CEV Model: Estimation and Applications in Option Pricing. An Introductory Tutorial with R and Python

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Outline

1. Introduction
2. Constant Elasticity of Variance (CEV) Model
3. CEV Process Simulation
4. CEV Density Function
5. CEV Option Pricing
6. CEV Model Estimation
7. Beckers' Estimation
8. Best Fit Approach
9. Emanuel and MacBeth β Estimation

Tutorial Overview

In this tutorial, we explored both the theoretical foundations and practical aspects of the Constant Elasticity of Variance (CEV) model. Particular attention was given to the nature of the model, its underlying rationale, and its key mathematical properties. We discussed how the CEV model extends the classical geometric Brownian motion by allowing volatility to depend non-linearly on the asset price, thereby capturing feature such as the leverage effect financial markets.

We presented the main option pricing formulas associated with the model, including closed-form solutions available under specific assumptions. In the final part of the tutorial, we focused on parameter estimation techniques, with a special emphasis on the estimation of the elasticity parameter β, which plays a central role in determining the shape of the volatility function.

To bridge theory and practice, we also introduced basic implementations in both R and Python, aiming to provide interactive tools that demonstrate the model's behavior and facilitate practical application in real-world financial contexts.

This tutorial draws primarily on the following reference:

- Randal, J. (1998). The constant elasticity of variance option pricing model. Master of Science in Statistics and Operations Research.

Volatility Smile and Leverage effect

Empirical studies of market-observed option prices have long highlighted systematic departures from the assumptions embedded in the classical **Black-Scholes** model, particularly the assumption of constant volatility. One of the most prominent anomalies is the *volatility smile*—the observed pattern where implied volatility varies with strike price and maturity, rather than remaining flat as predicted by the standard model.

This empirical evidence strongly suggests that asset price volatility is not a fixed parameter, but rather a dynamic quantity that may vary as a function of the underlying asset's level or other state variables.

A particularly influential empirical finding emerged from the work of **Fischer Black (1975)**, who documented an inverse relationship between asset price levels and their associated volatility. In many equities, volatility appeared to increase as the price of the underlying asset declined, and conversely, to decrease as the asset price rose.

This inverse dependency became a key focus for researchers seeking to construct more realistic models of asset price dynamics that could align more closely with observed market behavior. This phenomenon is known as ***leverage effect***.

Volatility Smile

An example of volatility smile with simulated data.

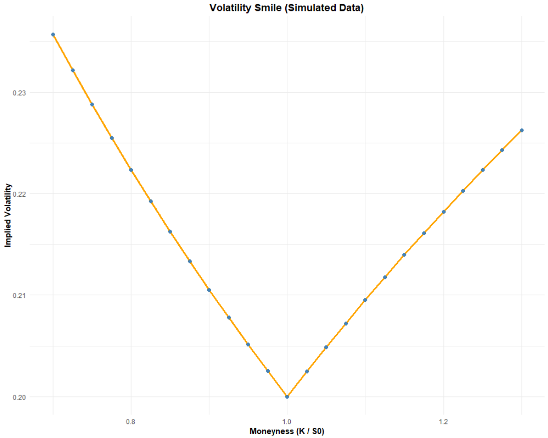

Figure: Volatility smile

The Leverage Effect

The leverage effect was first introduced by Black [2], who observed a systematic negative relationship between stock returns and future volatility. The economic intuition behind the leverage effect is relatively straightforward. When a firm's stock price declines, the market value of its equity falls, while the value of its outstanding debt generally remains unchanged. This leads to an increase in the firm's financial leverage. As leverage rises, the residual claim held by equity investors becomes riskier, since a smaller equity base must absorb the same level of underlying business risk. Consequently, the volatility of equity returns increases. It is precisely this mechanism—whereby falling prices raise leverage and, in turn, amplify volatility—that underpins the term "*leverage effect*." Following Black's initial observation, Christie [3] extended the analysis by linking equity return variance to financial leverage, drawing on the Modigliani–Miller framework. Christie demonstrated a strong positive relationship between leverage and equity variance, especially under varying interest rate environments. Later, Duffee [4] provided further empirical support by using a large panel of U.S. firms, showing that past negative returns were statistically associated with higher future volatility.

Leverage Effect

We can observe, the leverage effect from the following Figure. It is clear the increase in variability, during negative returns.

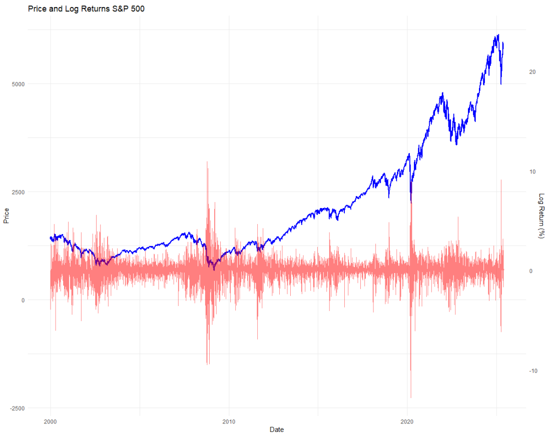

Figure: Price and Log Returns of S&P 500.

Leverage Effect

However, French et al. [5] argued that although the leverage effect is present, it might not fully explain all observed return-volatility dynamics, hinting at the presence of alternative mechanisms.
In contrast to the leverage effect, the volatility feedback hypothesis reverses the causal direction. Proposed by Bollerslev et al. [6], this theory suggests that increases in volatility today raise the equity risk premium demanded by investors, which in turn leads to lower future returns. While both frameworks can explain asymmetries in volatility and returns, they differ fundamentally in which variable is assumed to be the driver: returns in the leverage effect, or volatility in the feedback model. The leverage effect remains a foundational concept in understanding market dynamics during periods of distress. It highlights the interplay between capital structure and equity volatility, and helps explain why volatility tends to spike following negative market events. While competing theories like volatility feedback offer alternative interpretations, the leverage effect continues to inform the design of volatility models, risk assessment strategies, and corporate finance decisions.

Local Volatility Models

In the aftermath of the 1987 crash, empirical evidence challenged the assumptions of constant volatility embedded in the Black-Scholes model. Market practitioners observed a pronounced *volatility skew*, where implied volatilities varied with strike price and maturity. This inconsistency spurred the development of alternative models that could better capture the shape of the implied volatility surface.

Among the most prominent responses was the **Local Volatility Model**, a natural extension of Black-Scholes. It preserves the complete market framework while introducing a volatility function:

$$dS_t = (r - q)S_t\,dt + \sigma_\ell(t, S_t)S_t\,dW_t$$

Here, $\sigma_\ell(t, S_t)$ varies with both time and the underlying price, allowing the model to reproduce any observed implied volatility surface once calibrated.

The Dupire Formula

The cornerstone of local volatility modeling is the **Dupire Formula**, which provides a direct link between the local volatility function and the market-observed call price surface $C(K, T)$. Specifically, the local variance is given by:

$$\sigma_\ell^2(T, K) = \frac{\frac{\partial C}{\partial T} + (r - q)K\frac{\partial C}{\partial K} + qC}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

Given the implied volatility surface, one can compute the corresponding call price surface $C(K, T)$ via the Black-Scholes formula. To apply Dupire's equation, however, it is necessary to compute partial derivatives of this surface: first-order derivatives in time-to-maturity T, and both first and second-order derivatives in strike K.

In practice, this process is numerically challenging. Taking derivatives of market data can be unstable, especially when the implied volatility surface is noisy or lacks smoothness. As such, obtaining accurate local volatilities requires fitting a *sufficiently smooth* implied volatility surface beforehand, often using interpolation or parametric models.

Constant Elasticity of Variance (CEV) Model

In response to this empirical observation, the **Constant Elasticity of Variance (CEV)** model was developed. As recounted by **John Cox (1996)**, the model was created at the explicit request of Fischer Black, who sought a modification of the standard asset price model that would incorporate this inverse relationship between volatility and the asset price. The CEV can be considered as a parametric local volatility model.

The CEV model represents one of the earliest and most elegant generalizations of the geometric Brownian motion framework, allowing volatility to be directly linked to the level of the underlying asset price. It introduces a mechanism by which volatility becomes a non-linear, state-dependent function of the asset price.

This functional flexibility enables the model to capture features such as *leverage effects*, where declines in asset price are associated with increased volatility—an effect widely observed in equity markets.

Constant Elasticity of Variance (CEV) Model

Despite its simplicity, the CEV model significantly enhances the ability to fit market data.
While it does not fully capture all aspects of market dynamics—such as jumps, it serves as a foundational model in the evolution of option pricing theory. It paved the way for more advanced approaches, including stochastic volatility models (e.g., Heston) and local volatility frameworks (e.g., Dupire).
In summary, the CEV model bridges the gap between theoretical parsimony and empirical realism, representing a critical milestone in the development of modern financial modeling.

CEV Model: Definition and Dynamics

The Constant Elasticity of Variance (CEV) model assumes that the asset price S_t evolves according to the stochastic differential equation:

$$dS_t = \mu S_t \, dt + \delta S_t^{\beta/2} \, dB_t \quad \text{with} \quad \beta < 2$$

- When $\beta = 2$, we recover the standard Black-Scholes model with constant volatility.
- For $\beta < 2$, the volatility decreases as S_t increases, capturing the *leverage effect* observed in equity markets.
- Since the process can reach zero, an **absorbing barrier** at $S_t = 0$ is imposed:

$$\text{If } S_t = 0, \text{ then } S_u = 0 \quad \forall u > t$$

- This condition models firm bankruptcy and avoids unrealistic negative prices.

The model was further generalized by Emanuel and MacBeth (1982) to allow $\beta > 2$.

Constant Elasticity of Variance (CEV) Model

The instantaneous variance of the return process under the CEV model is:

$$\text{Var}\left(\frac{dS_t}{S_t}\middle| S_t\right) = \delta^2 S_t^{\beta-2}\,dt$$

- For $\beta < 2$, this variance decreases with increasing S_t.
- As $S_t \to 0$, volatility tends to infinity — consistent with high risk near default.
- As $S_t \to \infty$, the variance tends to zero — which contradicts models where return variance stabilizes at a constant.

Under a capital structure framework, the equity return variance is related to the firm's overall return R_a as:

$$\text{Var}(R_e) = \text{Var}(R_a)\left(1 + \frac{B}{S}\right)^2$$

As equity value S increases relative to debt B, the return variance decreases, matching the CEV behavior when $\beta < 2$.

Constant Elasticity of Variance

The CEV model derives its name from the property that the elasticity of the variance of the asset's return with respect to the asset price is constant. Formally, the variance of the asset's return is:

$$\mathrm{Var}\left(\frac{dS_t}{S_t}\middle| S_t\right) = \delta^2 S_t^{\beta-2}$$

Taking the relative rate of change of this variance with respect to a relative change in S_t, we obtain:

$$\frac{d\,\mathrm{Var}(dS_t/S_t)}{\mathrm{Var}(dS_t/S_t)} \bigg/ \frac{dS_t}{S_t} = \beta - 2$$

- This means that a proportional change in the asset price causes a proportional change in return variance by a constant factor.
- For example, a 20% increase in S_t leads to a $(\beta - 2) \times 20\%$ change in return variance.
- Since $\beta < 2$, this corresponds to a decrease in volatility when the price increases, and vice versa.

This feature makes the model consistent with observed market behavior where firms with falling stock prices exhibit rising volatility.

CEV Process Simulation

If we want to simulate the CEV process some remarks need to be made:

- In GBM, S_T can be written as a closed-form function of a standard normal variable.
- CEV models do not admit such closed-form solutions due to the nonlinearity in the volatility term.
- Consequently, direct simulation methods fail, and numerical solutions to the SDE must be employed.
- The standard method for discretizing such SDEs is the **Euler-Maruyama scheme** described in Kloeden & Platen (1992) and Mikosch (1994).

CEV Process Simulation

Given the CEV SDE:

$$dS_t = \mu S_t dt + \delta S_t^{\beta/2} dB_t$$

The Euler discretization over $[t, T]$ with partition in n sub-intervals, $t = t_0 < t_1 < \cdots < t_n = T$, not necessarily of equal length, gives:

$$S_{t_i} = S_{t_{i-1}} + \mu S_{t_{i-1}}(t_i - t_{i-1}) + \delta S_{t_{i-1}}^{\beta/2}(B_{t_i} - B_{t_{i-1}})$$

- Brownian increments $B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$.
- We write $B_{t_i} - B_{t_{i-1}} = \sqrt{t_i - t_{i-1}} \cdot Z_i$, where $Z_i \sim \mathcal{N}(0, 1)$ i.i.d.

The share price at time t_i is constructed recursively:

$$S_{t_i} = S_t + \sum_{j=1}^{i} \left[\mu S_{t_{j-1}} \Delta t_j + \delta S_{t_{j-1}}^{\beta/2} \sqrt{\Delta t_j} Z_j \right]$$

- Accurate simulation requires computing all previous S_{t_j} to determine S_{t_i}.
- The method discretizes a continuous process; thus, smaller Δt (larger n) improves accuracy.

CEV Process Simulation

Trade-off between accuracy and computational cost:

- Accuracy improves with smaller time steps ($\Delta t \to 0$).
- Computation time increases with number of steps n, due to sequential dependency.
- Linear interpolation is used to approximate the continuous-time process from discrete points:

$$S_t \approx S_{t_{i-1}} + \frac{t - t_{i-1}}{t_i - t_{i-1}} (S_{t_i} - S_{t_{i-1}})$$

CEV Process Simulation

One of the purpose is to ensure comparability of CEV and GBM paths for fair option pricing analysis.

MacBeth & Merville (1980):

- Align variance of return at initial time t_0.
- Require CEV process to satisfy:

$$dS_t = \mu S_t dt + \delta S_t^{\beta/2} dB_t \quad \text{with} \quad \delta = \sigma S_{t_0}^{1-\beta/2}$$

Beckers (1980):

- Align variance of terminal value S_T.
- Requires numerical inversion since $\mathrm{Var}(S_T \mid S_t)$ lacks closed form for general β.

CEV Process Simulation

CEV Simulation considering MacBeth & Merville (1980)

R function `CEV Process Simulation`

```
cevS.f <- function(S, tau, mu, sigma, beta, n, dB = rnorm(n, 0, sqrt(tau / n))) {
  # Initialize the vector of process values
  share <- numeric(n + 1)
  share[1] <- S

  for (i in 1:n) {
    delta <- sigma * share[i]^(1 - beta / 2)
    dS <- mu * share[i] * (tau / n) + delta * share[i]^(beta / 2) * dB[i]

    # Ensure the value stays positive
    if (share[i] + dS > 0) {
      share[i + 1] <- share[i] + dS
    } else {
      return(share[1:i])
      # return up to the point where the value remained positive
    }
  }
 return(share)
}
```

CEV Process Simulation

CEV Simulation considering MacBeth & Merville (1980)

Python function `CEV Process Simulation`

```
def cev_process(S0, tau, mu, sigma, beta, n, dB=None):
    dt = tau / n
    if dB is None:
        dB = np.random.normal(loc=0.0, scale=np.sqrt(dt), size=n)

    share = np.zeros(n + 1)
    share[0] = S0

    for i in range(n):
        delta = sigma * share[i]**(1 - beta / 2)
        dS = mu * share[i] * dt + delta * share[i]**(beta / 2) * dB[i]

        if share[i] + dS > 0:
            share[i + 1] = share[i] + dS
        else:
            return share[:i + 1]  # return up to the stopping point
```

Steps to Simulate a CEV Process

The simulation involves three key steps:

- **Step (1):** Compute δ based on the current share price S_t, the elasticity parameter β, and volatility σ. This corresponds to the alignment method of MacBeth and Merville (1980) discussed;
- **Step (2):** Calculate the share price increment dS_t^i using the CEV drift and diffusion terms;
- **Step (3):** Update the share price as $S_t^{i+1} = S_t^i + dS_t^i$. If the updated price is negative, the simulation stops and all subsequent values are set to zero.

CEV vs Black-Scholes

- Unlike the Black-Scholes model, the general CEV model does not allow for a closed-form expression for S_T similar to solution of GBM.
- According to Cox and Ross (1976), the CEV option pricing formula can still be derived using the same transformation techniques as Black-Scholes.
- The key idea is to transform the process C_t such that the resulting PDE and boundary conditions correspond to a known solution.

In particular:

- Both the Black-Scholes PDE and the CEV-related PDE can be transformed into the classical heat equation.
- Once transformed, standard analytical techniques can be used to solve them.
- This approach facilitates deriving an option price under the CEV framework.

Risk-Neutral Pricing

As already known the valuation of the option is based on **risk-neutral pricing**, pioneered by Cox and Ross.
The idea: Future cash flows P_T can be valued under a risk-neutral measure, assuming: The underlying asset has an instantaneous mean return r All future payments are discounted at rate r Hence, the present value is:

$$P_t = e^{-r\tau}\mathbb{E}^Q\left[P_T \mid \mathcal{F}_t\right]$$

Indeed, we recall,

- For options, the future payment is $P_T = (S_T - K)^+$.
- To compute $\mathbb{E}^Q\left[(S_T - K)^+ \mid \mathcal{F}_t\right]$, we require the transition density of $S_t \to S_T$.
- If such a density function can be identified, then the risk-neutral expected value and thus the option price can be evaluated.

In order to obtain the density we need to consider the Kolmogorov equations, indeed:

- The Kolmogorov equations describe the transition probabilities of a continuous-time Markov diffusion process.

Density Function under CEV

Following Cox and Miller (1965), the density function of the asset price within the CEV model, is given by:

$$f_{S_T|S_t}(s,\tau) = (2-\beta)\,\tilde{k}^{\frac{1}{2-\beta}}\left(\tilde{x}\tilde{z}^{1-2\beta}\right)^{\frac{1}{2(2-\beta)}} e^{-\tilde{x}-\tilde{z}} I_{\frac{1}{2-\beta}}\left(2(\tilde{x}\tilde{z})^{1/2}\right)$$

where

$$\tilde{k} = \frac{2\mu}{\delta^2(2-\beta)\left(e^{\mu(2-\beta)\tau}-1\right)}, \quad \tilde{x} = \tilde{k}S_t^{2-\beta}e^{\mu(2-\beta)\tau}, \quad \tilde{z} = \tilde{k}s^{2-\beta}$$

and $s > 0$, moreover $I_{\frac{1}{2-\beta}}$ is the modified Bessel function. This density function of S_T conditional on S_t, is a function of S_t and the time elapsed, τ, between t and T, but also depends on the CEV parameters.

Density Function under CEV

An alternative density is given by the following equation considering the power series expansion of the modified Bessel function in Abramowitz & Stegun (1968):

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^n}{n!\,\Gamma(\nu+n+1)}$$

giving the following transition density:

$$f_{S_T|S_t}(s,\tau) = (2-\beta)\,\tilde{k}^{\frac{1}{2-\beta}} \left(\tilde{x}\tilde{z}^{1-\beta}\right)^{\frac{1}{2-\beta}} e^{-\tilde{x}-\tilde{z}} \sum_{n=0}^{\infty} \frac{(\tilde{x}\tilde{z})^n}{n!\,\Gamma\left(n+1+\frac{1}{2-\beta}\right)}$$

$$= (2-\beta)\,\tilde{k}^{\frac{1}{2-\beta}} e^{-\tilde{x}-\tilde{z}} \sum_{n=0}^{\infty} \frac{\tilde{x}^{n+\frac{1}{2-\beta}}\,\tilde{z}^{n+\frac{1-\beta}{2-\beta}}}{n!\,\Gamma\left(n+1+\frac{1}{2-\beta}\right)}$$

Density Function under CEV

The density function given above is valid only for $S_T > 0$ we should then consider some adjustments.
Under the CEV model with $\beta < 2$, the probability of bankruptcy is:

$$\mathbb{P}(S_T = 0 \mid S_t) = G(\tilde{x}, \tfrac{1}{2-\beta})$$

where $G(y, \nu)$ is the survivor function at y of a Gamma distribution with shape ν and unit scale. $G(y,\nu) = \int_y^\infty \frac{e^{-x} x^{\nu-1}}{\Gamma(\nu)}\, dx$
With:

$$\tilde{x} = \tilde{k} S_t^{\beta-2} e^{\mu(2-\beta)\tau}, \quad \tilde{k} = \frac{2\mu}{\delta^2(2-\beta)\left(e^{\mu(2-\beta)\tau} - 1\right)}$$

Start from the conditional density $f_{S_T|S_t}(s,\tau)\, ds$ and change variable using:

$$\tilde{z} = \tilde{k} s^{2-\beta} \quad \Rightarrow \quad s = \left(\frac{\tilde{z}}{\tilde{k}}\right)^{\frac{1}{2-\beta}} \quad ds = \tilde{k}^{-1/(2-\beta)} \frac{1}{2-\beta} \tilde{z}^{-1+\frac{1}{2-\beta}}\, d\tilde{z}$$

Density Function under CEV

Substitute into the density expression:

$$f_{S_T|S_t}(s,\tau)\,ds = e^{-\tilde{x}-\tilde{z}} \sum_{n=0}^{\infty} \frac{\tilde{x}^{n+\frac{1}{2-\beta}}\,\tilde{z}^n}{n!\,\Gamma(n+1+\frac{1}{2-\beta})} d\tilde{z}$$

The probability of bankruptcy is:

$$\mathbb{P}(S_T = 0 \mid S_t) = G(\tilde{x}, \tfrac{1}{2-\beta})$$

Hence, the probability of bankruptcy between now and $T > t$, for a firm whose share price is a solution to the SDE of CEV, depends on the parameters μ, δ, and β, the initial share price S_t, and on the length of the time period $[t, T]$ in question. Analysis of the relationship between β and the probability of bankruptcy indicates that, for a fixed T, the probability $\mathbb{P}(S_T = 0)$ is relatively insensitive to changes in the share price at time t, but is sensitive to changes in the values of μ and δ. As expected, an increase in μ reduces the probability of bankruptcy for any β, while an increase in volatility (via the parameter δ) increases the probability of bankruptcy.

Density Function under CEV

The densities seen in the previous slide are quite complicated to be calculated, an alternative was proposed by Schroder (1989). In particular, Schroder represents the transition survivor function of S_T given S_t in terms of a non-central Chi-squared random variable.
The transition distribution function for a CEV share price S_T given S_t,

$$F_{S_T|S_t}(s,\tau) = \mathbb{P}(S_T < s \mid S_t)$$

is given by the equation:

$$F_{S_T|S_t}(s,\tau) = \begin{cases} 0, & s < 0 \\ G(x, \frac{1}{2-\beta}), & s = 0 \\ Q(2\tilde{x}; \frac{2}{2-\beta}, 2\tilde{y}), & s > 0 \end{cases}$$

where

$$\tilde{y} = \tilde{k} s^{2-\beta},$$

$\tilde{k}$ and $\tilde{x}$ are defined previously, and $Q(x;\nu,\lambda)$ is the survivor function at x for a non-central Chi-squared distribution with ν degrees of freedom and non-centrality parameter λ.

CEV Option Pricing

The price of a European option under the CEV model was first derived by John Cox (1975) in an unpublished note, later summarized in Cox (1996).

- Two special cases:
  - Absolute model
  - Square Root model
- The derivation is based on the **risk-neutral pricing method**, rather than the classical Black-Scholes approach.

The assumptions are:

- Constant and known risk-free interest rate r
- Stock price S_t follows the CEV process:

$$dS = \mu S\, dt + \delta S^{\beta/2}\, dZ$$

- Parameters δ and β are constant and known
- No dividends during the life of the option
- No transaction costs, taxes, or short-selling constraints
- Continuous trading and infinitely divisible assets

Comparison with Black-Scholes

All assumptions are the same as in the Black-Scholes model, except: The **stock price dynamics** are different (non-lognormal). Volatility depends on the stock price level via $S^{\beta/2}$. As with Black-Scholes, the option pricing PDE is derived under the **risk-neutral measure**.
Assuming that, S_t is the solution to the stochastic differential equation (SDE):

$$dS_t = \mu(S_t, t)S_t\,dt + \sigma(S_t, t)S_t\,dB_t$$

Then the price at time t of a European call option written on the stock with price S_t must satisfy the following partial differential equation (PDE):

$$\frac{1}{2}\sigma^2(S_t, t)S_t^2\frac{\partial^2 C}{\partial S^2} + rS_t\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0$$

subject to the terminal condition (payoff at maturity T):

$$C_T = (S_T - K)^+$$

where r is the continuously compounding risk-free interest rate.

CEV Option Pricing

The CEV option price partial differential equation (PDE) follows directly from the previous slide, by substituting the local volatility function $\sigma(S_t, t) = \delta S_t^{\frac{\beta}{2}-1}$. This leads to the PDE:

$$\frac{1}{2}\delta^2 S^\beta \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0,$$

subject to the terminal condition:

$$C_T = (S_T - K)^+.$$

As in the Black-Scholes framework, this PDE is independent of the drift term μ, and thus independent of investor risk preferences. Cox and Ross therefore conclude that the option can be valued under any investor preferences, in particular, under the assumption of risk-neutrality.
This independence arises because the option price satisfies a forward-backward parabolic PDE derived from the Kolmogorov equations. In the risk-neutral world, the drift μ is replaced by the risk-free rate r, allowing expectations to be computed under the risk-neutral measure.

Cev Option Pricing

Some remarks:

- The PDE is independent of the drift μ, and therefore investor preferences.
- Valuation under the **risk-neutral measure** is valid.
- The option price is given by:

$$C(S_t, \tau) = e^{-r\tau}\mathbb{E}^Q\left[(S_T - K)^+ \mid S_t\right]$$

- $\mathbb{E}^Q$ denotes expectation under the risk-neutral measure, replacing μ with r.

Cox (1996) presents a risk-neutral density function for $\beta < 2$, correcting and extending earlier work (e.g., Cox & Ross 1976). Particular issues noted for the Square Root CEV case ($\beta = 1$).

Risk-Neutral Density Function

In Cox (1996) we find that for $\beta < 2$, the risk-neutral transition density is:

$$f^{Q}_{S_T|S_t}(s,\tau) = (2-\beta)k^{\frac{1}{2-\beta}}\left(xz^{1-\beta}\right)^{\frac{1}{2(2-\beta)}} e^{-x-z} I_{\frac{1}{2-\beta}}\left(2\sqrt{xz}\right)$$

where:

$$x = kS_t^{2-\beta}e^{r(2-\beta)\tau}, \quad z = kS_T^{2-\beta}$$

$$k = \frac{2r}{\delta^2(2-\beta)(e^{r(2-\beta)\tau}-1)}$$

Using the Bessel function's series expansion, the risk-neutral density becomes:

$$f^{Q}_{S_T|S_t}(s,\tau) = (2-\beta)k^{\frac{1}{2-\beta}} e^{-x-z} \sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2-\beta}}\, z^{n+\frac{1-\beta}{2-\beta}}}{n!\,\Gamma\left(n+1+\frac{1}{2-\beta}\right)}$$

This formulation can be then used to derive the option price by Cox.

CEV Option Pricing

The solution of the previous PDE gives us the price of a European call option under the CEV model:

$$
\begin{aligned}
C_t = S_t \sum_{n=0}^{\infty} & \frac{e^{-x} x^n \, G\left(kK^{2-\beta},\ n+1+\frac{1}{2-\beta}\right)}{\Gamma(n+1)} \\
& - Ke^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-x} x^{n+\frac{1}{2-\beta}} \, G\left(kK^{2-\beta},\ n+1\right)}{\Gamma\left(n+1+\frac{1}{2-\beta}\right)}
\end{aligned}
$$

Where:

- $G(y, \nu)$: Gamma survivor function with ν as shape parameter and unit scale;
- $\Gamma(\cdot)$: Gamma function.

This option formula is the one from Cox.

CEV Option Pricing

Cox's option pricing formula for the CEV model with $\beta < 2$, as given in Equation , is consistently reported in papers by Beckers (1980), Emanuel & MacBeth (1982), MacBeth & Merville (1980) and Schroder (1989).

- Cox's full option pricing formula is mathematically elegant but computationally complex.
- Unlike the Black-Scholes formula, it requires:
  - Evaluation of infinite series
  - Gamma distribution functions
- In practice, approximations are necessary to make the model usable.

Empirical studies (e.g., Beckers, 1980; MacBeth & Merville, 1980) approximate the option price by truncating the infinite series:

$$C_t \approx S_t \sum_{n=n_1}^{n_2} g(x, n+1)\, G\left(kK^{2-\beta}, n+1+\frac{1}{2-\beta}\right) - Ke^{-r\tau} \sum_{n=n_1}^{n_2} g\left(x, n+1+\frac{1}{2-\beta}\right) G\left(kK^{2-\beta}, n+1\right)$$

- n_1, n_2: truncation limits ensuring convergence
- $g(x, \nu)$: gamma PDF, $G(y, \nu)$: gamma survivor function

CEV Option Pricing

Certain values of β allow for analytical or efficient numerical solutions:

- **Absolute CEV (β = 0):**
  - Volatility is constant in absolute terms (not in relative terms)
  - Closed-form solution exists
- **Square Root CEV (β = 1):**
  - Also called Cox–Ross model
  - No closed-form solution, but efficient numerical approximations exist

Schroder (1989) derives a general and efficient solution to the CEV option pricing problem. His method reformulates the CEV option price in terms of the non-central chi-squared distribution. Provides a closed-form expression valid for any $\beta < 2$. Makes empirical analysis of a wide range of CEV processes much more tractable.

CEV Option Pricing

According to Schroder the European call option price under the CEV model is:

$$C_t = S_t \cdot Q(2y;\, 2 + \tfrac{2}{2-\beta},\, 2x) - Ke^{-r\tau} \cdot \left[1 - Q(2x;\, \tfrac{2}{2-\beta},\, 2y)\right]$$

Where:

- $Q(y; \nu, \lambda)$: survivor function of a non-central chi-squared distribution, where ν re the degrees of freedom, and λ is the non-centrality parameter;
- $y = kK^{2-\beta}, \quad x = kS_t^{2-\beta} e^{r(2-\beta)\tau}$;
- $k = \dfrac{2r}{\delta^2(2-\beta)\left(e^{r(2-\beta)\tau} - 1\right)}$.

Moreover, the cumulative distribution of $S_T \mid S_t$ is:

$$F_{S_T|S_t}(s, \tau) = Q(2\tilde{x};\, \tfrac{2}{2-\beta},\, 2\tilde{y})$$

where:

$$\tilde{k} = \frac{2\mu}{\delta^2(2-\beta)(e^{\mu(2-\beta)\tau} - 1)}, \quad \tilde{x} = \tilde{k} S_t^{2-\beta} e^{\mu(2-\beta)\tau}, \quad \tilde{y} = \tilde{k} s^{2-\beta}$$

CEV Option Pricing for $\beta > 2$

In the case $\beta > 2$, the risk-neutral transition density of the share price $S_t \to S_T$ is:

$$f^Q_{S_T|S_t}(s, \tau \mid \beta > 2) = (\beta - 2)k^{\frac{1}{2-\beta}} \cdot (xz^{1-2\beta})^{\frac{1}{2(2-\beta)}}\, e^{-x-z}\, I_{1/(\beta-2)}(2\sqrt{xz})$$

- where all terms are defined as earlier in the model.

The corresponding option price is:

$$C_t = S_t Q\left(2x; \frac{2}{\beta-2}, 2y\right) - Ke^{-r\tau}\left[1 - Q\left(2y; 2 + \frac{2}{\beta-2}, 2x\right)\right], \quad (\beta > 2)$$

CEV Option Price $\beta < 2$

R function: CEV Option Price $\beta < 2$

```
cev_option_price <- function(S, K, r, tau, delta, beta, type = "call") {
  if (beta == 2) {
    stop("Use the Black-Scholes formula for beta = 2")
  }
    exponent <- 2 - beta
  k <- (2 * r) / (delta^2 * exponent * (exp(r * exponent * tau) - 1))
    x <- k * S^(exponent) * exp(r * exponent * tau)
  y <- k * K^(exponent)

  nu1 <- 2 + 2 / exponent
  nu2 <- 2 / exponent
    Q1 <- pchisq(2 * y, df = nu1, ncp = 2 * x, lower.tail = FALSE)
  Q2 <- pchisq(2 * x, df = nu2, ncp = 2 * y, lower.tail = FALSE)
    call_price <- S * Q1 - K * exp(-r * tau) * (1 - Q2)
    if (type == "call") {
    return(call_price)
  } else if (type == "put") {
    return(call_price + K * exp(-r * tau) - S)
  } else {
    stop("type must be 'call' or 'put'")
  }
}
```

CEV Option Price $\beta < 2$

Python function: `CEV Option Price` $\beta < 2$

```
import numpy as np
from scipy.stats import ncx2
def cev_option_price(S, K, r, tau, delta, beta, option_type='call'):
    if beta == 2:
        raise ValueError("Use Black-Scholes formula when beta = 2")
    exp_factor = 2 - beta
    k = (2 * r) / (delta**2 * exp_factor * (np.exp(r * exp_factor * tau) - 1))
    x = k * S**exp_factor * np.exp(r * exp_factor * tau)
    y = k * K**exp_factor
    nu1 = 2 + 2 / exp_factor
    nu2 = 2 / exp_factor
    Q1 = ncx2.sf(2 * y, df=nu1, nc=2 * x)
    Q2 = ncx2.sf(2 * x, df=nu2, nc=2 * y)
    call_price = S * Q1 - K * np.exp(-r * tau) * (1 - Q2)
    if option_type == 'call':
        return call_price
    elif option_type == 'put':
        put_price = call_price + K * np.exp(-r * tau) - S
        return put_price
    else:
        raise ValueError("option_type must be 'call' or 'put'")
```

CEV Option Price $\beta > 2$

R function: `CEV Option Price` $\beta > 2$

```
cev_option_price_beta_gt_2 <- function(S, K, r, tau, delta, beta, type = "call") {
  if (beta <= 2) {
    stop("This function is valid only for beta > 2")
  }
exponent <- beta - 2
  discount_factor <- exp(-r * tau)
k <- (2 * r) / (delta^2 * exponent * (exp(r * exponent * tau) - 1))
  x <- k * S^(2 - beta)
  y <- k * K^(2 - beta)
    nu1 <- 2 / exponent
  nu2 <- 2 + 2 / exponent
 Q1 <- pchisq(2 * x, df = nu1, ncp = 2 * y, lower.tail = FALSE)
  Q2 <- pchisq(2 * y, df = nu2, ncp = 2 * x, lower.tail = FALSE)
call_price <- S * Q1 - K * discount_factor * (1 - Q2)
 if (type == "call") {
    return(call_price)
  } else if (type == "put") {
    return(call_price + K * discount_factor - S)
  } else {
    stop("type must be 'call' or 'put'")
  }
}
```

CEV Option Price $\beta > 2$

Python function: `CEV Option Price` $\beta > 2$

```
import numpy as np
from scipy.stats import ncx2
def cev_option_price_beta_gt_2(S, K, r, tau, delta, beta, option_type='call'):
    if beta <= 2:
        raise ValueError("This function is valid only for beta > 2")
exponent = beta - 2
    discount_factor = np.exp(-r * tau)
k = (2 * r) / (delta**2 * exponent * (np.exp(r * exponent * tau) - 1))
    x = k * S**(2 - beta)
    y = k * K**(2 - beta)
    nu1 = 2 / exponent
    nu2 = 2 + 2 / exponent
    Q1 = ncx2.sf(2 * x, df=nu1, nc=2 * y)
    Q2 = ncx2.sf(2 * y, df=nu2, nc=2 * x)
    call_price = S * Q1 - K * discount_factor * (1 - Q2)
    if option_type == 'call':
        return call_price
    elif option_type == 'put':
        return call_price + K * discount_factor - S
    else:
        raise ValueError("option_type must be 'call' or 'put'")
```

Estimation vs. Calibration

Estimation:

- Involves choosing the *best model structure*, functional forms, and parameters to fit the data.
- Based on statistical methods and optimization criteria (e.g., maximum likelihood, GMM).
- Allows for model comparison and selection.
- *“What kind of model best explains the data?”*

Calibration:

- Assumes the model structure is already fixed.
- Involves adjusting parameter values so the model replicates selected empirical facts or moments.
- Often used when data is limited or formal estimation is infeasible.
- *“Given this model, what parameter values make it match observed behavior?”*

Philosophical Insight:

- Calibration assumes the model is structurally correct, only coefficients need tweaking.

Estimation of the CEV Model

- To apply the CEV model in practice, we must estimate two key parameters:
  - δ: scale parameter (analogous to volatility)
  - β: elasticity of variance
- Estimating β is essential to determine whether the CEV model provides a better fit to market data than the Black-Scholes model.
- Several empirical studies have focused on estimating β from observed asset price data.
- Unlike Black-Scholes, which assumes constant volatility and requires only a single parameter σ, the CEV model allows volatility to vary with the asset price:
$$\mathrm{Var}\left(\frac{dS_t}{S_t}\right) = \delta^2 S_t^{\beta-2}$$
- This flexibility makes the CEV model potentially more realistic in capturing market dynamics.

Beckers β Estimation

Model Setup

Starting from the stochastic differential equation (SDE) of the CEV model:

$$dS_t = \mu S_t \, dt + \delta S_t^{\beta/2} \, dB_t \quad \text{with} \quad \beta < 2$$

This implies that the conditional variance of the relative return is:

$$\mathrm{Var}\left(\frac{dS_t}{S_t} \mid S_t\right) = \delta^2 S_t^{\beta-2} dt$$

Assuming daily data ($dt = 1$), and taking logarithms, we obtain the linear relationship:

$$\ln\left(s\left(\frac{S_{t+1}}{S_t} \mid S_t\right)\right) = \ln \delta + \frac{\beta - 2}{2} \ln S_t$$

However, the standard deviation on the left-hand side is not directly observable in practice.

Beckers β Estimation

Approximation

For small daily returns, we can assume $\mu \approx 0$, leading to:

$$E\left[\ln\left(\frac{S_{t+1}}{S_t}\right)\right] \approx 0$$

Assuming approximate normality of returns, the standard deviation can be approximated by:

$$s\left(\frac{S_{t+1}}{S_t}\right) \approx \left|\ln\left(\frac{S_{t+1}}{S_t}\right)\right|$$

Substituting this into the earlier equation yields the regression model:

$$\ln\left|\ln\left(\frac{S_{t+1}}{S_t}\right)\right| = a + b\ln S_t + w_t$$

From the estimated slope b, the elasticity parameter is recovered as:

$$\beta = 2b + 2$$

Caveat

Beckers reports low R^2 and Durbin-Watson statistics in his empirical results, suggesting that the model may omit relevant explanatory variables.

Beckers β Estimation

R function for Beckers' Approximation for Estimating β

```
library(quantmod)

# Download sample stock data (e.g., AAPL)
getSymbols("AAPL", from = "2020-01-01", to = "2023-01-01")

# Extract weekly closing prices
prices <- Cl(AAPL)  # Only keep Close

# Compute weekly log returns
log_returns <- diff(log(prices))  # Length n - 1

# Filter out 0 returns before taking abs + log
log_returns <- log_returns[log_returns != 0]  # Avoid log(0)

# Align log prices at time t (drop first to match returns)
log_prices_t <- log(prices[-1])
log_prices_t <- log_prices_t[index(log_returns)]  # Align by date

# Create data frame
df <- data.frame(
  lhs = log(abs(log_returns)),  # log of absolute return
  logS = log_prices_t
)

# Clean up any NA or -Inf values (just in case)
df <- df[is.finite(df$lhs) & is.finite(df$logS), ]
colnames(df)=c("lhs","logS")
# Run regression
model <- lm(lhs ~ logS, data = df)
summary(model)

# Estimate beta
b_hat <- coef(model)["logS"]
beta_hat <- 2 * b_hat + 2
cat("Estimated beta:", round(beta_hat, 4), "\n")
```

Beckers β Estimation

Python function for Beckers' Approximation for Estimating β

```
import yfinance as yf
import pandas as pd
import numpy as np
import statsmodels.api as sm
import matplotlib.pyplot as plt
# Download weekly data
data = yf.download("AAPL", start="2020-01-01", end="2023-01-01", interval="1d")
# Flatten columns if MultiIndex
if isinstance(data.columns, pd.MultiIndex):
    data.columns = data.columns.get_level_values(0)
# Use 'Close' (or 'Adj Close' if available)
if "Adj Close" in data.columns:
    prices = data["Adj Close"].dropna()
else:
    prices = data["Close"].dropna()
# Compute log returns
log_returns = np.diff(np.log(prices.values))
# Remove zero returns to avoid log(0)
non_zero_mask = log_returns != 0
log_returns = log_returns[non_zero_mask]
log_prices_t = np.log(prices.values[1:])[non_zero_mask]
# Prepare DataFrame
df = pd.DataFrame({
    'lhs': np.log(np.abs(log_returns)),
    'logS': log_prices_t
}).replace([np.inf, -np.inf], np.nan).dropna()
# Regression
X = sm.add_constant(df['logS'])
model = sm.OLS(df['lhs'], X).fit()
print(model.summary())
# Estimate beta
b_hat = model.params['logS']
beta_hat = 2 * b_hat + 2
print(f"Estimated beta: {beta_hat:.4f}")
```

MacBeth and Merville β Estimation

Model Assumption:
From the CEV SDE

$$dS_t = \mu S_t dt + \delta S_t^{\beta/2} dW_t$$

one can derive the test statistic:

$$u_t = \frac{(dS_t - \mu S_t dt)^2}{\delta^2 S_t^{\beta} dt} \sim \chi_1^2$$

given values for μ, δ, and β.
Estimation Procedure:
A sample of daily returns is used to estimate μ, and a constant scaling is applied to absorb δ^2.
For a chosen β, the resulting u_t values are tested for goodness-of-fit against the χ_1^2 distribution.
Result:
The test fails to find any β such that the sample is consistent with χ_1^2, suggesting possible non-normality in dB_t or poor estimation of μ.

MacBeth and Merville β Estimation

Log-linear Regression Approach:
Taking logs of the Chi-squared expression yields:

$$\ln\left((dS_t - \mu S_t dt)^2\right) - \ln dt = 2\ln\delta + 2\beta\ln S_t + \ln(\chi_1^2)$$

This regression is used to obtain a point estimate for β, treating $\ln(\chi_1^2)$ as noise. Since $\mathbb{E}[\ln(\chi_1^2)] \neq 0$, only the intercept is affected.
Confidence Intervals:
A second regression, exploiting the inverse relationship between β and the test statistic u_t, is used to build confidence intervals for β.
These intervals are wide, and only one excludes the geometric Brownian motion case.
Final Estimation via Options:
For each stock, option prices are used to find implied δ values for various integer β. The β that produces approximately equal implied δ's is selected.
These estimates are consistent with the intervals but often far from the earlier point estimates.

MacBeth and Merville β Estimation

Key Observations:
- Estimating β via distributional assumptions (e.g., χ^2) is sensitive to normality and accurate drift estimation.
- Regression-based methods are more robust but yield imprecise estimates.
- Option-implied β via calibration can complement statistical estimation but may lead to conflicting results.

Implication:
Modeling volatility as a function of price introduces identifiability challenges that require a balance between statistical rigor and practical calibration.

MacBeth and Merville β Estimation

R function for MacBeth and Merville β Estimation

```
# Load required libraries
library(quantmod)

# 1. Download weekly stock data
getSymbols("AAPL", from = "2020-01-01", to = "2023-01-01")
prices <- Cl(AAPL)

# 2. Compute weekly log returns
log_returns <- diff(log(prices))
log_prices <- log(prices[-1])  # log(S_t)

# 3. Estimate drift µ as average log return
mu_hat <- mean(log_returns, na.rm = TRUE)

# 4. Prepare regression data
S_t <- exp(log_prices)
lhs <- na.omit(log((log_returns - mu_hat)^2))
rhs <- 2*log(S_t)

# 5. Run log-linear regression: ln[(S - µS)^2] = a +  ln(S)
model <- lm(lhs ~ rhs)

# 6. Extract point estimate of beta
beta_hat <- coef(model)["rhs"]
cat("Estimated beta from regression:", round(beta_hat, 4), "\n")
```

MacBeth and Merville β Estimation

Python function for MacBeth and Merville β Estimation

```
import yfinance as yf
import numpy as np
import statsmodels.api as sm

# 1. Download weekly adjusted close prices
data = yf.download("AAPL", start="2020-01-01", end="2023-01-01", interval="1d")
prices = data['Close'].dropna()

# 2. Compute log returns and lagged log prices
log_prices = np.log(prices)
log_returns = log_prices.diff().dropna()
log_prices_t = log_prices.shift(1).loc[log_returns.index]

# 3. Estimate mu
mu_hat = log_returns.mean()

# 4. Compute regression variables
lhs = np.log((log_returns - mu_hat)**2)
rhs = 2*log_prices_t


# 5. Run linear regression
X = sm.add_constant(rhs)
model = sm.OLS(lhs, X).fit()
print(model.summary())

# 6. Extract beta
beta_hat = model.params[1]
print(f"Estimated beta from regression: {beta_hat:.4f}")
```

Best Fit Approach

Given the CEV SDE:

$$dS_t = \mu S_t \, dt + \delta S_t^{\beta/2} \, dW_t$$

- μ: drift
- δ: volatility coefficient
- β: variance elasticity parameter (to estimate)
- W_t: standard Brownian motion

- Estimate β from observed data S_t
- Based on the idea that residuals should be normally distributed
- Use statistical goodness-of-fit to determine the most likely β

Best Fit Approach

From observed values, compute increments:

$$dS_t = S_{t+\Delta t} - S_t$$

Define the normalized residuals:

$$dW_t^{(\beta)} = \frac{dS_t - \mu S_t \Delta t}{\delta S_t^{\beta/2}}$$

If the model is correctly specified, these residuals should be approximately i.i.d. standard normal.

1. Select a grid of candidate values $\beta \in [\beta_{\min}, \beta_{\max}]$
2. For each β:
   - Compute $dW_t^{(\beta)}$
   - Test for normality (e.g., chi-square test)
   - Record the p-value
3. Choose the β with:

$$\text{max p-value, subject to } p \geq 0.05$$

Emanuel and MacBeth β Estimation

Emanuel and MacBeth (1982) extend the data of MacBeth & Merville (1980) by analyzing an additional year for the same six stocks.

- They propose a method to estimate the CEV model's elasticity parameter β.
- Although the method is not optimal, it provides empirical support for non-Black-Scholes models.

They assume the volatility of the underlying asset follows:

$$\sigma(S_t) = \delta S_t^{\frac{\beta - 2}{2}}$$

- δ: scale parameter
- β: elasticity parameter to be estimated

The instantaneous standard deviation is inferred from Black-Scholes implied volatility (ATM). Thus, the left hand side of the above formula, comes from the BS IV from ATM options.

Emanuel and MacBeth β Estimation

The estimation is based on the following steps:

- Use market prices of at-the-money call options.
- For each candidate β, compute CEV option prices.
- Minimize squared differences between CEV prices and market prices.

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{N} \left[C^{\text{CEV}}(S_i, K_i, T_i; \beta) - C_i^{\text{Market}} \right]^2$$

Some remarks:

- Black-Scholes implied volatilities are used as proxies for market volatility.
- Even if Black-Scholes is not strictly valid under CEV, ATM implied volatilities are considered reasonable.
- The CEV option pricing model is numerically computed.

Emanuel and MacBeth β Estimation

R function Emanuel and MacBeth β Estimation

```
black_scholes_call <- function(S, K, T, r, sigma) {
  if (sigma <= 0 || T <= 0) return(NA)
  d1 <- (log(S/K) + (r + 0.5*sigma^2)*T) / (sigma * sqrt(T))
  d2 <- d1 - sigma * sqrt(T)
  call_price <- S * pnorm(d1) - K * exp(-r * T) * pnorm(d2)
  return(call_price)
}
cev_option_price <- function(S, K, T, r, delta, beta) {
  # Local volatility according to the CEV model
  sigma_cev <- delta * S^((beta - 2)/2)
  return(black_scholes_call(S, K, T, r, sigma_cev))
}
objective_function <- function(beta, market_data, delta, r) {
  errors <- numeric(nrow(market_data))
  for (i in 1:nrow(market_data)) {
    row <- market_data[i, ]
    cev_price <- cev_option_price(S = row$S, K = row$K, T = row$T,
                                  r = r, delta = delta, beta = beta)
    if (is.na(cev_price)) {
      errors[i] <- Inf
    } else {
      errors[i] <- (cev_price - row$market_price)^2
    }
  }
  return(sum(errors))
}
market_data <- data.frame(
  S = c(100, 105, 98),
  K = c(100, 105, 100),
  T = c(0.5, 0.25, 1),
  market_price = c(7.5, 5.2, 9.1)
)
r <- 0.01        # risk-free interest rate
delta <- 0.25    # CEV parameter (chosen or estimated separately)
result <- optimize(f = objective_function, interval = c(0.5, 3),
                   market_data = market_data, delta = delta, r = r)
```

Emanuel and MacBeth β Estimation

Python function Emanuel and MacBeth β Estimation

```
import numpy as np
import pandas as pd
from scipy.stats import norm
from scipy.optimize import minimize_scalar
def black_scholes_call(S, K, T, r, sigma):
    if sigma <= 0 or T <= 0:
        return np.nan
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    call_price = S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)
    return call_price
def cev_option_price(S, K, T, r, delta, beta):
    sigma_cev = delta * S ** ((beta - 2) / 2)
    return black_scholes_call(S, K, T, r, sigma_cev)
def objective_function(beta, market_data, delta, r):
    errors = []
    for _, row in market_data.iterrows():
        cev_price = cev_option_price(S=row['S'], K=row['K'], T=row['T'],
                                     r=r, delta=delta, beta=beta)
        if np.isnan(cev_price):
            errors.append(np.inf)
        else:
            errors.append((cev_price - row['market_price']) ** 2)
    return np.sum(errors)
market_data = pd.DataFrame({
    'S': [100, 105, 98],
    'K': [100, 105, 100],
    'T': [0.5, 0.25, 1],
    'market_price': [7.5, 5.2, 9.1]
})
r = 0.01        # risk-free interest rate
delta = 0.25    # CEV parameter
result = minimize_scalar(
    objective_function,
    bounds=(0.5, 3),
    args=(market_data, delta, r),method='bounded')
```

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