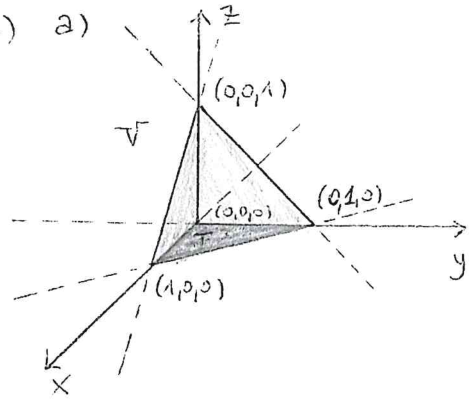
Soluzione $\boxed{\text{Scheda N. 12}}$

ES1) a)b)c)d)e) $\Pi_{x,z}(V)$ $\Pi_{y,z}(V)$ a pag. 3

1) a)

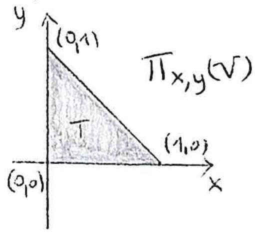


V è il TETRAEDRO di VERTICI (1,0,0) (0,1,0) (0,0,1) (0,0,0)

($z=1-x-y$ è un PIANO INCLINATO)

VOLUME $V=\int_T (1-x-y)\,dx\,dy$ dove

T = triangolo di vertici (0,0) (1,0) (0,1)



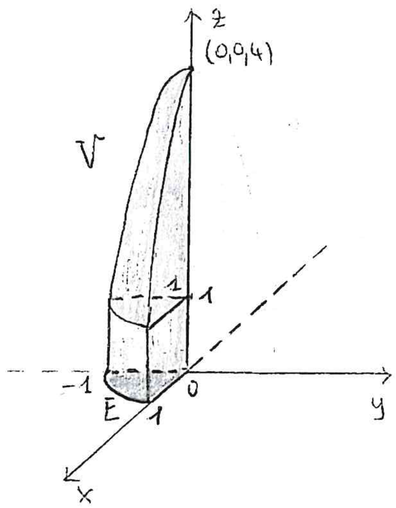
$$T=\{(x,y)\in\mathbb{R}^2 :\ 0\le x\le 1,\ 0\le y\le 1-x\}$$

$$\text{Vol } V=\int_0^1\left(\int_0^{1-x}(1-x-y)\,dy\right)dx=\int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}dx=$$

$$=\int_0^1\Big(\underbrace{1-x-x(1-x)-\tfrac{1}{2}(1-x)^2}_{1-x-x+x^2-\frac{1}{2}x^2-\frac{1}{2}+x}\Big)dx=\int_0^1\left(\frac{1}{2}x^2-x+\frac{1}{2}\right)dx=\left[\frac{x^3}{6}-\frac{x^2}{2}+\frac{1}{2}x\right]_0^1=$$

$$=\frac{1}{6}-\frac{1}{2}+\frac{1}{2}=\boxed{\frac{1}{6}}$$

b)



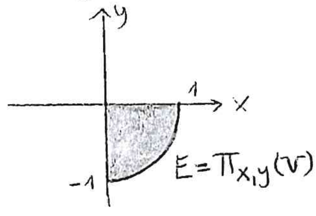
$z=4-3(x^2+y^2)$ è un paraboloide di $V(0,0,4)$, verso il basso, apertura $a=3$,

$\cap\, z=0$ su $x^2+y^2=\frac{4}{3}$ $\quad R=\frac{2}{\sqrt{3}}\simeq 1{,}15$

su $x^2+y^2=1 \rightarrow z_{par}=1$

Volume $V=\int_E (4-3(x^2+y^2))\,dx\,dy$ dove

$E:\ x^2+y^2\le 1,\ x\ge 0,\ y\le 0$

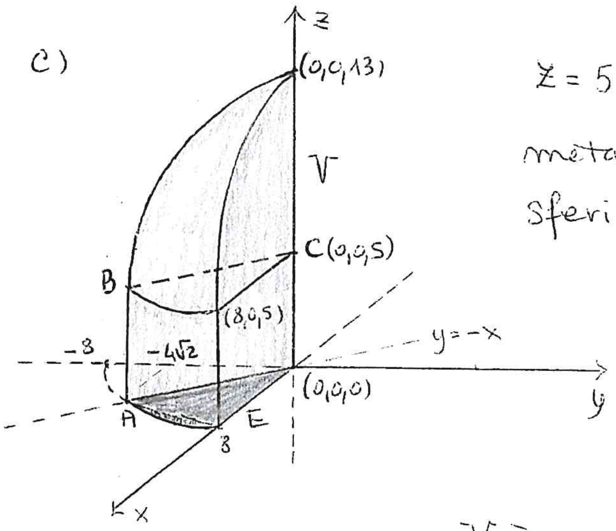


V è composto da un $\frac{1}{4}$ CILINDRO di $R=1$ e $h=1$ (per $0\le z\le 1$) e da $\frac{1}{4}$ di paraboloide pieno per $1\le z\le 4$

$$\text{vol } V \underset{\text{COORD. POLARI}}{=} \int_{\frac{3}{2}\pi}^{2\pi}\left(\int_0^1 (4-3\rho^2)\,\rho\,d\rho\right)d\theta=\frac{\pi}{2}\left[2\rho^2-3\frac{\rho^4}{4}\right]_0^1=$$

$$=\frac{\pi}{2}\left(2-\frac{3}{4}\right)=\frac{\pi}{2}\cdot\frac{5}{4}=\boxed{\frac{5}{8}\pi}$$

c)



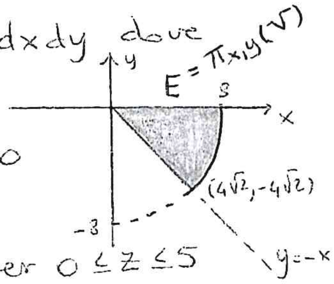
$A=(4\sqrt{2},-4\sqrt{2},0)$

$B=(4\sqrt{2},-4\sqrt{2},5)$

$z=5+\sqrt{64-x^2-y^2}$ è la metà superiore della superficie sferica di $C(0,0,5)$ e $R=8$ $(z_{max}=13)$

volume $V=\int\limits_E (5+\sqrt{64-x^2-y^2})\,dx\,dy$ dove

$E: x^2+y^2\le 64,\ -x\le y\le 0$



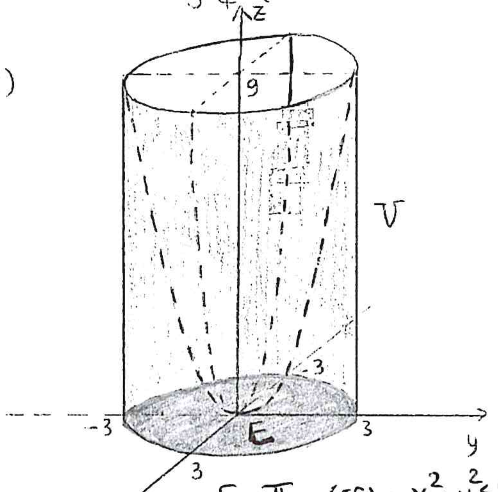
V è composto da $\frac{1}{8}$ di CILINDRO per $0\le z\le 5$ e da $\frac{1}{16}$ SFERA per $5\le z\le 13$

$$\text{Volume } V=\int_{\frac{7}{4}\pi}^{2\pi}\left(\int_0^8 (5+\sqrt{64-\rho^2})\cdot\rho\,d\rho\right)d\theta=\frac{\pi}{4}\cdot\int_0^8 (5\rho+\rho\sqrt{64-\rho^2})\,d\rho=$$

$$=\frac{\pi}{4}\cdot\left(\left[5\frac{\rho^2}{2}\right]_0^8-\frac{1}{2}\int_0^8(-2\rho)(64-\rho^2)^{1/2}d\rho\right)=\frac{\pi}{4}\left(160-\frac{1}{2}\left[\frac{(64-\rho^2)^{3/2}}{3/2}\right]_0^8\right)=$$

$$=40\pi-\frac{1}{3}\frac{\pi}{4}(0-8^3)=40\pi+\frac{128}{3}\pi=\boxed{\frac{248}{3}\pi}$$

d)



$E=\pi_{x,y}(V): x^2+y^2\le 9$

$z=x^2+y^2$ è il paraboloide di base

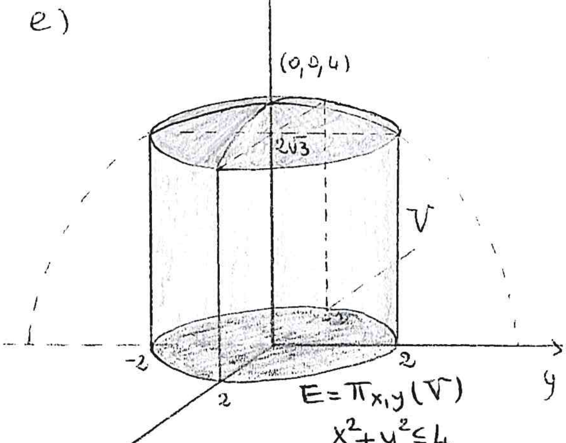
su $x^2+y^2=9 \rightarrow z_{par}=9$

V è il cilindro di $R=3$ $0\le z\le 9$, scavato del paraboloide $z=x^2+y^2$

volume $V=\int\limits_E (x^2+y^2)\,dx\,dy$ dove $E: x^2+y^2\le 9$

$$\text{vol } V=\int_0^{2\pi}\left(\int_0^3 \rho^3 d\rho\right)d\theta=2\pi\left[\frac{\rho^4}{4}\right]_0^3=\boxed{\frac{81}{2}\pi}$$

e)



$E=\pi_{x,y}(V)$
$x^2+y^2\le 4$

$z=\sqrt{16-x^2-y^2}$ è la metà superiore della superficie sferica di $C(0,0,0)$ e $R=4$

su $x^2+y^2=4$ $z_{sup\,sf}=\sqrt{16-4}=\sqrt{12}=2\sqrt{3}\simeq 3,5$

V è composto dal cilindro di $R=2$ per $0\le z\le 2\sqrt{3}$ sormontato da una calotta sferica per $2\sqrt{3}\le z\le 4$

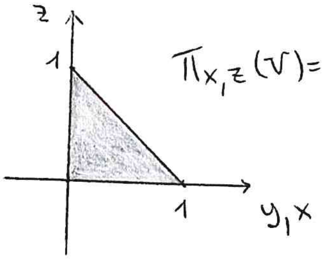
$$\text{Vol } V = \int_E \sqrt{16-x^2-y^2}\, dx\,dy \quad \text{con } E: x^2+y^2 \le 4$$

$$\text{vol } V = \int_0^{2\pi}\left(\int_0^2 \rho\sqrt{16-\rho^2}\, d\rho\right) d\theta = 2\pi\cdot\left(-\frac{1}{2}\right)\int_0^2 (-2\rho)(16-\rho^2)^{1/2}\, d\rho =$$

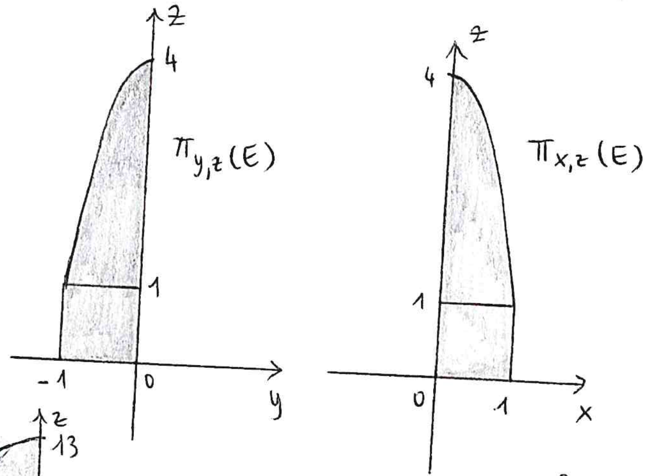
$$= -\pi\cdot\left[\frac{(16-\rho^2)^{3/2}}{3/2}\right]_0^2 = -\frac{2}{3}\pi\left[12^{3/2}-16^{3/2}\right] = \frac{2}{3}\pi\left[64-24\sqrt{3}\right] =$$

$$\sqrt{12} = 2\sqrt{3} \qquad = \boxed{\left(\frac{128}{3}-16\sqrt{3}\right)\pi}$$

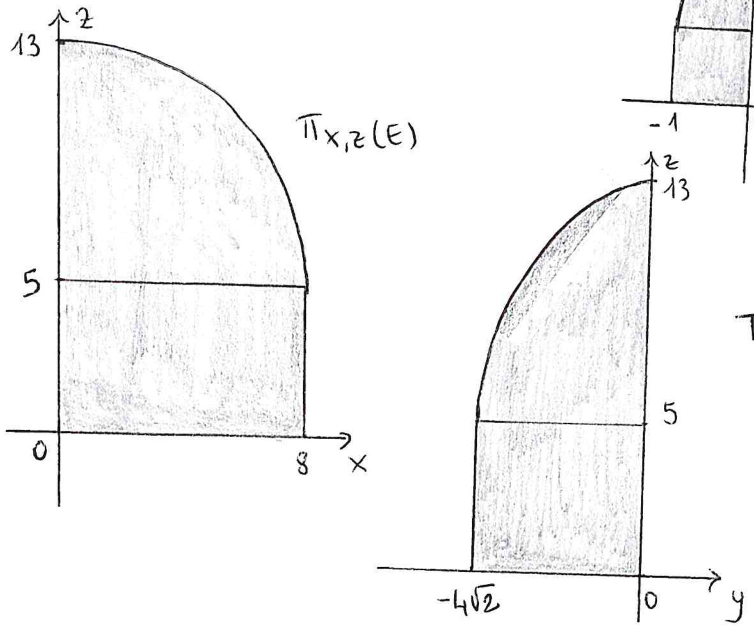
a) $\Pi_{x,z}(V) = \Pi_{y,z}(V)$



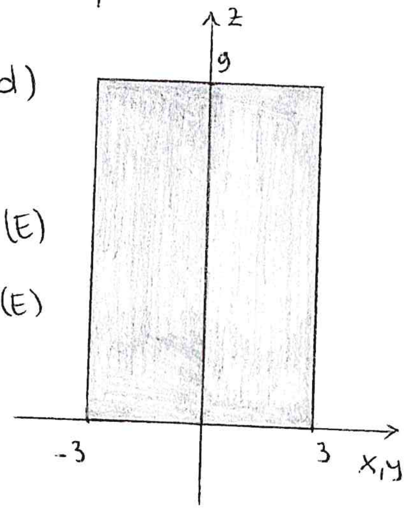
b) $\Pi_{y,z}(E)$ $\Pi_{x,z}(E)$



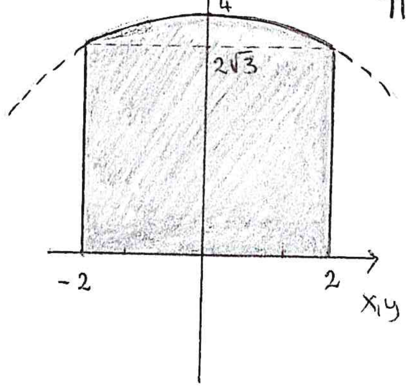
c) $\Pi_{x,z}(E)$ $\Pi_{y,z}(E)$



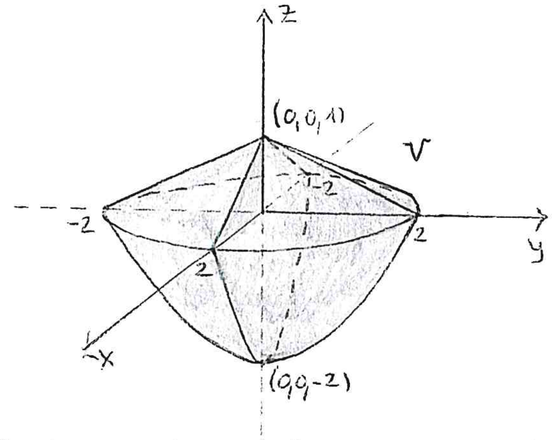
d) $\Pi_{x,z}(E) = \Pi_{y,z}(E)$



e) $\Pi_{x,z}(V)$ $\Pi_{y,z}(V)$



f)

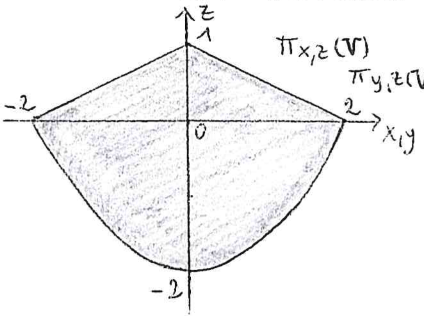


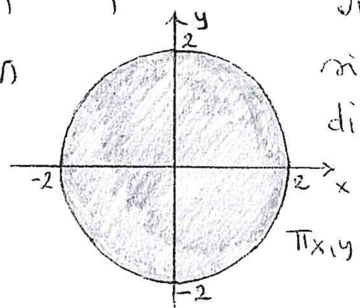
$z=\frac{1}{2}(x^2+y^2)-2$ è un PARABOLOIDE CIRCOLARE di $V(0,0,-2)$, verso l'alto, $a=\frac{1}{2}$, $\cap\, z=0$ su $R=2$

$z=1-\frac{1}{2}\sqrt{x^2+y^2}$ è un CONO CIRCOLARE di $V(0,0,1)$, verso il basso, $a=\frac{1}{2}<1$ $\rightarrow \hat{ap}\approx 63{,}4°$, $\cap\, z=0$ su $R=2$

In particolare cono e paraboloide si intersecano sulla circonferenza di $R=2$ a quota $z=0$.

V è il paraboloide pieno per $-2\le z\le 0$ sormontato dal cono pieno per $0\le z\le 1$



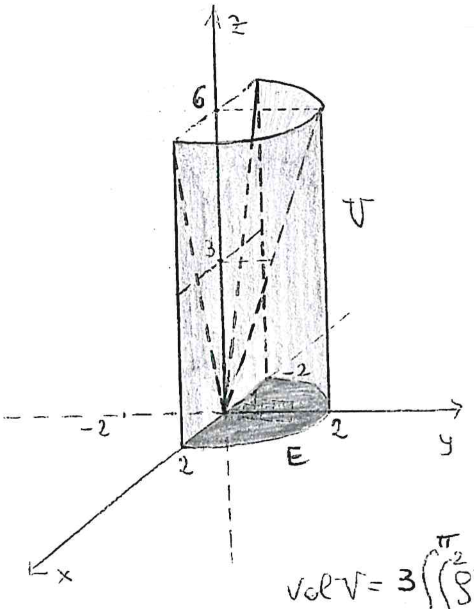


$\Pi_{x,y}(V)$

VOLUME di V a pag. 8

es g) h) $\Pi_{y,z}(V)$, $\Pi_{x,z}(V)$ a pag. 5

g)



$z=3\sqrt{x^2+y^2}$ è un CONO CIRCOLARE di $V(0,0,0)$, $a=3>1 \rightarrow \hat{ap}\approx 18{,}4°$

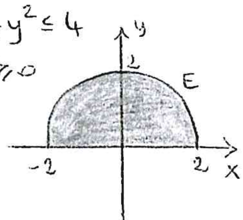
su $x^2+y^2=1 \rightarrow z=3$

su $x^2+y^2=4 \rightarrow z=6$

V è $\frac{1}{2}$ cilindro di $R=2$ e $h=6$ per $0\le z\le 6$, $y\ge 0$

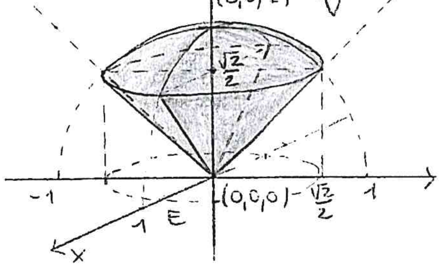
Scavato del cono -

$\text{vol } V=\int_E 3\sqrt{x^2+y^2}\,dx\,dy$ con $\underset{\Pi_{x,y}(V)}{\overset{\shortparallel}{E}}$: $x^2+y^2\le 4$, $y\ge 0$



$\text{vol } V=3\int_0^{\pi}\int_0^{2}\rho^2\,d\rho\,d\theta=3\pi\left[\frac{\rho^3}{3}\right]_0^2=3\pi\cdot\frac{8}{3}=\boxed{8\pi}$

h)



$z=\sqrt{x^2+y^2}$ è il cono di base ($\hat{ap}=45°$)

$x^2+y^2+z^2=1$ è la superficie sferica di $C(0,0,0)$ e $R=1$

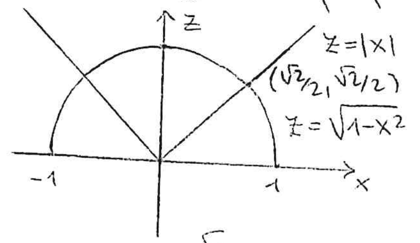
V è un CONO Base $R=\frac{\sqrt{2}}{2}$ $h=\frac{\sqrt{2}}{2}$ per $0\le z\le\frac{\sqrt{2}}{2}$ sormontato da una CALOTTA SFERICA ($\frac{\sqrt{2}}{2}\le z\le 1$)

$\Pi_{x,y}(V)$ a pag. 5

Cono è sup. sferica si $\cap$ sulla circonf di $R=\frac{\sqrt{2}}{2}$ a quota $z=\frac{\sqrt{2}}{2}$

$$\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+y^2+z^2=1 \end{cases} \rightarrow 2x^2+2y^2=1 \quad x^2+y^2=\frac{1}{2}$$

$z_{\text{cono}}=\frac{\sqrt{2}}{2}$ $\quad z_{\text{sup sferica}}=\frac{\sqrt{2}}{2}$



metà sup. della sup. sferica $z=\sqrt{1-x^2-y^2}$

$$\text{vol}\,V=\int_E \left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dx\,dy \quad \text{con } E = \Pi_{x,y}(V) \quad x^2+y^2\le\frac{1}{2} \quad R=\frac{\sqrt{2}}{2}$$

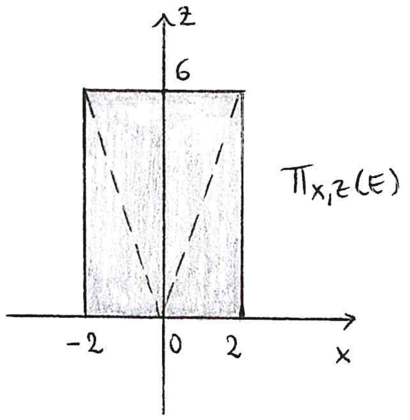
$$\text{vol}\,V=\int_0^{2\pi}\left(\int_0^{\frac{\sqrt{2}}{2}}\left(\sqrt{1-\rho^2}-\rho\right)\rho\,d\rho\right)d\theta = 2\pi\cdot\int_0^{\frac{\sqrt{2}}{2}}\left(\rho\sqrt{1-\rho^2}-\rho^2\right)d\rho =$$

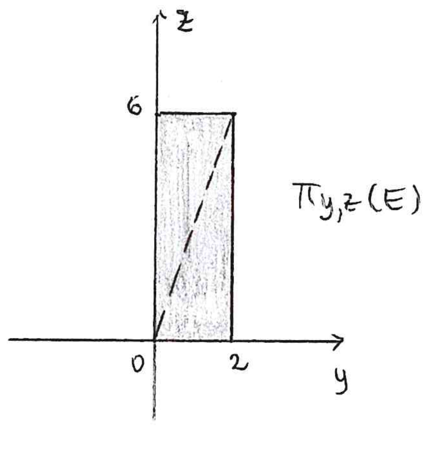
$$= 2\pi\cdot\left(-\frac{1}{2}\left[\frac{(1-\rho^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}/2}-\left[\frac{\rho^3}{3}\right]_0^{\frac{\sqrt{2}}{2}}\right) = 2\pi\left[-\frac{1}{3}\left(\left(\frac{1}{2}\right)^{3/2}-1\right)-\frac{2\sqrt{2}}{24}\right] =$$

$$\left(\frac{1}{2}\right)^{3/2} \downarrow \frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$$

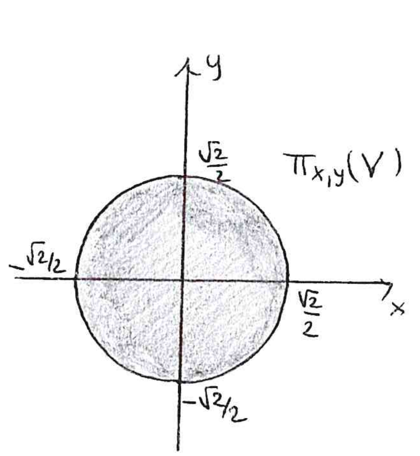
$$= 2\pi\left(-\frac{\sqrt{2}}{12}+\frac{1}{3}-\frac{\sqrt{2}}{12}\right) = \boxed{\frac{2}{3}\pi-\frac{\sqrt{2}}{3}\pi}$$

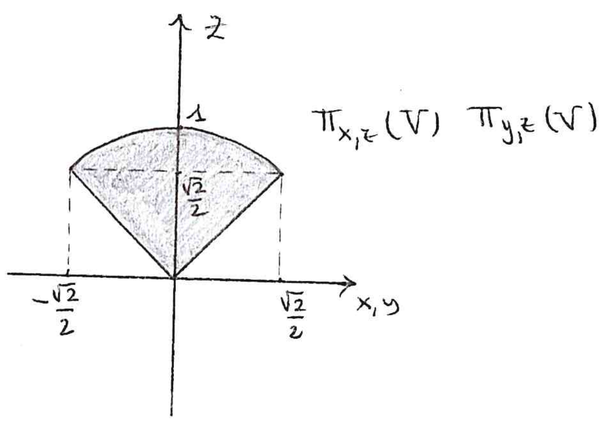
g)



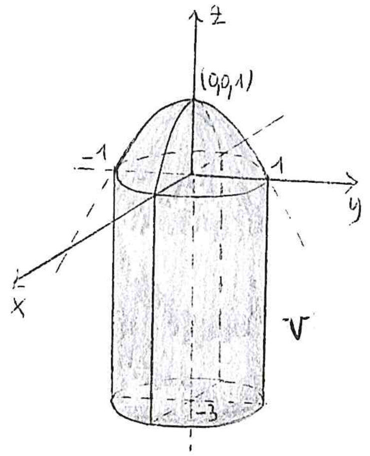


h)

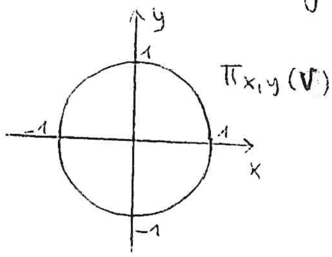


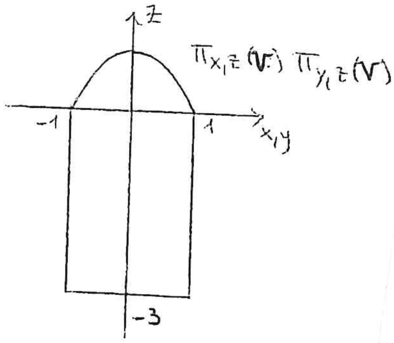


i)



$z = 1 - x^2 - y^2$ è un PARABOLOIDE CIRCOLARE di $V(0,0,1)$, $a = 1$, verso il basso, $\cap\, z = 0$ su $x^2 + y^2 = 1$





VOLUME di V =

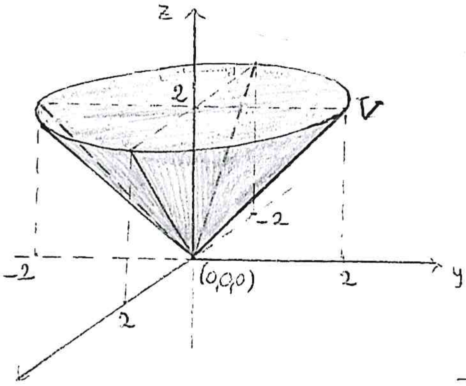
$$\int_{\Pi_{x,y}(V)} [1 - x^2 - y^2 - (-3)]\, dx\,dy =$$

V è un CILINDRO di $R = 1$ e $h = 3$ per $(-3 \le z \le 0)$ sormontato da un PARABOLOIDE per $0 \le z \le 1$

$$= \int_{x^2+y^2 \le 1} (1 - x^2 - y^2 + 3)\, dx\,dy = \int_0^{2\pi} \left(\int_0^1 (4 - \rho^2)\rho\, d\rho \right) d\theta = 2\pi \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^1 = 2\pi \cdot \frac{7}{4} =$$

$$= \boxed{\frac{7}{2}\pi}$$

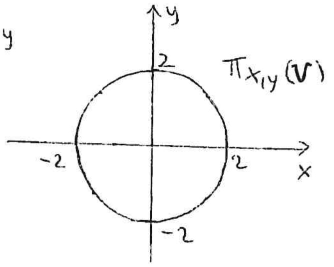
j)

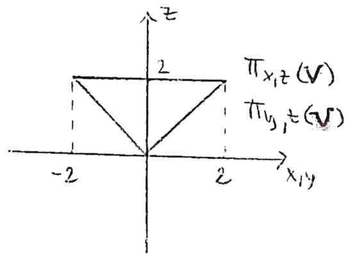


$z = \sqrt{x^2 + y^2}$ è il cono di base ($\hat{ap} = 45°$)

$z = 2$ su $R = 2$

V è un cono pieno: base $R = 2$, $h = 2$



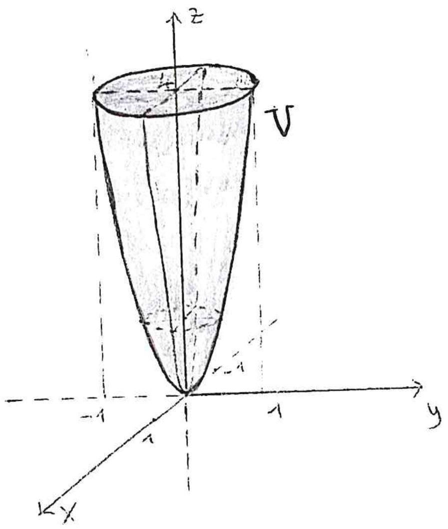


$$\text{Volume di } V = \int_{\Pi_{x,y}(V)} (2 - \sqrt{x^2+y^2})\, dx\,dy = \int_{x^2+y^2 \le 4} (2 - \sqrt{x^2+y^2})\, dx\,dy =$$

$$= \int_0^{2\pi} \left(\int_0^2 \underbrace{(2 - \rho)\rho}_{2\rho - \rho^2}\, d\rho \right) d\theta = \int_0^{2\pi} \left[\rho^2 - \frac{\rho^3}{3} \right]_0^2 d\theta = \int_0^{2\pi} \left(4 - \frac{8}{3} \right) d\theta =$$

$$= \frac{4}{3} \cdot 2\pi = \boxed{\frac{8}{3}\pi}$$

k)

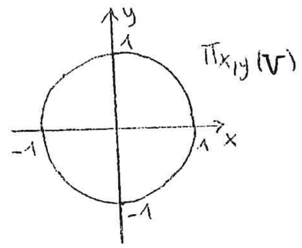


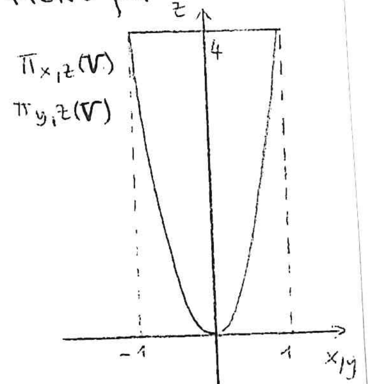
$z = 4(x^2+y^2)$ è un PARABOLOIDE CIRCOLARE di $V(0,0,0)$ verso l'alto

$a = 4 > 1$ (più stretto di $z = x^2+y^2$)

$z = 4$ su $x^2+y^2 = 1$

V è un PARABOLOIDE PIENO per $0 \le z \le 4$





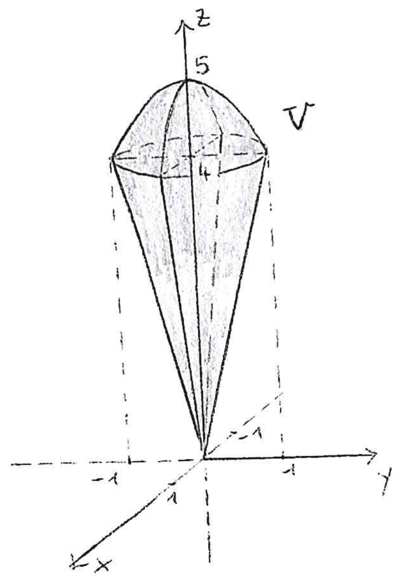
VOLUME di V =

$$\int_{\Pi_{x,z}(V)} [4-4(x^2+y^2)]\,dx\,dy = \int_0^{2\pi}\Big(\int_0^1 \underbrace{(4-4\rho^2)\rho}_{4\rho-4\rho^3}\,d\rho\Big)\,d\theta =$$

$\Pi_{x,z}(V)$ ↳ $x^2+y^2 \le 1$

$$= \int_0^{2\pi} \big[2\rho^2 - \rho^4\big]_0^1\,d\theta = \int_0^{2\pi}(2-1)\,d\theta = \boxed{2\pi}$$

l)



$z = 4\sqrt{x^2+y^2}$ è un CONO CIRCOLARE di $V(0,0,0)$, verso l'alto,

$a = 4 > 1 \to \hat{ap} \approx 14°$, $z = 4$ su $x^2+y^2 = 1$

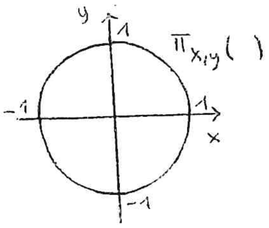
$z = 5-(x^2+y^2)$ è un PARABOLOIDE CIRCOLARE di $V(0,0,5)$, verso il basso,

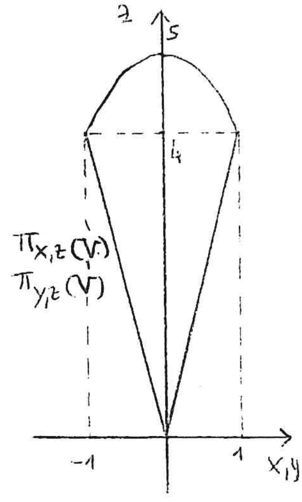
$a = 1$, $\cap\, z=0$ su $x^2+y^2 = 5$ $R = \sqrt{5}$

Cono e paraboloide si intersecano sicuramente $\begin{cases} z = 4\sqrt{x^2+y^2} \\ z = 5-(x^2+y^2) \end{cases}$

$R = \sqrt{x^2+y^2} \ge 0 \to 4R = 5-R^2 \quad R^2+4R-5=0 \quad R_1 = 1 \quad R_2 = -5 < 0$ (NON ACC.)

Si intersecano sulla circonferenza di $R = 1$ a quota $z = 4$





$\Pi_{x,y}(V): x^2+y^2 \le 1$

VOLUME di V =

$$= \int_{x^2+y^2 \le 1} \left(5-(x^2+y^2)-4\sqrt{x^2+y^2}\right) dx\,dy =$$

$$= \int_0^{2\pi} \left(\int_0^1 \underbrace{(5-\rho^2-4\rho)\rho}_{5\rho-\rho^3-4\rho^2} \, d\rho \right) d\theta = 2\pi \left[5\frac{\rho^2}{2} - \frac{\rho^4}{4} - 4\frac{\rho^3}{3} \right]_0^1 =$$

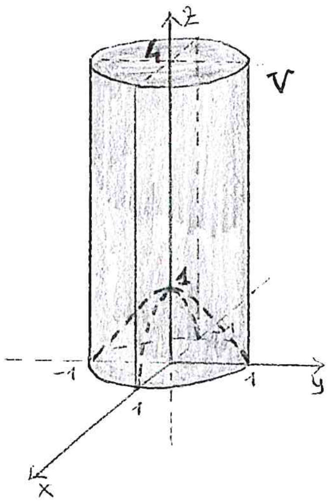
$$= 2\pi \left(\frac{5}{2} - \frac{1}{4} - \frac{4}{3} \right) = 2\pi \cdot \frac{11}{12} = \boxed{\frac{11}{6}\pi}$$

f) VOLUME di $V = \int_{\Pi_{x,y}(V)} \left[\left(1-\frac{1}{2}\sqrt{x^2+y^2}\right) - \left(\frac{1}{2}(x^2+y^2)-2\right) \right] dx\,dy =$

$$= \int_{x^2+y^2 \le 4} \left(3-\frac{1}{2}\sqrt{x^2+y^2}-\frac{1}{2}(x^2+y^2)\right) dx\,dy = \int_0^{2\pi} \left(\int_0^2 \underbrace{\left(3-\frac{1}{2}\rho-\frac{1}{2}\rho^2\right)\rho}_{3\rho-\frac{1}{2}\rho^2-\frac{1}{2}\rho^3} \, d\rho \right) d\theta =$$

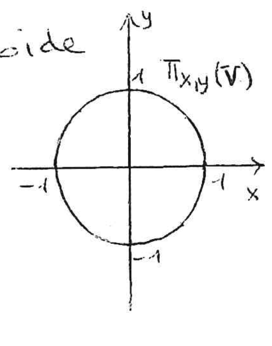
$$= \int_0^{2\pi} \left[3\frac{\rho^2}{2} - \frac{1}{2}\frac{\rho^3}{3} - \frac{\rho^4}{8} \right]_0^2 d\theta = \int_0^{2\pi} \left(\underbrace{6-\frac{4}{3}}_{4-4/3} -2 \right) d\theta = \frac{8}{3} \cdot 2\pi = \boxed{\frac{16}{3}\pi}$$

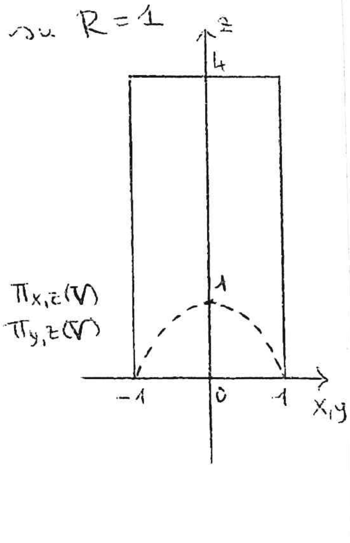
m)



V è il CILINDRO $R=1$ $h=4$ per $0 \le z \le 4$ privato del paraboloide per $0 \le z \le 1$

$z = 1-(x^2+y^2)$ è un PARABOLOIDE CIRCOLARE di $V(0,0,1)$, verso il basso, $a=1$, $\cap\, z=0$ su $R=1$



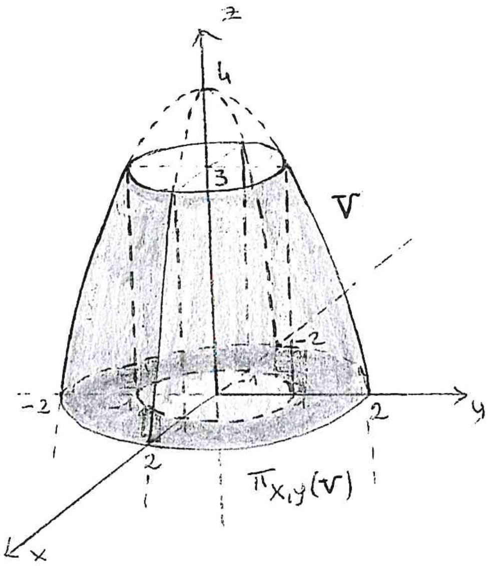


$\Pi_{x,y}(V): x^2+y^2 \le 1$

$$\text{Vol}\,V = \int_{x^2+y^2 \le 1} \left[4-(1-(x^2+y^2))\right] dx\,dy = \int_{x^2+y^2 \le 1} (3+(x^2+y^2))\,dx\,dy =$$

$$= \int_0^{2\pi} \left(\int_0^1 3\rho+\rho^3 \, d\rho \right) d\theta = 2\pi \cdot \left[3\frac{\rho^2}{2} + \frac{\rho^4}{4} \right]_0^1 = 2\pi \left[\frac{3}{2} + \frac{1}{4} \right] = 2\pi \cdot \frac{7}{4} = \boxed{\frac{7}{2}\pi}$$

n)



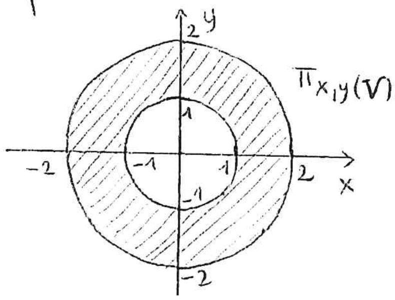
$z = 4-(x^2+y^2)$ è un

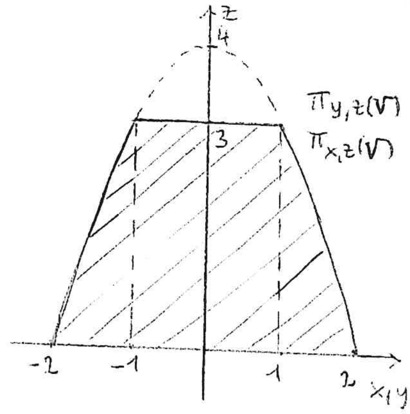
PARABOLOIDE CIRCOLARE di $V(0,0,4)$ verso il basso di apertura $a=1$, $\cap\ z=0$ $R=2$

su $x^2+y^2=1$ z par $=3$

V è il paraboloide per $0 \le z \le 3$

privato del CILINDRO di $R=1$



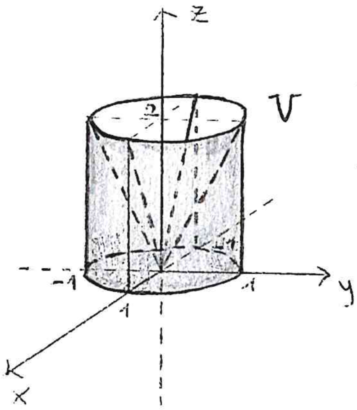


$\pi_{x,y}(V): 1 \le x^2+y^2 \le 4$

$$\text{Vol}\,V = \int_{1\le x^2+y^2\le 4} (4-x^2-y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\left(\int_1^2 (4-\rho^2)\rho\,d\rho\right)d\theta = 2\pi\left[2\rho^2-\frac{\rho^4}{4}\right]_1^2 = 2\pi\left[8-4-\underbrace{2+\frac{1}{4}}_{2+\frac{1}{4}}\right] = \boxed{\frac{9}{2}\pi}$$

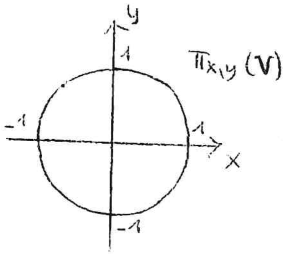
o)

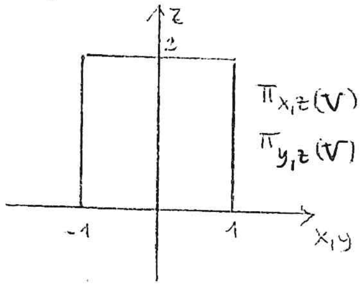


V è il CILINDRO di $R=1$ e $h=2$ per $0 \le z \le 2$ scavato del CONO

$z = 2\sqrt{x^2+y^2}$ è un CONO CIRCOLARE di $V(0,0,0)$ verso l'alto di apertura $a=2>1 \to \hat{ap} \simeq 26{,}6°$

su $x^2+y^2=1 \to z=2$



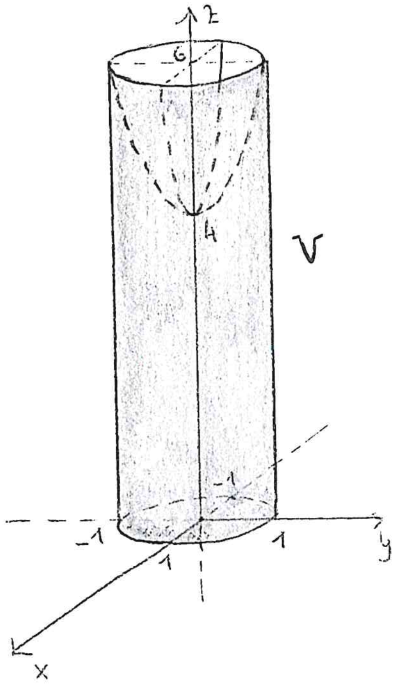


$$\text{Volume di } V = \int_{\pi_{x,y}(V)} 2\sqrt{x^2+y^2}\,dx\,dy =$$

$$= \int_{x^2+y^2\le 1} 2\sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\left(\int_0^1 2\rho\,\rho\,d\rho\right)d\theta = \int_0^{2\pi}\left[2\frac{\rho^3}{3}\right]_0^1 d\theta =$$

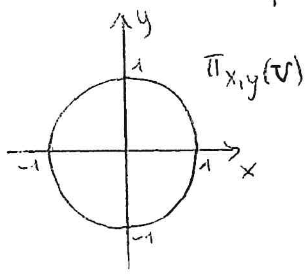
$$= \frac{2}{3}\cdot 2\pi = \boxed{\frac{4}{3}\pi}$$

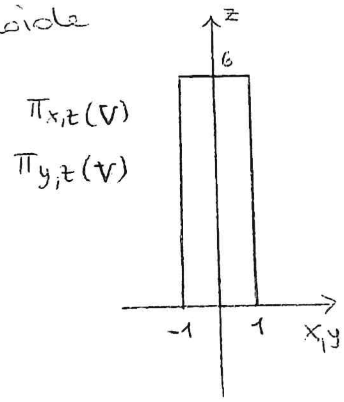
1)



$z = 2(x^2+y^2)+4$ è un paraboloide circolare di $V(0,0,4)$, verso l'alto, $a=2$, $\cap\, z=0\ \phi$, se $x^2+y^2=1 \to z=6$

V è il CILINDRO di $R=1$ e $h=6$ (per $0 \le z \le 6$) scavato del paraboloide



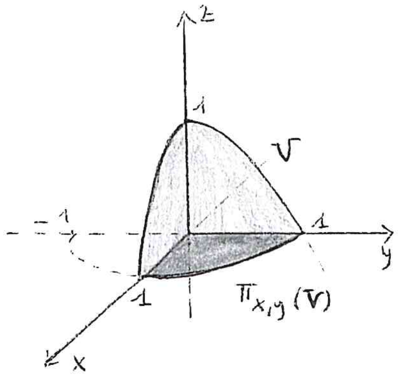


$\pi_{x,y}(V)$: $x^2+y^2 \le 1$

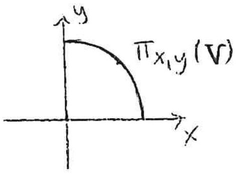
$$\text{Vol } V = \int_{x^2+y^2\le 1} (2(x^2+y^2)+4)\,dx\,dy =$$

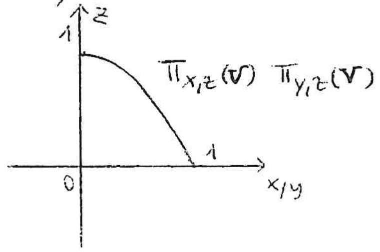
$$= \int_0^{2\pi}\left(\int_0^1 (2\rho^2+4)\cdot\rho\,d\rho\right)d\theta = 2\pi\cdot\left[\frac{\rho^4}{2}+2\rho^2\right]_0^1 = 2\pi\cdot\frac{5}{2} = \boxed{5\pi}$$

2)



$z = 1-x^2-y^2$ è il paraboloide circolare di $V(0,0,1)$ verso il basso, $a=1$, $\cap\, z=0$ su $R=1$



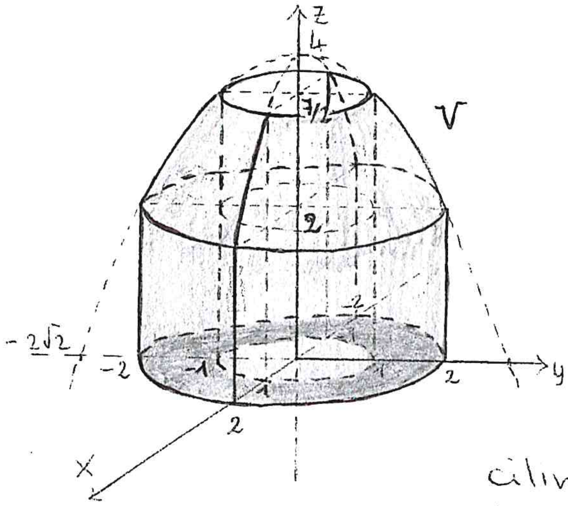


$$\text{volume di } V = \int_{\pi_{x,y}(V)} (1-x^2-y^2)\,dx\,dy =$$

$$= \int_{\substack{x^2+y^2\le 1\\ x\ge 0\ y\ge 0}} (1-x^2-y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 \underbrace{(1-\rho^2)\rho}_{\rho-\rho^3}\,d\rho\,d\theta = \int_0^{\pi/2}\left[\frac{\rho^2}{2}-\frac{\rho^4}{4}\right]_0^1 d\theta =$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{4}\right)d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{8}}$$

r)



$z = 4 - \frac{1}{2}(x^2+y^2)$ è il paraboloide circolare di $V(0,0,4)$, verso il basso, apertura $a = \frac{1}{2}$,

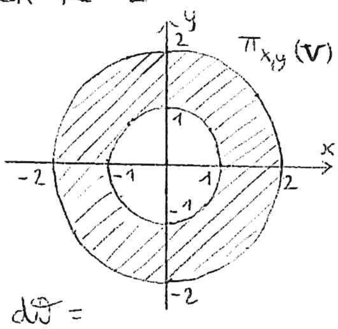
$\cap z = 0$ su $R = 2\sqrt{2}$

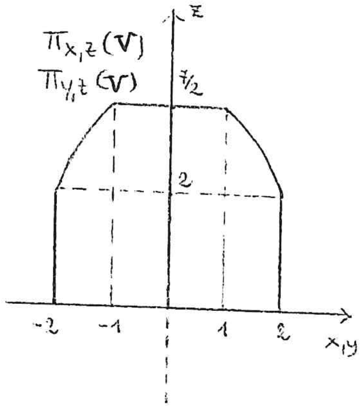
se $x^2+y^2 = 1 \rightarrow z = \frac{7}{2}$

se $x^2+y^2 = 4 \rightarrow z = 2$

V è composto per $0 \le z \le 2$ da un cilindro di $R=2$ e $h=2$ privato del cilindro di $R=1$ e $h=2$, sormontato per $2 \le z \le 7/2$ dal paraboloide privato del cilindro di $R=1$

$\Pi_{x,y}(V)$: $1 \le x^2+y^2 \le 4$



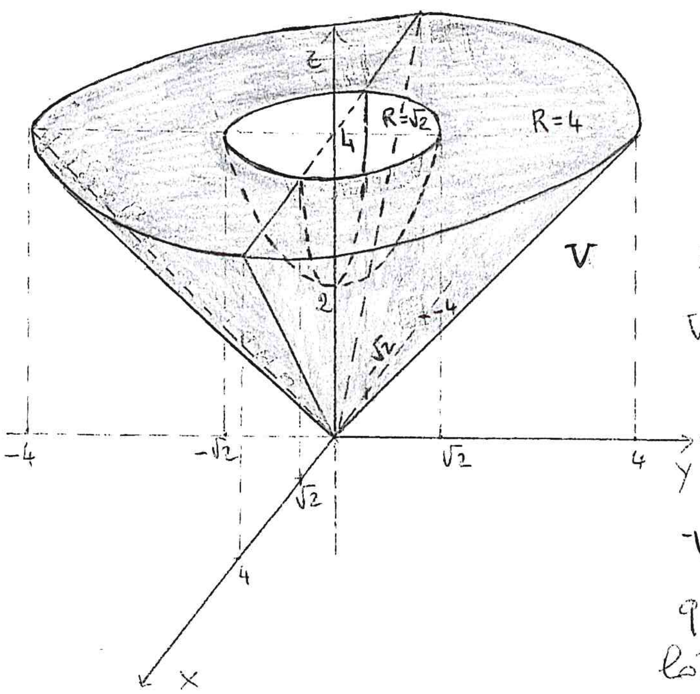


VOLUME di V =

$$= \iint_{1 \le x^2+y^2 \le 4} \left(4 - \frac{1}{2}(x^2+y^2)\right) dx\,dy = \int_0^{2\pi}\int_1^2 \underbrace{\left(4 - \frac{1}{2}\rho^2\right)\rho}_{(4\rho - \frac{1}{2}\rho^3)}\, d\rho\, d\theta =$$

$$= \int_0^{2\pi} \left[2\rho^2 - \frac{\rho^4}{8}\right]_1^2 d\theta = \left(8 - 2 - 2 + \frac{1}{8}\right) \cdot 2\pi = \left(4 + \frac{1}{8}\right) 2\pi = \boxed{\frac{33}{4}\pi}$$

s)



$z = \sqrt{x^2+y^2}$ è il cono di base ($\widehat{ap} = 45°$)

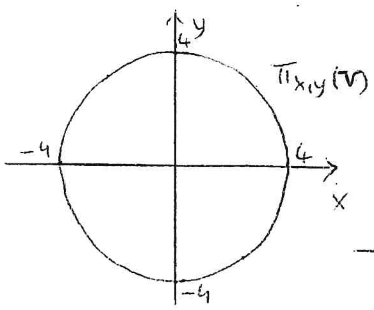
$z = 4$ su $R = 4$

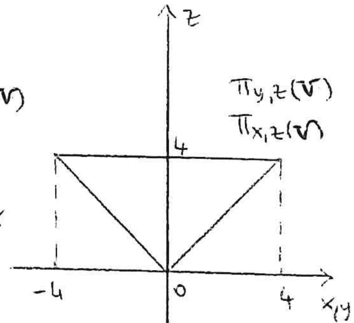
$z = 2 + x^2 + y^2$ è il paraboloide di $V(0,0,2)$, verso l'alto, $a = 1$ (è il paraboloide di base in alto di 2)

$z = 4 \rightarrow 4 = 2 + x^2+y^2 \rightarrow x^2+y^2 = 2$ $R = \sqrt{2}$

V è il cono pieno di base $R=4$ (a quota $z=4$) e $h=4$ privato del paraboloide (per $2 \le z \le 4$)

$$\Pi_{x,y}(V):\ x^2+y^2\le 16$$





$$\mathrm{vol}\,V=\int_{x^2+y^2\le 2}\left[2+x^2+y^2-\sqrt{x^2+y^2}\,\right]dx\,dy$$

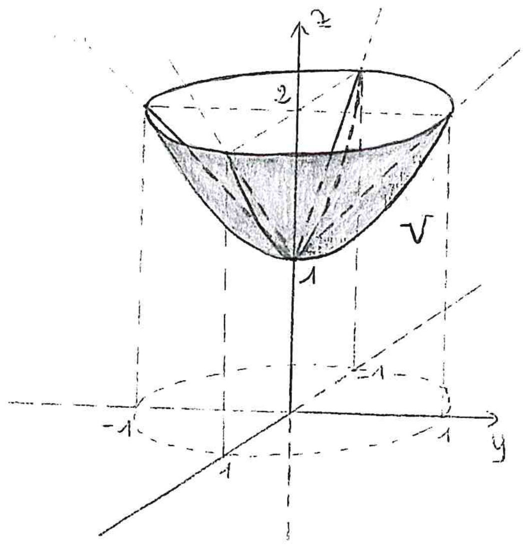
$$+\int_{2\le x^2+y^2\le 16}\left(4-\sqrt{x^2+y^2}\right)dx\,dy=\int_{x^2+y^2\le 2}\left(2+(x^2+y^2)-\sqrt{x^2+y^2}\right)dx\,dy\ +$$

$$+\int_{2\le x^2+y^2\le 16}\left(4-\sqrt{x^2+y^2}\right)dx\,dy=\int_0^{2\pi}\int_0^{\sqrt2}\underbrace{(2+\rho^2-\rho)\rho}_{2\rho+\rho^3-\rho^2}\,d\rho\,d\theta+\int_0^{2\pi}\int_{\sqrt2}^{4}\underbrace{(4-\rho)\rho}_{4\rho-\rho^2}\,d\rho\,d\theta=$$

$$=2\pi\left[\rho^2+\frac{\rho^4}{4}-\frac{\rho^3}{3}\right]_0^{\sqrt2}+2\pi\left[2\rho^2-\frac{\rho^3}{3}\right]_{\sqrt2}^{4}=2\pi\left(2+1-\cancel{\tfrac{2}{3}\sqrt2}\right)+2\pi\left(32-64/3\right.$$

$$\left.-4+\cancel{\tfrac{2}{3}\sqrt2}\right)=6\pi+\frac{40}{3}\pi=\boxed{\frac{58}{3}\pi}$$

t)



$z=1+\sqrt{x^2+y^2}$ è il CONO CIRCOLARE di $V(0,0,1)$, verso l'alto, $a=1$, $\hat{ap}=45°$ (è il CONO DI BASE IN ALTO di 1)

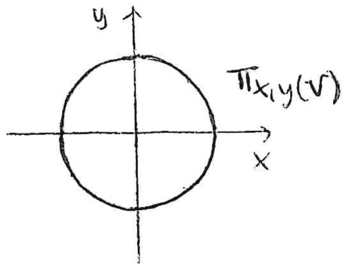
$z=1+x^2+y^2$ è il paraboloide circolare di $V(0,0,1)$, verso l'alto, $a=1$ (è il paraboloide di BASE in alto di 1)

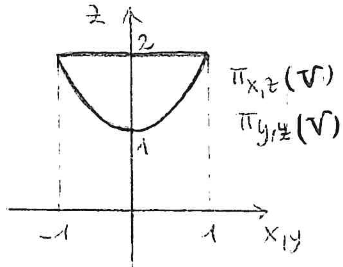
Paraboloide e cono si intersecano:

$$\begin{cases} z=1+\sqrt{x^2+y^2}\\ z=1+x^2+y^2\end{cases}\ \to\ 1+R=1+R^2\ \text{con}\ R=\sqrt{x^2+y^2}\ge 0\qquad R^2-R=0\ \begin{matrix}\nearrow R=0\\ \searrow R=1\end{matrix}$$

Cono e paraboloide si intersecano a quota $z=1$ nel punto $(0,0,1)$ $(R=0)$ e a quota $z=2$ nella circonf. di $R=1$

V è il paraboloide pieno (per $1\le z\le 2$) privato del cono $(1\le z\le 2)$.

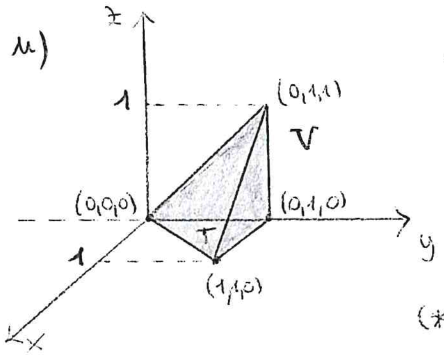




$$\text{VOLUME } V = \int_{\pi_{x,y}(V)} \left[\left(1+\sqrt{x^2+y^2}\right) - \left(1+x^2+y^2\right)\right] dx\,dy = \int_{x^2+y^2 \le 1} \left[\sqrt{x^2+y^2} - (x^2+y^2)\right] dx\,dy =$$

$$= \int_0^{2\pi} \left(\int_0^1 \underbrace{(\rho-\rho^2)\rho}_{\rho^2-\rho^3}\, d\rho \right) d\theta = \int_0^{2\pi} \left[\frac{\rho^3}{3} - \frac{\rho^4}{4} \right]_0^1 d\theta = \underbrace{\left(\frac{1}{3} - \frac{1}{4} \right)}_{\frac{1}{12}} 2\pi = \boxed{\frac{\pi}{6}}$$

n)

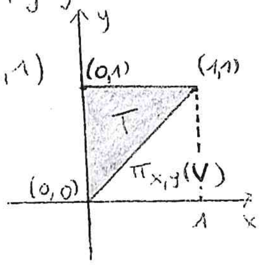


$V = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in T,\ 0 \le z \le -x+y \}$ (*)

T = triangolo di VERTICI (0,0) (0,1) (1,1)

$T = \{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x \le y \le 1 \}$

come NORM. risp. a x

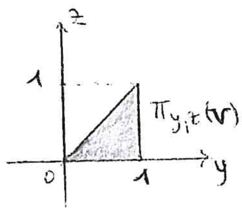


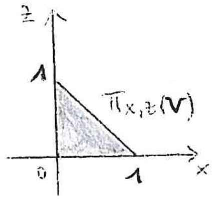
(*) piano per (1,1,0) (0,0,0) (0,1,1)

$z = ax+by+c \quad c=0 \quad \begin{cases} a+b=0 \\ b=1 \end{cases} \quad \begin{cases} a=-1 \\ b=1 \end{cases}$

$\boxed{z = -x+y}$

VOLUME V =





$$\int_T (-x+y)\,dx\,dy =$$

$$= \int_0^1 \left(\int_x^1 (-x+y)\,dy \right) dx = \qquad T_x = \{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x \le y \le 1 \}$$

$$= \int_0^1 \left[-xy + \frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left(-x + \frac{1}{2} + x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx =$$

$$= \left[\frac{x^3}{6} - \frac{x^2}{2} + \frac{1}{2}x \right]_0^1 = \frac{1}{6} - \frac{1}{2} + \frac{1}{2} = \boxed{\frac{1}{6}}$$

Si può anche usare $T_y = \{ (x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le y \}$.

v) $z = 4\sqrt{x^2+y^2}$ è il cono circolare di $V(0,0,0)$, verso l'alto, apertura $a=4$, $\hat{ap} \simeq 14°$

su $x^2+y^2=4 \to z=8$

$z=(x^2+y^2)+3$ è il paraboloide circolare di $V(0,0,3)$, verso l'alto, apertura $a=1$ (è il paraboloide di base in alto di 3)

su $x^2+y^2=4 \to z=7$

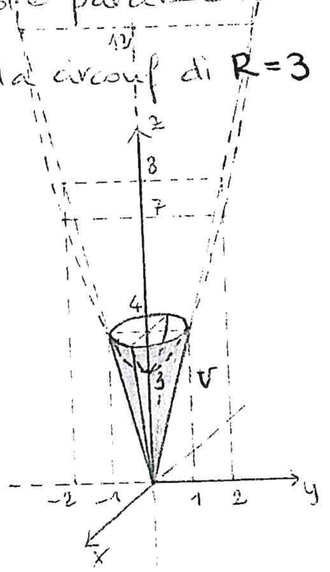
Il cono e il paraboloide si intersecano **2 volte**:

$$\begin{cases} z = 4\sqrt{x^2+y^2} \\ z = 3+(x^2+y^2) \end{cases}$$

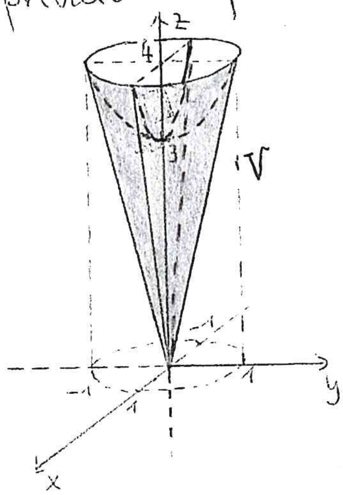
$4R = 3+R^2$ con $R=\sqrt{x^2+y^2}\geq 0$

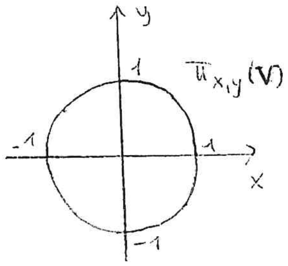
$R^2-4R+3=0 \quad R_{1,2} = \frac{2\pm\sqrt{4-3}}{1} = 2\pm 1 \to R_1=1, \ R_2=3$

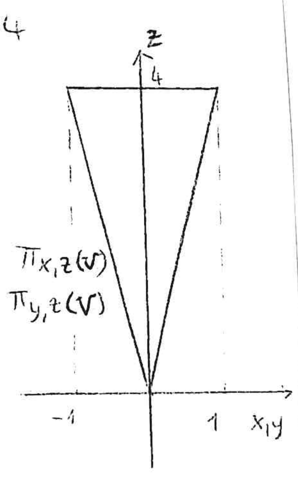
Cono e paraboloide si $\cap$ sulla circonf. di **$R=1$ a quota $z=4$** e sulla circonf. di **$R=3$ a quota $z=12$**



I punti di V sono quelli al di sopra del cono e al di sotto del paraboloide, internamente al cilindro di $R=2 \Rightarrow V$ è il cono per $0\leq z\leq 4$ privato del paraboloide per $3\leq z\leq 4$







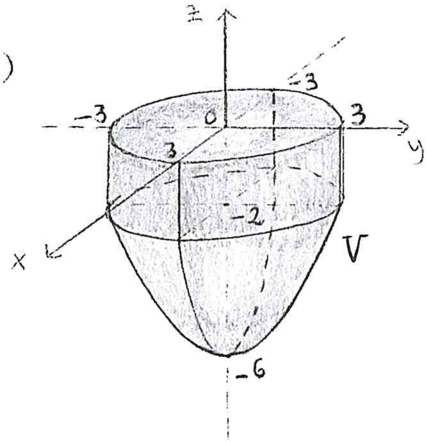
$\Pi_{x,y}(V)$: $x^2+y^2\leq 1$

VOLUME V =

$$= \int_{x^2+y^2\leq 1} \left(3+(x^2+y^2)-4\sqrt{x^2+y^2}\right)dx\,dy = \int_0^{2\pi}\left(\int_0^1 \overbrace{(3+\rho^2-4\rho)\rho}^{3\rho+\rho^3-4\rho^2}\,d\rho\right)d\theta = 2\pi\left[3\frac{\rho^2}{2}+\frac{\rho^4}{4}-4\frac{\rho^3}{3}\right]_0^1 =$$

$$= 2\pi\left(\frac{3}{2}+\frac{1}{4}-\frac{4}{3}\right) = 2\pi\cdot\frac{5}{12} = \boxed{\frac{5}{6}\pi}$$

w)

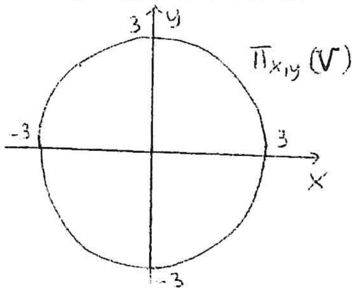


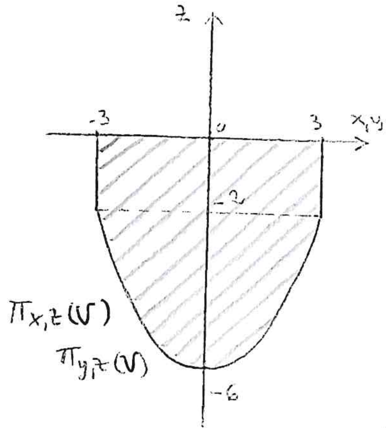
$z = -6 + \frac{4}{9}(x^2+y^2)$ è il paraboloide circolare di $V(0,0,-6)$, verso l'alto, apertura $a = \frac{4}{9} < 1$,

$\cap\, z=0$ su $x^2+y^2 = \frac{54}{4} = \frac{27}{2}$ $\quad R = \sqrt{\frac{27}{2}} = 3\frac{\sqrt{3}}{\sqrt{2}} \simeq 3{,}7$

Su $x^2+y^2 = 9 \to z_{par} = -2$

V è il paraboloide pieno per $-6 \le z \le -2$ sormontato da un cilindro di $R=3$ e $h=2$ per $-2 \le z \le 0$.





$\Pi_{x,y}(V) : x^2+y^2 \le 9$

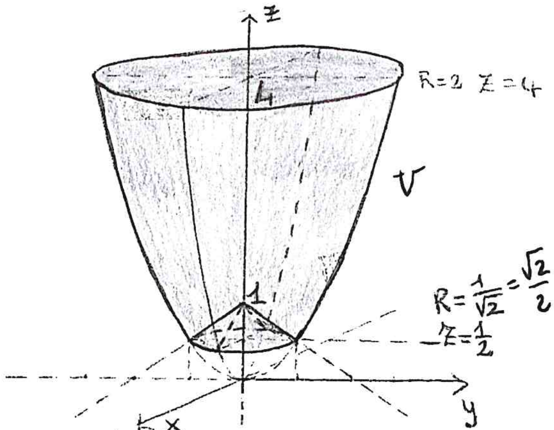
VOLUME V =

$$= \int_{x^2+y^2\le 9} \left[0 - \left(-6 + \frac{4}{9}(x^2+y^2)\right)\right] dx\,dy =$$

$$= \int_{x^2+y^2\le 9} \left(6 - \frac{4}{9}(x^2+y^2)\right) dx\,dy = \int_0^{2\pi}\left(\int_0^3 \left(6 - \frac{4}{9}\rho^2\right)\rho\, d\rho\right) d\theta = 2\pi \cdot \left[3\rho^2 - \frac{1}{9}\rho^4\right]_0^3 =$$

$$= 2\pi(27-9) = \boxed{36\pi}$$

x)



$z = x^2+y^2$ è il PARABOLOIDE di Base $z=4$ su $R=2$

$z = 1 - \frac{1}{\sqrt{2}}\sqrt{x^2+y^2}$ è il cono circolare di $V(0,0,1)$, verso il basso, $a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} < 1$

$\to \hat{ap} \simeq 54{,}7°$, $\cap\, z=0$ su $R = \sqrt{2}$

Cono e paraboloide si intersecano:

$$\begin{cases} z = x^2+y^2 \\ z = 1 - \frac{1}{\sqrt{2}}\sqrt{x^2+y^2} \end{cases} \to R^2 = 1 - \frac{1}{\sqrt{2}}R \quad \text{con } R = \sqrt{x^2+y^2} \ge 0$$

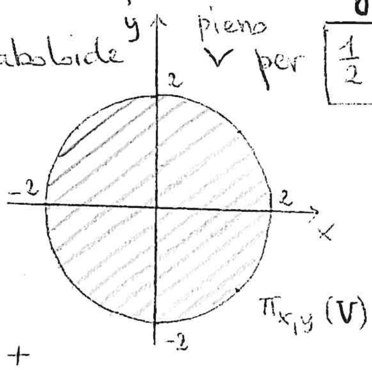
$$\boxed{R^2 + \frac{1}{\sqrt{2}}R - 1 = 0}$$

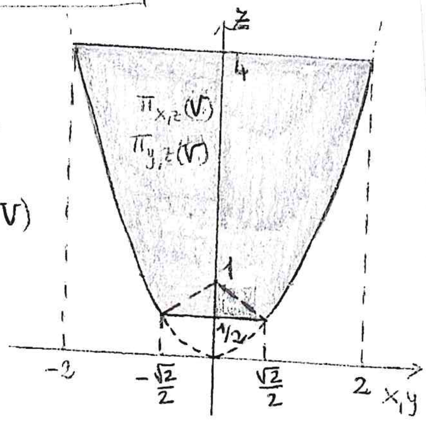
$R^2 + \frac{1}{\sqrt{2}}R - 1 = 0 \qquad R_{1,2} = \dfrac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2}+4}}{2}$

$$R_{1,2} = \frac{-\frac{1}{\sqrt{2}} \pm \frac{3}{\sqrt{2}}}{2} = \begin{cases} R_1 = -\frac{2}{\sqrt{2}} < 0 \text{ NON ACC.} \\ R_2 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \to R = \frac{\sqrt{2}}{2} \end{cases}$$

$z_{par} = R^2 = \frac{1}{2} \qquad z_{cono} = 1 - \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2} = 1 - \frac{1}{2} = \frac{1}{2}$

Cono e paraboloide si $\cap$ sulla circonferenza $x^2+y^2=\frac{1}{2}$ di $R=\frac{\sqrt{2}}{2}$ a quota $z=\frac{1}{2}$. V è il paraboloide pieno per $\boxed{\frac{1}{2}\le z\le 4}$ privato del cono per $\frac{1}{2}\le z\le 1$.





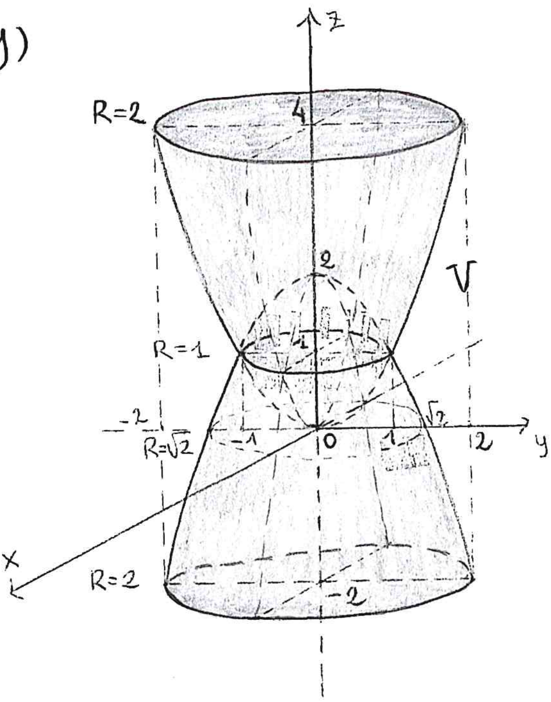
$$\pi_{x,y}(V): x^2+y^2\le 4$$

$$= \int_{x^2+y^2\le\frac{1}{2}} \left(4-\left(1-\frac{1}{\sqrt{2}}\sqrt{x^2+y^2}\right)\right)dx\,dy + \int_{\frac{1}{2}\le x^2+y^2\le 4} \left(4-(x^2+y^2)\right)dx\,dy = \int_{x^2+y^2\le\frac{1}{2}} \left(3+\frac{1}{\sqrt{2}}\sqrt{x^2+y^2}\right)dx\,dy + \int_{\frac{1}{2}\le x^2+y^2\le 4} \left(4-(x^2+y^2)\right)dx\,dy =$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \underbrace{\left(3+\frac{1}{\sqrt{2}}\rho\right)\rho}_{3\rho+\frac{1}{\sqrt{2}}\rho^2}\,d\rho\,d\vartheta + \int_0^{2\pi}\int_{\frac{1}{\sqrt{2}}}^{2} \underbrace{(4-\rho^2)\rho}_{4\rho-\rho^3}\,d\rho\,d\vartheta = 2\pi\left[3\frac{\rho^2}{2}+\frac{1}{\sqrt{2}}\frac{\rho^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} +$$

$$+ 2\pi\left[2\rho^2-\frac{\rho^4}{4}\right]_{\frac{1}{\sqrt{2}}}^{2} = 2\pi\underbrace{\left(\frac{3}{4}+\frac{1}{12}\right)}_{5/6} + 2\pi\left[8-4\underbrace{-1+\frac{1}{16}}_{\frac{49}{16}}\right] = \frac{5}{3}\pi + \frac{98}{16}\pi = \frac{\cancel{374}^{\,187}}{\cancel{48}_{\,24}}\pi = \boxed{\frac{187}{24}\pi}$$

y)



$z=x^2+y^2$ è il paraboloide di base

$z=4$ su $R=2$ $\rightarrow$ V_1 è il paraboloide pieno per $0\le z\le 4$

$z=2-(x^2+y^2)$ è il paraboloide circolare di $V(0,0,2)$, verso il basso, apertura $a=1$, $\cap\, z=0$ su $R=\sqrt{2}$ (è il paraboloide di base capovolto e alzato di 2), $z=-2$ su $R=2$

$\rightarrow$ V_2 è il paraboloide pieno per $-2\le z\le 2$

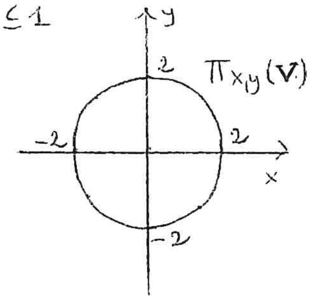
$V = V_1 \cup V_2$: dobbiamo unire i due insiemi, i due paraboloidi si intersecano

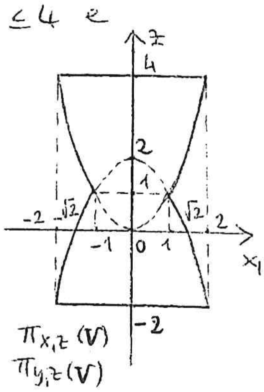
$$\begin{cases} z = x^2+y^2 \\ z = 2-(x^2+y^2) \end{cases} \rightarrow x^2+y^2 = 2-(x^2+y^2)$$

$$x^2+y^2=1$$

Sulla circonferenza di $R=1$ a quota $z=1$.

V è composto dal paraboloide di base pieno per $1 \le z \le 4$ e dall'altro paraboloide pieno per $-2 \le z \le 1$





$$\text{VOLUME } V = 2 \text{ VOLUME}(V \cap z \ge 1) =$$

$\pi_{x,y}(V)$ $x^2+y^2 \le 4$

$$= 2\Big(\underbrace{\int_{x^2+y^2\le 1} (4-1)\,dx\,dy}_{(*)} +$$

$$+ \int_{1\le x^2+y^2 \le 4} (4-(x^2+y^2))\,dx\,dy \Big) = 2\Big(3 \text{ area}(x^2+y^2 \le 1) + \int_0^{2\pi} \Big(\int_1^2 \underbrace{(4-\rho^2)\rho}_{4\rho-\rho^3}\,d\rho \Big) d\theta \Big) =$$

$$= 2\cdot\Big(3\pi + \int_0^{2\pi} \Big[2\rho^2 - \frac{\rho^4}{4}\Big]_1^2 d\theta\Big) = 2\Big(3\pi + \Big(8-4-2+\frac{1}{4}\Big)\cdot 2\pi\Big) =$$

$$= 2\cdot\Big(3\pi + \frac{9}{2}\pi\Big) = 2\cdot\Big(\frac{15}{2}\pi\Big) = \boxed{15\pi}$$

Senza la simmetria si può calcolare

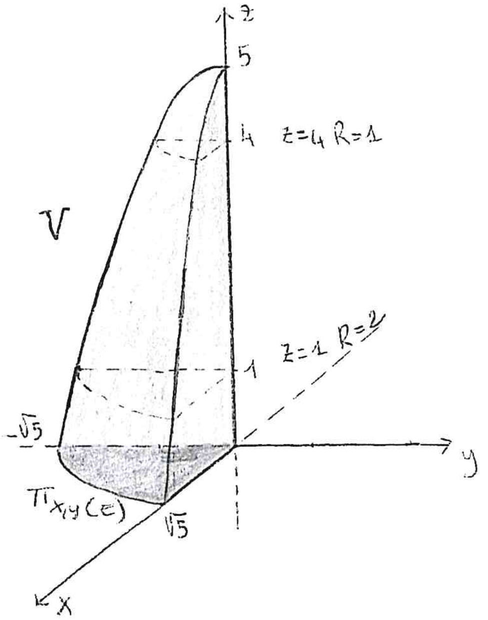
$$\text{VOLUME } V = \int_{x^2+y^2\le 1} (4-(-2))\,dx\,dy + \int_{1\le x^2+y^2\le 4} (4-(x^2+y^2))\,dx\,dy +$$

$$+ \int_{1\le x^2+y^2\le 4} \Big[(2-\underbrace{(x^2+y^2))-(-2)}_{4-(x^2+y^2)}\Big]\,dx\,dy$$

sono uguali

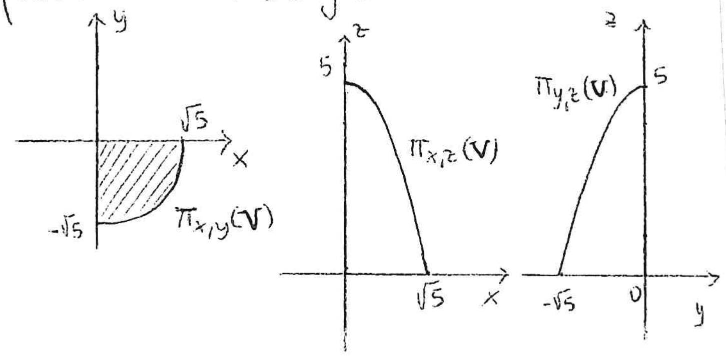
$= 2\ (*)$

Z)



$z = 5 - (x^2+y^2)$ è il paraboloide circolare di $V(0,0,5)$, verso il basso, apertura $a=1$, $\cap z=0$ su $R=\sqrt{5}$

Si considera il paraboloide pieno da $z=0$ al vertice, ma solo il quarto con $x \geq 0$ $y \leq 0$



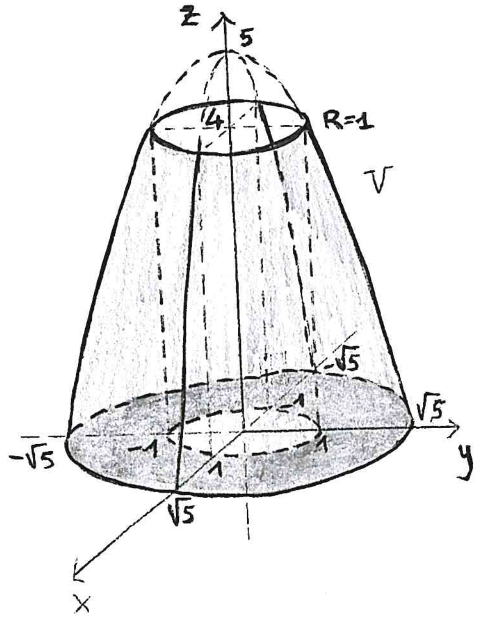
$\pi_{x,y}(V)$: $x^2+y^2 \leq 5$, $x \geq 0$, $y \leq 0$

$$\text{VOLUME di } V = \int_{\substack{x^2+y^2\leq 5 \\ x\geq 0 \; y\geq 0}} 5-(x^2+y^2)\,dx\,dy =$$

$$= \int_{\frac{3}{2}\pi}^{2\pi} \left(\int_0^{\sqrt{5}} \underbrace{(5-\rho^2)\rho}_{5\rho-\rho^3}\, d\rho \right) d\theta = \int_{\frac{3}{2}\pi}^{2\pi} \left[5\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^{\sqrt{5}} d\theta = \left(\frac{25}{2} - \frac{25}{4} \right) \frac{\pi}{2} =$$

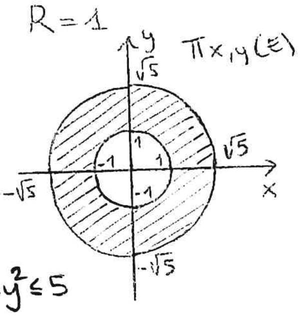
$$= \frac{25}{4} \cdot \frac{\pi}{2} = \boxed{\frac{25}{8}\pi}$$

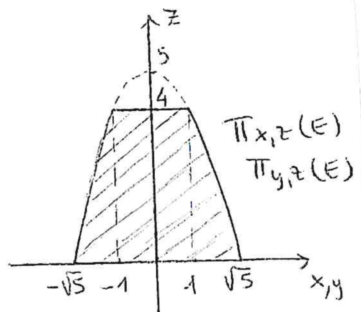
Z')



$z = 5-(x^2+y^2)$ è lo stesso paraboloide dell' es. Z) su $x^2+y^2=1 \rightarrow z$ par $=4$

V è il paraboloide pieno per $0 \leq z \leq 4$, privato del cilindro di R = 1





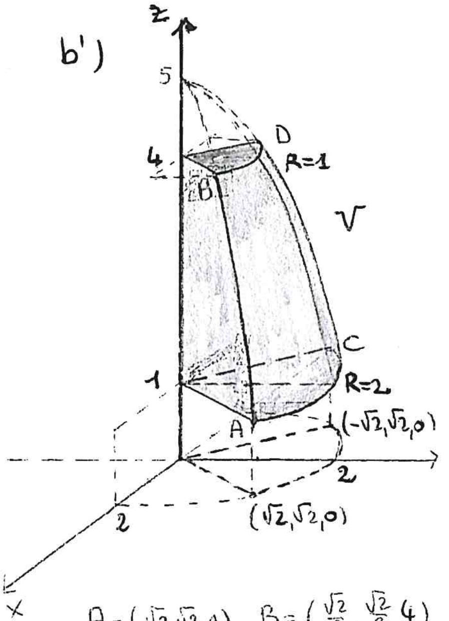
$\pi_{x,y}(V)$: $1 \leq x^2+y^2 \leq 5$

$$\text{VOLUME } V = \iint_{1\le x^2+y^2\le 5} [5-(x^2+y^2)]\,dx\,dy =$$

$$= \int_0^{2\pi}\left(\int_1^{\sqrt5} \underbrace{(5-\rho^2)\rho}_{5\rho-\rho^3}\,d\rho\right)d\theta = \int_0^{2\pi}\left[5\frac{\rho^2}{2}-\frac{\rho^4}{4}\right]_1^{\sqrt5} d\theta = \left(\frac{25}{2}-\frac{25}{4}-\frac{5}{2}+\frac{1}{4}\right)\cdot 2\pi =$$

$$= 4\cdot 2\pi = \boxed{8\pi}$$

b')



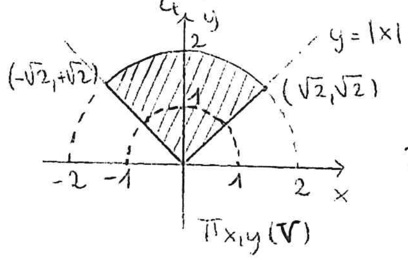
$A=(\sqrt2,\sqrt2,1)$ $B=\left(\frac{\sqrt2}{2},\frac{\sqrt2}{2},4\right)$

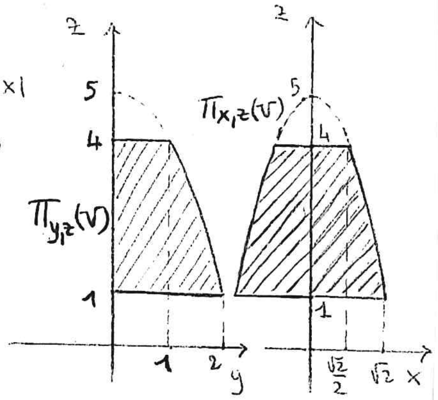
$C=(-\sqrt2,+\sqrt2,1)$ $D=\left(-\frac{\sqrt2}{2},+\frac{\sqrt2}{2},4\right)$

$Z = 5-(x^2+y^2)$ è lo stesso paraboloide degli es. z) e a')

$Z=1$ su $R=2$, $Z=4$ su $R=1$

La condizione $y\ge|x|$ individua $\frac{1}{4}$ del solido





$\Pi_{x,y}(V)$ $\quad x^2+y^2\le 4,\ y\ge|x|$

$$\text{VOLUME } V = \iint_{\substack{x^2+y^2\le 1\\ y\ge|x|}} (4-1)\,dx\,dy + \iint_{\substack{1\le x^2+y^2\le 4\\ y\ge|x|}} [(5-(x^2+y^2))-1]\,dx\,dy =$$

$$= 3\int_{\frac{\pi}{4}}^{\frac34\pi}\left(\int_0^1 \rho\,d\rho\right)d\theta + \int_{\frac{\pi}{4}}^{\frac34\pi}\left(\int_1^2 \underbrace{(4-\rho^2)\rho}_{4\rho-\rho^3}\,d\rho\right)d\theta =$$

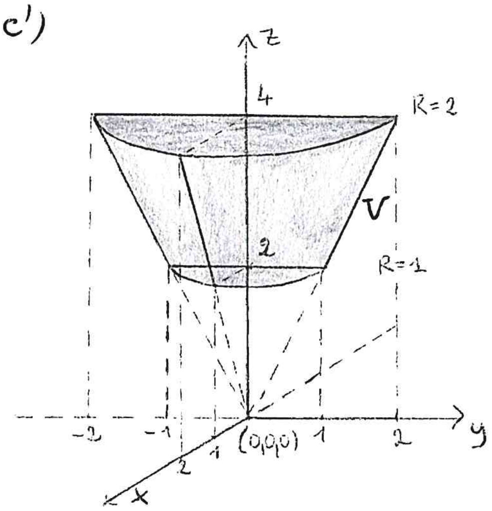
$$= 3\int_{\frac{\pi}{4}}^{\frac34\pi}\left[\frac{\rho^2}{2}\right]_0^1 d\theta + \int_{\frac{\pi}{4}}^{\frac34\pi}\left[2\rho^2-\frac{\rho^4}{4}\right]_1^2 d\theta = \frac32\left(\overbrace{\frac34\pi-\frac{\pi}{4}}^{\pi/2}\right) +$$

$$+ \left(8-4-2+\frac14\right)\left(\frac34\pi-\frac{\pi}{4}\right) =$$

$$= \frac34\pi + \frac98\pi = \boxed{\frac{15}{8}\pi}$$

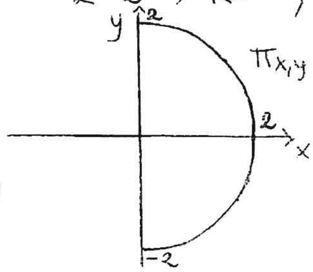
si può anche calcolare come $\frac14$ area$(x^2+y^2\le1) = \frac{\pi}{4}$

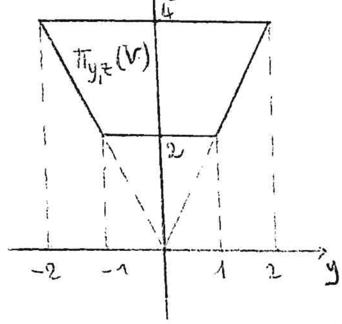
c')



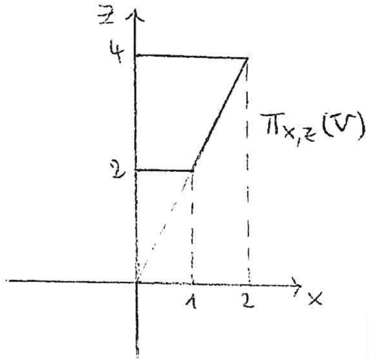
$z=2\sqrt{x^2+y^2}$ è il cono circolare di $V(0,0,0)$, verso l'alto, $a=2>1 \rightarrow \hat{ap} \simeq 26{,}6°$,

$z=2 \rightarrow R=1$, $z=4 \rightarrow R=2$





$\Pi_{x,y}(V)$: $x^2+y^2 \le 4$, $x \ge 0$



$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 1\\ x\ge 0}} (4-2)\,dx\,dy + \int_{\substack{1\le x^2+y^2\le 4\\ x\ge 0}} \left(4-2\sqrt{x^2+y^2}\right)dx\,dy =$$

$$= 2\,\frac{1}{2}\,\text{area}(x^2+y^2\le 1) + \int_{-\pi/2}^{\pi/2}\left(\int_1^2 \underbrace{(4-2\rho)\rho}_{4\rho-2\rho^2}\,d\rho\right)d\theta =$$

$$= \pi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2\rho^2 - 2\frac{\rho^3}{3}\right]_1^2 d\theta = \pi + \left(8-\frac{16}{3}-2+\frac{2}{3}\right)\cdot\pi =$$

$$= \pi + \left(6-\frac{14}{3}\right)\pi = \pi + \frac{4}{3}\pi = \boxed{\frac{7}{3}\pi}\ .$$

ES.2) a) $$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 16\\ 0\le y\le x}} \sqrt{16-x^2-y^2}\,dx\,dy = \int_0^{\pi/4}\left(\int_0^4 \sqrt{16-\rho^2}\cdot\rho\,d\rho\right)d\theta =$$

$\Pi_{x,y}(V)$: $x^2+y^2\le 16$, $0\le y\le x$

$$= -\frac{1}{2}\int_0^{\pi/4}\left(\int_0^4 (-2\rho)(16-\rho^2)^{1/2}d\rho\right)d\theta = -\frac{1}{2}\int_0^{\pi/4}\left[\frac{(16-\rho^2)^{3/2}}{3/2}\right]_0^4 d\theta =$$

$$= -\frac{1}{3}\int_0^{\pi/4}\left[0-(16)^{3/2}\right]d\theta = -\frac{1}{3}(-4^3)\cdot\frac{\pi}{4} = \boxed{\frac{16}{3}\pi}$$

$$-\frac{1}{3}(-64)\cdot\frac{\pi}{4}$$

b) $\Pi_{x,y}(V)$: $x^2+y^2\le 4$, $y\ge 0$ $$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 4\\ y\ge 0}} \left[5-\frac{5}{2}\sqrt{x^2+y^2}-\left(-\sqrt{4-x^2-y^2}\right)\right]dx\,dy =$$

$$= \int_0^{\pi}\left(\int_0^2 \left(5-\frac{5}{2}\rho+\sqrt{4-\rho^2}\right)\rho\,d\rho\right)d\theta = \int_0^{\pi}\left(\int_0^2\left(5\rho-\frac{5}{2}\rho^2\right)d\rho - \frac{1}{2}\int_0^2(-2\rho)(4-\rho^2)^{1/2}d\rho\right)d\theta =$$

$$= \left[5\frac{\rho^2}{2}-\frac{5\rho^3}{6}\right]_0^2\cdot\pi - \frac{1}{2}\int_0^{\pi}\left[\frac{(4-\rho^2)^{3/2}}{3/2}\right]_0^2 d\theta = \left(10-\frac{20}{3}\right)\pi - \frac{1}{3}\int_0^{\pi}\left(0-4^{3/2}\right)d\theta =$$

$$= \frac{10}{3}\pi + \frac{8}{3}\pi = \boxed{6\pi}$$

c) $\Pi_{x,y}(V)$ $x^2+y^2\leq 1$, $x\geq 0, y\leq 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\leq 1\\ x\geq 0\ y\leq 0}} \left[1-x^2-y^2-\left(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{x^2+y^2}\right)\right] dx\,dy = \int_{\frac{3}{2}\pi}^{2\pi}\left(\int_0^1 \underbrace{\left(\tfrac{1}{2}-\rho^2+\tfrac{1}{2}\rho\right)\rho}_{\frac{1}{2}\rho-\rho^3+\frac{1}{2}\rho^2}\,d\rho\right)d\theta =$$

$$= \int_{\frac{3}{2}\pi}^{2\pi}\left(\left[\frac{\rho^2}{4}-\frac{\rho^4}{4}+\frac{\rho^3}{6}\right]_0^1\right)d\theta = \left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{4}}+\tfrac{1}{6}\right)\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{12}}$$

d) $\Pi_{x,y}(V)$: $x^2+y^2\leq 1$, $x\geq 0$, $y\geq 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\leq 1\\ x\geq 0\ y\geq 0}} 4-3(x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\left(\int_0^1 \underbrace{(4-3\rho^2)\rho}_{4\rho-3\rho^3}\,d\rho\right)d\theta =$$

$$= \int_0^{\pi/2}\left[2\rho^2-3\frac{\rho^4}{4}\right]_0^1 d\theta = \left(2-\frac{3}{4}\right)\cdot\frac{\pi}{2} = \frac{5}{4}\cdot\frac{\pi}{2} = \boxed{\frac{5}{8}\pi}$$

e) $\Pi_{x,y}(V)$: $x^2+y^2\leq 4$, $y\geq 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\leq 4\\ y\geq 0}} \left[6-2\sqrt{x^2+y^2}-\left(2-\sqrt{4-x^2-y^2}\right)\right] dx\,dy =$$

$$= \int_0^{\pi}\left(\int_0^2 \left(4-2\rho+\sqrt{4-\rho^2}\right)\rho\,d\rho\right)d\theta = \int_0^{\pi}\left(\int_0^2 (4\rho-2\rho^2)\,d\rho\right)d\theta$$

$$-\frac{1}{2}\int_0^{\pi}\left(\int_0^2(-2\rho)(4-\rho^2)^{1/2}\,d\rho\right)d\theta = \int_0^{\pi}\left[2\rho^2-2\frac{\rho^3}{3}\right]_0^2 d\theta -$$

$$-\frac{1}{2}\int_0^{\pi}\left[\frac{(4-\rho^2)^{3/2}}{3/2}\right]_0^2 d\theta = \left(8-\frac{16}{3}\right)\pi-\frac{1}{3}\left(0-\underbrace{4^{3/2}}_{8}\right)\pi =$$

$$= \frac{8}{3}\pi+\frac{8}{3}\pi = \boxed{\frac{16}{3}\pi}$$

f) $\Pi_{x,y}(V)$: $x^2+y^2\leq 4$ $\qquad$ $\text{VOLUME } V = \int_{x^2+y^2\leq 4}\left(4-\frac{1}{2}\sqrt{x^2+y^2}\right)dx\,dy =$

$$= \int_0^{2\pi}\left(\int_0^2 \underbrace{\left(4-\tfrac{1}{2}\rho\right)\rho}_{4\rho-\frac{1}{2}\rho^2}\,d\rho\right)d\theta = \int_0^{2\pi}\left[2\rho^2-\frac{1}{2}\frac{\rho^3}{3}\right]_0^2 d\theta = \left(8-\frac{\cancel{8}^{\,4}}{\cancel{6}_{\,3}}\right)\cdot 2\pi = \frac{20}{3}\cdot 2\pi = \boxed{\frac{40}{3}\pi}$$

g) $\Pi_{x,y}(V): x^2+y^2 \le 1,\ y \ge 0, x \ge 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 1\\ x\ge 0,\, y\ge 0}} \left[2-x^2-y^2-\sqrt{x^2+y^2}\right] dx\,dy =$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^1 \underbrace{(2-\rho^2-\rho)\rho}_{2\rho-\rho^3-\rho^2} d\rho \right) d\theta = \int_0^{\pi/2} \left[\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{3} \right]_0^1 d\theta =$$

$$= \left(1-\frac{1}{4}-\frac{1}{3}\right)\frac{\pi}{2} = \left(\frac{12-3-4}{12}\right)\frac{\pi}{2} = \boxed{\frac{5}{24}\pi}$$

ES. 3) a) $\Pi_{x,y}(V): x^2+y^2 \le 4,\ x \ge 0,\ y \le 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\le 0}} [\underbrace{6-x^2-y^2-1}_{5-(x^2+y^2)}]dx\,dy = \int_{\frac{3}{2}\pi}^{2\pi} \left(\int_0^2 \underbrace{(5-\rho^2)\rho}_{5\rho-\rho^3}\, d\rho \right) d\theta =$$

$$= \int_{\frac{3}{2}\pi}^{2\pi} \left[5\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^2 d\theta = (10-4)\cdot\frac{\pi}{2} = 6\cdot\frac{\pi}{2} = \boxed{3\pi}$$

b) $\Pi_{x,y}(V): x^2+y^2 \le 16,\ y \ge 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 16\\ y\ge 0}} \left[3+\sqrt{16-x^2-y^2}-1\right] dx\,dy = \int_0^\pi \left(\int_0^4 (2+\sqrt{16-\rho^2})\rho\, d\rho \right) d\theta =$$

$$= \int_0^\pi \left(\int_0^4 2\rho\, d\rho \right) d\theta - \frac{1}{2}\int_0^\pi \left(\int_0^4 (-2\rho)(16-\rho^2)^{1/2} d\rho \right) d\theta =$$

$$= \int_0^\pi [\rho^2]_0^4 d\theta - \frac{1}{2}\int_0^\pi \left[\frac{(16-\rho^2)^{3/2}}{3/2} \right]_0^4 d\theta = 16\pi - \frac{1}{3}\left[0-16^{3/2}\right]\pi =$$

$$= 16\pi + \frac{1}{3}4^3\cdot\pi = 16\pi + \frac{64}{3}\pi = \boxed{\frac{112}{3}\pi}$$

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c) $\pi_{x,y}(V): x^2+y^2\le 9,\ x\ge 0, y\ge 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 1\\ x\ge0, y\ge 0}} 6\,dx\,dy + \int_{\substack{1\le x^2+y^2\le 9\\ x\ge 0, y\ge 0}} (9-3\sqrt{x^2+y^2})\,dx\,dy =$$

$$= 6 \text{ area}\left(\substack{x^2+y^2\le 1\\ x\ge 0\ y\ge 0}\right) + \int_0^{\pi/2}\Big(\int_1^3 \underbrace{(9-3\rho)\rho}_{9\rho-3\rho^2}\,d\rho\Big)\,d\theta =$$

$$= 6\,\frac{1}{4}\pi + \int_0^{\pi/2}\left[9\frac{\rho^2}{2}-\rho^3\right]_1^3 d\theta = \frac{3}{2}\pi + \Big(\overbrace{\frac{81}{2}-27-\frac{9}{2}}^{36}+1\Big)\frac{\pi}{2} =$$

$$= \frac{3}{2}\pi + 10\cdot\frac{\pi}{2} = \boxed{\frac{13}{2}\pi}$$

d) $\pi_{x,y}(V): x^2+y^2\le 16,\ y\le 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 16\\ y\le 0}} \left[8-\sqrt{16-x^2-y^2}-(-2)\right]dx\,dy =$$

$$= \int_\pi^{2\pi}\Big(\int_0^4 (10-\sqrt{16-\rho^2})\,\rho\,d\rho\Big)\,d\theta = \int_\pi^{2\pi}\Big(\int_0^4 10\rho\,d\rho\Big)\,d\theta + \frac{1}{2}\int_\pi^{2\pi}\Big(\int_0^4 (-2\rho)(16-\rho^2)^{1/2}d\rho\Big)\,d\theta =$$

$$= \int_\pi^{2\pi}\left[5\rho^2\right]_0^4 d\theta + \frac{1}{2}\int_\pi^{2\pi}\left[\frac{(16-\rho^2)^{3/2}}{3/2}\right]_0^4 d\theta = 80\pi + \frac{1}{3}\left(0-16^{3/2}\right)\pi =$$

$$= 80\pi - \frac{64}{3}\pi = \boxed{\frac{176}{3}\pi}$$

e) $\pi_{x,y}(V): x^2+y^2\le 4,\ x\ge 0, y\le 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 4\\ x\ge 0, y\le 0}} \left[\frac{11}{2}-\frac{3}{4}\sqrt{x^2+y^2}-\left(1+\frac{3}{4}(x^2+y^2)\right)\right]dx\,dy =$$

$$= \int_{\substack{x^2+y^2\le 4\\ x\ge 0, y\le 0}} \left(\frac{9}{2}-\frac{3}{4}\sqrt{x^2+y^2}-\frac{3}{4}(x^2+y^2)\right)dx\,dy =$$

$$= \int_{\frac{3}{2}\pi}^{2\pi} \Big(\int_0^2 \underbrace{\Big(\frac{9}{2} - \frac{3}{4}\rho - \frac{3}{4}\rho^2 \Big)\rho}_{\frac{9}{2}\rho - \frac{3}{4}\rho^2 - \frac{3}{4}\rho^3} \, d\rho \Big) d\theta =$$

$$= \int_{\frac{3}{2}\pi}^{2\pi} \Big[\frac{9}{4}\rho^2 - \frac{\rho^3}{4} - \frac{3}{16}\rho^4 \Big]_0^2 d\theta = (9-2-3)\cdot\frac{\pi}{2} = \boxed{2\pi}$$

f) $\pi_{x,y}(V)$: $x^2+y^2 \le 4,\ x \le 0,\ y \le 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 1 \\ x\le 0,\, y\le 0}} (5-2)\,dx\,dy + \int_{\substack{1\le x^2+y^2\le 4 \\ x\le 0,\, y\le 0}} (\overbrace{5-(1+x^2+y^2)}^{4-(x^2+y^2)})\,dx\,dy =$$

$$= 3 \text{ area } (x^2+y^2\le 1, x\le 0, y\le 0) + \int_{\pi}^{3/2\pi} \Big(\int_1^2 \underbrace{(4-\rho^2)\rho}_{4\rho-\rho^3}\, d\rho \Big) d\theta =$$

$$= 3\cdot\frac{\pi}{4} + \int_{\pi}^{\frac{3}{2}\pi} \Big[2\rho^2 - \frac{\rho^4}{4} \Big]_1^2 d\theta = \frac{3}{4}\pi + (\underbrace{8-4-2+\tfrac{1}{4}}_{9/4})\frac{\pi}{2} = \frac{3}{4}\pi + \frac{9}{8}\pi = \boxed{\frac{15}{8}\pi}$$

g) $\pi_{x,y}(V)$: $x^2+y^2 \le 16,\ x \ge 0,\ y \ge 0$

$$\text{VOLUME } V = \int_{\substack{x^2+y^2\le 4 \\ x\ge 0,\, y\ge 0}} (6-3)\,dx\,dy + \int_{\substack{4\le x^2+y^2\le 16 \\ x\ge 0,\, y\ge 0}} \Big(6-\frac{3}{2}\sqrt{x^2+y^2}\Big)\,dx\,dy =$$

$$= 3 \text{ area } (x^2+y^2\le 4, x\ge 0, y\ge 0) + \int_0^{\pi/2} \Big(\int_2^4 \underbrace{\Big(6-\frac{3}{2}\rho\Big)\rho}_{6\rho-\frac{3}{2}\rho^2}\, d\rho \Big) d\theta =$$

$$= 3\,\frac{1}{4}\pi\cdot 4 + \int_0^{\pi/2} \Big[3\rho^2 - \rho^3/2 \Big]_2^4 d\theta = 3\pi + (\underbrace{48-32-12+4}_{8})\frac{\pi}{2} = \boxed{7\pi}$$

h) $\pi_{x,y}(V)$: $x^2+y^2 \le 4,\ x \le 0,\ y \le 0$ $\quad$ VOLUME $V = \displaystyle\int_{\substack{x^2+y^2\le 4 \\ x\le 0,\, y\le 0}} \Big[4 - \frac{1}{2}(x^2+y^2) - 1 \Big]\,dx\,dy =$

$$= \int_{\pi}^{\frac{3}{2}\pi} \Big(\int_0^2 \underbrace{\Big(3-\frac{1}{2}\rho^2\Big)\rho}_{3\rho-\frac{1}{2}\rho^3}\, d\rho \Big) d\theta = \int_{\pi}^{\frac{3}{2}\pi} \Big[3\frac{\rho^2}{2} - \frac{1}{8}\rho^4 \Big]_0^2 d\theta = (6-2)\cdot\frac{\pi}{2} = \boxed{2\pi}$$