulability measure and the distance from joint limits (same as before).

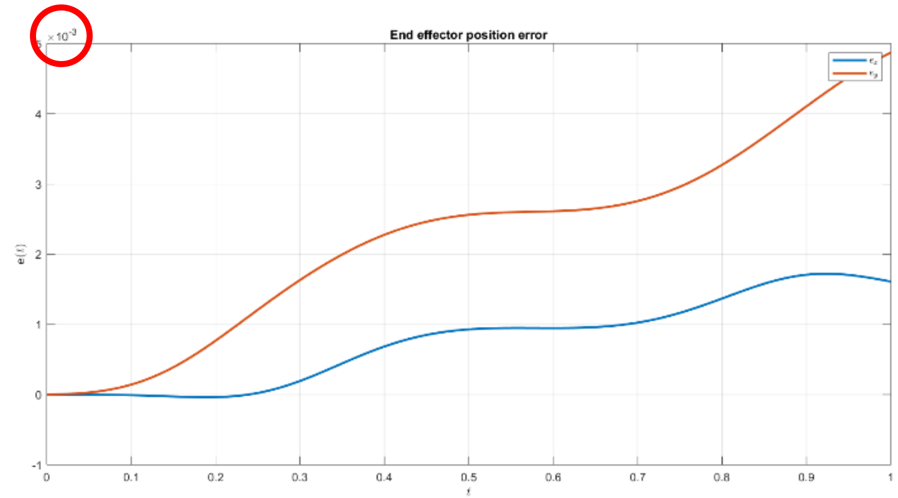
Weight of secondary cost index: $k_0 = 200$.

The magnitude of the error is relevant: $0.06\ m$.

Case 4b: hybrid secondary cost index

$$\Delta t = 0.001\ s \qquad \mathcal{J} = \int_{t_i}^{t_f} \left\{ k_0 \left[\sqrt{det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)} - \frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right) \right] + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} \right\} dt$$



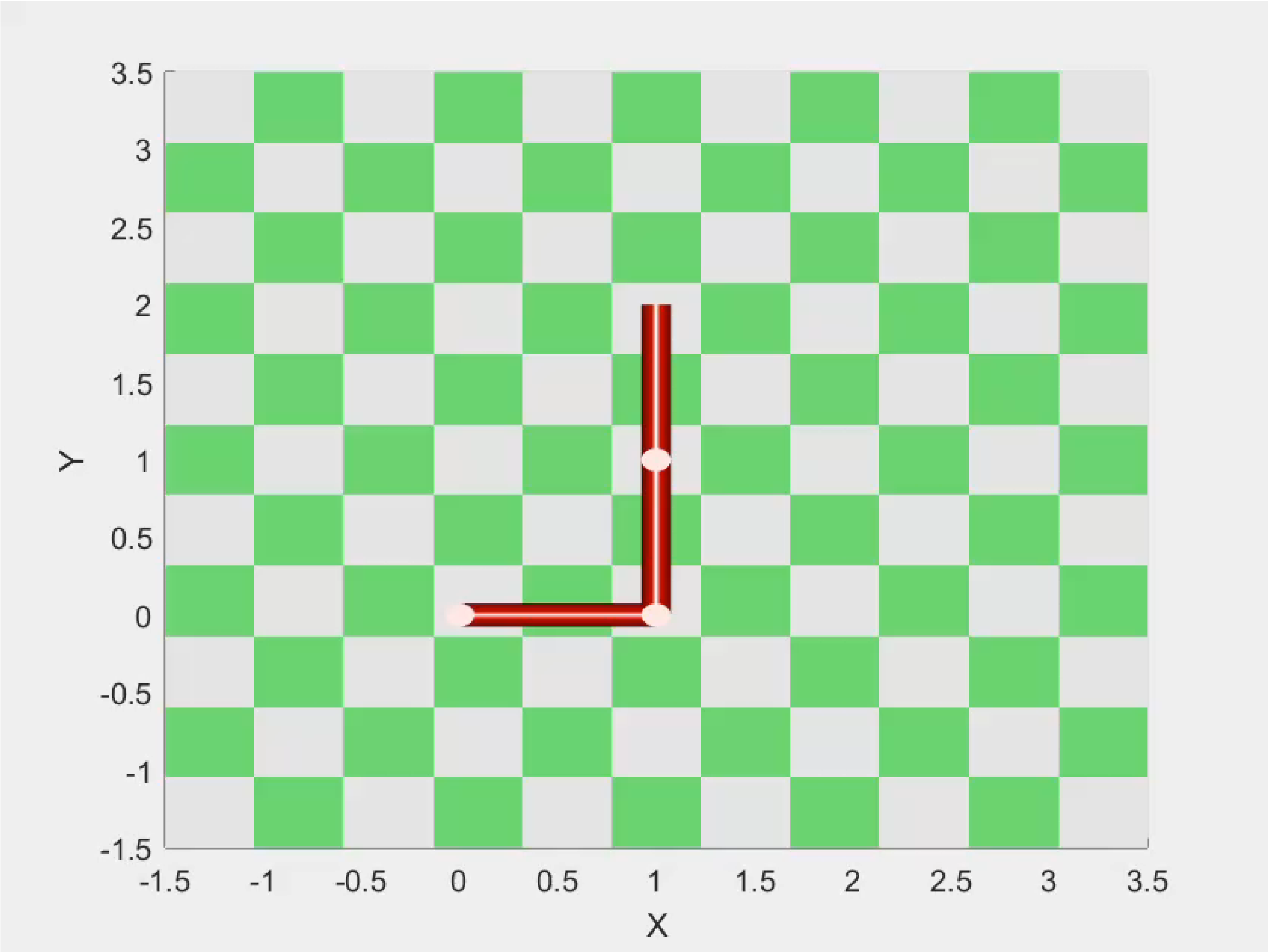


In case 4a, the absence of a controller can not recover the error produced by the discretization: the sampling time has been reduced.

Weight of secondary cost index: $k_0 = 200$ (as before).

The magnitude of the error is reduced: $0.006\ m$.

Case 4b: hybrid secondary cost index



Conclusions

The resolution of kinematic redundancy problem is considered as an optimal control problem and it is solved by the necessary conditions of the Pontryagin theory.

We have achieved the optimal solution described by $2n$ first order partial differential equations with their boundary conditions.

To verify the effectiveness of the proposed algorithm, four simulations on a 3R robot manipulator were shown, with different cost indices and different weights.