Robotics 2

July 10, 2023

Exercise 1

A robot with n degrees of freedom and dynamics (with no gravity)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

is redundant with respect to a m-dimensional task ($m < n$) described at the second-order differential level by

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

where the $m \times n$ task Jacobian $\boldsymbol{J}$ is assumed to be full row rank. In the redundant case, the joint torque $\boldsymbol{\tau} \in \mathbb{R}^n$ can always be decomposed as

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} + \left(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{H}(\boldsymbol{q})\right)\boldsymbol{\tau}_0,$$

where $\boldsymbol{F} \in \mathbb{R}^m$ is the task-space generalized force performing work on $\dot{\boldsymbol{y}}$, matrix $\boldsymbol{H}$ is any generalized inverse of $\boldsymbol{J}^T$ (i.e., such that $\boldsymbol{J}^T\boldsymbol{H}\boldsymbol{J}^T = \boldsymbol{J}^T$), and $\boldsymbol{\tau}_0 \in \mathbb{R}^n$.

With the robot in the state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, prove the following two statements.

a) In order for an arbitrary $\boldsymbol{\tau}_0 \neq \boldsymbol{0}$ not to produce any task acceleration ($\ddot{\boldsymbol{y}} = \boldsymbol{0}$), the only choice for $\boldsymbol{H}$ is

$$\boldsymbol{H}(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q}), \tag{1}$$

namely the weighted pseudoinverse of $\boldsymbol{J}^T$, with the inverse of the robot inertia as weight.

b) Based on (1), the m-dimensional dynamic model of the robot in the task space is given by

$$\boldsymbol{M}_{\boldsymbol{y}}(\boldsymbol{q})\ddot{\boldsymbol{y}} + \boldsymbol{c}_{\boldsymbol{y}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{F} \tag{2}$$

with the $m \times m$ task-space inertia matrix $\boldsymbol{M}_{\boldsymbol{y}}$ and the task-space Coriolis and centrifugal terms $\boldsymbol{c}_{\boldsymbol{y}}$ given respectively by

$$\boldsymbol{M}_{\boldsymbol{y}}(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \quad \boldsymbol{c}_{\boldsymbol{y}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{M}_{\boldsymbol{y}}(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right).$$

Exercise 2 [skip this if you accepted the midterm grade]

Consider the 3-dof planar robot in Fig. 1, with one prismatic and two revolute joints, moving in a vertical plane. The coordinates $\boldsymbol{q}$ to be used are defined in the figure. Each link of the robot has uniformly distributed mass $m_i > 0$, $i = 1, 2, 3$, with center of mass on its geometric axis, and a diagonal barycentric inertia matrix. The prismatic joint has a limited range $q_2 \in [-L_2, L_2]$, while the revolute joints are unlimited. The robot is commanded by a joint force/torque $\boldsymbol{\tau} \in \mathbb{R}^3$.

a) Derive the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$.

b) Derive the gravity term $\boldsymbol{g}(\boldsymbol{q})$ and find all unforced equilibrium configurations (i.e., with $\boldsymbol{\tau} = \boldsymbol{0}$).

c) Assume that the gravity acceleration g_0 and the kinematic quantities L_2 and L_3 are known, while all other dynamic parameters are unknown. Provide a linear parametrization of the gravity vector $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})\,\boldsymbol{a}_g$, in terms of a vector $\boldsymbol{a}_g \in \mathbb{R}^p$ of unknown dynamic coefficients and a $3 \times p$ regressor matrix $\boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})$. Discuss the minimality of p.

d) Provide a symbolic expression (in terms of the robot dynamic parameters and joint limits) of a constant upper bound $\alpha > 0$ for the norm of the gradient of the gravity vector, i.e., such that $\|\partial \boldsymbol{g}(\boldsymbol{q})/\partial \boldsymbol{q}\| \leq \alpha$ for all feasible $\boldsymbol{q}$.

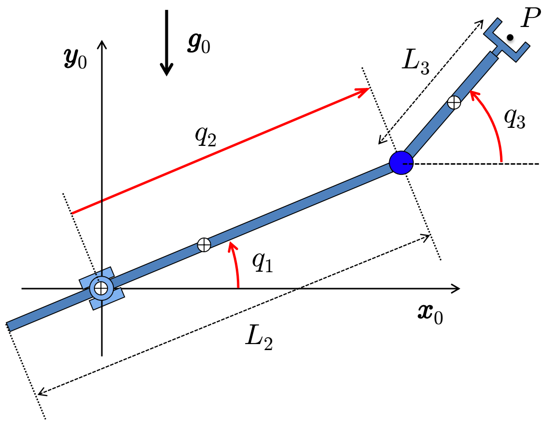


Figure 1: A planar RPR robot, with the definition of the coordinates to be used $\boldsymbol{q} = (q_1\ q_2\ q_3)^T$.

Exercise 3

Consider the robotic task of inserting a sphere in a cylindrical hole having the same size (zero clearance), as shown in Fig. 2. Assuming rigid and frictionless contacts, define a task frame, the natural constraints imposed by the geometry on the generalized velocity/force quantities expressed in this task frame, and the artificial constraints that can be taken as reference values by a hybrid force-velocity control law for the execution of this sphere-in-hole task with minimum effort.

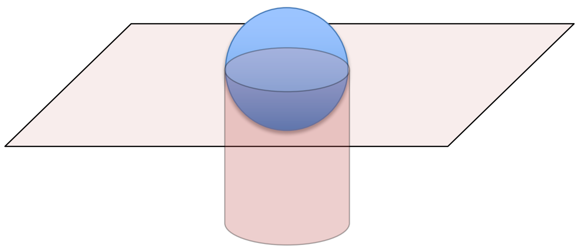


Figure 2: Sphere-in-hole task.

Provide a basis for the space of admissible twists $\boldsymbol{V} = (\boldsymbol{v}^T \boldsymbol{\omega}^T)^T \in \mathbb{R}^6$ and a complementary basis for the space of reaction wrenches $\boldsymbol{F} = (\boldsymbol{f}^T \boldsymbol{m}^T)^T \in \mathbb{R}^6$. Discuss how measurements that are inconsistent with the geometric model are being handled by an hybrid force-velocity control law, and give two examples of such inconsistent measurements, one related to motion and one related to interaction.

[180 minutes; open books (for the complete exam)]
[90 minutes, open books (for students with accepted midterm grade)]

Solution

July 10, 2023

Exercise 1

For compactness, drop the dependencies of matrices and vectors on the current $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. The proofs go as follows.

a) Substitute the acceleration $\ddot{\boldsymbol{q}}$ from the dynamic model in the expression of task acceleration $\ddot{\boldsymbol{y}}$:

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{c}) + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}.$$

Plugging in this the given decomposition of the joint torques leads to

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{J}^T\boldsymbol{F} + (\boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{H})\,\boldsymbol{\tau}_0) - \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}. \tag{3}$$

The possible contribution of $\boldsymbol{\tau}_0$ to $\ddot{\boldsymbol{y}}$ is weighted by the matrix $\boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{H})$. Choosing for $\boldsymbol{H}$ the expression (1) provides

$$\begin{aligned}\boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{H}) &= \boldsymbol{J}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{J}\boldsymbol{M}^{-1}) \\ &= \boldsymbol{J}\boldsymbol{M}^{-1} - \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{J}\boldsymbol{M}^{-1} - \boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{O}.\end{aligned}$$

Thus, with this choice $\boldsymbol{\tau}_0$ will not contribute to $\ddot{\boldsymbol{y}}$.

b) With the choice (1), from eq. (3) we have

$$\ddot{\boldsymbol{y}} = \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\boldsymbol{F} - \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}.$$

Premultipying by the inverse of $\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T$, we obtain

$$(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}\ddot{\boldsymbol{y}} = \boldsymbol{F} - (\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1}(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}})$$

or

$$\boldsymbol{M}_{\boldsymbol{y}}\ddot{\boldsymbol{y}} + \boldsymbol{c}_{\boldsymbol{y}} = \boldsymbol{F},$$

which is the task-space dynamics with the given definitions of $\boldsymbol{M}_{\boldsymbol{y}}$ and $\boldsymbol{c}_{\boldsymbol{y}} = \boldsymbol{M}_{\boldsymbol{y}}(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{c} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}})$.

Exercise 2

a) The kinetic energies of the three links are

$$\begin{aligned}T_1 &= \frac{1}{2}I_1\,\dot{q}_1^2, \\ T_2 &= \frac{1}{2}I_2\,\dot{q}_1^2 + \frac{1}{2}m_2\left(\dot{q}_2^2 + \left(q_2 - \frac{L_2}{2}\right)^2\dot{q}_1^2\right), \\ T_3 &= \frac{1}{2}I_3\,\dot{q}_3^2 + \frac{1}{2}m_3\left(\dot{q}_2^2 + q_2^2\,\dot{q}_1^2 + \left(\frac{L_3}{2}\right)^2\dot{q}_3^2\right. \\ &\qquad\qquad\qquad\left. + L_3\big(q_2\cos(q_3 - q_1)\,\dot{q}_1\dot{q}_3 - \sin(q_3 - q_1)\,\dot{q}_2\dot{q}_3\big)\right).\end{aligned}$$

From $T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, we obtain

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + m_2\Big(q_2 - \dfrac{L_2}{2}\Big)^2 + m_3\,q_2^2 & 0 & m_3q_2\Big(\dfrac{L_3}{2}\Big)\cos(q_3 - q_1) \\ 0 & m_2 + m_3 & -m_3\Big(\dfrac{L_3}{2}\Big)\sin(q_3 - q_1) \\ m_3q_2\Big(\dfrac{L_3}{2}\Big)\cos(q_3 - q_1) & -m_3\Big(\dfrac{L_3}{2}\Big)\sin(q_3 - q_1) & I_3 + m_3\Big(\dfrac{L_3}{2}\Big)^2 \end{pmatrix}.$$

b) The potential energies of the three links due to gravity are

$$U_1 = 0, \qquad U_2 = m_2 g_0 \left(q_2 - \frac{L_2}{2}\right) \sin q_1, \qquad U_3 = m_3 g_0 \left(q_2 \sin q_1 + \frac{L_3}{2} \sin q_3\right).$$

From $U = U_1 + U_2 + U_3$, we obtain

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} \left((m_2 + m_3) q_2 - m_2 \dfrac{L_2}{2}\right) \cos q_1 \\ (m_2 + m_3) \sin q_1 \\ m_3 \dfrac{L_3}{2} \cos q_3 \end{pmatrix}.$$

The unforced equilibrium configurations $\boldsymbol{q}_e$ are four, namely

$$\boldsymbol{q}_e = \begin{pmatrix} \{0, \pi\} \\ \dfrac{m_2}{m_2 + m_3} \dfrac{L_2}{2} \\ \pm \dfrac{\pi}{2} \end{pmatrix},$$

depending on the presence of a half-rotation or not for q_{e1} and on the chosen sign for q_{e3}. Note that, being $q_{e2} < L_2/2$, the CoM of link 2 will always be placed opposite to the third link with respect to the first joint axis.

c) The linear parametrization of the gravity term is

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} g_0 \left(q_2 - \dfrac{L_2}{2}\right) \cos q_1 & g_0\, q_2 \cos q_1 \\ g_0 \sin q_1 & g_0 \sin q_1 \\ 0 & g_0 \dfrac{L_2}{2} \cos q_3 \end{pmatrix} \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q}) \boldsymbol{a}_{\boldsymbol{g}},$$

with $p = 2$ dynamic parameters (minimal).

d) The gradient of the gravity vector is

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = g_0 \begin{pmatrix} -\left((m_2 + m_3) q_2 - m_3 \dfrac{L_2}{2}\right) \sin q_1 & (m_2 + m_3) \cos q_1 & 0 \\ (m_2 + m_3) \cos q_1 & 0 & 0 \\ 0 & 0 & -m_3 \dfrac{L_3}{2} \sin q_3 \end{pmatrix},$$

which is a symmetric matrix (thus, with real eigenvalues), but not positive semi-definite (thus, with eigenvalues that can also be negative). Multiplying by the transpose of this matrix, we obtain

$$\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)$$
$$= \begin{pmatrix} \left((m_2 + m_3) q_2 - m_3 \dfrac{L_2}{2}\right)^2 s_1^2 + (m_2 + m_3)^2 c_1^2 & -\left((m_2 + m_3) q_2 - m_3 \dfrac{L_2}{2}\right)(m_2 + m_3)\, s_1 c_1 & 0 \\ -\left((m_2 + m_3) q_2 - m_3 \dfrac{L_2}{2}\right)(m_2 + m_3)\, s_1 c_1 & (m_2 + m_3)^2 c_1^2 & 0 \\ 0 & 0 & m_3^2 \left(\dfrac{L_3}{2}\right)^2 s_3^2 \end{pmatrix},$$

where the shorthand notation has been used for trigonometric functions. The three real (non-negative) eigenvalues of this matrix are the two roots of

$$\lambda^2 - \left(\left((m_2+m_3)q_2 - m_3\,\frac{L_2}{2} \right)^2 s_1^2 + 2\,(m_2+m_3)^2\,c_1^2 \right) \lambda + (m_2+m_3)^4\,c_1^4 = 0$$

or

$$\begin{aligned} \lambda_{1,2} = & \frac{1}{2} \left(\left((m_2+m_3)q_2 - m_3\,\frac{L_2}{2} \right)^2 s_1^2 + 2\,(m_2+m_3)^2\,c_1^2 \right) \\ & \pm \frac{1}{2} \sqrt{\left((m_2+m_3)q_2 - m_3\,\frac{L_2}{2} \right)^4 s_1^4 + 4\left((m_2+m_3)q_2 - m_3\,\frac{L_2}{2} \right)^2 (m_2+m_3)^2\, s_1^2 c_1^2} \end{aligned} \tag{4}$$

and

$$\lambda_3 = m_3^2 \left(\frac{L_3}{2} \right)^2 s_3^2.$$

In order to determine a constant upper bound α such that

$$\left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right\| = \sqrt{\lambda_{\max} \left\{ \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right) \right\}} \leq \alpha$$

for all values of $\boldsymbol{q}$, we should proceed by overbinding (trigonometric) terms in λ_1, i.e., in the largest positive eigenvalue in (4) when taking the + sign, and in λ_3, using also the prismatic joint limitation $|q_2| \leq L_2$. As a result, one has

$$\begin{aligned} 0 < \lambda_1 \leq & \frac{1}{2} \left(\left((m_2+m_3)L_2 - m_3\,\frac{L_2}{2} \right)^2 + 2\,(m_2+m_3)^2 \right) \\ & + \frac{1}{2} \sqrt{\left((m_2+m_3)L_2 - m_3\,\frac{L_2}{2} \right)^4 + 4\left((m_2+m_3)L_2 - m_3\,\frac{L_2}{2} \right)^2 (m_2+m_3)^2} = \lambda_{1,max} \end{aligned}$$

and

$$0 \leq \lambda_3 \leq m_3^2 \left(\frac{L_3}{2} \right)^2.$$

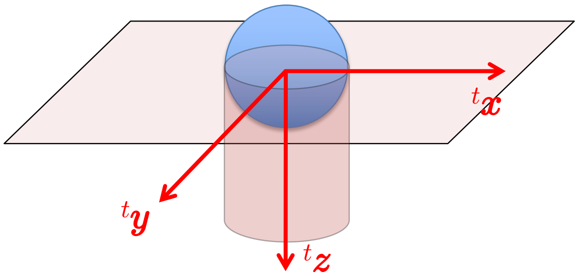
Thus

$$\alpha = \max \left\{ \sqrt{\lambda_{1,max}},\, m_3 \left(\frac{L_3}{2} \right) \right\}.$$

Exercise 3

See the summary in Fig. 3. The assigned zero values to selected artificial constraints represent the execution of the task with minimum effort, whereas $v_d > 0$ is the desired speed of insertion. Basis for the required spaces are given by

$$\boldsymbol{V} = \boldsymbol{T} \begin{pmatrix} v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \quad \text{with } \boldsymbol{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



natural constraints	artificial constraints
$v_x = 0$	$f_x = f_{x,d} = 0$
$v_y = 0$	$f_y = f_{y,d} = 0$
$f_z = 0$	$v_z = v_d$
$m_x = 0$	$\omega_x = \omega_{x,d} = 0$
$m_y = 0$	$\omega_y = \omega_{y,d} = 0$
$m_z = 0$	$\omega_z = \omega_{z,d} = 0$

Figure 3: The task frame for the sphere-in-hole problem, with the natural and artificial constraints.

and

$$\boldsymbol{F} = \boldsymbol{Y} \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \quad \text{with } \boldsymbol{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which are in the simple form of columns of 0/1 selection matrices. Indeed, $\boldsymbol{T}^T\boldsymbol{Y} = \boldsymbol{O}$. Examples of measurements that would be inconsistent with the model are a velocity ${}^t v_x \neq 0$ (e.g., due to contact compliance) or a force ${}^t f_z \neq 0$ (due to sliding friction at the contact).

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