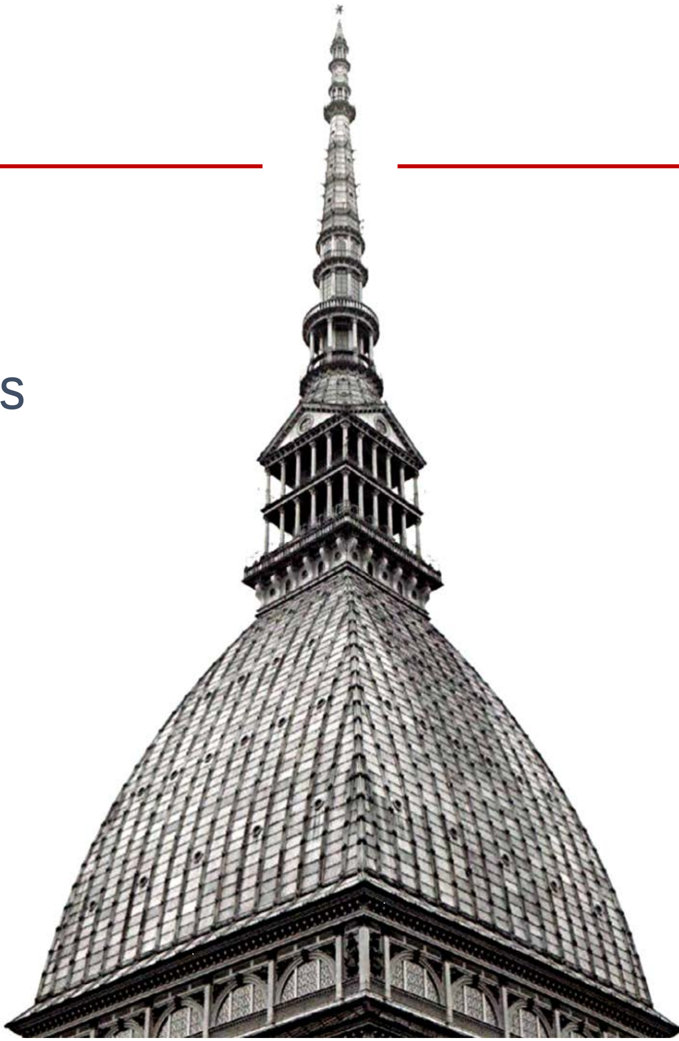
POLITECNICO DI TORINO



Chapters

Index of contents

1. Introduction (1/2)

The following equations come into play:

✓ moment equilibrium: $\tau_{ik} = \tau_{ki}$ i.e. simmetry of stress tensor

✓ force equilibrium: $\sum_i \frac{\partial \sigma_{ik}}{\partial X_i} + \Phi_k = 0$ (3 equations : $k = 1,2,3$)

✓ x_i (i=1,2,3) : cartesian coordinates

✓ compatibility (kinematic): i.e. strains come from a displacement field

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial X_i} \quad (i = 1, 2, 3)$$

$$\gamma_{ik} = \left(\frac{\partial u_i}{\partial X_k} + \frac{\partial u_k}{\partial X_i} \right) \quad (ik = 12, 13, 23)$$

✓ material (constitutive) equations, for isotropy

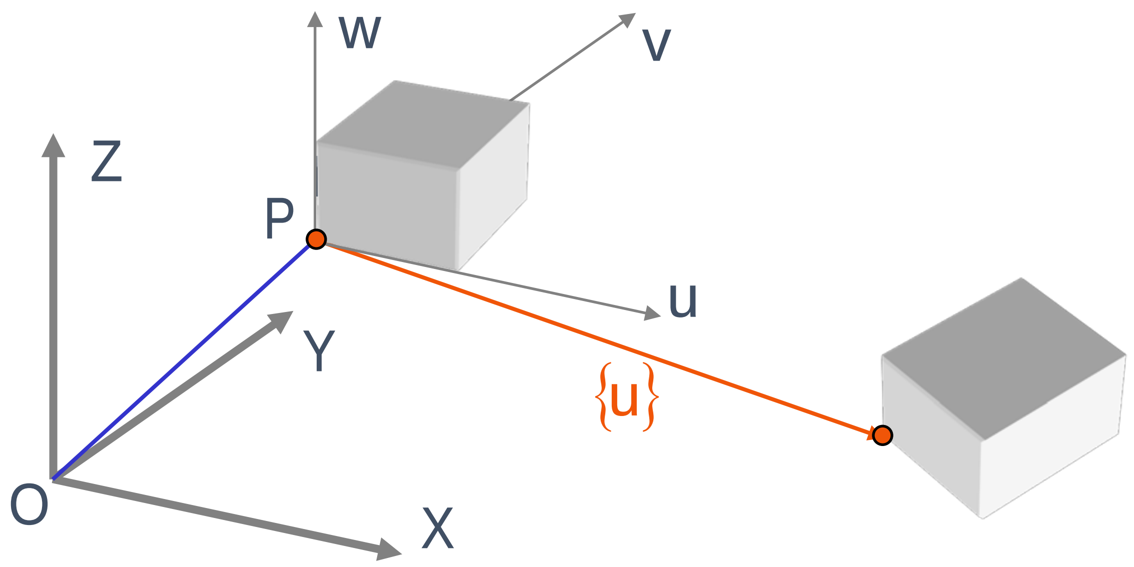
$$\varepsilon_{ii} = \frac{1}{E} \left[\sigma_{ii} - \nu \sigma_{jj} - \nu \sigma_{kk} \right] \quad (i = 1, 2, 3)$$

$$\gamma_{ik} = \frac{1}{G} \tau_{ik} \quad (ik = 12, 13, 23)$$

1. Introduction (2/2)

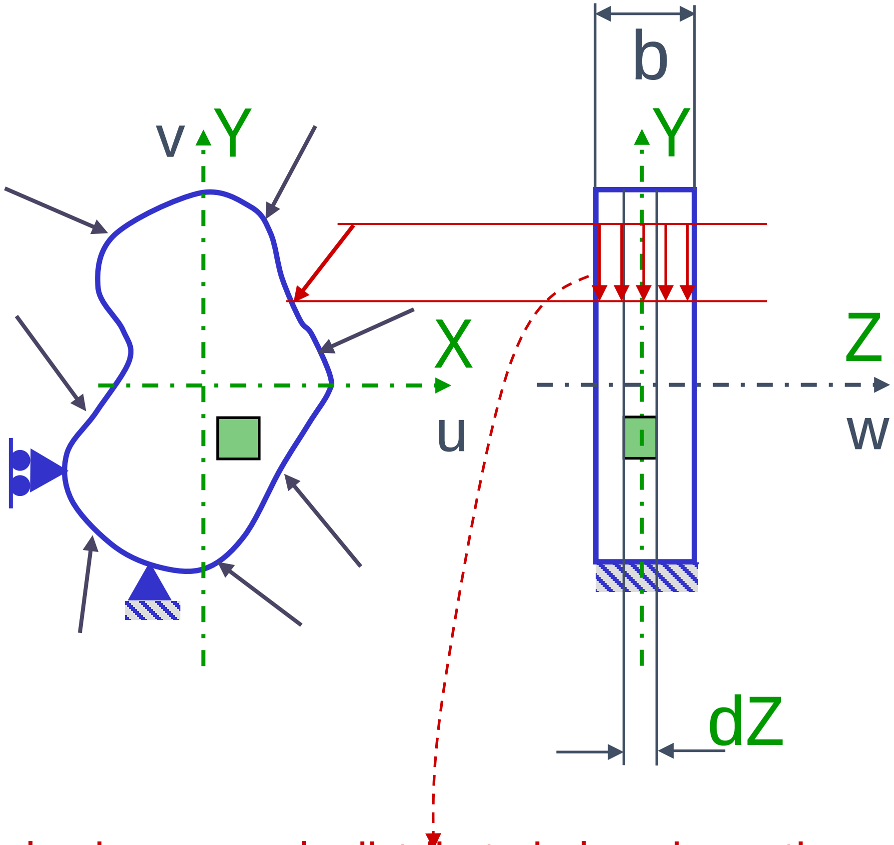
The same equations can be written, according to convenience, with either numerical or alphabetical symbols; in what follows we shall use the second type of notation. Then:

$X_1 \to X \qquad X_2 \to Y \qquad X_3 \to Z \qquad u_1 \to u \qquad u_2 \to v \qquad u_3 \to w$



2. Assumptions for the plane problem (1/5)

Plane body, constant thickness, loaded in the plane



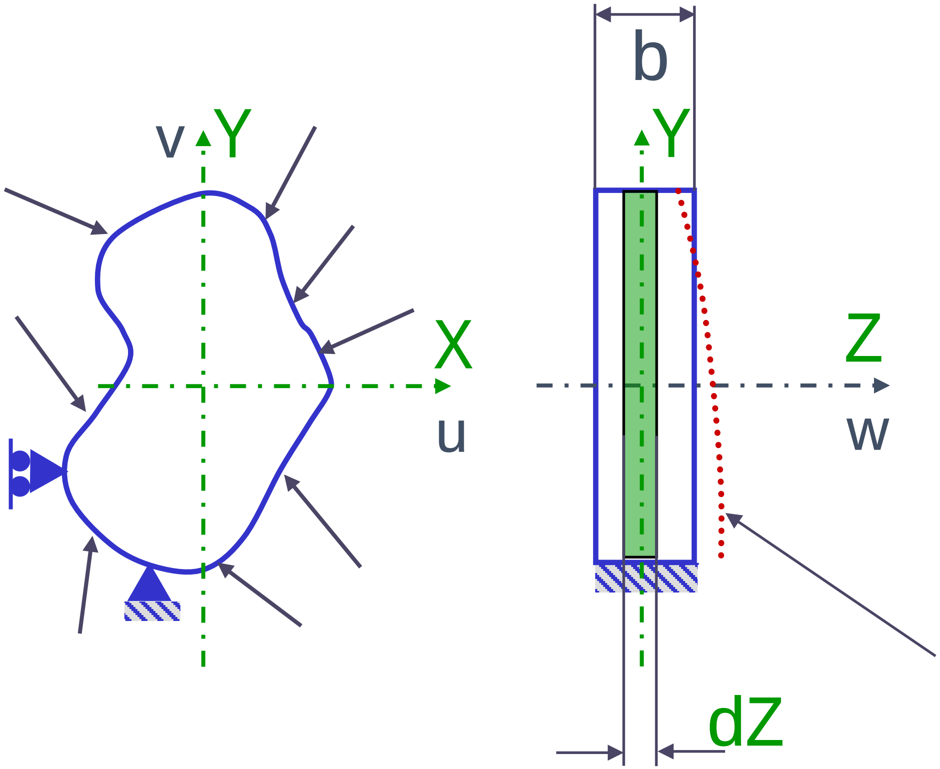
loads are evenly distributed along b, on the lateral surface

Assumptions:

- ✓ zero volume force Φ_z along Z
- ✓ volume forces Φ_x , Φ_y do not vary along Z
- ✓ homogeneous and isotropic material
- ✓ all "slices" of thickness dZ at any level Z are equally loaded and equally constrained at their boundary contour in the plane X, Y

2. Assumptions for the plane problem (2/5)

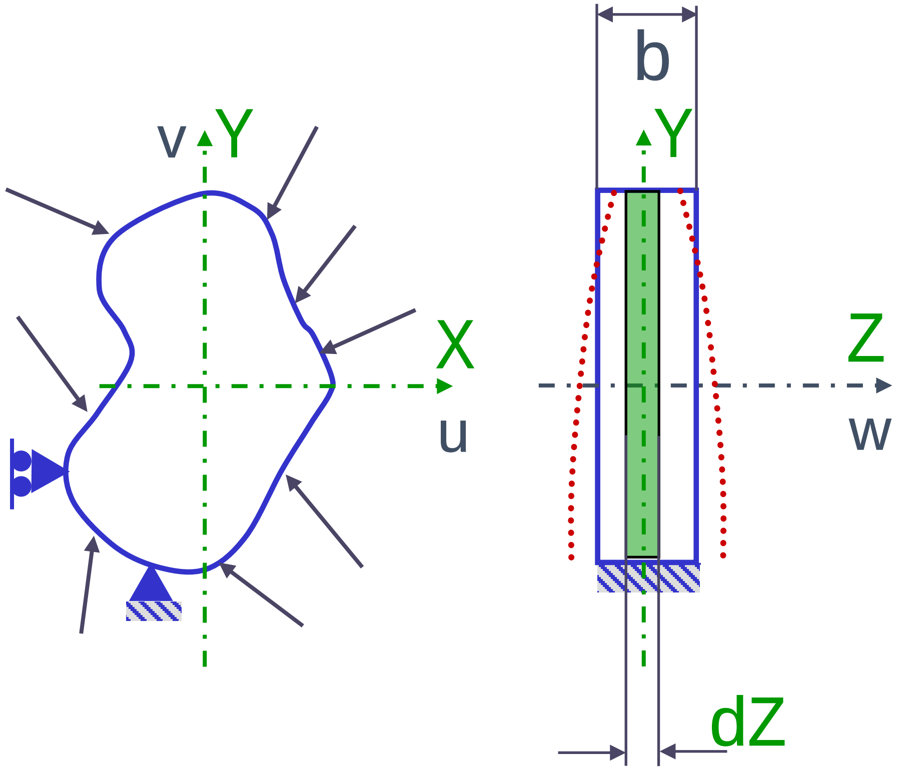
If thickness b is small compared to the in-plane dimensions, we can assume that the median infinitesimal slice is representative of all neighbouring slices: this means that u,v displacements do not vary along Z:



$$\frac{\partial u}{\partial Z} = \frac{\partial v}{\partial Z} = 0$$

However, the strain ε_{zz} varies with X,Y the w displacement along Z will vary with X,Y ...

... the consequence is that angular strains become non-zero:



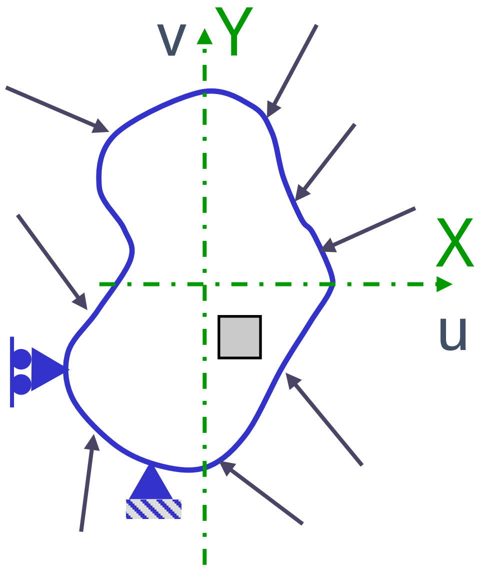
$$\gamma_{xz} = \left(0 + \frac{\partial w}{\partial X} \right) \qquad \gamma_{yz} = \left(0 + \frac{\partial w}{\partial Y} \right)$$

which is in conflict with the fact that $\tau_{xz}=\tau_{yz}=0$ at the lateral boundaries $Z=\pm\, b/2$

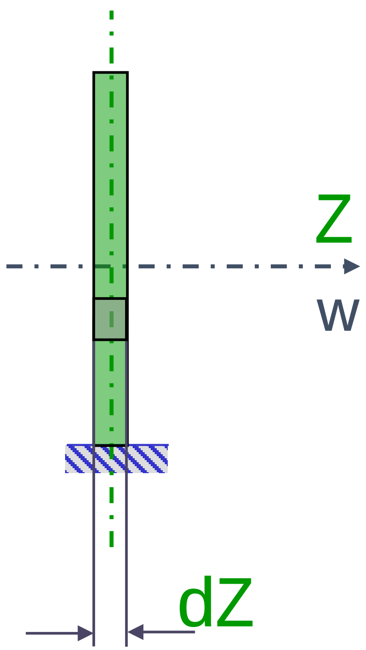
This contradiction may appear when lateral boundaries are free, i.e., in a “plane stress” condition; however, it depends on the effective w displacement field given by the solution of the elastic problem.

2. Assumptions for the plane problem (4/5)

We shall now accept this approximation and study two possibile cases: “plane stress” and “plane strain”.

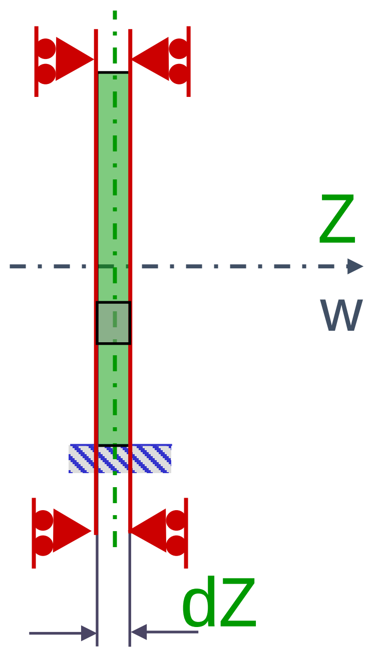


plane stress: $\sigma_{zz} = 0$



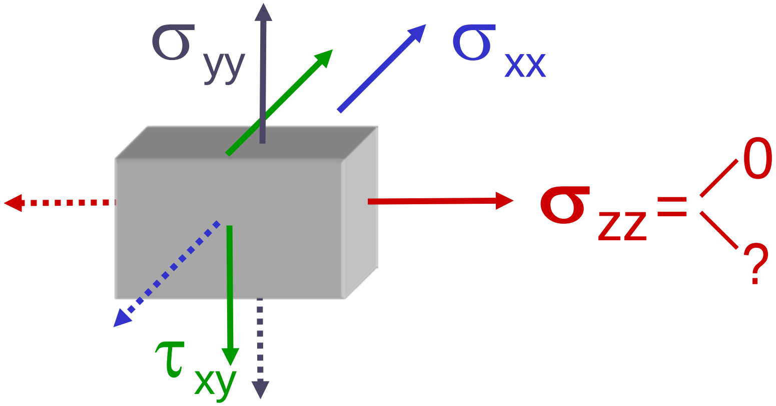
lateral free surfaces, then also no tangential stress

plane strain: $\varepsilon_{zz} = 0$



two lateral perfecly sliding constraints, i.e. producing no tangential stress

All stresses acting on an infinitesimal element at co-ordinates X, Y are:



In both cases:
$$\begin{cases} \tau_{zx} = 0 \\ \tau_{zy} = 0 \end{cases}$$

In plane stress: $\sigma_{zz} = 0$

In plane strain: $\varepsilon_{zz} = 0$

Remark that τ_{xz} and τ_{yz} are missing.

Note: in this case strains and stresses along Z are principal, because $\tau_{xz}=\tau_{yz}= \gamma_{xz}=\gamma_{yz} = 0$; then we may write σ_z and ε_z instead of σ_{zz} and ε_{zz} .

According to compatibility:

$$\gamma_{xy} = \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial X \partial Y}\left(\gamma_{xy}\right) = \frac{\partial^2}{\partial X \partial Y}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right) = \underline{\frac{\partial^3}{\partial X \partial Y^2}(u) + \frac{\partial^3}{\partial X^2 \partial Y}(v)}$$

$$\frac{\partial^2}{\partial Y^2}\left(\frac{\partial u}{\partial X}\right) + \frac{\partial^2}{\partial X^2}\left(\frac{\partial v}{\partial Y}\right)$$

else:

A) $$\frac{\partial^2}{\partial X \partial Y}\left(\gamma_{xy}\right) = \frac{\partial^2 \varepsilon_{xx}}{\partial Y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial X^2}$$

Equilibrium equations are now:

$$\frac{\partial \sigma_{xx}}{\partial X} + \frac{\partial \tau_{yx}}{\partial Y} + \Phi_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial X} + \frac{\partial \sigma_{yy}}{\partial Y} + \Phi_y = 0 \qquad \frac{\partial \boldsymbol{\sigma}_z}{\partial Z} = 0$$

The first and the second are derived to permit combination with the compatibility equation:

$$\frac{\partial^2 \boldsymbol{\sigma}_{xx}}{\partial X^2} + \frac{\partial^2 \boldsymbol{\tau}_{yx}}{\partial X \partial Y} + \frac{\partial \boldsymbol{\Phi}_x}{\partial X} = 0$$

$$\frac{\partial^2 \boldsymbol{\sigma}_{yy}}{\partial Y^2} + \frac{\partial^2 \boldsymbol{\tau}_{xy}}{\partial X \partial Y} + \frac{\partial \boldsymbol{\Phi}_y}{\partial Y} = 0$$

Their sum: B)

$$\frac{\partial^2 \sigma_{xx}}{\partial X^2} + 2\frac{\partial^2 \tau_{yx}}{\partial X \partial Y} + \frac{\partial^2 \sigma_{yy}}{\partial Y^2} + \frac{\partial \Phi_x}{\partial X} + \frac{\partial \Phi_y}{\partial Y} = 0$$

Compatibility and equilibrium equations together give:

$$\begin{cases} \text{A)} \quad \dfrac{\partial^2 \boldsymbol{\varepsilon}_{xx}}{\partial Y^2} + \dfrac{\partial^2 \boldsymbol{\varepsilon}_{yy}}{\partial X^2} = \dfrac{\partial^2 \boldsymbol{\gamma}_{yx}}{\partial X \partial Y} \\ \text{B)} \quad \dfrac{\partial^2 \boldsymbol{\sigma}_{xx}}{\partial X^2} + \dfrac{\partial^2 \boldsymbol{\sigma}_{yy}}{\partial Y^2} = -2 \dfrac{\partial^2 \boldsymbol{\tau}_{yx}}{\partial X \partial Y} - \left(\dfrac{\partial \boldsymbol{\Phi}_x}{\partial X} + \dfrac{\partial \boldsymbol{\Phi}_y}{\partial Y} \right) \end{cases}$$

The purpose is to obtain the final equations in terms of stresses.

Then equation A) will be transformed by means of the material equations, which can be written in the following form:

$$\varepsilon_{xx} = \frac{1}{\tilde{E}} \left(\sigma_{xx} - \tilde{\nu} \sigma_{yy} \right) \qquad \varepsilon_{yy} = \frac{1}{\tilde{E}} \left(\sigma_{yy} - \tilde{\nu} \sigma_{xx} \right) \qquad \gamma_{xy} = \frac{1}{G} \tau_{xy}$$

which represent both plane stress and plane strain cases ... ...

... ...by just setting the values for elastic constants as follows:

plane stress:

$$\begin{cases} \tilde{E} \rightarrow E \\ \tilde{\nu} \rightarrow \nu \end{cases} \quad \text{when} \quad \sigma_z = 0$$

plane strain:

$$\begin{cases} \tilde{E} = \dfrac{E}{1-\nu^2} \\ \tilde{\nu} = \dfrac{\nu}{1-\nu} \end{cases} \quad \text{when} \quad \varepsilon_z = 0$$

while in both cases: $G = \dfrac{E}{2(1+\nu)} = \dfrac{\tilde{E}}{2(1+\tilde{\nu})} \equiv \tilde{G}$

Summary of important relations

$$E = \tilde{E} \cdot \left(1-\nu^2\right) = \tilde{E} \cdot \frac{1+2\tilde{\nu}}{(1+\tilde{\nu})^2};$$

$$G \equiv \tilde{G} = \frac{E}{2(1+\nu)} = \frac{1}{2}\tilde{E} \cdot \frac{1+2\tilde{\nu}}{(1+\tilde{\nu})^2} \cdot \frac{1+\tilde{\nu}}{1+2\tilde{\nu}} = \frac{\tilde{E}}{2(1+\tilde{\nu})}$$

$$\longrightarrow \tilde{\nu} + \frac{\tilde{E}}{2\tilde{G}} = 1$$

$$\tilde{\nu} = \frac{\nu}{1-\nu} \rightarrow \nu = \frac{\tilde{\nu}}{1+\tilde{\nu}}$$

$$1-\nu = \frac{1}{1+\tilde{\nu}}$$

$$1+\nu = \frac{1+2\tilde{\nu}}{1+\tilde{\nu}}$$

$$\left(1-\nu^2\right) = \frac{1+2\tilde{\nu}}{(1+\tilde{\nu})^2}$$

From: with:

$$\begin{cases} \varepsilon_{xx} = \dfrac{1}{\tilde{E}}\left(\sigma_{xx} - \tilde{\nu}\sigma_{yy}\right) \\ \varepsilon_{yy} = \dfrac{1}{\tilde{E}}\left(\sigma_{yy} - \tilde{\nu}\sigma_{xx}\right) \\ \tau_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases}$$

$$\begin{cases} \text{A)} & \dfrac{\partial^2 \varepsilon_{xx}}{\partial Y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial X^2} = \dfrac{\partial^2 \gamma_{yx}}{\partial X \partial Y} \\ \text{B)} & \dots \quad \dots \end{cases}$$

one obtains:

$$\begin{cases} \text{A)} & \left(\dfrac{\partial^2 \sigma_{xx}}{\partial Y^2} + \dfrac{\partial^2 \sigma_{yy}}{\partial X^2}\right) - \tilde{\nu}\left(\dfrac{\partial^2 \sigma_{xx}}{\partial X^2} + \dfrac{\partial^2 \sigma_{yy}}{\partial Y^2}\right) = \dfrac{\tilde{E}}{G}\dfrac{\partial^2 \tau_{yx}}{\partial X \partial Y} \\ \text{B)} & \left(\dfrac{\partial^2 \sigma_{xx}}{\partial X^2} + \dfrac{\partial^2 \sigma_{yy}}{\partial Y^2}\right) = -2\dfrac{\partial^2 \tau_{yx}}{\partial X \partial Y} - \left(\dfrac{\partial \Phi_x}{\partial X} + \dfrac{\partial \Phi_y}{\partial Y}\right) \end{cases}$$

Multiplying eq. B) times $(\tilde{E} / 2G)$ and summing to A):

$$\left(\frac{\partial^2 \boldsymbol{\sigma}_{xx}}{\partial Y^2}+\frac{\partial^2 \boldsymbol{\sigma}_{yy}}{\partial X^2}\right)+\left(\frac{\partial^2 \boldsymbol{\sigma}_{xx}}{\partial X^2}+\frac{\partial^2 \boldsymbol{\sigma}_{yy}}{\partial Y^2}\right)\left(-\tilde{\boldsymbol{\nu}}+\frac{\tilde{E}}{2G}\right)=$$

$$=-\frac{\tilde{E}}{2G}\left(\frac{\partial \boldsymbol{\Phi}_x}{\partial X}+\frac{\partial \boldsymbol{\Phi}_y}{\partial Y}\right)$$

which does not contain the term $\boldsymbol{\tau}_{yx}$

It is easily checked that: $-\tilde{\boldsymbol{\nu}}+\frac{\tilde{E}}{2G}=1$ then: ... ...

... then:

$$\left(\frac{\partial^2 \sigma_{xx}}{\partial Y^2}+\frac{\partial^2 \sigma_{yy}}{\partial X^2}\right)+\left(\frac{\partial^2 \sigma_{xx}}{\partial X^2}+\frac{\partial^2 \sigma_{yy}}{\partial Y^2}\right)=-\frac{\tilde{E}}{2G}\left(\frac{\partial \Phi_x}{\partial X}+\frac{\partial \Phi_y}{\partial Y}\right)$$

... or:

$$\underbrace{\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)}_{\Delta^2:\ \text{Laplacian}}\boldsymbol{\sigma}_{xx}+\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\boldsymbol{\sigma}_{yy}=-\frac{\tilde{E}}{2G}\left(\frac{\partial \boldsymbol{\Phi}_x}{\partial X}+\frac{\partial \boldsymbol{\Phi}_y}{\partial Y}\right)$$

which is written in compact form:

$$\boldsymbol{\Delta}^2\boldsymbol{\sigma}_{xx}+\boldsymbol{\Delta}^2\boldsymbol{\sigma}_{yy}=-\frac{\tilde{E}}{2G}\left(\frac{\partial \boldsymbol{\Phi}_x}{\partial X}+\frac{\partial \boldsymbol{\Phi}_y}{\partial Y}\right)$$

$$\begin{cases} \dfrac{\tilde{E}}{2G}=1+\nu & (\boldsymbol{\sigma}_z=0) \\[2ex] \dfrac{\tilde{E}}{2G}=\dfrac{1}{1-\nu}=1+\tilde{\nu} & (\boldsymbol{\varepsilon}_z=0) \end{cases}$$

4. Solution for stress (1/4)

In the special case in which volume forces are constant:

$$\left(\frac{\partial \Phi_x}{\partial X}+\frac{\partial \Phi_y}{\partial Y}\right)=0 \qquad \Rightarrow \qquad \Delta^2\left(\sigma_{xx}+\sigma_{yy}\right)=0$$

an equation is obtained which:

✓ holds for plane stress and plane strain

✓ does not depend on the material elastic constants.

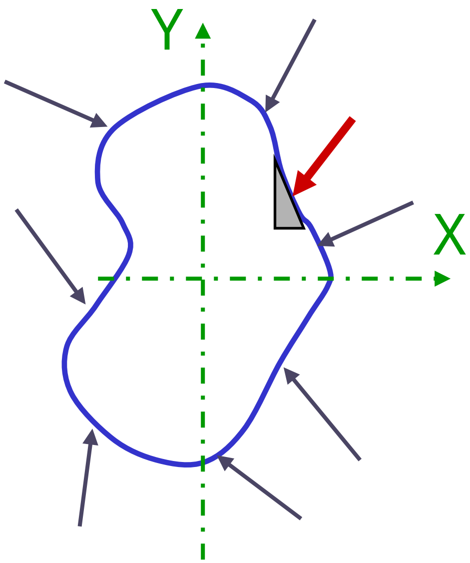
Together with the two (force) equilibrium equations:

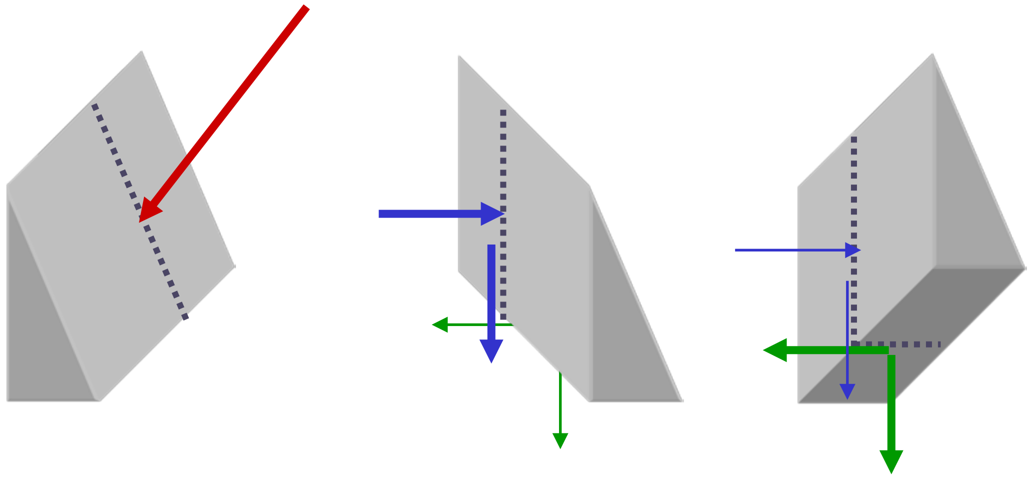
$$\frac{\partial \sigma_{xx}}{\partial X}+\frac{\partial \tau_{yx}}{\partial Y}+\Phi_x=0 \qquad \frac{\partial \sigma_{yy}}{\partial Y}+\frac{\partial \tau_{yx}}{\partial X}+\Phi_y=0$$

it forms a solving system of three equations in the three unknown stresses.

If boundary conditions are given in terms of stresses:

$$\{t\} = [\sigma]\{n\}$$



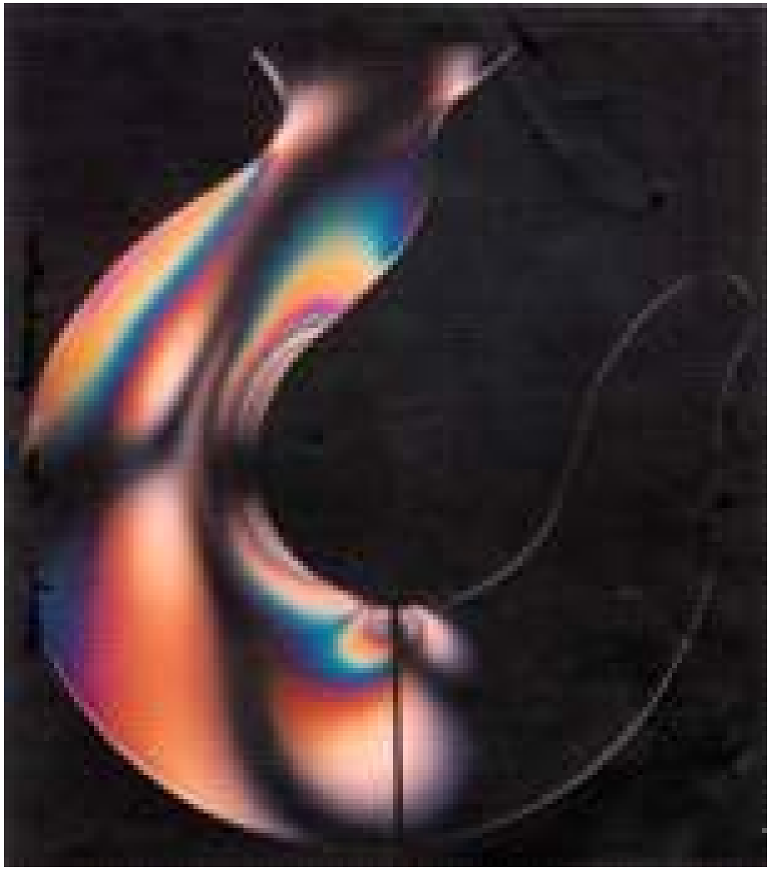


the final solution does not depend on material parameters.

This has an important consequence, of practical significance: stresses can be investigated on a model of equal shape, equally loaded, but made of a different – more convenient – material.

4. Solution for stress (3/4)

This is what happens in (plane stress) photoelasticity, a technique for studying and measuring the stresses and strains in an object by means of strain / stress dependent birefringence.



Plane model of a crane hook, with fringes of optical interference.

This method is mostly used in cases where mathematical methods become cumbersome. It was much more employed before the advent of finite element numerical analysis.

4. Solution for stress (4/4)

The other important conclusion, however, is that in-plane stresses σ_{xx}, σ_{yy}, τ_{xy} are the same for plane stress and plane strain.

Of course, σ_z and ε_z will differ in the two cases.

In practice, plane strain is present in infinitely long cylinders or bars (of any cross sectional shape – including thick walled tubes), where the lateral loading is invariant along the z-direction. Where therefore the displacement field takes the form:

$$\begin{cases} u = u(X, Y) \\ v = v(X; Y) \\ w = 0 \end{cases}$$