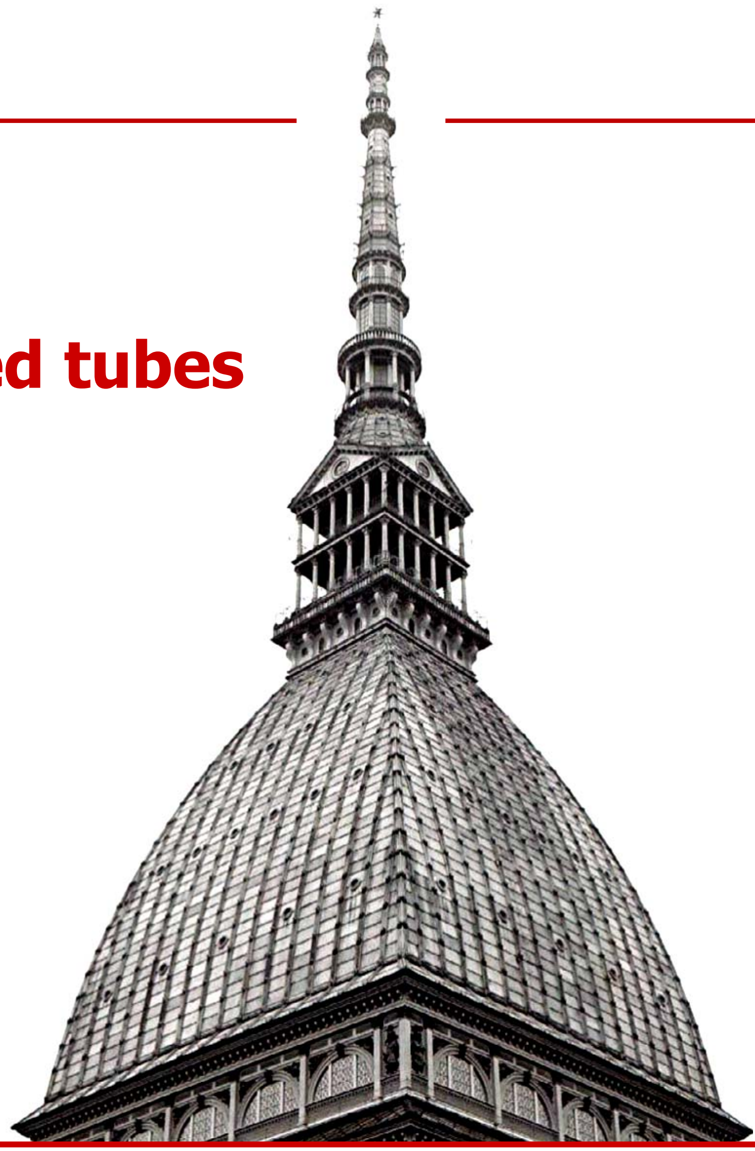
Daniele Botto
Costruzione di Motori per Aeromobili
Discs and thick walled tubes - Chapter 2

POLITECNICO DI TORINO

Chapters



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1. Forces and stresses in polar coordinates (1/4)

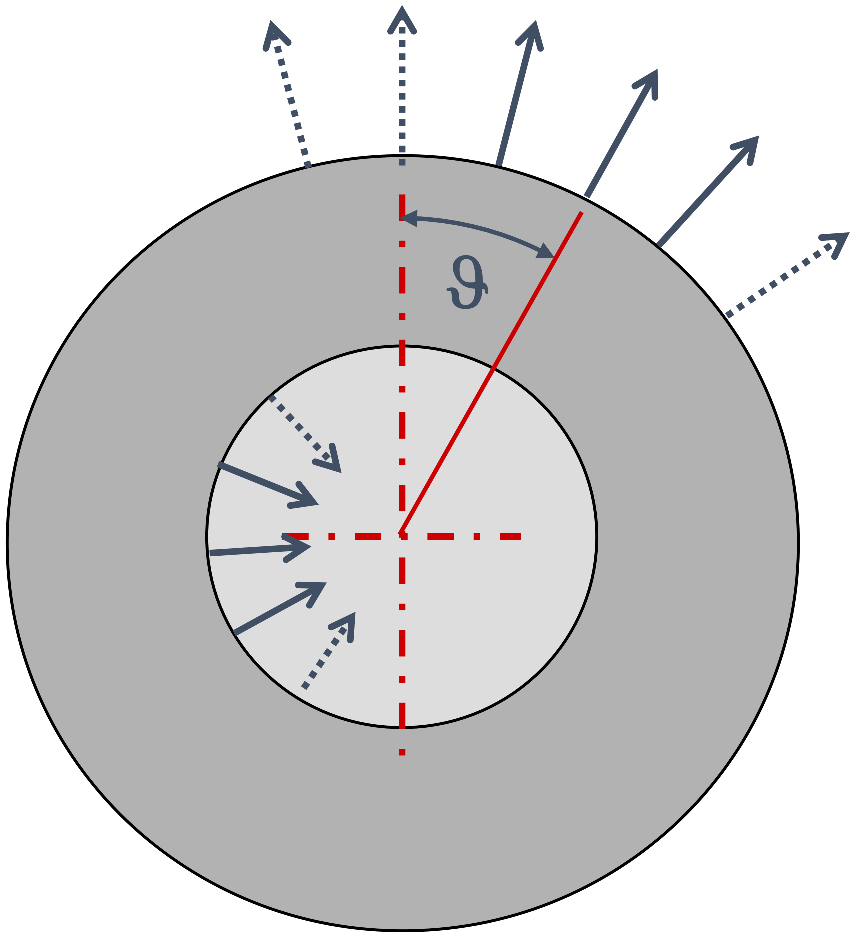
Problem of a solid with circular symmetry (axial-symmetry):

- ✓ Circular geometry
- ✓ Axi-symmetrical loads
- ✓ Isotropic material

⇩

The stress field is axi-symmetrical

i.e. no variable (stress component, displacement, strain) will depend on angle ϑ.



At this early stage only the constant thickness case will be considered; this allows to enucleate basic properties on the basis of an easy analytical solution in closed form.

1. Forces and stresses in polar coordinates (2/4)

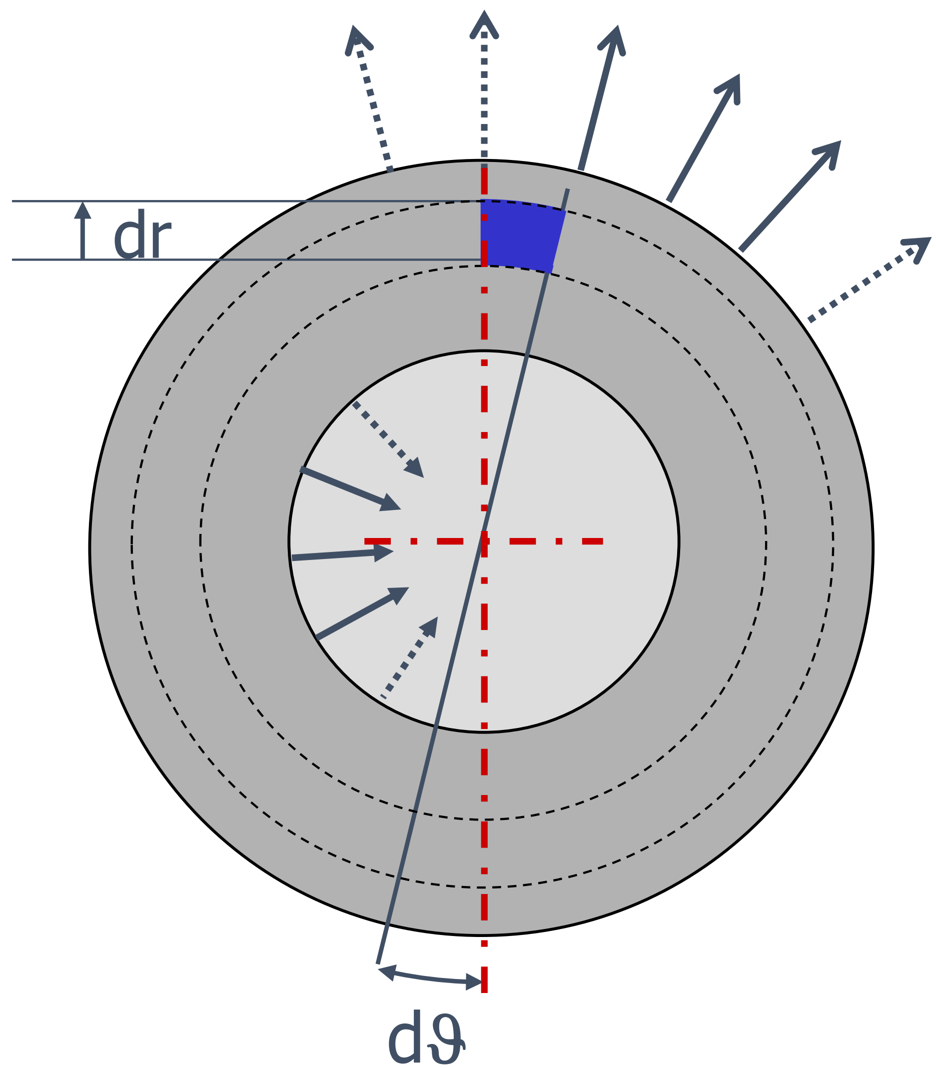
Due to axial symmetry, a polar coordinate system is used, which makes the analysis more convenient.

Equations of:

✓equilibrium

✓compatibility

will be written for the infinitesimal material element between two radii at angular distance $d\vartheta$ and two circles at radial distance dr.

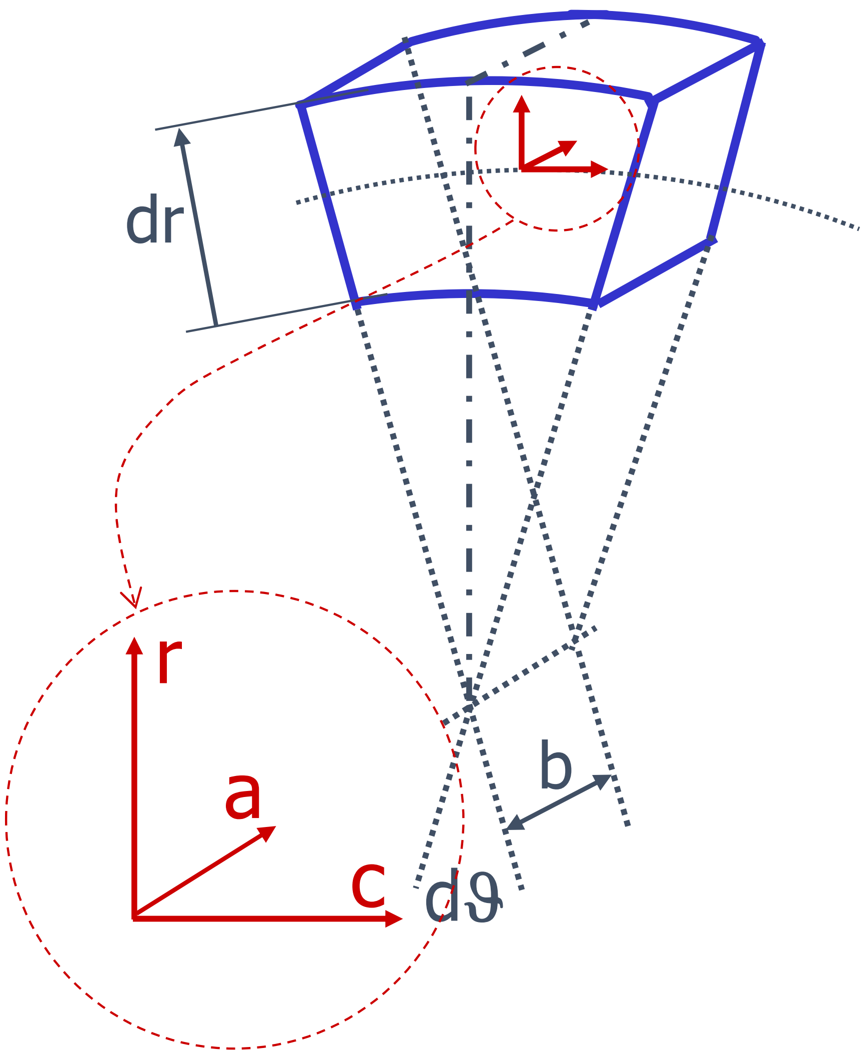


1. Forces and stresses in polar coordinates (3/4)

The following axes are conveniently defined through the center of mass of the element:

- ✓ axis a parallel to the revolution axis
- ✓ radial axis r
- ✓ circumferential axis c, tangent to the circle through the center of mass.

It will be easy to see that these local cartesian axes r, c, a are principal.



1. Forces and stresses in polar coordinates (4/4)

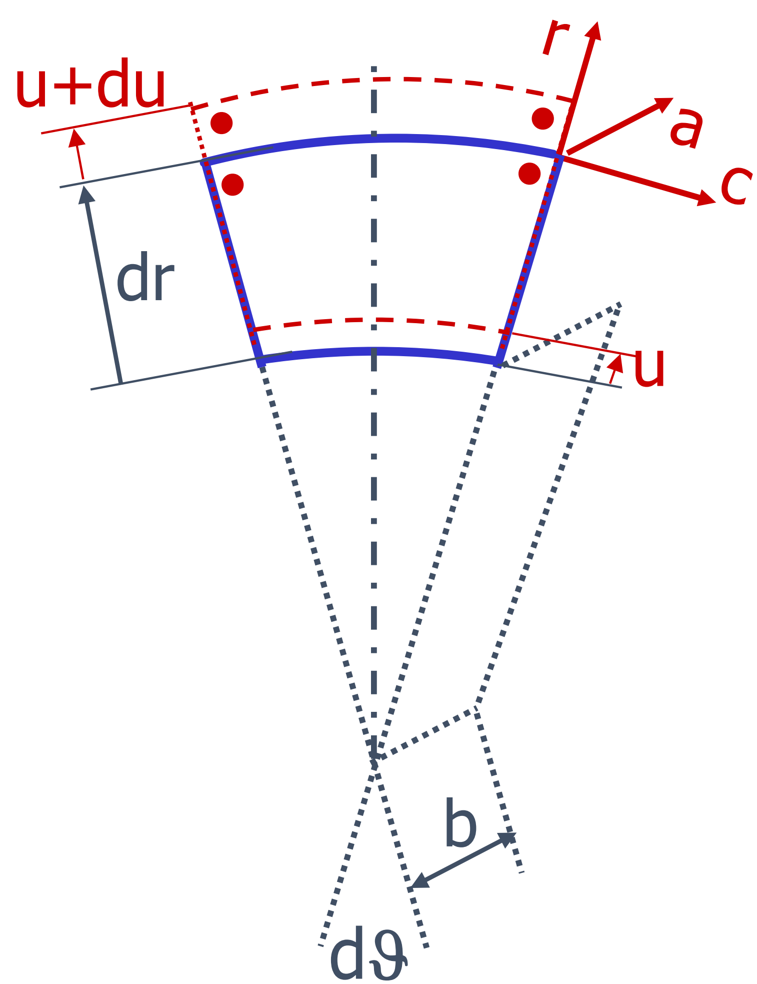
The figure on the right portrays the element in plane motion.

The in-plane displacement of any point in the cross section is one-dimensional, since it can only move along the radial r direction. Due to that, the shear strain γ_{rc} is zero …

(because two neighbouring elements will move radially of the same amount u, preserving angles)

… then also shear stress τ_{rc} =0.

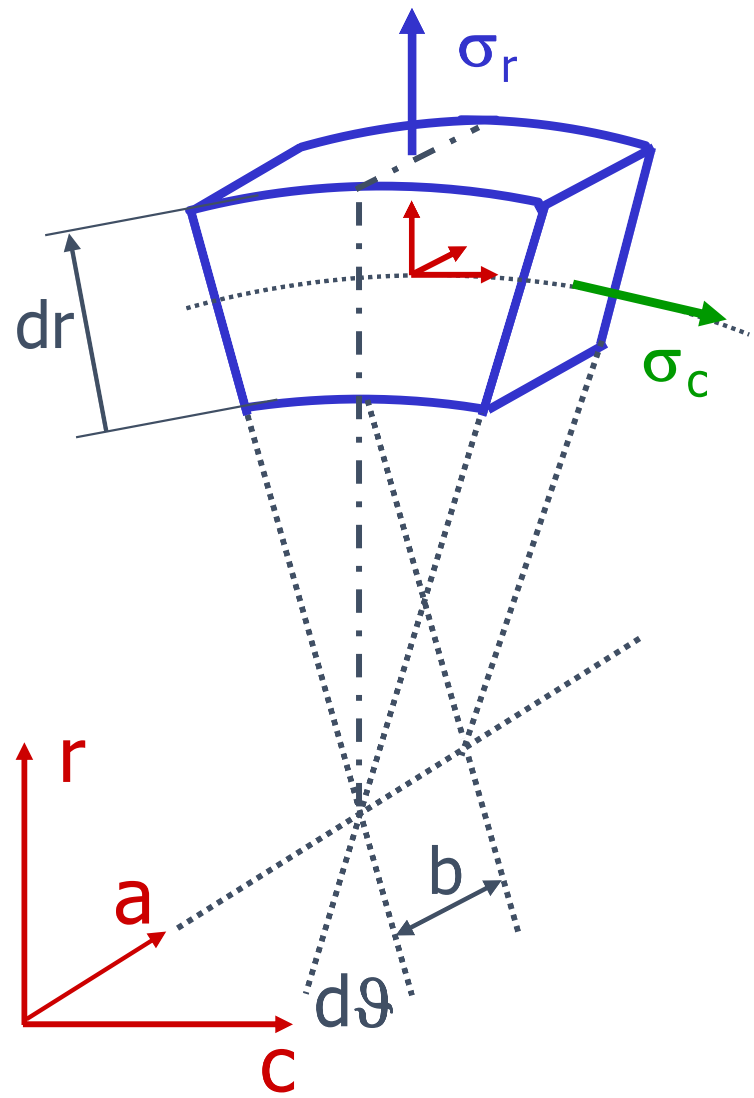
It follows that the normal stresses σ_r and σ_c , both independent of ϑ , are principal.



2. Equilibrium (1/5)

In the case of plane stress, stresses acting on the element are as in the figure on the right; all of them principal, including σ_a=0 .

Equations of equilibrium in space are six.



2. Equilibrium (2/5)

However, five of them are already satisfied due to axi-symmetry; in three of them:

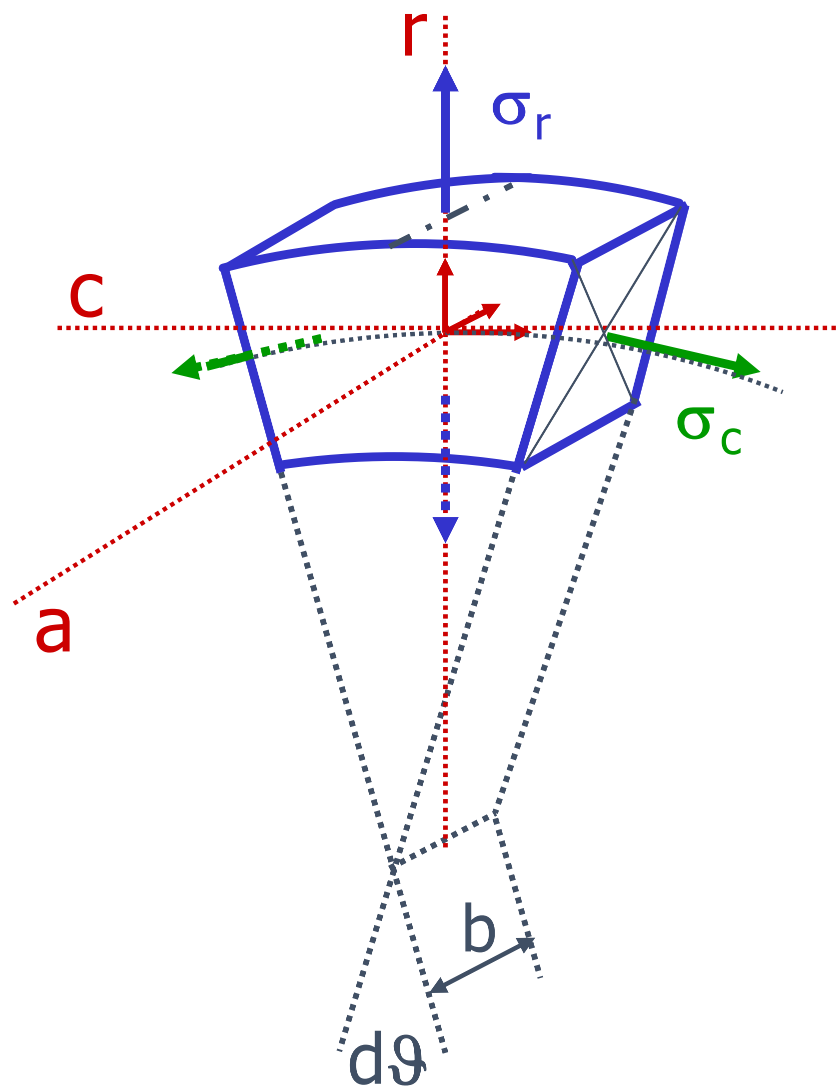
- ✓ force equilibrium along axis a
- ✓ moment equilibrium about r
- ✓ moment equilibrium about c

because all forces orthogonal to plane r-c are zero …

… while in the remaining two, the:

- ✓ total moment about a
- ✓ total force along c

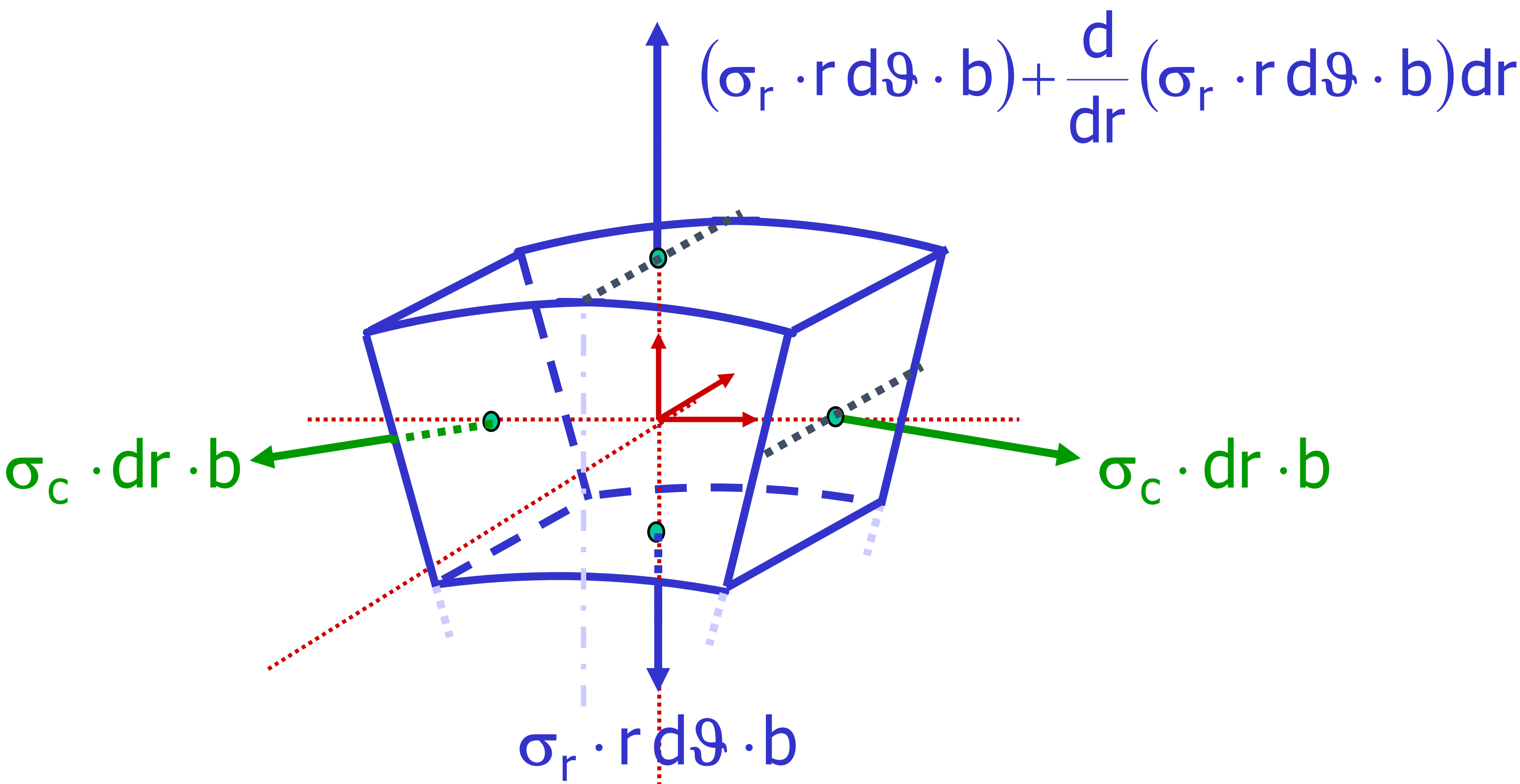
are zero because all forces in plane r-c are mirror-symmetric left and right of the radius through the centre.



Suggestion: try to write the equilibrium equations !

2. Equilibrium (3/5)

The sixth equation, force equilibrium along axis r, gives:



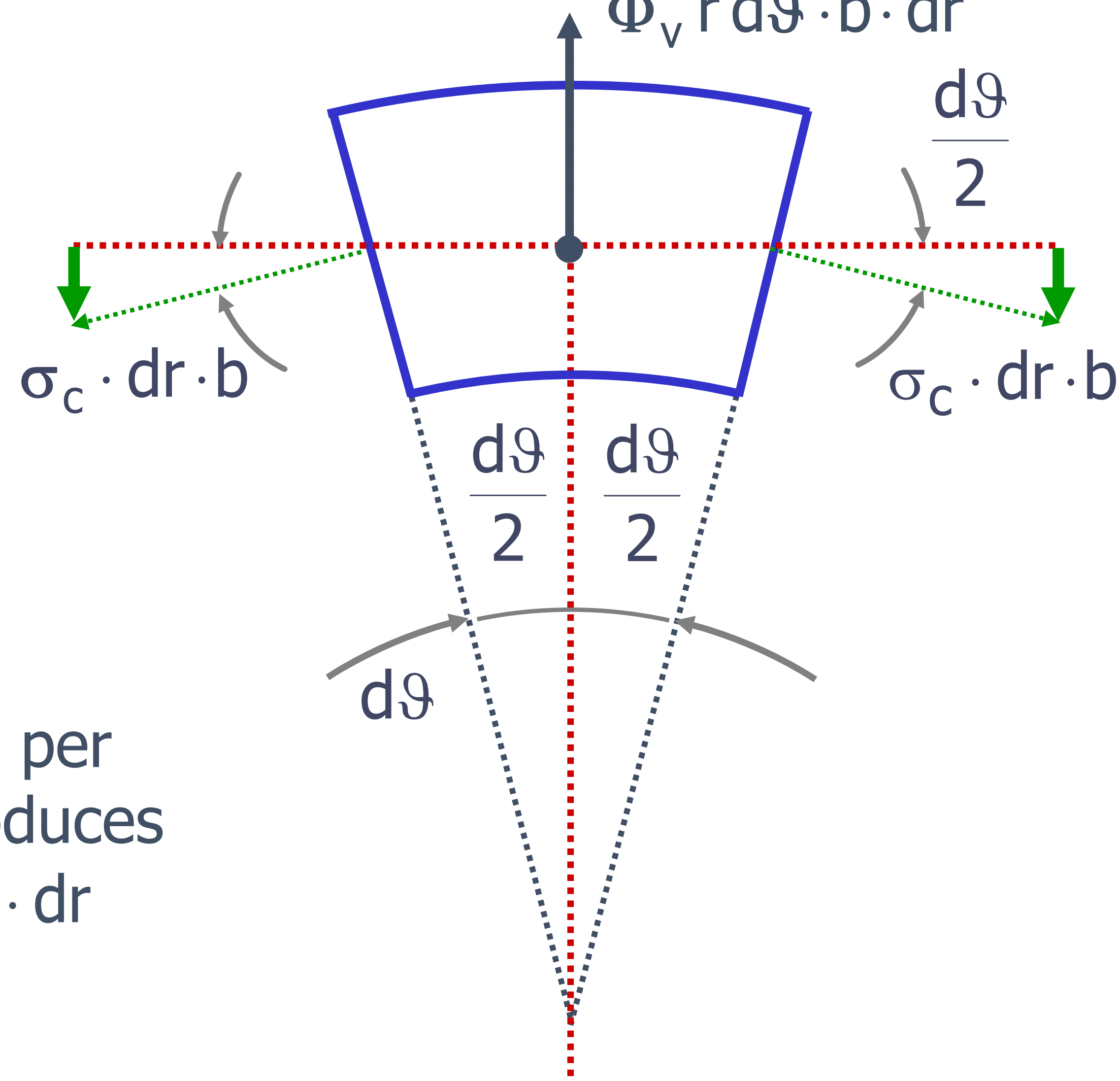
a) a net radial component due to σ_r : $\frac{d}{dr}(\sigma_r \cdot r\, d\vartheta \cdot b)dr$

2. Equilibrium (4/5)

b) the radial resultant force due to circumferential (hoop) stress σ_c , which is:

$$2 \cdot \sigma_c \cdot dr \cdot b \cdot \sin\left(\frac{d\vartheta}{2}\right) \cong$$

$$\cong \cancel{2} \cdot \sigma_c \cdot dr \cdot b \cdot \frac{d\vartheta}{\cancel{2}}$$



c) there may also be a force per unit volume Φ_v which produces a radial force: $\Phi_v \cdot r\, d\vartheta \cdot b \cdot dr$

2. Equilibrium (5/5)

Consider now the simplest case, where thickness b is constant and Φ_v=0 ; the sum of the three contributions:

$$\sigma_r : \quad \frac{d}{dr}\left(\sigma_r \cdot r\, d\vartheta \cdot b\right) dr \equiv b \frac{d}{dr}\left(\sigma_r \cdot r\, d\vartheta\right) dr$$

$$\sigma_c : -\sigma_c \cdot dr \cdot b \cdot d\vartheta$$

$$\Phi_v : \quad \Phi_v \cdot r\, d\vartheta \cdot b \cdot dr = 0$$

gives:

$$\cancel{b} \cdot d\cancel{\vartheta} \cdot \frac{d}{dr}\left(\sigma_r \cdot r\right) d\cancel{r} - \sigma_c \cdot \cancel{b} \cdot d\cancel{\vartheta} \cdot d\cancel{r} = 0 \quad \text{or:}$$

$$\frac{d}{dr}\left(\sigma_r \cdot r\right) - \sigma_c = 0$$

first form

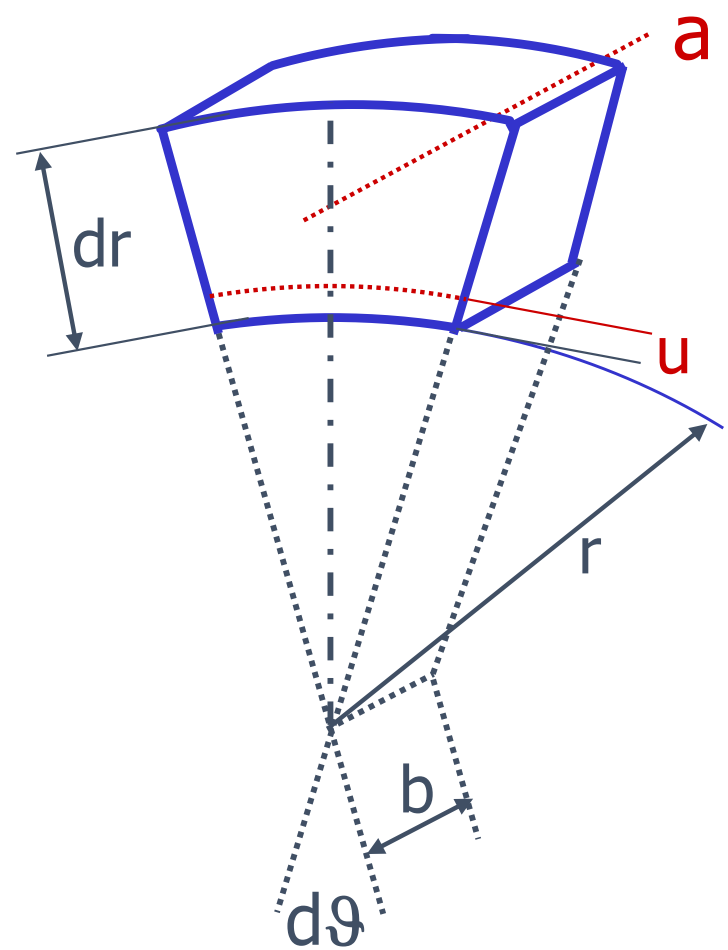
second form

$$\frac{d\sigma_r}{dr} \cdot r + \left(\sigma_r - \sigma_c\right) = 0$$

3. Compatibility

Due to symmetry, the only displacement in the r-c plane is u (a displacement along axis a is produced by transversal Poisson deformation)

$$\left.\begin{aligned} \varepsilon_r &= \frac{du}{dr} \\ \varepsilon_c &= \frac{d\vartheta(u+r)-d\vartheta r}{d\vartheta r} \equiv \frac{u}{r} \end{aligned}\right\} \Rightarrow \varepsilon_r = \frac{d}{dr}(\varepsilon_c \cdot r)$$



$$\frac{d}{dr}(\varepsilon_c \cdot r) - \varepsilon_r = 0 \qquad \text{(first form)}$$

$$\frac{d\varepsilon_c}{dr} \cdot r - (\varepsilon_r - \varepsilon_c) = 0 \quad \text{(second form)}$$

4. Material (constitutive) equations (1/3)

At this stage the plane stress case is developed.

The three-dimensional Hooke equations for elasticity, when $\sigma_a=0$

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}\left(\sigma_r - \nu\sigma_c - \nu\cancel{\sigma_a}\right) \\ \varepsilon_c = \dfrac{1}{E}\left(\sigma_c - \nu\sigma_r - \nu\cancel{\sigma_a}\right) \end{cases} \longrightarrow \begin{cases} \sigma_r = \dfrac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_c\right) \\ \sigma_c = \dfrac{E}{1-\nu^2}\left(\varepsilon_c + \nu\varepsilon_r\right) \end{cases}$$

(first form) (second form)

while the third material equation is:

$$\varepsilon_a = \frac{1}{E}\left(\cancel{\sigma_a} - \nu\sigma_r - \nu\sigma_c\right) = -\frac{\nu}{E}\left(\sigma_r + \sigma_c\right)$$

4. Material (constitutive) equations (2/3)

By subtraction the following auxiliary equations are obtained:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_c)$$

$$\varepsilon_c = \frac{1}{E}(\sigma_c - \nu\sigma_r)$$

(first form)

$$(\varepsilon_c - \varepsilon_r) = \frac{1}{E}(1+\nu)(\sigma_c - \sigma_r)$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_c)$$

$$\sigma_c = \frac{E}{1-\nu^2}(\varepsilon_c + \nu\varepsilon_r)$$

(second form)

$$(\sigma_r - \sigma_c) = \frac{E}{1-\nu^2}(1-\nu)(\varepsilon_r - \varepsilon_c)$$

4. Material (constitutive) equations (3/3)

One might consider also thermal expansion, which adds to linear strain (in the case of isotropic material) an equal term in all directions i=1,2,3 :

$$\varepsilon_i^T = \alpha^* \cdot \Delta T$$

linear thermal expansion coefficient

temperature increase

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_c\right) + \alpha^* \Delta T$$

$$\varepsilon_c = \frac{1}{E}\left(\sigma_c - \nu\sigma_r\right) + \alpha^* \Delta T$$

5. Solution for plane stress (1/7)

Equilibrium, compatibility and material equations are now combined to solve this plane stress problem in polar coordinates.

The solution will be now found for the simplest case:

- constant thickness
- no volume force
- no thermal expansion

Two approaches are possible, according to final equation which is more convenient to obtain:

"displacement" final equation	"stress" final equation
in this case one must start with equilibrium equations, transform stresses into strains and then get a solving equation for the displacement u; convenient when boundary conditions are on displacements	in this case one must start with compatibility equations, transform strains into stresses and then get a solving equation for the radial stress; convenient when boundary conditions are on stresses

5. Solution for plane stress (2/7)

1) Equilibrium
(second form)

$$r \cdot \frac{d\sigma_r}{dr} + (\sigma_r - \sigma_c) = 0$$

2) Material
(second form)

$$(\sigma_r - \sigma_c) = \frac{E}{1-\nu^2}(1-\nu)(\varepsilon_r - \varepsilon_c)$$

3) Again, material eq.
(second form)

$$\frac{d\sigma_r}{dr} = \frac{E}{1-\nu^2}\frac{d}{dr}(\varepsilon_r + \nu\,\varepsilon_c)$$

1) Compatibility
(second form)

$$r \cdot \frac{d\varepsilon_c}{dr} + (\varepsilon_c - \varepsilon_r) = 0$$

2) Material
(first form)

$$(\varepsilon_c - \varepsilon_r) = \frac{1}{E}(1+\nu)(\sigma_c - \sigma_r)$$

3) Again, material eq.
(first form)

$$\frac{d\varepsilon_c}{dr} = \frac{1}{E}\frac{d}{dr}(\sigma_c - \nu\sigma_r)$$

5. Solution for plane stress (3/7)

Applying 2) e 3) into 1):

4) $$r\left(\frac{d\varepsilon_r}{dr}+\nu\frac{d\varepsilon_c}{dr}\right)+$$
$$+(1-\nu)(\varepsilon_r-\varepsilon_c)=0$$

Applying 2) e 3) into 1):

4) $$r\left(\frac{d\sigma_c}{dr}-\nu\frac{d\sigma_r}{dr}\right)+$$
$$+(1+\nu)(\sigma_c-\sigma_r)=0$$

Remark that in both cases elastic modulus E no longer appears

Material and equilibrium have been applied; now:

5) Compatibility (second form)

$$(\varepsilon_r-\varepsilon_c)=r\frac{d\varepsilon_c}{dr}$$

Material and compatibility have been applied; now:

5) Equilibrium (second form)

$$r\frac{d\sigma_r}{dr}+(\sigma_r-\sigma_c)=0$$

putting 5) into 4):

6) $$r\left(\frac{d\varepsilon_r}{dr}+\nu\frac{d\varepsilon_c}{dr}\right)+$$
$$+(1-\nu)r\frac{d\varepsilon_c}{dr}=0$$

6') $$r\left(\frac{d\varepsilon_r}{dr}+\frac{d\varepsilon_c}{dr}\right)=0$$

$$\downarrow$$

$\varepsilon_r+\varepsilon_c=$ constant !

7) $$\varepsilon_r+\varepsilon_c=M'$$

putting 5) into 4):

6) $$r\left(\frac{d\sigma_c}{dr}-\nu\frac{d\sigma_r}{dr}\right)+$$
$$+(1+\nu)r\frac{d\sigma_r}{dr}=0$$

6') $$r\left(\frac{d\sigma_c}{dr}+\frac{d\sigma_r}{dr}\right)=0$$

$$\downarrow$$

$\sigma_r+\sigma_c=$ constant !

7) $$\sigma_r+\sigma_c=A'$$

Remark that material parameters have disappeared!

5. Solution for plane stress (5/7)

8) Compatibility
(first form)

$$\frac{d}{dr}\left(r\,\varepsilon_c\right)=\varepsilon_r$$

Applying 8) into 7)

9) $$\frac{d}{dr}\left(r\,\varepsilon_c\right)+\varepsilon_c=M'$$

or:

$$\frac{1}{r}\frac{d}{dr}\left[\left(r\,\varepsilon_c\right)r\right]=M'$$

8) Equilibrium
(first form)

$$\frac{d}{dr}\left(r\,\sigma_r\right)=\sigma_c$$

Applying 8) into 7)

9) $$\frac{d}{dr}\left(r\,\sigma_r\right)+\sigma_r=A'$$

or:

$$\frac{1}{r}\frac{d}{dr}\left[\left(r\,\sigma_r\right)r\right]=A'$$

5. Solution for plane stress (6/7)

The nonlinear differential equations have, thus, been reduced to simple integrals which are speedily calculated:

$$\frac{d}{dr}\left[r^2 \varepsilon_c\right] = M' r$$

$$r^2 \varepsilon_c = M' \frac{r^2}{2} + N$$

$$\varepsilon_c = \frac{1}{2} M' + \frac{N}{r^2}$$

$$\varepsilon_c = \frac{1}{2} M' + \frac{N}{r^2}$$

$$M = \frac{1}{2} M'$$

$$\frac{d}{dr}\left[r^2 \sigma_r\right] = A' r$$

$$r^2 \sigma_r = A' \frac{r^2}{2} + B$$

$$\sigma_r = \frac{1}{2} A' + \frac{B}{r^2}$$

$$\sigma_r = \frac{1}{2} A' + \frac{B}{r^2}$$

$$A = \frac{1}{2} A'$$

note: $\sigma_r + \sigma_c = 2A$

5. Solution for plane stress (7/7)

$$\frac{u}{r} = M + \frac{N}{r^2}$$

10) $$u = Mr + \frac{N}{r}$$

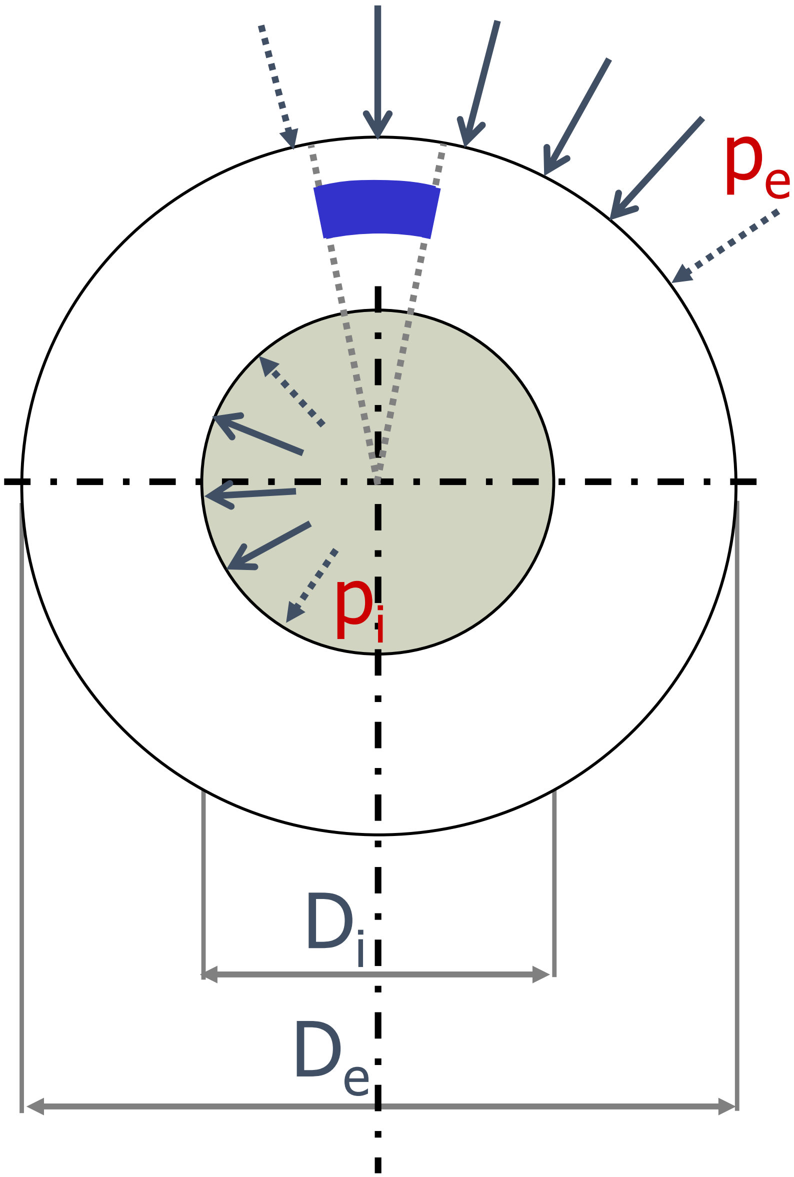
10) $$\sigma_r = A + \frac{B}{r^2}$$

Finally, with: $\sigma_r + \sigma_c = A' = 2A$

11) $$\sigma_c = A - \frac{B}{r^2}$$

Equations 10) contain two integration constants which must now be determined through boundary conditions.

6. Stresses in constant thickness discs (1/4)



Quite frequently, boundary conditions are given in terms of pressures applied at the inner and outer boundaries.

$$r = \frac{D_i}{2} \rightarrow \sigma_r = -p_i$$

$$r = \frac{D_e}{2} \rightarrow \sigma_r = -p_e$$

6. Stresses in constant thickness discs (2/4)

Radial stresses from the general solution, eq. 10 of slide 7 sect. 5, are then:

$$\begin{cases} -p_i = A + \dfrac{B}{r_i^2} \\ -p_e = A + \dfrac{B}{r_e^2} \end{cases}$$

$$-p_i r_i^2 = A r_i^2 + B$$

$$-p_e r_e^2 = A r_e^2 + B$$

$$p_e - p_i = B\left(\frac{1}{r_i^2} - \frac{1}{r_e^2}\right)$$

6. Stresses in constant thickness discs (3/4)

$$-p_i r_i^2 = A r_i^2 + B$$

$$-p_e r_e^2 = A r_e^2 + B$$

$$\left(p_e r_e^2 - p_i r_i^2\right) = A\left(r_i^2 - r_e^2\right)$$

$$A = \frac{p_e r_e^2 - p_i r_i^2}{\left(r_i^2 - r_e^2\right)}$$

$$p_e - p_i = B\left(\frac{1}{r_i^2} - \frac{1}{r_e^2}\right)$$

$$B = \frac{p_e - p_i}{\left(\frac{1}{r_i^2} - \frac{1}{r_e^2}\right)}$$

Integration constants have thus been determined.

6. Stresses in constant thickness discs (4/4)

It is practically preferable to "engineer" all formulas by using diameters:

$$A = p_i \frac{\frac{D_i^2}{D_e^2}}{1 - \frac{D_i^2}{D_e^2}} - p_e \frac{1}{1 - \frac{D_i^2}{D_e^2}}$$

$$B = \frac{(p_e - p_i)\left(\frac{D_i}{2}\right)^2}{1 - \frac{D_i^2}{D_e^2}}$$

$$\sigma_r = A + \frac{B}{r^2} = -p_i \frac{\frac{D_i^2}{D^2} - \frac{D_i^2}{D_e^2}}{1 - \frac{D_i^2}{D_e^2}} - p_e \frac{1 - \frac{D_i^2}{D^2}}{1 - \frac{D_i^2}{D_e^2}}$$

$$\sigma_c = A - \frac{B}{r^2} = p_i \frac{\frac{D_i^2}{D^2} + \frac{D_i^2}{D_e^2}}{1 - \frac{D_i^2}{D_e^2}} - p_e \frac{1 + \frac{D_i^2}{D^2}}{1 - \frac{D_i^2}{D_e^2}}$$

7. Displacements in constant thickness discs

Displacements are calculated starting from the circumferential compatibility equation:

$$\mathrm{u} = r \cdot \varepsilon_c = r \cdot \frac{1}{E}(\sigma_c - \nu\sigma_r)$$

then:

$$\mathrm{u} = \frac{D}{2}\frac{p_i}{E}\left[\frac{\frac{D_i^2}{D^2}(1+\nu) + \frac{D_i^2}{D_e^2}(1-\nu)}{1 - \frac{D_i^2}{D_e^2}}\right] - \frac{\mathrm{D}}{2}\frac{p_e}{E}\left[\frac{\frac{D_i^2}{D^2}(1+\nu) + (1-\nu)}{1 - \frac{D_i^2}{D_e^2}}\right]$$

8. Solid discs (1/2)

A special case, which is worth mentioning because it occurs in some cases, is the so called "solid disc", i.e., without a central hole.

In such special case:

$$\sigma_{r,c} = A \pm \frac{B}{r^2} \Rightarrow B = 0, \text{ otherwise} \begin{cases} \sigma \to \pm\infty \\ r \to 0 \end{cases} \Rightarrow \sigma_r = \sigma_c = A \equiv -p_e$$

$$\sigma_a = 0$$

in-plane stresses are constant over the disc

$$u = M \cdot r + \frac{N}{r} \Rightarrow N = 0, \text{ otherwise} \begin{cases} u \to \infty \\ r \to 0 \end{cases} \Rightarrow u = M\frac{D}{2}$$

$$u = r \cdot \varepsilon_c = \frac{D}{2} \cdot \frac{1}{E}\left(\sigma_c - \nu\sigma_r\right) = -p_e \frac{D}{2} \frac{(1-\nu)}{E}$$

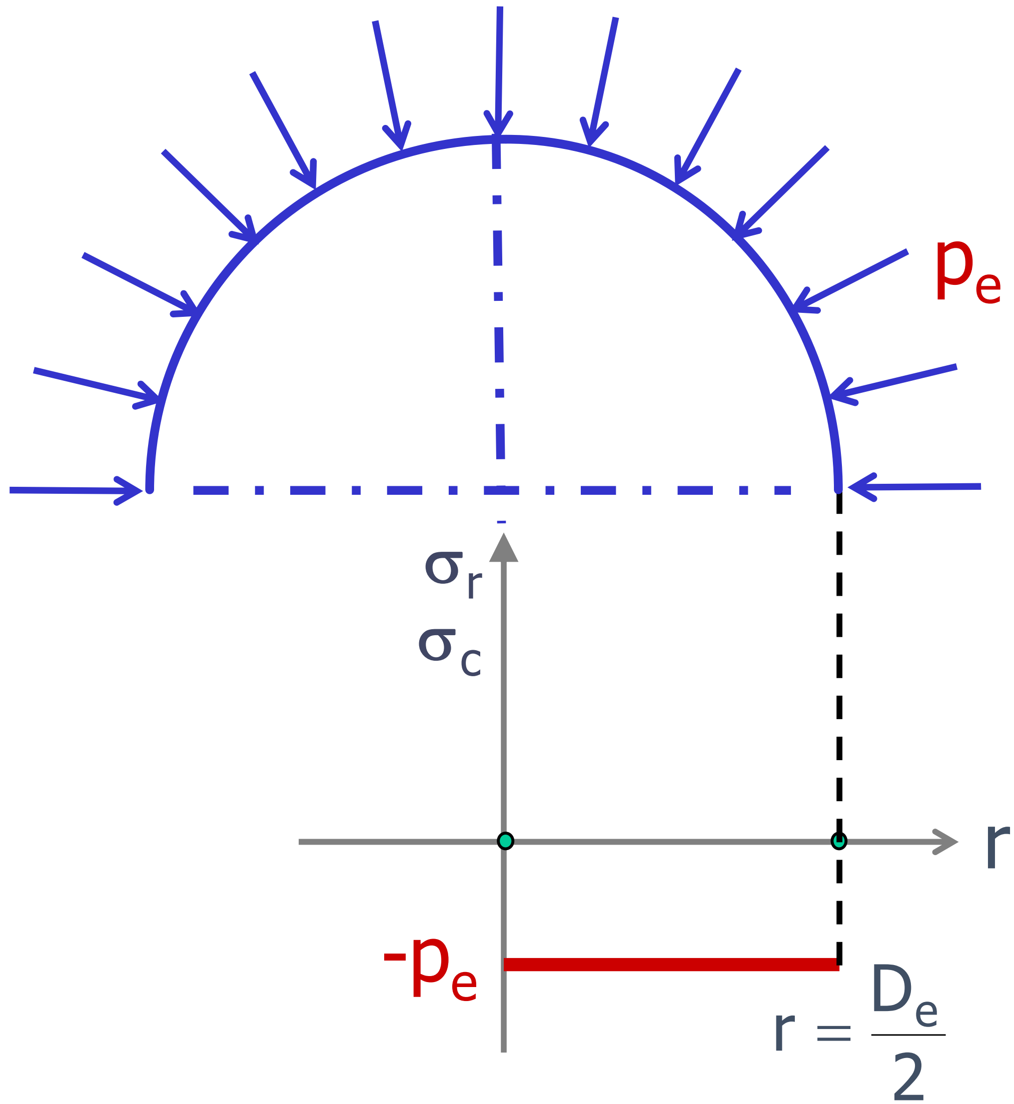
linearly variable over the disc radius (note: $M \equiv \varepsilon_c$)

8. Solid discs (2/2)

Radial displacement:

$$u = -p_e \frac{D}{2}\frac{(1-\nu)}{E}$$

Stresses: $\sigma_r = \sigma_c = A \equiv -p_e$



Axial displacement:

$$\varepsilon_a = -\frac{\nu}{E}(\sigma_r + \sigma_c) = p_e \frac{2\nu}{E}$$

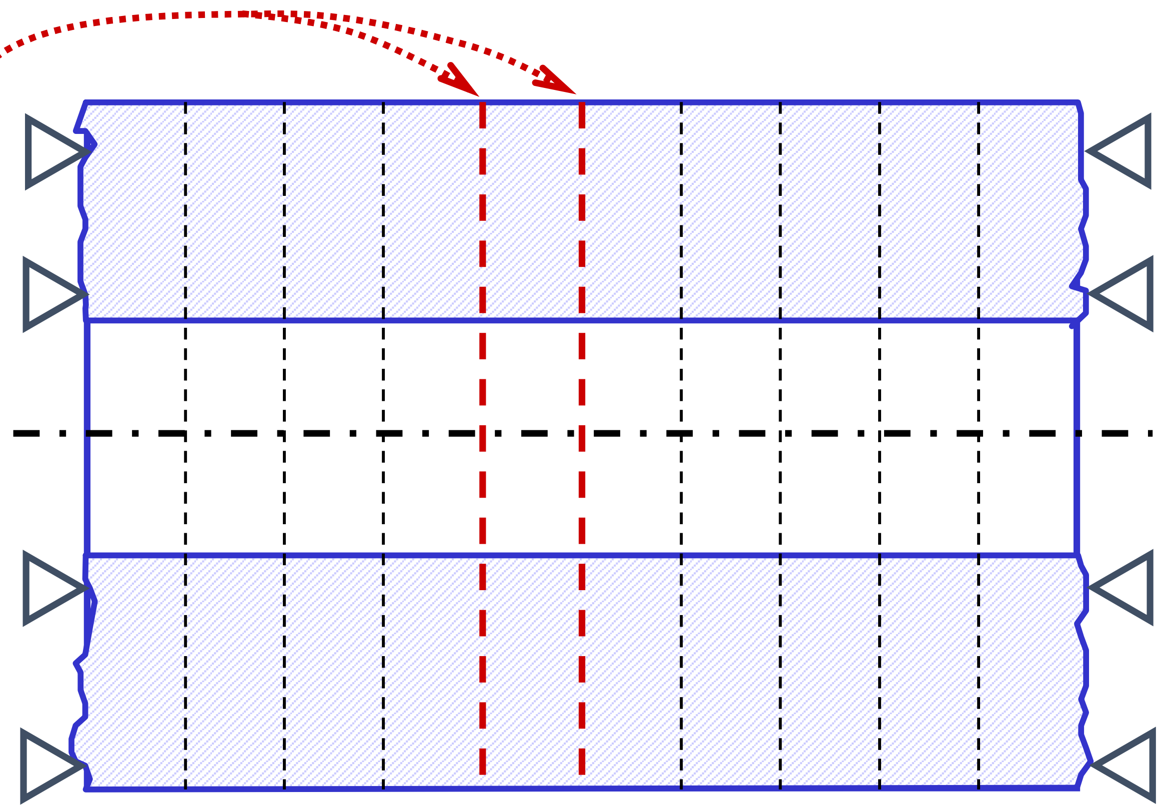
in this case the axial strain is constant, i.e., the axial displacement: ε_c times thickness b, is constant over the whole disc.

9. From plane stress to plane strain (1/6)

At one extreme, a very thin disc can be considered in plane stress, i.e, $\sigma_a=0$.

The other extreme is the long thick pipe, that we shall now investigate in plane strain, i.e., $\varepsilon_a=0$.

So, any two sections which were plane before application of loads will remain plane and at unchanged distance after load application.



9. From plane stress to plane strain (2/6)

In the case of plane strain, the following material equations hold. Equilibrium and compatibility are just the same as in plane stress.

$$\varepsilon_r = \frac{1-\nu^2}{E}\left(\sigma_r - \frac{\nu}{1-\nu}\sigma_c\right)$$

$$\varepsilon_c = \frac{1-\nu^2}{E}\left(\sigma_c - \frac{\nu}{1-\nu}\sigma_r\right)$$

We could repeat the procedure of sect. 5 of this chapter, only to discover that material parameters disappear and in-plane stresses σ_r and σ_c are just equal to those already obtained for plane stress.

9. From plane stress to plane strain (3/6)

This is predicted by the general treatment of Chapter 1, where in fact it was demonstrated that in the special case in which volume forces are constant (here they were taken zero) and, as in our case, boundary conditions are given in terms of stresses, a solution is obtained which:

- holds for plane stress and plane strain
- does not depend on the material elastic constants.

However, there is a simpler "engineering" way to this result, that we shall explore next.

This is based on the observation that the "plane stress and constant thickness" disc:

$$\varepsilon_a = -\frac{\nu}{E}\left(\sigma_r + \sigma_c\right) = -\frac{\nu}{E}2A$$

9. From plane stress to plane strain (4/6)

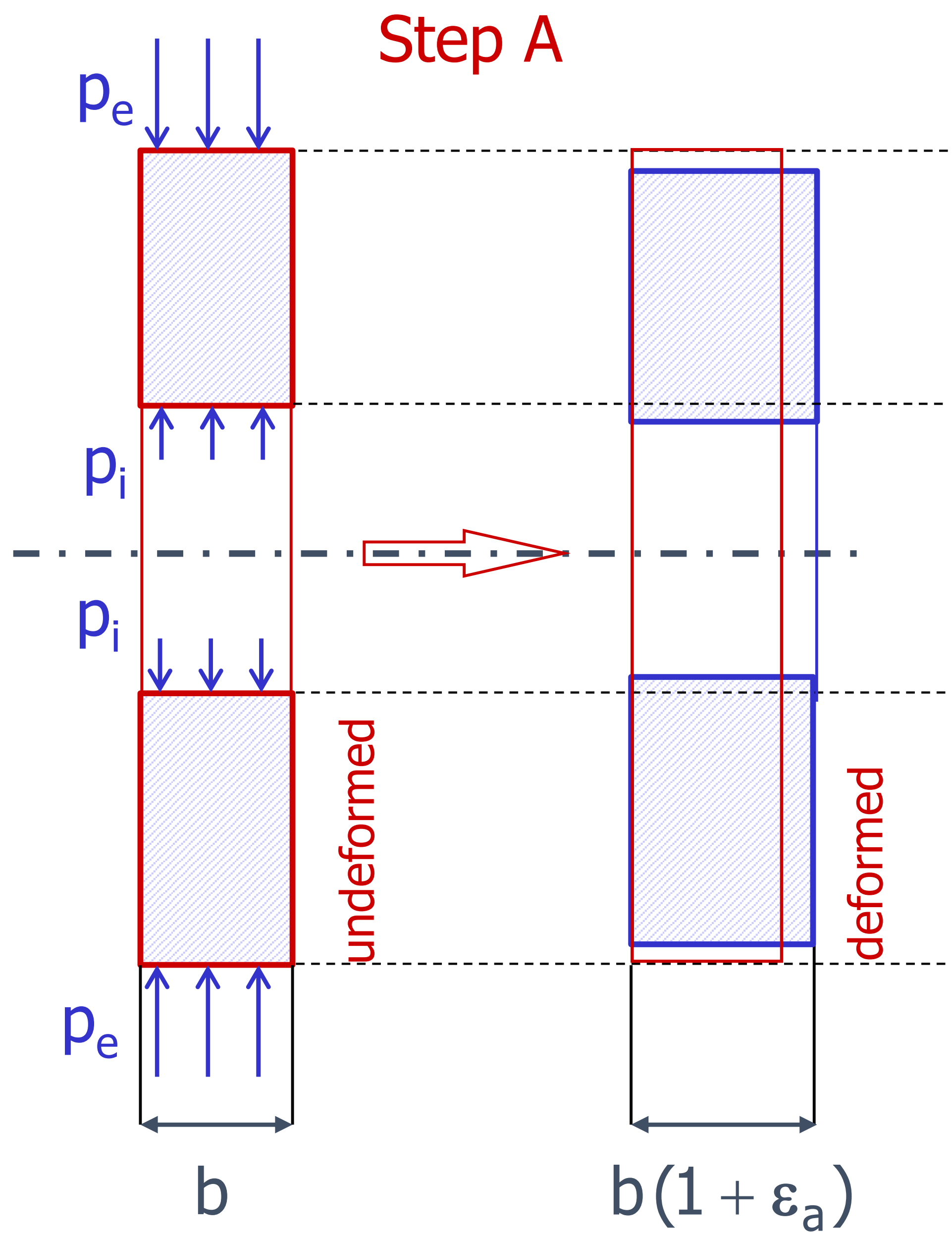
i.e., although σ_r and σ_c are variable over the radius, they combine in a way that ε_a is constant.

If we consider a plane stress disc with the same thickness of a slice of plane strain tube, subjected to the same inner and outer pressures p_i and p_e:

This will be "Step A", which produces an axial expansion:

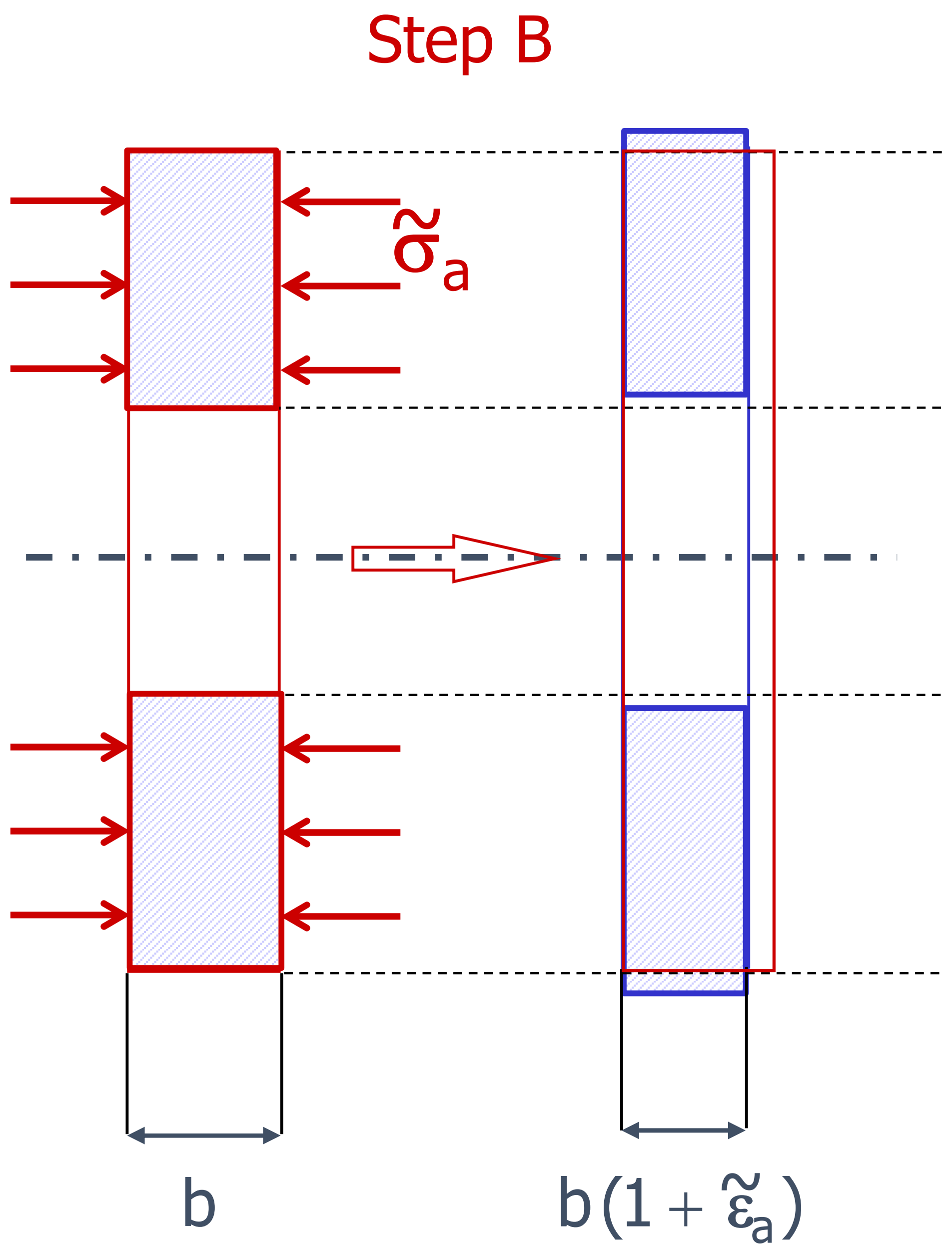
$$\Delta b = b\ \varepsilon_a$$

Radial and circumferential "disc" stresses σ_r and σ_c as calculated.



"Step B" consists of applying to the lateral surface a constant stress $\tilde{\sigma}_a$, that is a pure axial stress, producing a uniform strain $\tilde{\varepsilon}_a = \tilde{\sigma}_a / E$

Radial and circumferential stresses σ_r and σ_c are zero, for "Step B".



9. From plane stress to plane strain (6/6)

Summing displacements and stresses:

A (disc σ_a=0)	B (pure compress.)	A+B (tube ε_a=0)
axial strain: $\varepsilon_a = -\frac{\nu}{E} 2A$	$\tilde{\varepsilon}_a = \frac{\tilde{\sigma}_a}{E}$	$\varepsilon_{tot} = \frac{(\tilde{\sigma}_a - 2\nu A)}{E} = 0$
axial stress σ_a=0	$\tilde{\sigma}_a = 2\nu A$	$\sigma_{a,tot} = \sigma_a + \tilde{\sigma}_a = 2\nu A$
radial and circumf. stresses σ_r(disc) σ_c(disc)	$\tilde{\sigma}_r = 0$ $\tilde{\sigma}_c = 0$	$\sigma_{r,tot} = \sigma_r(\text{disc})$ $\sigma_{c,tot} = \sigma_c(\text{disc})$

10. Special cases (1/8)

Tube or disc under outer pressure

$p_i = 0 \; ; \; p_e \neq 0$

$$\sigma_r = -p_e \frac{1 - \dfrac{D_i^2}{D^2}}{1 - \dfrac{D_i^2}{D_e^2}}$$

$$\sigma_c = -p_e \frac{1 + \dfrac{D_i^2}{D^2}}{1 - \dfrac{D_i^2}{D_e^2}}$$

Tube or disc under inner pressure

$p_i \neq 0 \; ; \; p_e = 0$

$$\sigma_r = -p_i \frac{\dfrac{D_i^2}{D^2} - \dfrac{D_i^2}{D_e^2}}{1 - \dfrac{D_i^2}{D_e^2}}$$

$$\sigma_c = p_i \frac{\dfrac{D_i^2}{D^2} + \dfrac{D_i^2}{D_e^2}}{1 - \dfrac{D_i^2}{D_e^2}}$$

10. Special cases (2/8)

Tube or disc under outer pressure

$$A = -p_e \frac{1}{1 - \frac{D_i^2}{D_e^2}}$$

Tube or disc under inner pressure

$$A = p_i \frac{\frac{D_i^2}{D_e^2}}{1 - \frac{D_i^2}{D_e^2}}$$

Example: $D_i = D_e/2$

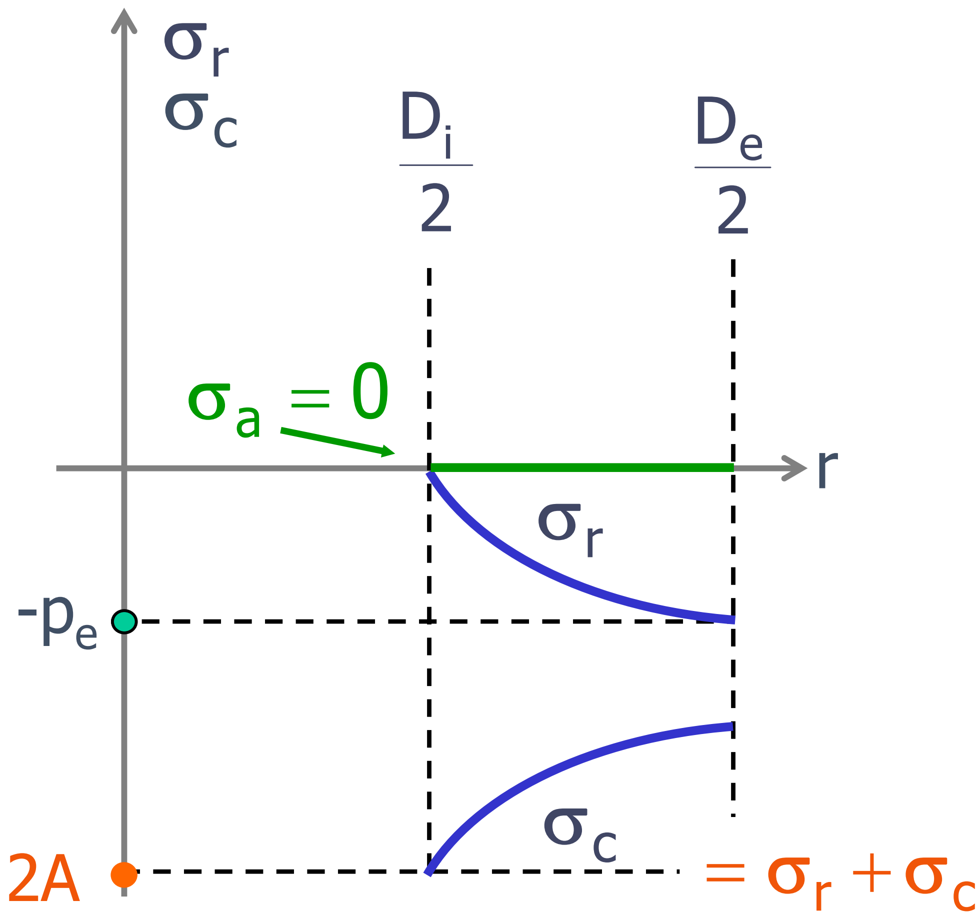
$$A = -p_e \frac{4}{3}$$

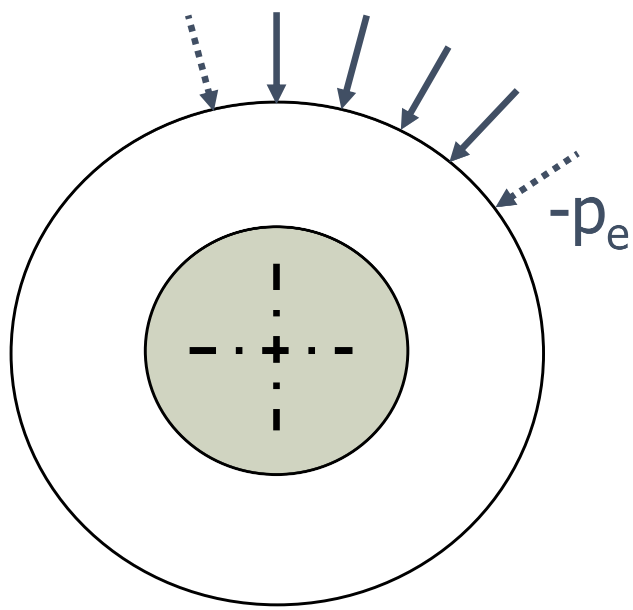
$$A = p_i \frac{1}{3}$$

10. Special cases (3/8) - outer pressure

In the case of outer pressure the stress diagrams are as follows:

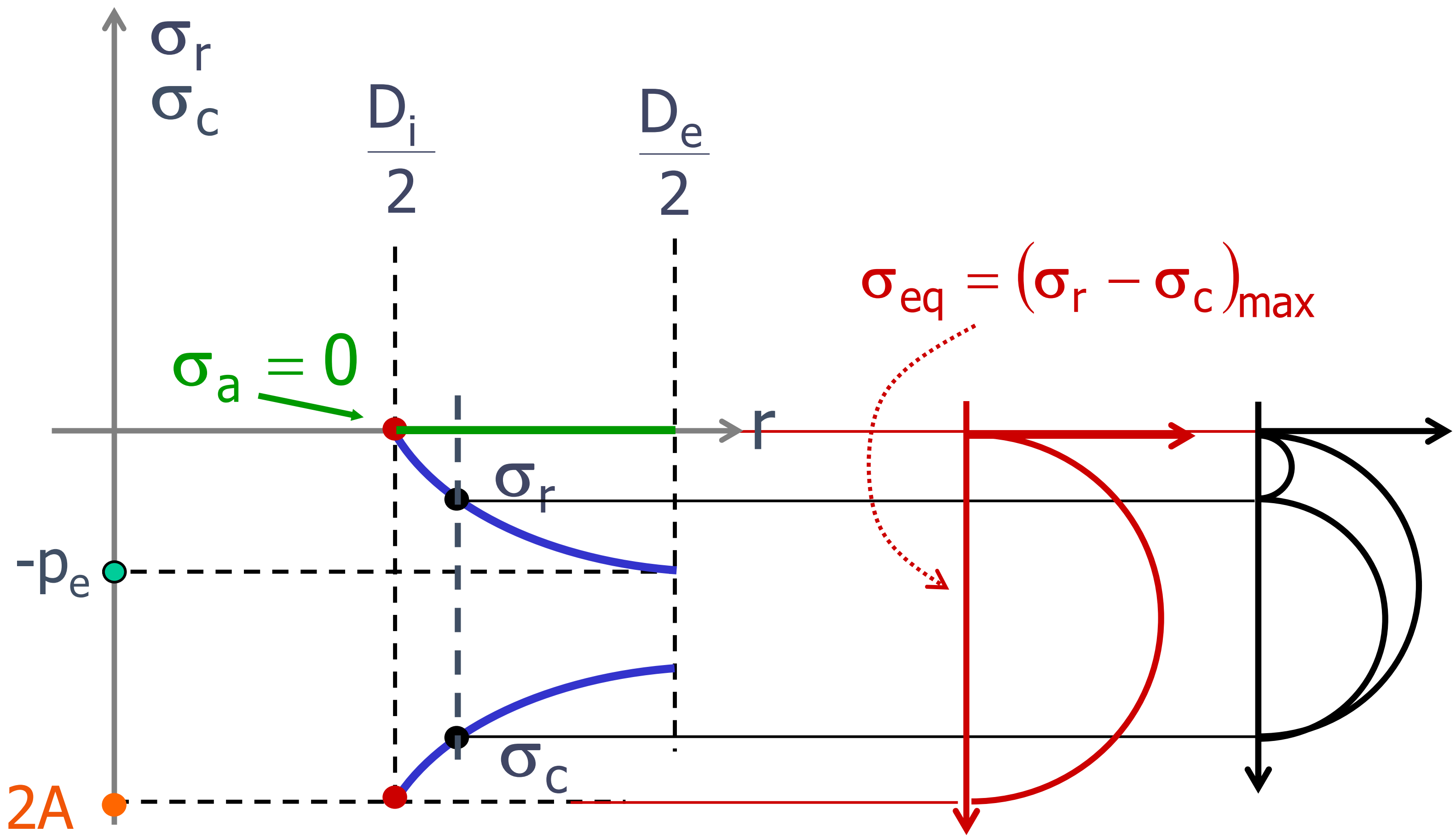
(example: $D_i/D_e=1/2$)





10. Special cases (4/8) - outer pressure

Mohr circles help find the most stressed position along the radius, which occurs at the inner radius, where the Mohr diameter is maximum:



10. Special cases (5/8) – outer pressure

Equivalent stresses according to Tresca in the most stressed location:

for a **ductile material**
at the design point, inner radius $D = D_i$:

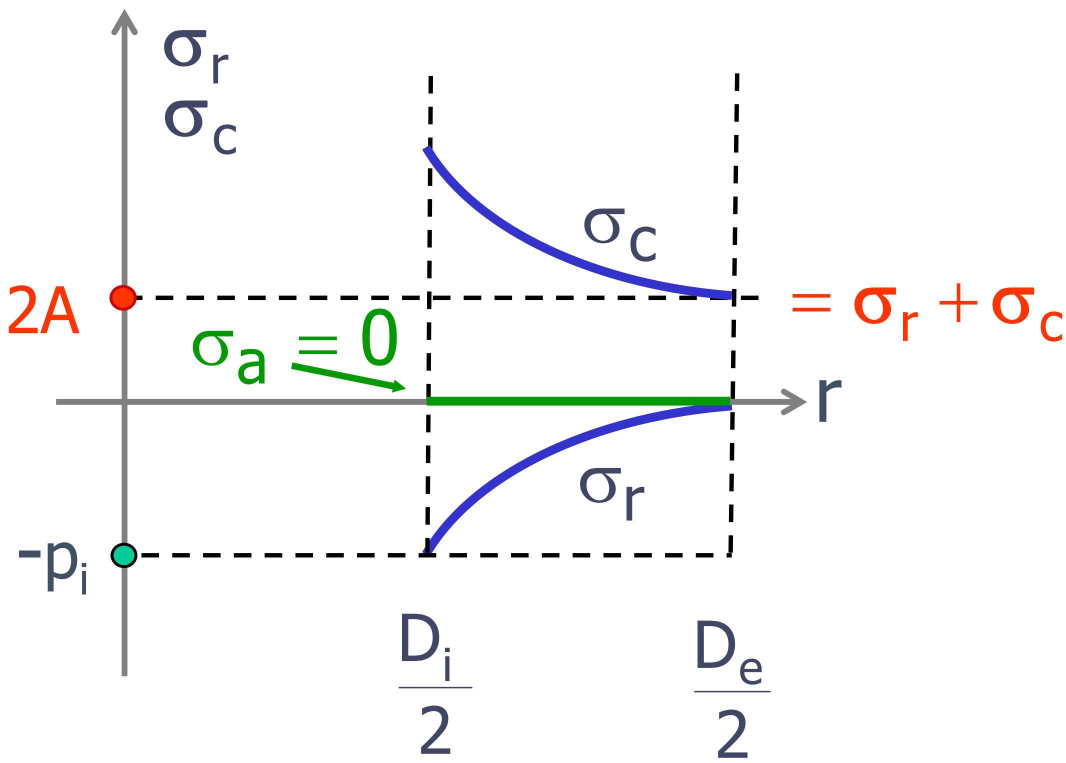
$$\sigma_{eq} = (\sigma_r - \sigma_c)_{max} = |\sigma_c(r = r_i)| =$$

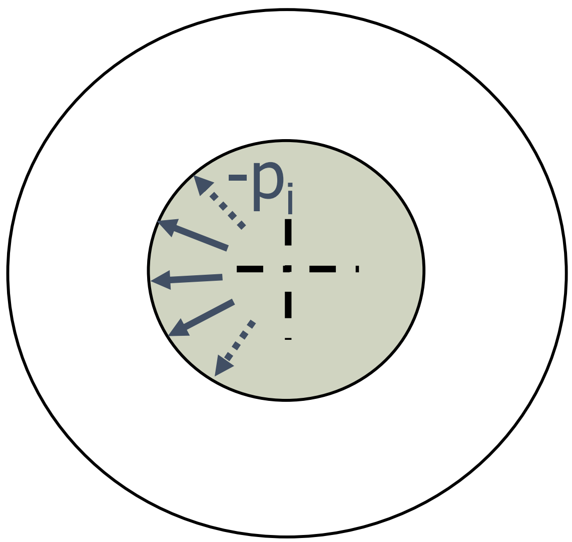
$$= |2A| \equiv p_e \frac{2}{1 - \frac{D_i^2}{D_e^2}}$$

for a **brittle material**:
there is no tensile stress,
the criterion is not applicable.

In the case of inner pressure the stress diagrams are as follows:

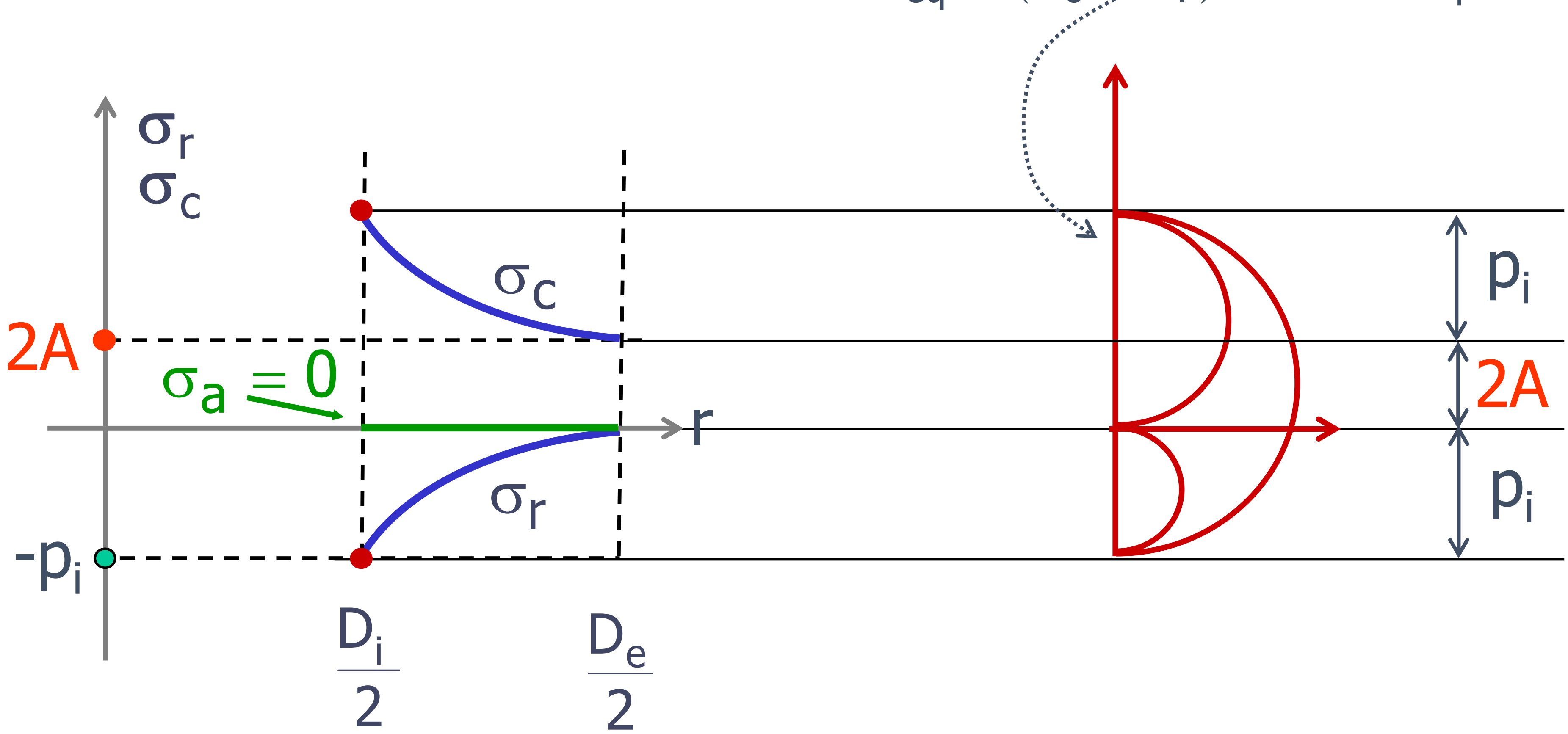
(example: Di/De=1/2)





Mohr circles help find the most stressed position along the adius, which also here occurs at the inner radius, where the Mohr diameter is maximum:

$$\sigma_{eq} = (\sigma_c - \sigma_r) = 2A + 2p_i$$



10. Special cases (8/8) – inner pressure

Equivalent stresses according to Tresca in the most stressed location:

for a **ductile material** at the design point, inner radius $D = D_i$:

$$\sigma_{eq} = \left(\sigma_c - \sigma_r\right) =$$

$$= 2A + 2p_i = 2\,p_i \frac{\dfrac{D_i^2}{D_e^2}}{1 - \dfrac{D_i^2}{D_e^2}} + 2p_i =$$

$$= p_i \frac{2}{1 - D_i^2 / D_e^2}$$

for a **brittle material**: at the design point, inner radius:

$$\sigma_{eq} = \sigma_c = p_i \frac{1 + \dfrac{D_i^2}{D_e^2}}{1 - \dfrac{D_i^2}{D_e^2}}$$

11. Thin shells (1/3)

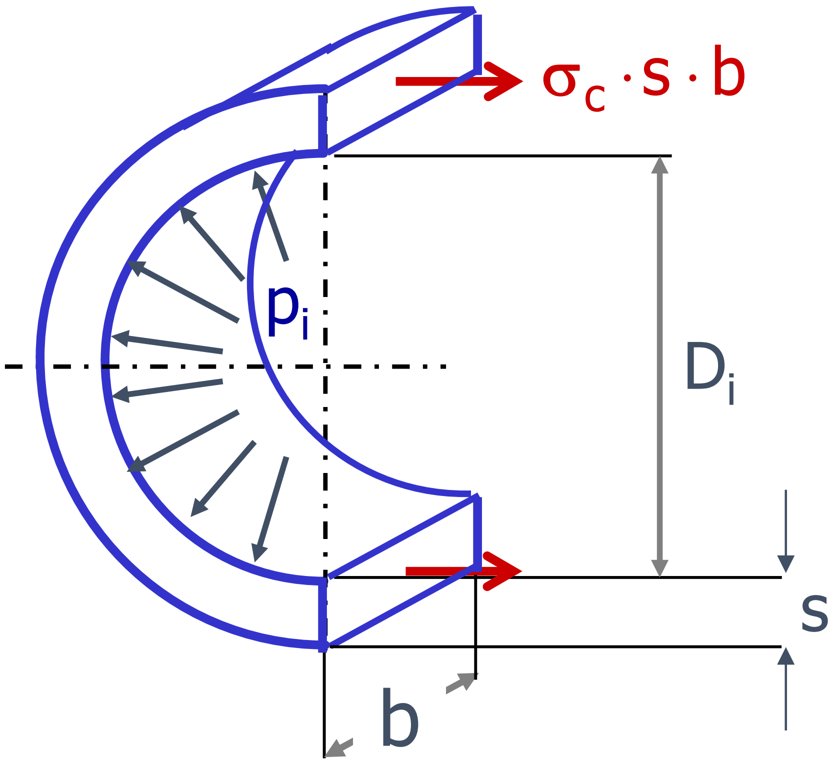
The case of pressurised thin shells can be treated with a simplification of the thick tube formula (inner pressure). Starting from the formula for equivalent stress at the inner radius:

$$\sigma_{eq} = p_i \frac{2}{1 - D_i^2 / D_e^2} = 2p_i \frac{D_e^2}{D_e^2 - D_i^2} = 2p_i \frac{D_e^2}{(D_e - D_i)(D_e + D_i)} =$$

$$= 2p_i \frac{D_e^2}{2s\,(D_e + D_i)} = p_i \frac{{D_e}^2}{2s \frac{D_e + D_i}{2}} = p_i \frac{D_e}{2s} \frac{D_e}{D_m}$$

… where s, shell thickness, is $(D_e - D_i)/2$.

The Boyle-Mariotte formula for thin shells is readily obtained from the transversal equilibrium of half shell after assuming that, due to the small thickness, σ_c can be taken constant over r:



$$p_i \cdot D_i \cancel{b} = 2\sigma_c \cdot s \cdot \cancel{b}$$

$$\sigma_c = p_i \cdot \frac{D_i}{2s}$$

11. Thin shells (3/3)

Equivalent stress at the inner diameter, where the stress difference is maximum:

$$\sigma_{eq} = \underbrace{\sigma_c - \sigma_r}_{\text{in } D=D_i} = \sigma_c + p_i = p_i \frac{D_e}{D_e - D_i} = p_i \frac{D_e}{2s}$$

Formulas for thick … … … … and thin shells:

$$\sigma_{eq} = p_i \frac{D_e}{2s} \frac{D_e}{D_m} \qquad \sigma_{eq} = p_i \frac{D_e}{2s}$$

tend to the same value if $s \ll D_i$, D_e and D_m, and $\frac{D_e}{D_m} \approx 1$