Robotics 2

March 24, 2023

Exercise 1

The torque controlled 2R planar robot in Fig. 1 moves in a vertical plane, performing a one-dimensional trajectory task $r = r_d(t) \in \mathbb{R}$ for its end-effector that is specified *only* along the y-direction. The links have equal length l and equal and uniformly distributed mass m, with barycentric inertia $I_c = ml^2/12$ (links are thin rods). When the robot is in the configuration $\bar{\boldsymbol{q}} = (\pi/4, -\pi/2)$ [rad] and has a joint velocity $\dot{\bar{\boldsymbol{q}}} = (1, -1)$ [rad/s], a task acceleration $\ddot{r}_d$ is assigned.

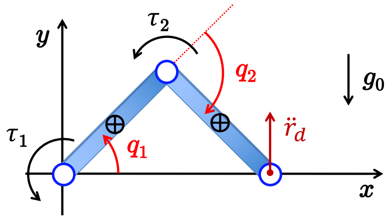

Figure 1: A 2R planar robot in a one-dimensional task.

Determine in a parametric way (with respect to l, m, $\ddot{r}_d$, and $\boldsymbol{K}_D$) the torque $\boldsymbol{\tau} \in \mathbb{R}^2$ that executes the task and instantaneously minimizes the cost

$$H = \frac{1}{2} \left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right)^T \boldsymbol{M}^{-1}(\boldsymbol{q}) \left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right), \qquad \boldsymbol{\tau}_0 = -\boldsymbol{K}_D \, \dot{\boldsymbol{q}},$$

where $\boldsymbol{M}(\boldsymbol{q})$ is the inertia matrix of the robot and matrix $\boldsymbol{K}_D > 0$ is diagonal. Evaluate then numerically the obtained solution torque $\boldsymbol{\tau}$ using the following data:

$$l = 0.5 \text{ [m]}, \qquad m = 3 \text{ [kg]}, \qquad \ddot{r}_d = 1 \text{ [m/s}^2\text{]}, \qquad \boldsymbol{K}_D = \text{diag}\{2, 2\} \text{ [Nm·s]}.$$

Exercise 2

Consider the same robot of Fig. 1, now with all *kinematic* and *dynamic* parameters that appear explicitly in the model having uncertain/unknown values. In this situation, design an adaptive control law that is able to guarantee global asymptotic tracking of a desired smooth joint trajectory $\boldsymbol{q}_d(t)$.

Exercise 3

With reference to the situation in Fig. 2, a robot should move a triangular-shaped tool in continuous contact with the bottom surface of a rectangular guide, in which the tool is perfectly inserted, of a metallic object in order to remove excessive material and debris (*deburring*). Assign a task frame so as to define in a decoupled way the natural and artificial constraints associated to this hybrid task. Assume that no friction is present at any of the contacts. Specify then qualitatively how desired values should be assigned to the variables that can be controlled, in order to perform the task at best and with minimum effort.

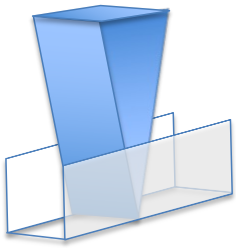

Figure 2: The set-up of a hybrid force-motion task.

[180 minutes; open books]

Solution

March 24, 2023

Exercise 1

The solution is obtained by solving a LQ optimization problem in the standard form. Using the expression of the dynamic model

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})=\boldsymbol{\tau},$$

the objective function

$$H=\frac{1}{2}\left(\boldsymbol{\tau}-\boldsymbol{\tau}_0\right)^T\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}-\boldsymbol{\tau}_0\right)$$

becomes a complete quadratic form in the joint acceleration $\ddot{\boldsymbol{q}}$

$$\begin{aligned} H &= \frac{1}{2}\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_0\right)^T\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_0\right)\\ &= \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}}+\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_0\right)^T\ddot{\boldsymbol{q}}+\frac{1}{2}\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_0\right)^T\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_0\right), \end{aligned}$$

to be locally minimized under the acceleration-level task constraint

$$\boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}-\ddot{\boldsymbol{r}}=\boldsymbol{0}.$$

As a result, the joint acceleration solution is

$$\ddot{\boldsymbol{q}}=\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0-\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{g}(\boldsymbol{q})\right)+\boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\left(\ddot{\boldsymbol{r}}-\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}-\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0-\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{g}(\boldsymbol{q})\right)\right)$$

where

$$\boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})=\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}$$

is the inertia-weighted pseudoinverse of $\boldsymbol{J}$. Then, the corresponding command torque is

$$\boldsymbol{\tau}=\boldsymbol{\tau}_0+\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\left(\ddot{\boldsymbol{r}}-\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}-\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0-\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{g}(\boldsymbol{q})\right)\right) \tag{1}$$

or, equivalently,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\left(\ddot{\boldsymbol{r}}-\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}+\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})\right)\right)\\ &\quad+\left(\boldsymbol{I}-\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\right)\boldsymbol{\tau}_0, \end{aligned}$$

with $\boldsymbol{\tau}_0=-\boldsymbol{K}_D\dot{\boldsymbol{q}}$.

In order to evaluate the torque (1) for the case at hand, we need first to find the expression of the terms associated to the second-order differential kinematics for the given motion task along the y-direction, namely $\ddot{r}_d\in\mathbb{R}$, the 1×2 task Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q})=\begin{pmatrix} lc_1+lc_{12} & lc_{12}\end{pmatrix},$$

and its time derivative

$$\dot{\boldsymbol{J}}(\boldsymbol{q})=\begin{pmatrix} -ls_1\dot{q}_1-ls_{12}\left(\dot{q}_1+\dot{q}_2\right) & -ls_{12}\left(\dot{q}_1+\dot{q}_2\right)\end{pmatrix},$$

so that

$$\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}=-ls_1\dot{q}_1^2-ls_{12}\left(\dot{q}_1+\dot{q}_2\right)^2.$$

The task Jacobian $\boldsymbol{J}$ loses rank (it vanishes!) for $c_1=c_{12}=0$, namely when $\boldsymbol{q}=\boldsymbol{q}_s=(\pm\pi/2,0)$. This is not our case since $\bar{\boldsymbol{q}}_0=(\pi/4,-\pi/2)\neq\boldsymbol{q}_s$.

We compute then the terms in the dynamic model of the robot, taking into account that the links are considered as a thin rods with uniform mass distribution.

Kinetic energy

For the first link, we have

$$T_1 = \frac{1}{2}\, m \left(\frac{l}{2}\right)^2 \dot q_1^2 + \frac{1}{2}\left(\frac{1}{12}\, m\, l^2\right)\dot q_1^2 = \frac{1}{2}\,\frac{m\, l^2}{3}\,\dot q_1^2.$$

For the second link, being

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\left(l \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} + \frac{l}{2}\begin{pmatrix} c_{12} \\ s_{12} \end{pmatrix}\right) = l \begin{pmatrix} -s_1 \dot q_1 - \frac{1}{2}\, s_{12}\,(\dot q_1 + \dot q_2) \\ c_1 \dot q_1 + \frac{1}{2}\, c_{12}\,(\dot q_1 + \dot q_2) \end{pmatrix},$$

it is

$$\|\boldsymbol{v}_{c2}\|^2 = l^2\left(\dot q_1^2 + \frac{1}{4}\,(\dot q_1 + \dot q_2)^2 + c_2\, \dot q_1 (\dot q_1 + \dot q_2)\right),$$

and thus

$$\begin{aligned} T_2 &= \frac{1}{2}\, m\, \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\left(\frac{1}{12}\, m\, l^2\right)(\dot q_1 + \dot q_2)^2 \\ &= \frac{1}{2}\, m\, l^2 \left(\frac{4}{3}\,\dot q_1^2 + \frac{1}{3}\,\dot q_2^2 + \frac{2}{3}\,\dot q_1 \dot q_2 + c_2 \dot q_1 (\dot q_1 + \dot q_2)\right). \end{aligned}$$

Finally,

$$T = T_1 + T_2 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Inertia matrix

Factoring out the common symbolic term $m\, l^2$, one has

$$\boldsymbol{M}(\boldsymbol{q}) = m\, l^2 \begin{pmatrix} \frac{5}{3} + c_2 & \frac{1}{3} + \frac{c_2}{2} \\ \frac{1}{3} + \frac{c_2}{2} & \frac{1}{3} \end{pmatrix},$$

with $\det \boldsymbol{M}(\boldsymbol{q}) = m\, l^2 (16 - 9\, c_2^2)/36$ and inverse

$$\boldsymbol{M}^{-1}(\boldsymbol{q}) = \frac{1}{m\, l^2} \cdot \frac{36}{16 - 9\, c_2^2} \begin{pmatrix} \frac{1}{3} & -\left(\frac{1}{3} + \frac{c_2}{2}\right) \\ -\left(\frac{1}{3} + \frac{c_2}{2}\right) & \frac{5}{3} + c_2 \end{pmatrix}.$$

Velocity terms

Using the Christoffel symbols, the Coriolis and centrifugal terms are computed as follows:

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \frac{m\, l^2}{2}\begin{pmatrix} 0 & -s_2 \\ -s_2 & -s_2 \end{pmatrix} \quad \Rightarrow \quad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} = -\frac{m\, l^2}{2}\, s_2\,(2\dot q_1 + \dot q_2)\,\dot q_2, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \frac{m\, l^2}{2}\begin{pmatrix} s_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{m\, l^2}{2}\, s_2\, \dot q_1^2. \end{aligned}$$

Therefore

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{m\, l^2}{2}\begin{pmatrix} -s_2\,(2\dot q_1 + \dot q_2)\,\dot q_2 \\ s_2\, \dot q_1^2 \end{pmatrix} = \frac{m\, l^2}{2}\begin{pmatrix} -s_2 \dot q_2 & -s_2\,(\dot q_1 + \dot q_2) \\ s_2 \dot q_1 & 0 \end{pmatrix}\begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}},$$

where the factorization with a matrix $\boldsymbol{S}$ satisfies the skew-symmetry of $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ (a useful property needed in Exercise 2).

Potential energy and gravity terms

For the first and second links, we have

$$U_1 = mg_0 \frac{l}{2} s_1, \qquad U_2 = mg_0 \left(l s_1 + \frac{l}{2} s_{12} \right),$$

Therefore

$$U = U_1 + U_2 \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}) = \frac{\partial U}{\partial \boldsymbol{q}} = mg_0 l \begin{pmatrix} \frac{3}{2} c_1 + \frac{1}{2} c_{12} \\ \frac{1}{2} c_{12} \end{pmatrix}.$$

We can now evaluate at $\bar{\boldsymbol{q}} = (\pi/4, -\pi/2)$ [rad] and $\bar{\dot{\boldsymbol{q}}} = (1, -1)$ [rad/s] all the needed terms using the given robot data ($l = 0.5$ [m], $m = 3$ [kg]), i.e.,

$$\begin{aligned} \boldsymbol{J}(\bar{\boldsymbol{q}}) &= \begin{pmatrix} 0.7071 & 0.3536 \end{pmatrix} & \dot{\boldsymbol{J}}(\bar{\boldsymbol{q}})\bar{\dot{\boldsymbol{q}}} &= -0.3536 \\ \boldsymbol{M}(\bar{\boldsymbol{q}}) &= \begin{pmatrix} 1.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix} & \boldsymbol{M}^{-1}(\bar{\boldsymbol{q}}) &= \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix} \\ \boldsymbol{c}(\bar{\boldsymbol{q}}, \bar{\dot{\boldsymbol{q}}}) &= \begin{pmatrix} -0.375 \\ -0.375 \end{pmatrix} & \boldsymbol{g}(\bar{\boldsymbol{q}}) &= \begin{pmatrix} 20.8102 \\ 5.2025 \end{pmatrix} \\ \boldsymbol{\tau}_0 = \boldsymbol{K}_D \bar{\dot{\boldsymbol{q}}} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} & \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\bar{\boldsymbol{q}}) &= \begin{pmatrix} 0.5657 \\ 1.6971 \end{pmatrix}, \end{aligned}$$

Finally, substituting these values in (1), the following joint torque is obtained for a desired task acceleration $\ddot{r}_d = 1$ [m/s^2]:

$$\boldsymbol{\tau} = \begin{pmatrix} 11.8985 \\ 8.9492 \end{pmatrix} \text{ [Nm]}.$$

Exercise 2

By the assumption on mass distribution of the two links of the planar 2R robot, it is clear from the obtained dynamic terms in Exercise 1 that the model can be linearly parametrized in terms of only two (mixed) dynamic/kinematic coefficients, namely $a_1 = ml^2$ and $a_2 = mg_0 l$ (the gravity acceleration g_0 can be included or excluded from a_2, with minimal changes in the following). Moreover, if the common length l of the two links (and g_0) were known, then the single dynamic parameter m would be sufficient.

Thus, being

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau}, \qquad \text{with } \boldsymbol{a} = \begin{pmatrix} ml^2 \\ mlg_0 \end{pmatrix},$$

an adaptive control law that guarantees asymptotic stabilization of the trajectory error $\boldsymbol{e}(t) = \boldsymbol{q}_d(t) - \boldsymbol{q}(t)$ to zero in a global way is given by

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \hat{\boldsymbol{a}} + \boldsymbol{K}_P\, \boldsymbol{e} + \boldsymbol{K}_D\, \dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma} \boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, (\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \end{aligned} \tag{2}$$

with $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e}$, diagonal gain matrices $\boldsymbol{\Lambda} > 0$, $\boldsymbol{\Gamma} > 0$, $\boldsymbol{K}_D > 0$, and $\boldsymbol{K}_P = \boldsymbol{K}_D \boldsymbol{\Lambda} > 0$, and where

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \hat{\boldsymbol{a}} = \hat{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}),$$

being the $\boldsymbol{S}$ matrix that factorizes the Coriolis and centrifugal terms such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric. Therefore, the 2×2 regressor matrix $\boldsymbol{Y}$ to be used in the adaptive control law (2) is

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \left(\frac{5}{3} + c_2\right) \ddot{q}_{1r} + \left(\frac{1}{3} + \frac{c_2}{2}\right) \ddot{q}_{2r} - \frac{s_2}{2} \left(\dot{q}_2 \dot{q}_{1r} + (\dot{q}_1 + \dot{q}_2)\dot{q}_{2r}\right) & \frac{3\, c_1}{2} + \frac{c_{12}}{2} \\ \left(\frac{1}{3} + \frac{c_2}{2}\right) \ddot{q}_{1r} + \frac{1}{3}\ddot{q}_{2r} + \frac{s_2}{2}\, \dot{q}_1 \dot{q}_{1r} & \frac{c_{12}}{2} \end{pmatrix}.$$

Exercise 3

The chosen task frame and the associated natural and artificial constraints are shown in Fig. 3. We have set to zero the reference values $f_{y,d} = m_{x,d} = m_{z,d} = 0$ so as to minimize the internal forces between the tool and those parts of the object for which deburring is not necessary. On the other hand, a non-zero value $f_{z,d} < 0$ is chosen both for maintaining the tool contact with the bottom of the guide and for guaranteeing a sufficient force to remove the debris that are present there. The speed $v_{x,d} > 0$ characterizes the progress of the deburring task. Finally, $\omega_{y,d}(t)$ is left free and is available for deflecting the tool from the normal direction to the bottom surface of the guide when necessary, e.g., for allowing a better execution of the deburring operation.

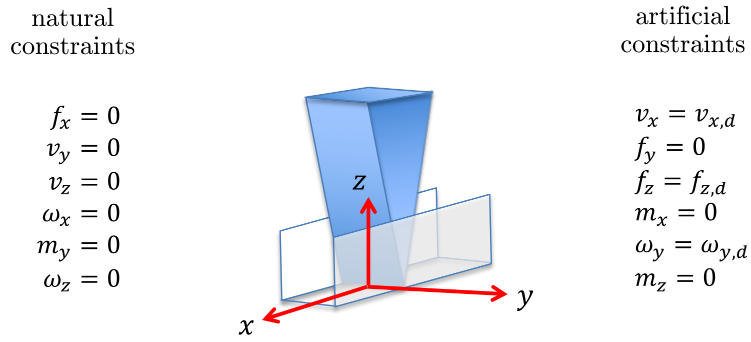

Figure 3: The chosen task frame and the associated natural and artificial constraints.

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