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On Limits in Complete Semirings

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Abstract. Various special cases of complete semirings are presented in a systematic way. We give new characterizations and a lot of examples.

0. Introduction

The notion of a complete semiring ([1], [2]) came up when classical formal language and automata theory was generalized to formal power-series (in noncommuting variables, with coefficients in a semiring). In the subsequent research, most authors considered more general concepts of summability (see [13] for an overview). On the other hand, it turned out that it may be useful to have additional axioms for complete semirings. Some of these axioms are motivated by the interconnection with the algebraic limit concept of Kuich and Salomaa [10], and so is the new one we introduce to define ω-finitary semirings. The main part of this paper is concerned with an overview of these and some related notions. We give a complete discussion of their interconnections, including new characterizations and counterexamples. Then we deal with topological summability as was introduced by Krob [6]. We connect this concept with the notions considered previously. (This was partially done by Krob himself.) Some problems remain open here. Finally, we consider finite resp. idempotent semirings. In these special cases, some notions happen to coincide.

The paper is organized as follows. Section 1 contains the basic definitions. (Apart from the basic topological concepts used in Sections 6 and 7, the paper is self-contained.) The subsequent sections then introduce step by step the various notions and discuss their interconnections. A first summary is given in Section 5. Topological semirings are discussed in Section 6. The final section considers finite resp. idempotent semirings. The information is always summarized in diagrams showing all implications and counterexamples.

We finally remark that all our semirings have both an absorbing zero and a unity. This is justified by the fact that we have automata theory in mind. (For example, the very important notion of a normalized automaton cannot be defined without the presence of the two neutral elements, cf. [10].)

1. Preliminaries

In this paper, a *semiring* is defined as an algebra $(A, +, \cdot, 0, 1)$, where $(A, +, 0)$ is a commutative monoid, $(A, \cdot, 1)$ is a monoid, and the operations are connected by the following laws:

$$(a+b)c = ac + bc, c(a+b) = ca + cb \,, \tag{1.1}$$

$$0 \cdot a = a \cdot 0 = 0 \,. \tag{1.2}$$

(Because of (1.2), the zero 0 is called absorbing.) In the following we only write $(A, +, \cdot)$ for such a semiring. A semiring is called *idempotent* if it satisfies $1 + 1 = 1$. A *complete* semiring $(A, +, \cdot, \Sigma)$ is a semiring $(A, +, \cdot)$ where, for every index set I and every family $(x_i, \; i \in I)$ of elements of A, the *sum* $\sum_{i \in I} x_i$ is defined and satisfies the following conditions:

$$\sum_{i \in \emptyset} x_i = 0 \,, \qquad \sum_{i \in \{1\}} x_i = x_1 \,, \qquad \sum_{i \in \{1,2\}} x_i = x_1 + x_2 \,, \tag{1.3}$$

$$\sum_{j \in J} \Big(\sum_{i \in I_j} x_i \Big) = \sum_{i \in I} x_i \,, \quad \text{if } I = \bigcup_{j \in J} I_j \text{ is a partition} \,, \tag{1.4}$$

$$\sum_{i \in I} z x_i = z \Big(\sum_{i \in I} x_i \Big) \,, \qquad \sum_{i \in I} x_i z = \Big(\sum_{i \in I} x_i \Big) z \,. \tag{1.5}$$

Informally speaking, $\sum$ extends the addition and satisfies the infinite laws of associativity and distributivity. Observe that, by (1.3) and (1.4),

$$\sum_{i \in I} x_i = \sum_{j \in J} \sum_{i \in \{\varphi(j)\}} x_i = \sum_{j \in J} x_{\varphi(j)} \tag{1.6}$$

for every bijection $\varphi : J \to I$.

Important examples of complete semirings are $\mathbb{B}$, $\mathbb{N}^{(\infty)}$ and $\mathbb{R}_+^{(\infty)}$, where $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$ is the Boolean semiring and $\mathbb{N}^{(\infty)}$ resp. $\mathbb{R}_+^{(\infty)}$ are obtained from $\mathbb{N}$ resp $\mathbb{R}_+$ (the nonnegative integers resp. reals) by adjoining an element ∞ satisfying $a + \infty = \infty + a = a \cdot \infty = \infty \cdot a = \infty$ for $a \neq 0$. (Observe that $\infty \cdot 0 = 0 \cdot \infty = 0$ by (1.2).) The infinite summation is then defined by

$$\sum_{i \in I} x_i = \sup_{F \subseteq I, F \text{ finite}} \sum_{i \in F} x_i \,,$$

where sup is taken with respect to the usual total order (in $\mathbb{B}$, we have $0 \leq 1$).

In a complete semiring, the sums $a^* = \sum_{i \in \mathbb{N}} a^i$ resp. $a^+ = \sum_{i \in \mathbb{N} \setminus \{0\}} a^i$ are called the *star* resp. the *quasiinverse* of $a \in A$.

If A is a complete semiring and $\sum$ is a finite or countable alphabet, then the semiring $(A\langle\langle \sum^* \rangle\rangle, +, \cdot)$ of formal power-series over the free monoid generated by $\sum$ is turned into a complete semiring by transferring the infinite summation of A pointwise to $A\langle\langle \sum^* \rangle\rangle$. A similar statement holds for the semiring of matrices $A^{J \times J}$ for an arbitrary index set J.

For any A, $A^{\mathbb{N}}$ denotes the set of sequences of elements of A. We will write $\alpha = (\alpha(n))$ for $\alpha \in A^{\mathbb{N}}$. Moreover, we will use the notations $(a_0, a_1, \ldots)$, $(a_n, \; n \in \mathbb{N})$, or simply (a_n) for sequences in $A^{\mathbb{N}}$.

A set of *convergent sequences* is a set $D \subseteq A^{\mathbb{N}}$ satisfying the following closure properties for α, α_1, $\alpha_2 \in D$, $c \in A$ (cf. [10]):

(1) $\eta = (1, 1, \ldots) \in D$,
(2) $\alpha_1 + \alpha_2 \in D$,
(3) $c\alpha, \alpha c \in D$,
(4) $\alpha_c = (c, \alpha(0), \alpha(1), \ldots) \in D$.

A *limit function* is a mapping $\lim : D \to A$ satisfying
(1) $\lim \eta = 1$,
(2) $\lim(\alpha_1 + \alpha_2) = \lim \alpha_1 + \lim \alpha_2$,
(3) $\lim c\alpha = c \lim \alpha$, $\lim \alpha c = (\lim \alpha)c$,
(4) $\lim \alpha_c = \lim \alpha$.

The least set satisfying (1)–(4) is the set D_d of ultimately constant sequences. The only possible limit function is denoted by $\lim_d$. A *notion of convergence* is a pair $(D, \lim)$. $(D_d, \lim_d)$ is called the *discrete convergence.* If A is also a partially ordered set (with order $\leq$), then any lim is called *compatible* with $\leq$ if $a_n \leq b_n$ for all $n \in \mathbb{N}$ (and two convergent sequences (a_n) and (b_n)) implies $\lim a_n \leq \lim b_n$. The discrete convergence is compatible with any order.

2. Basic Notions

All notions in this section are defined for semirings $(A, +, \cdot)$. Zerosumfree semirings were introduced by Kuich and Salomaa [10]. As the name indicates, the notion is defined by the following condition:

$$x + y = 0 \quad \text{implies} \quad x = y = 0 \; . \tag{2.1}$$

The next result is easily obtained.

Proposition 2.1. *Every complete semiring is zerosumfree.*

Proof. Observe first that, for all I,

$$\sum_{i \in I} 0 = \sum_{i \in I} 0 \cdot 1 = 0 \cdot \sum_{i \in I} 1 = 0 \; . \tag{2.2}$$

Assume now that $x + y = 0$. Then by (1.6),

$$0 = \sum_{i \in \mathbb{N}} (x + y) = x + \sum_{i \in \mathbb{N} \setminus \{0\}} x + \sum_{i \in \mathbb{N}} y = x + \sum_{i \in \mathbb{N}} (x + y) = x \; . \qquad \blacksquare$$

Equation (2.2) will be used in the sequel without further mention.

A semiring $(A, +, \cdot)$ together with a partial order $\leq$ on A is called here a *partially ordered (p.o.)* semiring $(A, +, \cdot, \leq)$ if the following conditions hold for all $a, b, c \in A$.

$$0 \leq a \quad \text{and} \tag{2.3}$$

$$a \leq b \quad \text{implies} \quad a + c \leq b + c, ac \leq bc, ca \leq cb \; . \tag{2.4}$$

From a more general point of view, a p.o. semiring as defined here should be called *positively* partially ordered, which we leave out for brevity. (See [14] for an overview and further references.) In the following, we only consider partial orders on semirings that make them p.o. in the above sense. The next result is well-known and will be very useful in the sequel.

Lemma 2.2. *Let $(A,+,\cdot,\leq)$ be a p.o. semiring. Then $a+x+y=a$ implies $a+x=a$ for all $a,x,y\in A$. Conversely, a semiring $(A,+,\cdot)$ satisfying the latter is a p.o. semiring $(A,+,\cdot,\sqsubseteq)$ if one defines $a\sqsubseteq b$ by $a+c=b$ for some $c\in A$.*

Proof. The first statement follows from $a\leq a+x\leq a+x+y=a$. Conversely, due to the assumption, $\sqsubseteq$ is a partial order on A which clearly satisfies (2.3) and (2.4). ■

The partial order defined in Lemma 2.2 is called the *natural* order on $(A,+,\cdot)$. We will always denote it by $\sqsubseteq$. It is the weakest partial order on $(A,+,\cdot)$, i.e., $a\sqsubseteq b$ implies $a\leq b$ for any partial order $\leq$ on A. (This is a direct consequence of (2.3) and the definition of $\sqsubseteq$.) Clearly, the usual order on the semirings $\mathbb{B}$, $\mathbb{N}^{(\infty)}$ and $\mathbb{R}_+^{(\infty)}$ is their natural order. As we show next, a semiring may actually have more than one partial order.

Example 2.1. Consider the (complete) semiring $A=\mathbb{R}_+^{(\infty)}$ and, for $r\geq 1$, the (complete) subsemiring $A_r=\mathbb{N}\cup[r,\infty]$. Then A_r has continuously many partial orders if $r>1$.

Proof. Clearly, A_r is (totally) ordered by the restriction $\leq$ of the usual order on A. Hence, it is also naturally ordered by Lemma 2.2. We denote the corresponding order by $\sqsubseteq_r$. Since, for $1\leq s\leq r$, A_r is also a subsemiring of A_s, $\sqsubseteq_s$ is a partial order on A_r. This proves the claim. ■

Every partially ordered semiring is zerosumfree, since $x+y=0$ implies $0\leq x\leq x+y=0$. Despite Proposition 2.1, a complete semiring is not necessarily partially ordered.

Example 2.2 (cf. [7]). Consider $A=\mathbb{Z}_2=\{\overline{0},\overline{1}\}$, the field of the residue classes of the integers (modulo 2). Adjoin a new zero element 0 according to (1.2). Then adjoin an element ∞ as described in Section 1 and define $\sum'$ on $A_1=\mathbb{Z}_2\cup\{0,\infty\}$ by

$$\sum_{i\in I}{}' a_i=\begin{cases}\sum\limits_{a_i\neq 0} a_i & \text{if } \{i\mid a_i\neq 0\} \text{ is finite,}\\ \infty & \text{otherwise}.\end{cases} \tag{2.5}$$

Then $(A_1,+,\cdot,\sum')$ is complete, but not partially ordered. We have $\overline{0}+\overline{1}+\overline{1}=\overline{0}$, but $\overline{0}+\overline{1}=\overline{1}\neq\overline{0}$, contradicting Lemma 2.2. (In fact, $\mathbb{Z}_2$ may be replaced by any ring with unity if the first equation is generalized to $\overline{0}+\overline{1}+(-\overline{1})=\overline{0}$.) ■

We finally remark that an idempotent semiring always has exactly one partial order, defined by $a\leq b$ iff $a+b=b$ (cf. [10], Exercise 5.4). Clearly $\leq$ coincides with the natural order.

3. d-Complete Semirings

We now start the discussion of the interconnection between limit notions and infinite sums. Recall that the discrete convergence $(D_d,\lim_d)$ can be defined in any semiring and is the "smallest" convergence. In a complete semiring, one may then ask whether $\sum$ is compatible with $\lim_d$. We call a complete semiring *d-complete* (cf. Goldstern [4]), if

$$\sum_{0\leq i\leq n} a_i=a \quad \text{for all } n\geq n_0 \text{ implies } \sum_{i\in\mathbb{N}} a_i=a\,. \tag{3.1}$$

Thus A being d-complete means that if the sequence of finite partial sums of a family $(a_i, i \in \mathbb{N})$ is ultimately constant, then the sum of this family equals that constant value. Using the notation of Section 1, this can also be expressed as

$$\lim_d \left(\sum_{0 \le i \le n} a_i \right) = \sum_{i \in \mathbb{N}} a_i \ . \tag{3.2}$$

The next example shows that a complete semiring is not necessarily d-complete.

Example 3.1 (cf. [4]). Consider the semiring $\mathbb{B}^{(\infty)} = \mathbb{B} \cup \{\infty\}$ with the extensions of the operations as in Section 1. Define $\sum'$ by (2.5). Then $(\mathbb{B}^{(\infty)}, +, \cdot, \sum')$ is complete, but not d-complete. Choose $a_i = 1$, $i \in \mathbb{N}$. Then $\sum'_{0 \le i \le n} a_i = 1$ for all n, but $\sum'_{i \in \mathbb{N}} a_i = \infty$. ■

The definition (3.1) uses a special order of the elements a_i which contrasts the "infinite commutativity" of $\sum$. The following characterization is more symmetric. (Condition (iii) is the axiom (PO3) of [5, p. 61].)

Proposition 3.1. *Assume that A is a complete semiring. Then the following statements are equivalent:*

(i) *A is d-complete.*

(ii) *$a + x_i = a$ for all $i \in \mathbb{N}$ implies $a + \sum_{i \in \mathbb{N}} x_i = a$.*

(iii) *If $(a_i, i \in I)$ is an at most countable family and $\sum_{i \in F} a_i = \sum_{i \in E} a_i$ for some finite $E \subseteq I$ and all finite F with $E \subseteq F \subseteq I$, then $\sum_{i \in I} a_i = \sum_{i \in E} a_i$.*

(iv) *If $(a_i, i \in I)$ is an at most countable family and for every finite $E \subseteq I$ there is a finite $F(E)$ with $E \subseteq F(E) \subseteq I$ and $\sum_{i \in F(E)} a_i = a$ (for some fixed a), then $\sum_{i \in I} a_i = a$.*

Proof. (i)$\Longrightarrow$(iv). We may assume that $I = \mathbb{N}$. We define the sequence $(F_n, n \in \mathbb{N})$ of finite index sets by $F_0 = F(\{0\})$, $F_{n+1} = F(F_n \cup \{\min(\mathbb{N} \backslash F_n)\})$. Clearly, $F_n \subsetneq F_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} F_n = \mathbb{N}$. Next we define the family $(c_n, n \in \mathbb{N})$ by

$$c_0 = \sum_{i \in F_0} a_i, \quad c_{n+1} = \sum_{i \in F_{n+1} \backslash F_n} a_i, \quad n \ge 0 \ .$$

Then we have $\sum_{i \le n} c_i = \sum_{i \in F_n} a_i = a$. Thus we obtain

$$\sum_{i \in I} a_i = \sum_{i \in \bigcup_{n \in \mathbb{N}} F_n} a_i = \sum_{n \in \mathbb{N}} c_n = a$$

since A is d-complete.

The implication (iv) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii). Suppose that $a + x_i = a$ for all $i \in \mathbb{N}$. We define the family $(a_i, i \in \mathbb{N})$ by $a_0 = a$, $a_{i+1} = x_i$, $i \geq 0$, and choose $E = \{0\}$. Then for all finite $F \supseteq E$, we have $\sum_{i \in F} a_i = \sum_{i \in E} a_i$ by induction on the cardinality of F. Thus we obtain

$$a + \sum_{i \in \mathbb{N}} x_i = \sum_{i \in \mathbb{N}} a_i = \sum_{i \in E} a_i = a\ .$$

(ii) $\Longrightarrow$ (i). Assume that $\sum_{i \leq n} a_i = a$ for $n \geq n_0$. We define $x_i = a_{n_0+i+1}$, $i \in \mathbb{N}$. Then $a + x_i = \sum_{j \leq n_0+i} a_j + a_{n_0+i+1} = \sum_{j \leq n_0+i+1} a_j = a$ for all i. Hence we obtain

$$\sum_{i \in \mathbb{N}} a_i = \sum_{i \leq n_0} a_i + \sum_{i > n_0} a_i = a + \sum_{i \in \mathbb{N}} x_i = a\ . \qquad \blacksquare$$

Finally, we establish the connection with the previous section.

Proposition 3.2 (Goldstern [4]). *Every d-complete semiring is partially ordered.*

Proof. Similar to that of Proposition 2.1, using Lemma 2.2. ∎

As d-completeness is a weak assumption, it is rather "natural" to conjecture that a complete semiring is partially ordered. However, this is not the case in general, as has already been shown in Example 2.2. Moreover, the converse of Proposition 3.2 does not hold, as is shown by Example 3.1. The semiring $\mathbb{B}^{(\infty)}$ is idempotent and totally ordered by $0 \leq 1 \leq \infty$.

4. l-Complete Semirings

This section continues the discussion about the interrelation between infinite sums and limit notions. (As already pointed out in the introduction, we use the algebraic limit concept according to [10], for topological considerations see Section 6.) This time we want to *define* a notion of convergence $(D, \lim)$ by the use of the infinite summation $\sum$. (Note that this is somewhat converse to real or complex analysis, where (topological) limits are used to define (some) countably infinite sums.) A preliminary step in this direction has been taken in the previous section. In any d-complete semiring, the notion of discrete convergence may be introduced according to (3.2). Clearly, (3.2) makes sense also for families (a_i) that do not satisfy the precondition of (3.1). Hence we choose the set of convergent sequences to be

$$D = \Big\{(b_n, n \in \mathbb{N}) \mid \text{there are } (a_i, i \in \mathbb{N}) \text{ and } n_0 \in \mathbb{N} \text{ with } b_n = \sum_{0 \leq i \leq n} a_i \text{ for } n \geq n_0\Big\}\ . \tag{4.1}$$

The definition of $\lim : D \to A$ is now a direct generalization of (3.2):

$$\lim(b_n) = \sum_{i \in \mathbb{N}} a_i\ , \tag{4.2}$$

where the families (a_i) and (b_n) are from (4.1). As we want lim to be well-defined, we need an additional assumption on $\sum$. We call a complete semiring *l-complete* (cf. Kuich [8]), if

$$\sum_{0 \leq i \leq n} a_i = \sum_{0 \leq i \leq n} a'_i \text{ for all } n \geq n_0 \text{ implies } \sum_{i \in \mathbb{N}} a_i = \sum_{i \in \mathbb{N}} a'_i\ . \tag{4.3}$$

As (4.3) is a generalization of (3.1), every l-complete semiring is d-complete. (The converse is not true, see the end of this section.) Hence it is partially ordered by the natural order of $\sqsubseteq$. Then by definition of D and $\sqsubseteq$, D coincides with the set of all ultimately monotonically increasing sequences. This gives additional motivation for the definition of D. In any l-complete semiring, the pair $(D, \lim)$ defined above is called the *natural convergence.*

The interrelation between the limit function and the partial order will be investigated in more detail in the next section. Our next result gives a more symmetric characterization of l-complete semirings. Condition (ii) is a strengthened version of the axiom (PO4) in [5, p. 61].

Proposition 4.1. *The following two statements are equivalent for a complete semiring A.*

(i) *A is l-complete.*

(ii) *If $(a_i, i \in I)$ and $(b_i, i \in I)$ are at most countable families and for every finite $E \subseteq I$, there is a finite $F(E)$ with $E \subseteq F(E) \subseteq I$ and $\sum_{i\in F(E)} a_i = \sum_{i\in F(E)} b_i$, then $\sum_{i\in I} a_i = \sum_{i\in I} b_i$.*

Proof. Similar to that of Proposition 3.1. ∎

Propositions 3.1 and 4.1 provide means to generalize the notions d-complete and l-complete to non-countable sums. The reader may want to verify that the semirings of Examples 4.1 and 5.2 satisfy Proposition 3.1 (ii)–(iv), resp. 4.1 (ii) for arbitrary index sets I.

We conclude this section with some examples. The semirings $\mathbb{B}$, $\mathbb{N}^{(\infty)}$ and $\mathbb{R}_+^{(\infty)}$ are l-complete. The semiring $\mathbb{B}^{(\infty)}$ of Example 3.1 is not l-complete, since it is not d-complete. The following is an example of a semiring that *is* d-complete, but not l-complete.

Example 4.1. Consider the complete semiring $A_1 = \mathbb{R}_+^{(\infty)} \times \mathbb{B}^{(\infty)}$ and its complete subsemiring $A_2 = A_1 \backslash \{(0,1),(0,\infty)\}$. Then a complete semiring A_3 is obtained from A_2 by identifying the elements $(\infty, 0)$, $(\infty, 1)$ and (∞,∞). Observe that A_3 was constructed in such a way that the equation $(a,b)+(x,y) = (a,b)$ always has only the solution $x = y = 0$ for $a < \infty$, whereas each $(x,y) \in A_3$ is a solution for $a = \infty$. From this it follows that A_3 is d-complete by Proposition 3.1 (ii). Define now $c_0 = d_0 = (1,1)$, $c_i = (1/2^i, 0)$, $d_i = (1/2^i, 1)$, $i \geq 1$. Then

$$\sum_{0\leq i\leq n} c_i = \sum_{0\leq i\leq n} d_i \text{ for all } n \in \mathbb{N}, \quad \text{but } \sum_{i\in\mathbb{N}} c_i = (2,1) \neq (2,\infty) = \sum_{i\in\mathbb{N}} d_i \,.$$

This shows that A_3 is not l-complete. ∎

5. (ω-)Finitary and (ω-)Continuous Semirings

As was already seen in Section 3, rather weak assumptions on the summation $\sum$ make a complete semiring partially ordered. In this section we investigate the interrelation between partial order and summation in more detail. *Finitary* semirings were introduced by Goldstern [4] as p.o. complete semirings $(A, +, \cdot, \leq, \sum)$ satisfying the condition

$$\sum_{i\in F} a_i \leq c \text{ for all finite } F \subseteq I \text{ implies } \sum_{i\in I} a_i \leq c \tag{5.1}$$

for families $(a_i, i \in I)$. Equivalently, every infinite sum is the supremum of all finite partial sums. (Thus $\sum$ is completely determined by $+$ and $\leq$.)

Examples of finitary semirings are $\mathbb{B}$, $\mathbb{N}^{(\infty)}$ and $\mathbb{R}_+^{(\infty)}$ (where $\leq$ is the usual total order). We want to point out that the notion depends heavily on the partial order under consideration. Hence we will always speak of finitary semirings *with respect to an order* $\leq$. Then we have the following

FACT 5.1. *Assume that A is finitary w.r.t. some partial order $\leq$. Then A need not be finitary w.r.t. the natural order $\sqsubseteq$.*

As an example, consider the finitary semiring $A = \mathbb{R}_+^{(\infty)} \times \mathbb{R}_+^{(\infty)}$ and its subsemiring $A_1 = A\backslash\{0\} \times (0,\infty]$. A_1 is complete under the restricted summation and hence finitary w.r.t. the pointwise partial order $\leq$. Consider now the family $a_i = (1/2^i, 1/2^i)$, $i \in \mathbb{N}$, and $c = (2,3)$. Then

$$\sum_{i\in F} a_i + \left(\sum_{i\in\mathbb{N}\backslash F} a_i + (0,1) \right) = c \ ,$$

implying $\sum\limits_{i\in F} a_i \sqsubseteq c$. But

$$\sum_{i\in\mathbb{N}} a_i = (2,2) \not\sqsubseteq (2,3) \ ,$$

since $(0,1)$ is not in A_1. Hence A_1 is not finitary w.r.t. $\sqsubseteq$. ∎

In view of the previous sections, it makes sense to restrict attention to countable sums. Thus we call a p.o. complete semiring $(A, +, \cdot, \leq, \sum)$ *ω-finitary* (with respect to $\leq$) if (5.1) holds for all at most countable index sets I. Again, the notion depends on the particular order. (The finitary semiring used to prove Fact 5.1 is not even ω-finitary with respect to $\sqsubseteq$). Moreover, as one may expect, an ω-finitary semiring is not necessarily finitary.

Example 5.1 (cf. [4]). We rename the elements 1 and ∞ of the semiring $\mathbb{B}^{(\infty)}$ (Example 3.1) as $\aleph_0$ and $\aleph_1$ to indicate that the elements will now resemble cardinal numbers. (Addition and multiplication remain unchanged.) We define

$$\sum_{i\in I} a_i = \begin{cases} \aleph_1 & \text{if } \{i \mid a_i \neq 0\} \text{ is not countable or some } a_i = \aleph_1 \\ \sum\limits_{a_i\neq 0} a_i & \text{if } \{i \mid a_i \neq 0\} \text{ is finite} \\ \aleph_0 & \text{otherwise .} \end{cases}$$

It is easily verified that $A = \{0, \aleph_0, \aleph_1\}$ is ω-finitary with respect to $0 \leq \aleph_0 \leq \aleph_1$. Consider now the family $(a_i, i \in \mathbb{R})$ with $a_i = \aleph_0$ for all i. Then $\sum\limits_{i\in F} a_i = \aleph_0$ for all finite $F \neq \emptyset$, but $\sum\limits_{i\in\mathbb{R}} a_i = \aleph_1$. Thus A is not finitary w.r.t. $\leq$ (which is the only order, since A is idempotent). ∎

If one considers limits in a partially ordered semiring, it is reasonable to require that the limit function is compatible with the partial order (cf. [10], Section 5). The next result shows that ω-finitary semirings have exactly this property. (This was the reason for introducing the notion.)

Proposition 5.2. *Assume that A is partially ordered (by $\leq$) and complete. Then A is ω-finitary (w.r.t. $\leq$) iff A is l-complete and the natural convergence is compatible with the order.*

Proof. Assume first that A is ω-finitary and $\sum_{0\leq i\leq n} a_i \leq \sum_{0\leq i\leq n} b_i$ for all $n \in \mathbb{N}$. Then $\sum_{0\leq i\leq n} a_i \leq \sum_{0\leq i\leq n} b_i + \sum_{i\geq n+1} b_i = \sum_{i\in\mathbb{N}} b_i$ and hence $\sum_{i\in\mathbb{N}} a_i \leq \sum_{i\in\mathbb{N}} b_i$. By symmetry, this implies that A is l-complete. Then by the last inequality, lim is compatible with $\leq$. Conversely, assume that $\sum_{i\in F} a_i \leq c$ for all finite $F \subseteq I$ (and I countable; we may assume $I = \mathbb{N}$). We define the sequences (b_n) and (c_n) by $b_n = \sum_{0\leq i\leq n} a_i$, $c_n = c$, $n \in \mathbb{N}$. Then $b_n \leq c_n$ for all n and hence $\sum_{i\in I} a_i = \lim b_n \leq \lim c_n = c$. ■

The next example shows that, in general, an l-complete semiring is not ω-finitary (w.r.t. any order).

Example 5.2. Consider the complete semiring $\mathbb{Q}_+^{(\infty)}$, the nonnegative rational numbers with an element ∞ as described in Section 1 and define $\sum'$ by (2.5). Clearly, A is l-complete. However, A is not ω-finitary w.r.t. the usual order $\leq$. For example, $\sum'_{i\in F} 1/2^i \leq 2$ for all finite $F \subseteq \mathbb{N}$, but $\sum'_{i\in\mathbb{N}} 1/2^i = \infty$. Observe that $\leq$ coincides with the natural order and recall that $\sqsubseteq$ is always the weakest order in a semiring. As $\sqsubseteq$ is already a total order in our case, there cannot be any other order on A. Thus A is not ω-finitary w.r.t. any order. ■

The next two results deal with an ω-finitary semiring A. The order $\leq$ is assumed to be fixed, but need *not* be the natural order.

Proposition 5.3. *a^*b is the least solution of $y = ay + b$.*

Proof. In [10], the limit concept was introduced as a means to define a "star" operation for matrices and power-series in arbitrary semirings. It is defined w.r.t. a given notion $(D, \lim)$ of convergence. Then $a^* = \lim\left(\sum_{0\leq i\leq n} a^i\right)$, provided this limit exists. Our definition of the star in a complete semiring is a special case of the above one if the semiring is l-complete and $(D, \lim)$ is chosen to be the natural convergence (cf. [8], p. 214). The statement now follows from [10], Theorem 5.11, by our Proposition 5.2 and the remark preceding it. ■

This statement need not be true for a p.o. l-complete semiring $(A, +, \cdot, \leq, \sum)$. Consider $\mathbb{Q}_+^{(\infty)}$ and the equation $y = \frac{1}{2}y+1$. Then $y = \left(\frac{1}{2}\right)^* = \infty$ is a solution, but $y = 2$ is the minimal solution.

Proposition 5.4. *Denote the natural convergence by $(D, \lim)$. Assume that there is another notion of convergence $(D', \lim')$ that is also compatible with $\leq$. Then $\lim \alpha = \lim' \alpha$ for all sequences $\alpha \in D \cap D'$.*

Proof. By definition of D, there is an $n_0 \in \mathbb{N}$ and a sequence $\overline{\alpha}$ satisfying

$$\alpha(n) = \sum_{0\leq i\leq n} \overline{\alpha}(i) \tag{1}$$

for $n \geq n_0$. Set $a = \lim \alpha$, $a' = \lim' \alpha$. For arbitrary $b \in A$ and $j \geq 0$, define the ultimately constant sequence

$$\beta^{(b,j)} = (\alpha(0), \alpha(1), \ldots, \alpha(j-1), b, b, \ldots) \in D \cap D' .$$

By (1) we have $\alpha \leq \beta^{(a,n_0)}$ and hence

$$a' = \lim{}' \alpha \leq \lim{}' \beta^{(a,n_0)} = a \ . \tag{2}$$

To establish the reverse inclusion, we show that

$$\alpha(n) \leq a' \ \text{ for } \ n \geq n_0 \ . \tag{3}$$

Consider, for $j \in \mathbb{N}$, the sequence $\beta_j = \beta^{(\alpha(j),j)}$. Then $\beta_j \leq \alpha$ for $j \geq n_0$. This implies

$$\alpha(j) = \lim{}' \beta_j \leq \lim{}' \alpha = a' \ ,$$

which is (3). (Observe that j remains fixed in the above computation.) Condition (3) implies $\alpha \leq \beta^{(a',n_0)}$. Hence

$$a = \lim \alpha \leq \lim \beta^{(a',n_0)} = a' \ . \tag{4}$$

Now $a = a'$ by (2) and (4). ■

The remainder of this section deals with the natural order. This is motivated by the fact that all semirings that are important in applications are finitary w.r.t. $\sqsubseteq$ (e.g. $\mathbb{B}$, $\mathbb{N}^{(\infty)}$, $\mathbb{R}_+^{(\infty)}$ and the matrix and power-series semirings derived from them, but also semirings occurring in connection with optimization problems, e.g. "shortest path", cf. [11] and the references given there). According to Sakarovitch [12] and Krob [7] we call these semirings *continuous.* Similarly, ω-continuous semirings are ω-finitary w.r.t. $\sqsubseteq$, cf. Kuich, [9] and the "d-continuous" semirings of [7]. Again, this notion is strictly weaker than continuity, cf. Example 5.1. The former authors do not consider the more general notion "finitary" at all, but as the example in the proof of Fact 5.1 shows, finitary semirings may occur as complete subsemirings of continuous ones.

As the natural order is completely determined by the addition, the same holds for the infinite summation in a continuous semiring by the remark immediately after (5.1). Clearly, a similar statement is true for countable sums and ω-continuity. As a consequence, we have the next result, which is again important for applications in automata theory.

Theorem 5.5. *Assume that A is a complete semiring, J an index set and $\sum$ an alphabet. Then A is continuous iff $A^{J\times J}$ (resp. $A\langle\langle\sum^*\rangle\rangle$) is, and in this case, summation in $A^{J\times J}$ (resp. $A\langle\langle\sum^*\rangle\rangle$) is the pointwise extension of the summation in A.*

Proof. As addition is defined pointwise in $A^{J\times J}$ and $A\langle\langle\sum^*\rangle\rangle$, so is the natural order. Now everything follows by the above remark and the fact that A is isomorphic to a subsemiring of $A^{J\times J}$ (resp. $A\langle\langle\sum^*\rangle\rangle$). ■

Again, a similar statement holds for ω-continuous semirings and countable summation.

We think that restriction to continuous (or at least ω-continuous) semirings makes the axiomatization of automata theory more satisfactory. We want to illustrate this by an example.

Example 5.3. Consider the matrix

$$M = \begin{pmatrix} \frac{1}{2}\varepsilon & x \\ 0 & x \end{pmatrix} \in \left(\mathbb{R}_+^{(\infty)}\langle\langle\{x\}^*\rangle\rangle\right)^{2\times 2} \ ,$$

which can be viewed as the transition matrix of a finite automaton, cf. [10]. Transfer the infinite summation from the continuous semiring $\left(\mathbb{R}_+^{(\infty)}, +, \cdot, \sqsubseteq, \sum\right)$ pointwise to $\mathbb{R}_+^{(\infty)}\langle\langle\{x\}^*\rangle\rangle$ and then in turn to $\left(\mathbb{R}_+^{(\infty)}\langle\langle\{x\}^*\rangle\rangle\right)^{2\times 2}$. Then the latter semiring is also continuous and induction on n (to obtain M^n) yields

$$M^* = \begin{pmatrix} 2\varepsilon & 2x^+ \\ 0 & x^* \end{pmatrix} .$$

On the other hand, Theorem 2.5 of [8] allows to compute the star of any matrix $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ over an l-complete semiring as $M^* = \begin{pmatrix} a^* & a^*bc^* \\ 0 & c^* \end{pmatrix}$.

Observe now that the nonnegative reals may also be equipped with the infinite summation according to (2.5) (cf. [6]). It is easily seen that $\mathbb{R}_+^{(\infty)}$ is l-complete also with respect to $\sum'$. If now power-series and matrices are summed pointwise according to $\sum'$, then application of the above mentioned result yields

$$M^* = \begin{pmatrix} \infty \cdot \varepsilon & \infty \cdot x^+ \\ 0 & x^* \end{pmatrix} . \qquad \blacksquare$$

With respect to this example, it is meaningful to restrict our attention to continuous semirings. (And as most sums occurring in automata theory are countable, restriction to ω-continuous ones will suffice, cf. [9]).

Continuous semirings were defined by a condition connecting summation and natural order. Surprisingly, it turns out that a characterization can be given that does not mention the partial order. This is the last result of the section.

Proposition 5.6. *Assume that A is a complete semiring. Then A is continuous (resp. ω-continuous) iff the following condition holds for all I (resp. for all at most countable I) and all $c \in A$:*

If for all finite $F \subseteq I$ there is a b_F such that $\sum_{i \in F} a_i + b_F = c$,

then there is also some b satisfying $\sum_{i \in I} a_i + b = c$. (5.7)

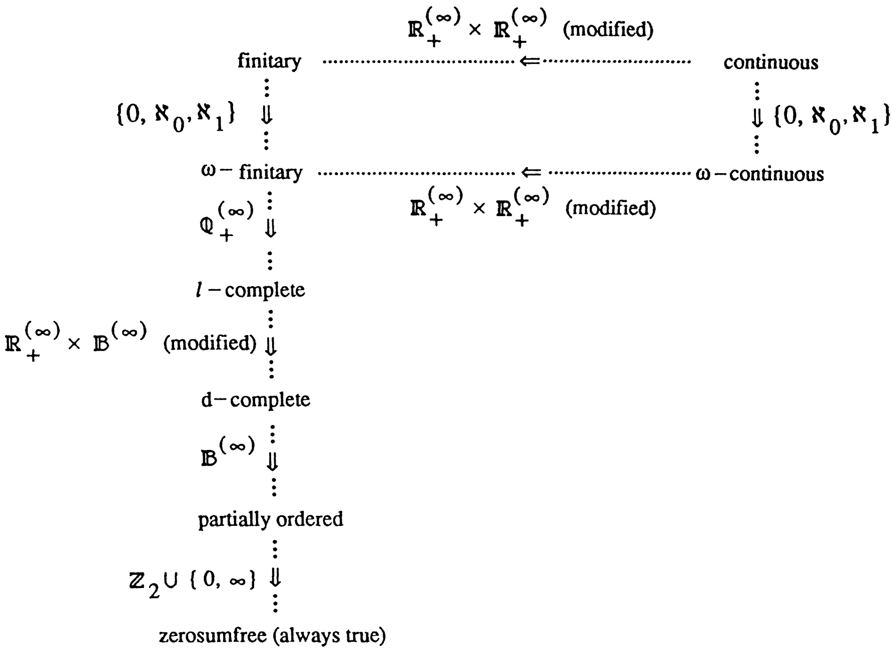
Proof. Observe that (5.7) is obtained from (5.1) by substituting the definition of the natural order. So we only have to show that every semiring satisfying (5.7) is naturally ordered. We use Lemma 2.2. Assume that $a + x + y = a$. Define $a_0 = a$, $a_{2i+1} = x$, $a_{2i+2} = y$, $i \geq 0$. Then the family $(a_i, i \in \mathbb{N})$ satisfies (5.7) with $c = a$, since by induction $a + n \cdot x + n \cdot y = a$ for all $n \in \mathbb{N}$. Hence

$$a = b + \sum_{i \in \mathbb{N}} a_i = b + \sum_{i \in \mathbb{N}} a_i + x = a + x ,$$

where the second equality follows as in Proposition 2.1. $\blacksquare$

The results of Sections 2–5 are summarized in Fig. 5.1. All implications are strict and are accompanied by a counterexample for the reverse direction.

Figure 5.1.



Properties of complete semirings

We invite the reader to verify that all our examples (except possibly the ones involving the reals) have minimal cardinality (see also Section 7).

6. Topological Complete Semirings

Every complete semiring A has a "natural" class of nets (in the topological sense). Consider an arbitrary family $(a_i, i \in I)$ of elements of A. Then $\{F \subseteq I \mid F \text{ is finite}\}$ is directed by set inclusion and thus $\left(\sum_{i\in F} a_i; F \subseteq I, F \text{ finite}\right)$ is a net. One may then ask whether $\sum_{i\in I} a_i$ is the limit of this net w.r.t. some T_2-topology. This was done by Krob [6] (cf. Def. II.4 and Def. III.2) who called a semiring $(A, +, \cdot, \mathcal{T})$ t-complete if $\mathcal{T}$ is a T_2-topology on A such that every net of the form described above has a limit and addition and multiplication are continuous. Every t-complete semiring is also a complete semiring $(A, +, \cdot, \sum)$ if $\sum_{i\in I} a_i$ is defined as the limit of $\left(\sum_{i\in F} a_i; F \subseteq I, F \text{ finite}\right)$, see [6], Proposition III.4. Thus for every neighbourhood V of $\sum_{i\in I} a_i$ there is a finite F_0 such that

$$\sum_{i\in F} a_i \in V \quad \text{for all finite } F \supseteq F_0 \,. \tag{6.1}$$

A reader familiar with topology may already have noticed that the characterizations of Propositions 3.1 and 4.1 reflect the above described view of sums as limits of nets. (For example, Proposition 3.1 (iii) is a statement on eventually constant nets, and Proposition 4.1 (ii) deals with nets that frequently coincide.) The following is now immediate by the uniqueness of limits in a T_2-space.

Proposition 6.1. *Every t-complete semiring is l-complete.* ∎

Thus every t-complete semiring is naturally ordered by Proposition 3.2, cf. also [6], Prop. II.9.

The above result cannot be strengthened, in general, see Example 6.1. The converse is also not true, as is seen by the following example.

Example 5.1 (continued). The semiring $A = \{0, \aleph_0, \aleph_1\}$ is ω-continuous, but not t-complete. This is again shown by the family $(a_i, i \in \mathbb{R})$ with $a_i = \aleph_0$ for all i. Assume that A is t-complete and consider an arbitrary neighbourhood V of $\aleph_1$. As $\sum_{i \in \mathbb{R}} a_i = \aleph_1$, but $\sum_{i \in F} a_i = \aleph_0$ for all non-empty, finite F, V must contain $\aleph_0$. This shows that the T_2-axiom is not satisfied. ∎

Krob [6] also considered "dt-complete" semirings, which are defined by restricting the nets defined above to at most countable index sets. (This means that only countable sums can be defined according to (6.1). Hence the notion does not really fit the framework of this paper.) Clearly, a dt-complete semiring also satisfies Proposition 4.1 (ii). By [7], Proposition V.1, the semiring of the above example *is* dt-complete. Seen from this point of view, the following example is more satisfactory.

Example 5.2 (continued). The l-complete semiring $\mathbb{Q}_+^{(\infty)}$ is not dt-complete. This can be seen by considering neighbourhoods of ∞. Assume that there is a T_2-topology on $\mathbb{Q}_+^{(\infty)}$ satisfying (6.1) for all at most countable index sets I. We show that addition is not continuous. Let a be an arbitrary fixed positive rational number. Then by (T_2), there is an open set V with $\infty \in V$, $a \notin V$. Recall that $\infty + \infty = \infty$. If the addition were continuous there would be open neighbourhoods U_1 and U_2 of ∞ with $U_1 + U_2 \subseteq V$. Assume furthermore that for all $b < a/2$, there is an $x(b)$ with $b < x(b) < a/2$ and $x(b) \notin U_1$. Then we define the sequences (c_n) and (d_n) by $c_0 = x(a/4)$, $c_{n+1} = x(c_n)$, $d_0 = c_0$, $d_{n+1} = c_{n+1} - c_n > 0$ for $n \geq 0$. Thus we have $\sum'_{i \in \mathbb{N}} d_n = \infty$, but $\sum'_{0 \leq i \leq n} d_i = c_n \notin U_1$ for all n, contradicting (6.1). Hence for some $b_0 < a/2$, $(b_0, a/2) \cap \mathbb{Q} \subseteq U_1$. Now if there is a $b_1 \in U_2$ with $a/2 < b_1 < a - b_0$, then $a = (a - b_1) + b_1 \in U_1 + U_2 \subseteq V$, contradicting $a \notin V$. But if there is no such b_1, then U_2 violates (6.1), as can be seen by an argument similar to the above one. ∎

The question arises if there are assumptions on a complete semiring that force it to be t-complete. The continuous semirings being promising candidates, only a partial result has been obtained so far. We call a semiring *naturally totally ordered* (n.t.o.), if the natural order is a total order. Observe that $\mathbb{B}$, $\mathbb{N}^{(\infty)}$ and $\mathbb{R}_+^{(\infty)}$ are n.t.o.

Theorem 6.2 (Krob, [7], Prop. V.1 and Cor. V.2). *Every naturally totally ordered continuous semiring is t-complete.* ∎

Again, the converse is not true. The following example combines ideas from [7] and [4]. It shows that in general a t-complete semiring does not satisfy any of the axioms of the previous sections that is stronger than "l-complete".

Example 6.1. Let $A = \mathbb{N} \cup \{\omega - n \mid n \in \mathbb{N}\}$ and denote $\omega - 0$ by ω. Addition and multiplication on $\mathbb{N}$ are defined as usual and are extended by

$$(\omega - n_1) + (\omega - n_2) = \omega, \quad (\omega - n_1) + n_2 = \begin{cases} \omega - (n_1 - n_2) & \text{if } n_1 > n_2 \\ \omega & \text{otherwise}, \end{cases}$$
$$(\omega - n_1) \cdot (\omega - n_2) = \omega, \quad (\omega - n_1) \cdot n_2 = n_2 \cdot (\omega - n_1) = \omega \ \text{ if } \ n_2 \geq 2 .$$

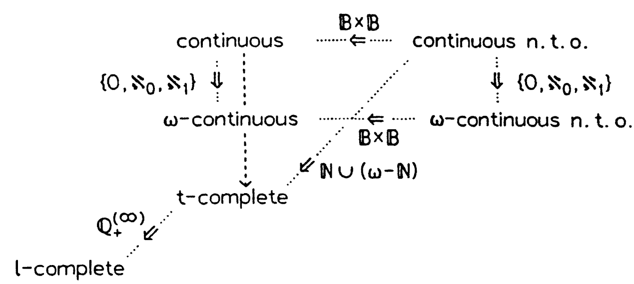
Here we have $n_1, n_2 \in \mathbb{N}$. We define infinite summation as in Example 2.2. Moreover, we introduce a topology on A by giving the base $\mathcal{B} = \{\{a\} \mid a \in A \backslash \{\omega\}\} \cup \{([n,\infty) \cap \mathbb{N}) \cap \{\omega\} \mid n \in \mathbb{N}\}$. Then A is t-complete, cf [7], p. 66 – the proof of the continuity of the addition easily transfers to the multiplication. Clearly, A is also totally ordered by the natural order. Observe furthermore that $\sum_{0 \leq i \leq n} i \sqsubseteq \omega - 1$ for all n, but $\sum_{i \in \mathbb{N}} i = \omega$. This shows that A is not ω-continuous. By the argument of Example 5.2, A is not ω-finitary w.r.t. any order. ∎

We remark that, as all our semirings, A is unitary. So the above example confirms a conjecture of Krob [7, p. 78].

Clearly, a continuous semiring is not necessarily n.t.o. A nice example is $A = \mathbb{B} \times \mathbb{B}$, which is isomorphic to the powerset of $\{x, y\}$ with union and intersection as operations. (This semiring is also t-complete, as will be seen in Section 7.)

We summarize the results of this section graphically. The main question (whether a continuous semiring is always topological) remains open. The broken arrow denotes the corresponding conjecture posed in [7, p. 61].

Figure 6.1.



Some more properties of complete semirings

7. Idempotent and Finite Semirings

Looking at the various counterexamples of Figures 5.1 and 6.1, a lot of them are "small" (i.e. contain three or four elements only) and, moreover, idempotent, whereas some others involve the real numbers. In this section we show that at least the latter cannot be replaced by "very simple" ones, i.e. some notions that are different in the general case turn out to be equivalent for finite or idempotent semirings. Moreover, considering idempotent semirings is motivated also by applications (e.g. formal language theory, cf. [10]).

The first result shows that d-completeness is a strong notion under our additional assumptions.

Proposition 7.1. *Let A be d-complete. If A is finite or idempotent, then A is ω-finitary w.r.t. every partial order.*

Proof. If A is finite, the statement follows from the observation that every ultimately monotonic sequence is ultimately constant. If A is idempotent, d-completeness implies that $\sum_{i\in\mathbb{N}} a = a$ for all $a \in A$. Recall that the only possible order is defined by $a \leq b$ iff $a + b = b$. If $\sum_{i\in F} a_i \leq c$ for all finite $F \subseteq I$ and I countable, then

$$\left(\sum_{i\in I} a_i\right) + c = \sum_{i\in I} a_i + \sum_{i\in I} c = \sum_{i\in I}(a_i + c) = \sum_{i\in I} c = c\,. \qquad \blacksquare$$

Remark. The condition $\sum_{i\in\mathbb{N}} a = a$ for all $a \in A$ is called "Axiom (ID)" by Hebisch [5]. It implies that A is idempotent ([5], Bemerkung 11.4). Together with the above proof this shows that (ID) implies A being ω-continuous, which extends [5], Satz 11.7.

Next, we deal with finitary semirings. As an idempotent semiring has exactly one partial order, the notions "finitary" and "continuous" coincide under this assumption. A similar result holds for finite semirings.

Proposition 7.2. *Assume that A is finite and finitary w.r.t. some partial order $\leq$. Then A is finitary w.r.t. any partial order (and hence continuous).*

Proof. Consider an arbitrary family $(a_i, i \in I)$. At first one can show that the set $X = \left\{\sum_{i\in F} a_i \mid F \subseteq I, F \text{ finite}\right\}$ has a maximum $\max_{\leq'} X$ w.r.t. each partial order $\leq'$ on A, and that each of these coincides with $\max_{\sqsubseteq} X$. Since A is finitary w.r.t. $\leq$, we have $\sum_{i\in I} a_i = \max X$, where the maximum is taken w.r.t. $\leq$ and hence w.r.t. any partial order $\leq'$ on A. $\blacksquare$

We finally consider t-complete semirings. This notion proves to be very strong under our assumptions, too. (Recall that every semiring of this kind is partially ordered.)

Proposition 7.3. *Let A be t-complete. If A is finite or idempotent, then A is finitary w.r.t. any partial order.*

Proof. If A is idempotent, the net $\left(\sum_{a\in F} a \mid F \subseteq I, F \text{ finite}\right)$ converges to a. This yields $\sum_{i\in I} a = a$ for all $a \in A$ and each I, and one can go on in the pattern of the proof of Proposition 7.1. If A is finite, recall that $X = \left\{\sum_{i\in F} a_i \mid F \subseteq I, F \text{ finite}\right\}$ has the same maximum $b = \max X$ w.r.t. any partial order. Since the net corresponding to X converges to b, we obtain $\sum_{i\in I} a_i = \max X$ which completes the proof. $\blacksquare$

The converse is also true for finite semirings, but the question remains open for idempotent ones (see also below, Theorem 7.5).

Proposition 7.4. *Every finite finitary semiring is t-complete.*

Proof. Consider the discrete topology. By the proof of Proposition 7.2, this topology satisfies (6.1). Clearly the semiring operations are continuous. ∎

Again we summarize our results in figures. All questions are settled for finite semirings. As regards the idempotent ones, it remains open, whether every continuous semiring is t-complete.

Figure 7.1.

$$\begin{array}{ccc}
\text{finitary} \Leftrightarrow \text{continuous} \Leftrightarrow \text{t-complete} & \overset{\mathbb{B}\times\mathbb{B}}{\Leftarrow} & \text{continuous n. t. o.} \\
\{0,\aleph_0,\aleph_1\}\ \Downarrow & & \Downarrow\ \{0,\aleph_0,\aleph_1\} \\
\omega\text{-finitary} \Leftrightarrow \omega\text{-continuous} \Leftrightarrow \text{l-complete} \Leftrightarrow \text{d-complete} & \underset{\mathbb{B}\times\mathbb{B}}{\Leftarrow} & \omega\text{-continuous n. t. o.} \\
\mathbb{B}^{(\infty)}\ \Downarrow & & \\
\text{partially ordered} & & \\
\mathbb{Z}_2\cup\{0,\infty\}\ \Downarrow & & \\
\text{zerosumfree (always true)} & &
\end{array}$$

Properties of finite, complete semirings

Figure 7.2.

$$\begin{array}{ccc}
\text{finitary} \Leftrightarrow \text{continuous} \ \underset{\Leftarrow}{\dashrightarrow}\ \text{t-complete} & \overset{\mathbb{B}\times\mathbb{B}}{\Leftarrow} & \text{continuous n. t. o.} \\
\{0,\aleph_0,\aleph_1\}\ \Downarrow & & \Downarrow\ \{0,\aleph_0,\aleph_1\} \\
\omega\text{-finitary} \Leftrightarrow \omega\text{-continuous} \Leftrightarrow \text{l-complete} \Leftrightarrow \text{d-complete} & \underset{\mathbb{B}\times\mathbb{B}}{\Leftarrow} & \omega\text{-continuous n. t. o.} \\
\mathbb{B}^{(\infty)}\ \Downarrow & & \\
\text{partially ordered, zerosumfree (always true)} & &
\end{array}$$

Properties of idempotent, complete semirings

Remark. The results about idempotent semirings are partially due to Hebisch [5]. Krob [7, p. 77], has a remark on finite complete monoids (cf. below). It claims that for such a monoid $(M, +, \sum)$ the properties "continuous", "d-continuous" (a notion similar to our "ω-continuous"), "t-complete", and "dt-complete" are equivalent. This would imply the same statement for finite complete semirings, which our Example 5.1 shows to be incorrect.

We conclude our paper with a partial result supporting Krob's conjecture mentioned above. All notions used in this paper transfer to (additively written) commutative monoids by omitting the axioms that involve multiplication. (This notion is called "complete monoid" in [6].) Idempotence of a monoid M means $a + a = a$ for all $a \in M$. With these definitions we can give the following characterization.

Theorem 7.5. *Let $(M, +, \sum)$ be a complete idempotent monoid. Then M is t-complete iff it is continuous.*

Proof. By Proposition 7.3, we only have to construct the topology for a continuous monoid. Define, for $c \in M$, the sets $U_c = \{x \in M \mid x \leq c\}$ and $V_c = M \backslash U_c$. Note that in general $V_c \neq \{x \in M \mid x > c\}$, since the order is only a partial one. Now a subbase for our topology is given by $\{U_c \mid c \in M\} \cup \{V_c \mid c \in M\}$. The topology satisfies (T_2), since if $a \neq b$, then for example $a \not\leq b$ and $a \in V_b$, $b \in U_b$. By continuity, (6.1) is satisfied for U_c and V_c. Clearly, (6.1) transfers then to arbitrary open sets. It only remains to show that addition is continuous. If $a + b \in U_c$, then $U_a + U_b \subseteq U_c$. If $a + b \in V_d$, then $a \in V_d$ or $b \in V_d$, since otherwise $a + b \leq d + d = d$. (The idempotence of M is used only here!) Now if $a \in V_d$, then $V_d + A \subseteq V_d$, and V_d resp. A are neighbourhoods of a and b, respectively. ∎

The topologies used by Krob ([7], Prop. III.3 and Cor. III.4) to prove Theorem 6.2 are finer than ours when transferred to *partially* ordered semirings. (This follows from [7], Proposition III.2 and the fact that M is continuous.) Thus it is possible to show the continuity of the multiplication. On the other hand, the topologies are too fine in the general case, since (6.1) may be violated.

The above characterization shows that as regards monoids, the broken arrow in Fig. 7.2 may be replaced by an equivalence sign. Thus the discussion is complete in this case.

We remark that some rather strange phenomena occur when dealing with complete monoids instead of semirings. For example, the very "natural" equation (2.2) does not hold in general, and also the zerosumfreeness in Fig. 7.2 then follows from the idempotence, but not from completeness axioms. See [6] for details and examples.

Conclusion

We gave a systematic overview of various axioms for complete semirings. It turned out that any of these forces the semiring to be partially ordered, which appears thus to be a basic feature of complete semirings. Moreover, in most cases, both a "symmetric" definition (reflecting the infinite commutativity of the summation) and a characterization using sequences were given. This shows the notions to be "sound". The main open question is whether a continuous semiring is always topologically complete.

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