Homework 1: Linear Algebra and Floating Point Arithmetic

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Abstract

This report provides a comprehensive analysis of the conditioning of random, Vandermonde, and Hilbert matrices, and its effects on the accuracy of computed solutions for linear systems. The findings are supported by graphical representations derived from numerical experiments.

1 Random Matrices

In the initial analysis of random matrices, the condition numbers were observed to increase with matrix size, suggesting a trend towards numerical instability for larger matrices. The figure below illustrates the condition numbers obtained from a single random matrix generation for each size.

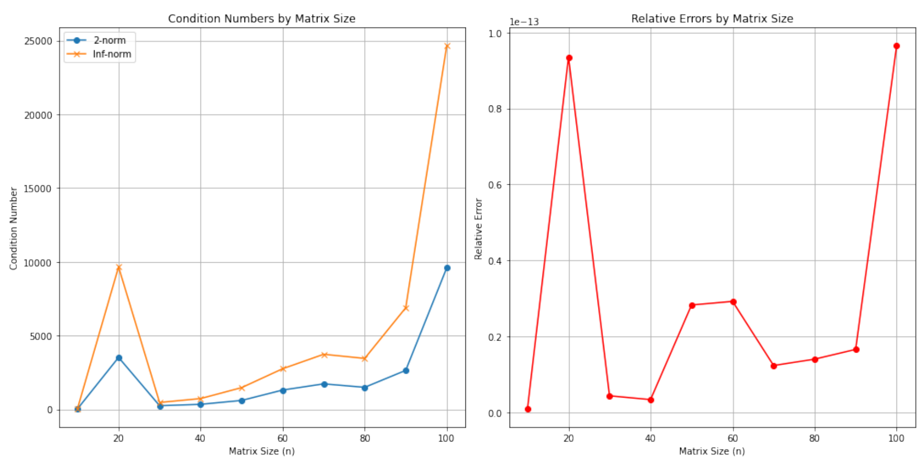

Figure 1: Initial condition numbers of random matrices, indicating potential instability as matrix size increases.

Recognizing that such results might overly depend on the particular instance of random matrix generation, a more robust approach was employed. By generating multiple matrices for each size and conducting 30 trials, a clearer and more general picture was obtained. The median of the condition numbers and relative errors across these trials presents a more reliable analysis, reducing the impact of outliers and random anomalies. The resultant graph, as seen below, demonstrates the trends with a better representation of the expected behavior for random matrices.

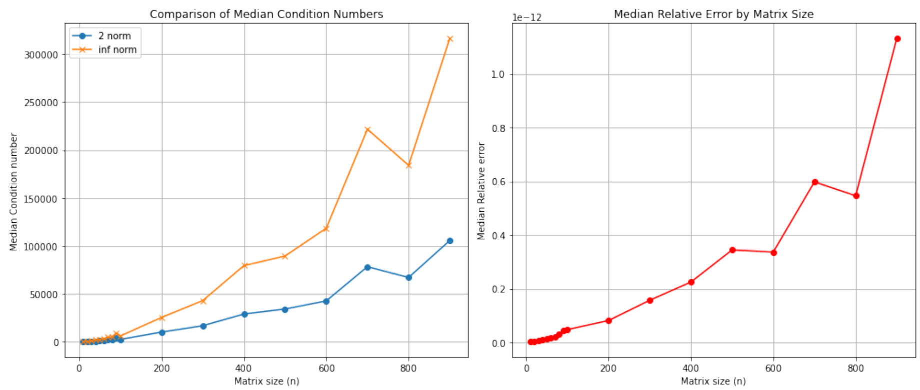

Figure 2: Median condition numbers from 30 trials of random matrix generation, offering a more stable and general representation of matrix behavior.

2 Vandermonde Matrices

Vandermonde matrices are particularly interesting due to their structure, which inherently leads to a rapid increase in condition numbers as the size of the matrix grows. When considering the condition numbers in both the 2-norm and infinity-norm, it is observed that they exhibit exponential growth. This phenomenon is attributed to the geometric progression in the columns of the Vandermonde matrix, which, as the matrix size increases, amplifies the effects of round-off errors and makes the matrix increasingly ill-conditioned.

The similarity in the trend between $K_2(A)$ and $K_\infty(A)$ arises from the fact that both norms reflect how the progressive powers within the matrix can lead to columns that are nearly linearly dependent. However, the infinity-norm condition number is particularly sensitive to the accumulation of these powers in the rows, which can lead to even higher condition numbers.

Ill-conditioning in the context of Vandermonde matrices is thus clearly defined by both norms, although the specific values may differ. The ill-conditioning indicates that solving a linear system with such a matrix is susceptible to large errors in the presence of small perturbations, a fact that becomes increasingly problematic as the matrix size increases.

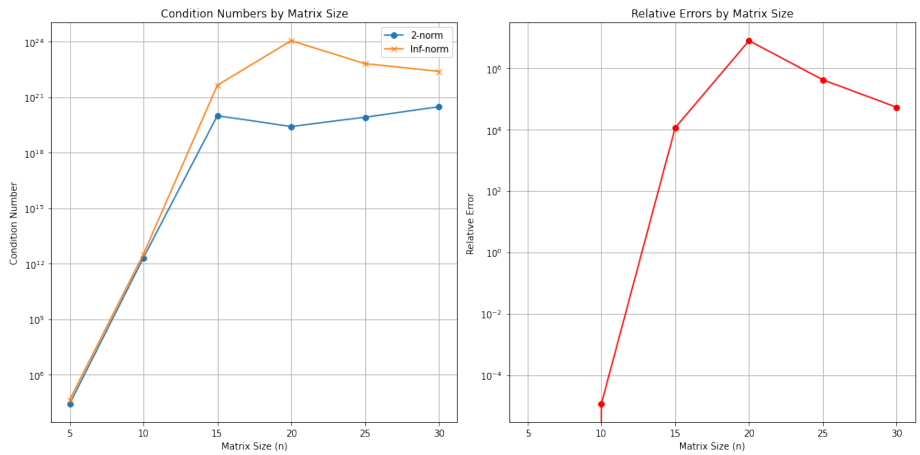

Figure 3: Exponential growth of condition numbers in Vandermonde matrices, signaling their proneness to numerical instability.

Interestingly, in an extended analysis with matrices of significantly larger dimensions, such as 1000, 2000, or 3000, a stabilization in the growth of both condition numbers and relative errors was observed. This behavior suggests a potential saturation effect in numerical computations, particularly due to the limits of floating-point precision, and highlights an intriguing area for further research, especially in understanding the behavior of large-scale Vandermonde matrices in numerical contexts.

3 Hilbert Matrices

Hilbert matrices showcase an extreme case of ill-conditioning, which is evident even for matrices of small dimensions. The entries of a Hilbert matrix are fractions that become increasingly small as one moves across the rows and down the columns. Both $K_2(A)$ and $K_\infty(A)$ for Hilbert matrices are observed to grow rapidly, which is indicative of their susceptibility to numerical instability.

The definition of ill-conditioning is indeed dependent on the norm used. While both the 2-norm and infinity-norm capture the growth in condition numbers, the infinity-norm particularly highlights the cumulative effect of the reciprocal entries along the rows of a Hilbert matrix. This can often result in even larger condition numbers compared to the 2-norm, reflecting the inherent sensitivity of Hilbert matrices to perturbations.

The relationship between the condition number and the relative error in the computed solution is evident in Hilbert matrices. The high condition numbers correlate with significant relative errors, underscoring the fact that even minute perturbations in input or during computation can lead to considerable inaccuracies in the solution.

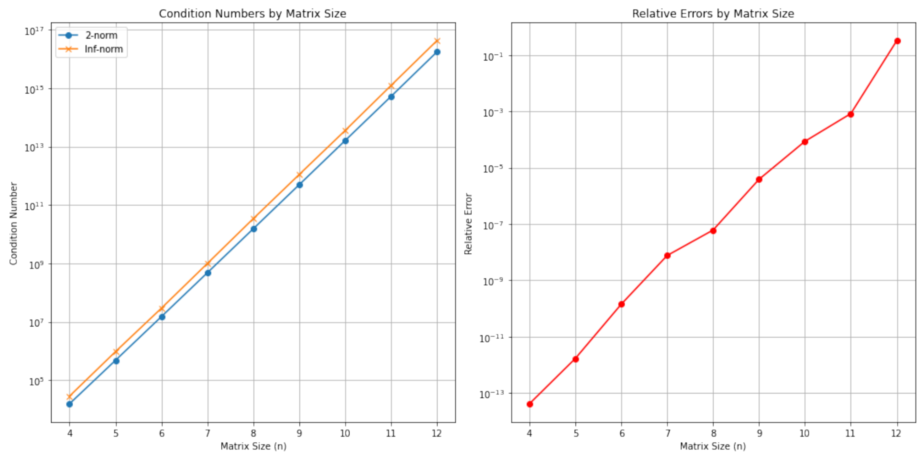

Figure 4: The rapid escalation in condition numbers for Hilbert matrices, underlining their extreme ill-conditioning and the consequent numerical challenges.

4 Conclusions

- A key finding across all three types of matrices is the similarity in the trend of the condition numbers, irrespective of the norm used. It was consistently observed that as the matrix size increases, both the 2-norm and infinity-norm condition numbers tend to rise, indicating a progression towards ill-conditioning. This phenomenon underscores a fundamental characteristic of matrix behavior, where larger matrices are more prone to numerical instability and sensitivity to input perturbations.

- Moreover, while the specific measure of ill-conditioning depends on the norm chosen, the general tendency of a linear system to become more ill-conditioned with increasing matrix size is a widely observed trait. This tendency is not solely dictated by the chosen norm but is intrinsically linked to the inherent structure of the matrix. Therefore, the definition of ill-conditioning, while quantitatively dependent on the norm, qualitatively remains a consistent aspect of matrix analysis.