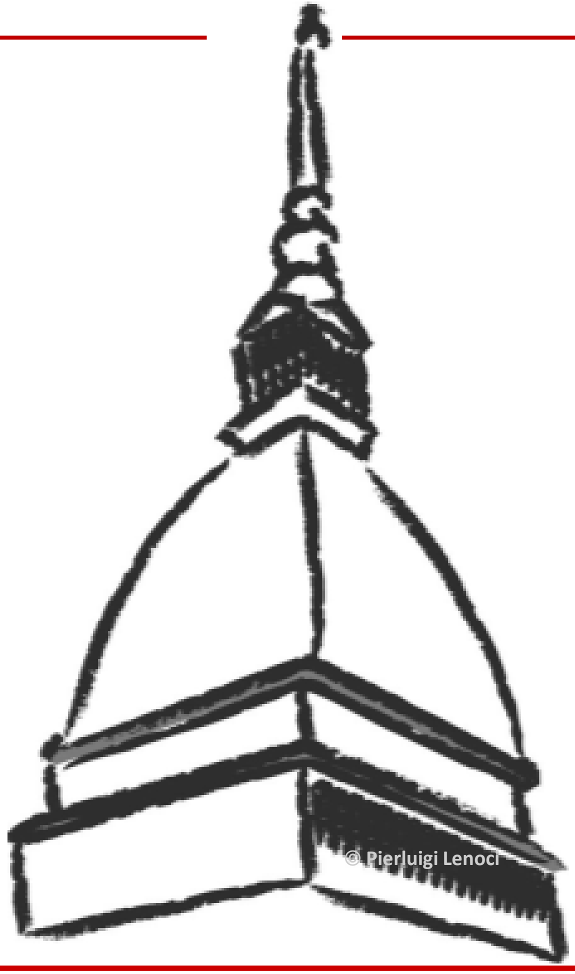
Daniele Botto

Costruzione di Motori per Aeromobili

Hertz contacts & bearings - Chapter 1

POLITECNICO DI TORINO

Chapters



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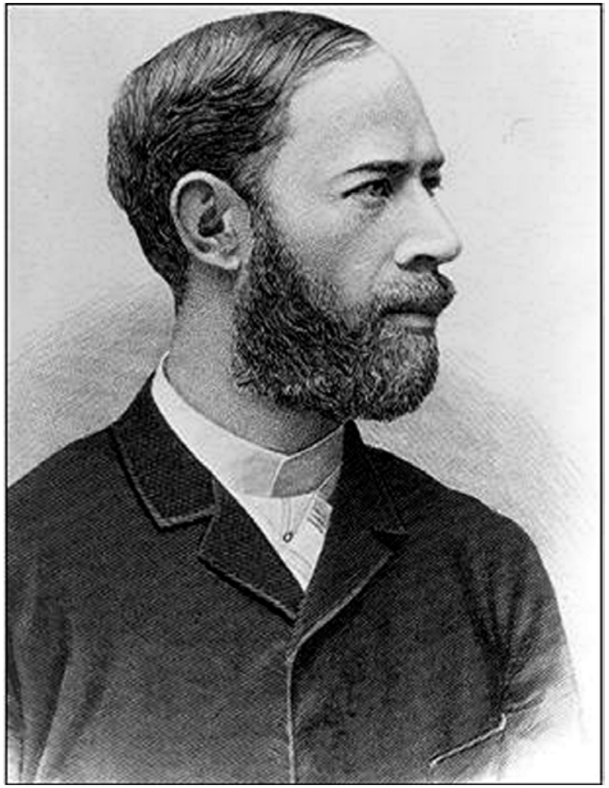
1. Introduction

The problem of elastic contact between two elastic bodies was first solved by Heinrich Rudolf Hertz (Feb. 22, 1857 – Jan. 1, 1894).

Although well known for his contributions to the field of electrodynamics, in 1881–1882, Hertz published also two papers* on what was to become known as the field of contact mechanics.

His purpose was to determine matematically how bodies in “point contact” when undeformed will behave when loaded one against the other.

He obtained his results within the classical theory of elasticity.



* H. Hertz, “Über die Berührung fester elastischer Körper,” *Gesammelte Werke* (P. Lenard, ed.), Bd. 1, (J.A. Barth, Leipzig, 1895) pp. 155-173. Originally published in Journal für die reine und angewandte Mathematik, 92,156-171 (1881)

Sections 2, 3 - The geometry of the contacting bodies

Sections 2 and 3 are just a preliminary to Hertz's solution of the contact problem.

The purpose of Section 2 is to define the second order approximation of each of the two bodies in contact taken separately. Moreover, to put into evidence the connection between the second derivatives at the theoretical contact point and the radii which are usually specified in the mechanical drawings of contacting bodies. In addition, an intuitive sign definition of engineering value is put forward.

Section 3 studies the two bodies in contact, the attention is on their distance measured along the common normal at the theoretical contact point. The treatment allows to determine the principal axes of such distance, and the principal values γ_X, γ_Y of its quadratic form.

Appendix II shows an alternative way to come to the same result; this way is longer, less elegant, perhaps more elementary.

2. Local geometry (1/7)

Contact conditions:

a) Geometry:
- two bodies in contact
- in a non-singular point of their surfaces (regular at least to the second derivative); then (first derivatives) the common tangent plane exists

b) Material:
- elastic, isotropic
- no friction

c) Hypothesis (empirically observed, ex post mathematically proved)
- small contact surface (length and width small compared to curvature radii of bodies in contact)

2. Local geometry (2/7)

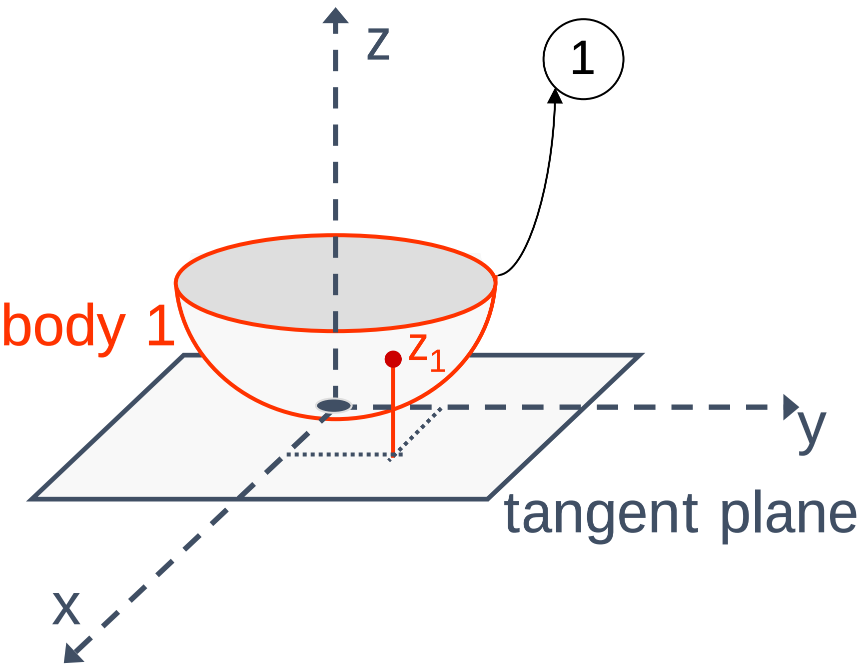
Let us define surface properties for body 1:

$$z_1 = z_1(x, y)$$

$$z_1(0,0) = 0$$

$$\frac{\partial z_1}{\partial x}(0,0) = 0$$

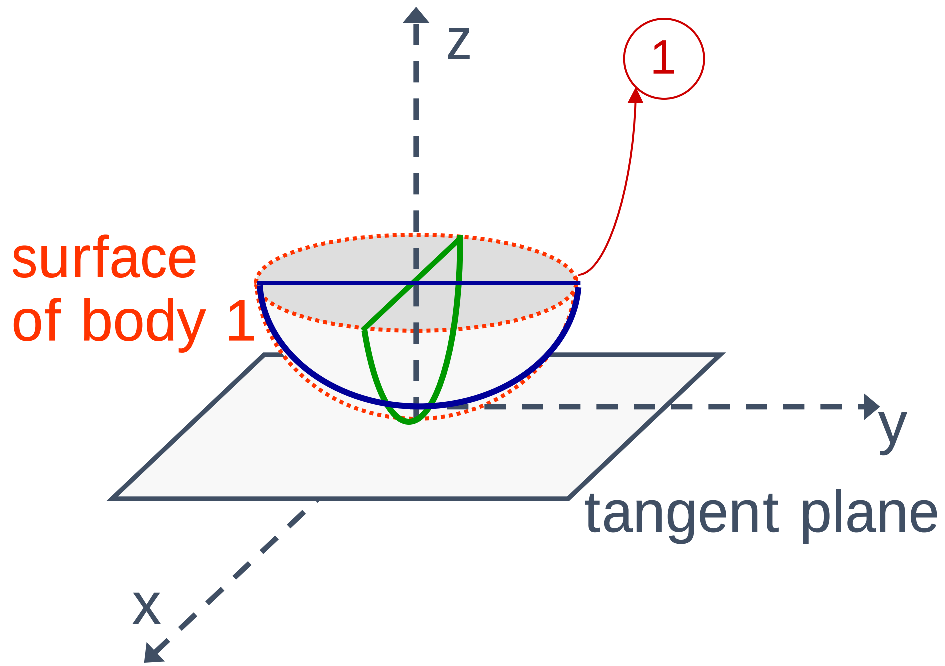
$$\frac{\partial z_1}{\partial y}(0,0) = 0$$



which developed up to the 2nd order about point (0,0):

$$z_1 \cong \underset{\substack{\downarrow \\ z_1(0,0)}}{0} + \underset{\substack{\downarrow \\ \frac{\partial z_1}{\partial x}}}{0} \cdot x + \underset{\substack{\downarrow \\ \frac{\partial z_1}{\partial y}}}{0} \cdot y + \frac{1}{2}\left[\frac{\partial^2 z_1}{\partial x^2} x^2 + 2\frac{\partial^2 z_1}{\partial x \partial y} xy + \frac{\partial^2 z_1}{\partial y^2} y^2\right] + \ldots.$$

This truncated series expansion is quadratic form, which is conveniently written in matrix notation; note that it is symmetrical :



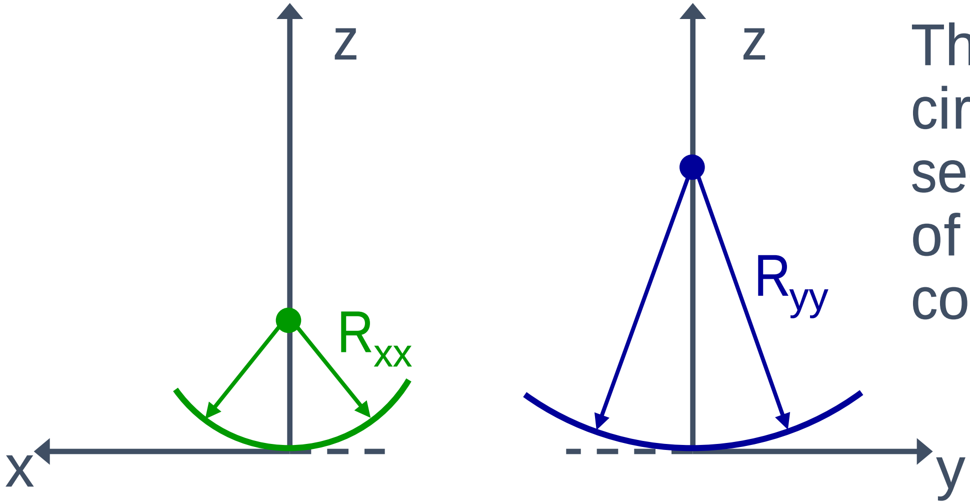
$$z_1 \cong \{x \quad y\} \begin{bmatrix} \alpha_{xx} & \alpha_{xy} \\ \alpha_{xy} & \alpha_{yy} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\alpha_{xx} = \frac{1}{2}\frac{\partial^2 z_1}{\partial x^2} \rightarrow \frac{1}{2}\cdot\frac{1}{R_{xx}}$$ → curvature radius of section with plane xz

$$\alpha_{yy} = \frac{1}{2}\frac{\partial^2 z_1}{\partial y^2} \rightarrow \frac{1}{2}\cdot\frac{1}{R_{yy}}$$ → curvature radius of section with plane yz

$$\alpha_{xy} = \frac{1}{2}\frac{\partial^2 z_1}{\partial x \partial y} \rightarrow$$ more complex meaning

Remark: α_{xx} and α_{yy} are reciprocals of diameters !!



These two figures show the circles which approximate to the second order the cross sections of the surface of body 1 with the coordinate planes x-z and y-z .

The symmetry of the quadratic form implies that there always exists a cartesian (orthogonal) reference system (X, Y), rotated with respect to axes (x,y), which makes the form diagonal:

$$z_1 = \{X, Y\} \begin{bmatrix} \alpha_X & 0 \\ 0 & \alpha_Y \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} \quad \text{or else} \quad z_1 = \alpha_X X^2 + \alpha_Y Y^2$$

R_x, R_y: principal curvature radii: $\quad \alpha_X = \dfrac{1}{2R_X} \qquad \alpha_Y = \dfrac{1}{2R_Y}$

Note 1:

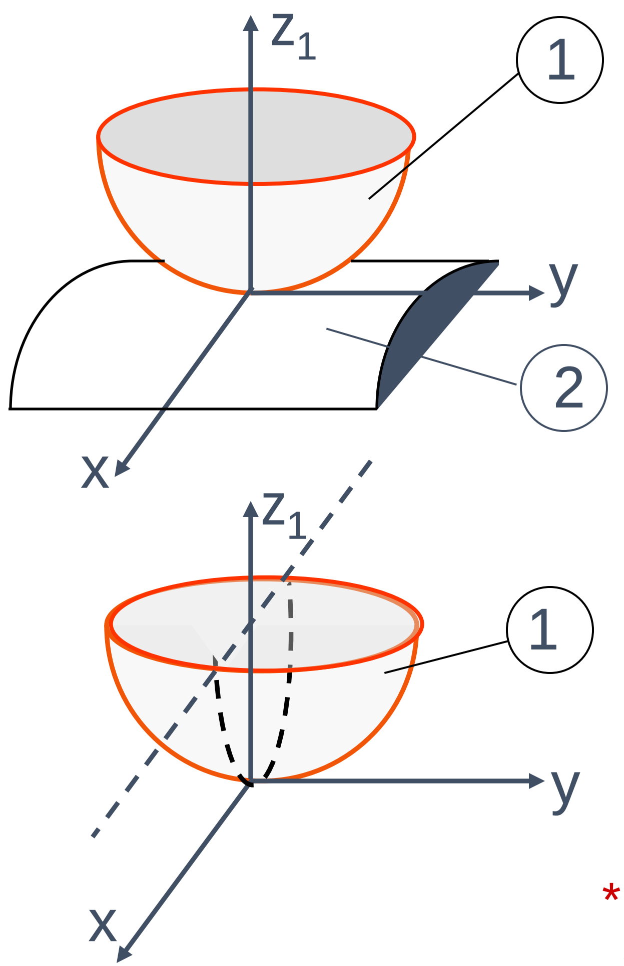
It was demonstrated (Gauss) that the principal curvatures ($1/R_X$, $1/R_Y$), and then also radii (R_X, R_Y), are “extreme”: i.e., they are the maximum and the minimum radius of the section curve that we obtain when rotating the section plane about axis z.

Note 2:

{x, y} symbol for “any” axes

capital letters {X, Y} for “principal” axes.

Two bodies with surfaces 1 and 2 in contact are tangent through the same common plane:

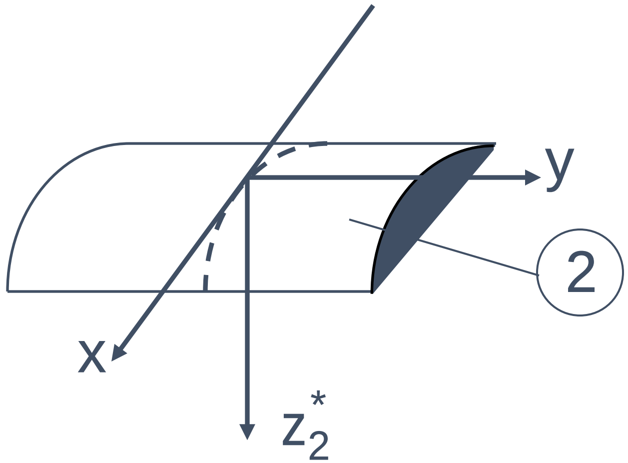


To facilitate an intuitive practical approach to sign definitions, it is expedient to define axis z so that curvatures are positive when the body is convex*; this is already the case of axis z_1 for body 1:

$$z_1 = \{x, y\} \begin{bmatrix} \alpha_{xx} & \alpha_{xy} \\ \alpha_{xy} & \alpha_{yy} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\alpha_{xx} = \frac{1}{2} \frac{1}{R_{xx,1}}; \quad \alpha_{yy} = \frac{1}{2} \frac{1}{R_{yy,1}}$$

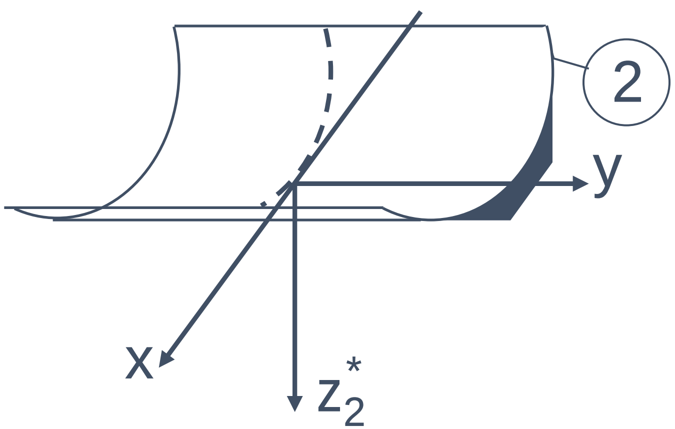
*a surface section is convex at a point of a body when it is seen by an observer - outside the body material - on the opposite side of the tangent plane to the body in that point)



For body 2 the axis z_2^* is chosen inside (downwards) so that with positive curvatures the surface coordinates are positive (downwards)

$$z_2^* = \{x \ y\} \begin{bmatrix} \beta_{xx} & \beta_{xy} \\ \beta_{xy} & \beta_{yy} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\beta_{xx} = \frac{1}{2} \cdot \frac{1}{R_{xx,2}} ; \ \beta_{yy} = \frac{1}{2} \cdot \frac{1}{R_{yy,2}}$$



Were the body concave, the coefficients would be negative.

The practical rule is that for each body its own axis z (here z_2^*) is chosen "inside" its material.

Consider that in practice coefficients α_{ij} and β_{ij} will not be calculated as derivatives, but instead found by halving the reciprocals of radii taken from a mechanical drawing.

Therefore the couple convex-positive, concave-negative is a kind of intrinsic property related to phisical appearance.

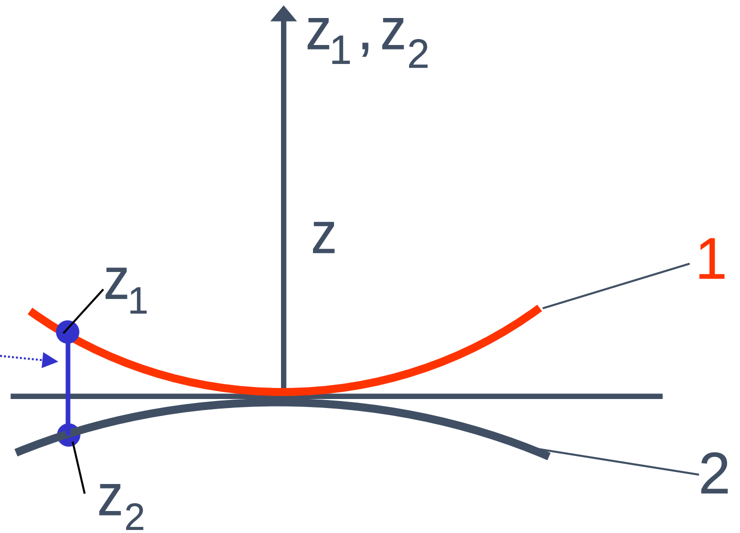
We must now express both bodies in the same reference system, that we may choose to be (x, y, z) of body 1, then:

$$z_2 = -z_2^* \qquad z_2 = \{x \; y\}\begin{bmatrix} -\beta_{xx} & -\beta_{xy} \\ -\beta_{xy} & -\beta_{yy} \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

A common reference system is convenient to calculate the vertical distance between the two bodies 1 and 2:

$$z_1 - z_2 = z_1 + z_2^* =$$

$$= \{x \quad y\}\begin{bmatrix} \alpha_{xx} + \beta_{xx} & \alpha_{xy} + \beta_{xy} \\ \alpha_{xy} + \beta_{xy} & \alpha_{yy} + \beta_{yy} \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$



... which can be written:

$$z_1 - z_2 = \begin{Bmatrix} x & y \end{Bmatrix} \begin{bmatrix} \gamma_{xx} & \gamma_{xy} \\ \gamma_{xy} & \gamma_{yy} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

matrix of relative curvatures

Also this matrix is symmetric, then it exists a rotated reference system with axes $\{X \;\; Y\}$ in which the matrix has diagonal form:

$$\begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix}$$

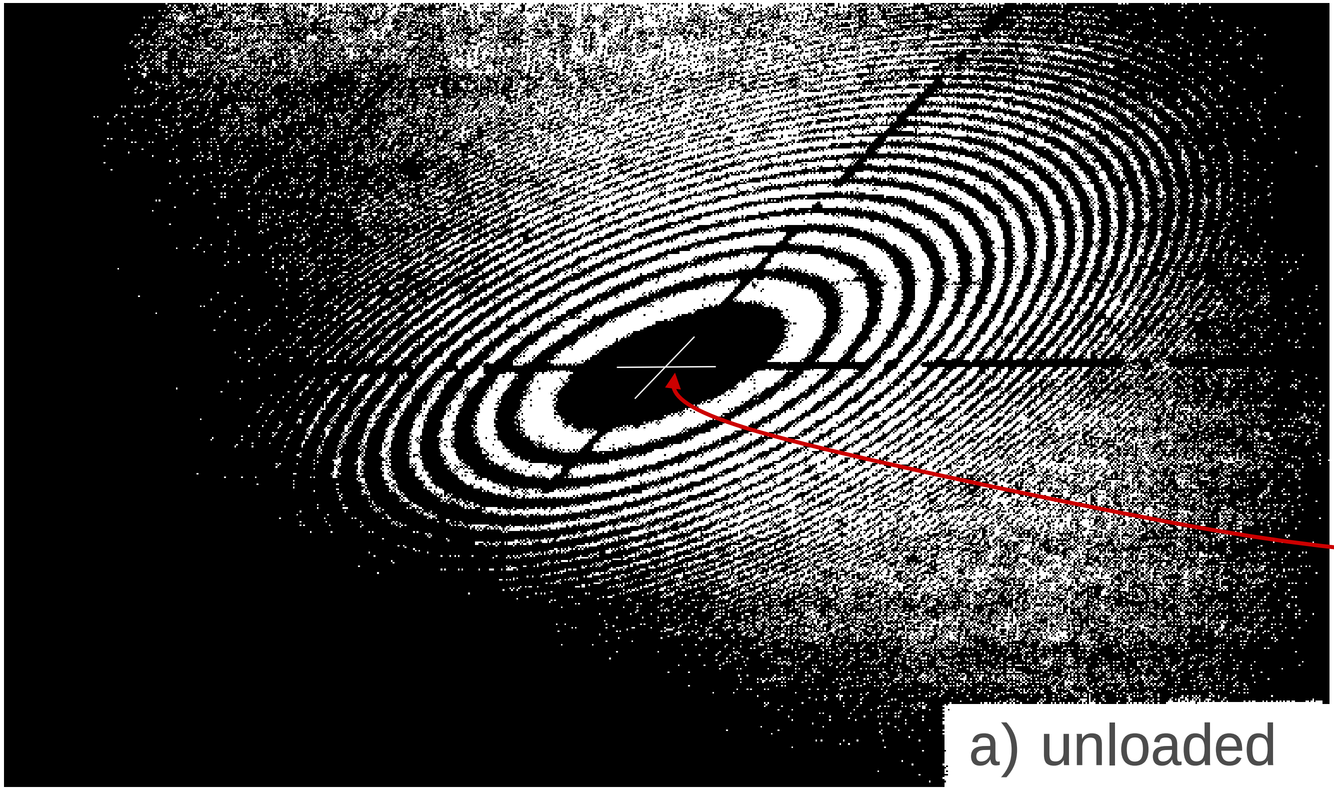
and the coefficients on the principal diagonal are the “principal relative curvatures”. They always exist, they are always placed on two mutually orhogonal axes X,Y .

There are infinite different couples of body shapes z_1, z_2 whose relative distance is represented by the same relative curvature matrix.

The iso-distance curves between 1 and 2 are ellipses:

$z_1 - z_2 = h = \gamma_X X^2 + \gamma_Y Y^2$ Indeed, this quadratic form must satisfy:

$\gamma_X \geq 0, \ \ \gamma_Y \geq 0$ because, if a contact is physically possible, then $z_1 - z_2 \geq 0$ at all points (X,Y), i.e., compenetration does not occur.

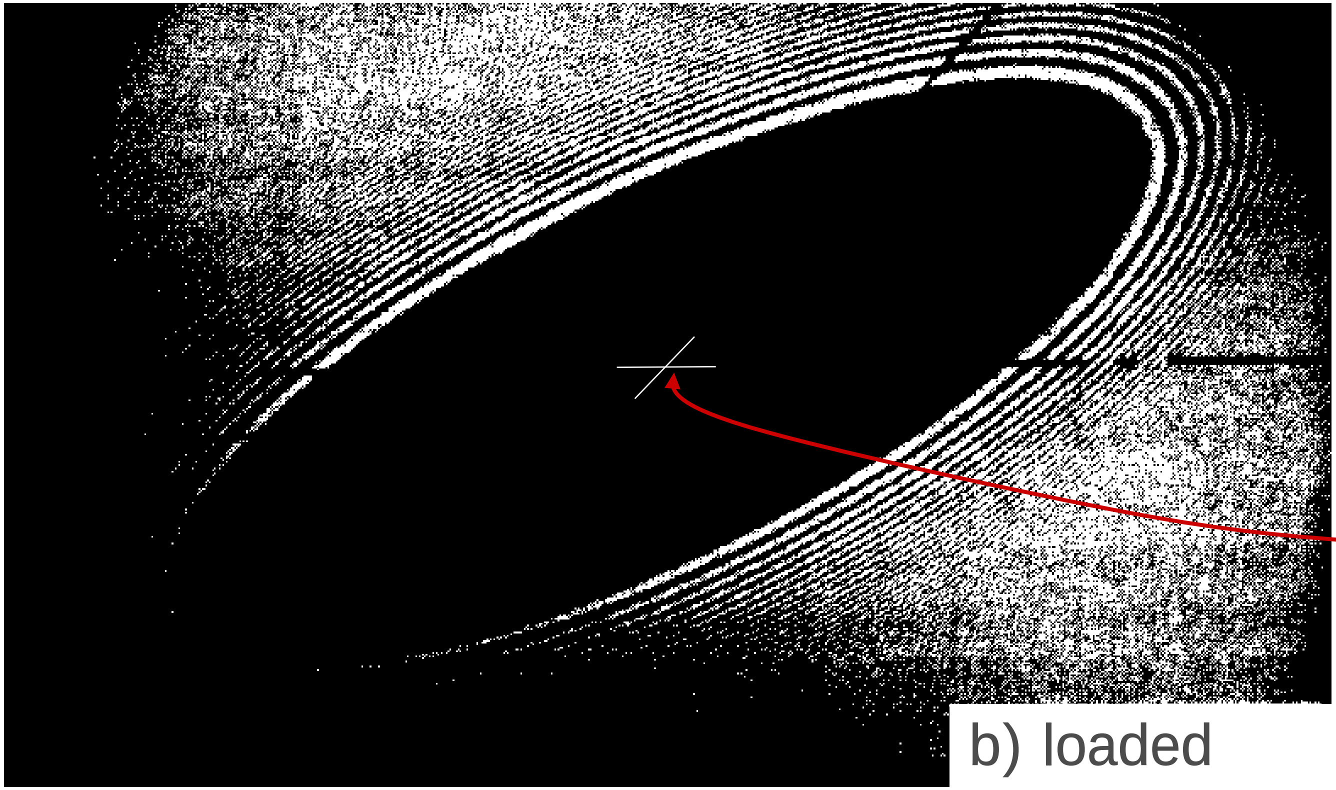


a) unloaded

Interference fringes of two transparent bodies in contact, unloaded

central "initial" contact point

To develop his theory Hertz used his observation of elliptical Newton's rings formed upon placing a glass sphere upon a lens. Observation suggests that when two bodies 1, 2 are pressed one against the other they will touch over a surface having an elliptical contour.



Interference fringes of two transparent bodies in contact, loaded

the black ellipse is very nearly the contact area

3. Contact geometry (5/17)

When a plano-convex lens with its convex surface is placed on a plane glass sheet, an air film of gradually increasing thickness outward is formed between the lens and the sheet.

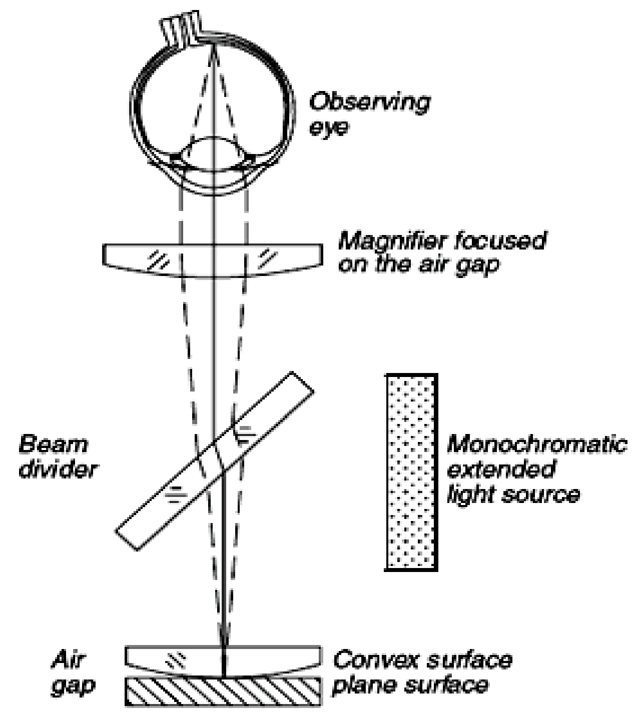
The thickness of film at the point of contact is zero. If monochromatic light is allowed to fall normally on the lens, and the film is viewed in reflected light, alternate bright and dark concentric rings are seen around the point of contact.

These rings were first discovered by Newton, that is why they are called Newton's rings. They are formed due to interference between the light waves reflected from the top and bottom surfaces of the air film formed between the lens and glass sheet, where an air film of varying thickness is formed .

When a light ray is incident on the upper surface of the lens, it is reflected as well as refracted.

When the refracted ray strikes the glass sheet, it undergoes a phase change of 180° on reflection.

Interference occurs between the two waves which interfere constructively if path difference between them is $(m+1/2)\lambda$ (m integer) and destructively if path difference between them is $m\lambda$ producing alternate bright and dark rings.



3. Contact geometry (6 / 17)

In general, the principal axes of bodies 1, 2 do not coincide, i.e., the two bodies are rotated one against the other about their common normal:

We shall call:

X_1, Y_1 : the unique couple of principal axes of body 1

X_2, Y_2 : the unique couple of principal axes of body 2

ϑ : rotation angle between X_1 and X_2

x, y : one of the infinite couples of non-principal axes of difference $z_1 - z_2$

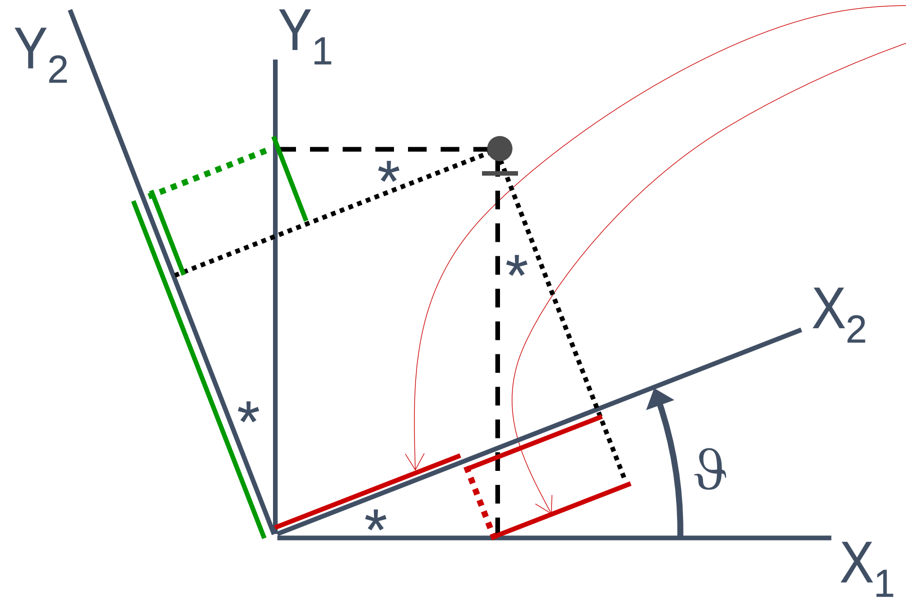
X, Y : the unique couple of principal axes of difference $z_1 - z_2$

Each surface 1 and 2 is expressed, in the simplest way, in its own principal system:

$$z_1 = \alpha_X X_1^2 + \alpha_Y {Y_1}^2$$

$$z_2 = -\left(\beta_X {X_2}^2 + \beta_Y {Y_2}^2\right)$$

In general, the two systems may not coincide, then an axis rotation is necessary to express the physical rotation between the two bodies:



$$\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} \boxed{\cos\vartheta} & \boxed{\sin\vartheta} \\ \boxed{-\sin\vartheta} & \boxed{\cos\vartheta} \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}$$

reminder : the inverse is equal to the transpose

$$\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix}$$

... which substituted into the quadratic form of body 2:

$$-z_2 = \{X_2 \;\; Y_2\}\begin{bmatrix}\beta_X & 0 \\ 0 & \beta_Y\end{bmatrix}\begin{Bmatrix}X_2 \\ Y_2\end{Bmatrix}$$

remember that : $\{b\} = [M]\{a\} \rightarrow \{b\}^T = \{a\}^T[M]^T$

yields :

$$-z_2 = \boxed{\{X_1 \;\; Y_1\}\begin{bmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{bmatrix}}\begin{bmatrix}\beta_X & 0 \\ 0 & \beta_Y\end{bmatrix}\boxed{\begin{bmatrix}\cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta\end{bmatrix}\begin{Bmatrix}X_1 \\ Y_1\end{Bmatrix}}$$

Then:

$$-z_2 = \{X_1 \;\; Y_1\}\left[\begin{array}{c|c}\beta_X\cos^2\vartheta + \beta_Y\sin^2\vartheta & (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta \\ \hline (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta & \beta_X\sin^2\vartheta + \beta_Y\cos^2\vartheta\end{array}\right]\begin{Bmatrix}X_1 \\ Y_1\end{Bmatrix}$$

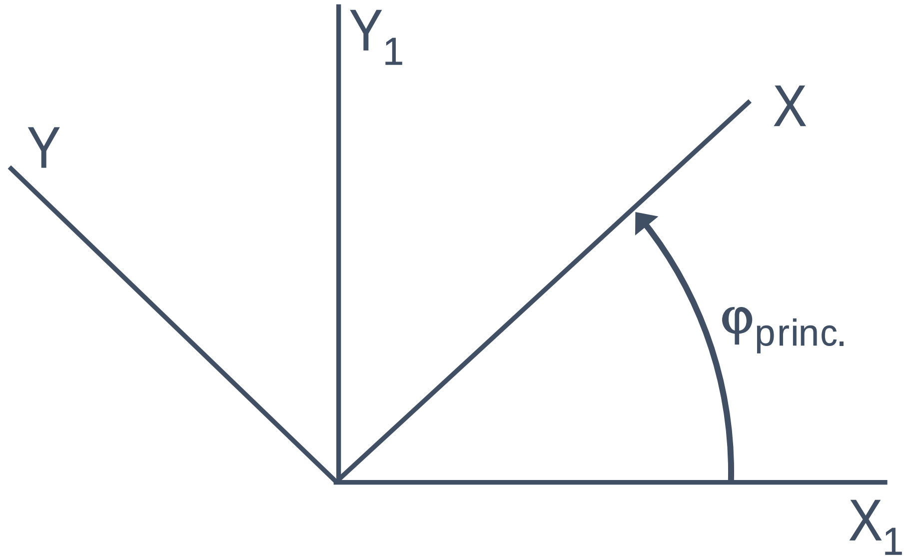
It is now possible to calculate the difference z_1-z_2 in the common reference system, that we have chosen to be $\{X_1, Y_1\}$:

$$(z_1 - z_2) =$$

$$= \{X_1 \;\; Y_1\} \left[\begin{array}{c|c} \alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta & (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta \\ \hline (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta & \alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta \end{array} \right] \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}$$

We now look for a rotated reference system (X,Y) where this matrix is diagonal.

The principal system is rotated by the angle $\varphi_{princ.}$ relative to System (X_1,Y_1).



* Once again: each couple (α_Y,$\mathbf{a}_Y$) , (β_X, β_Y) , (γ_X, γ_Y) is defined in its own principal axes

3. Contact geometry (10/17)

In the principal reference system X,Y the relative distance (z_1-z_2) is represented by:

$$(z_1 - z_2) = \{X \quad Y\} \begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

A diagonal matrix has the property that if (and only if) the vector (X,Y) to be transformed is on a **principal axis** X or Y, i.e., (X,0) or (0,Y), then the transformed vector:

$$\begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix}$$ is parallel to (X,Y); then there are two and only two cases

$\lambda_2=\gamma_Y$

$\lambda_1=\gamma_X$

$$\begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \lambda \begin{Bmatrix} X \\ Y \end{Bmatrix} \qquad \begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix} \begin{Bmatrix} X \\ 0 \end{Bmatrix} = \gamma_X \begin{Bmatrix} X \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix} \begin{Bmatrix} 0 \\ Y \end{Bmatrix} = \gamma_Y \begin{Bmatrix} 0 \\ Y \end{Bmatrix}$$

This vector property is independent of the reference system adopted to describe it, then it is valid also if axes (X,0), (0,Y) are expressed in (X_1, Y_1) coordinates:

$$\begin{bmatrix} \alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta & (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta \\ (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta & \alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}_{\text{princ.}} = \lambda \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}_{\text{princ.}}$$

3. Contact geometry (11/17)

The values of γ_X, γ_Y are then the solution of the eigenvalue problem:

$$\begin{bmatrix} (\alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta) - \lambda & (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta \\ (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta & (\alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta) - \lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}_{princ.} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where the (eigen)vector $\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}_{princ.}$ is either the principal axis X or Y

expressed in the reference system (X_1, Y_1), with eigenvalues for λ .

It is known that finding the solution requires the determinant of the matrix to be zero, then:

$\lambda^2 - \lambda\ \mathbf{I} + \mathbf{II} = 0$ (*)

where **I** and **II** are respectively the first and the second invariant of the matrix, and the roots for λ are the eigenvalues.

3. Contact geometry (12/17)

The first (I) and second (II) invariants of the matrix:

$$\begin{bmatrix} \alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta & (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta \\ (\beta_X - \beta_Y)\sin\vartheta\cos\vartheta & \alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta \end{bmatrix}$$

are:

$$\mathrm{I} = \left[\alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta\right] + \left[\alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta\right] =$$

$$= \alpha_X + \alpha_Y + \beta_X + \beta_Y$$

$$\mathrm{II} = \left(\alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta\right)\left(\alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta\right) -$$

$$-\left((\beta_X - \beta_Y)\sin\vartheta\cos\vartheta\right)^2$$

While for matrix: $\begin{bmatrix} \gamma_X & 0 \\ 0 & \gamma_Y \end{bmatrix}$ they are: $\mathrm{I} = \gamma_X + \gamma_Y$, $\mathrm{II} = \gamma_X \cdot \gamma_Y$ and, being invariants (for rotations of axes) they have the same value.

Therefore:

$$\mathbf{I} = \gamma_X + \gamma_Y = \alpha_X + \alpha_Y + \beta_X + \beta_Y$$

$$\mathbf{II} = \gamma_X \cdot \gamma_Y = \left(\alpha_X + \beta_X \cos^2\vartheta + \beta_Y \sin^2\vartheta\right)\left(\alpha_Y + \beta_X \sin^2\vartheta + \beta_Y \cos^2\vartheta\right) - \\ - \left(\left(\beta_X - \beta_Y\right)\sin\vartheta\cos\vartheta\right)^2$$

Remember that at this stage (above) we write their expressions in (X_1,Y_1) coordinates, and not in principal (X, Y) coordinates.

The algebraic second degree equation (*) of sl.11 has roots:

$$\begin{cases} \lambda_1 = \dfrac{\mathbf{I} + \sqrt{\mathbf{I}^2 - 4\,\mathbf{II}}}{2} \\ \lambda_2 = \dfrac{\mathbf{I} - \sqrt{\mathbf{I}^2 - 4\,\mathbf{II}}}{2} \end{cases} \Rightarrow \begin{matrix}\text{with sl.10} \\ \lambda_1, \lambda_2 \\ \text{are } \gamma_X, \gamma_Y\text{:}\end{matrix} \Rightarrow \begin{cases} \gamma_X = \dfrac{\mathbf{I} + \sqrt{\mathbf{I}^2 - 4\,\mathbf{II}}}{2} \\ \gamma_Y = \dfrac{\mathbf{I} - \sqrt{\mathbf{I}^2 - 4\,\mathbf{II}}}{2} \end{cases}$$

We may check that the argument $I^2 - 4\,II$ of the square root is:

$$I^2 - 4\,II =$$

$$= \alpha_X^2 + \alpha_Y^2 + \beta_X^2 + \beta_Y^2 + 2\alpha_X\alpha_Y + 2\alpha_X\beta_X + 2\alpha_X\beta_Y + 2\alpha_Y\beta_X + 2\alpha_Y\beta_Y + 2\beta_X\beta_Y -$$

$$- 4\alpha_X\alpha_Y - 4(\sin\vartheta)^2(\alpha_X\beta_X + \alpha_Y\beta_Y) - 4(\cos\vartheta)^2(\alpha_X\beta_Y + \alpha_Y\beta_X) - 4\beta_X\beta_Y =$$

$$= (\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2(\alpha_X\beta_Y + \alpha_Y\beta_X)\left(1 - 2(\cos\vartheta)^2\right) +$$

$$+ 2(\alpha_X\beta_X + \alpha_Y\beta_Y)\left(1 - 2(\sin\vartheta)^2\right) =$$

$$= (\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2(\alpha_X\beta_Y + \alpha_Y\beta_X)\left(1 + 2(\sin\vartheta)^2\right) +$$

$$+ 2(\alpha_X\beta_X + \alpha_Y\beta_Y)\left(1 - 2(\sin\vartheta)^2\right) =$$

$$= (\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2\left(1 - 2(\sin\vartheta)^2\right)(\alpha_X\beta_X + \alpha_Y\beta_Y - \alpha_X\beta_Y - \alpha_Y\beta_X) =$$

$$= (\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2\cos 2\vartheta\,(\alpha_X - \alpha_Y)(\beta_X - \beta_Y)$$

again written in the original (X_1, Y_1) coordinates,
while in principal (X, Y) coordinates it is:

$$I^2 - 4\,II = (\gamma_X + \gamma_Y)^2 - 4\gamma_X\gamma_Y = (\gamma_X - \gamma_Y)^2$$

Taking $\gamma_X + \gamma_Y$ from sl. 13 and extracting $\gamma_X - \gamma_Y$ from sl. 13, we finally conclude that:

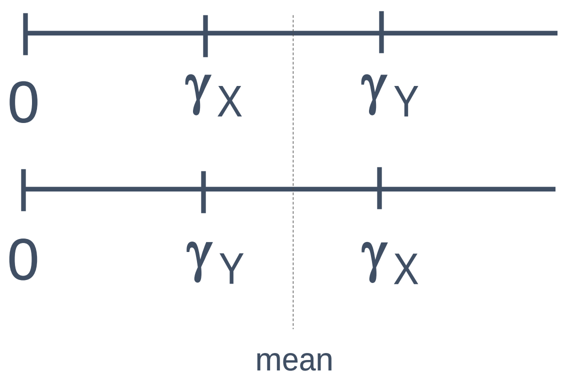
$$\begin{cases} \gamma_X + \gamma_Y = \alpha_X + \alpha_Y + \beta_X + \beta_Y \\ \gamma_X - \gamma_Y = \pm\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2\cos 2\vartheta(\alpha_X - \alpha_Y)(\beta_X - \beta_Y)} \end{cases}$$

Then the roots of sl. 13: are:

$$\begin{cases} \gamma_X = \dfrac{\mathrm{I} + \sqrt{\mathrm{I}^2 - 4\,\mathrm{II}}}{2} \\ \gamma_Y = \dfrac{\mathrm{I} - \sqrt{\mathrm{I}^2 - 4\,\mathrm{II}}}{2} \end{cases} \Rightarrow \begin{cases} \gamma_X = \dfrac{1}{2}\left[(\alpha_X + \alpha_Y + \beta_X + \beta_Y) \pm \sqrt{\ldots}\right] \\ \gamma_Y = \dfrac{1}{2}\left[(\alpha_X + \alpha_Y + \beta_X + \beta_Y) \mp \sqrt{\ldots}\right] \end{cases}$$

The consequence of: $\gamma_X \geq 0$, $\gamma_Y \geq 0$ (sl. 3) is that $\gamma_X + \gamma_Y$ is always higher than the absolute value of $\gamma_X - \gamma_Y$.

There are two possibilities ... ... and in either case:

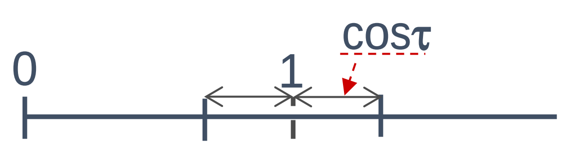


$$0 \leq \frac{\left|\gamma_X - \gamma_Y\right|}{\gamma_X + \gamma_Y} \leq 1$$

this fraction is given the name "**cos τ**", due to limits $0 \div 1$; it is used as a governing parameter

The meaning of $\cos\tau$, which will later be used as the independent parameter against which results of the theory will be plotted, is not misterious at all: it is simply an exotic way to name the relative deviation from the mean, roots-related:

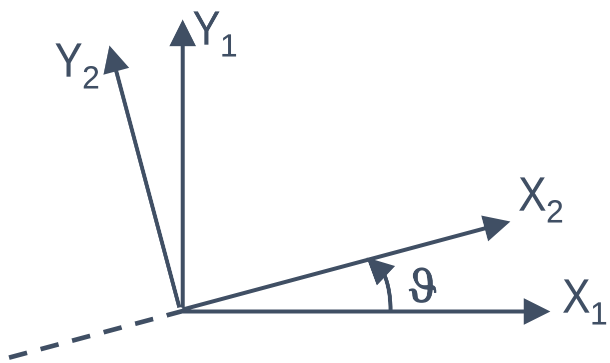
$$\lambda_1, \lambda_2 \equiv \gamma_X, \gamma_Y = \frac{(\gamma_X + \gamma_Y) \pm \left|\gamma_X - \gamma_Y\right|}{2} = \frac{(\gamma_X + \gamma_Y)}{2}\left[1 \pm \frac{\left|\gamma_X - \gamma_Y\right|}{\gamma_X + \gamma_Y}\right] = \frac{(\gamma_X + \gamma_Y)}{2}\left[1 \pm \cos\tau\right]$$

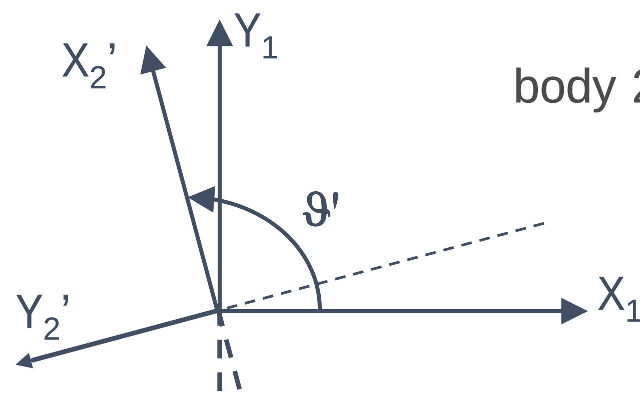


3. Contact geometry (17 / 17)

Optional slide

Note: in the formula for $\cos\tau$ we find, inside the square root, $\cos 2\theta$ because the name given to axes is arbitrary, then the two situations represented below are physically equivalent, i.e., they “must” produce the same value in the formula for $\cos\tau$.





$$\text{body 2} \Rightarrow \begin{cases} \beta_{X'} = \beta_Y \\ \beta_{Y'} = \beta_X \end{cases}$$

$$\begin{cases} \vartheta' = \vartheta + 90° \\ \cos 2\vartheta' = -\cos 2\vartheta \end{cases}$$

$$\cos\tau = \frac{|\gamma_X - \gamma_Y|}{(\gamma_X + \gamma_Y)} = \frac{\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_{X'} - \beta_{Y'})^2 + 2(\alpha_X - \alpha_Y)(\beta_{X'} - \beta_{Y'})\cos 2\vartheta'}}{\alpha_X + \alpha_Y + \beta_{X'} + \beta_{Y'}} =$$

$$= \frac{\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_Y - \beta_X)^2 - 2(\alpha_X - \alpha_Y)(\beta_Y - \beta_X)\cos 2\vartheta}}{\alpha_X + \alpha_Y + \beta_Y + \beta_X} =$$

$$= \frac{\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2(\alpha_X - \alpha_Y)(\beta_X - \beta_Y)\cos 2\vartheta}}{\alpha_X + \alpha_Y + \beta_X + \beta_Y}$$

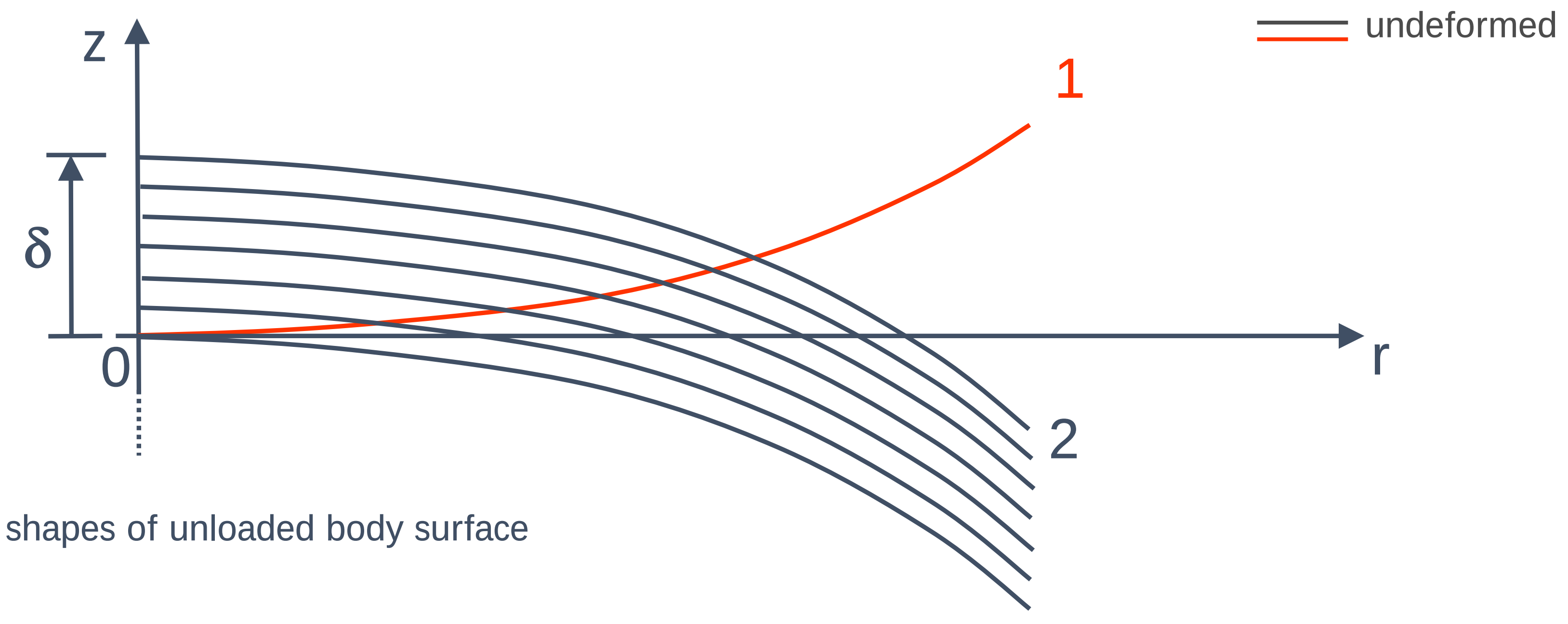
Sections 4 to 7 - Hertz's solution of the contact problem

Section 4 shows how the two bodies in contact will elastically deform one against the other when a force will press them. Advantage is taken of the fact that the relative distance z_1-z_2 is more easily expressed in terms of the relative principal axes found in Sect.3. Formulas and diagrams are given both for "point" contact and "line contact".

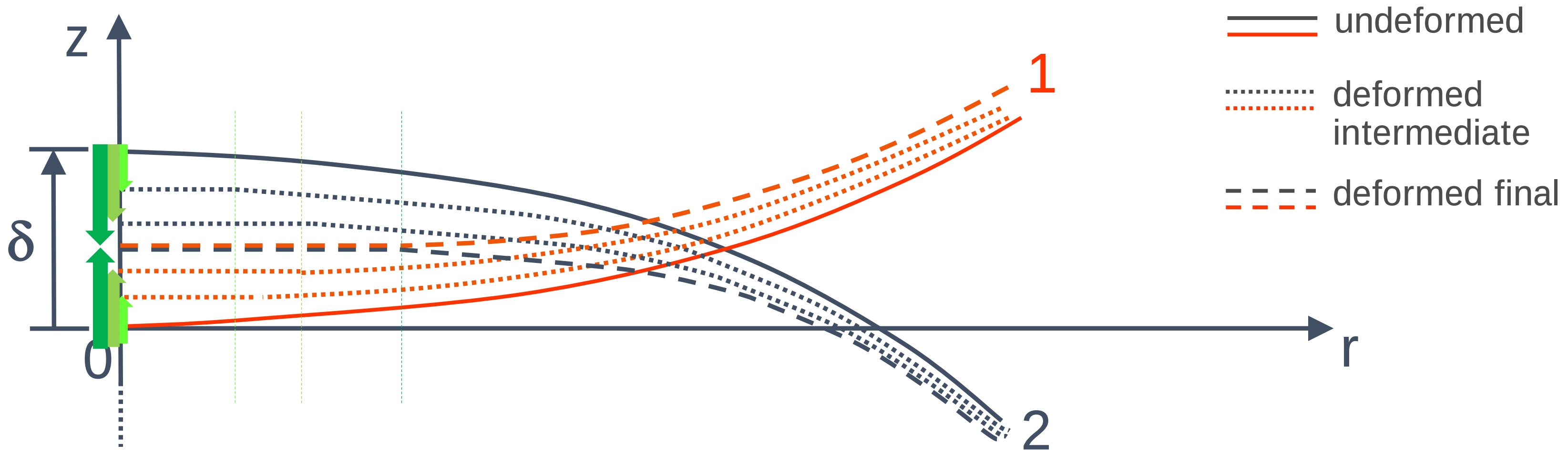
Sections 5 and 6 show how general formulas can be transformed for engineering use. Certain diagrams are presented which help to quickly investigate the orders of magnitude of contact pressures in simple reference cases.

Section 7 summarises reference formulas for practical use, and gives reference orders of magnitude of expected contact pressures.

"Approach" of body 2 against body 1 by a total value δ .



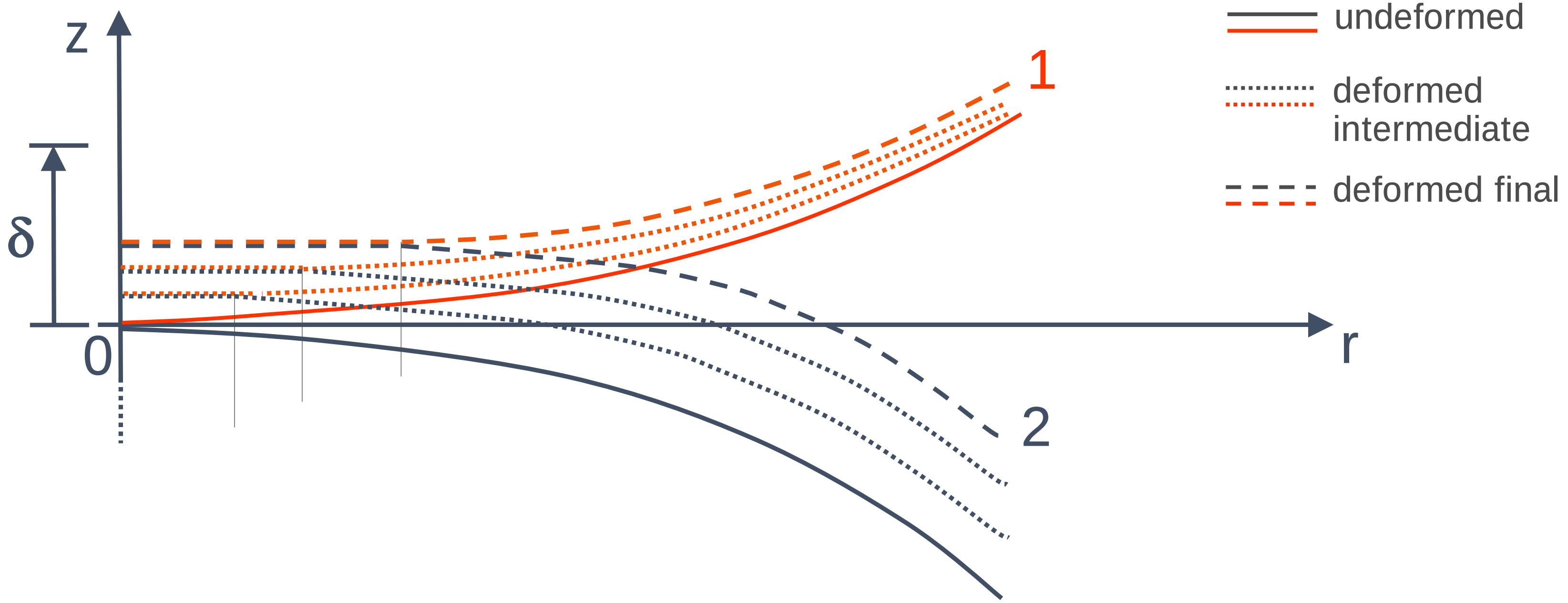
Elastic deformation of bodies to mutually adapt them: here we show the application of an increasing force (the resultant is represented) until the interference is fully recovered.



Stresses and strains disappear at great distance from the contact zone; where by "great" we mean much larger than the (greater) dimension of the contact area.
We may choose to take the two "far enough" points to be the centres of the circles which locally approximate the surface cross-sections (see Sect. 6 sl. 5).

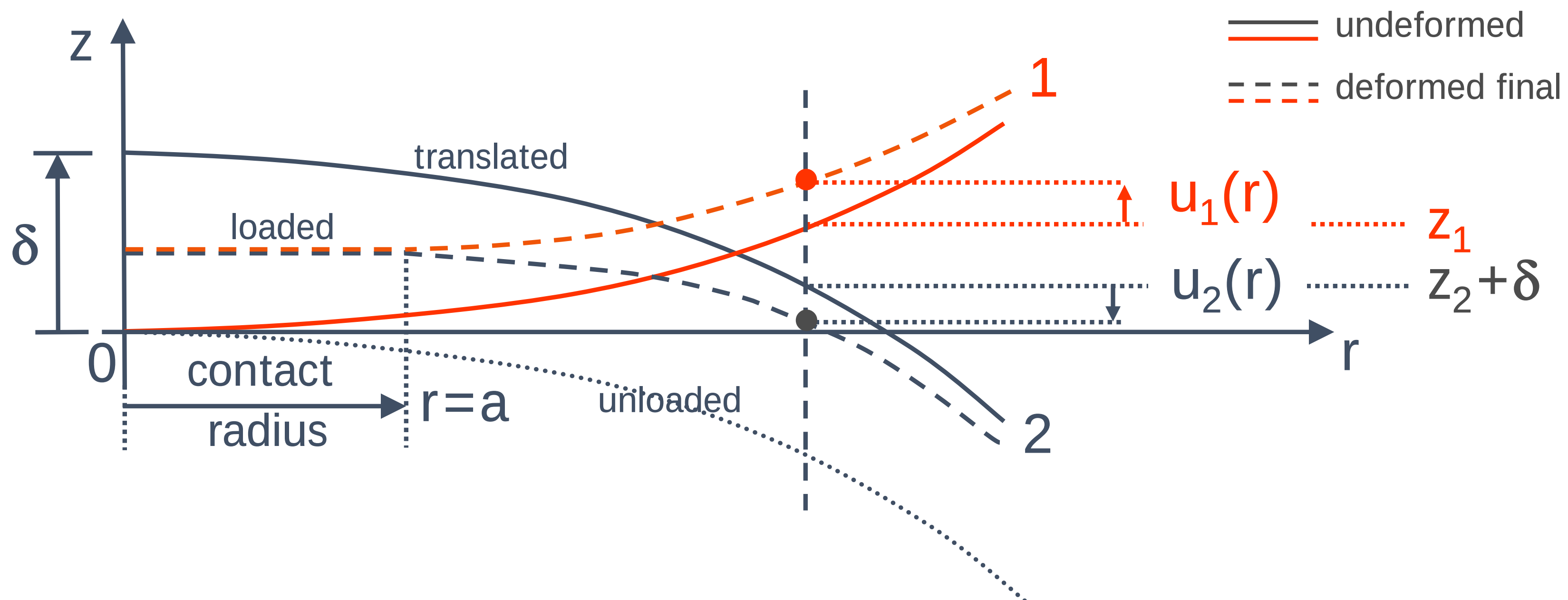
In reality the loading process takes place gradually.

Here we show (forces omitted) the progressive generation of the displacement and of the contact area (which increases with increasing force).



Stresses and strains disappear at great distance from the contact zone; where by “great” we mean much larger than the (greater) dimension of the contact area.
We may choose to take the two “far enough” points to be the centres of the circles which locally approximate the surface cross-sections (see Sect. 6 sl. 5).

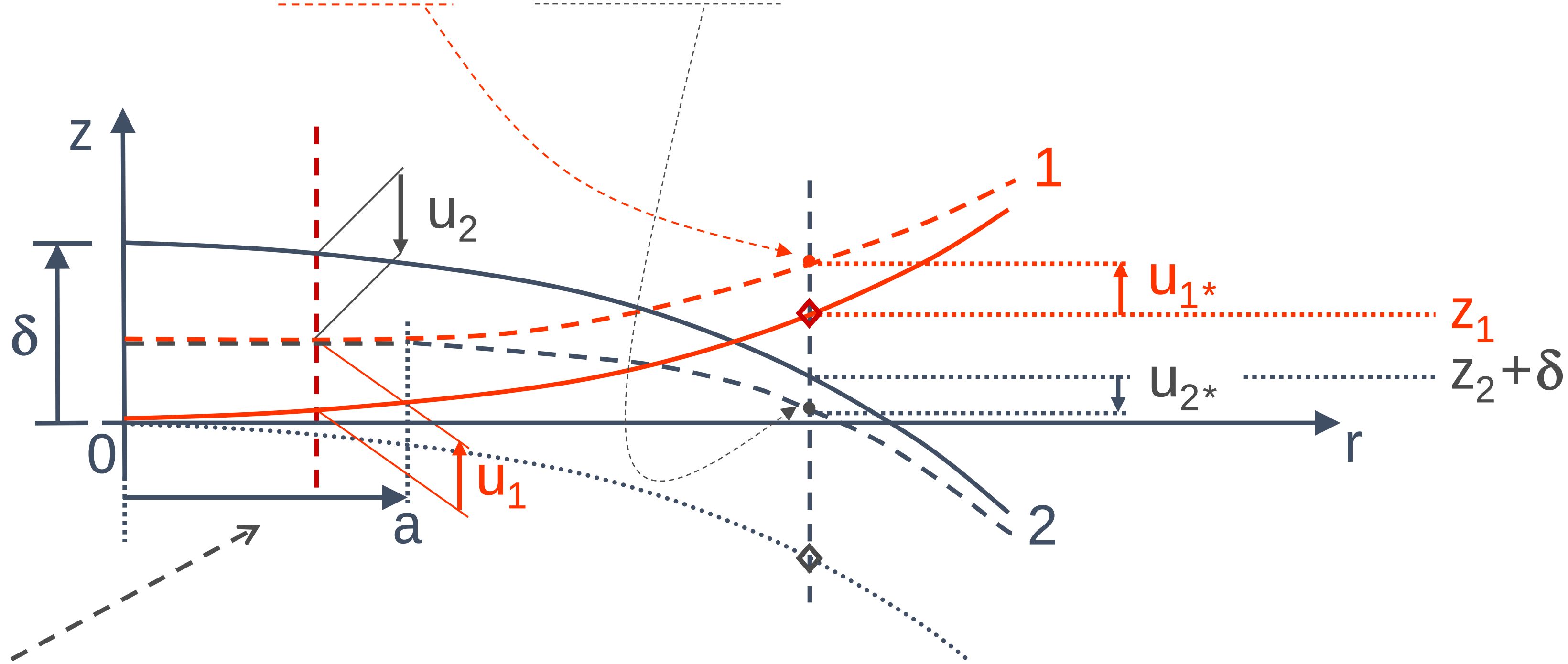
In summary: the “approach” of the two bodies is defined as δ, the relative rigid movement of two points far enough from the zone where local hertzian stresses and strains change the undeformed shape.



At any radius r, the point of the undeformed surface z_1 moves to z_1+u_1; while z_2 to comply with its rigid “approach” δ (which is an interference) without compenetration moves elastically to $z_2+\delta-u_2$.

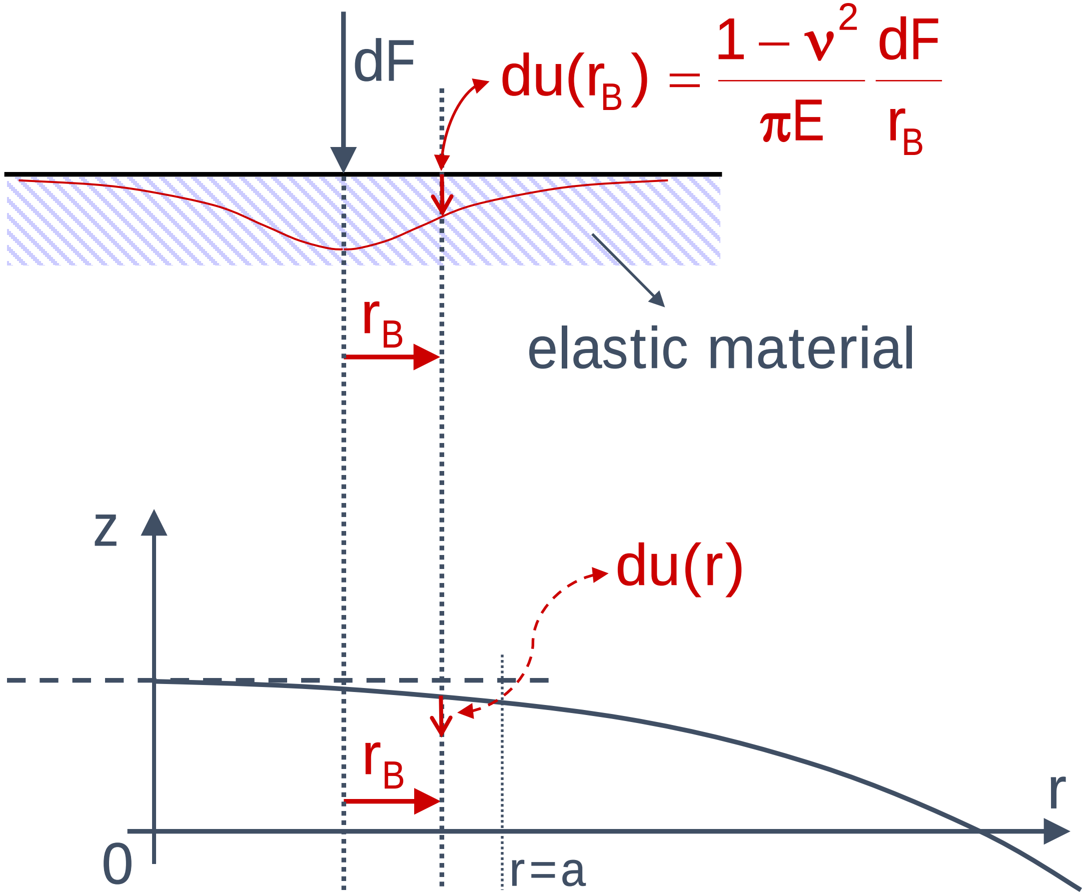
Displacements u_1 e u_2 depend on the distribution of contact pressure p.

When two corresponding points (same r) are outside (* : r≥a) the contact zone: $z_1 + u_{1*} > z_2 + \delta - u_{2*} \;\rightarrow\; u_{1*} + u_{2*} > \delta - (z_1 - z_2)$



Where, on the contrary, there is contact, i.e. r≤a: $u_1 + u_2 = \delta - (z_1 - z_2)$
with $(z_1\text{-}z_2)$ as from Sect. 3 sl. 3, 10 .

Hertz started from Boussinesq* solution of the “elastic half-space” loaded by a concentrated force:



*J. Boussinesq, Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques, Gauthiers-Villars, Paris, 1885

In our case the elastic half space is limited in any section plane by a curve of local radius R; however, as a<<R, the curve can be approximated by its tangent.

So the solution is the one valid for the plane half space!

The figure on the left shows a special case where force and displacement are in the same plane (r,z).

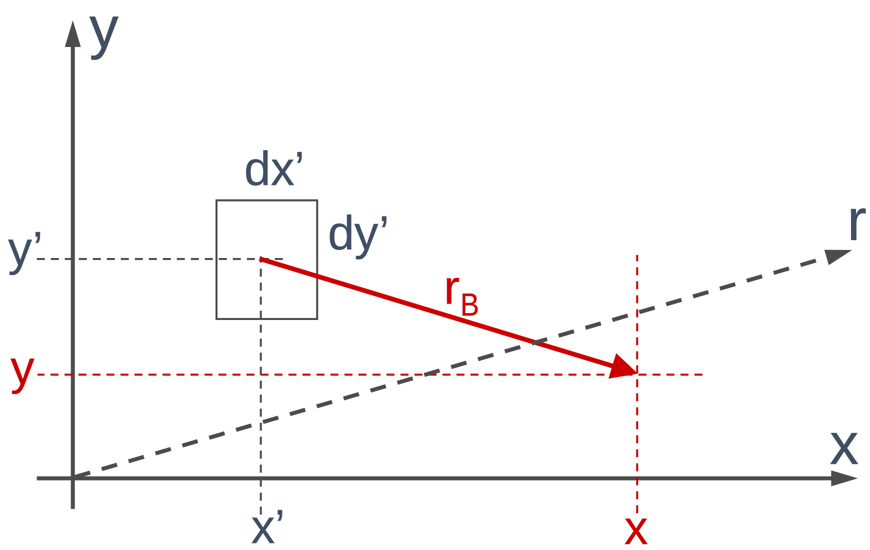
On each area element dA=dx' dy' inside the contact surface at a point (x',y'), an infinitesimal force dF is applied:

$$dF = p(x',y')\, dx'dy'$$

which produces at any other point of coordinates (x,y) a contribution to displacements:

$$du(x,y) = \frac{1-\nu^2}{\pi E}\frac{1}{r_B}\cdot p(x',y')\, dx'dy'$$

$$r_B = \sqrt{(x-x')^2+(y-y')^2}$$



The figure on the left shows the force application point (x',y') and the displacement point (x,y) projected on the tangent plane .

If the contact area (and its boundary) were known, and the pressure distribution function p(x',y') were known as well, we could calculate:

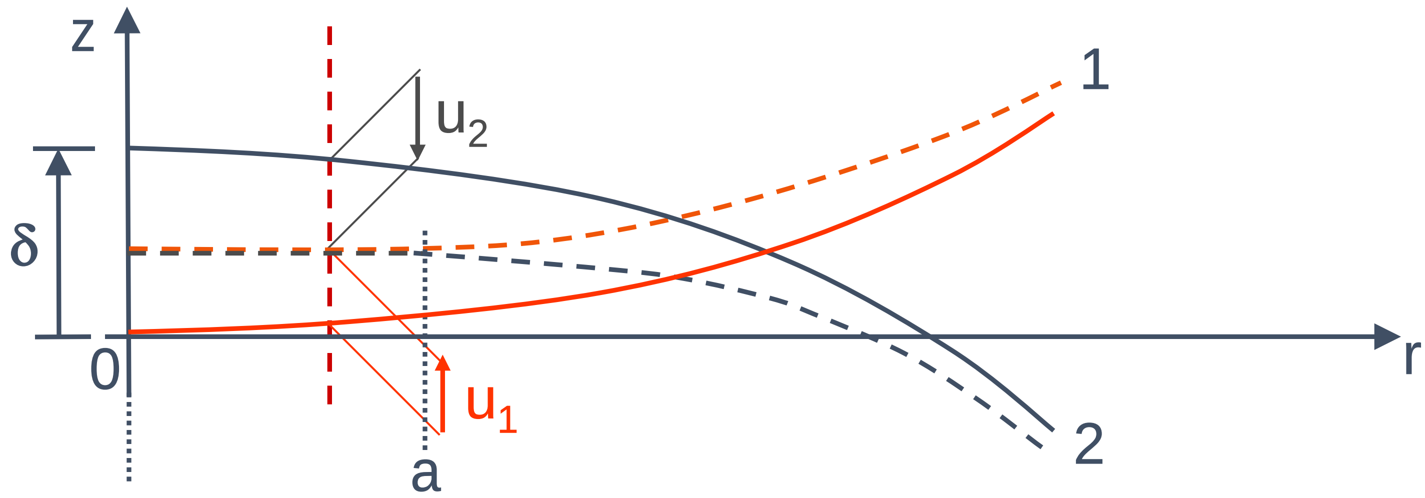
$$u(x,y) = \frac{1-\nu^2}{\pi E} \iint\limits_{\text{contact area}} \frac{p(x',y')}{r_B}\, dx'\, dy'$$

The contact pressure is the same for the two bodies, displacements u were defined positive along the pressure orientation, then the ratio of displacements u_1 and u_2 is:

$$\frac{u_1(x,y)}{u_2(x,y)} = \frac{\dfrac{1-\nu_1{}^2}{E_1}}{\dfrac{1-\nu_2{}^2}{E_2}}$$

Being the solution related to the plane half spaces, it follows that for equal materials the surface displacements will be the same, and so strain and stresses inside !!!

Therefore in the deformed zone in contact, $0 \le r \le a$, the distance $u_1 + u_2 = \delta - (z_1 - z_2)$ between the two undeformed curves is split into two parts in the proportion u_1/u_2 (=1 for equal materials):



In the contact zone:

in principal X,Y axes for the relative curvature

$$\underbrace{\frac{1}{\pi}\left(\frac{1-\nu_1^{\;2}}{E_1}+\frac{1-\nu_2^{\;2}}{E_2}\right)\iint \frac{p(X',Y')}{r_B}\,dX'dY'}_{u_1+u_2} = \underbrace{\delta - \gamma_X X^2 - \gamma_Y \cdot Y^2}_{\delta-(z_1-z_2)}$$

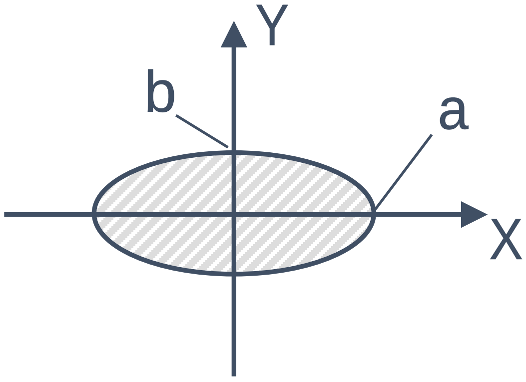
Solving this problem is difficult, as both pressure and contact area boundary are to be found. Hertz found that with contact force F:

$$F = \iint_{area} p(x,y) \cdot dx \cdot dy \qquad \text{then:} \qquad p = \frac{3}{2}\frac{F}{\pi ab} \cdot \sqrt{1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2}}$$

with: a, b semi axes of the elliptic contact surface.

The pressure distributed on the contact area is zero on its boundary:

$$1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 0$$



Moreover

$$p_{mean} = \frac{F}{\pi ab} \quad ; \quad p_{max} = \frac{3}{2}\frac{F}{\pi ab}$$

The values of semi axes depends on the solution of an integral equation, which is normally given in numerical form through tables or diagrams. They are given in the form:

$$a = a^* \cdot f \qquad b = b^* \cdot f$$

$$\text{with: } f = \sqrt[3]{\frac{3}{2} \cdot \frac{F}{2(\alpha_X + \alpha_Y + \beta_X + \beta_Y)} \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right)}$$

where parameters a^* and b^* are non dimensional coefficients.

The approach δ due to Hertz deformations is:

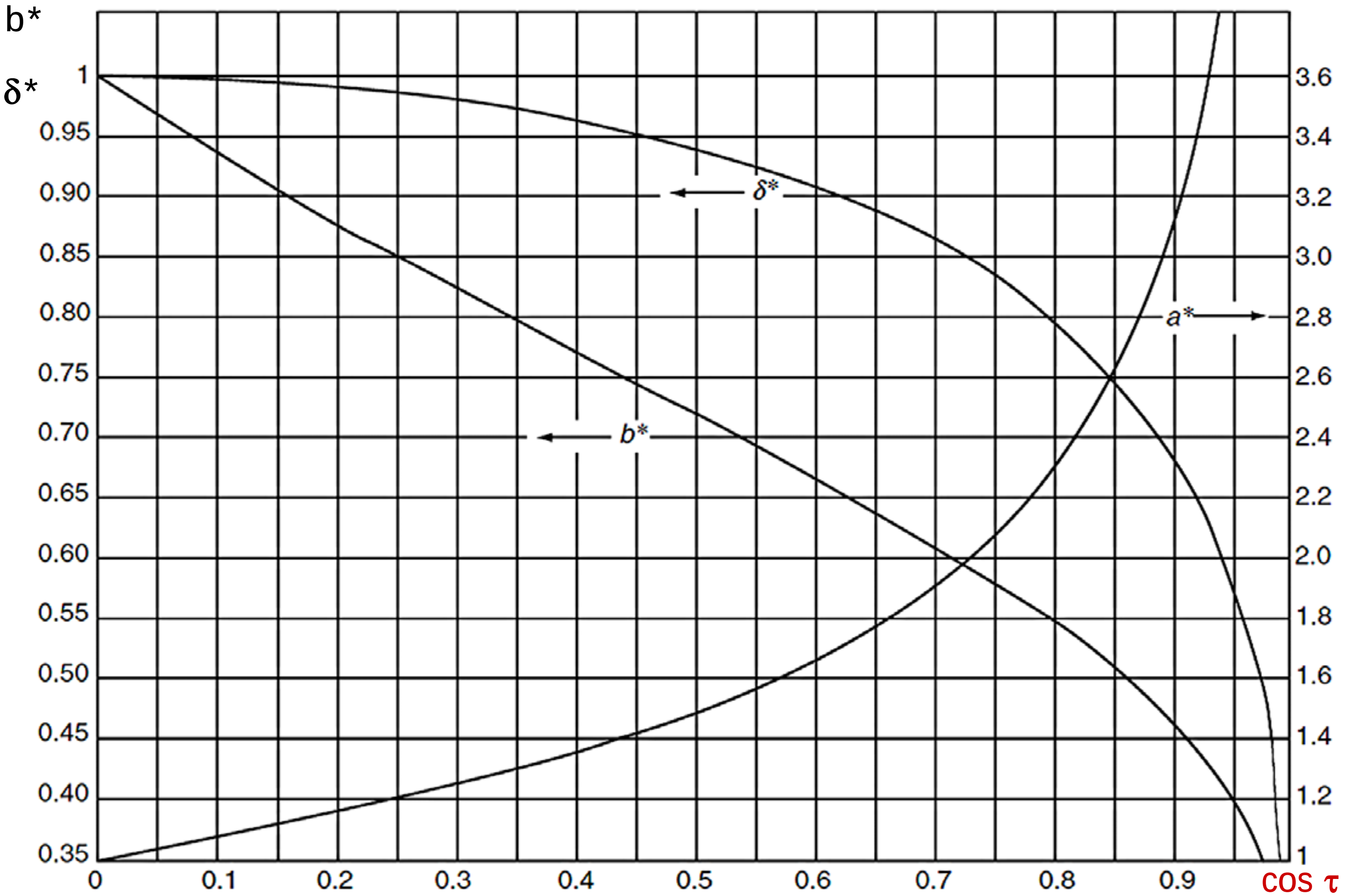
$$\delta = \delta^* (\alpha_X + \alpha_Y + \beta_X + \beta_Y) f^2$$

with δ^* a non dimensional coefficient.

Coefficients a*, b*, δ* are expresses as functions of **cos τ ,** usually given in a diagram like the one in next slides:

Diagrams a^*, b^*, δ^*

$$\cos\tau = \frac{\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2(\alpha_X - \alpha_Y)(\beta_X - \beta_Y)\cos 2\vartheta}}{\alpha_X + \alpha_Y + \beta_X + \beta_Y}$$

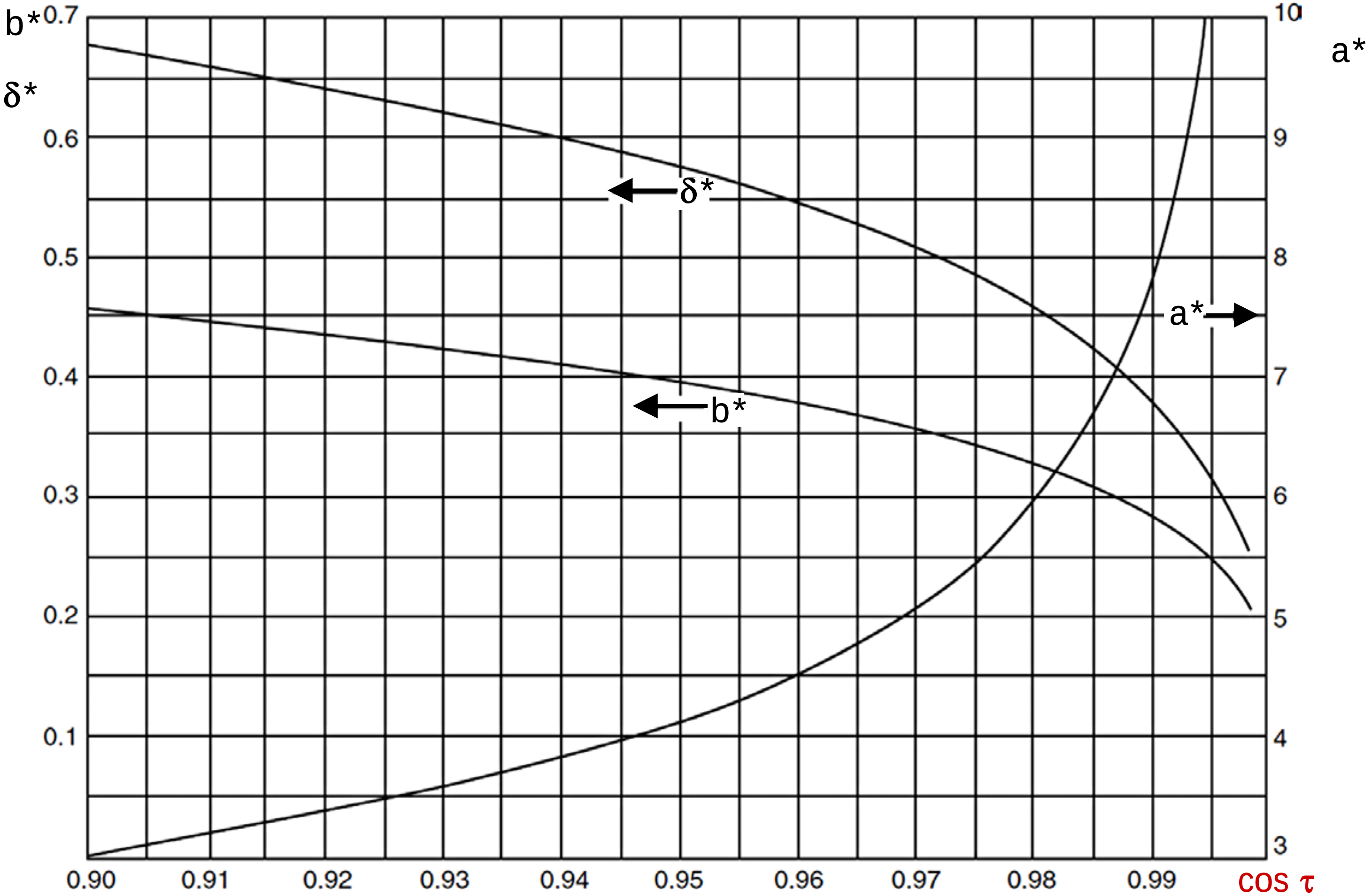


Remark:

these coefficients have been defined so that in the case of sphere on sphere contact, i.e. for $\cos\tau = 0$, they are all equal to 1

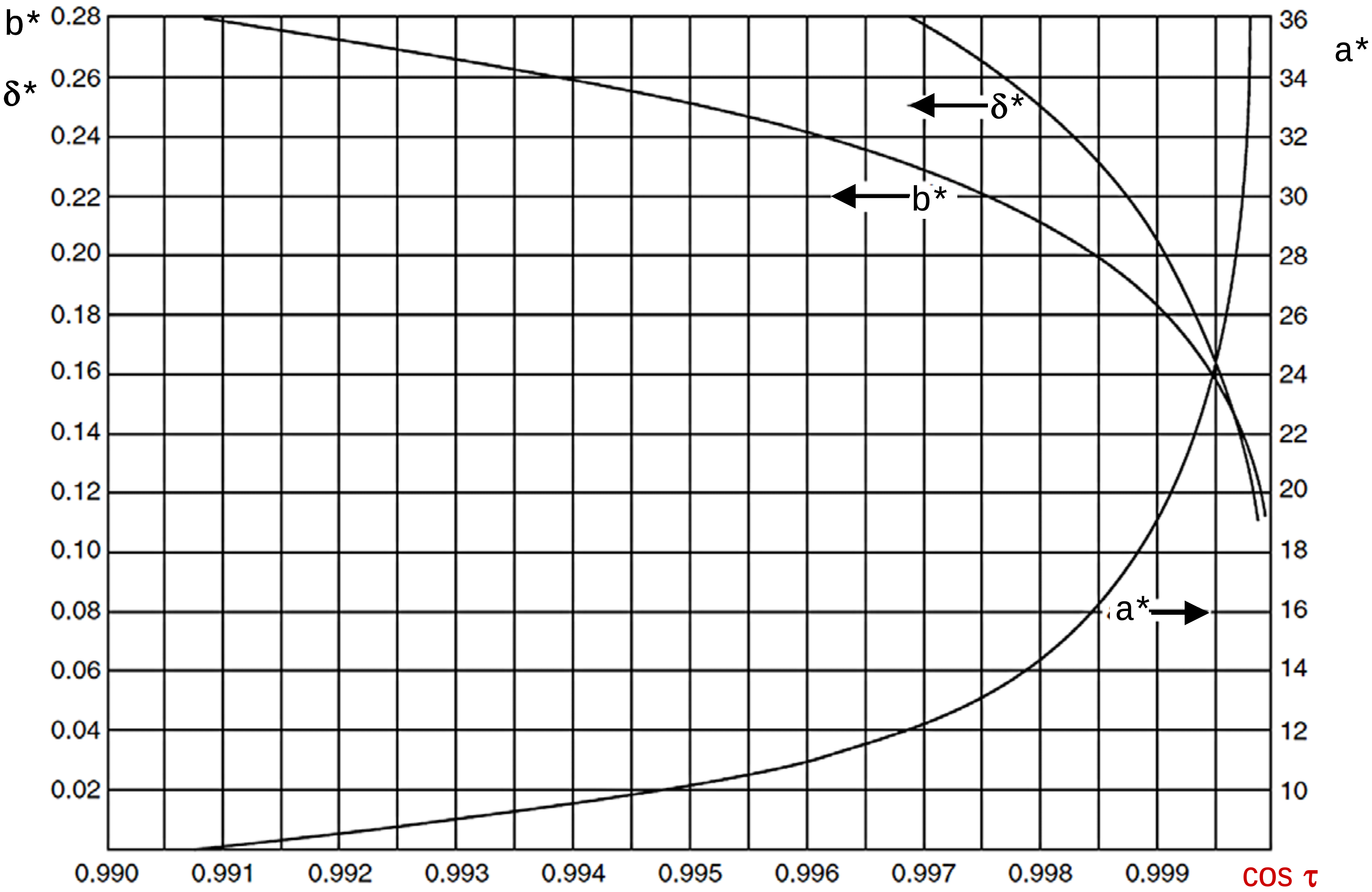
(from: Harris T.A., Kotzalas M.N., *Rolling Bearing Analysis V Ed. - Advanced Concepts of Bearing Technology*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2007)

Detail for 0.9≤ cosτ <1



(from: Harris T.A., Kotzalas M.N., *Rolling Bearing Analysis V Ed. - Advanced Concepts of Bearing Technology*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2007)

Detail for 0.99≤ cosτ <1



(from: Harris T.A., Kotzalas M.N., *Rolling Bearing Analysis V Ed. - Advanced Concepts of Bearing Technology*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2007)

4. The elastic contact solution (12/14)

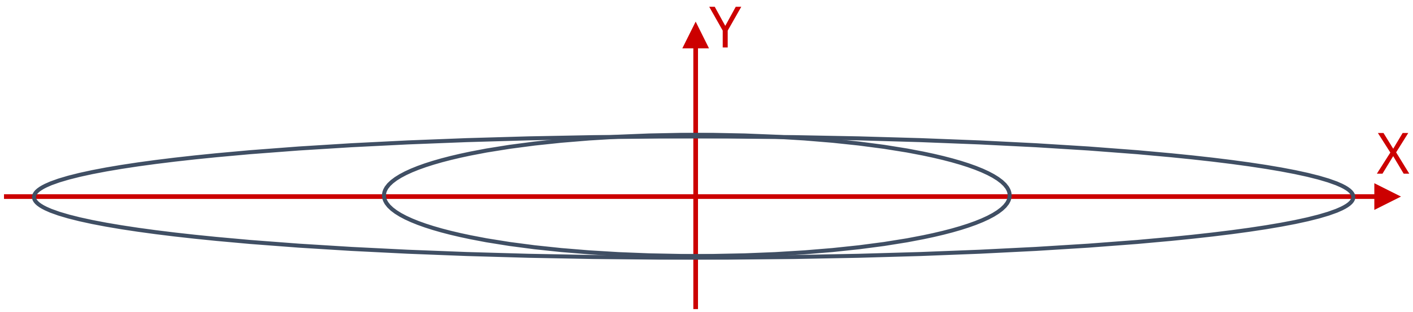
Remark 1 - These formulas and the related diagrams cannot be extended to the case of contact of cylindrical bodies with parallel axes. In this case

$$\alpha_X = \beta_X \to 0 \qquad \theta = 0 \text{ , } \cos\tau \to 1$$

then

$$a^* \to \infty \text{ ; } b^* \to 0$$

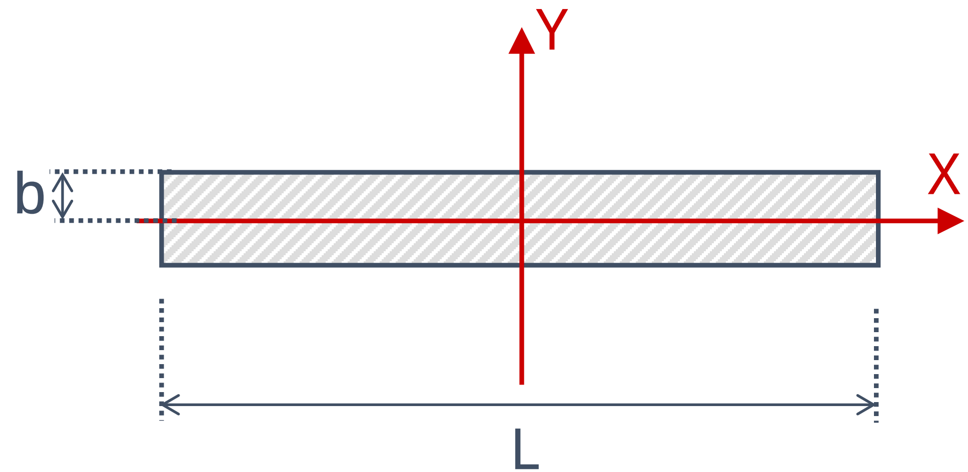
i.e. the contact area is finite because one semi axis tends to infinity and the other to zero, but it requires that the major axis to be infinite while in cylindrical contacts it is limited.



Remark 2 - Appendix I, Sect.10, gives practical formulas to calculate a^*, b^* and δ^*.

However, practical cylindrical contacts have a finite length L, and a finite semi-width which we call b, given by the formula below.

Formulas for the cylindrical contact are:

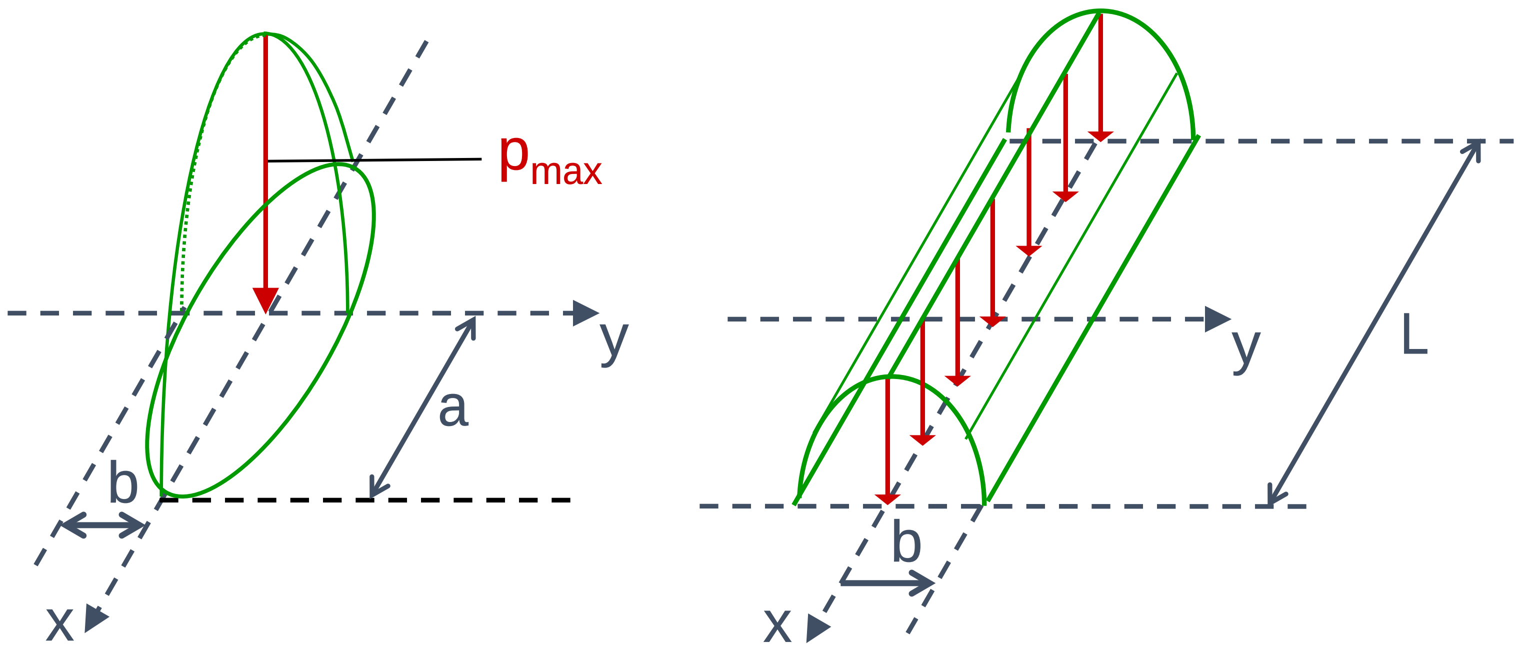


$$p = \frac{2F}{\pi L b} \cdot \sqrt{1 - \left(\frac{Y}{b}\right)^2}$$

$$p_{max} = \frac{2F}{\pi L b}$$

$$b = \sqrt{\frac{4}{\pi}\frac{F}{L}\frac{1}{L \cdot 2(\alpha_X + \beta_X)}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)}$$

$$p_{mean} = \frac{F}{2Lb} \Rightarrow p_{max} = \frac{4}{\pi} \cdot p_{mean}$$



In summary, in the figures above we see the stress (pressure) distributions on the contact surface for “point” and “line” contact.

It is now convenient to "engineer" formulas, i.e., write them with the traditional "workshop" symbols and for the most frequent case of **steel on steel** contact. Starting from the general formulas:

$$\begin{cases} a = a^* \cdot f \\ b = b^* \cdot f \end{cases} \qquad f = \sqrt[3]{\frac{3}{2} \cdot \frac{F}{2(\alpha_X + \alpha_Y + \beta_X + \beta_Y)} \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right)}$$

$$\delta = \delta^* (\alpha_X + \alpha_Y + \beta_X + \beta_Y)\ f^2$$

we can introduce for shorter formulas: $(\alpha_X + \alpha_Y + \beta_X + \beta_Y) = \sum \frac{1}{\tilde{d}}$

For the special case of steel: $E_1 = E_2 = 2 \cdot 10^5$ MPa, $\nu = 0.3$
we get:

$$f = 1{,}90 \cdot 10^{-2} \sqrt[3]{\frac{F}{\sum \frac{1}{\tilde{d}}}} \Rightarrow \begin{cases} \delta = \delta^*\, 3{,}60\ 10^{-4} \sqrt[3]{F^2 \sum \frac{1}{\tilde{d}}} \\ p_{max} = \frac{3}{2} \frac{F}{\pi ab} = \frac{3}{2} \frac{F}{\pi a^* b^* f^2} = \frac{3}{2\pi a^* b^*} \frac{\sqrt[3]{\left(\sum \frac{1}{\tilde{d}}\right)^2}}{3{,}60\ 10^{-4}} \sqrt[3]{F} \end{cases}$$

5. Engineered formulas for sphere on sphere (2/9)

any material

It is interesting to have formulas for the case **sphere/on sphere**, for **any material** (remark, equal for the two spheres !):

$$\cos\tau = 0 \Rightarrow a^* = b^* = \delta^* = 1$$

$$a = b = f = \frac{0{,}880}{\sqrt[3]{E}\,\sqrt[3]{\left(1 \pm \frac{d}{D}\right)}}\sqrt[3]{F\,d} \Rightarrow \begin{cases} \delta = f^2 = \dfrac{1{,}55}{\sqrt[3]{E^2}}\sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)}\sqrt[3]{\dfrac{F^2}{d}} \\[2ex] p_{max} = 0{,}616\,\sqrt[3]{E^2}\,\sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)^2}\sqrt[3]{\dfrac{F}{d^2}} \end{cases}$$

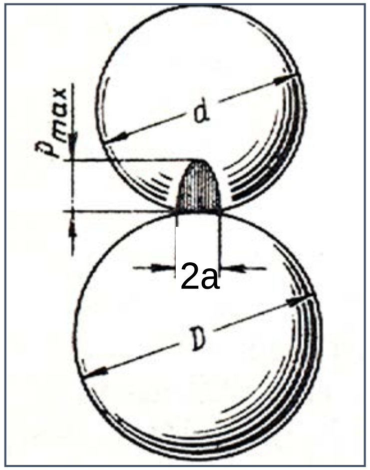
All constants in these formulas are non-dimensional, therefore they are independent of the system of units.

steel on steel

In this page formulas are given for the special, frequent and of great technical interest:

steel on steel, **sphere/on sphere**: $\cos\tau = 0 \Rightarrow a^* = b^* = \delta^* = 1$

$$\alpha_x = \alpha_y = \frac{1}{d} \qquad \beta_x = \beta_y = \frac{1}{D} \qquad \Rightarrow \sum \frac{1}{\tilde{d}} = \frac{2}{d}\left(1 \pm \frac{d}{D}\right)$$



$$a = b = f = \frac{1{,}51 \cdot 10^{-2}}{\sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)}} \sqrt[3]{F\,d}$$

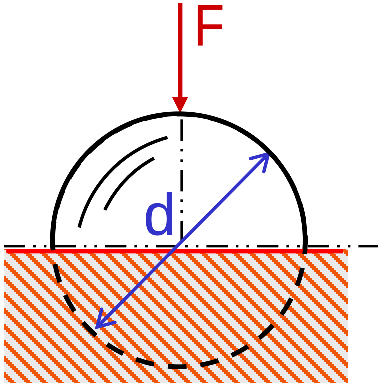
$$\begin{cases} + : D \text{ convex} \\ - : D \text{ concave} \end{cases}$$

$$p_{max} = 2107 \sqrt[3]{\left(1 \pm \frac{d}{D}\right)^2} \sqrt[3]{\frac{F}{d^2}} \qquad \delta = 4{,}53 \cdot 10^{-4} \sqrt[3]{\left(1 \pm \frac{d}{D}\right)} \sqrt[3]{\frac{F^2}{d}}$$

Caution!! All constants in these formulas are dimensional because they depend on the material. Therefore they are valid for forces in N and dimension in mm !!

Following Orlov* it is now convenient to define a reference pressure "p_{ref}" which may take into account the applied load and the dimension of the body to which it is applied.

$$p_{ref} = \frac{F}{\frac{\pi}{4} \cdot d^2}$$



As shown on the left, p_{ref} it is the mean pressure through the central, i.e. the greatest, cross section of the sphere of diameter d:

$$A_{ref} = \pi\, d^2/4$$

i.e., also the mean pressure on a sliding pad having the same dimension of the sphere;

p_{ref} is the lowest possible mean pressure, given the body dimension d and the applied load F.

It will be seen in the next slide that p_{ref} is the parameter which governs p_{mean} and p_{max} on it, and the approach δ.

* P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

steel on steel

The mean pressure p_{mean} on the contact surface is:

$$p_{mean} = \frac{F}{\pi ab} = \frac{F\left(\frac{1}{d} \pm \frac{1}{D}\right)^{2/3}}{\pi\left(1{,}51 \cdot 10^{-2}\right)^2 F^{2/3}} = \frac{\sqrt[3]{\pi/4}}{\pi\left(1{,}51 \cdot 10^{-2}\right)^2} \sqrt[3]{p_{ref}} \sqrt[3]{\left(1 \pm \frac{d}{D}\right)^2}$$

... and the approach is:

$$\delta = 3{,}60\,10^{-4} \sqrt[3]{F^2 \sum \frac{1}{\tilde{d}}} = 4{,}53 \cdot 10^{-4} \sqrt[3]{\frac{F^2}{d}\left(1 \pm \frac{d}{D}\right)}$$

then:

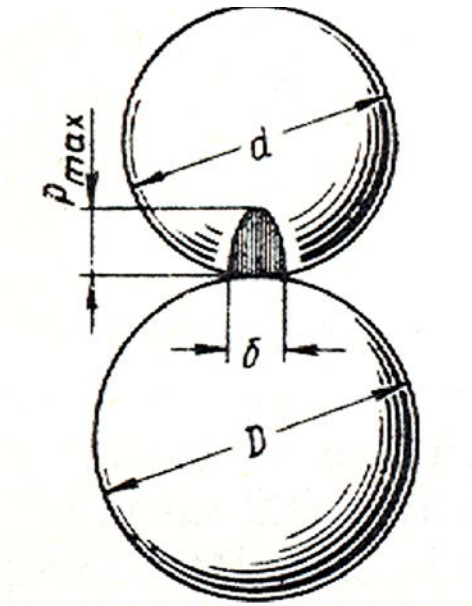
$$p_{max} = \frac{3}{2} p_{mean} = 1930 \ \underline{\sqrt[3]{P_{ref}}} \ \underline{\sqrt[3]{\left(1 \pm \frac{d}{D}\right)^2}} \qquad \frac{\delta}{d} = 3{,}86 \cdot 10^{-4} \ \underline{\sqrt[3]{p_{ref}^2}} \ \underline{\sqrt[3]{\left(1 \pm \frac{d}{D}\right)}}$$

load and sphere-size

geometrical ratio factor

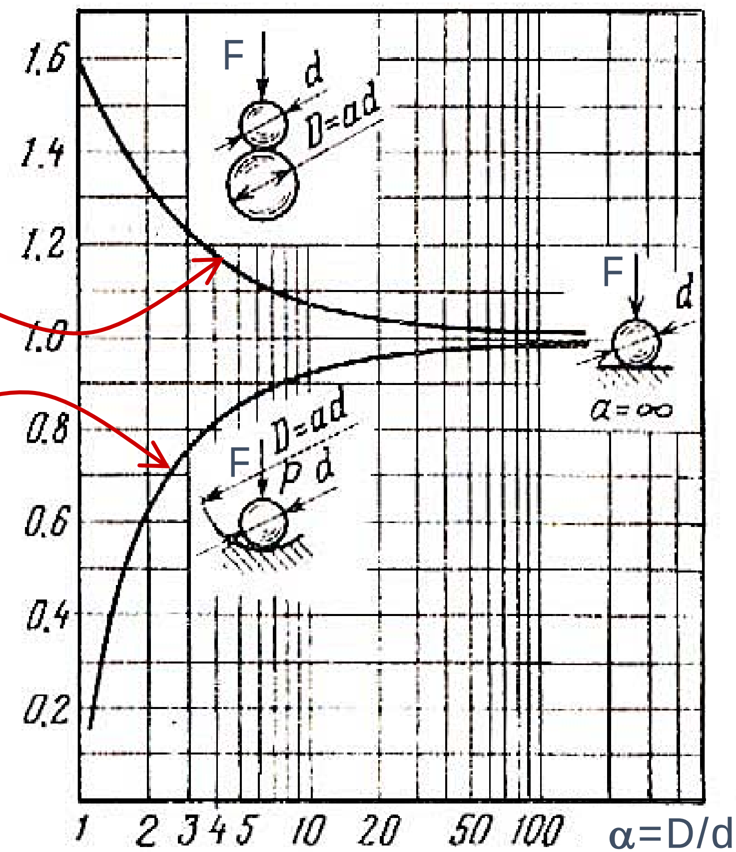
The figures in the next slides will analyse the contribution of the different factors into play .

The geometrical ratio factor:



$$\sqrt[3]{\left(1+\frac{d}{D}\right)^2}$$

$$\sqrt[3]{\left(1-\frac{d}{D}\right)^2}$$



This shows the effect of D/d for a given p_{ref}, i.e., F and d.

The figure shows that the worst case is when two spheres (convex-convex contact) have equal diameter, D=d.

figures from: P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

5. Engineered formulas for sphere on sphere (7 / 9)

The diagram on the right allows a quick estimate of stresses and areas.

Example: for p_{ref}=1 and sphere on plane contact, $\alpha=\infty$, $p_{max} \approx 2000\ p_{ref}$ (rounded, red point ●) the contact area is loaded by a mean pressure at 2/3 p_{max} , so it is over 1300 times smaller than the area of the sphere central cross section; then the contact radius is about 36 times smaller than the sphere radius.

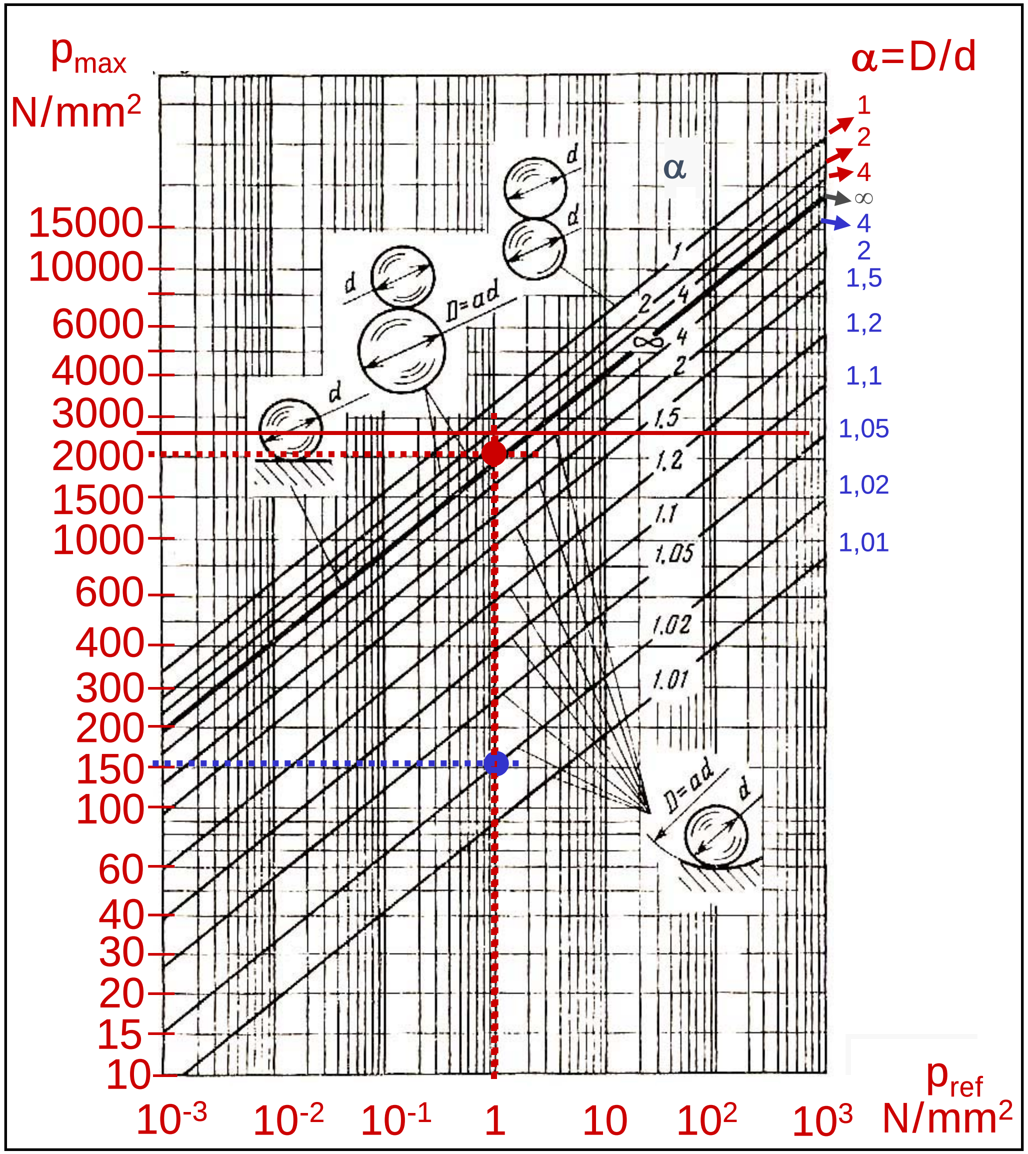


figure from: P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

5. Engineered formulas for sphere on sphere (8 / 9)

Example: for p_{ref}=1 and sphere on plane contact, $\alpha=\infty$, $p_{max}\approx 2000\ p_{ref}$ (rounded, point ●)
the contact area is loaded by a mean pressure at 2/3 p_{max} , so it is over 1300 times smaller than the area of the sphere central cross section; then the contact radius is about 36 times smaller than the sphere radius.

$\Rightarrow$

We may come to the same conclusion through the following formula for the Hertz contact area:

$A_H=\pi a^2$ vs. A_{ref} i.e.:

$$\frac{A_H}{A_{ref}} = 7{,}72 \cdot 10^{-4} \frac{\sqrt[3]{p_{ref}^2}}{\sqrt[3]{\left(1 \pm \frac{d}{D}\right)^2}}$$

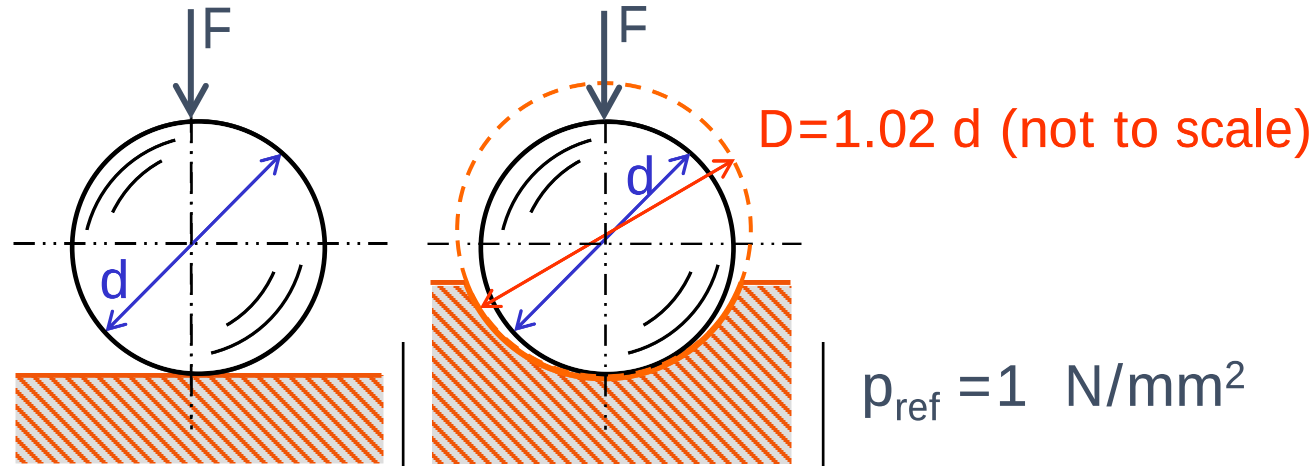
If we take p_{ref}=1 and d/D=0 :

$$\frac{A_H}{A_{ref}} = 7{,}72 \cdot 10^{-4}$$

then: $A_H= A_{ref}$ /1295

By chance, a value $p_{ref} = 1$ N/mm^2 is quite realistic: in fact it produces contact stresses of the order of magnitude of those that a bearing steel will take in operating conditions.

Two reference cases are examined, sphere on plane and sphere in a spherical cavity at the threshold of Hertz theory validity.



			$p_{ref} = 1$ N/mm^2
p_{max}	≈ 2000	≈ 150	N/mm^2
i.e., p_{max}	≈ 2000	≈ 150	times p_{ref}

6. Engineered formulas for cylinder on cylinder (1/8)

In the case of the cylinder / cylinder contact:

$$b = \sqrt{\frac{4}{\pi}\frac{F}{L}\frac{d}{\left(1 \pm \frac{d}{D}\right)}\left(\frac{1-\nu^2}{E}\right)} = \frac{1{,}08}{\sqrt{E}}\frac{1}{\sqrt{\left(1 \pm \frac{d}{D}\right)}}\sqrt{\frac{F}{L}d}$$

$$p_{mean} = \frac{F}{2Lb} = 0{,}465\sqrt{E}\sqrt{\left(1 \pm \frac{d}{D}\right)}\sqrt{\frac{F}{Ld}}$$

$$p_{max} = \frac{4}{\pi}\frac{F}{2Lb} = 0{,}591\sqrt{E}\sqrt{\left(1 \pm \frac{d}{D}\right)}\sqrt{\frac{F}{Ld}}$$

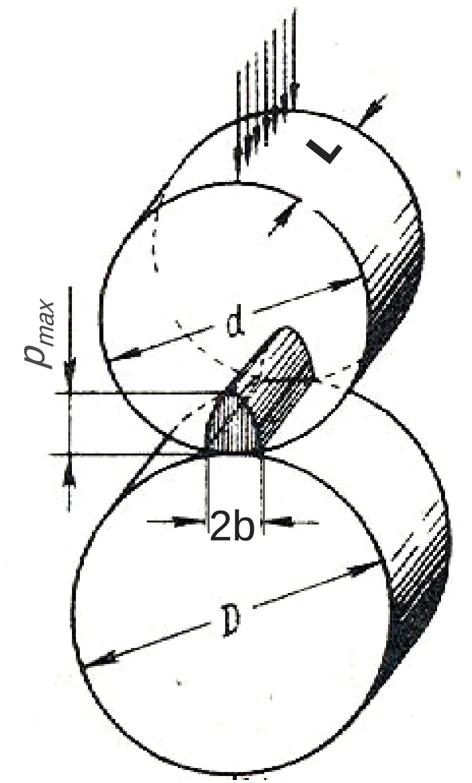


figure from: P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

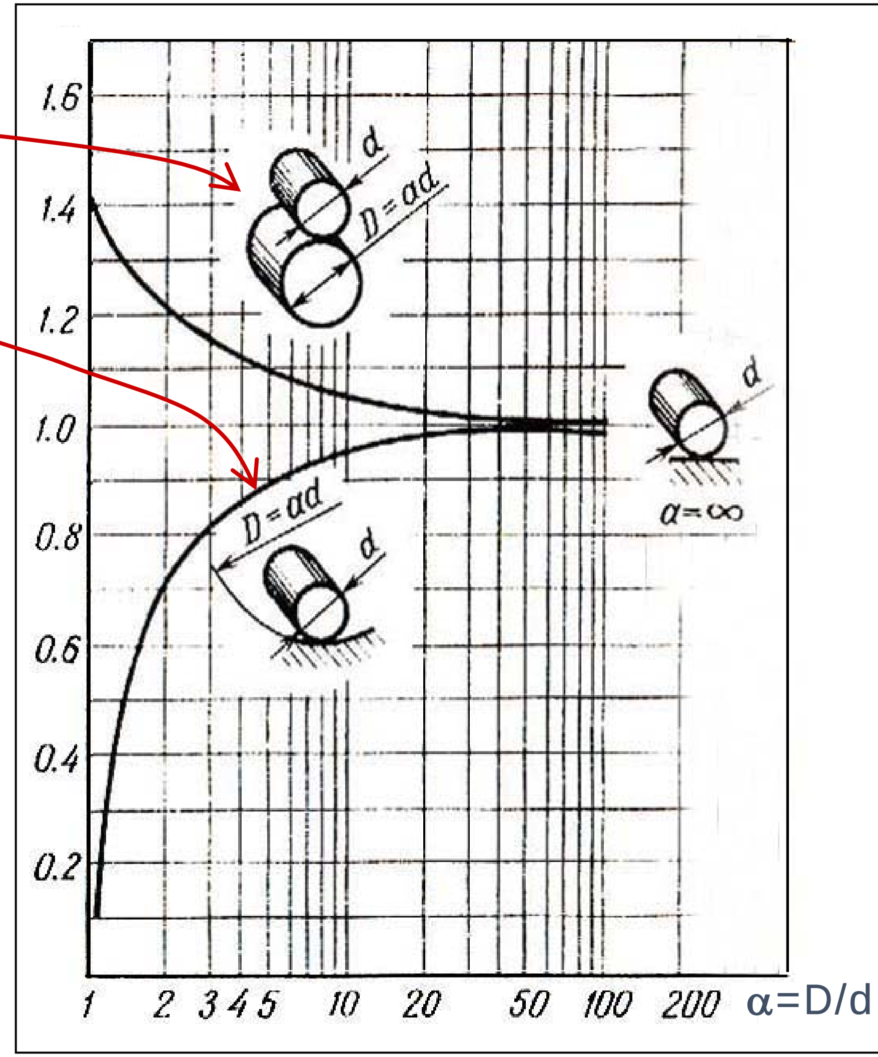
In the case cylinder/cylinder, steel on steel:

$$p_{mean} = \frac{F}{2Lb} = \underbrace{207}_{0,464\sqrt{E}}\sqrt{\frac{F}{Ld}}\sqrt{\left(1 \pm \frac{d}{D}\right)}$$

The geometrical ratio factor:

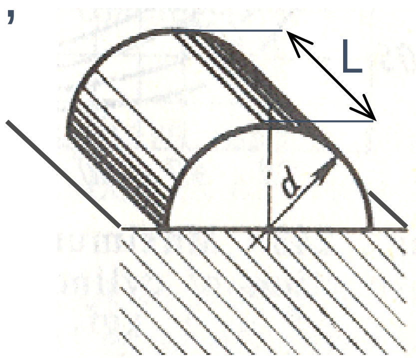
+

–



Here again a "reference pressure": $p_{ref} = \dfrac{F}{L\,d}$

may be defined, together with the areas ratio A_H/A_{ref}:



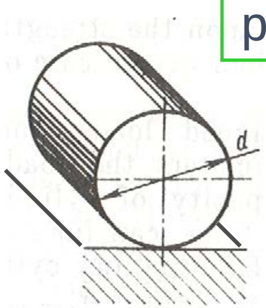
figures from: P. Orlov, *Fundamentals of Machine Design*, Mir, Moscow, 1977/1980

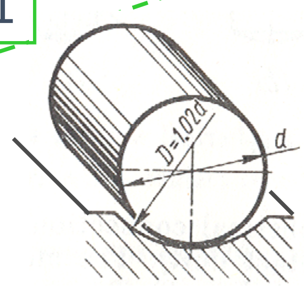
$$p_{max} = \frac{4}{\pi} p_{mean} =$$

$$= \underbrace{264}_{0{,}591\sqrt{E}} \sqrt{p_{ref}} \sqrt{\left(1 \pm \frac{d}{D}\right)}$$

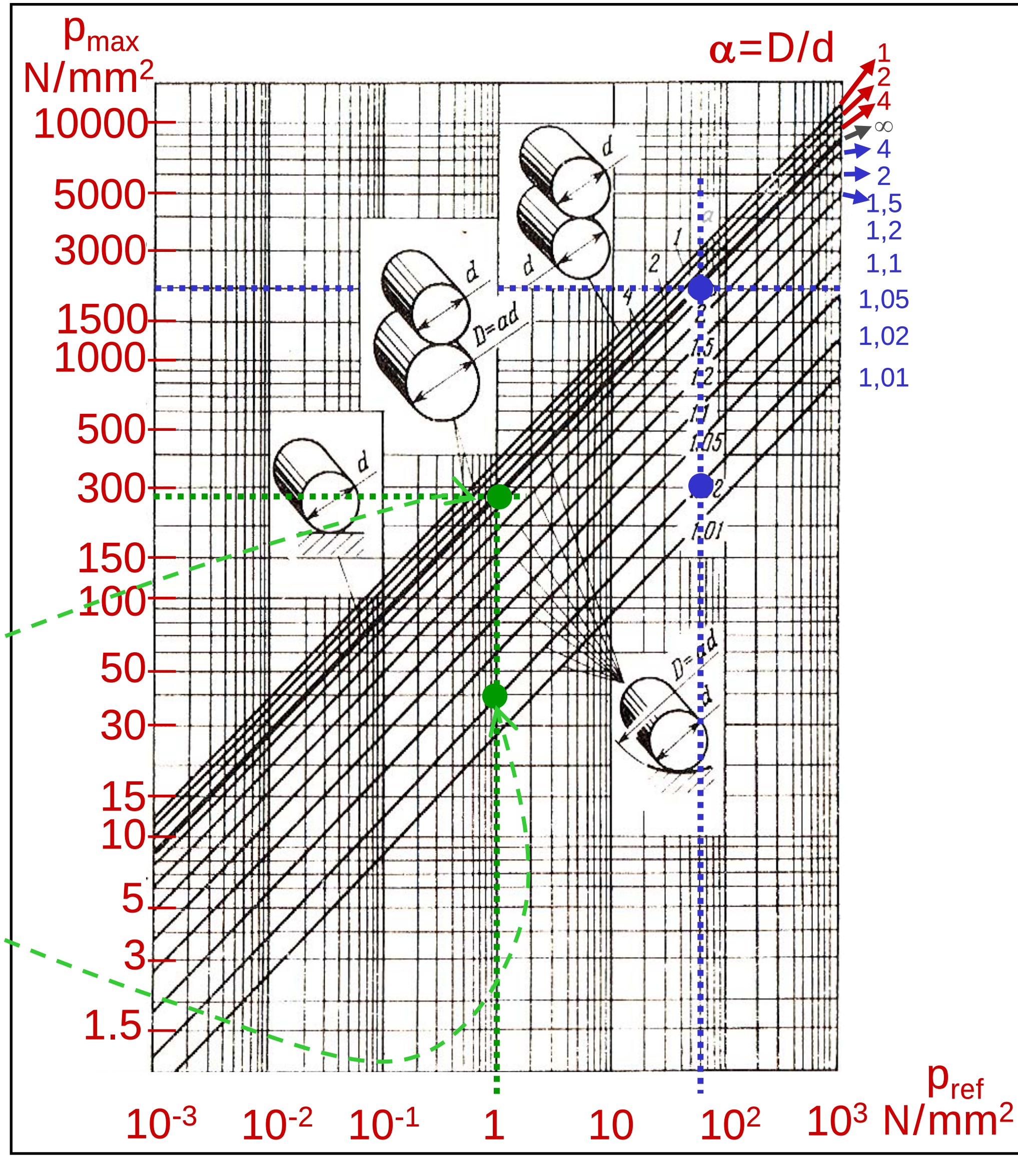
$$\frac{A_H}{A_{ref}} = 4{,}83 \cdot 10^{-3} \frac{\sqrt{p_{ref}}}{\sqrt{1 \pm \frac{d}{D}}}$$

$\boxed{p_{ref} = 1}$



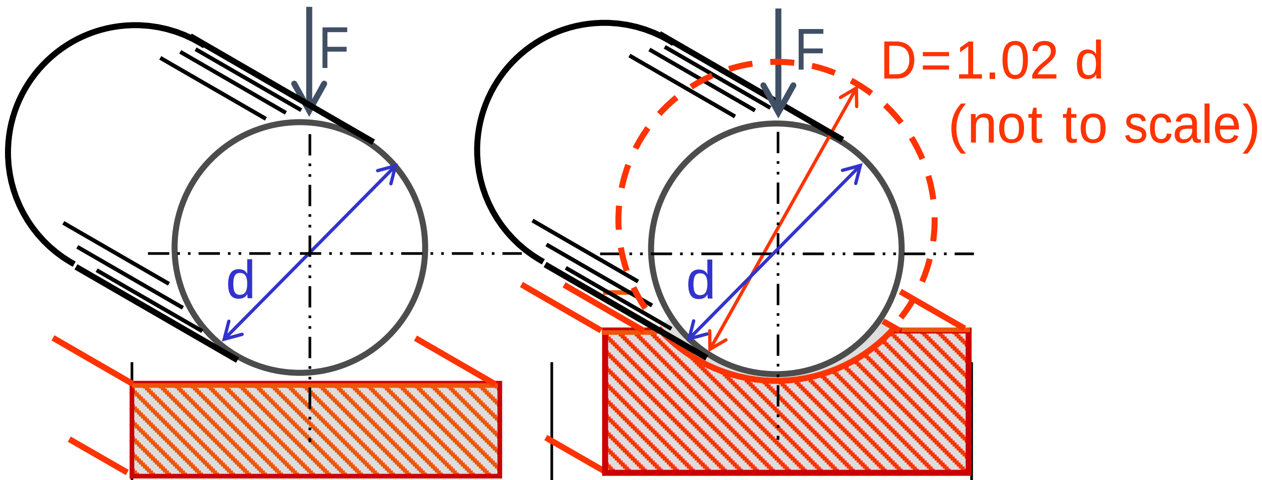


$p_{max} \cong 285$ $p_{max} \cong 40$



figures from: P. Orlov, *Fundamentals of Machine Design*, Mir, Moscow, 1977/1980

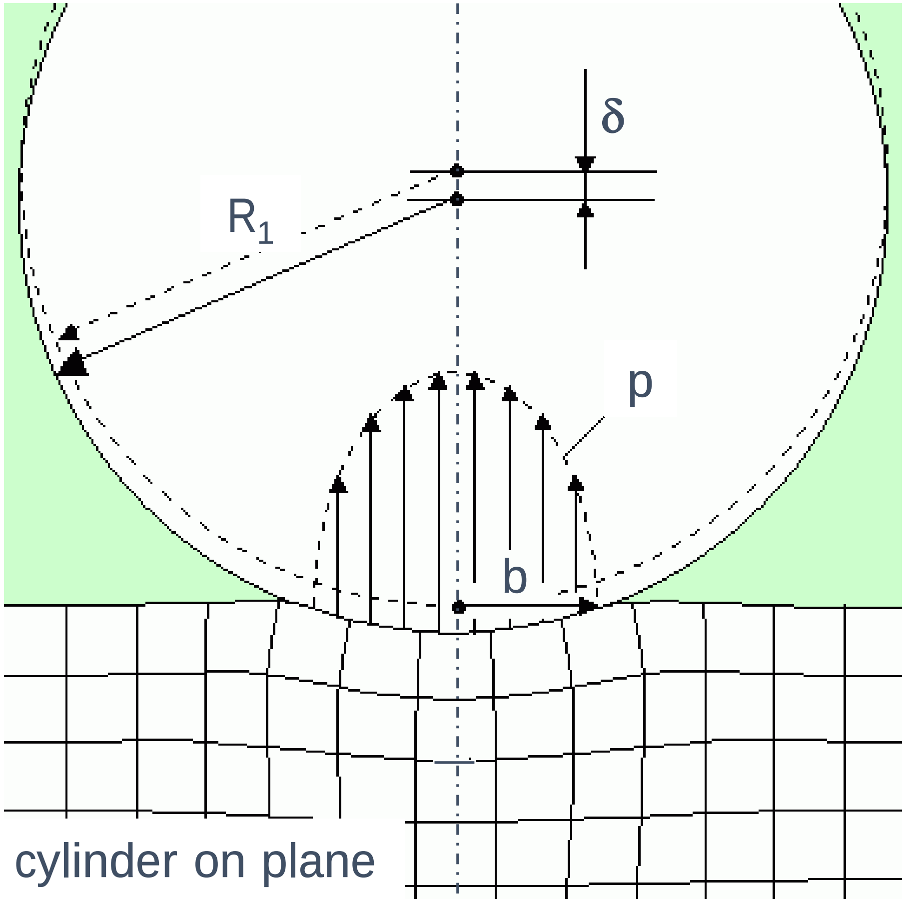
See here cylinder p_{max} for $p_{ref}=1$ and for $p_{ref}=60$ N/mm^2 (where p_{max} is about 2000 MPa) and compare with the sphere case of slide 50 of this chapter.



p_{ref}=1 MPa	p_{max}	≈ 285	≈ 40	N/mm^2
	i.e., p_{max}	≈ 285	≈ 40	times p_{ref}
p_{ref}=60 MPa	p_{max}	≈ 2000	≈ 300	N/mm^2
	i.e., p_{max}	≈ 33	≈ 5	times p_{ref}

The problem of cylinder to cylinder approach.

The treatment is considerably more difficult than its “point contact” counterpart.



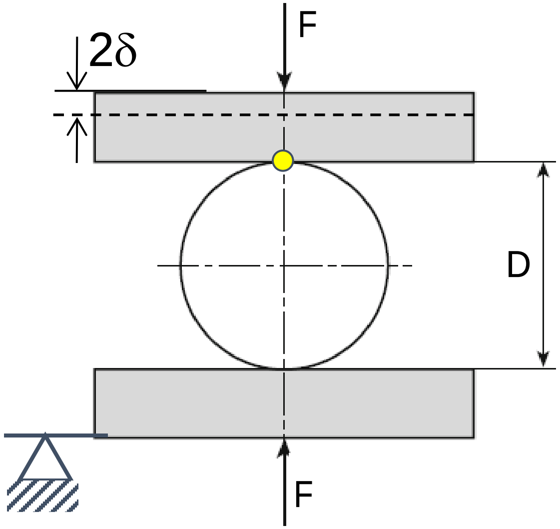
cylinder on plane

The approach problem for the elastic cylinder / cylinder contact is quite formidable: the literature reports quite a few different formulas, all of them characterised by the fact that the applied force appears in implicit form, which is not convenient for subsequent manipulations when calculating multiple simultaneous contacts, as we shall see in rolling bearings.

6. Engineered formulas for cylinder on cylinder (6 / 8)

It is more convenient a power expression, similar in structure to the one already obtained for the point contact; formulas of this type have been obtained from interpolations of experimental data or from fitting of more complex formulas.

Harris* quotes the Palmgren (1923) formula valid for steel on steel, obtained compressing a bearing-type crowned roller between two flat plates.



The value given by the following empirical formula represents the contribution δ of the single contact to the total "approach" 2δ i.e. relative displacement of the two flat plates:

$$\delta = 3{,}84 \; 10^{-5} \left(\frac{F}{L}\right)^{0{,}9} L^{0{,}1} \qquad [F]=N, \; [L]=mm, \; [\delta]=mm$$

* T.Harris, *Rolling Bearing Analysis*, III ed., 1991, John Wiley & Sons

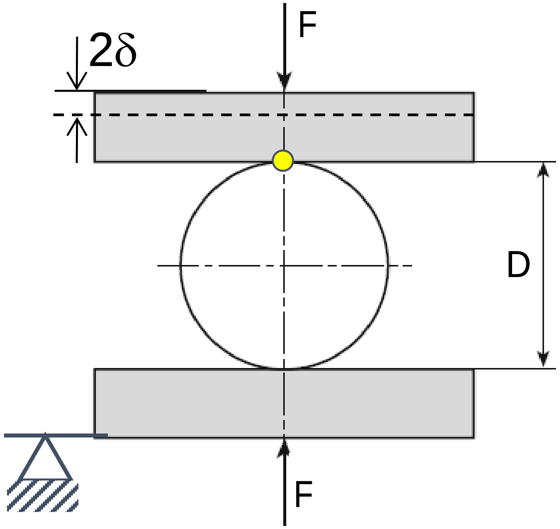
According to Brändlein* recent theoretical investigations have established that for the single contact:

$$\delta = \frac{4{,}05}{10^5} \frac{F^{0{,}925}}{L^{0{,}85}} \qquad [F]=N, \quad [L]=mm, \quad [\delta]=mm$$

where L is, of course, not the roller axial length but is the entire roller length minus possible either recesses in the raceways or minus the two corner radii of the roller.

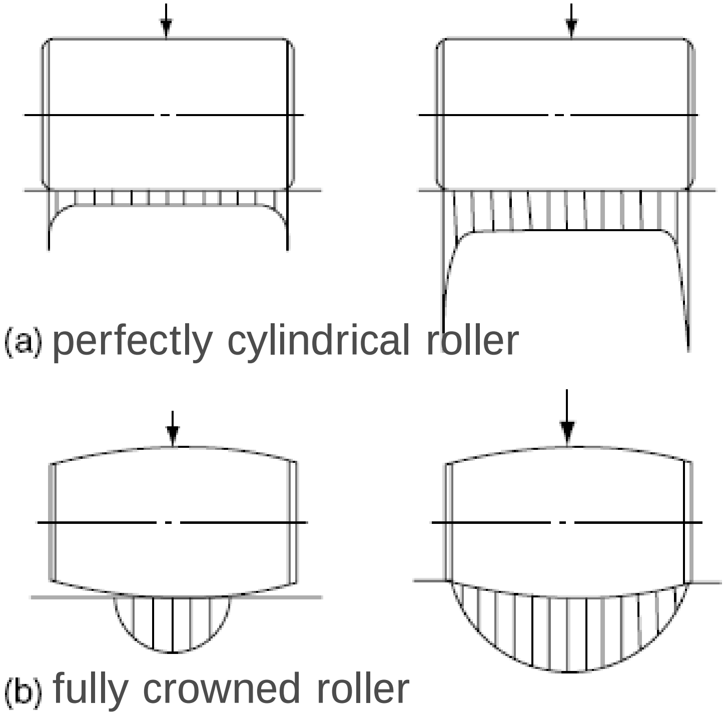
Note that δ does not depend on diameter D.

The total displacement 2δ of the two contacts of a roller with the plates (or the raceways) is twice as large as the value obtained as above for one contact.



It should be noted, however, that the total approach of plates in reality is the sum of the two local contact displacements plus the compression of the cylinder.

* Brändlein J., *Ball and Roller Bearings : Theory, Design, and Application*, 1999, John Wiley & Sons



(from: Harris T.A., Kotzalas M.N., *Rolling Bearing Analysis V Ed. - Advanced Concepts of Bearing Technology*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2007)

As anticipated in sl.6 of this Section, the problem is further complicated by the fact that in order to avoid local peaks of stress at the two ends of a cylindrical roller on a flat raceway, the roller is "crowned", i.e., is given a slight modification of the profile which eliminates peaks.

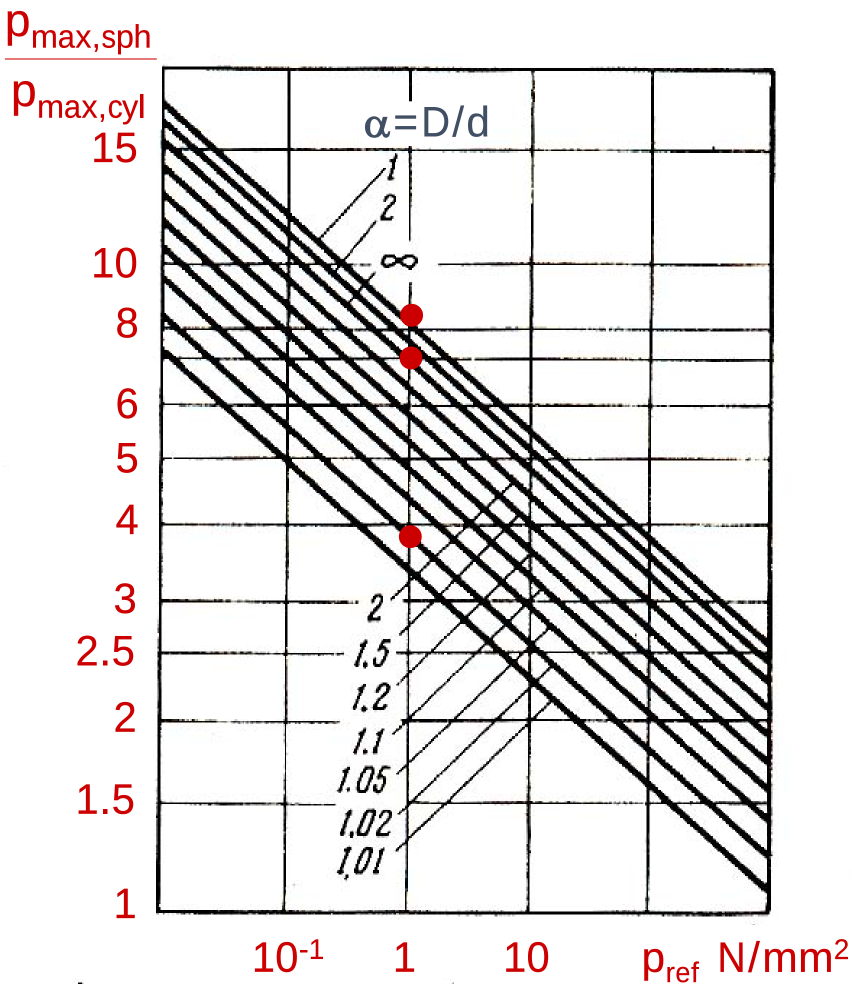
The figure on the left shows the diagrams of the maximum contact pressure along the centre of the contact area.

7. Sphere vs. cylinder and summary of formulas (1/6)

Max pressure comparison: for the technically representative reference value p_{ref} = 1 N/mm^2, the values for p_{max} are:

contact geometry	p^{sph}_{max} (MPa)	contact geometry	p^{cyl}_{max} (MPa)	$p^{sph}_{max} / p^{cyl}_{max}$
sphere / equal sph.	≈3200	cylinder / equal cyl.	≈400 ●	≈8
sphere / plane	≈2000	cylinder / plane	≈285 ●	≈7
sphere / cup x 1.02	≈150	cylinder / bushing x 1.02	≈40 ●	≈3.75

A comparison between sphere and cylinder contacts, in the case of steel, leads to the diagram below which illustrates the ratio:

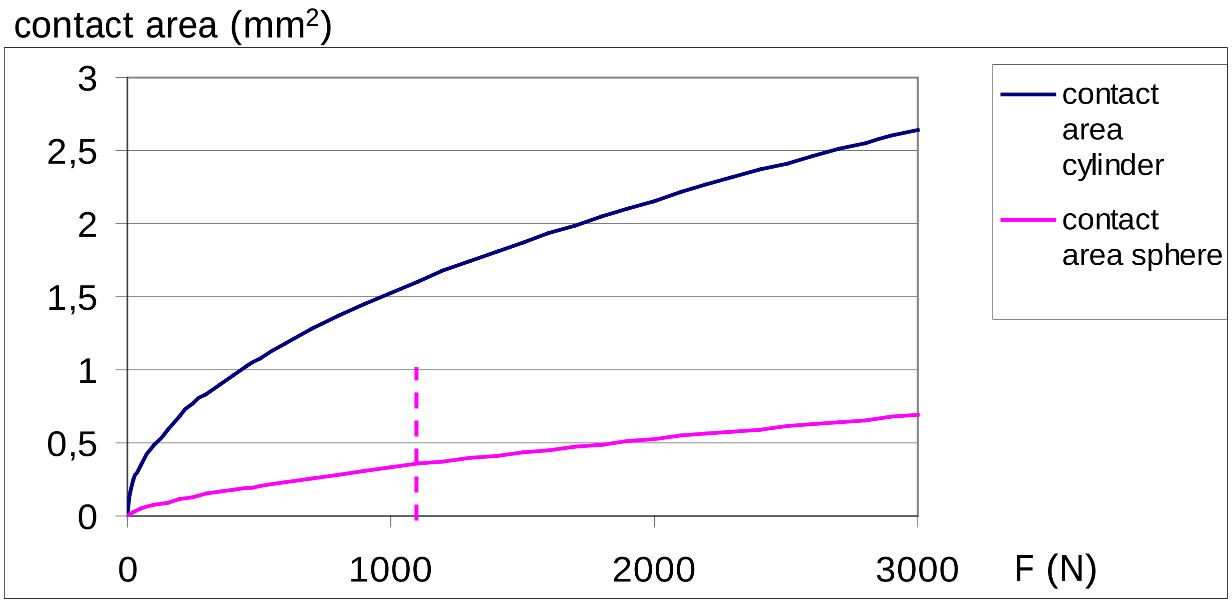


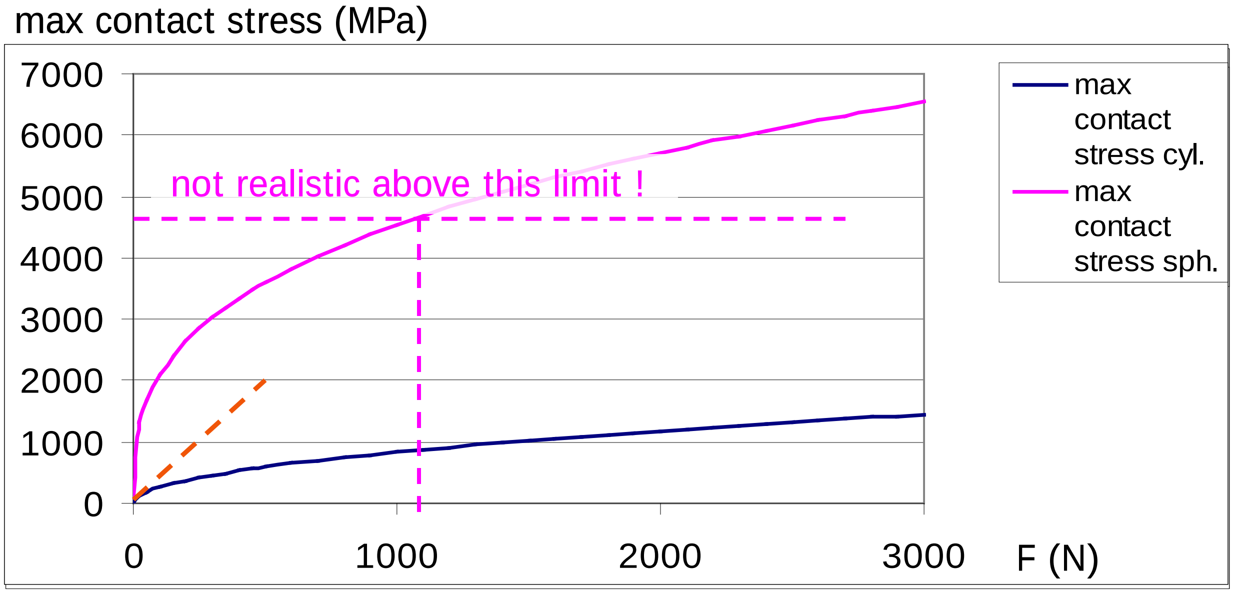
$$\frac{p_{max,sph}}{p_{max,cyl}} = \frac{1900\sqrt[3]{p_{ref}} \cdot \sqrt[3]{\left(1 \pm \frac{d}{D}\right)^2}}{270\sqrt{p_{ref}} \cdot \sqrt{\left(1 \pm \frac{d}{D}\right)}}$$

p_{ref} takes into account applied force and dimension d of the smaller body, α=d/D taken into account the the diameter of the second body against which the first is pressed.

Dots • mark the results shown in the previous slide.

figure from: P. Orlov, *Fundamentals of Machine Design*, Mir, Moscow, 1977/1980



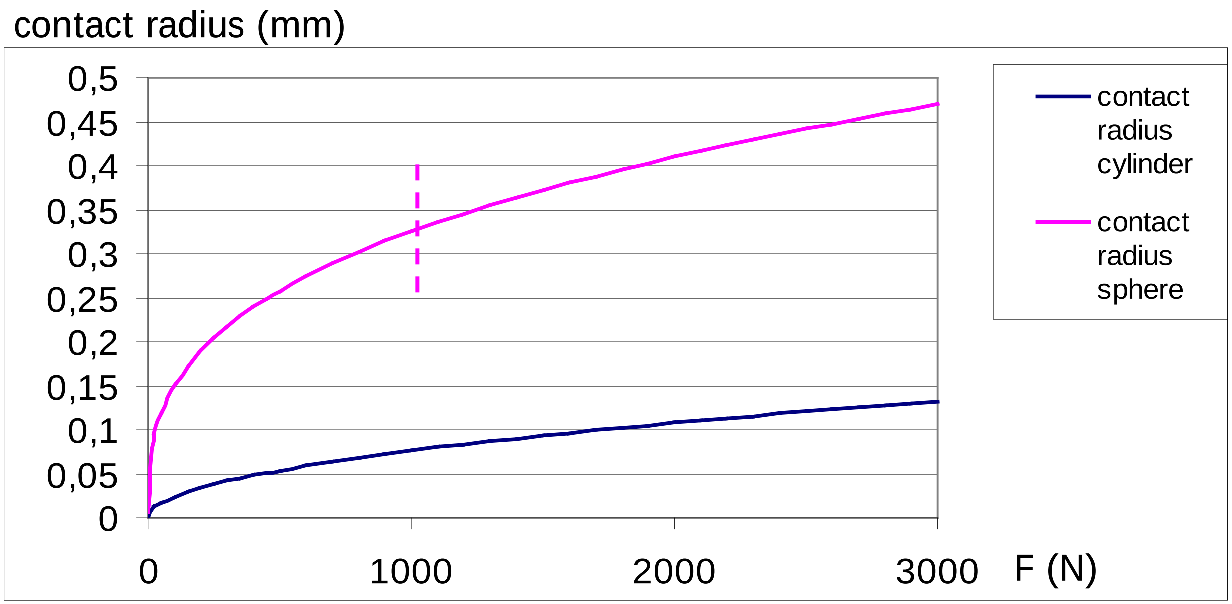


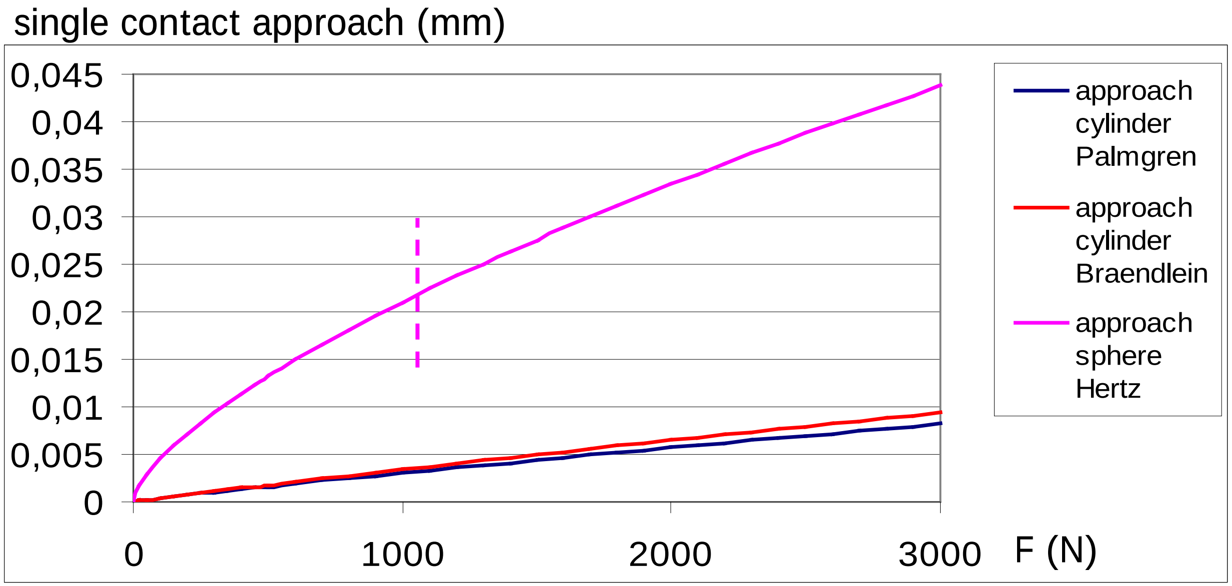
Familiarise with orders of magnitude:

compare a sphere, diameter d=10 mm, with a cylinder d=10, L=10,

both in steel and both pressed against a flat steel plate.

Remark however that the formulation through p_{ref} introduced in Sect. 5 sl. 4 is much more powerful, in that it condenses force and dimension in one parameter





Continue the comparison between a sphere, diameter d=10 mm (A_{ref}=78,5 mm^2), and a cylinder d=10, L=10 (A_{ref}=100 mm^2).

Both in steel and both pressed against a flat steel plate.

Main elastic formulas steel on steel	Sphere on sphere	Cylinder on cylinder
Contact radius "a" or contact semi-width "b" mm	$a = \dfrac{1{,}51 \cdot 10^{-2}}{\sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)}} \sqrt[3]{F\,d}$	$b = \dfrac{2{,}41 \cdot 10^{-3}}{\sqrt{\left(1 \pm \dfrac{d}{D}\right)}} \sqrt{\dfrac{F}{L} d}$
Mean contact pressure MPa	$p_{mean} = \dfrac{2}{3} p_{max}$	$p_{mean} = \dfrac{\pi}{4} p_{max}$
Max contact pressure MPa	$p_{max} = 2107 \sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)^2} \sqrt[3]{\dfrac{F}{d^2}}$	$p_{max} = 264 \sqrt{\left(1 \pm \dfrac{d}{D}\right)} \sqrt{\dfrac{F}{Ld}}$
Single contact approach mm (for cylinder: cyl. on plate)	$\delta = 4{,}53 \cdot 10^{-4} \sqrt[3]{\left(1 \pm \dfrac{1}{D}\right)} \sqrt[3]{\dfrac{F^2}{d}}$	$\delta = \dfrac{4{,}05}{10^5} \dfrac{F^{0,925}}{L^{0,85}}$ Brändlein

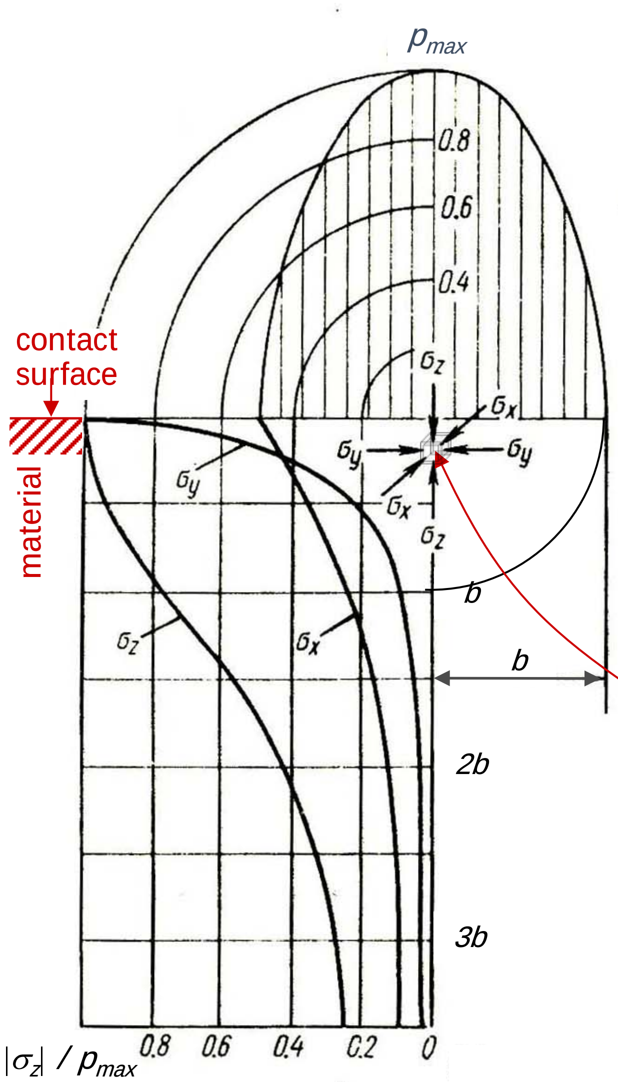
Main elastic formulas steel on steel	Sphere on sphere	Cylinder on cylinder
Definition of p_{ref} and A_{ref}:	$p_{ref} = \dfrac{4F}{\pi d^2}$; $A_{ref} = \dfrac{\pi}{4} d^2$	$p_{ref} = \dfrac{F}{L d}$; $A_{ref} = Ld$
Hertz contact area A_H mm	$\dfrac{A_H}{A_{ref}} = 7{,}72 \cdot 10^{-4} \dfrac{\sqrt[3]{p_{ref}^2}}{\sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)^2}}$	$\dfrac{A_H}{A_{ref}} = 4{,}83 \cdot 10^{-3} \dfrac{\sqrt{p_{ref}}}{\sqrt{1 \pm \dfrac{d}{D}}}$
Max contact pressure MPa	$p_{max} = 1930 \sqrt[3]{P_{ref}} \sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)^2}$	$p_{max} = 264 \sqrt{p_{ref}} \sqrt{\left(1 \pm \dfrac{d}{D}\right)}$
Single contact approach, mm (for cylinder: cyl. on plate)	$\dfrac{\delta}{d} = 3{,}86 \cdot 10^{-4} \sqrt[3]{p_{ref}^2} \sqrt[3]{\left(1 \pm \dfrac{d}{D}\right)}$	$\delta = \dfrac{4{,}05}{10^5} \dfrac{F^{0{,}925}}{L^{0{,}85}}$ Brändlein

Sections 8, 9 - Stresses and static contact design

Section 8 illustrates which is the stress field under the surface of each of the two bodies in contact. Stresses are shown within the elastic range. The stress state is three dimensional and compressive. It is shown at which depth the maximum equivalent stress occurs, and which is its value compared to the maximum pressure at the contact surface, which is used as the benchmark value. It is found that the equivalent (yelding) stress is only around 60% of p_{max}.

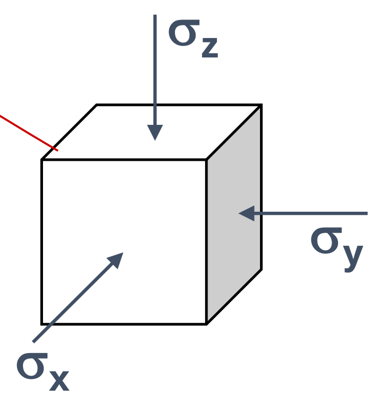
Section 9 introduces the plastic approach produced by loads in excess of the maximum load which can be taken elastically.
The acceptable amount of plastic set, although limited, produces a considerable increase of contact load, therefore being of great interest to the designer.

A caveat! These Sections deal with static design and yielding, but the reader is warned that contact failure in most technical applications is rather due to rolling fatigue, which will be treated in the context of rolling bearings, Chapter 3.



Surface and sub-surface stresses
for the cylinder contact

Experimental evidence shows that failure starts at points below the surface. It is then of interest to determine subsurface stresses, which are here shown for the case of the cylinder contact.

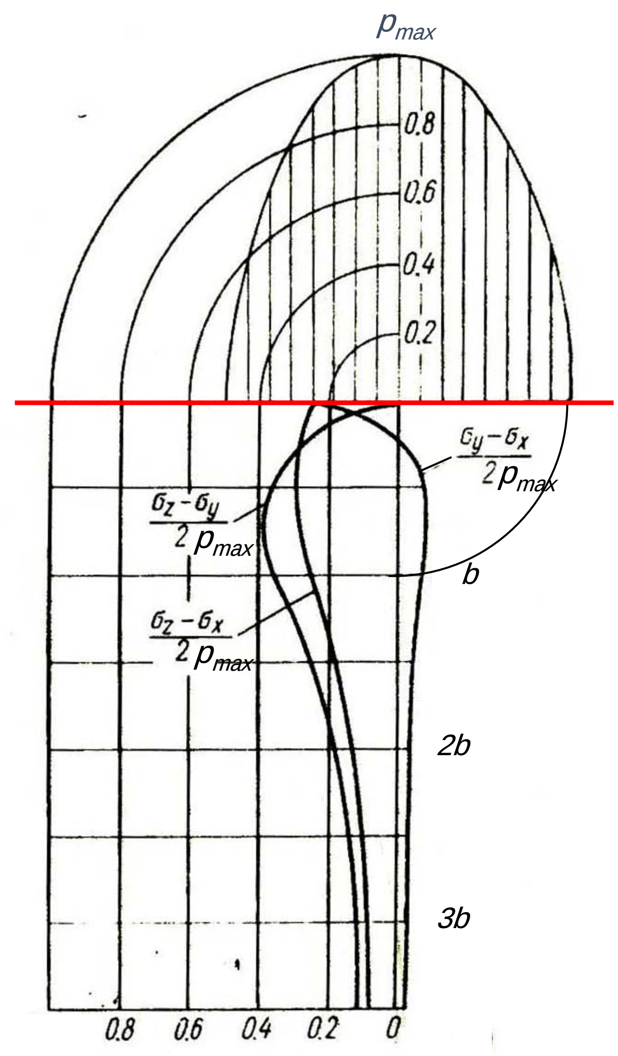


Axis z is on the centre of the contact; symmetry implies that along this axis shear stresses τ_{ij} (on any infinitesimal element) are zero. Then normal stresses are principal.

P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

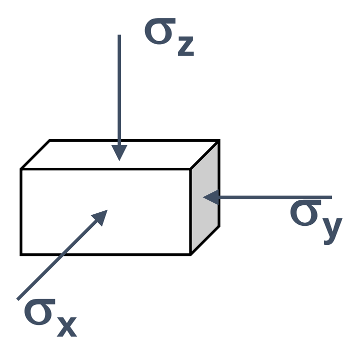
The figure on the right, not very accurate, shows maximum shear stresses on the three principal planes. Their double are Mohr’s diameters on the principal planes.

The highest diameters is, according to Tresca, the equivalent stress; it is $(\sigma_z - \sigma_y)$, which has a maximum at about depth $z \cong 0{,}65$ b where b is the contact semiwidth.

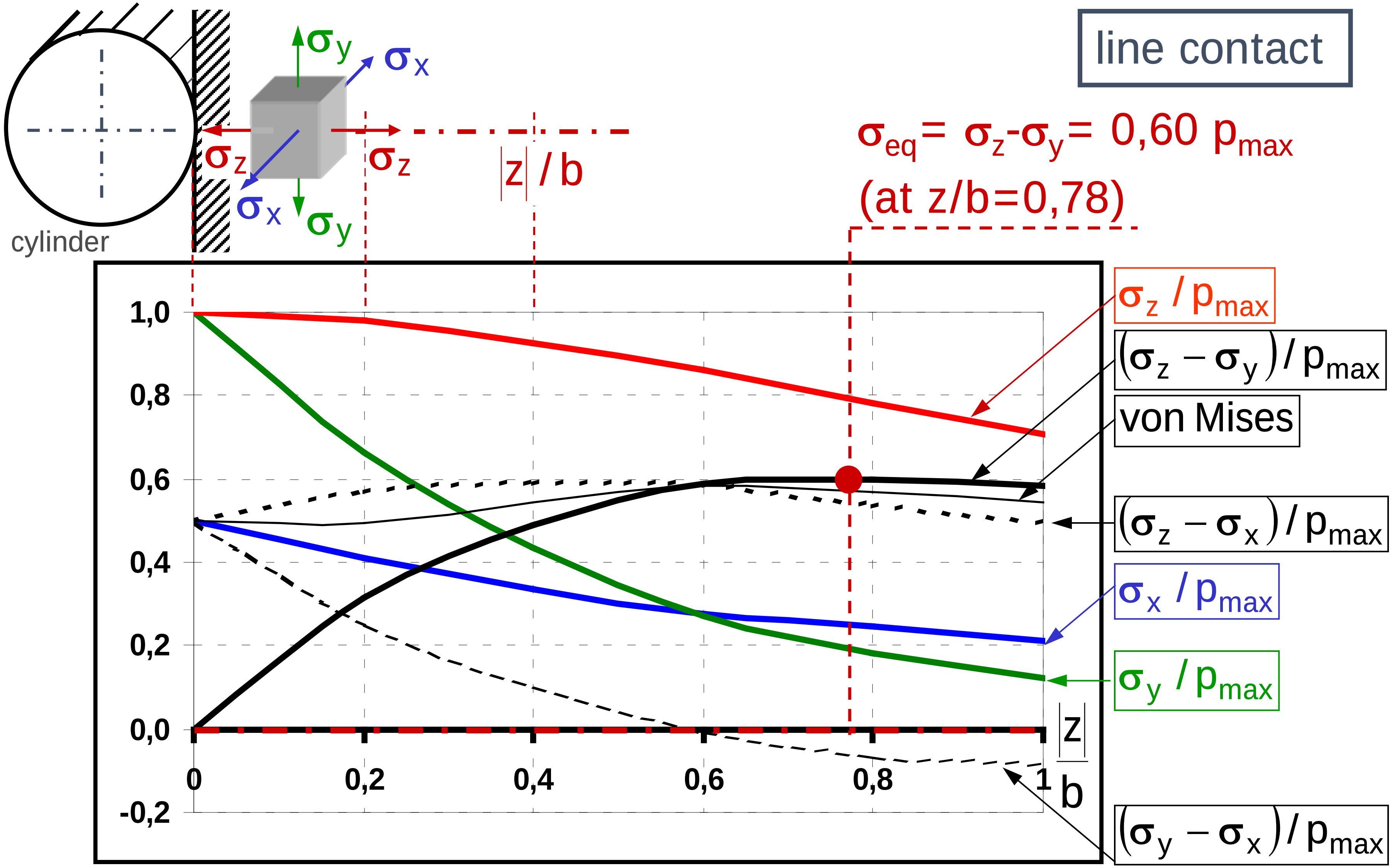


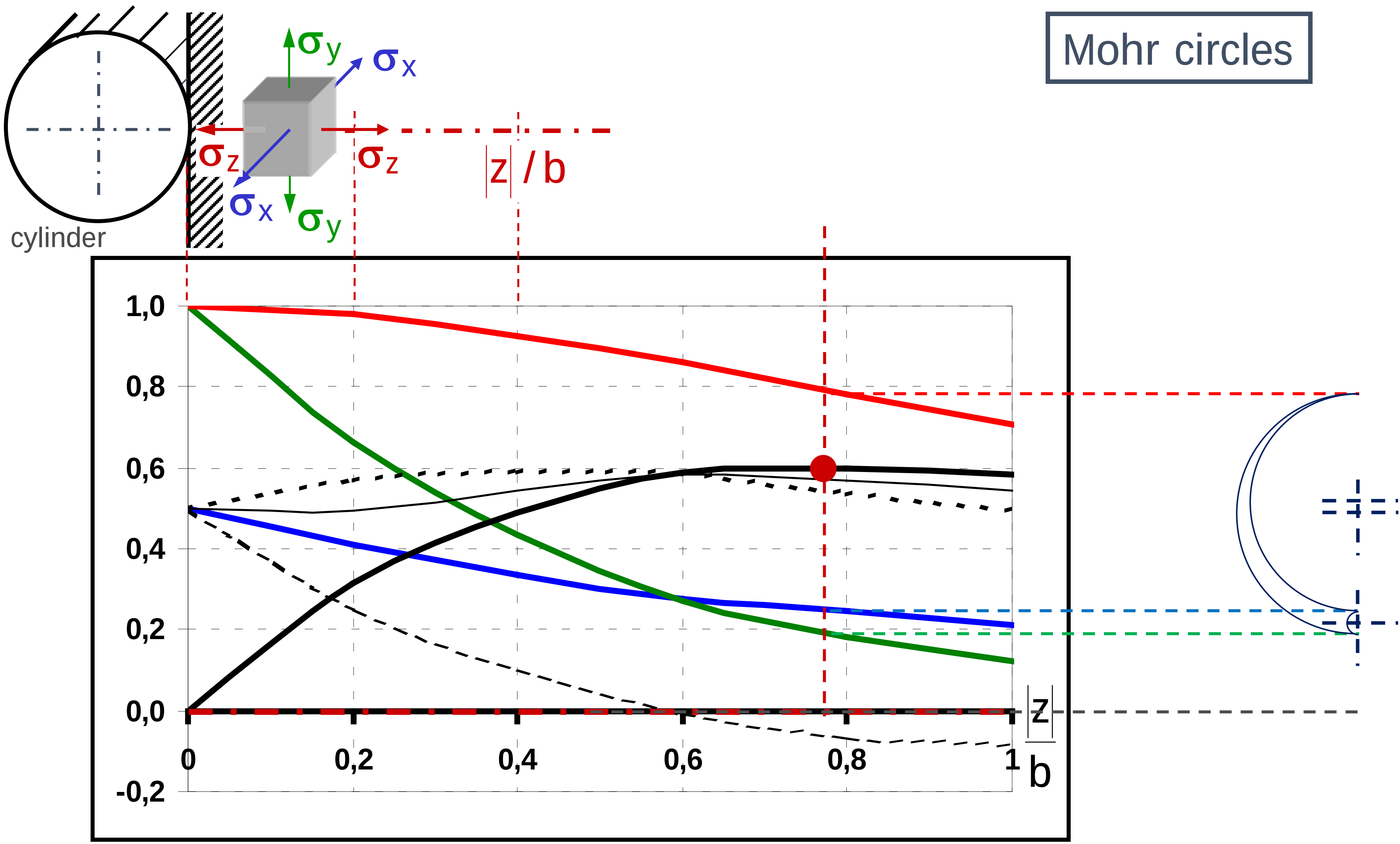
P. Orlov, *Fundamentals of Machine Design* , Mir, Moscow, 1977/1980

The next three slides will show in detail sub-surface stresses for three representative cases.
Note that stress diagrams are equal for the same z/b .



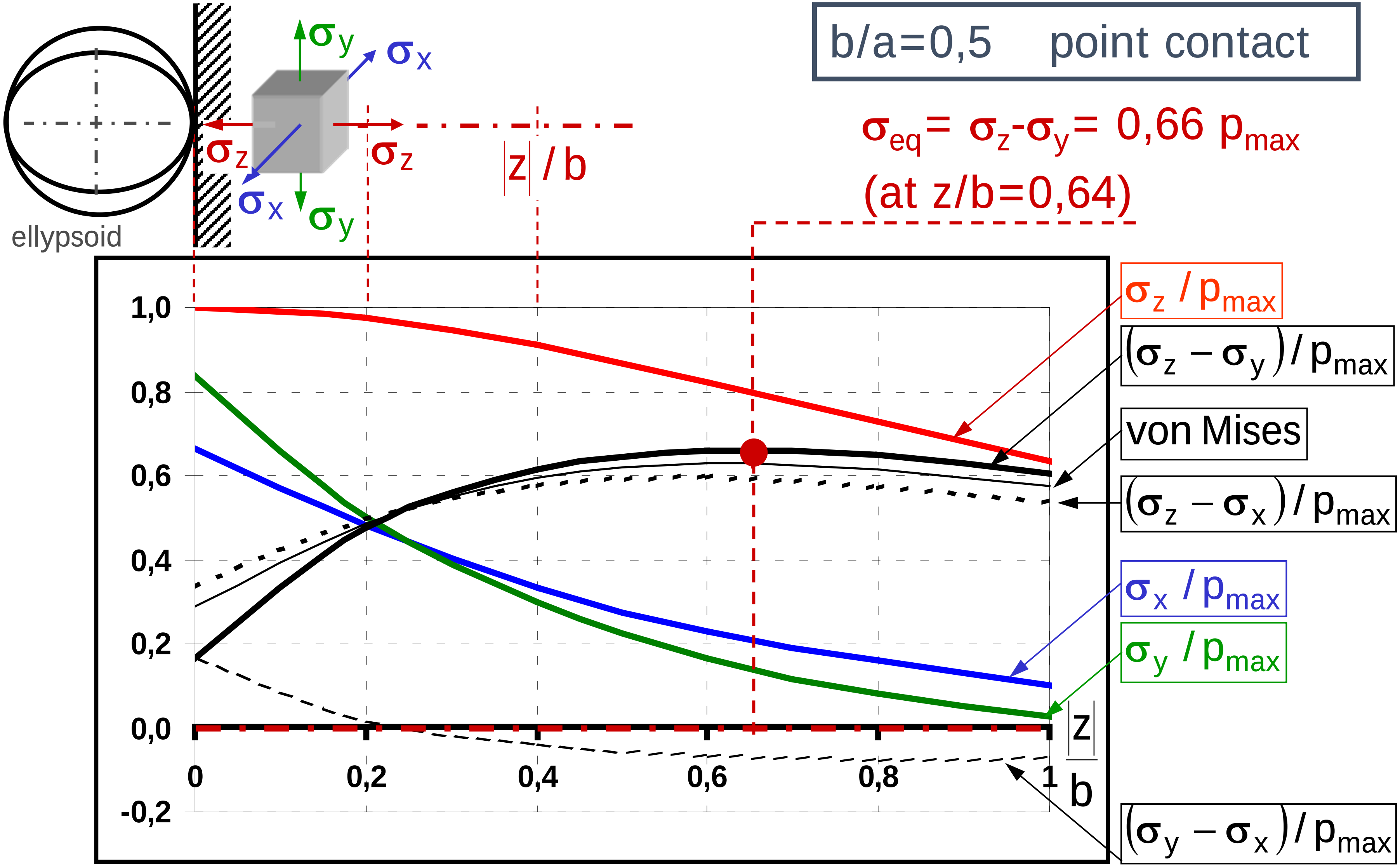
8. Maximum subsurface stresses (3/6)





cylinder
σy
σx
σz
σz
σx
σy
|z| / b
Mohr circles
1,0
0,8
0,6
0,4
0,2
0,0
-0,2
0
0,2
0,4
0,6
0,8
1
|z| / b

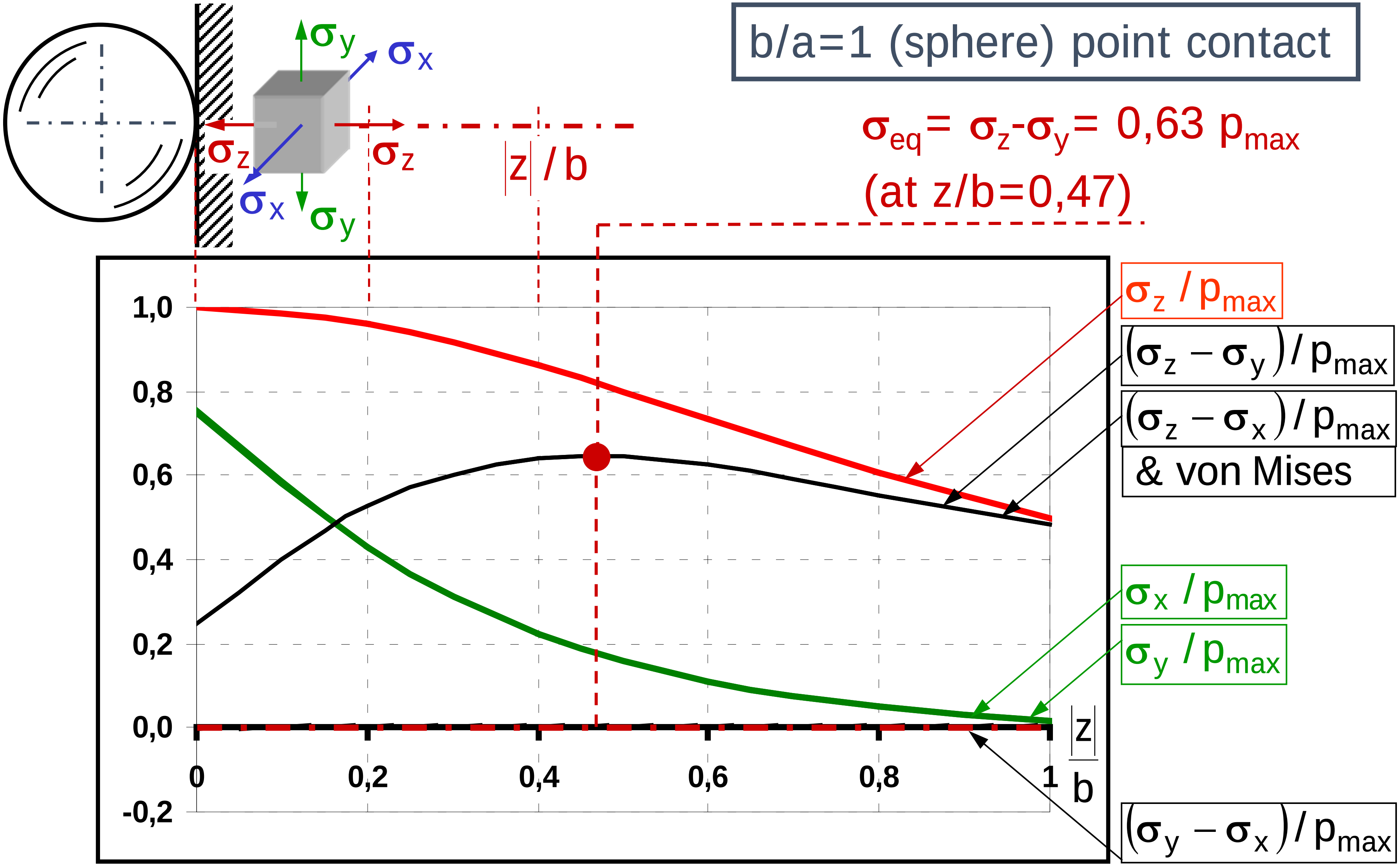
8. Maximum subsurface stresses (4/6)



b/a=1 (sphere) point contact

$\sigma_{eq} = \sigma_z - \sigma_y = 0{,}63\ p_{max}$

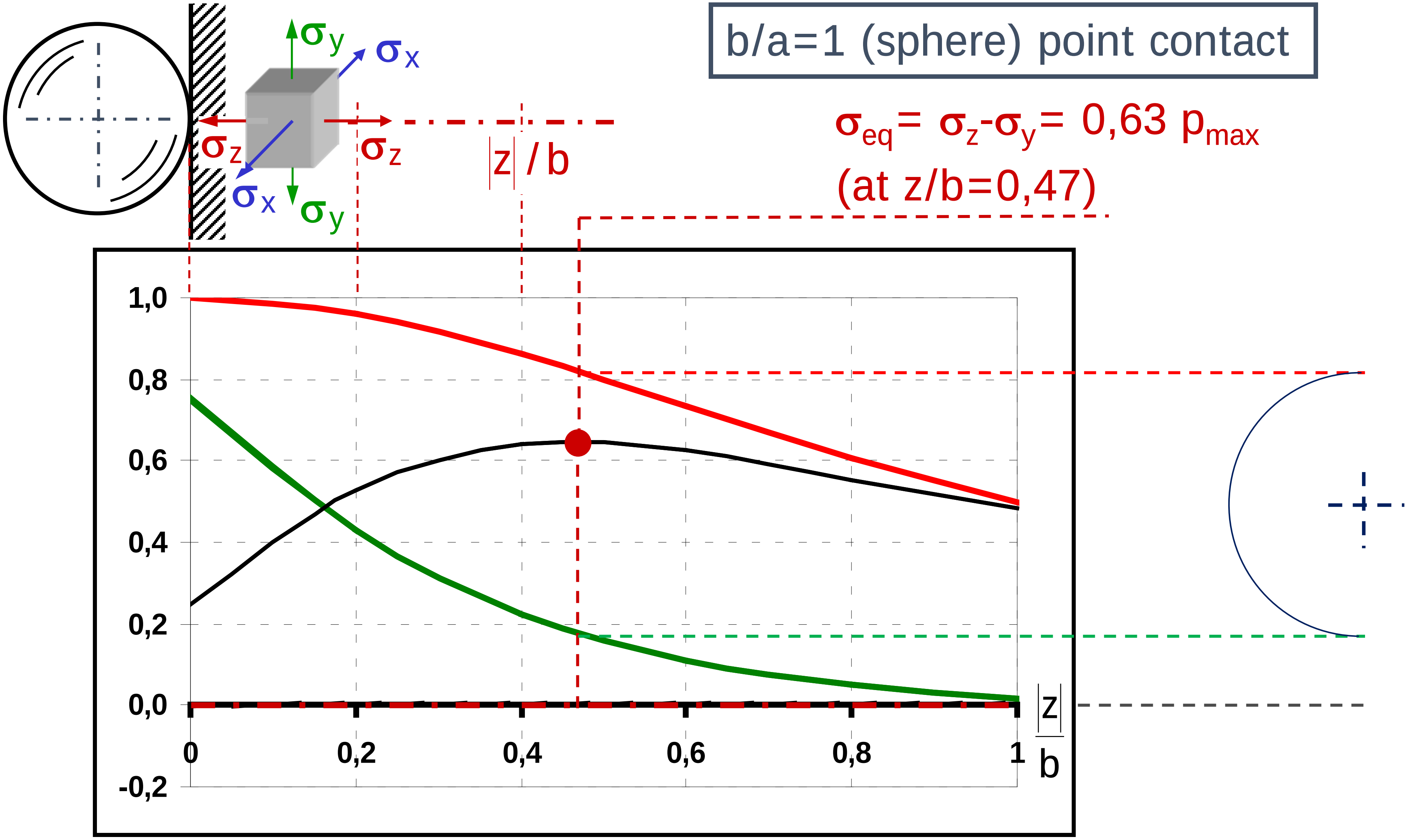
(at z/b=0,47)



b/a=1 (sphere) point contact

$\sigma_{eq} = \sigma_z - \sigma_y = 0,63\ p_{max}$

(at z/b=0,47)



It is quite difficult to state precisely at which depth the maximum stress is, as the curves are quite flat, moreover it is not clear whether we should take Tresca or von Mises. Fortunately, they produce almost equal results:

with Tresca:

-) cylinder / cylinder contact: $\sigma_{eq} = \sigma_z - \sigma_y = 0{,}60\ p_{max}$ (at $z/b = 0{,}79$)

-) sphere / sphere contact: $\sigma_{eq} = \sigma_z - \sigma_y = 0{,}62\ p_{max}$ (at $z/b = 0{,}48$)

with von Mises:

-) cylinder / cylinder contact: $\sigma_{eq} = 0{,}59\ p_{max}$ (at $z/b \cong 0{,}6$)

-) sphere / sphere contact: $\sigma_{eq} = 0{,}63\ p_{max}$ (at $z/b = 0{,}47$)

9. Allowable static stresses and contact design (1/6)

Maximum contact stress at the center of contact surface, at yield onset, is obtained as follows:

-) cylinder / cylinder:

$$\sigma_{eq} = \sigma_z - \sigma_y = 0{,}60\ p_{max} = R_e$$

$$\text{i.e.}\quad p_{max} = R_e/0.60 \cong 1{,}7\ R_e$$

-) sphere / sphere:

$$\sigma_{eq} = \sigma_z - \sigma_y = 0{,}63\ p_{max} = R_e$$

$$\text{i.e.}\quad p_{max} = R_e/0.63 \cong 1{,}6\ R_e$$

Yield onset in bearing steels is around R_e=1800÷2000 N/mm^{2}.

Thus, for a sphere on sphere and R_e=2000 MPa, maximum pressure will produce yield when $p_{max} \cong$ 3200 MPa.

When yield stress is exceeded, then a permanent (plastic) deformation is produced. The value of permanent deformation in point contact (sphere !) was obtained from empirical data by Palmgren (1959) by testing bearing quality steel hardened between 63,5 and 65,6 Rockwell C. The following interpolation formula represents the results:

$$\delta_s = 1.3\ 10^{-7}\ \frac{F^2}{d}\ 4\left(\alpha_x + \beta_x\right)\left(\alpha_y + \beta_y\right) \qquad \text{(units are N and mm)}$$

where d is the ball diameter.

For a sphere of diameter d on a raceway of diameters D_1 and D_2 this formula can be further elaborated:

$$\frac{\delta_s}{d} = 3.2\ 10^{-7}\ p_{ref}^2 \left(1 \pm \frac{d}{D_1}\right)\left(1 \pm \frac{d}{D_2}\right)$$

It seems that permanent deformation is split into two almost equal parts on ball and raceway.

9. Allowable static stresses and contact design (3/6)

Experience has shown that a small amount of permanent deformation (permanent set) can be tolerated in rolling bearings.

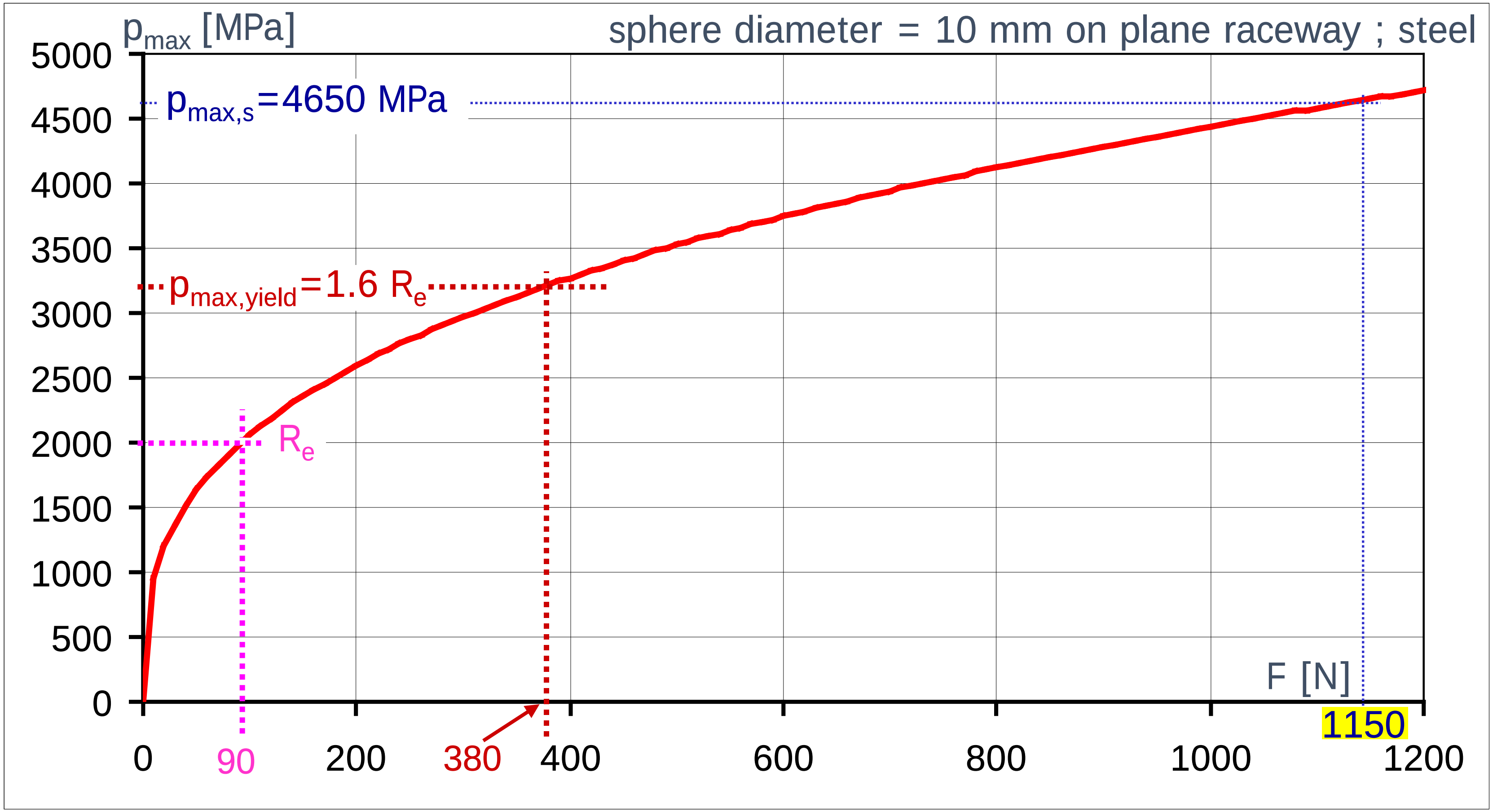
The value not to be exceeded is established, from experience, at: $\delta_s = d/10^4$.

This can be tolerated in the majority of bearing applications without reducing service life, however at the price of a somewhat noisier operation.

If the deformations become significantly larger than that limit, then the bearing shows vibrations and becomes noisier, although bearing friction does not increase significantly. Service life is reduced.

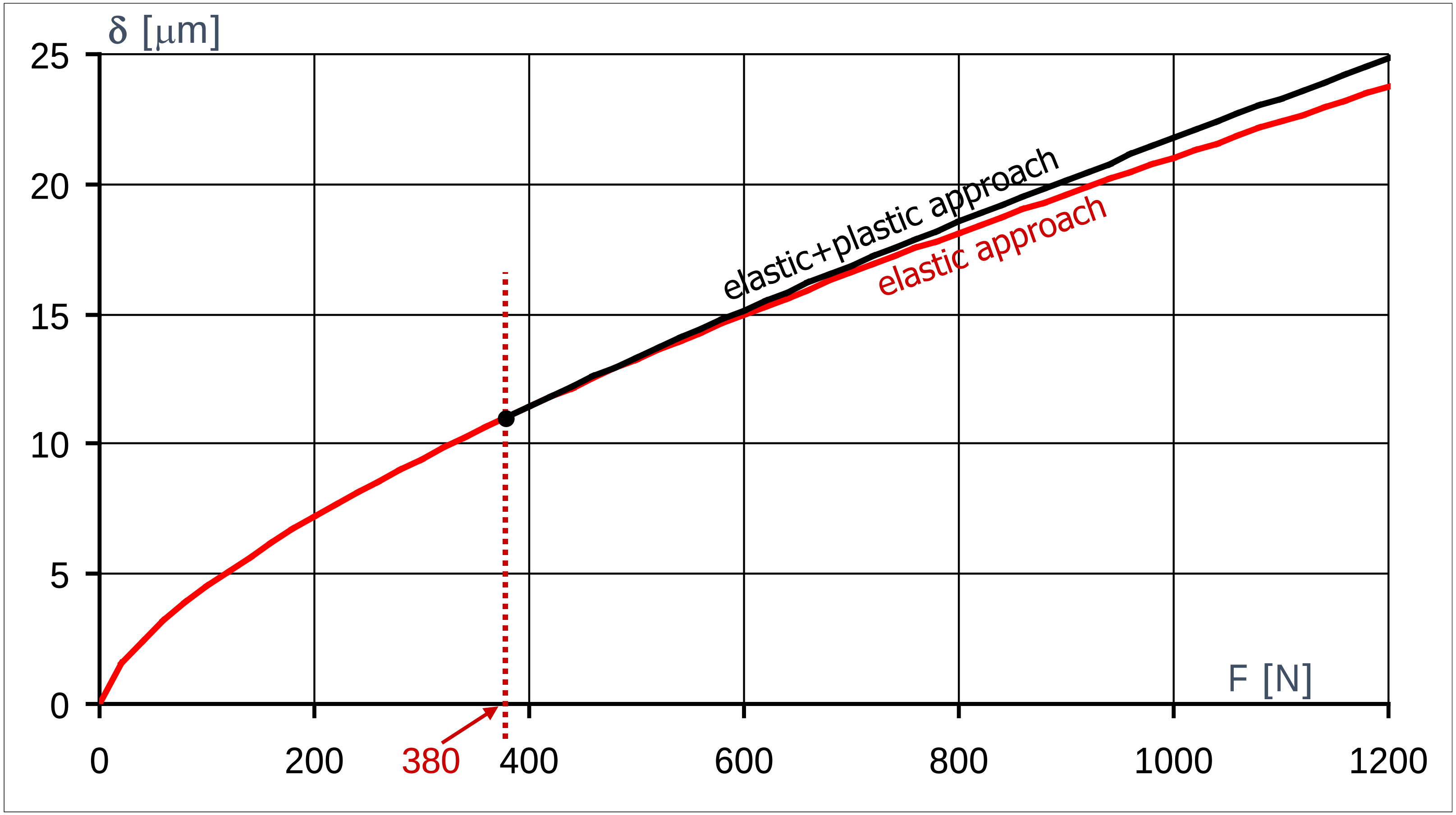
Acceptance of that amount of permanent set has advantages on allowable contact load which we shall now explore.

The next two slides show how much the elastic limit can be exceeded and which are the consequences on the compressive load which can be applied. The example chosen for demonstration is the sphere-on-plane case.

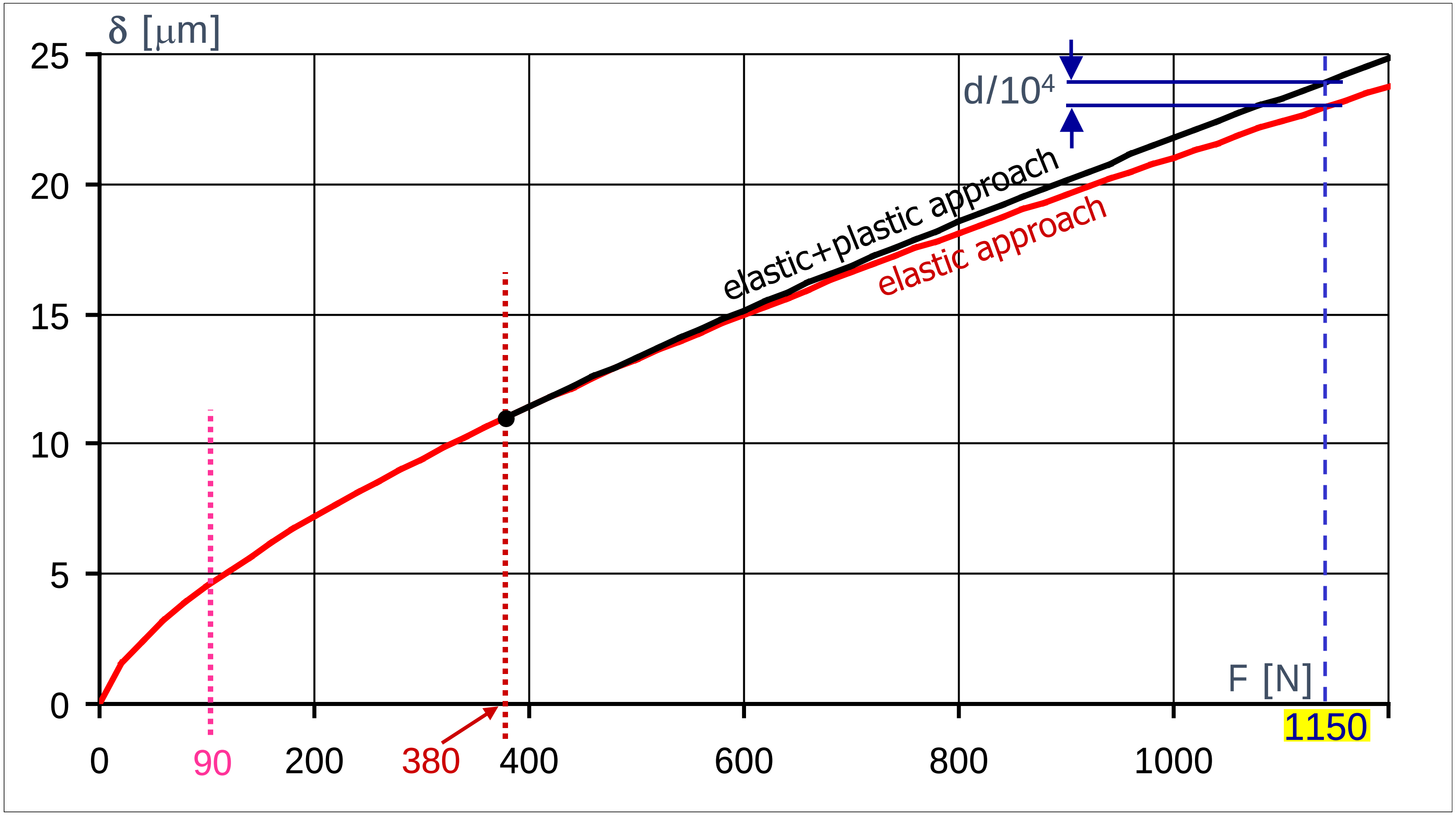


The plot is the max contact pressure p_{max} against the contact force F producing it.

This shows a sphere on plane case, where $p_{max}=R_e$ would (incorrectly) lead to a max allowable load F=90 N, σ_{eq}=2000 MPa while instead $p_{max,yield}=1.6\ R_e$ corresponds to F=380 N, i.e. to the elastic limit; however ... look at $p_{max,s}$



This plot shows the elastic approach for the same case of the previous figure. The red curve is given by the elastic approach. However, when F=380 N is reached, p_{max} reaches the value which produces yield in the subsurface. Then the plastic component of Sect.9 sl.2 must be added, the elastic+plastic approach being given by the black curve.



Acceptance of a plastic residual indentation of $\delta_s = d/10^4 = 10^{-3}$ mm leads to 1150 N, which (see sl.4) corresponds to a value p_{max}, as calculated by the purely elastic formula (then "nominal") of 4650 MPa (in fact, real stresses are lower due to the fact that the real elastic-plastic contact area is larger than predicted by the elastic formula).

Appendices

Appendix I provides a useful set of formulas which help avoiding the tedious task of reading and interpolating from tables or diagrams the values of parameters a^*, b^* and δ^* necessary for Hertz contact. They can be readily programmed in software or electronic sheets.

Appendix II links the geometry developments of Section 3 to matrix invariants and to eigenvalues / eigenvectors, thus providing a perhaps faster and certainly more elegant alternative approach.

10. Appendix I: Brewe-Hamrock approximation (1/7)

When writing software to calculate Hertz contact stresses and elastic approach it may be inconvenient to use graphs of a*, b* and δ* like those of this Chapter, Sect. 4, sl. 9, 10, 11. It may be expedient to use the Brewe and Hamrock* approximation. Using a least squares regression they expressed the elliptic integrals $\mathcal{E}$, $\mathcal{F}$:

$$\mathcal{E} \approx 1{,}0003 + \frac{0{,}5968}{\left(\frac{R_Y}{R_X}\right)} \qquad \mathcal{F} \approx 1{,}5277 + 0{,}6023 \ln\left(\frac{R_Y}{R_X}\right)$$

and $\kappa = a/b$, the ratio of the projected contact ellipse semi-axes (where "a" is the major semi-axis, "b" is the minor):

$$\kappa = \frac{a}{b} \approx 1{,}0339 \left(\frac{R_Y}{R_X}\right)^{0{,}636} \qquad \text{with: } \begin{cases} R_X^{-1} = 2\gamma_X \\ R_Y^{-1} = 2\gamma_Y \end{cases}$$

* Brewe, D. and Hamrock, B., Simplified solution for elliptical-contact deformation between two elastic solids, ASME Trans. J. Lub. Tech., 101(2), 231–239, 1977

The coefficients a*, b* and δ* are then calculated:

$$a^* = \left(\frac{2\kappa^2\mathcal{E}}{\pi}\right)^{1/3} \qquad b^* = \left(\frac{2\mathcal{E}}{\pi\kappa}\right)^{1/3} \qquad \delta^* = \frac{2\mathcal{F}}{\pi}\left(\frac{\pi}{2\kappa^2\mathcal{E}}\right)^{1/3}$$

Additional remarks - Let us remember that there is a term (see Sect. 3 sl. 15 and Sect. 5 sl. 3) which we name "curvature sum":

$$2\sum\frac{1}{\tilde{d}} = \sum\frac{1}{\tilde{r}} = 2(\gamma_X + \gamma_Y) = 2(\alpha_X + \alpha_Y + \beta_X + \beta_Y)$$

and a term which we can now name "curvature difference":

$$2(\gamma_X - \gamma_Y) = \pm 2\sqrt{(\alpha_X - \alpha_Y)^2 + (\beta_X - \beta_Y)^2 + 2(\alpha_X - \alpha_Y)(\beta_X - \beta_Y)\cos 2\vartheta}$$

which in the most common technical case $\vartheta = 0$:

$$2(\gamma_X - \gamma_Y) = 2[(\alpha_X - \alpha_Y) + (\beta_X - \beta_Y)]$$

10. Appendix I: Brewe-Hamrock approximation (3/7)

from which the positive ausiliary function:

$$\frac{|\gamma_X - \gamma_Y|}{\gamma_X + \gamma_Y} = \cos\tau = \frac{|(\alpha_X - \alpha_Y) + (\beta_X - \beta_Y)|}{\alpha_X + \alpha_Y + \beta_X + \beta_Y} \quad (\text{for } \vartheta = 0)$$

Note: it may be worth reminding that each system (α_Y, α_Y) , (β_X, β_Y) , (γ_X, γ_Y) was defined in its own principal axes. However when $\vartheta = 0$ axes X, Y are the same for all systems, as in the example of sl.2 of this Section.

In this case: $\begin{cases} 2\gamma_X = R_X^{-1} = 2(\alpha_X + \beta_X) \\ 2\gamma_Y = R_Y^{-1} = 2(\alpha_Y + \beta_Y) \end{cases}$

therefore: $$\cos\tau = \frac{|(\alpha_X + \beta_X) - (\beta_Y + \alpha_Y)|}{(\alpha_X + \alpha_Y) + (\beta_X + \beta_Y)} = \frac{\left|(R_X^{-1}) - (R_Y^{-1})\right|}{(R_X^{-1}) + (R_Y^{-1})} = \frac{\frac{R_Y}{R_X} - 1}{\frac{R_Y}{R_X} + 1}$$

This formula is valid only when choosing $R_y \geq R_X$, because $\cos\tau \geq 0$

.

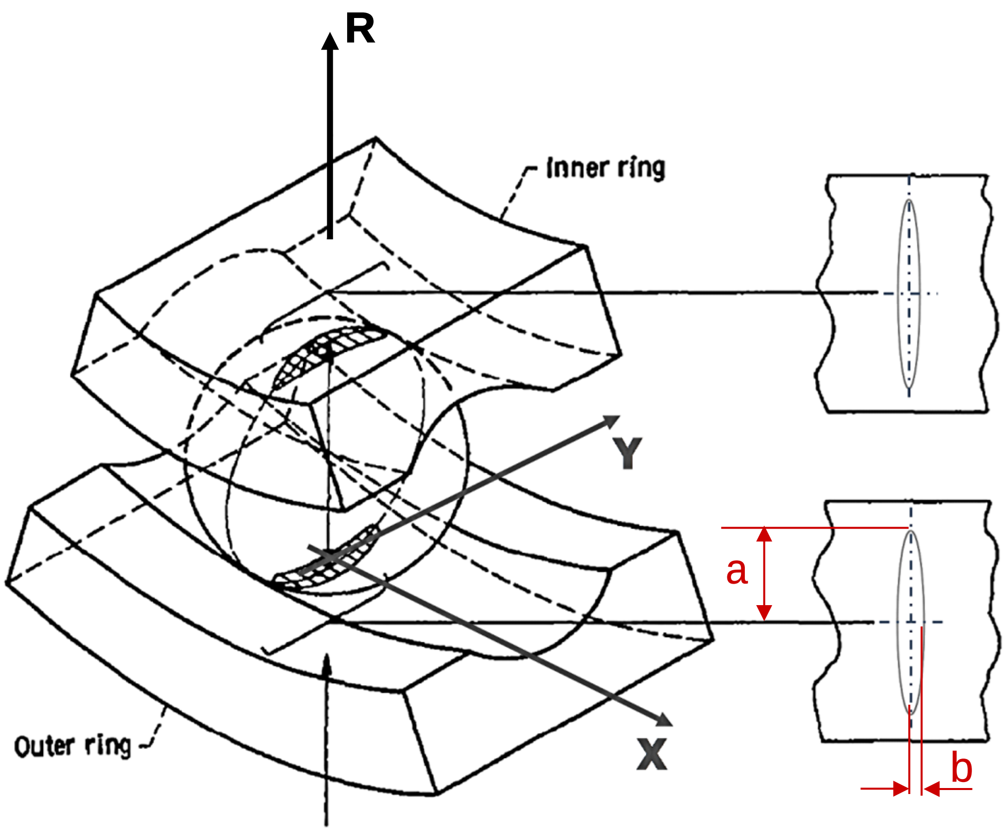
R_X and R_Y may be called "directional equivalent radii", a concept which it is interesting to illustrate in the case of a bearing.

In Ch. 2 Sect. 2 sl. 1 the figure below appears, and contact conditions calculated.

Pay attention: in this case all principal axes of bodies in contact coincide!

Take as an example the case of contact at the outer ring.

Name α the ball and β the raceway reciprocals of diameters.



R_X and R_Y are:

$$\begin{cases} R_X^{-1} = 2\gamma_X = 2(\alpha_X + \beta_X) \\ R_Y^{-1} = 2\gamma_Y = 2(\alpha_Y + \beta_Y) \end{cases}$$

Ball:

$$\alpha_X = \alpha_Y = 1 / d_s$$

Outer ring contact:

$$\beta_{Xo} = -1 / D_o$$

$$\beta_{Yo} \cong -1 / (1{,}05\ d_s)$$

10. Appendix I: Brewe-Hamrock approximation (5/7)

In this case then:

$$R_X^{-1} = 2(\alpha_X + \beta_X) = 2\left(\frac{1}{d_s} - \frac{1}{D_o}\right) =$$

$$R_X = \frac{1}{2} d_s \frac{1}{\left(1 - \frac{d_s}{D_o}\right)}$$

"Directional equivalence" along Y means that the in the cross section in the **X-R** plane it is equivalent to having a ball of radius R_X pressed on a flat body $\beta'_X=0$.

Example: with $d_s=10$, $D_o=40$ mm:

$R_X=6,67$ mm

i.e., the ball against a large but concave raceway (outer ring) is equivalent to a slightly larger ball against a plane, as if:

$$R_X=2(\alpha'_X + 0)=2\,\frac{1}{13,3}$$

$$R_Y^{-1} = 2(\alpha_Y + \beta_Y) = 2\,\frac{0,05}{1,05\,d_s}$$

$$R_Y = \frac{1}{2} d_s \frac{1,05}{0,05}$$

"Directional equivalence" along Y means that the in the cross section in the **Y-R** plane it is equivalent to having a ball of radius R_Y pressed on a flat body $\beta'_Y=0$.

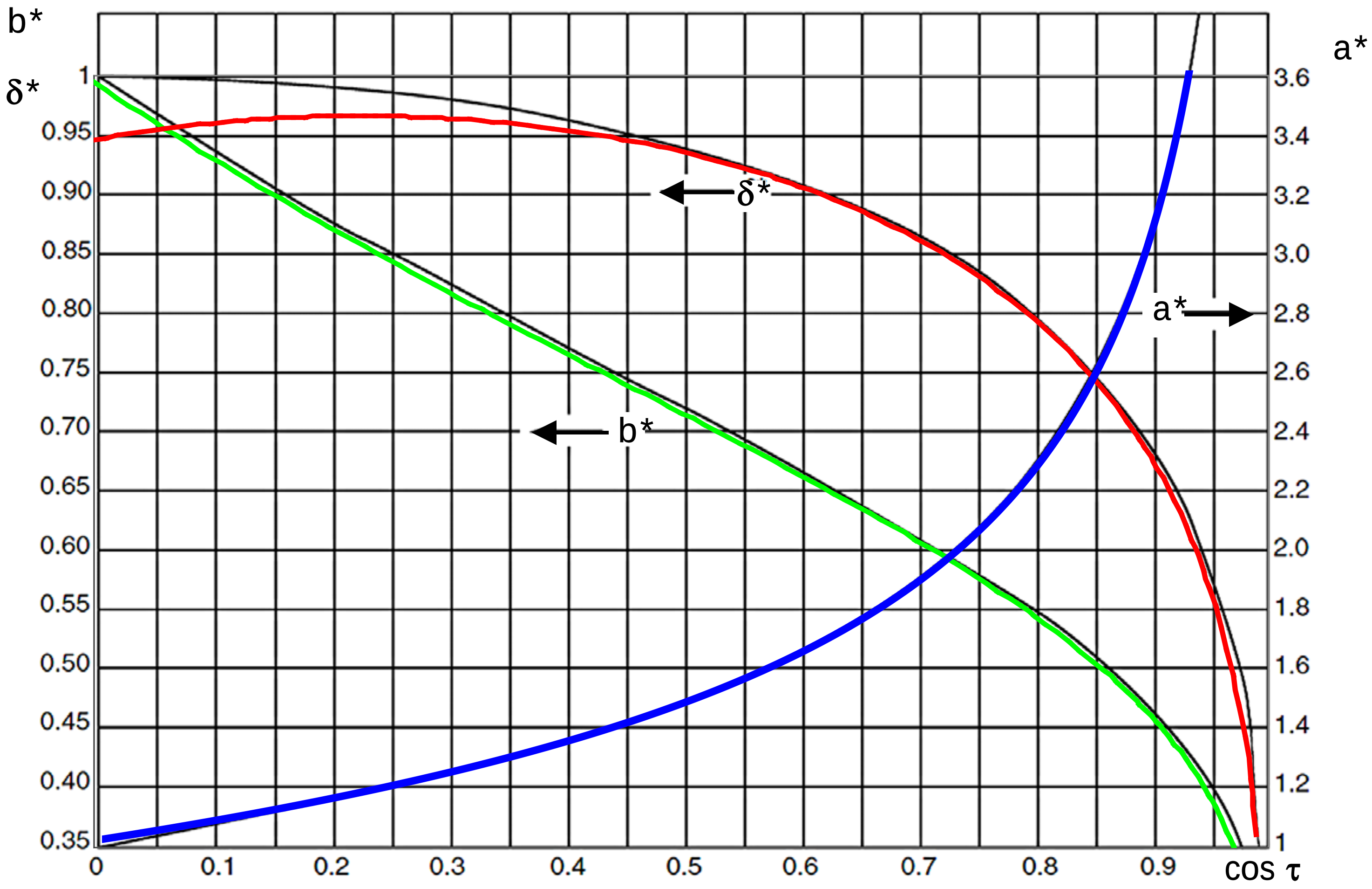
Example: with $d_s=10$, $D_o=40$ mm:

$R_Y=105$ mm

i.e., the highly conforming contact is equivalent to a much larger ball against a plane, as if:

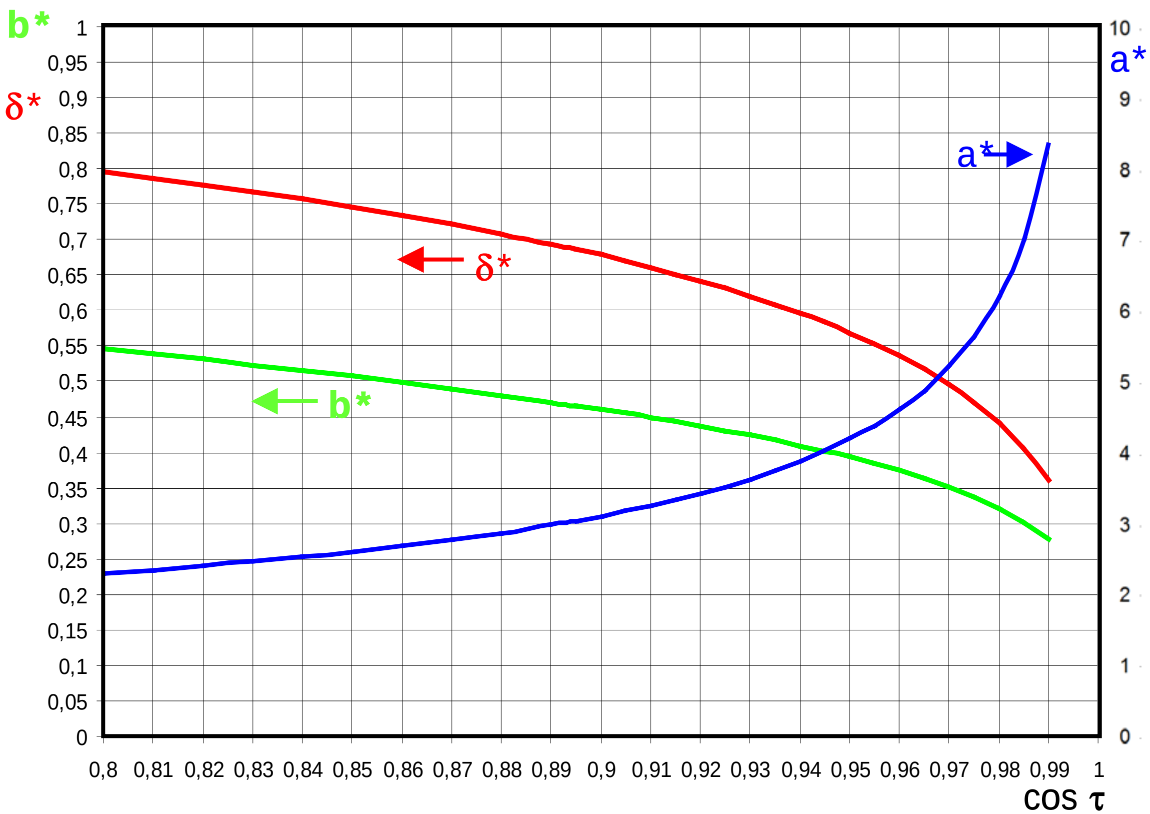
$$R_Y=2(\alpha'_Y + 0)=2\,\frac{1}{210}$$

Rather than the ratio R_Y/R_X it is more convenient to use the function $\cos\tau$ to plot a*, b* and δ*. The figure below compares exact results of Sect.4 sl.9 (black) with those from Brewe-Hamrock formulas.



Note that the approximation is fully acceptable for engineering cases, however for δ* lower than 0,5 it tends to worsen up to a 5% error at $\cos\tau=1$. Instead of taking value 1, as they should, at $\cos\tau=0$ we get the approximate values:

a*= 1,028
b*= 0,994
δ*= 0,946



Ry/Rx	a*	b*	δ*	cosτ
9,0	2,28	0,55	0,80	0,8
9,5	2,33	0,54	0,79	0,81
10,1	2,39	0,53	0,78	0,82
10,8	2,45	0,52	0,77	0,83
11,5	2,52	0,52	0,76	0,84
12,3	2,59	0,51	0,75	0,85
13,3	2,67	0,50	0,73	0,86
14,4	2,76	0,49	0,72	0,87
15,7	2,86	0,48	0,71	0,88
17,2	2,97	0,47	0,69	0,89
18,0	3,03	0,47	0,69	0,895
19,0	3,10	0,46	0,68	0,9
20,1	3,17	0,46	0,67	0,905
21,2	3,24	0,45	0,66	0,91
22,5	3,32	0,44	0,65	0,915
24,0	3,41	0,44	0,64	0,92
25,7	3,51	0,43	0,63	0,925
27,6	3,62	0,42	0,62	0,93
29,8	3,73	0,42	0,61	0,935
32,3	3,86	0,41	0,60	0,94
35,4	4,01	0,40	0,58	0,945
39,0	4,18	0,39	0,57	0,95
43,4	4,37	0,38	0,55	0,955
49,0	4,60	0,37	0,54	0,96
56,1	4,87	0,36	0,52	0,965
65,7	5,20	0,35	0,50	0,97
79,0	5,62	0,34	0,47	0,975
99,0	6,19	0,32	0,44	0,98
132,3	6,99	0,30	0,41	0,985
199,0	8,31	0,28	0,36	0,99

This graph and this table provide some of the values that can be obtained from the formulas of sl.2 of this Appendix I .

In Sect.3, page 12, we have seen how to find the principal relative half-curvatures γ_X and γ_Y from the solutions of an eigenvalue problem.

This Appendix will treat the same problem as a trigonometrical one and will of course get to the same (happy) end.

On the one hand, the body profile is easily written in the principal reference frame (for the curvature),

$$z_1 = \frac{1}{2}\frac{1}{R_{X1}}X_1^2 + \frac{1}{2}\frac{1}{R_{Y1}}Y_1^2$$

$$z_2 = -\left(\frac{1}{2}\frac{1}{R_{X2}}X_2^2 + \frac{1}{2}\frac{1}{R_{Y2}}Y_2^2\right)$$

where the use of capital letters X and Y means a principal reference frame (different for each body) is used.

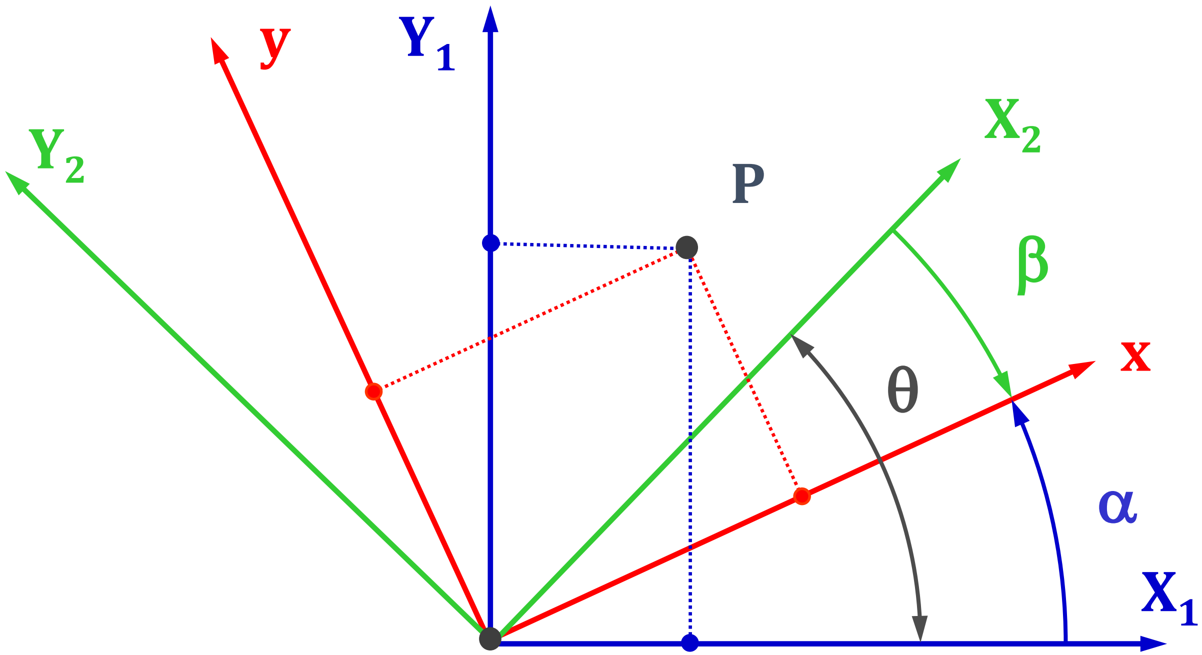
On the other hand, the difference $h_0 = z_1 - z_2$ must be written in a common reference frame, namely {x, y}.

In a rotated reference frame, the rotation matrix is:

$$\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

with the sign convention that both angles α and β are positive as in the sketch



The bodies profile are rewritten in the common reference frame

$$\begin{cases} X_1 = x\cos\alpha - y\sin\alpha \\ Y_1 = x\sin\alpha + y\cos\alpha \end{cases}$$

$$z_1 = \frac{1}{2}\frac{1}{R_{X1}}X_1^2 + \frac{1}{2}\frac{1}{R_{Y1}}Y_1^2 = \frac{1}{2}\frac{1}{R_{X1}}[x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha] +$$

$$+\frac{1}{2}\frac{1}{R_{Y1}}[x^2\sin^2\alpha + y^2\cos^2\alpha + 2xy\sin\alpha\cos\alpha]$$

$$z_2 = -\left(\frac{1}{2}\frac{1}{R_{X2}}X_2^2 + \frac{1}{2}\frac{1}{R_{Y2}}Y_2^2\right) = -\frac{1}{2}\frac{1}{R_{X2}}[x^2\cos^2\beta + y^2\sin^2\beta + 2xy\sin\beta\cos\beta] +$$

$$-\frac{1}{2}\frac{1}{R_{Y2}}[x^2\sin^2\beta + y^2\cos^2\beta - 2xy\sin\beta\cos\beta]$$

$$\begin{cases} X_2 = x\cos\beta + y\sin\beta \\ Y_2 = -x\sin\beta + y\cos\beta \end{cases}$$

The bodies profile are rewritten in the common reference frame

$$z_1 = \left(\frac{1}{2}\frac{1}{R_{X1}}\cos^2\alpha + \frac{1}{2}\frac{1}{R_{Y1}}\sin^2\alpha\right)x^2 + \left(\frac{1}{2}\frac{1}{R_{X1}}\sin^2\alpha + \frac{1}{2}\frac{1}{R_{Y1}}\cos^2\alpha\right)y^2 + \\ -\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)2\sin\alpha\cos\alpha\, xy$$

$$z_2 = -\left(\frac{1}{2}\frac{1}{R_{X2}}\cos^2\beta + \frac{1}{2}\frac{1}{R_{Y2}}\sin^2\beta\right)x^2 - \left(\frac{1}{2}\frac{1}{R_{X2}}\sin^2\beta + \frac{1}{2}\frac{1}{R_{Y2}}\cos^2\beta\right)y^2 + \\ -\left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)2\sin\beta\cos\beta\, xy$$

The distance h_0 between the bodies in contact (geometric contact, i.e. without external loading) is

$$h_0 = z_1 - z_2 = \gamma_{xx}x^2 + \gamma_{yy}y^2 + 2\gamma_{xy}xy$$

The coupling term γ_{xy} is

$$2\gamma_{xy} = \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right) 2\sin\beta\cos\beta - \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right) 2\sin\alpha\cos\alpha$$

Or, by using the double-angle trigonometric formulae

$$2\sin\beta\cos\beta = \sin 2\beta$$
$$2\sin\alpha\cos\alpha = \sin 2\alpha$$

$$2\gamma_{xy} = \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\sin 2\beta - \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\sin 2\alpha$$

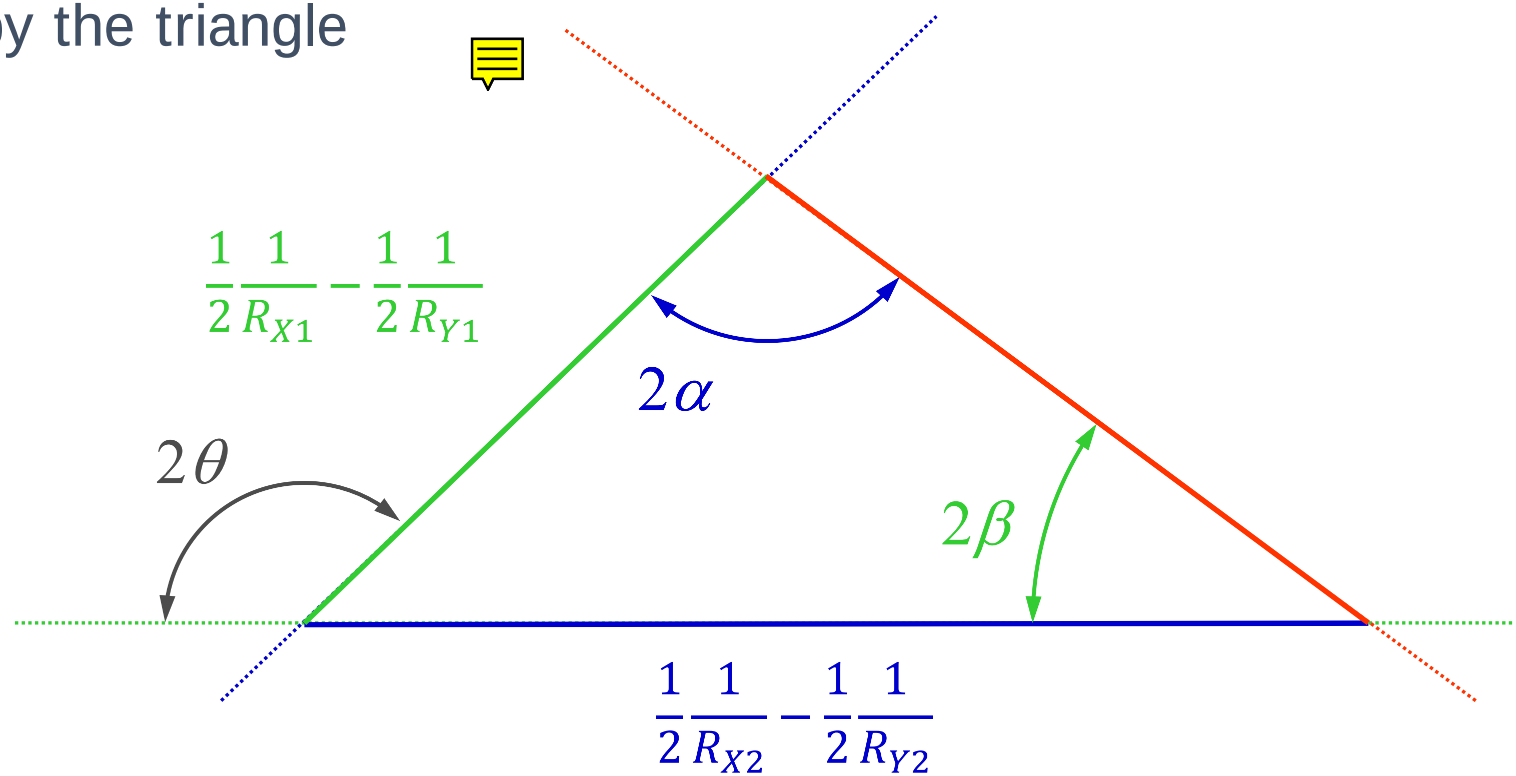
In order that the reference frame (xy) is a principal reference frame the coupling γ_{xy} term must be zero

$$2\gamma_{xy} = \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\sin 2\beta - \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\sin 2\alpha = 0$$

A null coupling term $\gamma_{xy} = 0$ means

$$\frac{\sin 2\beta}{\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)} = \frac{\sin 2\alpha}{\left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)}$$

This condition can be easily graphically represented using the law of sines by the triangle



It is easily to find the difference between the relative curvature $\gamma_X - \gamma_Y$.
Now capital letters are being used as the reference frame is principal

$$h_0 = z_1 - z_2 = \gamma_{xx}x^2 + \gamma_{yy}y^2 + \gamma_{xy}xy = \gamma_X X^2 + \gamma_Y Y^2$$

with

$$\gamma_X = \left(\frac{1}{2}\frac{1}{R_{X1}}\cos^2\alpha + \frac{1}{2}\frac{1}{R_{Y1}}\sin^2\alpha\right) + \left(\frac{1}{2}\frac{1}{R_{X2}}\cos^2\beta + \frac{1}{2}\frac{1}{R_{Y2}}\sin^2\beta\right)$$

$$\gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}}\sin^2\alpha + \frac{1}{2}\frac{1}{R_{Y1}}\cos^2\alpha\right) + \left(\frac{1}{2}\frac{1}{R_{X2}}\sin^2\beta + \frac{1}{2}\frac{1}{R_{Y2}}\cos^2\beta\right)$$

Then

$$\gamma_X + \gamma_Y = \frac{1}{2}\left(\frac{1}{R_{X1}} + \frac{1}{R_{Y1}} + \frac{1}{R_{X2}} + \frac{1}{R_{Y2}}\right)$$

$$\gamma_X - \gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)(\cos^2\alpha - \sin^2\alpha) + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)(\cos^2\beta - \sin^2\beta)$$

By using the Werner formulas

$$\gamma_X - \gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)(\cos^2\alpha - \sin^2\alpha) + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)(\cos^2\beta - \sin^2\beta)$$

$$\cos^2\alpha \rightarrow \frac{1}{2}\cos(2\alpha) \qquad \sin^2\alpha \rightarrow -\frac{1}{2}\cos(2\alpha) \qquad \cos^2\beta \rightarrow \frac{1}{2}\cos(2\beta) \qquad \sin^2\beta \rightarrow -\frac{1}{2}\cos(2\beta)$$

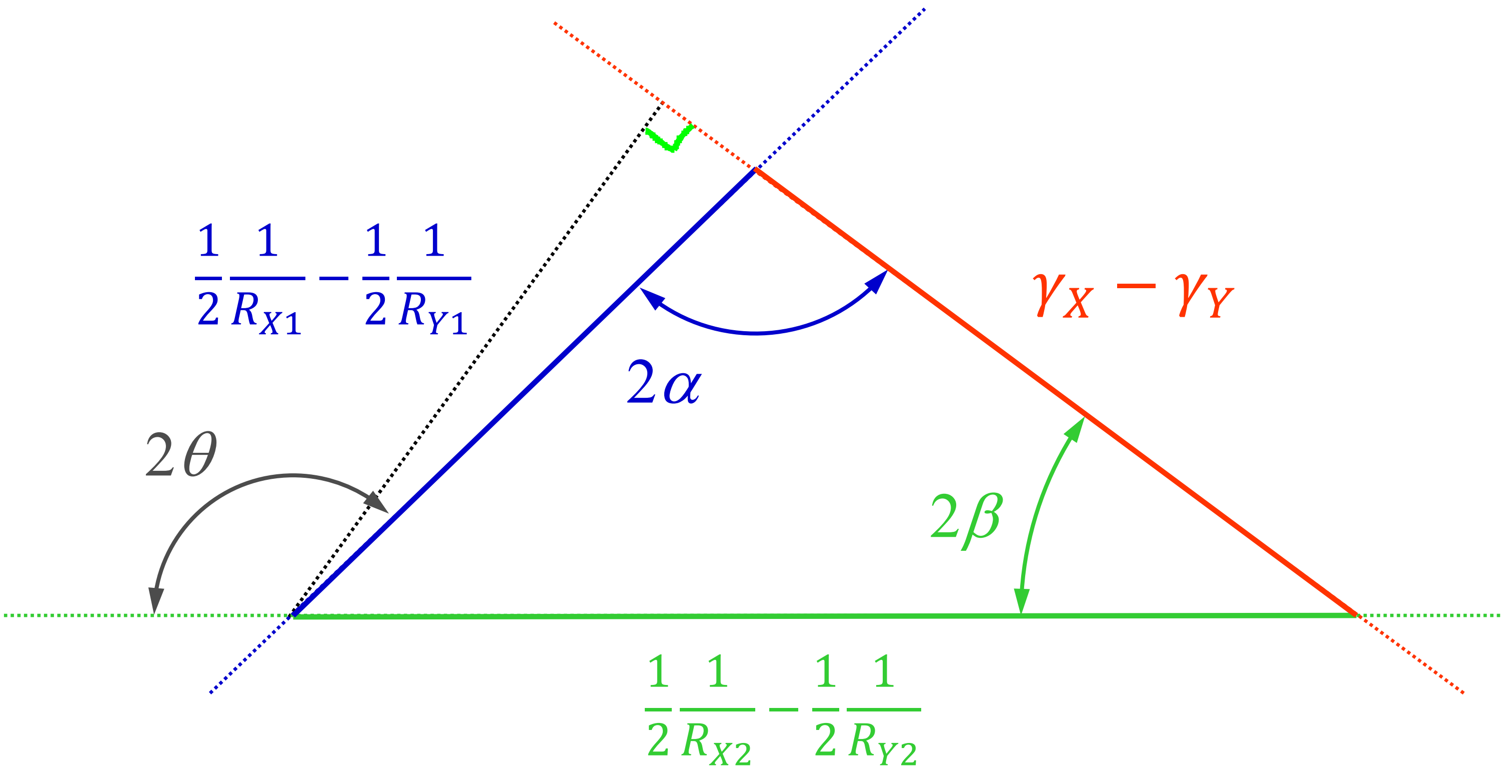
The equation becomes

$$\gamma_X - \gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\cos 2\alpha + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\cos 2\beta$$

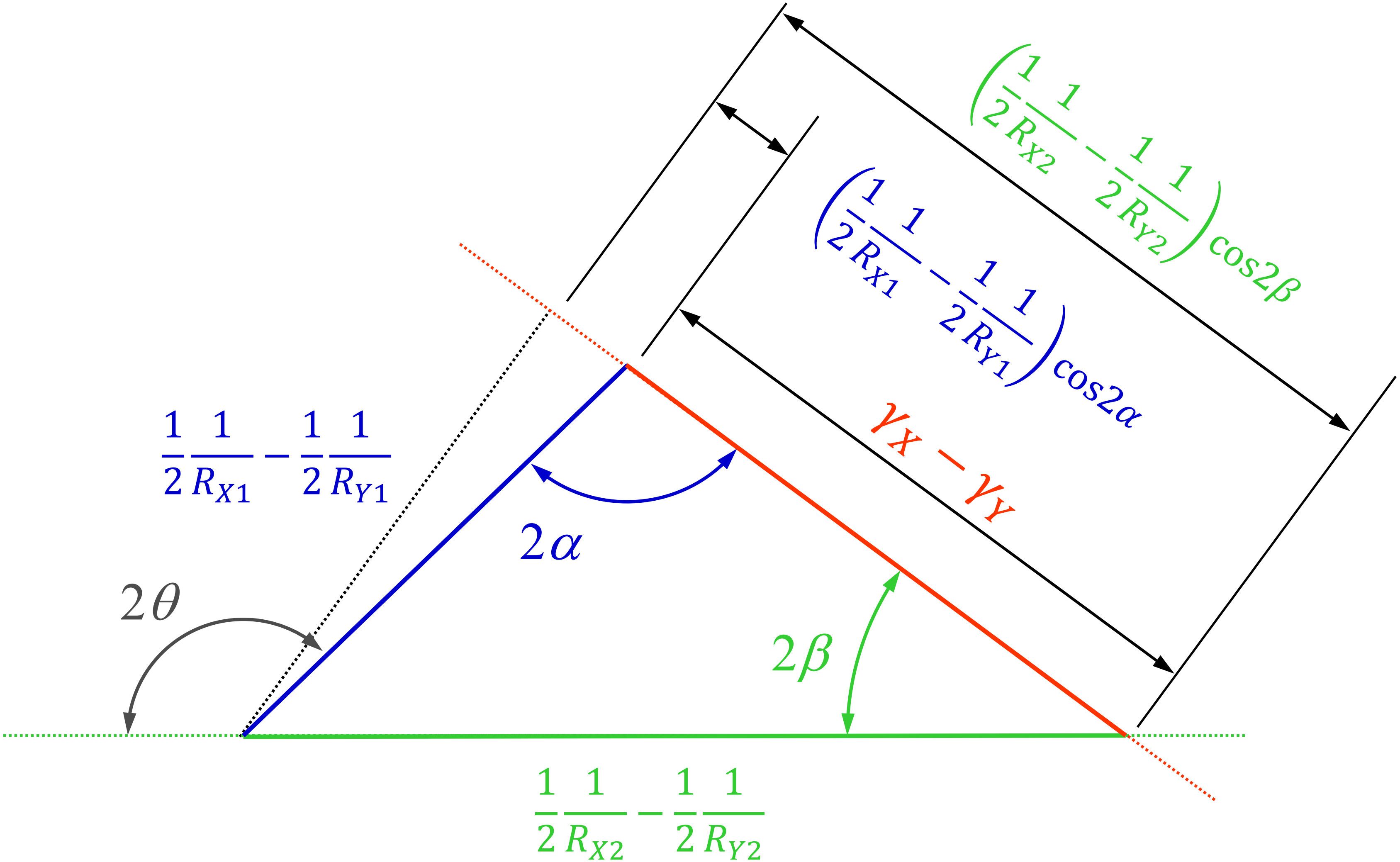
And it is easily to see that the difference

$$\gamma_X - \gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\cos 2\alpha + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\cos 2\beta$$

is geometrical represented by the third side of the triangle.



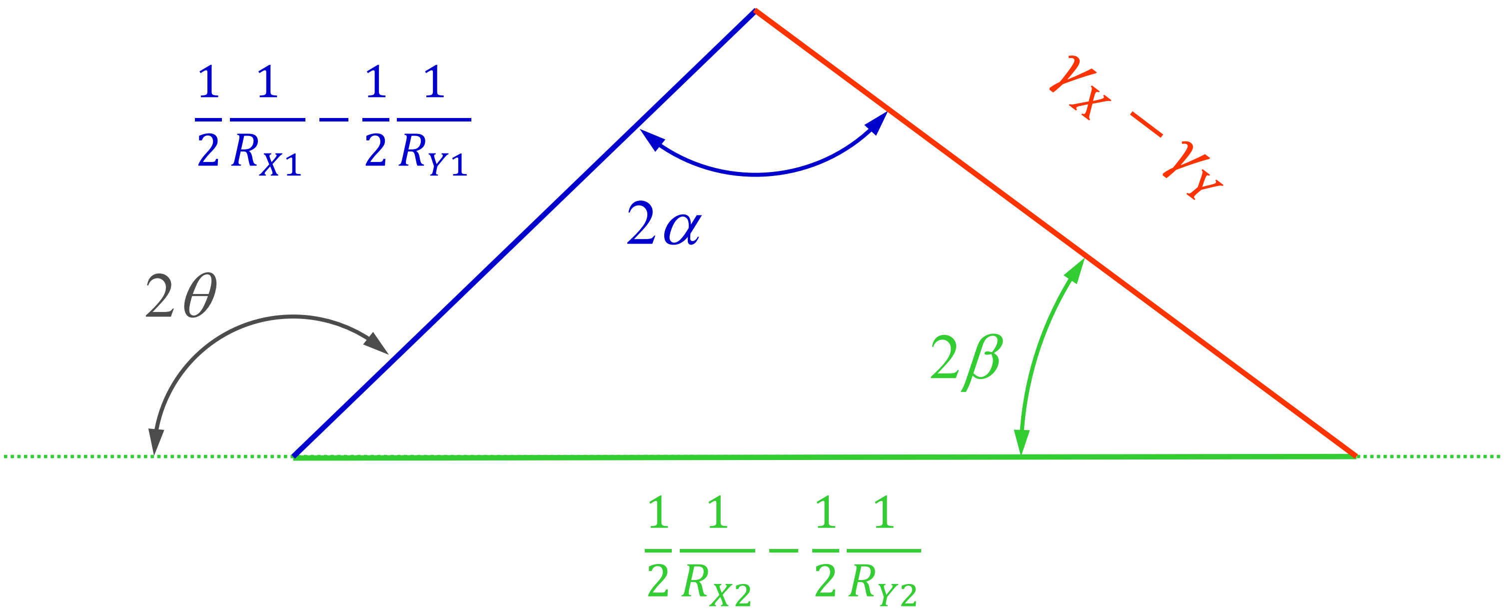
Also remembering that $\cos 2\alpha = -\cos(\pi - 2\alpha)$ so that the shorter line segment is subtracted from the longer segment.



The difference written in this form

$$\gamma_X - \gamma_Y = \left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\cos 2\alpha + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\cos 2\beta$$

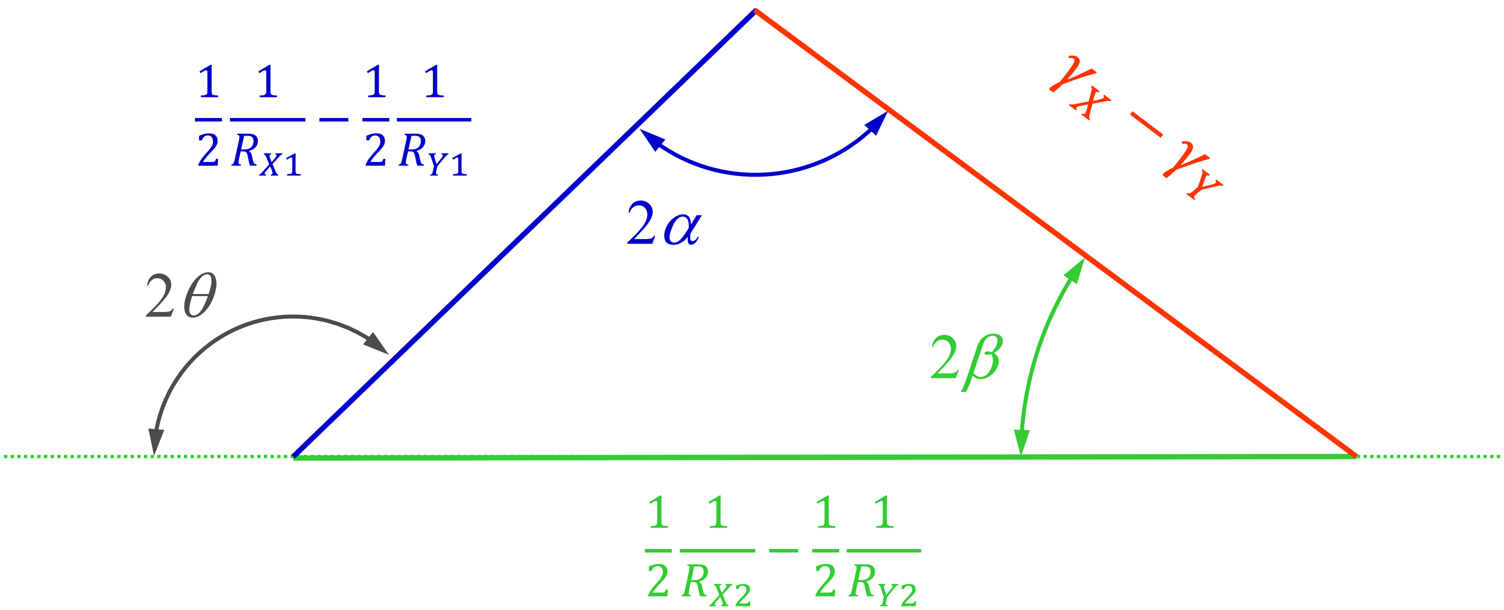
Is not useful, because angles α and β are unknown.
The side $(\gamma_X - \gamma_Y)$ of the triangle can be also computed by using the Carnot's theorem



Then, by applying the Carnot's theorem

$$|\gamma_X - \gamma_Y| =$$

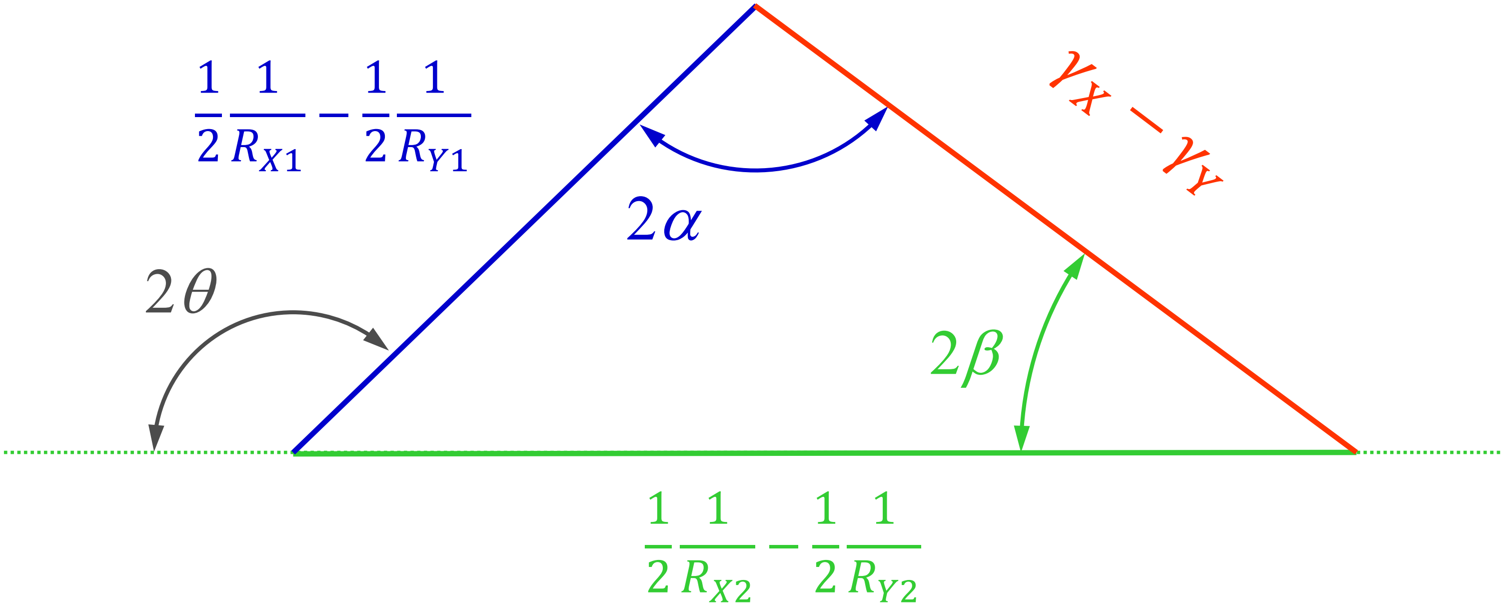
$$= \sqrt{\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)^2 + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)^2 - 2\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\cos(\pi - 2\theta)}$$



Or

$$|\gamma_X - \gamma_Y| =$$

$$= \sqrt{\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)^2 + \left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)^2 + 2\left(\frac{1}{2}\frac{1}{R_{X1}} - \frac{1}{2}\frac{1}{R_{Y1}}\right)\left(\frac{1}{2}\frac{1}{R_{X2}} - \frac{1}{2}\frac{1}{R_{Y2}}\right)\cos 2\theta}$$



While the sum of the relative principal curvature was easily obtained as

$$\gamma_X + \gamma_Y = \frac{1}{2}\left(\frac{1}{R_{X1}} + \frac{1}{R_{Y1}} + \frac{1}{R_{X2}} + \frac{1}{R_{Y2}}\right)$$