Theory of disjunctive attacks, Part I

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Abstract

We discuss properties of what we call argumentation systems with disjunctive attacks, where we have an argument x attacking a set of arguments H, without specifically attacking any specific $y \in H$. In other words, x attacks the conjunction $\bigwedge_{y \in H} y$.

Keywords: Argumentation network, disjunctive attack, representation theorem, conjunctive attack, talmudic logic, collective argumentation, higher level attack.

1 Orientation and discussion

An argumentation network has the form $\mathcal{A} = (S, R)$, where $S \neq \varnothing$ is a set of atomic arguments and $R \subseteq S \times S$ is the attack relation. The objects serving as targets of attacks are single elements $y \in S$. In practice as well as in theory, there are cases where attacks target non-empty sets of elements. This means we need a relation $\rho \subseteq S \times (2^S - \{\varnothing\})$.[1] We recall several examples where such attacks arise.

REMARK 1.1
Although this article is motivated through argumentation theory, the focus is on the notion of an 'attack'. Thus we are more concerned with the properties of the attack relation itself rather than the status of its relata as arguments. We shall therefore use the term 'argument' liberally to cover entities that may be seen as directly (and possibly asymmetrically) conflicting with each other. These could be interpreted as assertions, propositions, results, readings, information, potential states of affairs, decisions, promises and so on. The corresponding notion of attack will be different for different interpretations: e.g. decisions might *overrule* other decisions, whereas a reading might *cast doubt on* another reading Our main interest here is modelling disjunctive attacks, however 'attack' might be interpreted.

Note that the range of possible understandings of 'attack', in our initial motivating examples of Section 1, significantly exceeds our actual technical study of later sections made in the (quite specific) formal argumentation framework of 'Caminada labellings'. Our aim in this Part 1 is to remain in the framework of abstract argumentation and provide a meaningful generalizations to the disjunctive case.

E1. Elections

There are many cases of fraud in counting votes in elections. There are ballot boxes attached to geographical voting areas and people vote by putting a ballot paper in the box. Sometimes the

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[1] Where $A - B = \{x \mid x \in A \;\&\; x \notin B\}$.

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number of votes exceeds the number of registered voters. Thus, the legitimacy of the votes as a set in the ballot box is attacked without attacking any specific item. (If the item is written upon or is blank it is specifically attacked.)

E2. Fibring argumentation networks

In 2009 [12], we introduced the notion of fibring networks. The idea is that given a $y \in S$, we can substitute a network $\mathcal{A}_y = (S_y, R_y)$ for y. Any attack on y, say by x, will be an attack on $\mathcal{A}_y$. There are options of how to define this attack. If, e.g., $\mathcal{A}_y$ intends to justify y (so we have $y \in S_y$) by presenting an extension $E_y \subseteq S_y$ containing y, then the attack of x is on E_y. For example, the attack might be in the form of claiming that E_y is not the grounded extension, and that the ground extension does not contain y, and therefore the choice of E_y as a justification for y really amount to an arbitrary choice of y.

E3. Attacking a conjunction

Given $\mathcal{A} = (S, R)$ the elements of S may be instantiated by arguments from a logic **L**. We instantiate the element x by a theory Δ_x. We may have that Δ_x equals x and Δ_y equals y and xRy because in **L**, $\Delta_x \vdash_{\mathbf{L}} \neg \bigwedge \Delta_y$. Many logics can prove the negation of a conjunction without proving the negation of any specific conjunct. For example in classical logic

$$\vdash \neg(A \wedge \neg A)$$

but neither $\vdash \neg A$ or $\vdash \neg\neg A$. So again , we have here a disjunctive attack on Δ_y.

E4. Disjunctive attacks in Talmudic logic

Mr Smith is a rich old man who wants to donate a very rare classic painting to one of two national museums. He committed the donation in a letter to the charity commission and said hat he would inform them which museum he would choose in a few days. Mr Smith unfortunately died before he made that choice. We are now left with an unclear legal situation regarding the ownership.

Let
$a =$ the painting does belong to museum a.
$b =$ the painting does belong to museum b.

We have, of course, that a and b are mutually exclusive. Therefore, we have that truth $\top$ disjunctively attacks the set $\{a, b\}$. Talmudic logic debate distinguishes several views on this scenario. The facts on the ground are that Mr Smith's estate claims that there was no donation, as no museum was chosen. The museums claim that there was a donation and there can be a deal worked out, such as co-ownership or sharing, or we can let the estate continue and choose a museum, we can flip a coin, or …whatever. Talmudic logic debate offers two main views on this.

View 1. Quantum like view. This view is that ownership is super-imposed on both museums in the same sense as, nowadays, modern quantum mechanics treats the two slits experiment. A single electron is sent towards two slits a and b and the electron passes through both slits as a wave and interferes with itself. So even though logically the electron can pass only through one slit, as a wave it passes through both. The people holding this view are divided in their verdict.

Option 1. Since a and b are mutually exclusive, there is no donation.

Option 2. There is a donation but both museums should waive their 'ownership' back to the estate, for otherwise we are stuck. After that, the estate can re-donate the painting if they want to.

Of course these options have implications towards estate tax duties, etc.

View 2. Fuzzy probabilistic view. There was a donation and we view the scenario as if there was a choice of museum, except that we do not know what it was, i.e. we treat the case as if Mr Smith did choose a museum, wrote a letter but died and the letter was lost.

So we are expected to provide some mechanism to divide/allocate the painting. For example

(1) share ownership 50/50;
(2) make a case for one museum over the other. For example, if the painting was in the special area of museum a, so we claim that there is a high probability that a was chosen;
(3) recommend other arrangements, such as time sharing, etc.

Note that in this story there is an attack on the set $\{a, b\}$ without there being any specific attacks on a or on b. The story can continue as follows.

Suppose each of a and of b, independently attack c. We have several options for reasoning here

(1) c must be out, since either a or b is in.
(2) c must be in since the disjunctive attack is super-imposed on both a and b.
(3) c is undecided since we do not know exactly what is going on with $\{a, b\}$.

1.1 Motivating complete extensions for disjunctive attacks

We now address the problem of how to define extensions for a network with disjunctive attacks. We believe a good starting point is to check if we can modify the Caminada labelling approach for traditional Dung networks and extend the definition to the case of disjunctive attacks. We shall be using some key examples to discuss various possible options for understanding the nature of disjunctive attacks and try to discover some useful intuitive patterns.

The basic traditional (Dung) pattern for a network of the form (S, R) with $R \subset S \times S$, is that of Figure 1.

In this figure, x is directly attacked by each of $y_1, \ldots, y_k$. When we say x is 'directly attacked' by y, we mean that yRx holds and no other side elements are involved. So for a disjunctive attack relation $\rho \subseteq S \times (2^S - \{\varnothing\})$, a direct attack of y on x means $y\,\rho\,\{x\}$.

The traditional legitimate Caminada labelling $\lambda : S \mapsto \{\text{in, out, und}\}$ for the situation of Figure 1—exemplifying a network (S, R)—requires 3 conditions.

(C1) $\lambda(x) =$ in, if either x has no attackers, or all of its attackers are out (i.e. there are no y_i s.t. yRx or for all such y_i $\lambda(y_i) =$ out).

(C2) $\lambda(x) =$ out, if for some $y_i, \lambda(y_i) =$ in.

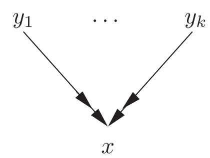

FIG. 1.

(C3) $\lambda(x) =$ undecided, if for all $i, \lambda(y_i) \neq$ in and for some $i, \lambda(y_i) =$ und.

Let us ask what the rationale of clause (C3) is.

(C3) is the mathematical complement of (C1) and (C2), i.e. (C3) $= \neg$ (C1) $\wedge \neg$ (C2). If (C1) does not hold then some y_i is not out. If (C2) does not hold then all y_i are not in. Therefore $\neg$ (C1) $\wedge \neg$ (C2) means that none of the y_i are in and one of them, say y_0 is nether in nor out, i.e. $y_0 =$ und, where und $= \neg$ in $\wedge \neg$ out. So in this case, since neither (C1) nor (C2) hold for x, then x is undecided. This is (C3).

Remark 1.2

We comment here in anticipation of Section 5. Given a traditional network (S, R) with $R \subset S \times S$, let us call any pair $(y, x) \in R$ a *direct attack formation* on x. This formation is in if y is in; it is out if y is out; and it is undecided if y is undecided. Thus condition (C2) on λ is a local condition. It tells us when the labelling of a single attack formation is legitimate for λ. Condition (C1) is a global condition, it concerns the totality of all attack formations on x. It tells us when the totality of attack formation for x is to be considered legitimate for λ.

We now want to motivate conditions on λ for the case of disjunctive attacks. There may be more than one way to generalize our conditions for the disjunctuve case. So we have to consider some examples and use our common sense and intuition. The different options to generalize may not be the same mathematically.

Let us sharpen the problem we are facing also by looking at it from the set theoretic point of view. When we have a network with disjunctive attacks, the attack relation is between single points z and sets of points H. The traditional set theoretic concept of complete extension for a network (S, R), with $R \subseteq S \times S$ applies to a set of points $E \subseteq S$. The traditional definition of such an extension E involves the notion of 'E is conflict free and E protects all of its members and E contains all points which it protects' protecting a point x is defined using R. So what does this mean in the context of disjunctive attacks? We are given the relation $z \rho H$ of 'z attacks a set H', not the relation of zRx 'z attacks a point x'. So we need to define the notion of 'protect' using the geometric relation of $x \rho H$, which is given to us. The way we understand/derive the notion of 'protect' in this context is the way we view/understand the notion of disjunctive attacks.

Now consider Figure 2. This figure describes a single (indirect) attack of z on x, namely $z \rho \{x, u_1, \ldots, u_r\}$. This is a single attack formation on x (see Remark 1.2). The attack is not direct because $u_1, \ldots, u_r$ are also involved. x is in danger because $u_1, \ldots, u_r$ and z might all be in and so x would have to be out. We must give conditions for when this attack formation is legitimate for λ.

We offer 3 initial views:

The minimalist approach: $z =$ in requires one of $\{x, u_1, \ldots, u_r\}$ to be out. If more of them are out then it is because of other attacks.

This is a local condition, which is parallel to condition (C2) above for the traditional case. The minimalist approach does not say anything more about the global conditions corresponding to the totality of the attack formations on x, (parallel to (C1) of the traditional network).

The conjunctive approach: $z =$ in requires that if all but one y in $\{x, u_1, .., u_r\}$ are in then this y is out. For example, if all of $\{u_1, \ldots, u_r\}$ are in then x is out. It is called the conjunctive approach because it requires that for each $y \in \{x, u_1, \ldots, u_r\}$ we have that $(\{z, x, u_1, \ldots, u_r\} - \{y\})$ attacks y.

Again, notice that this is a local condition on the legitimacy of the local attack formation. However since it converts the local attack formation into conjunctive attacks, a global condition may be implicitly implied.

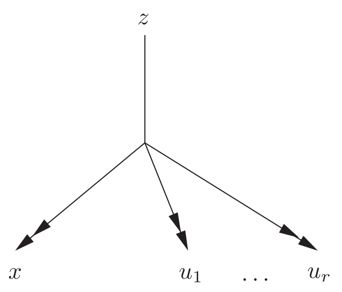

FIG. 2.

The open approach: For some fixed non-empty subset H of $\{x, u_1, .., u_r\}$, $z =$ in requires all members of H to be out.

This reduces the attack formation of Figure 2, to several traditional attack formations of the form $\{(z,h) \mid h \in H\}$, and therefore it explicitly implies both local and global conditions on λ.

Some examples would help.

EXAMPLE 1.3 (The minimalist approach)
Consider Figure 3. The attacks on x compose of only one attack:

$$\text{Attack}(x) : z\,\rho\,\{x, u\}.$$

The attacks on u are two attacks.

$$\text{Attack}_1(u) : z\,\rho\,\{x, u\}$$
$$\text{Attack}_2(u) : u\,\rho\,\{u\}$$

There are also attacks $y\,\rho\,\{z\}$ an $z\,\rho\,\{y\}$.

Let us list the extensions λ_i we can have:

λ_1: $y =$ in, $z =$ out, $x =$ in, $u =$ und (In this case , since z is out, we get x, being unattacked, is in and u, being self attacking, is und).

λ_2: $y =$ out, $z =$ in, $x =$ in, $u =$ out (In this case there is a minimalist attack by z on $\{x, u\}$. The minimalist approach 'offers' u to be out. Therefore z no longer has any attack interest on x and so x is in).

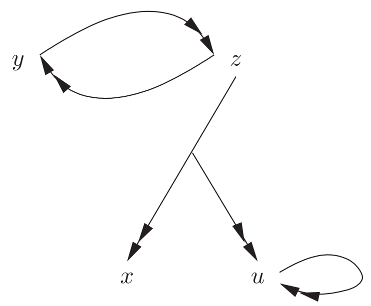

FIG. 3.

λ_3: $y=$ out, $z=$ in, $x=$ out, $u=$ und (In this case there is a minimalist attack by z on $\{x,u\}$. The minimalist approach 'offers' x to be out. Therefore z no longer has any attack interest on u. But u is self attacking and so u is und).

λ_4: $y=$ und, $z=$ und, $x=$ und, $u=$ und (In this case z is undecided. z is undecided because of the loop $\{y,z\}$. z might be in or it might be out. If z is in then x might be in as in λ_1 or λ_2 and u is out or und. If z is out then x could be out as in λ_3 and u is und. Either way since z is und we must have $x=u=$und , because of the above options).

What we cannot have in the minimalist approach is the extension λ_5:

λ_5: $y=$ out, $z=$ in, $x=$ out, $u=$ out (we cannot have this extension because more than one of the attacked is out!)

We now have to examine the view of a disjunctive attack of x on a set H as simply attacking the conjunction of the elements of H and does not care how many of H are out. This is the Open (non-minimalist) approach. According to this view, in Example 1.5, the extension λ_5 is also acceptable.

To continue our investigations and the examination of various view for disjunctive attacks and further compare with the conjunctive approach, we need to introduce formally conjunctive attacks, and we therefore move to the next subsection.

1.2 Comparison with conjunctive attacks

We saw in Section 1.1 that the disjunctive attack of x on a set H can be viewed as an open (non-minimalist) attack on the conjunction $\bigwedge_{y\in H} y$. So if $x=$ in, then at least one of the $y\in H$ must be out. This also means (taking the contra-positive of the condition) that if all $y\in H$ are in then x cannot be in. So we ask can disjunctive attacks be reduced to conjunctive attacks?

We need to describe conjunctive attacks (see [12, 25]). Figure 4 describes the basic configuration for conjunctive attacks

The relation of attack is between sets H of nodes and single nodes x. The labelling conditions are

(CC1*) The attack of H on x is in, if the labels of all $y\in H$ are in.
(CC2*) The attack of H on x is out, if some $y\in H$ is labelled out.
(CC3*) The attack of H on x is undecided, if none of the labels of $y\in H$ is out and at least one $y\in H$ is labelled undecided.

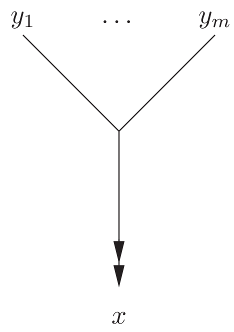

FIG. 4.

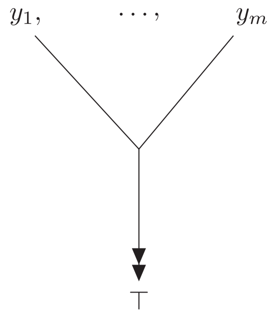

FIG. 5.

REMARK 1.4

(1) Note a connection between conjunctive attacks and disjunctive attacks. Consider Figure 4 again. Suppose we insist (i.e. put a constraint on labellings) that $x = \text{in}$. In this case we must have that one of the y_i is out. This may not be possible if the entire argumentation network is composed of just Figure 4, then all of y_i are unattacked and so are in, so x must be out. The constraint $x = \text{in}$ cannot be satisfied.

(2) We need to define what we mean by finding an extension relative to a constraint. Let $(S, \mathbb{R})$ be a conjunctive network with $\mathbb{R} \subseteq (2^S - \{\varnothing\}) \times S$.
Let $\lambda_1 : E \mapsto \{\text{in}, \text{out}, \text{und}\}$ where $E \subseteq S$, then any λ is a legitimate extension subject to the constraints of λ_1 if λ is legitimate (e.g. according to (CC1*)-(CC3*) above) and $\lambda_1 \subseteq \lambda$. Such a λ may not exist

(3) We note that constrains of the form $\lambda_1(x) = \text{in}$ can be simply implemented by the substitution/instantiation $x = \top$. So Figure 4 becomes Figure 5.

(4) The question arises, given a network with disjunctive attacks can we replace them with conjunctive attacks using $\top$? This will be investigated in Section 4. Much depends on the version of disjunctive attack which is involved. The disjunctive attack obtained from the configuration of Figure 5 seems to be similar or maybe equal to the open non-minimalist approach.
Let us see what values $y_1, \dots, y_m$ can get which will be accpetable with the value $x = \text{in}$ $(x = \top)$.

   (a) If one or more of y_i are out, then this is acceptable.
   (b) If at least one y_i is undecided and none of y_i is out, this is not acceptable, because x should be undecided in this case and the constraint on x is $x = \text{in}$ $(x = \top)$.

(5) In anticipation of Section 4, let us ask ourselves one more question. Suppose it is undecided whether in Figure 5 the attack arrow is live or dead (in higher level attacks, the arrow itself might be attacked by some undecided attacker, thus becoming undecided). What are the acceptable options in this case? Can $y_1 = \text{und}$ and $y_j = \text{in}$ for $j \geq 2$, be acceptable?
Well, if there is an attack on $\top$, then y_1 must be out (since all the other $y_j, j \geq 2$ are in). If there is no attack then y_1 can be any value as far as this particular figure is concerned. So we may say that this set of values is acceptable if the attack is undecided.
To compare with Figure 4, we ask whether, if we insist on $x = \text{und}$, is setting the values $y_1 = \text{und}$ and $y_j = \text{in}$ for $j \geq 2$ acceptable? The answer is that it is.

EXAMPLE 1.5
We now consider the simple example of Figure 6

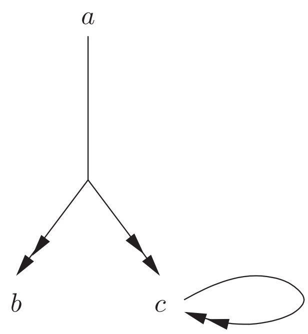

FIG. 6.

This figure describes a disjunctive attack. The extensions are (according to the minimalist view) the following:

λ_1: $a =$ in, $b =$ out, $c =$ und
λ_2: $a =$ in, $b =$ in, $c =$ out.

The following candidate λ_3 is *not* an extension.

λ_3: $a =$ in, $b =$ und, $c =$ und

The obvious reduction of a disjunctive attack, perceived according to the minimalist view, to a system with conjunctive attacks is to convert

$$x \twoheadrightarrow \{y_1, \ldots, y_m\}$$

into the family of attacks

$$\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m, x\} \twoheadrightarrow y_i.$$

$$x \twoheadrightarrow H$$

converts to the family

$$\{x\} \cup (H - \{y\}) \twoheadrightarrow y,$$

for each $y \in H$.

Figure 7 does the conversion for Figure 6. Does it capture the right meaning? The answer is no.

In Figure 6, the extension λ_3 is allowed, but λ_1 is not allowed.

The reader might ask, what is then the meaning of disjunctive attacks as perceived according to the minimalist view in terms of conjunctive attacks? We notice that in Figure 7, we put both conjunctive attacks in the same figure, so let us change that. Let us try to separate the disjunctive attack into two figures, each with a single conjunctive attack. Look at Figures 8 and 9.

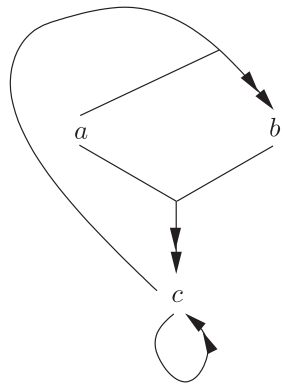

FIG. 7.

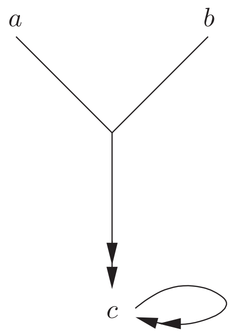

FIG. 8.

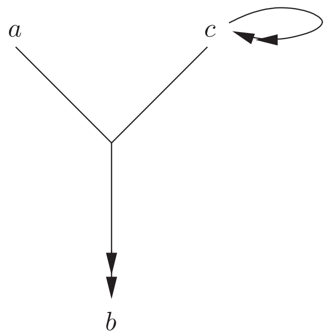

FIG. 9.

Can we say that λ is an extension of Figure 6 iff it is an extension of one of Figures 8 or 9? Well, λ_2 corresponds to Figure 8 but λ_1 does not correspond to Figure 9, because in this figure we must have $b =$ und.[2]

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[2]We shall give a definition of a complete extension for disjunctive attacks in Section 3, which allows for the truth of the statement 'λ is an extension of Figure 6 iff it is an extension of one of Figures 8 or 9'. This is actually an example of the conjunctive approach.

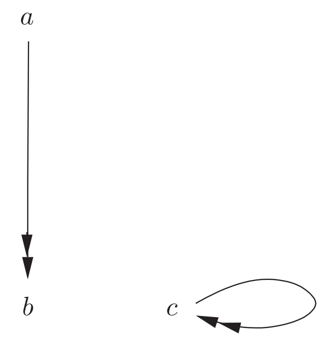

FIG. 10.

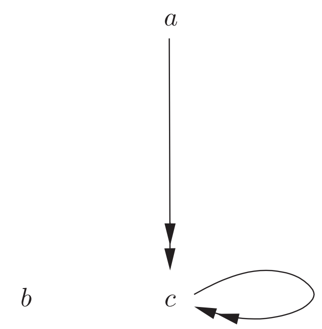

FIG. 11.

REMARK 1.6

Our examination with the previous Example 1.5, gives us a new idea. Let us try a different split. This is still for the minimalist approach to the disjunctive network of Figure 6. The attack in Figure 6 says

- $a =$ in $\rightarrow$ ($b =$ out $\vee$ $c =$ out), reading $\vee$ as an exclusive disjunction.

which is equivalent to the exclusive disjunction of two attacks.

- ($a =$ in $\rightarrow b =$ out) $\vee$ ($a =$ in $\rightarrow c =$ out).

So let us split Figure 6 into two figures, Figure 10 and Figure 11.
λ_1 is the extension of Figure 10 and λ_2 is the extension of Figure 11. Indeed, we have

- λ is an extension for Figure 6 iff λ is an extension of one of the split figures 10 or 11.

This seems to be a good idea. In fact, we shall give a new definition and a new point of view and we shall seek a possible theorem in a later section. At this point, we do not even know if every disjunctive network has an extension.

REMARK 1.7

This remark tries to refine further our understanding of the minimalist approach in contrast to the open (non-minimalist) approach. Consider the network of Figure 12

In this figure we must have $x =$ in and $b' =$ in and so $b =$ out.

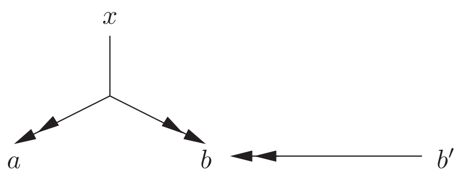

FIG. 12.

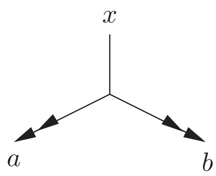

FIG. 13.

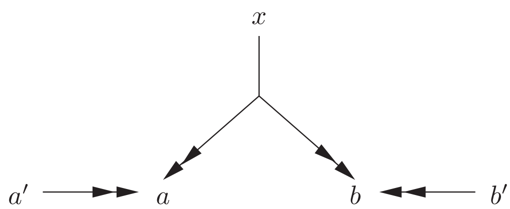

FIG. 14.

Now x being in requires, because of the disjunctive attack $x\rho\{a,b\}$, that one of a or b be out. We already have $b=$ out. Our question is the following:

Question: Do we view a as being attacked by x and therefore we are not obligated to let $a=$ in? Or do we view a as not attacked (since $b=$ out) as in the minimalist approach, and therefore a must be in?

Put differently, consider Figure 13. Can we accept $x=$ in, $a=b=$ out as a legitimate extension? It is an acceptable extension in the open non-minimalist approach, but then what about $x=$ in, $a=$ und, $b=$ out?

In the minimalist view we regard a as not attacked because $b=$ out already, then a should be in. So this is not an extension. So this view practically says that $x\rho\{a,b\}$ requires exactly one of $\{a,b\}$ to be out. In contrast, in the open view, a can still be attacked, this possibility is not excluded, but then, can we also say in the open view that $a=$ und?

In the minimalist view what do we do with Figure 14?

We can change our formulation and say that when $x\rho H$, we want exactly one $y\in H$ to be out, unless more are explicitly attacked by other nodes.

This is not good either, because consider Figure 15. In this figure a,b will be explicitly attacked if we choose λ such that $\lambda(\beta)=\lambda(\delta)=$ out. Should we not choose such a λ?

It seems we might get ourselves into a conceptual mess. We might play safe and say in case of $x\rho H$ that all elements of H are considered attacked, whether successfully or not. So in Figure 12, we also allow for $x=$ in, $b'=$ in, $b=$ out and $a=$ in, or $a=$ out, but still, do we allow $a=$ und?

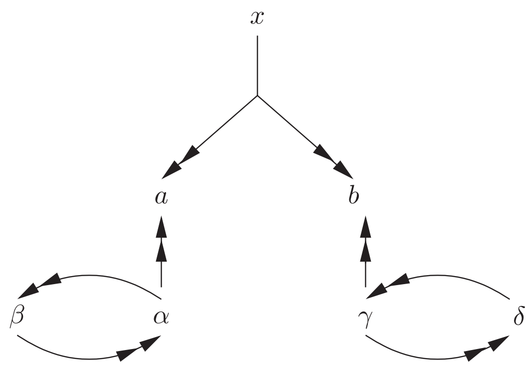

FIG. 15.

2 Formal definitions of disjunctive attacks

Following the discussions in Section 1, it is time to give mathematical definitions for acceptable labellings for networks with disjunctive attacks. We shall deal primarily with networks where the disjunctive attacks emanate from single points but the results are equally valid for networks with conjunctive–disjunctive attacks, where the disjunctive attacks emanate from sets of points. As we have seen in Section 1, there are several options, we choose here the one that is, conceptually speaking, most independent from the particular properties of the traditional Caminada labelling. It is the one inspired by Remark 1.6

DEFINITION 2.1

(1) A conjunctive argumentation network has the form $(S, \mathbb{R})$, where $S \neq \varnothing$ is the set of arguments and $\mathbb{R} \subseteq (2^S - \{\varnothing\}) \times S$ is the attack relation.
(2) A legitimate labelling $\lambda : S \mapsto$ {in, out, und} is a function satisfying conditions (CC1)–(CC3) below:

(CC1) $\lambda(x) =$ in if for each H such that $H\mathbb{R}x$ there exists a $y \in H$ such that $\lambda(y) =$ out.
(CC2) $\lambda(x) =$ out if for some H such that $H\mathbb{R}x$ we have that for al $y \in H, \lambda(y) =$ in.
(CC3) $\lambda(x) =$ und, if neither (CC1) nor (CC2) hold. Namely

(a) For some $H\mathbb{R}x$, we have that for all $y \in H, \lambda(y) \neq$ out.
(b) For all H such that $H\mathbb{R}x$, there eis a $y \in H$ such that $\lambda(y) \neq$ in.

PROPOSITION 2.2
Let $(S, \mathbb{R})$ be a conjunctive network. Then there exist legitimate labelling for it.

PROOF. Well known, by [18, 19]. ∎

DEFINITION 2.3

(1) A disjunctive argumentation network has the form (S, ρ), where S is a non-empty set and $\rho \subseteq S \times (2^S - \{\varnothing\})$. See Figure 16.
(2) Say f is a choice function on S, if $f(y, Y) \subseteq Y$ and $f(y, Y) \neq \varnothing$ for $y \rho Y$.
(3) Let $\lambda : S \mapsto$ {in, out, und}. We say that λ is a legitimate Caminada–Gabbay labelling for (S, ρ) iff there is a choice function f such that the following conditions hold:
(D1) $\lambda(x) =$ in iff for every y, Y s.t. $y \rho Y$ and $x \in Y$, $\lambda(y) =$ out or $x \notin f(y, Y)$.

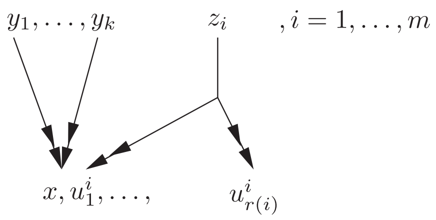

FIG. 16.

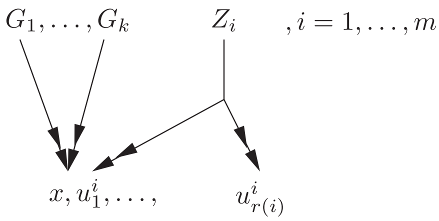

FIG. 17.

(D2) $\lambda(x) =$ out iff for some y, Y, $\lambda(y) =$ in and $x \in f(y, Y)$.

(D3) $\lambda(x) =$ und iff there are no z, Z s.t. $\lambda(z) =$ in and $x \in f(z, Z)$, and there are also y, Y s.t. $x \in f(y, Y)$ and $\lambda(y) =$ und.

It is straightforward to show that $(D3)$ holds at x iff neither $(D1)$ nor $(D2)$ hold at x.

REMARK 2.4

We now define the notion of conjunctive–disjunctive networks. There are presented in Definition 2.5 and Figure 17. The reduction theorems of this section will be formulated and proved for ordinary disjunctive networks of Definition 2.3 and Figure 16, but they equally hold for conjunctive–disjunctive networks. This is because the reduction process of these theorems simply eliminates the disjunction attack part in favour of the familiar point to point (resp. set to point) attacks.

In [8], Bochman calls conjunctive–disjunctive networks by the name 'collective argumentation networks', and he handles them differently from us, and we shall compare the two approaches in Section 5

DEFINITION 2.5

(1) A conjunctive–disjunctive argumentation network has the form (S, ϱ), where S is a non-empty set and $\varrho \subseteq (2^S - \{\varnothing\}) \times (2^S - \{\varnothing\})$. See Figure 16.

(2) Say f is a choice function on S, if $f(G, Y) \subseteq Y$ and $f(G, Y) \neq \varnothing$ for $G \varrho Y$.

(3) Let $\lambda : S \mapsto \{\text{in, out, und}\}$. We say that λ is a legitimate Caminada–Gabbay labelling for (S, ϱ) iff there is a choice function f such that the following conditions hold:

(CD1) $\lambda(x) =$ in iff for every G, Y s.t. $G \varrho Y$ and $x \in Y$, either $\lambda(y) =$ out for some $y \in G$ or $x \notin f(G, Y)$.

(CD2) $\lambda(x) =$ out iff for some G, Y, $\lambda(y) =$ in for all $y \in G$ and $x \in f(G, Y)$.

(CD3) $\lambda(x) =$ und iff there are no G, Z such that: $\lambda(z) =$ in for all $z \in G$, $x \in f(G, Z)$, and there are also G', Y s.t. $x \in f(G', Y)$ and for all $y \in G'$ $\lambda(y) \neq$ out with some $\lambda(y) =$ und.

It is largely straightforward to show that $(CD3)$ holds at x iff neither $(CD1)$ nor $(CD2)$ hold at x.

REMARK 2.6
Definition 2.3 committed us to one way of understanding disjunctive attacks. There are other ways, such as the 'attack' view of Section 1 leading to the minimalist view and the set theoretic view of Section 3. We also have the higher level view of the Example 4.1 in Section 4; and moreover the Talmudic approach of [1].

How all of these options are related needs to be investigated. The position we adopted in Definition 2.3 is a refinement of the open view, as inspired by Remark 1.6, but made specific and presented as a reductionist position. We view the attack of $y\rho Y$ as a pre-meditated attack on each element of some non-empty subset $f(y,Y)\subseteq Y$). This subset is not known and so we must consider all possible subsets $f(y,Y)$ for all pairs $y\rho Y$. This type of attack is well known in modern employment practices (especially in academia). If a manager wants to dismiss an employee or a group of employees, in many cases they cannot simply be sacked. The manager has to dismiss the entire section and let them all reapply and then, naturally, some will not be successful. So if a university wants to close a department, they would disjunctively 'attack' the whole school, and make them reapply even though it is already known (only to the manager) that department will be 'out'.

We now proceed to prove that every finite disjunctive network has at least one open view reductionist extension (as defined in Definition 2.3). Our proof uses induction on the number of disjunctive attacks in the network. The inductive step reduces the number of disjunctive attacks by eliminating one of them in favour of direct attacks. So if we have a network $\mathcal{A}=(S,\rho)$, with disjunctive attacks, we take one such attack say $s\twoheadrightarrow H, x\in S$, where $H\subseteq S$ and $x\rho H$ and split it into the direct attacks $\{x\twoheadrightarrow y|y\in H\}$. We reduce $\mathcal{A}=(S,\rho)$ into the set $\{\mathcal{A}_y|y\in H\}$, where $\mathcal{A}_y=(S,\rho_y)$ with $\rho_y=\big(\rho-\{(x,H)\}\big)\cup\{(x,\{y\})\}$. (In the same way that the network of Figure 6 was split into two networks of Figure 10 and Figure 11). Using this method we will be able to completely characterize all the extensions of a disjunctive network in terms of extensions of traditional Dung networks.

DEFINITION 2.7
(1) Let $\mathbb{F}(S)$ be a family of networks of the form $(S,R_i), i\in I$, based on the same S. Call $\mathbb{F}(S)$ a flat bundle of networks based on S. Let λ be a Caminada labelling on S. We say that λ is an extension for the bundle $\mathbb{F}(S)$ iff λ is a legal Caminada labelling for at least one of the networks of $\mathbb{F}(S)$.

DEFINITION 2.8
Let $A=(S,\rho)$ be a disjunctive network.

(1) A function $\mathbf{f}$ is said to be a selection function for $\mathcal{A}$ if for every pair (x,H) such that $x\rho H$ holds, the function $\mathbf{f}$ selects a set of pairs $(x,\{y\}), y\in H$.
(2) Define $\mathbb{F}(\mathcal{A})$, the disjunctive flattening of $\mathcal{A}$, to be the family of networks of the form $\mathcal{A}_\mathbf{f}=(S,R_\mathbf{f})$, where $R_\mathbf{f}=\bigcup_{x\in S,H\subseteq S}\mathbf{f}(x,H)$.
In other words the $R_\mathbf{f}$ are all the ways of selecting pairs $(x,\{y\})$ on the condition that for each x and each H, if $(x,H)\in\rho$ (i.e. x attacks H) then there is at least one $y\in H$ s.t. $(x,\{y\})\in R_\mathbf{f}$
(3) Note that if all the attacked sets H contain only one element then $\mathcal{A}_\mathbf{f}=\mathcal{A}$.

THEOREM 2.9 (Representation theorem)
Let $\mathcal{A}=(S,\rho)$ be a *finite* disjunctive network and let $\mathbb{F}(\mathcal{A})$ be its disjunctive flattening. Then for any Caminada labelling λ we have:
λ is an extension of $\mathcal{A}$ iff λ is an extension for $\mathbb{F}(\mathcal{A})$.

PROOF.
Part 1. We argue that if λ is legitimate for $\mathcal{A}$ then it is legitimate for some $\mathcal{A}'\in\mathbb{F}(\mathcal{A})$.

If λ is legitimate for $\mathcal{A}$ then there is a choice function f such that the conditions ($D1$)–($D3$) of 16 are met by λ and f. Define $\mathcal{A}' = (S, \rho')$ where:

$$\rho' = \{(x, \{y\}) | x \rho X \text{ and } y \in f(x, X) \text{ for some } X\}.$$

Then clearly $\mathcal{A}' \in \mathbb{F}(\mathcal{A})$. Define f' over $\mathcal{A}'$ so that $f(x, \{x\}) = x$. We claim that λ and f' meet conditions ($D1$)–($D3$) at all nodes, and is hence legitimate. Let x be an arbitrary node in $\mathcal{A}'$, there are three cases.

Case $\lambda(x) = \text{in}$**.** Then ($D1$) holds in $\mathcal{A}$: for every y, Y s.t. $y \rho Y$ and $x \in Y$, $\lambda(y) =$ out or $x \notin f(y, Y)$. But then if $\{y, \{x\}\} \in \rho'$, $x \in f(y, Y)$ for some Y and so $\lambda(y) =$ out.

Case $\lambda(x) = \text{out}$**.** Then ($D2$) holds in $\mathcal{A}$: for some y, Y, $\lambda(y) =$ in and $x \in f(y, Y)$. Then, $(y, \{x\}) \in \rho'$ and condition ($D2$) is met as $\lambda(y) =$ in and $x \in f'(x)$.

Case $\lambda(x) = \text{und}$**.** Then ($D3$) holds in $\mathcal{A}$: (i) there are no z, Z s.t. $\lambda(z) =$ in and $x \in f(z, Z)$, and (ii) there are also y, Y s.t. $x \in f(y, Y)$ and $\lambda(y) =$ und. Then from the definition of ρ' and (i) there can be no z s.t. $\lambda(z) =$ in and $z \rho' \{x\}$; also, from (ii), for some y, $\lambda(y) =$ und and $(y, \{x\}) \in \rho'$, then since by definition $x \in f'(y, \{x\})$, ($D3$) is met.

Part 2. We argue that if λ is legitimate for some $\mathcal{A}' \in \mathbb{F}(\mathcal{A})$ then λ is legitimate for $\mathcal{A}$.

If λ is legitimate for $\mathcal{A}' \in \mathbb{F}(\mathcal{A})$ then there is a unique choice function f such that the conditions ($D1$)–($D3$) of 16 are met by λ and f (as $x \rho X$ implies X is a singleton). If $(x, X) \in \mathcal{A}$ then set $f'(x, X) = \{y \in X | (x, \{y\}) \in \rho'\}$. We claim that λ and f' meet conditions ($D1$)–($D3$) at all nodes, and is hence legitimate. Let x be an arbitrary node in $\mathcal{A}$, again there are three cases.

Case $\lambda(x) =$ **in.** Then ($D1$) holds in $\mathcal{A}'$. But then if $\{y, Y\} \in \rho$ and $x \in f(y, Y)$ then $(y, \{x\}) \in \rho'$ and so $\lambda(y) =$ out.

Case $\lambda(x) =$ **out.** Then ($D2$) holds in $\mathcal{A}'$: Then for some y, $(y, \{x\}) \in \rho'$ and so for some Y s.t. $x \in Y$, $(y, Y) \in \rho$. Moreover $x \in f'(y, Y))$ and so condition ($D2$) is met since $\lambda(y) =$ in.

Case $\lambda(x) =$ **und.** Then ($D3$) holds in $\mathcal{A}$: (i) there are no z s.t. $\lambda(z) =$ in and $z \rho' \{x\}$, and (ii) there is also a y s.t. $y \rho' \{x\}$ and $\lambda(y) =$ und. Then from (i) there can be no z, Z s.t. $\lambda(z) =$ in, $x \in f'(z, Z)$; also, from (ii) for some y, Y, $\lambda(y) =$ und, $(y, Y) \in \rho$ and, since $(y, \{x\}) \in \rho'$, $x \in f'(y, Y)$ and so ($D3$) is met. ∎

EXAMPLE 2.10
Consider Figure 18.

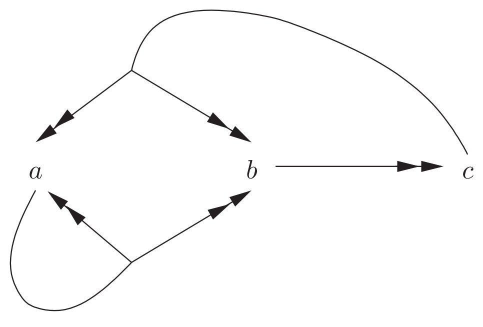

FIG. 18.

In this figure we have $a\,\rho\,\{a,b\}, b\,\rho\,\{c\}, c\,\rho\,\{a,b\}$.

There are 9 possible flattening networks for this figure. The first 5, which we shall denote by $\mathbf{g}_1,\ldots,\mathbf{g}_5$, involve the flattening of either $a \twoheadrightarrow \{a,b\}$ into $\{a \twoheadrightarrow a$ and $a \twoheadrightarrow b\}$ or of $c \twoheadrightarrow \{a,b\}$ into $\{c \twoheadrightarrow a$ and $c \twoheadrightarrow b\}$ or both. The second batch of 4 more, which we denote by $\mathbf{f}_1,\ldots,\mathbf{f}_4$ involve the flattening of any $x \twoheadrightarrow \{y,z\}$ into either $x \twoheadrightarrow y$ or $x \twoheadrightarrow z$ but not both. This is a requirement in Talmudic Logic. They require the choice of one only, i.e. $x \twoheadrightarrow \{y_1,\ldots,y_k\}$ can be flattened (or 'collapsed', as they call it in [1]) to either of $\{x \twoheadrightarrow y_i\}$.

So let us list what we have. The function g are as follows:

$$\begin{aligned}
&\mathbf{g}_1 : a \twoheadrightarrow a, b \twoheadrightarrow c, c \twoheadrightarrow a, c \twoheadrightarrow b\\
&\mathbf{g}_1 : a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow a, c \twoheadrightarrow b\\
&\mathbf{g}_1 : a \twoheadrightarrow a, a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow a\\
&\mathbf{g}_1 : a \twoheadrightarrow a, a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow b\\
&\mathbf{g}_1 : a \twoheadrightarrow a, a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow a, c \twoheadrightarrow b.
\end{aligned}$$

The 5 flattening networks corresponding to $\mathbf{g}_1,\ldots,\mathbf{g}_5$ are Figures 19, 20, 21, 22 and 23.

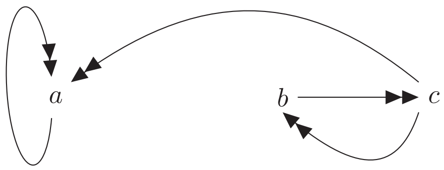

FIG. 19.

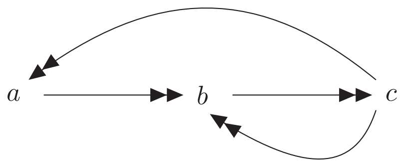

FIG. 20.

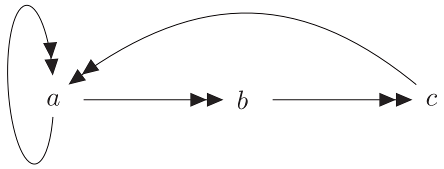

FIG. 21.

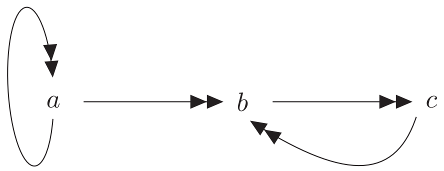

FIG. 22.

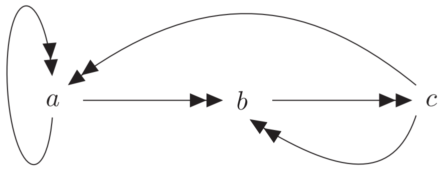

FIG. 23.

The extensions are as follows:

$\mu_{1,1}$: $a = b =$ out, $c =$ in
$\mu_{1,2}$: $a =$ und, $b =$ in, $c =$ in
μ_2: $a = b = c =$ und
μ_3: $a = b = c =$ und
$\mu_{4,1}$: $a =$ und, $b =$ out, $c =$ in
$\mu_{4,2}$: $a = b = c =$ und
μ_5: $a = b = c =$ und.

The functions **f** are as follows:

$$\mathbf{f}_1 : a \twoheadrightarrow a, b \twoheadrightarrow c, c \twoheadrightarrow a$$
$$\mathbf{f}_2 : a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow a$$
$$\mathbf{f}_3 : a \twoheadrightarrow a, b \twoheadrightarrow c, c \twoheadrightarrow b$$
$$\mathbf{f}_4 : a \twoheadrightarrow b, b \twoheadrightarrow c, c \twoheadrightarrow b.$$

The four flattening networks corresponding to $\mathbf{f}_1, \ldots, \mathbf{f}_4$ are Figures 24, 25, 26 and 27.
The extensions are as follows:

λ_1: $a =$ und, $b =$ in, $c =$ out
λ_2: $a = b = c =$ und
λ_3: $a =$ und, $b =$ out, $c =$ in
λ_4: $a =$ in, $b =$ out, $c =$ in.

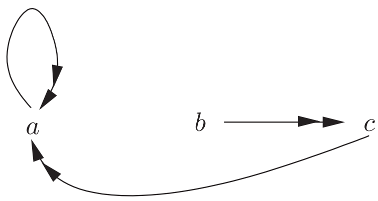

FIG. 24.

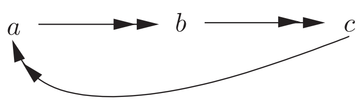

FIG. 25.

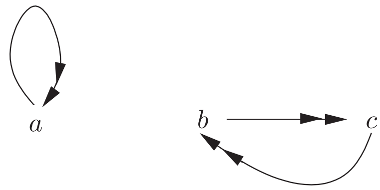

FIG. 26.

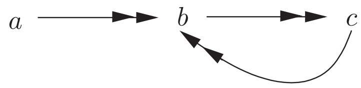

FIG. 27.

REMARK 2.11

We saw that every finite disjunctive network $\mathcal{A}=(S,\rho)$ can be reduced to a flat bundle family $\mathbb{F}(\mathcal{A})$ of traditional networks of the form $\mathcal{A}_i=(S,R_i), i\in I$.

The question arises, can we go also in the other direction, namely given a flat bundle family $\mathbb{F}$ of traditional networks on the same S, say for simplicity two of them, $\mathcal{A}_1=(S,R_1)$ and $\mathcal{A}_2=(S,R_2)$, is there a single disjunctive network $\mathcal{A}=(S,\rho)$ such that for any labelling λ on S we have the equivalence below?

- λ is legitimate for $\mathcal{A}$ off λ is legitimate for $\mathcal{A}_1$ or for $\mathcal{A}_2$.

Let us look at an example. Consider (S,R_1) of Figure 28 and (S,R_2) of Figure 29.

So

$$R_1=\{(x,a),(x,b)\}$$
$$R_2=\{(b,x),(x,x),(x,a)\}.$$

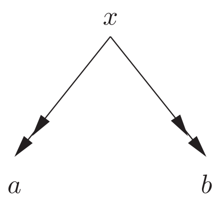

FIG. 28.

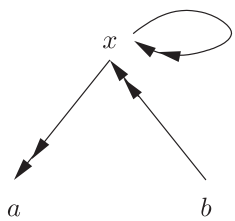

FIG. 29.

The extension for Figure 28 is

$$\lambda_1 : x = \text{ in}, a = \text{ out}, b = \text{ out}$$

and for Figure 29 we have

$$\lambda_2 : x = \text{ out}, a = \text{ in}, b = \text{ in}.$$

We want to find an (S, ρ) which has exactly λ_1 and λ_2 as extensions. We insist that ρ must be defined using R_1 and R_2 in a systematic, not *ad hoc* way.

To do that we need new notation.

When we write $x \rho \{a, b\}$ we mean a disjunctive attack on $\{a, b\}$, meaning

- $x = \text{ in} \to (a = \text{ out}) \vee (b = \text{ out})$.

Let us agree that when we write $x \rho [[a, b]]$, we mean the conjunction of $x \twoheadrightarrow a$ and $x \twoheadrightarrow b$, namely:

- $x = \text{ in} \to (a = \text{ out}) \wedge (b = \text{ out})$.

Using this notation, we can write

$$\begin{array}{l} R_1 \text{ as } x \rho_1 [[a, b]] \\ R_2 \text{ as } b \rho_2 [[x]] \text{ and } x \rho_2 [[x, a]]. \end{array}$$

This would mean, according to what we agreed

- $x = \text{ in } \to ([[a, b]] = \text{ out}) \vee ([[x, a]] = \text{ out})$

or, if we continue

- $x = \text{ in } \to ((a = \text{ out}) \wedge (b = \text{ out})) \vee ((x = \text{ out}) \wedge (a = \text{ out}))$.

Figure 30 describes this configuration.

So the reduction of (S, R_1) and (S, R_2) to a simple level disjunctive network (S, ρ_3) is fully displayed in Figure 31.

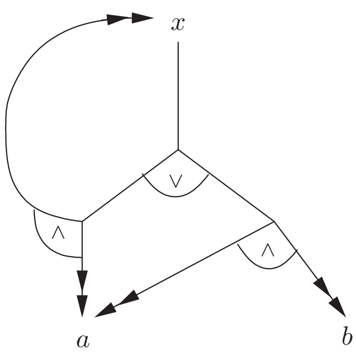

FIG. 30.

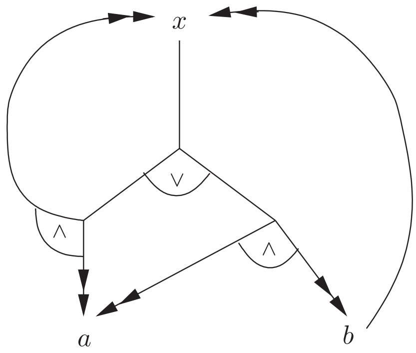

FIG. 31.

We have the network (S,ρ_3):

$$x\,\rho_3\,\{[[a,b]],[[x,a]]\}$$
$$b\,\rho_3\,\{[[x]]\}.$$

Once we have this notation, we can iterate and use it to combine networks which have disjunctive attacks in them. For example, combine Figures 32 and 33.

We have (S,ρ_1) and (S,ρ_2).

$$\rho_1=\{(x,\{a,b\}),(b,\{x\})\}$$
$$\rho_2=\{(x,\{a\}),(a,\{b\}),(a,\{x,b\})\}.$$

We have the combined network (S,ρ_3)

$$x\,\rho_3\,\{[[\{a,b\}]],[[\{a\}]]\}$$
$$a\,\rho_3\,\{[[\{b\},\{b,x\}]]\}$$
$$b\,\rho_3\,\{[[\{x\}]]\}.$$

There is a suitable representation theorem for this case but we leave this to a continuation paper, [20].

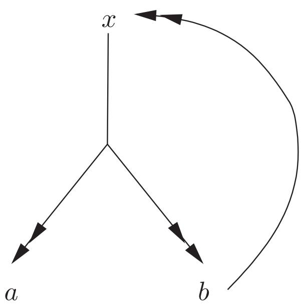

FIG. 32.

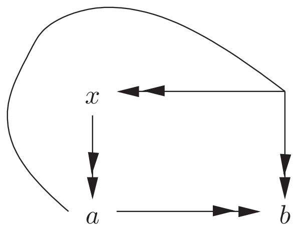

FIG. 33.

3 Traditional definition of disjunctive extensions

In our investigation of options for viewing disjunctive attacks, we use the Caminada labelling approach. We owe it to the community also to try and see what options we have for complete extensions for disjunctive networks also within the framework of traditional set theoretic approach.

There are two ingredients for the traditional approach

(1) the notion of conflict freeness of a set H.
(2) the notion of a set H protecting a node x by attacking its attackers.

With disjunctive attacks we have a problem because nodes can be attacked indirectly by other nodes. So if x attacks H, any element of H is attacked indirectly.

There is another problem, when we want to compare with labellings. Assume we did manage to define the notion of complete extensions E set theoretically. How do we get labellings from it? The obvious way is to say to be a member of E is to be in. Whatever E attacks is out and what is left is undecided. This is fine but we are back to question of what to take as an attack on single elements. So let us try something new.

We use the traditional set theoretic method of defining complete extensions.

DEFINITION 3.1
Let f be a choice function on a disjunctive network (S,ρ) as defined in 2.3

(1) say E is *conflict free for* f iff for all $x\in E$, if $x\rho H$ then $f(x,H)\cap E=\varnothing$.
(2) We say that E *protects* α *for* f iff for any z,Z s.t. $z\rho Z$ and $\alpha\in f(z,Z)$ there exists a β,H where $\beta\rho H$ and $z\in f(\beta,H)$.
(3) We say that E *protects itself in* f iff E protects each of its elements for f.
(4) We say $E\subseteq S$ is a *complete extension for* f if E is conflict free for f, protects itself (for f) and contains all the elements it protects (for f).

THEOREM 3.2
Given a disjunctive network $\mathcal{A}$ and a choice function f (2.3), λ is a legitimate Caminada–Gabbay labelling over $\mathcal{A},f$ (2.3) iff there an E which is a complete extension for f and:

(T1) $\lambda(x)=$ in iff $x\in E$
(T2) $\lambda(x)=$ out iff there are y,Y such that $x\in Y$, $y\rho Y$, $y\in E$ and $x\in f(y,Y)$.
(T3) $\lambda(x)=$ und otherwise

PROOF.
Suppose λ is a legitimate labelling. We argue that $E=\{x\mid\lambda(x)=\text{in}\}$ is a required set. First we show that E is a complete extension for f.

- Clearly if $x \in E$ and $y \in f(x,H) \cap E$ then λ breaches $(D1)$ of 2.3 E must be conflict free.
- Also if E protects x then $y \in f(z,Z)$ for some $z \in E$ whenever $x \in f(y,Y)$; but given that $z \in E$ iff $\lambda(z) = \text{in}$ this is precisely the right-hand side of condition $(D1)$. Thus $\lambda(x) = \text{in}$ and so $x \in E$.
- Finally, if $x \in E$ and $x \in f(y,Y)$ for some y, Y. Then since $\lambda(x) = \text{in}$ we must have $\lambda(y) = \text{out}$ which implies by $(D2)$ that there are z, Z s.t. $\lambda(z) = \text{in}$ and $y \in f(z,Z)$. But, by definition, $\lambda(z) = \text{in}$ iff $z \in E$. Thus E protects itself for f.

Thus E is a complete extension for f. Furthermore $(T1)$ is met by definition of E, and then replacing '$y \in E$' in $(T2)$ with '$\lambda(x) = \text{in}$' makes it identical to $(D2)$. Now since λ is a map to $\{\text{in}, \text{out}, \text{und}\}$ we may conclude that if λ is legitimate then it meets the conditions $(T1)$–$(T3)$ for $E = \{x \mid \lambda(x) = \text{in}\}$.

Suppose E is a complete extension for f. We argue that any λ satisfying $(T1)$–$(T3)$ is legitimate according to 2.3.

If $\lambda(x) = \text{in}$ then, by $(T1)$, $x \in E$. Then since E is complete for f, if $x \in f(y,Y)$ then E protects x, i.e. there are β, H such that $y \in f(\beta, H)$ and $\beta \in E$; but this is the right-hand side of $(D1)$. Conversely, suppose x satisfies the r.h.s. of $(D1)$: for every y, Y s.t. $x \in f(y,Y)$, $\lambda(y) =$ out. Then for any such y, by $(T2)$, there are z, Z such that $y \in Z$, $z \in E$ and $y \in f(z,Z)$. But this means that E protects x and so $x \in E$ and $\lambda(x) = \text{in}$ by $(T1)$

$\lambda(x) = \text{out}$ if and only if by $(T1)$ and $(T2)$ there are y, Y such that $x \in Y$, $y \rho Y$, $\lambda(y)$in and $x \in f(y,Y)$. This is the r.h.s. of $(D2)$.

Finally, since conditions $(D1)$ and $(D2)$ are met by λ, it follows from $(T3)$ that λ is indeed a map to $\{\text{in}, \text{out}, \text{und}\}$ and so $(D3)$ is met as well. ∎

Not only does this provide us with an alternative way of characterizing the notion of disjunctive attack defined by 2.3, but it also suggests a means of characterizing disjunctive attacks without appealing to choice functions f or flattenings (2.8).

Definition 3.3
Let $\mathcal{A} = (S, \rho)$ be a disjunctive network and let $E \subseteq S$:

(1) We say E is *conflict free* iff for no $x \in E$ and $H \subseteq E$ do we have $x \rho H$.
(2) We say that *E protects α* iff for any $z \rho H \cup \{\alpha\}$ there exists a $\beta \in E$, $E' \subseteq E$ and $H' \subseteq H$ such that $\beta \rho H' \cup E' \cup \{z\}$.
(3) We say that *E guarantees α* iff for any $z \rho H \cup \{\alpha\}$ there exists a $\beta \in E$, $E' \subseteq E$ and $y \in H \cup \{z\}$ such that $\beta \rho E' \cup \{y\}$.
(4) We say that *E protects [guarantees] itself* iff E protects [guarantees] each of its elements.
(5) We say E is a *complete [total] extension* if E is conflict free, protects [guarantees] itself and contains all the elements it protects [guarantees].

Lemma 3.4
Let $\mathcal{A} = (S, \rho)$ be a disjunctive network.

(1) If $E \subseteq S$ is conflict free, protects itself and protects α, then $E \cup \{\alpha\}$ is conflict free and protects itself.
(2) If $E \subseteq S$ is conflict free, guarantees itself and guarantees α, then $E \cup \{\alpha\}$ is conflict free and guarantees itself.

Proof. Suppose E is conflict free, protects itself and also protects α, then it is immediate that $E \cup \{\alpha\}$ protects itself.

To show $E \cup \{\alpha\}$ is conflict free we assume otherwise and derive a contradiction. There are 3 possibilities and we obtain a contradiction for each.

Case 1: For some $E_1 \subseteq E$ we have $\alpha \rho E_1$. Since E protects itself, it protects every member x of E_1, and since $E_1 \subseteq E$ we can conclude that there is a y and $H \subseteq E$ such that $y \rho H \cup \{\alpha\}$. But E protects α, so there is a $\beta \in E$, $H_1 \subseteq H$ and $H_2 \subseteq E$ such that $\beta \rho H_1 \cup H_2 \cup \{y\}$. This contradicts that E is conflict free as $H_1 \cup H_2 \cup \{y\} \subseteq E$.

Case 2: For some $H \subseteq E$ and $u \in E$ we have $u \rho H \cup \{\alpha\}$. Since E protects α, there are $\beta \in E$, $E' \subseteq E$ and $H' \subseteq H$ such that $\beta \rho H' \cup E' \cup \{u\}$. Again this contradicts the assumption that E is conflict free.

Case 3: For some $H \subseteq E$ we have $\alpha \rho H \cup \{\alpha\}$. Since E protects α, there are $\beta \in E$, $E' \subseteq E$ and $H' \subseteq H$ such that $\beta \rho E' \cup H' \cup \{\alpha\}$. But this is now Case 2 which is impossible.

The argument is similar for the case where E is conflict free, guarantees itself and guarantees α. ∎

Corollary 3.5

Let $\mathcal{A} = (S, \rho)$ be a disjunctive network. Then there exist sets E which are conflict free, protect themselves and contain all elements which they protect.

Proof. Start with the empty set $\varnothing$ and keep augmenting it using Lemma 3.4. ∎

Example 3.6

Consider Figure 13. In this figure $E = \{x\}$ is conflict free and protects itself. It also does not protect a for to do so we would require some subset of $\{x, b\}$ to be attacked by x (and analogously, E does not protect b). So $E = \{x\}$ is a complete extension, and by similar reasoning it is also total.

Moreover $\{x, a\} = E \cup \{a\}$ is a complete extension and so is $\{x, b\} = E \cup \{b\}$. To see that, e.g., $\{x, a\}$ protects (and in fact guarantees) a: for x attacks $\{a, b\}$ where $\{a\} \subseteq \{x, a\} = E$.

Since $\{x, a, b\}$ is not conflict free it follows that the complete, and also total, extensions are $\{x\}$, $\{x, a\}$ and $\{x, b\}$.

Talmudic disjunctive attack $x \rho H$ wants *exactly one* $y \in H$ to be out. Talmudic logic thinks of it as a collapse of $x \rho H$ to a single $x \rho \{y\}$ for $y \in H$. So Talmudic Logic [1] would nor regard the complete extension $\{x\}$ as a candidate for the set of 'in' elements of the network. Talmudic disjunctive argumentation actually requires two values of undecided, and the interested reader can consult [1].

Example 3.7

Consider Figure 34. In this network, since a is attacked by nothing $E = \{a\}$ protects (and guarantees) itself. But E also protects d. To see this note that the only (disjunctive) attacker of d is b and b attacks $\{c, d\}$, thus setting $\{c\} = H = H'$ in 3.3.2 we see that $\{a\}$ protects d. Moreover since $\{a, d\}$ protects neither b nor c, $\{a, d\}$ is a complete extension whereas $\{a\}$ is not.

On the other hand, $E = \{a, d\}$ does not guarantee d, for nothing attacks $\{d\}$ or $\{a, d\}$. Thus $\{a, d\}$ is not a total extension. Moreover $\{a\}$ does not guarantee any of the other nodes, so $\{a\}$ is a total extension.

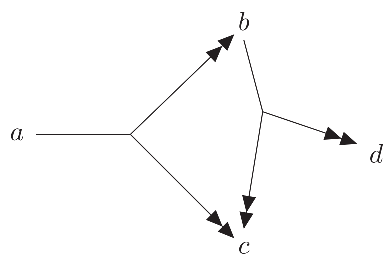

Fig. 34.

DEFINITION 3.8
Let (S,ρ) be a disjunctive network, define its *conjunctive flattening* to be the conjunctive network $(S,\mathbb{R})$ where for $H\subseteq S$ and $x\in S$:

$$H\mathbb{R}x \text{ iff } \exists y\in H \text{ s.t. } y\,\rho\,((H-\{y\})\cup\{x\}).$$

THEOREM 3.9
Given a disjunctive network $\mathcal{A}$ and its conjunctive flattening $\mathcal{A}'$, λ is a legitimate labelling over $\mathcal{A}'$ (Definition 2.1) iff there is a total extension E (on $\mathcal{A}$) such that:

(T1) $\lambda(x)=$ in iff $x\in E$
(T2) $\lambda(x)=$ out iff there are y,Y such that $x\in Y$, $y\,\rho\,Y$, $Y-\{x\}\subseteq E$.
(T3) $\lambda(x)=$ und otherwise

PROOF. The argument is very similar to that of 3.2.

Suppose λ is a legitimate labelling of $\mathcal{A}'$. We argue that $E=\{x\,|\,\lambda(x)=\text{in}\}$ is a required set. First we show that E is a total extension.

- If $x\in E$, $x\,\rho\,H$ and $H\subseteq E$ then for some $y\in H$, $((H-\{y\})\cup\{x\})\mathbb{R}y$. But then $(CC1)$ of 3.1 is violated (as x, y and H are all in). Thus E must be conflict free.
- Also if E guarantees x then, if $H\mathbb{R}x$ we must have some $y\in H$ s.t. $y\,\rho\,H\cup\{x\}$. But then there is an $E'\subseteq E$, and an $\alpha\in E$ s.t. $\alpha\,\rho\,E'\cup\{z\}$ for some $z\in H\cup\{y\}$. But then by $(T2)$ and the definition of E, $\lambda(z)=$out and clearly $z\in H$. So by $(CC1)$ $\lambda(x)=$in and so $x\in E$.
- Finally, suppose $x\in E$ and $y\,\rho\,H\cup\{x\}$. Then $H\cup\{y\}\mathbb{R}x$ and since $\lambda(x)=$in we must have $\lambda(z)=$ out for some $z\in H\cup\{y\}$. By $(CC2)$ there is an E' s.t. $\lambda(v)=$in for all $v\in E'$ and $E'\mathbb{R}z$. From this we may conclude that $E'\subseteq E$ and $\beta\,\rho\,((E'-\{\beta\})\cup\{z\})$ for some $\beta\in E'$. But, since $z\in H\cup\{y\}$ it follows that E guarantees x.

Thus E is a total extension and we may conclude the argument as we did in 3.2: $(T1)$ is met by definition of E; moreover $(T2)$ then becomes equivalent to $(CC2)$ given the definition of $\mathbb{R}$; and since λ is a map to $\{\text{in},\text{out},\text{und}\}$ we may conclude that if λ is legitimate then it meets the conditions $(T1)$–$(T3)$ for $E=\{x\,|\,\lambda(x)=\text{in}\}$.

Suppose E is a total extension of $\mathcal{A}$. We argue that any λ satisfying $(T1)$–$(T3)$ is legitimate (for $\mathcal{A}'$) according to 2.3.

If $\lambda(x)=$in then, by $(T1)$, $x\in E$. Then since E is total, if $y\,\rho\,Y\cup\{x\}$ then E guarantees x, i.e. there are β,H such that $\beta\,\rho\,H\cup\{z\}$ for $\beta\in E$, $H\subseteq E$ and some $z\in Y\cup\{y\}$. Then $H\cup\{\beta\}\mathbb{R}z$ where the members of $H\cup\{\beta\}$ are all in. This implies the right-hand side of $(CC1)$.

Conversely, suppose x satisfies the r.h.s. of $(CC1)$: for every H s.t. $H\,\rho\,x$, $\lambda(y)=$ out for some $y\in H$. Then for any such y, by $(T2)$, there are z,Z such that $z+Z\subseteq E$ and $z\,\rho\,Z\cup\{x\}$. But this means that E guarantees x and so $x\in E$ and $\lambda(x)=$in by $(T1)$.

$\lambda(x)=$out if and only if by $(T1)$ and $(T2)$ there are y,Y such that $y\,\rho\,Y\cup\{x\}$ and $\lambda(z)$in for all $z\in Y\cup\{y\}$. This is equivalent to the r.h.s. of $(CC2)$ by the definition of $\mathbb{R}$.

Finally, since conditions $(CC1)$ and $(CC2)$ are met by λ, it follows from $(T3)$ that λ is indeed a map to $\{\text{in},\text{out},\text{und}\}$ and so $(CC3)$ is met as well. ∎

4 Disjunctive attacks viewed as higher level attacks

It is possible to define disjunctive attacks according to the open reductionist view of Definition 2.3 by using conjunctive attacks together with higher level attacks and the constant $\top$. The general question is what approaches to disjunctive attack we can reproduce using higher level attacks and $\top$.

The idea is as follows.

(1) We note (following Remark 1.4) that $\top$ must always be in and so any attack on $\top$ cannot be in. In fact it must be out , it cannot be undecided either.
(2) Let H be a set of attackers conjunctively attacking $\top$. So the attack must be out. So at least one member of H must be out. So this is the same as $\top$ disjunctively attacking H according to the open reductionist view.
(3) So we have now that $\top \twoheadrightarrow H$ disjunctively is implemented as $H \twoheadrightarrow \top$ conjunctively. How do we implement $x \twoheadrightarrow H$ disjunctively by using $\top$? We need a trick.
(4) Consider $\top \twoheadrightarrow (\top \twoheadrightarrow H)$. This is an attack of $\top$ at a higher level , attacking $\top$'s own attack on H. Since $\top$ is always in, it cancels its own attack, so we have nothing attacking H.
(5) If now x attack this cancellation attack, we get that x activates the disjunctive attack on H. This means that $x \twoheadrightarrow H$ is the same as $x \twoheadrightarrow (\top \twoheadrightarrow (\top \twoheadrightarrow H))$. This by item 3 above is equivalent to $x \twoheadrightarrow (\top \twoheadrightarrow (H \twoheadrightarrow \top))$.

EXAMPLE 4.1
Using higher level attacks, see [3, 13], and $\top$, we can implement disjunctive attacks using conjunctive attacks. See Figure 35. This figure implements the disjunctive attack of Figure 36. We have that $\{b,c\} \twoheadrightarrow \top)$ and $\top \twoheadrightarrow (\{b,c\} \twoheadrightarrow \top)$ and $a \twoheadrightarrow (\top \rightarrow (\{b,c\} \twoheadrightarrow \top))$

REMARK 4.2

(1) We need to explain what we mean by the term 'implement', used in Example 4.1.
Let (S,ρ) be a disjunctive network. Let $x \in S, H \subseteq S$ and assume $x \rho H$ holds. We write $\psi = [x \twoheadrightarrow H]$ to mean the configuration containing the nodes $\psi_1 = H \cup \{x\}$ and $\psi_2 = \{(x,H)\}$ among them. So formally we can write $\psi = (\psi_1, \psi_2)$.
(2) We now describe a different configuration on the nodes $\psi_1 = \{x\} \cup H \cup \{\top\}$ denoted by

$$\phi = [x \twoheadrightarrow (\top \twoheadrightarrow (H \twoheadrightarrow \top))].$$

This configuration involves attacks of points on point, or on sets, or on attacks. It contains the following relations on the nodes

$$\phi_2 = \{(H,\top), (\top,(H,\top)), (x,(\top,(H,\top)))\}.$$

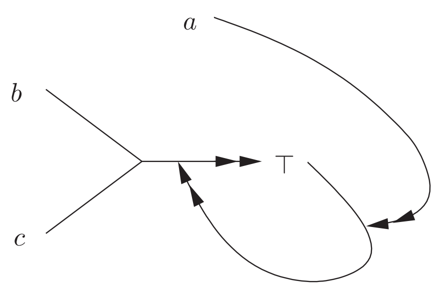

FIG. 35.

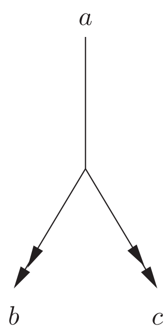

FIG. 36.

So formally $\phi = (\phi_1, \phi_2)$.

(3) Given a disjunctive network (S, ρ) we describe a higher level network $(S \cup \{\top\}, \overline{\rho})$ where

$$\begin{aligned} \overline{\rho} \quad \subseteq \quad & (2^S - \{\varnothing\}) \times S \\ & \cup S \times \big((2^S - \{\varnothing\}) \times S\big) \\ & \cup S \times \big(S \times \big((2^S - \{\varnothing\}) \times S\big)\big) \end{aligned}$$

constructed as follows. Let $\overline{\rho} = \bigcup_{\psi = [x\rho H]} \phi_2$, as defined for ψ in item 2 above.

PROPOSITION 4.3

Let $x \twoheadrightarrow H$ be an open reductionist attack of x on H viewed according to Definition 2.3. Translate $\psi = [x \twoheadrightarrow H]$, using higher level attacks and $\top$, into

$$\phi = \big[x \twoheadrightarrow \big(\top \twoheadrightarrow (H \twoheadrightarrow \top)\big)\big].$$

Then $x \twoheadrightarrow H$ and its translation are equivalent according to the discussion of Remark 4.2.

PROOF. We distinguish three cases on x.

Case $x = \text{in}$**.** In this case $x \twoheadrightarrow H$, by $(D2)$ of 2.3, requires that there is an f such that the non-empty subset $f(x, H)$ is out (i.e. every one of its members is out). But since x is in, then in ϕ we have $H \twoheadrightarrow \top$ is active and so some non-empty subset E of H must be out.

Case $x = \text{out}$**.** In this case H is free (at least from x). But, on the other hand, in this case the attack of $\top$ on $H \twoheadrightarrow \top$ is active and so H does not attack $\top$ and then H is free (at least from the attack on $\top$).

Case $x = \text{und}$**.** In this case it is undecided whether there is a disjunctive attack from x to H. In ϕ, x being undecided means it is undecided whether the attack of $H \twoheadrightarrow \top$ is cancelled or not. This means that it is undecided whether the attack $H \twoheadrightarrow \top$ is in or not. See also item 5 of Remark 1.4. ∎

REMARK 4.4

We can implement Figure 35 using auxiliary points as indicated in Figure 37. We need auxiliary points $\alpha(a,b)$ representing the attack double arrow of $\{b,c\} \twoheadrightarrow \top$. We need $\beta(a,b), \gamma(a,b)$ acting as intermediaries for the attack of $\top$ on $\alpha(b,c)$ and thus a can implement the attack on $\top \to \alpha(b,c)$ by attacking $\gamma(b,c)$. We further need for each node x its negation node $\neg x$, for reasons explained in the next Remark 4.5.

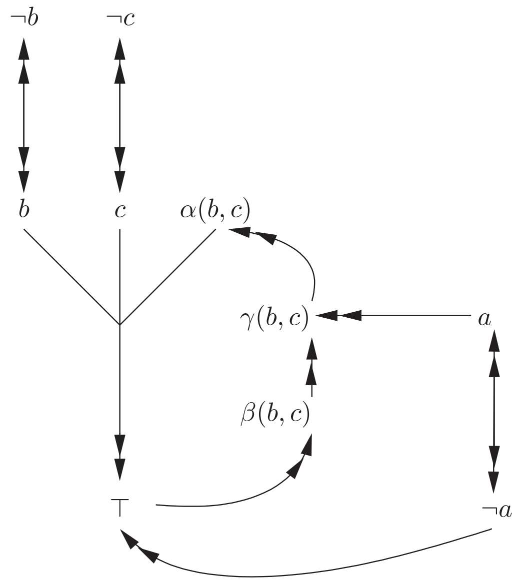

FIG. 37.

REMARK 4.5
In this section we used $\top$ as a node in argumentation network. As noted in [16], if we do that, we may fail to have extensions at all for the network. (See, e.g., a network with $\{x, \top\}$ and x attacks $\top$, x is not attacked and so it is supposed to be in but it attacks $\top$ and so it must be out). So we have to make sure that networks like that of Figure 37 which arise from disjunctive networks by adding points and adding $\top$ are assured to have extensions. If this can be shown then we would also get that disjunctive networks always have extensions. We shall get to these issues later in the continuation paper, [20].

5 Comparison with collective argumentation

This section compares our discussion with that of [8]. We first recall a connection with logic programming. Consider the basic situation of Figure 1. In this figure we have an attack of $y_1, \ldots, y_k$ on x. It has been observed in [21], and no doubt by many other researchers, that this attack can be translated into logic programming as the clause

$$\bigwedge_i \neg y_i \to x. \qquad (\dagger)$$

In fact, any logic program $\mathcal{P}$ satsifying (i) and (ii) below defines an argumentation network (S, R) where

(i) Each literal is the head of exactly one clause.
(ii) The body of any clause is comprised of negated atoms.

Let S be the set of atoms of $\mathcal{P}$. We define yRx if $\neg y$ appears in the body of the clause with head x.

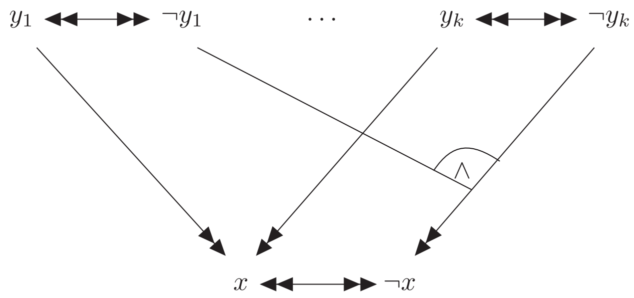

FIG. 38.

Bochman [8] views (†) as a conjunctive attack of $\bigwedge_i \neg y_i$ on $\neg x$, namely $\{\neg y_1,\dots,\neg y_k\} \twoheadrightarrow \neg x$. Figure 38 shows the equivalence.[3]

Note also that one of the approaches to disjunctive attacks presented in this article was to view $z \twoheadrightarrow H$ as $z \twoheadrightarrow \bigwedge_{y\in H} y$. Further note that if we have (†) for several x_j, we get

$$\bigwedge_i \neg y_i \to \bigwedge_j x_j$$

which can be written, following Bochman in [8], as

$$\bigwedge_i \neg y_i \twoheadrightarrow \neg \bigwedge_j x_j$$

or

$$\bigwedge_i \neg y_i \to \bigvee_j x_j. \tag{‡}$$

we thus get conjunctive–disjunctive attacks. Bochman calls these collective attacks.

In [8] Bochman takes the route outlined above, considering networks with collective attacks, with a view to connecting argumentation with disjunctive logic programming. We shall limit our discussion to a comparison of our article with Bochman's approach.

Bochman views $G \twoheadrightarrow H$ for $G,H \subseteq S$ as a sort of consequence relation and therefore seeks semantics and axiomatic properties for it. For example, Bochman observes that the consequence relation could have properties such as *monotonicity*

$$\text{If } G \twoheadrightarrow H, \text{ then } G\cup G' \twoheadrightarrow H\cup H',$$

symmetry

$$\text{If } G \twoheadrightarrow H_1\cup H_2, \text{ then } G\cup H_1 \twoheadrightarrow H_2.$$

and *normality*[4]

[3]Bochman's paper [8] predates (2003) Nielsen and Parson's paper [21] (2007) which studies conjunctive attacks and also Gabbay's [12] (2009) which also introduced conjunctive and disjunctive attacks.

[4]See also Remark 5.1.

(1) the empty set is not attacked (*affirmativity*).
(2) If $G \twoheadrightarrow H_1 \cup H_2$, then $G \twoheadrightarrow H_1$ or $G \twoheadrightarrow H_2$ (*locality*).

If we have $\neg$ in the language with $\neg\neg x \leftrightarrow x$ and $\neg\neg x \twoheadleftarrow\!\!\twoheadrightarrow x$, then Bochman writes a proper consequence relation $\Vdash$ where

$$G \Vdash H \text{ iff } G \twoheadrightarrow \neg H,$$

where for a set H, $\neg H$ is $\{\neg y \mid y \in H\}$.

A difference, however, is that where we interpret arguments as having one of 3 possible values (in, out and und), Bochman defines his consequence relation over a 4-valued interpretation.

A *4-valued interpretation* over collective argumentation network $\mathcal{A}$ is a function $v : A \mapsto 2^{\{\mathbf{t},\mathbf{f}\}}$ assigning each argument a subset of $\{\mathbf{t},\mathbf{f}\}$

- An attack $G \twoheadrightarrow H$ *holds* in an interpretation v of $\mathcal{A}$ if either $\mathbf{t} \notin v(y)$, for some $y \in G$, or $\mathbf{f} \in v(x)$, for some $x \in H$.
- An interpretation v is a *model* of a collective argumentation theory $\mathcal{A}$ if every attack from $\mathcal{A}$ holds in v.

The 4 values are intended to provide greater expressive capabilities of the language and, in addition to the acceptance and rejection of arguments, permit ‘situations in which an argument is both accepted and rejected, or, alternatively, neither accepted, nor rejected’ ([8, p. 7]). Bochman goes on to point out that ‘this interpretation is nothing other than the well-known *Belnap’s interpretation of four-valued logic* (see [7])’.

Very roughly, we have only 3 values: *in*, corresponding to $\{\mathbf{t}\}$; *out*, corresponding to $\{\mathbf{f}\}$; and *und* corresponding to $\varnothing$. But even if we ignore the value $\{\mathbf{t},\mathbf{f}\}$ as unacceptable to us,[5] and look at Bochman’s definition of a legitimate labelling (i.e. an interpretation v modelling a collective network $\mathcal{A}$), it deals with local formations only. For example, let λ denote interpretations v that never assign the value $\{\mathbf{t},\mathbf{f}\}$. Then in our notation, for the case of the local formation of Figure 2, Bochman’s definition is that

- an attack holds for λ (is legitimate for the labelling λ) if either $\lambda(z) \neq \{\mathbf{t}\}$ (in) or $\lambda(x) = \{\mathbf{f}\}$ (out) for some $x \in \{\lambda(x), \lambda(u_1), \ldots, \lambda(u_r)\}$.

This is our minimalist condition.

It suits the purposes of [8] to leave things at that. Bochman does not give a global condition, he basically says that a labelling is a model (i.e. is legitimate) if every attack formation is locally legitimate. Global conditions can, however, be put forward in Bochman’s approach by axioms and principles on the attack relation $\twoheadrightarrow$. For example, the principle of symmetry corresponds to what we called in Section 1 the ‘conjunctive approach’. See our Example 1.5, and Remark 1.6.[6]

[5] Bochman’s framework can also rule out the value $\{\mathbf{t},\mathbf{f}\}$, he calls it *negative argumentation* ([8, Section 6]). It is characterized in his system by the rule:

$$G \twoheadrightarrow G \cup H \text{ implies } G \twoheadrightarrow H.$$

So e.g., if we insist on normality (defined above), then models where x is assigned $\{\mathbf{t},\mathbf{f}\}$ will invalidate this rule for $H = \varnothing$ and $G = \{x\}$.

[6] We suspect that such axioms may not be sufficient for characterizing all our proposed approaches, especially the reduction approach. We suspect that one also needs reduction principles of the form

$$G \twoheadrightarrow H \text{ iff } \bigvee_j G \twoheadrightarrow_j H,$$

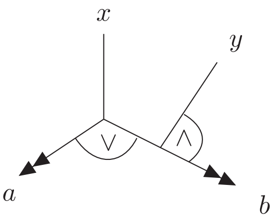

FIG. 39.

We can view our efforts as coming from a different geometric point of view. We look for global conditions by making use of the geometry of (S,ρ) (or even (S,ϱ)).[7] Having the geometry available allows us to generalize more, as we did in Remark 2.11 and Figure 30. Bochman could adopt our notation and generalize his consequence relation. However, we doubt whether it is possible to represent the situation of Figure 39 in terms of a consequence relation between subsets of S.

As a further difference, consider Figure 6. In this case, all approaches agree that a should be in (or **t**). However, even if we ignore Bochman's value $\{\mathbf{t},\mathbf{f}\}$ and equate und with $\varnothing$, Bochman's approach would allow $c=\mathbf{f}$ and $b=\varnothing$. But this is not allowed in any of the interpretations we discuss in this article. This is because our interpretations are based, not on all possible models of a consequence relation, but on the attacks of the network. So in Figure 6, if a is in and c is out, it is not a question what what value could be consistently assigned to b, but whether it is still attacked or not: if it is, then it is out, otherwise it is in.

REMARK 5.1
We would also like to compare our open reductionist approach with Bochman's notion of normality. The reader might ask if the notion of normal consequence expresses the notion of open reductionist approach, both saying

$$G \twoheadrightarrow H \text{ iff } \exists y_{\in H}(G \twoheadrightarrow \{y\}).$$

The answer is yes indeed, they are the same. Definition 2.3 requires G to be a singleton and was extended in Definition 2.5 to conjunctive–disjunctive networks.

Of course the main difference is the starting point of view. Bochman's starting point is a fundamental distinction between non-monotonic and logical aspects of argumentation. What we discuss in our comparison with Bochman is only the logical part of the collective argumentation. Our approach, however, also contains the standard non-monotonic semantics in the form of Dung-like semantics. Bochman would say that we are mixing logical and non-monotonic semantics (analogous to what we call local and global conditions respectively). From our point of view, we are just trying to generalize the traditional approach to argumentation networks to handle disjunctive attacks.

where $\twoheadrightarrow$ and $\twoheadrightarrow_j$ are different attack relations. This is like defining a consequence relation $\Vdash$ over two logics $\Vdash_1$ and $\Vdash_2$, where $A \Vdash B_1 \vee B_2$ if $A \Vdash_i B_j$ for some $i,j \in \{1,2\}$.

[7] We can draw an analogy here with Revision Theory. The axiomatic approach of [8] is like providing AGM postulates, while our geometric approach is like providing an algorithmic truth maintenance system.

6 Conclusion and discussion

In this article, we studied in detail the notion of disjunctive attacks introduced in 2009 in [12]. These are attacks of points $x \in S$ on sets $H \subseteq S$. The notion is new. Although some papers used attacks of points x on sets H, these were defined in terms of attacks of points on points, namely

- $x \twoheadrightarrow H$ iff $\exists y \in H(x \twoheadrightarrow y)$.

This is not the same as disjunctive attacks. We recognize, however, the paper of Bochman [8], discussed in Section 5.

The main result, both conceptual and mathematical, is the representation theorem, Theorem 2.9, reducing disjunctive attacks to the notion of flat bundles and semantics for such bundles. The representation theorem turns the concept of disjunctive attacks into meta-level concept. Whenever we have networks and semantics for them, e.g. abstract Dialectical Frames (ADF) or instantiated frames (ASPIC) or any other notions, we can always take flat bundles of them as in Definition 2.7, define the semantics as done there and prove a representation theorem.

Comparing with Talmudic Disjunctive attacks of [1], there are two differences:

(1) Talmudic disjunctive attacks require 4 values, {in, out, und, wave}. The value ‘wave’ represents being part of a state which is like a superposition state of an electron in the quantum mechanics two slit experiment.
(2) When x is attacking a set H, the attack can either collapse to attacking a single y in H, i.e $x \twoheadrightarrow H$ collapses to exactly one $x \twoheadrightarrow y$, or it can create a superposition of all the elements of H with each such element getting the value “wave”. See [1] for details.

In our article $x \twoheadrightarrow H$ always collapses to $\{x \twoheadrightarrow y \mid y \in f(x,H)\}$, when we take the view of disjunctive attacks given by Definition 2.3 (but this is not necessarily the case for the view given by 3.1).

We shall investigate these concepts further together with their connection to higher level attacks in a subsequent paper, [20].

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References

[1] M. Abraham, I. Belfer, D. Gabbay and U. Schild. *Quantum and Disjunctive attacks in Talmudic Logic*. To appear.
[2] P. Baroni, M. Giacomin and G. Guida. SCC-recursiveness: a general schema for argumentation semantics. *Artificial Intelligence*, **168**, 162–210, 2005.
[3] P. Baroni, F. Cerutti, M. Giacomin and G. Guida. Encompassing attacks to attacks in abstract argumentation frameworks. In *European Conferences on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU)*, pp. 83–94, 2009.
[4] P. Baroni, F. Cerutti, M. Giacomin and G. Guida. AFRA: argumentation framework with recursive attacks. *International Journal of Approximate Reasoning*, **52**, 19–37, 2011.
[5] H. Barringer, D. M. Gabbay and J. Woods. Temporal dynamics of support and attack networks: from argumentation to zoology. In *Mechanizing Mathematical Reasoning*, D. Hutter and W. Stephan, eds, pp. 59–98, 2005.

[6] H. Barringer, D. Gabbay and J. Woods. Temporal, numerical and metalevel dynamics in argument networks. In *Argumentation and Computation*, **3**, 143–202, 2012.

[7] N. D. Belnap, Jr. A useful four-valued logic. In *Modern Uses of Multiple-Valued Logic*, M. Dunn and G. Epstein, eds, pp. 8–41. D. Reidel, 1977.

[8] A. Bochman. Collective argumentation and disjunctive logic programming. *Journal of Logic Computation*, **13**, 405–428, 2003.

[9] G. Brewka, P. Dunne and S. Woltran. Relating the semantics of abstract dialectical frameworks and standard AFs. In *Proceedings IJCAI-11*, pp. 780–785. AAAI Press, 2011.

[10] G. Brewka and S. Woltran. Abstract dialectical frameworks. In *Proceedings of the 20th International Conference on the Principles of Knowledge Representation and Reasoning (KR 2010)*, pp. 102–111. AAAI press, 2010.

[11] M. Caminada and D. M. Gabbay. A logical account of formal argumentation. *Studia Logica*, **93**, 109–145, 2009.

[12] D. M. Gabbay. Fibring argumentation frames. *Studia Logica*, **93**, 231–295, 2009.

[13] D. M. Gabbay. Semantics for higher level attacks in extended argumentation frames. Part 1: Overview. *Studia Logica*, **93**, 355–379, 2009.

[14] D. M. Gabbay. The equational approach to CF2 semantics, short version. In *Proceedings COMMA 2012, Computational Models of Argument*, B. Verheij, S. Szeider and S. Woltran, eds, pp. 141–153. IOS Press, 2012.

[15] D. M. Gabbay. *Meta-Logical Investigations in Argumentation Networks*. Research Monograph College Publications, 2013, 770 pp.

[16] D. M. Gabbay. Theory of semi-instantiation in abstract argumentation. To appear *Logica Universalis*.

[17] D. Gabbay and M. Gabbay. The attack as strong negation, Part II. In Preparation.

[18] D. Gabbay and M. Gabbay. The attack as strong negation, Part 1. To appear in *Logic Journal of the IGPL*. http://arxiv.org/abs/1506.03266.

[19] D. Gabbay and M. Gabbay. The attack as intuitionistic negation. To appear in *Logic Journal of the IGPL*.

[20] D. Gabbay and M. Gabbay. Theory of disjunctive attacks, Part II, in preparation.

[21] D. Gabbay and A. Garcez. Logical modes of attack in argumentation networks. *Studia Logica*, **93**, 199–230, 2009.

[22] D. M. Gabbay and O. Rodrigues. Probabilistic argumentation, an equational approach. To appear in *Logica Universalis*, http://arxiv.org/abs/1503.05501.

[23] D. D. Hanh, P. M. Dung and P. M. Thang. Inductive defense for Modgil's extended argumentation framework. *J Logic Computation*. doi: 10.1093/logcom/exq018. First published online: June 16, 2010.

[24] H. Jakobovits and D. Vermeir. Robust semantics for argumentation frameworks. *Journal of Logic and Computation*, **9**, 215–261, 1999.

[25] S. H. Nielsen and S. Parsons. A generalisation of Dung's abstract framework for argumentation: arguing with sets of attacking arguments. Lecture Notes in Computer Science (LNCS) Vol. 4766, *Argumentation in Multi-Agent Systems*, pp. 54–73. Springer, 2007.

[26] F. Toni. Assumption based argumentation-tutorial. *Argument & Computation*, **5**, 89–117, 2014.

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Appendix

A Summary of the main approaches to disjunctive attacks

A.1 Approach based of point to point attacks

The basis for this definition is a traditional network (S, R) with $R \subseteq S \times S$. We define $x \rho H$, for $H \subseteq S$ to be:

$$x \rho H \text{ iff } \exists y \in H \text{ s.t. } xRy.$$

This definition is used in Assumption Based Argumentation in [26] and [25].

A.2 Minimalist approach

We have (S, ρ) with $\rho \subseteq S \times (2^S - \{\varnothing\})$. This is discussed in Section 1, see Examples 1.3 and 1.5. On this approach if $x \rho H$ and x is in then exactly one of the H should be out.

A.3 The conjunctive approach

This approach reads $x \rho H$ as describing a number of conjunctive attacks: all the $H - \{y\} \twoheadrightarrow y$ for $y \in H$, where $\twoheadrightarrow$ denotes a conjunctive attack. See Definition 3.3 and Theorem 3.9.

A.4 The open reductionist approach

This approach says $x \rho H$ iff for some $G \subseteq H$ we have $x \rho \{y\}$ for each $y \in G$.

See Definition 2.8 and Theorem 2.9. This is similar to the minimalist approach except we are not restricted to 'exactly one' of H being out if it is disjunctively attacked.

A.5 Traditional set theoretic definitions

These approaches use different notions of one node 'protecting' another from attack. See Definition 3.3 and Example 3.7.

A.6 Talmudic approach

See [1]. The Talmudic approach allows for quantum superposition-like attacks on the attacked sets. There is a hint in Example 34 and a comment in Section 5.