Residuation for Lexicographic Orders

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Abstract

Residuation theory concerns the study of partially ordered algebraic structures, most often just monoids, equipped with a weak inverse for the monoidal operator. One of its area of application has been Constraint Programming, whose key requirement is the presence of an aggregator operator for combining preferences. Given a residuated monoid of preferences, the paper shows how to build a new residuated monoid of (possibly infinite) tuples based on the lexicographic order.

Keywords: Residuation theory, Lexicographic orders, Soft constraints.

1. Introduction

Residuation theory [1] concerns the study of partially ordered algebraic structures, most often just monoids, equipped with an operator that behaves as a weak inverse to the monoidal one, without the structure being necessarily a group. One of the recent area of application of residuation theory is constraint programming. Roughly, a soft constraint satisfaction problem is given by a relation on a set of variables and a preference score to each assignment of the variables [2, 3]. A key requirement is the presence of an aggregator operator for combining preferences, making their set a monoid: a large body of work has been devoted to enrich such a structure, guaranteeing that resolution techniques are generalised by a parametric formalism for designing metrics and algorithms.

The relevance of residuated monoids for local consistency algorithms, where in order "move" costs they must be "subtracted" somewhere and "added" elsewhere, has been spotted early on [4, 5], and various extensions has been proposed [6]. Also, residuated monoids found applications in languages based on the Linda paradigm, such as soft concurrent constraint programming, where a process "tells" and "asks" constraints to a centralised store [7]. Lexicographic orders are useful in applications that involve multiple objectives and attributes, and they have been extensively investigated in the literature on soft constraints. However, usually the connection has been established by encoding a lexicographic *hard* constraint problem, where the preference structure is a Boolean algebra, into a soft constraint formalism. However, while lifting the algebraic structure of a preference set to the associated set of (possibly infinite) tuples with a point-wise order is straightforward, doing the same for the lexicographic

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order is not, and it cannot be directly achieved for the formalisms in [2, 3]. The solution advanced in [8, 9] is to drop some preference values from the domain carrier of the set of tuples. The present work builds on this proposal by dealing with sets of preferences that form residuated monoids, systematise and extending the case of infinite tuples tackled in [10] to tuples of any length.

The paper extends those results in [11] concerning lexicographic orders, and it has the following structure. In Section 2 we present the background on partially ordered residuated monoids, the structure we adopt to model preferences. In Section 3 we consider the collapsing elements of a monoid, which are used to define an ad-hoc algebraic structure representing (possibly infinite) lexicographically ordered tuples of elements of the chosen monoid, as given in Section 4. The latter section also presents our main construction, introducing residuation for these lexicographically ordered monoids. Finally, in Section 5 we wrap up the paper with concluding remarks and ideas about future works.

2. Preliminaries

This section recalls some of the basic algebraic structures needed for defining the set of preference values. We propose elements of *ordered monoids* to serve as preferences, which allows us to compare and compose preference values.

2.1. Ordered Monoids

The first step is to define an algebraic structure for modelling preferences. We refer to [6] for an introduction and a comparison with other proposals.

Definition 1 (Orders)**.** *A partial order (PO) is a pair $\langle A, \leq \rangle$ such that A is a set and $\leq \subseteq A \times A$ is a reflexive, transitive, and anti-symmetric relation. A join semi-lattice (simply semi-lattice, SL) is a POs such that any finite subset of A has a least upper bound (LUB); a complete lattice (CL) is a PO such that any subset of A has a LUB.*

The LUB of a subset $X \subseteq A$ is denoted $\bigvee X$, and it is unique. Note that we require the existence of $\bigvee \emptyset$, which is the bottom of the order, denoted as $\bot$, and sometimes we will talk about a PO with bottom element (POB). The existence of LUBs for any subset of A (thus including $\emptyset$) guarantees that CLs also have greatest lower bounds (GLBs) for any subset X of A: it will be denoted by $\bigwedge X$. Whenever it exists, $\bigvee A$ corresponds to the top of the order, denoted as $\top$.

Definition 2 (Ordered monoids)**.** *A (commutative) monoid is a triple $\langle A, \otimes, \mathbf{1} \rangle$ such that $\otimes : A \times A \to A$ is a commutative and associative function and $\mathbf{1} \in A$ is its* identity *element, i.e., $\forall a \in A. a \otimes \mathbf{1} = a$.*

A partially ordered monoid (POM) is a 4-tuple $\langle A, \leq, \otimes, \mathbf{1} \rangle$ such that $\langle A, \leq \rangle$ is a PO and $\langle A, \otimes, \mathbf{1} \rangle$ a monoid. A semi-lattice monoid (SLM) and a complete lattice monoid (CLM) are POMs such that their underlying PO is a SL, a CL respectively.

For ease of notation, we use the infix notation: $a \otimes b$ stands for $\otimes(a, b)$.

Example 1 (Power set)**.** *Given a (possibly infinite) set V of variables, we consider the monoid $\langle 2^V, \cup, \emptyset \rangle$ of (possibly empty) subsets of V, with union as the monoidal operator. Since the operator is idempotent (i.e., $\forall a \in A.\, a \otimes a = a$), the natural order ($\forall a, b \in A.\, a \leq b$ iff $a \otimes b = b$) is a partial order, and it coincides with subset inclusion: in fact, $\langle 2^V, \subseteq, \cup, \emptyset \rangle$ is a CLM.*

In general, the partial order $\leq$ and the multiplication $\otimes$ can be unrelated. This is not the case for distributive CLMs.

Definition 3 (Distributivity)**.** *A SLM $\langle A, \leq, \otimes, \mathbf{1}\rangle$ is finitely distributive if $\forall X \subseteq_f A. \forall a \in A. \quad a \otimes \bigvee X = \bigvee\{a \otimes x \mid x \in X\}$. A CLM is distributive is the equality holds also for any subset.*

In the following, we will sometimes write $a \otimes X$ for the set $\{a \otimes x \mid x \in X\}$.

Remark 1. *Note that $a \leq b$ is equivalent to $\bigvee\{a, b\} = b$ for all $a, b \in A$. Hence, finite distributivity implies that $\otimes$ is monotone with respect to $\leq$ (i.e., $\forall a, b, c \in A. a \leq b \Rightarrow a \otimes c \leq b \otimes c$) and that $\perp$ is the zero element of the monoid (i.e., $\forall a \in A. a \otimes \perp = \perp$). The power-set CLM in Example 1 is distributive.*

Example 2 (Extended integers)**.** *The extended integers $\langle \mathbb{Z} \cup \{\pm\infty\}, \leq, +, 0\rangle$, where $\leq$ is the natural order. For $k \in \mathbb{Z}$ we have $-\infty \leq k \leq +\infty$, $+$ is the natural addition. For $k \in \mathbb{Z} \cup \{+\infty\}$ we have $\pm\infty + k = \pm\infty$, $+\infty + (-\infty) = -\infty$, and 0 the identity element constitutes a distributive CLM, and $+\infty$ and $-\infty$ are respectively the top and the bottom element of the CL.*

Remark 2. *Finitely distributive SLMs precisely corresponds to* tropical *semirings by defining the (idempotent) sum operator as $a \oplus b = \bigvee\{a, b\}$ for all $a, b \in A$. If, moreover, $\mathbf{1}$ is the top of the SLM we end up with* absorptive *semirings [12], which are known as c-semirings in the soft constraint jargon [2]. Together with monotonicity, imposing $\mathbf{1}$ to coincide with $\top$ means that preferences are negative (i.e., $a \leq \mathbf{1}$ for all $a \in A$). Distributive CLMs are also known as* quantales *[13].*

Remark 3. *Given two distributive CLMs, it is easy to show that their Cartesian product, whose elements are pairs and where the partial order and the monoidal operator are defined point-wise, is a distributive CLM. In particular, in the following we consider the Cartesian product of $\langle \mathbb{Z} \cup \{\pm\infty\}, \leq, +, 0\rangle$ with itself: its set of elements is $(\mathbb{Z} \cup \{\pm\infty\})^2$, the identity element is $(0, 0)$, and the top and bottom elements are $(+\infty, +\infty)$ and $(-\infty, -\infty)$, respectively.*

2.2. Residuated monoids

We first introduce *residuation*, which allows us to define a "weak" inverse operator with respect to the monoidal operator $\otimes$. In this way, besides aggregating values together, it is also possible to remove one from another. Residuation theory [12] is concerned with the study of sub-solutions of the equation $b \otimes x = a$, where x is a "divisor" of a with respect to b. The set of sub-solutions of an equation contains also the possible solutions, whenever they exist, and in that case the maximal element is also a solution.

Definition 4 (residuation)**.** *A residuated POM is a 5-tuple $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ such that $\langle A, \leq\rangle$ is a PO, $\langle A, \otimes, \mathbf{1}\rangle$ is a monoid, and $\ominus : A \times A \to A$ is a function such that: $\forall a, b, c \in A.\ b \otimes c \leq a \iff c \leq a \ominus b$.*

In the following, we will sometimes write $a \ominus X$ and $X \ominus a$ for the set $\{a \ominus x \mid x \in X\}$ and $\{x \ominus a \mid x \in X\}$, respectively.

Remark 4. *It is easy to show that residuation is monotone on the first argument and anti-monotone on the second. In fact, in a SML $\bigvee(X \ominus a) \leq \bigvee X \ominus a$, and the same in a CLM with respect to infinite sub-sets. However, the equality does not hold, e.g. in the Cartesian product of the CLM $\langle N \cup \{\infty\}, \geq, +, 0\rangle$ with itself. Also, $a \ominus \bigvee X \leq \bigwedge(a \ominus X)$ whenever the latter exists, as it does in CLMs.*

Remark 5. *As for distributivity, given two residuated POMs, it is easy to show that their Cartesian product is a residuated POM.*

Residuation implies distributivity (see e.g. [6, Lemma 2.2]).

Lemma 1. *Let $\langle A, \leq, \otimes, \mathbf{1} \rangle$ be a residuated POM. Then it is monotone. If additionally it is a SLM (CLM), then it is finitely distributive (distributive).*

Conversely, it is noteworthy that CLMs are always residuated, and the following folklore fact holds.

Lemma 2. *Let $\langle A, \leq, \otimes, \mathbf{1} \rangle$ be a distributive CLM. It is residuated and $\forall a, b \in A.\ a \ominus b = \bigvee \{c \mid b \otimes c \leq a\}$.*

3. The ideal of collapsing elements

As shown in [8], the first step for obtaining SLMs based on a lexicographic order is to restrict the carrier of the monoid.

Definition 5. *Let $\langle A, \otimes, \mathbf{1} \rangle$ be a monoid. Its sub-set $I(A)$ of* cancellative *elements is defined as $\{c \mid \forall a, b \in A.\ a \otimes c = b \otimes c \implies a = b\}$.*

We recall a well-known fact.

Lemma 3. *Let $\langle A, \otimes, \mathbf{1} \rangle$ be a monoid. Then $I(A)$ is a sub-monoid of A and $C(A) = A \setminus I(A)$ is a prime ideal of A.*

Explicitly, $C(A) = \{c \mid \exists a, b \in A.\ a \neq b \wedge a \otimes c = b \otimes c\}$. Being an ideal means that $\forall a \in A, c \in C(A).\ a \otimes c \in C(A)$, and being prime further states that $\forall a, b \in A.\ a \otimes b \in C(A) \implies a \in C(A) \vee b \in C(A)$. All the proofs are straightforward, and we denote $C(A)$ as the set of *collapsing* elements of A.

Note that an analogous closure property does not hold for LUBs.

Example 3. *Consider the monoid of natural numbers $\langle \mathbb{N}, +, 0 \rangle$ and the (non distributive) CLM with elements $\mathbb{N} \cup \{\bot, \top\}$ obtained by lifting the flat order (i.e., $a \not\leq b$ for any $a, b \in \mathbb{N}$ as well as $a + \bot = \bot = \top + \bot$ and $a + \top = \top$ for any $a \in \mathbb{N}$). Then, $I(\mathbb{N} \cup \{\bot, \top\}) = \mathbb{N}$ is not closed under finite LUBs.*

Now, let us consider the distributive CLM with elements $\mathbb{N} \cup \{\infty\}$ obtained by lifting the natural order induced by addition. We have that $I(\mathbb{N} \cup \{\infty\}) = \mathbb{N}$ is a (finitely distributive) SLM, yet it is not closed with respect to infinite LUBs.

We now present a simple fact that is needed later on.

Lemma 4. *Let A_1, A_2 be POMs and $A_1 \times A_2$ their Cartesian product. Then we have $C(A_1 \times A_2) = C(A_1) \times A_2 \cup A_2 \times C(A_2)$.*

Example 4. *Let us consider the tropical SLM $\langle \mathbb{N} \cup \{\infty\}, \geq, +, 0 \rangle$ and the Cartesian product with itself. Clearly, $C(\mathbb{N} \times \mathbb{N})$ is not closed under finite LUBs: it suffices to consider $X = \{\langle \infty, 3 \rangle, \langle 4, \infty \rangle\} \subseteq C(\mathbb{N} \times \mathbb{N})$, since $\bigvee X = \langle 3, 4 \rangle \notin C(\mathbb{N} \times \mathbb{N})$. Neither is $C(\mathbb{N} \times \mathbb{N})$ closed under residuation, since the top element is not necessarily collapsing. Indeed, in $C(\mathbb{N} \times \mathbb{N})$ we have $\langle \infty, 4 \rangle \ominus \langle \infty, 3 \rangle = \langle 0, 1 \rangle$.*

Remark 6. *In an absorptive CLM A we have that $a \ominus b = 1$ whenever $b \leq a$. Hence $C(A)$ is usually not closed under residuation, since 1 is cancellative.*

3.1. A different view on collapsing elements

When the first presentation of lexicographic SLMs was provided [8], a different set of collapsing elements was considered.

Definition 6 ([8])**.** *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a POM. Its sub-set $C'(A)$ is defined as $\{c \mid \exists a, b \in A.\ a < b \wedge a \otimes c = b \otimes c\}$.*

Clearly, $C'(A) \subseteq C(A)$. However, we can replicate Lemma 3.

Lemma 5. *Let $\langle A, \otimes, \mathbf{1}\rangle$ be a monoid. Then $C'(A)$ is an ideal of A. If $\otimes$ is monotone, then $I'(A) = A \setminus C'(A)$ is a sub-monoid of A and $C'(A)$ a prime ideal of A.*

Explicitly, $I'(A) = \{c \mid \forall a, b \in A.\ a \otimes c = b \otimes c \implies a \not< b\}$. The definitions we encounter in the next section could then be rephrased using $I'(A)$ and $C'(A)$ with minimal adjustments, thus confirming the proposal in [8].[1]

However, what is in fact noteworthy is that the two approaches are coincident whenever distributivity holds, as shown by the lemma below.

Lemma 6. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a finitely distributive SLM. Then $C'(A) = C(A)$.*

Proof. We already noted that $C'(A) \subseteq C(A)$ always holds. Now, let a, b, c such that $a \neq b \wedge a \otimes c = b \otimes c$: it suffices to consider $a \vee b$, noting that it must be either $a < a \vee b$ or $b < a \vee b$ and that by distributivity $(a \vee b) \otimes c = a \otimes c = b \otimes c$. □

Remark 7. *Consider the (non distributive) CLM $\langle [0 \dots n] \cup \{\bot, \top\}, +, 0\rangle$ obtained by lifting the initial segment $[0 \dots n]$ of the natural numbers with the flat order (as done for the CLM of natural numbers in Example 3). Here addition is capped, so that e.g. $n + m = n$ for all m. Hence, $C([0 \dots n] \cup \{\bot, \top\}) = [1 \dots n] \cup \{\bot, \top\}$ that is, all elements except 0. Instead, $C'([0 \dots n] \cup \{\bot, \top\})) = \{\bot, \top\}$.*

4. On lexicographic orders

We now move to lexicographic orders, considering the results in Section 3.

Proposition 1. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a POM with bottom element $\bot$. Then we can define a family $\langle Lex_k(A), \leq_k, \otimes^k, \mathbf{1}^k\rangle$ of POMs with bottom element $\bot^k$ such that $\otimes^k$ is defined point-wise, $Lex_1(A) = A$ and $\leq_1 = \leq$, and*

- *$Lex_{k+1}(A) = I(A)Lex_k(A) \cup C(A)\{\bot\}^k$,*
- *$a_1 \dots a_k \leq_k b_1 \dots b_k$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 \dots a_k \leq_{k-1} b_2 \dots b_k$.*

Proof. Monoidality of $\otimes^k$ as well as reflexivity and symmetry of $\leq_k$ are straightforward. As for transitivity, let $a_1 \dots a_k \leq_k b_1 \dots b_k$ and $b_1 \dots b_k \leq_k c_1 \dots c_k$. If $a_1 \leq b_1 < c_1$ or $a_1 < b_1 \leq c_1$, it follows immediately; if $a_1 = b_1 = c_1$, then it holds by induction. □

[1]And in fact, the lemma holds also for a property that is weaker than monotonicity: it suffices that $\forall a, b, c.\ a \leq b \implies (a \otimes c \leq b \otimes c) \vee (b \otimes c \leq a \otimes c)$.

Note that $Lex_k(A)$ is contained in the k-times Cartesian product A^k, and the definitions of $\otimes^k$, $\mathbf{1}^k$, and $\perp^k$ coincide. Also, the bottom element is needed for padding the tuples, in order to make simpler the definition of the order.

We can provide an alternative definition for such POMs.

Lemma 7. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a POM with bottom element $\perp$. Then $Lex_{k+1}(A) = \bigcup_{i\leq k} I(A)^i A\{\perp\}^{k-i}$ for all k.*

Proof. The proof is by induction on k. For $k = 0$ it is obvious. Let us assume it true for $k = n$, that is, $Lex_{n+1}(A) = \bigcup_{i\leq n} I(A)^i A\{\perp\}^{n-i}$. Now, $Lex_{n+2}(A) = I(A)Lex_{n+1}(A) \cup C(A)\{\perp\}^{n+1} = \bigcup_{i\leq n} I(A)^{i+1} A\{\perp\}^{n-i} \cup C(A)\{\perp\}^{n+1} = \bigcup_{i\leq n+1} I(A)^i A\{\perp\}^{n+1-i}$ and we are done, where the latter equality holds since $I(A)\{\perp\}^{n+1} \in \bigcup_{i\leq n} I(A)^{i+1} A\{\perp\}^{n-i}$ and $A = I(A) \cup C(A)$. □

Now, given a tuple a of elements in A^k, for $i \leq k$ we denote with a_i its i-th component and with $a_{|i}$ its prefix $a_1 \dots a_i$, with the obvious generalisation for a set $X \subseteq A^k$, noting that $a_1 = a_{|1}$.

Theorem 1. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a finitely distributive SLM (distributive CLM). Then so is $\langle Lex_k(A), \leq_k, \otimes^k, \mathbf{1}^k\rangle$ for all k.*

Proof. We inductively define the LUB $\bigvee X$ of a set $X \subseteq Lex_k(A)$ as

- $(\bigvee X)_1 = \bigvee X_1 = \bigvee\{y \mid y \in X_{|1}\}$
- $(\bigvee X)_{i+1} = \bigvee\{y \mid (\bigvee X)_1 \dots (\bigvee X)_i y \in X_{|i+1}\}$

Now, $\bigvee X$ is clearly a suitable candidate, since $x \leq_k \bigvee X$ for all $x \in X$. Minimality is proved inductively by exploiting the analogous definition of $\leq_k$.

Concerning distributivity, we need to show that $(\bigvee a \otimes^k X)_n = a_n \otimes (\bigvee X)_n$ holds for all $n \leq k$. We proceed by induction on n. If $n = 1$, this boils down to the distributivity of the underlying monoid, since

$$(\bigvee a \otimes^k X)_1 = \bigvee (a \otimes^k X)_1 = \bigvee\{y \mid y \in (a \otimes^k X)_{|1}\} =$$

$$\bigvee\{y \mid \exists z \in X_{|1}.\ y = a_1 \otimes z\} = \bigvee\{a_1 \otimes z \mid z \in X_{|1}\}$$

and

$$a_1 \otimes (\bigvee X)_1 = a_1 \otimes \bigvee\{y \mid y \in X_{|1}\} = \bigvee\{a_1 \otimes y \mid y \in X_{|1}\}$$

Now, let us assume that it holds for n. Now we have

$$(\bigvee a \otimes^k X)_{n+1} = \bigvee\{y \mid (\bigvee a \otimes^k X)_1 \dots (\bigvee a \otimes^k X)_n y \in (a \otimes^k X)_{|n+1}\} =$$

$$\bigvee\{y \mid \exists z \in X_{n+1}.\ y = a_{n+1} \otimes z \wedge a_1 \otimes (\bigvee X)_1 \dots a_n \otimes (\bigvee X)_n y \in (a \otimes^k X)_{|n+1}\} =$$

$$\bigvee\{a_{n+1} \otimes z \mid (\bigvee X)_1 \dots (\bigvee X)_n z \in X_{|n+1}\}$$

The latter equality obviously holds if $a_1 \dots a_n$ are cancellative, yet it holds also otherwise since in that case $a_{n+1} = \perp$, hence both sides coincide with $\perp$. Finally

$$a_{n+1} \otimes (\bigvee X)_{n+1} = a_{n+1} \otimes \bigvee\{y \mid (\bigvee X)_1 \dots (\bigvee X)_n y \in X_{|n+1}\} =$$

$$\bigvee\{a_{n+1} \otimes y \mid (\bigvee X)_1 \dots (\bigvee X)_n y \in X_{|n+1}\}$$

□

4.1. On lexicographic residuation

Since $Lex_k(A)$ is a CLM if so is A then $Lex_k(A)$ is also residuated.

Example 5. *Let us consider the usual tropical CLM of natural numbers with inverse order, and the CLM* $Lex_2(\mathbb{N})$*. Clearly* $C(\mathbb{N}) = +\infty$*. We then have for example that* $(3,6) \ominus_2 (4,2) = \bigvee\{(x,y) \mid (4+x, 2+y) \leq_2 (3,6)\} = (0,0)$*. Indeed,* $(4+x, 2+y) \leq_2 (3,6)$ *holds for any possible choice of* (x,y)*, since* $4+x < 3$ *for all* x*, hence* $(0,0)$ *as the result. Note that for the CLM obtained via the Cartesian product* $\mathbb{N} \times \mathbb{N}$*, the result would have been* $(0,4)$*.*

Indeed, this can be proved in general for POMs. First, we need some additional definitions and technical lemmas.

Definition 7. *Let* $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ *be a residuated POM with bottom and* $a, b \in Lex_k(A)$*. Then we define* $\gamma(a,b) = min\{i \mid (a_i \ominus b_i) \in C(A)\}$ *and* $\delta(a,b) = min\{i \mid (a_i \ominus b_i) \otimes b_i < a_i\}$*, with the convention that the result is* $k+1$ *whenever the set if empty.*

Lemma 8. *Let* $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ *be a residuated POM with bottom and* $a, b \in Lex_k(A)$*. Then either* $\delta(a,b) = k+1$ *or* $\delta(a,b) \leq \gamma(a,b)$*.*

Proof. If $\gamma(a,b) < \delta(a,b) \leq k$ then $a_{\gamma(a,b)} = (a_{\gamma(a,b)} \ominus b_{\gamma(a,b)}) \otimes b_{\gamma(a,b)}$. Since $C(A)$ is an ideal of A, it holds that $a_{\gamma(a,b)} \in C(A)$, which in turn implies that $a_{\delta(a,b)} = \bot$, hence a contradiction. □

We can then present the definition of residuation for lexicographic POMs only for the cases identified by the proposition above.

Proposition 2. *Let* $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ *be a residuated POM with bottom and* $a, b \in Lex_k(A)$*. If* $\delta(a,b) = \gamma(a,b) = k+1$ *then their residuation* $a \ominus_k b$ *in* $Lex_k(A)$ *exists and it is given by* $(a_1 \ominus b_1) \ldots (a_k \ominus b_k)$*.*

Proof. First of all, note that $a \ominus_k b \in I(A)^k \subseteq Lex_k(A)$. So, given $c \in Lex_k(A)$, we need to prove that $b \otimes^k c \leq_k a$ iff $c \leq_k a \ominus_k b$.

[$\mathbf{b \otimes^k c \leq_k a}$]. Let $l = min\{i \mid b_i \otimes c_i < a_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, k\} < \delta(a,b)$.

We have that $b_j \otimes c_j \leq a_j$ for all $j \leq m$, hence $c_j \leq a_j \ominus b_j$ for all $j \leq m$. If $c_n < a_n \ominus b_n$ for some $n \leq m$ we are done. Otherwise, if $m = k < l$ then $c_n = a_n \ominus b_n$ for all $n \leq k$ and we are done. Finally, if $m = l \leq k$ then $b_l \otimes c_l = b_l \otimes (a_l \ominus b_l) = a_l$ since $l < \delta(a,b)$, a contradiction.

[$\mathbf{c \leq_k a \ominus_k b}$]. Let $l = min\{i \mid c_i < (a \ominus_k b)_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, k\} < \delta(a,b)$.

We have that $c_j \leq (a \ominus_k b)_j = a_j \ominus b_j$ for all $j \leq m$, hence $b_j \otimes c_j \leq a_j$ for all $j \leq m$. If $b_n \otimes c_n < a_n$ for some $n \leq m$ we are done. Otherwise, if $m = k < l$ then $b_n \otimes c_n = a_n$ for all $n \leq k$ and we are done. Finally, if $m = l \leq k$ then $b_l \otimes c_l = a_l = b_l \otimes (a_l \ominus b_l)$ since $l < \delta(a,b)$, hence either a contradiction if $b_l \in I(A)$ or $b_p \otimes c_p = \bot = a_p$ for all $p > l$ if $b_l \in C(A)$ and consequently $a_l \in C(A)$. □

Note that $a \ominus_k b$ here coincides with the residuation $a \ominus^k b$ on the Cartesian product. Furthermore, we have that $(a \ominus_k b) \otimes^k b = a$.

Proposition 3. *Let $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ be a residuated POM with bottom and $a, b \in Lex_k(A)$. If $\delta(a,b) < \gamma(a,b)$ then their residuation $a \ominus_k b$ in $Lex_k(A)$ exists and it is given by $(a_1 \ominus b_1) \dots (a_{\delta(a,b)} \ominus b_{\delta(a,b)})(\bigvee Lex_{k-\delta(a,b)}(A))$.*

Proof. First of all, note that $(a \ominus_k b)_{|\delta(a,b)} \in I(A)^{\delta(a,b)} \subseteq Lex_{\delta(a,b)}(A)$. So, given $c \in Lex_k(A)$, we need to prove that $b \otimes^k c \leq_k a$ iff $c \leq_k a \ominus_k b$.

[$\mathbf{b \otimes^k c \leq_k a}$]. Let $l = min\{i \mid b_i \otimes c_i < a_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, \delta(a,b)\}$.

We have that $b_j \otimes c_j \leq a_j$ for all $j \leq m$, hence $c_j \leq a_j \ominus b_j$ for all $j \leq m$. If $c_n < a_n \ominus b_n$ for some $n \leq m$ we are done. Otherwise, if $m = \delta(a,b) \leq l$ then $c_j \leq a_j \ominus b_j$ for all $j \leq \delta(a,b)$ and we are done. Finally, if $m = l < \delta(a,b)$ then $b_l \otimes c_l = b_l \otimes (a_l \ominus b_l) = a_l$, a contradiction.

[$\mathbf{c \leq_k a \ominus_k b}$]. Let $l = min\{i \mid c_i < (a \ominus_k b)_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, \delta(a,b)\}$.

We have that $c_j \leq (a \ominus_k b)_j = a_j \ominus b_j$ for all $j \leq m$, hence $b_j \otimes c_j \leq a_j$ and $b_j \otimes c_j \leq b_j \otimes (a_j \ominus b_j)$ for all $j \leq m$, the latter by monotonicity of $\otimes$. If $b_n \otimes c_n < a_n$ for some $n \leq m$ we are done. Otherwise, if $m = \delta(a,b) \leq l$ then $b_{\delta(a,b)} \otimes (a_{\delta(a,b)} \ominus b_{\delta(a,b)}) < a_{\delta(a,b)} = b_{\delta(a,b)} \otimes c_{\delta(a,b)} \leq b_{\delta(a,b)} \otimes (a_{\delta(a,b)} \ominus b_{\delta(a,b)})$, hence a contradiction. Finally, if $m = l < \delta(a,b)$ then $b_l \otimes c_l = a_l = b_l \otimes (a_l \ominus b_l)$, hence either a contradiction if $b_l \in I(A)$ or $b_p \otimes c_p = \bot = a_p$ for all $p > l$ if $b_l \in C(A)$ and consequently $a_l \in C(A)$. $\square$

Additionally, please note that $\bigvee Lex_n(A)$ can be easily characterised: it coincides with $\top^n$ if $\top \in I(A)$, and with $\top\bot^{n-1}$ otherwise.

Proposition 4. *Let $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ be a residuated POM with bottom element $\bot$ and $a, b \in Lex_k(A)$. If $\delta(a,b) = \gamma(a,b) \leq k$ or $\gamma(a,b) < \delta(a,b)$ then their residuation $a \ominus_k b$ in $Lex_k(A)$ exists: it is given by $(a_1 \ominus b_1) \dots (a_{\gamma(a,b)} \ominus b_{\gamma(a,b)})\bot^{k-\gamma(a,b)}$.*

Proof. First of all, note that $(a \ominus_k b)_{|\gamma(a,b)} \in I(A)^{\gamma(a,b)-1}C(A) \subseteq Lex_{\gamma(a,b)}(A)$. Also, $\delta(a,b) > \gamma(a,b)$ implies that $\delta(a,b) = k+1$ and $a_{\gamma(a,b)} \in C(A)$. Given $c \in Lex_k(A)$, we need to prove that $b \otimes^k c \leq_k a$ iff $c \leq_k a \ominus_k b$.

[$\mathbf{b \otimes^k c \leq_k a}$]. Let $l = min\{i \mid b_i \otimes c_i < a_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, \gamma(a,b)\} \leq \delta(a,b)$.

We have that $b_j \otimes c_j \leq a_j$ for all $j \leq m$, hence $c_j \leq a_j \ominus b_j$ for all $j \leq m$. If $c_n < a_n \ominus b_n$ for some $n \leq m$ we are done. Otherwise, if $m = \gamma(a,b) \leq l$ then $c_{\gamma(a,b)} \in C(A)$ and we are done since $c_p = \bot = (a \ominus_k b)_p$ for all $p > \gamma(a,b)$. Finally, if $m = l < \gamma(a,b)$ then $b_l \otimes c_l = b_l \otimes (a_l \ominus b_l) = a_l$ since $l < \delta(a,b)$, a contradiction.

[$\mathbf{c \leq_k a \ominus_k b}$]. Let $l = min\{i \mid c_i < (a \ominus_k b)_i\}$, with the convention that the result is $k+1$ whenever the set if empty. Also, let $m = min\{l, \gamma(a,b)\} \leq \delta(a,b)$.

We have that $c_j \leq (a \ominus_k b)_j = a_j \ominus b_j$ for all $j \leq m$, hence $b_j \otimes c_j \leq a_j$ and $b_j \otimes c_j \leq b_j \otimes (a_j \ominus b_j)$ for all $j \leq m$, the latter by monotonicity of $\otimes$. If $b_n \otimes c_n < a_n$ for some $n \leq m$ we are done. Otherwise, if $m = \gamma(a,b) < l$ then $c_{\gamma(a,b)} \in C(A)$ and we are done since $b_p \otimes c_p = \bot \leq a_p$ for all $p > \gamma(a,b)$. Finally, if $m = l \leq \gamma(a,b)$ then $b_l \otimes (a_l \ominus b_l) \leq a_l = b_l \otimes c_l \leq b_l \otimes (a_l \ominus b_l)$ since $l \leq \delta(a,b)$, hence either a contradiction if $b_l \in I(A)$ or $b_p \otimes c_p = \bot = a_p$ for all $p > l$ if $b_l \in C(A)$ and consequently $a_l \in C(A)$. $\square$

From the propositions above it is straightforward to derive Theorem 2, which states that, given a residuated POM, it is possible to define a lexicographic order on its tuples, which is a residuated POM as well.

Theorem 2. *Let $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ be a residuated POM with bottom element $\perp$. Then so is $\langle Lex_k(A), \leq_k, \otimes^k, \ominus_k, \mathbf{1}^k\rangle$ for all k, with $\ominus_k$ defined as*

$$a \ominus_k b = \begin{cases} (a_1 \ominus b_1) \ldots (a_k \ominus b_k) & \text{if } k+1 = \gamma(a,b) = \delta(a,b) \\ (a_1 \ominus b_1) \ldots (a_{\gamma(a,b)} \ominus b_{\gamma(a,b)}) \perp^{k-\gamma(a,b)} & \text{if } k+1 \neq \gamma(a,b) \leq \delta(a,b) \\ (a_1 \ominus b_1) \ldots (a_{\delta(a,b)} \ominus b_{\delta(a,b)})(\bigvee Lex_{k-\delta(a,b)}(A)) & \text{otherwise} \end{cases}$$

4.2. Infinite tuples

We can now move to POMs whose elements are tuples of infinite length.

Proposition 5. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a POM with bottom element $\perp$. Then we can define a POM $\langle Lex_\omega(A), \leq_\omega, \otimes^\omega, \mathbf{1}^\omega\rangle$ with bottom element $\perp^\omega$ such that $\otimes^\omega$ is defined point-wise and* **i)** *$Lex_\omega(A) = I(A)^\omega \cup I(A)^* A\{\perp\}^\omega$,* **ii)** *$a \leq_\omega b$ if $a_{\leq k} \leq_k b_{\leq k}$ for all k.*

A straightforward adaptation of Proposition 1. Thus, we can define a POM of infinite tuples simply by lifting the family of POMs of finite tuples.

Remark 8. *Note that the seemingly obvious POM structure cannot be lifted to $\bigcup_k Lex_k(A) = I(A)^* A\{\perp\}^*$: it would be missing the identity of the monoid.*

Proposition 6. *Let $\langle A, \leq, \otimes, \mathbf{1}\rangle$ be a finitely distributive SLM (distributive CLM). Then so is $\langle Lex_\omega(A), \leq_\omega, \otimes^\omega, \mathbf{1}^\omega\rangle$.*

Also a straightforward adaptation, this time of Theorem 1.

Proposition 7. *Let $\langle A, \leq, \otimes, \ominus, \mathbf{1}\rangle$ be a residuated POM with bottom. Then so is $\langle Lex_\omega(A), \leq_\omega, \otimes^\omega, \ominus_\omega, \mathbf{1}^\omega\rangle$, with $\ominus_w$ defined as*

$$a \ominus_\omega b = \begin{cases} (a_1 \ominus b_1) \ldots (a_k \ominus b_k) \ldots & \text{if } \infty = \gamma(a,b) = \delta(a,b) \\ (a_1 \ominus b_1) \ldots (a_{\gamma(a,b)} \ominus b_{\gamma(a,b)}) \perp^\omega & \text{if } \infty \neq \gamma(a,b) \leq \delta(a,b) \\ (a_1 \ominus b_1) \ldots (a_{\delta(a,b)} \ominus b_{\delta(a,b)})(\bigvee Lex_\omega(A)) & \text{otherwise} \end{cases}$$

It follows from Theorem 2, via the obvious extension of Lemma 8. Note that $\bigvee Lex_\omega(A)$ is $\top^\omega$ if $\top \in I(A)$, and $\top\perp^\omega$ otherwise.

5. Conclusions

In this paper we considered a framework for soft constraint formalisms based on a residuated monoid of partially ordered preferences. More specifically, our focus was to show that the framework can include also lexicographic orders.

Despite the practical relevance of these orders, the valued structures for constraints proposed in the literature, such as the pioneering [2, 3], cannot straightforwardly deal with such a class of preferences, and their modelling required some ingenuity [8, 9]. Generalising such proposals by using residuated monoids [4] allows the extension of the classical solving algorithms that need preference removal, such as arc consistency, where values need to be moved from binary to unary constraints [6]. The paper shows that the resolution techniques devised for this richer framework can tackle also lexicographic orders.

Our current work is on the design of specific heuristics to specialise such techniques to constraint satisfaction problems involving these orders.

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