Optimization via Gradient Descent

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import math
#np.seterr(all='raise')
```

. For each of the functions above, run the GD method with and without the backtracking, trying different values for the step size α > 0 when you are not using backtracking. Observe the different behavior of GD.

. To help visualization, it is convenient to plot the error vector that contains the ||∇f(xk)||2, to check that it goes to zero. Compare the convergence speed (in terms of the number of iterations k) in the different cases.

. For each of the points above, fix x0= (0, 0, . . . , 0)T, kmax = 100, while choose your values for tolf and tolx. It is recommended to also plot the error ||xk − x∗||2 varying k when the true x∗ is available.

. Only for the non-convex function defined in 5, plot it in the interval [−3, 3] and test the convergence point of GD with different values of x0 and different step-sizes. Observe when the convergence point is the global minimum and when it stops on a local minimum or maximum.

. Hard (optional): For the functions 1 and 2, plot the contour around the minimum and the path defined by the iterations (following the example seen during the lesson). See plt.contour to do that.

In [ ]:
```
def backtracking(f, grad_f, x):
    alpha = 1
    c = 0.8
    tau = 0.25
    # f(x) - c * alpha * np.linalg.norm(grad_f(x), 2) ** 2 : represents the estimate of
    # if f(x - alpha * grad_f(x)) is greater then the estimate, it means that we aren't
    # the estimate is greater then what we obtain from the computation of the gradient.
    # f(x - alpha * grad_f(x)) <= f(x) - c * alpha * np.linalg.norm(grad_f(x), 2) ** 2
    while f(x - alpha * grad_f(x)) > f(x) - c * alpha * np.linalg.norm(grad_f(x), 2) **
        alpha = tau * alpha

        if alpha < 1e-3: #lower bound for the value of alpha
            break
    return alpha

    # per riassumere, la stima della minima decrescita dovrà essere maggiore della decre
    # condizione quello che si fa è decrementare il valore di alpha fino a che non la si
```

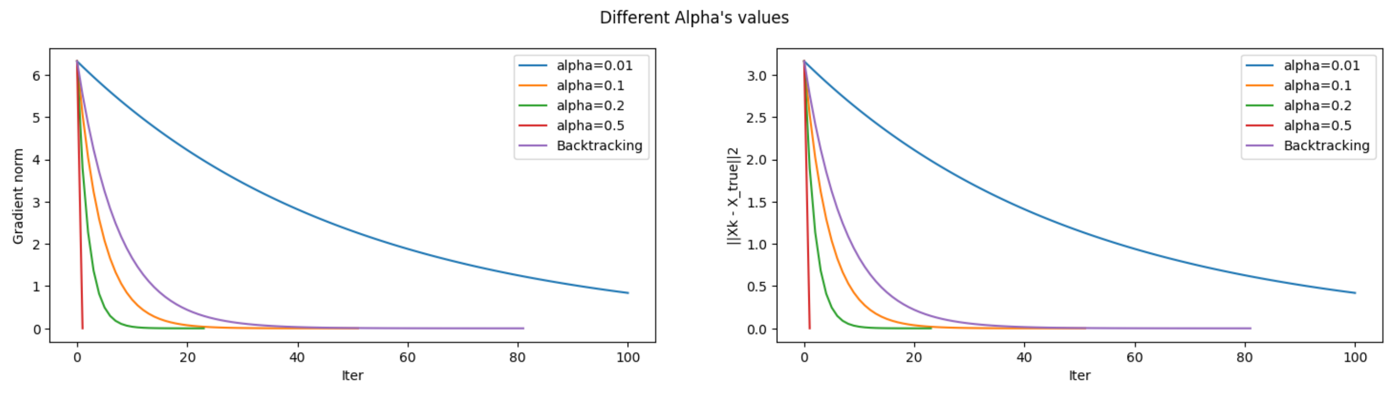
In [ ]:
```
def gd(fn, grad_fn, x0, k_max, tol_f, tol_x, alpha=None): #as loss function we use the f
    curr_x, prev_x = x0, np.inf
    curr_k = 0
    grad_x0, curr_grad = grad_fn(x0), grad_fn(curr_x)
    history_x = [x0]
    history_f = [fn(x0)]
    history_grad = [grad_x0]
    history_err = [np.linalg.norm(grad_x0, 2)]
    use_backtracking = alpha is None # we define this boolean. the condition that define
    # if alpha is None we assign its value using alpha = backtracking(fn, grad_fn, curr_
    # of the function
    while (curr_k < k_max and
            not (np.linalg.norm(curr_grad, 2) < tol_f*np.linalg.norm(grad_x0, 2)) and #h
            not (np.linalg.norm(curr_x - prev_x, 2) < tol_x)): # how much the current an
        if use_backtracking:
            alpha = backtracking(fn, grad_fn, curr_x)
        prev_x = curr_x
        curr_x = curr_x - alpha*grad_fn(curr_x)

        curr_grad = grad_fn(curr_x)
        curr_k += 1

        history_x.append(curr_x)
        history_f.append(fn(curr_x))
        history_grad.append(curr_grad)
        history_err.append(np.linalg.norm(curr_grad, 2))

    return history_x, curr_k, history_f, history_grad, history_err
```

In [ ]:
```
def testVaryingAlpha(fn, grad_fn, input_size, x_true, k_max=100, try_alpha=[0.01, 0.1, 0
    plt.figure(figsize=(18, 4))
    plt.suptitle("Different Alpha's values")
    ax1 = plt.subplot(1, 2, 1)
    ax2 = plt.subplot(1, 2, 2)
    for alpha in try_alpha:
        try:
            history_x, curr_k, history_f, history_grad, history_err = gd(fn, grad_fn, np
            label_alpha = "Backtracking" if alpha is None else f"alpha={alpha}"
            ax1.plot(range(0, len(history_err)), history_err, label=label_alpha) # plot
            ax2.plot(range(0, len(history_x)), [np.linalg.norm(x - x_true, 2) for x in h
        except Exception as e:
            print(f"alpha={alpha}: {e}")
    ax1.set_xlabel("Iter")
    ax1.set_ylabel("Gradient norm")
    ax1.legend()
    ax2.set_xlabel("Iter")
    ax2.set_ylabel("||Xk - X_true||2")
    ax2.legend()
    plt.show()

#just for function4
def testVaryingLambda(fn, input_size, x_true, n=5, k_max=100, try_lambda=[0, 0.1, 0.25,
    plt.figure(figsize=(18, 4))
    plt.suptitle(f"Lambda's values (alpha defined using backtracking)")
    ax1 = plt.subplot(1, 2, 1)
    ax2 = plt.subplot(1, 2, 2)
    for lamb in try_lambda:
        f, grad_f, x_true, input_size = fn(n=n, lamb=lamb)
        history_x, curr_k, history_f, history_grad, history_err = gd(f, grad_f, np.zeros
        ax1.plot(range(0, len(history_err)), history_err, label=f"lamb={lamb}")
        ax2.plot(range(0, len(history_x)), [np.linalg.norm(x - x_true, 2) for x in histo
    ax1.set_xlabel("Iter")
    ax1.set_ylabel(f"Gradient norm")
    ax1.legend()
    ax2.set_xlabel("Iter")
    ax2.set_ylabel("||Xk - X_true||2")
    ax2.legend()
    plt.show()


def showContour(fn, grad_fn, x_true, x0, k_max, tol_f, tol_x, alpha=None, contour_area=(
    history_x, curr_k, history_f, history_grad, history_err = gd(fn, grad_fn, x0, k_max,
    x = np.linspace(contour_area[0], contour_area[1], 1000)
    y = np.linspace(contour_area[0], contour_area[1], 1000)
    x_contour, y_contour = np.meshgrid(x, y)
    z_contour = fn((x_contour, y_contour))

    history_f.sort()
    to_visualize_levels = [history_f[0]] + [history_f[i] for i in range(1, len(history_f
    # to_visualize_levels = history_f
    contour_graph = plt.contour(x, y, z_contour, levels=to_visualize_levels)
    plt.clabel(contour_graph, inline=1, fontsize=10)
    plt.scatter([a[0] for a in history_x], [a[1] for a in history_x], marker="o") #gradi
    plt.scatter(x_true[0], x_true[1], marker="x", c="red", label="Optima") # solution
    plt.legend()
    plt.show()
```

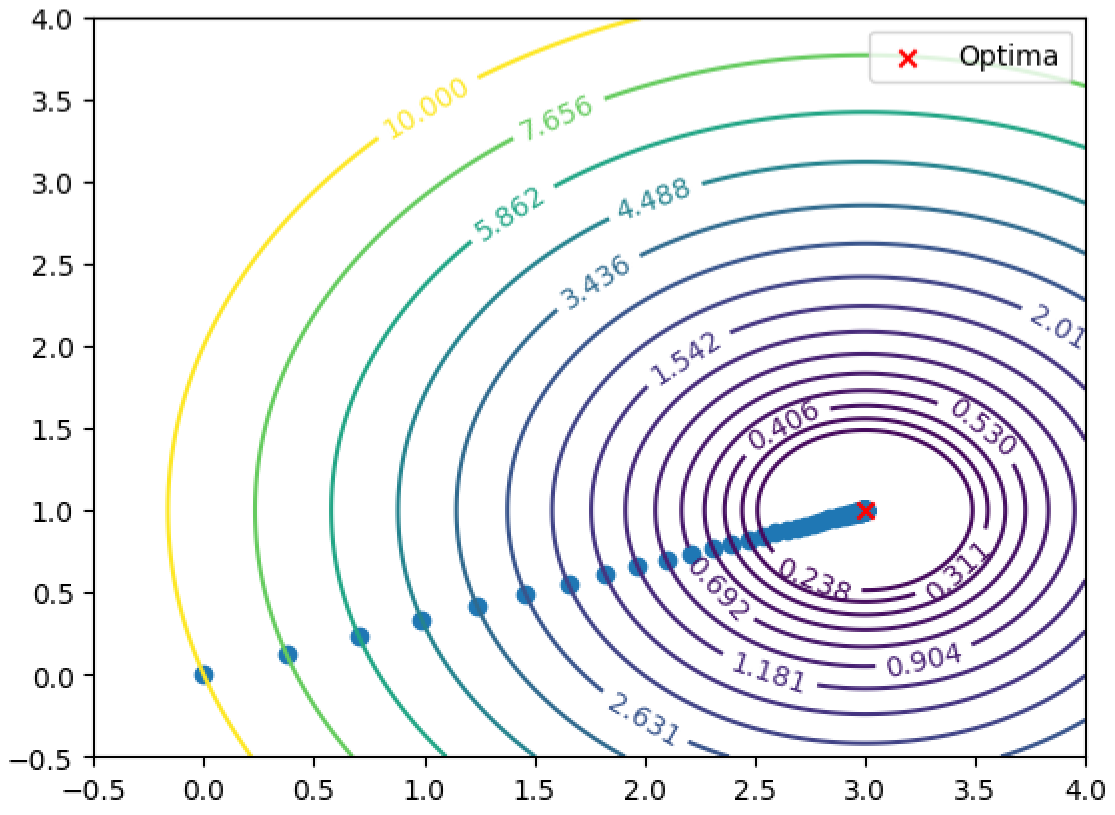
In [ ]:
```
def function1():
    def f(x):
        x1, x2 = x
        return (x1 - 3)**2 + (x2 - 1)**2

    def grad_f(x):
        x1, x2 = x
        return np.array([ 2*(x1-3), 2*(x2-1) ])

    return f, grad_f, np.array([3, 1]), 2


def function2():
    def f(x):
        x1, x2 = x
        return 10*(x1 - 1)**2 + (x2 - 2)**2

    def grad_f(x):
        x1, x2 = x
        return np.array([ 20*(x1-1), 2*(x2-2) ])

    return f, grad_f, np.array([1, 2]), 2


def function3(n=5):
    x_true = np.ones((n,)) # define x_true as a vector of ones
    A = np.vander(np.linspace(0, 1, n))
    b = A @ x_true
    def f(x):
        return (1/2) * np.linalg.norm(A@x - b, 2)**2

    def grad_f(x):
        return (x.T @ A.T @ A - b.T @ A)

    return f, grad_f, x_true, n


def function4(n=5, lamb=0.1):
    x_true = np.ones((n,))
    A = np.vander(np.linspace(0, 1, n))
    b = A @ x_true

    def f(x):
        return (1/2)*np.linalg.norm(A@x - b, 2)**2 + (lamb/2)*np.linalg.norm(x)**2

    def grad_f(x):
        return (x.T @ A.T @ A - b.T @ A) + (lamb*x.T)

    return f, grad_f, x_true, n


def function5():
    def f(x):
        return x**4 + x**3 - 2*x**2 - 2*x

    def grad_f(x):
        return 4*x**3 + 3*x**2 - 4*x - 2

    return f, grad_f, None, 1
```

function 1

```
In [ ]: f, grad_f, x_true, input_size = function1()
        testVaryingAlpha(f, grad_f, input_size, x_true)
```

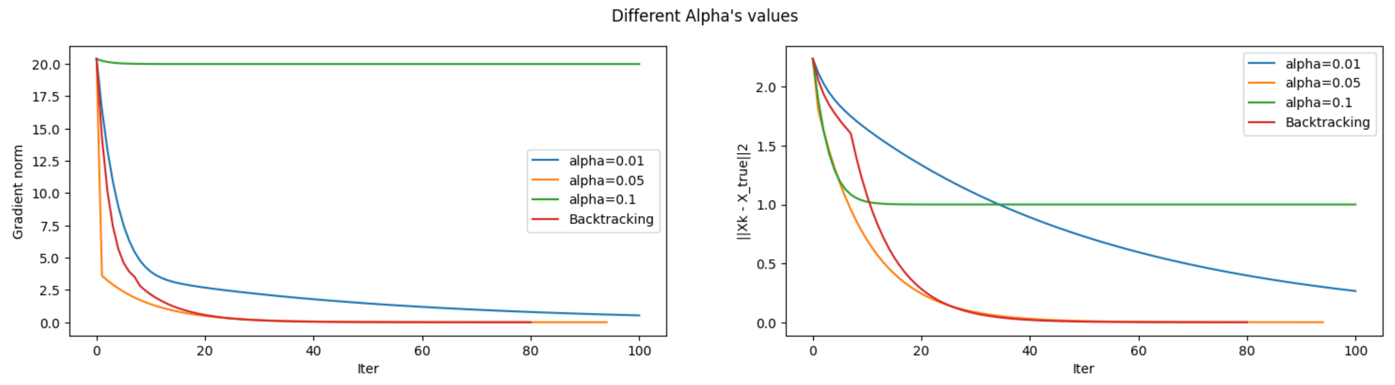


```
In [ ]: showContour(f, grad_f, x_true,x0 = np.zeros((input_size,)), k_max = 200, tol_f = 1e-5, t
```

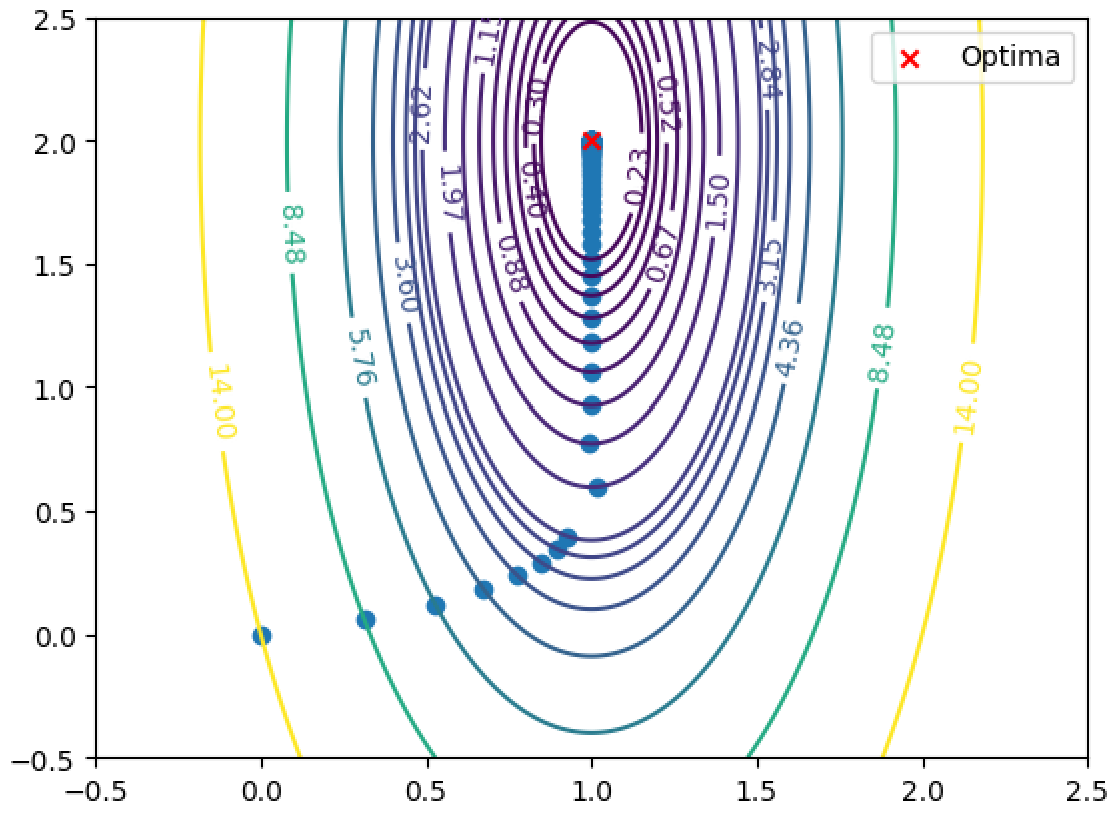


function 2

```
In [ ]: f, grad_f, x_true, input_size = function2()
        testVaryingAlpha(f, grad_f, input_size, x_true, try_alpha=[0.01, 0.05, 0.1, None])
        #testVaryingX0(f, grad_f, input_size, x_true)
        #testVaryingTolerance(f, grad_f, input_size, x_true)
```



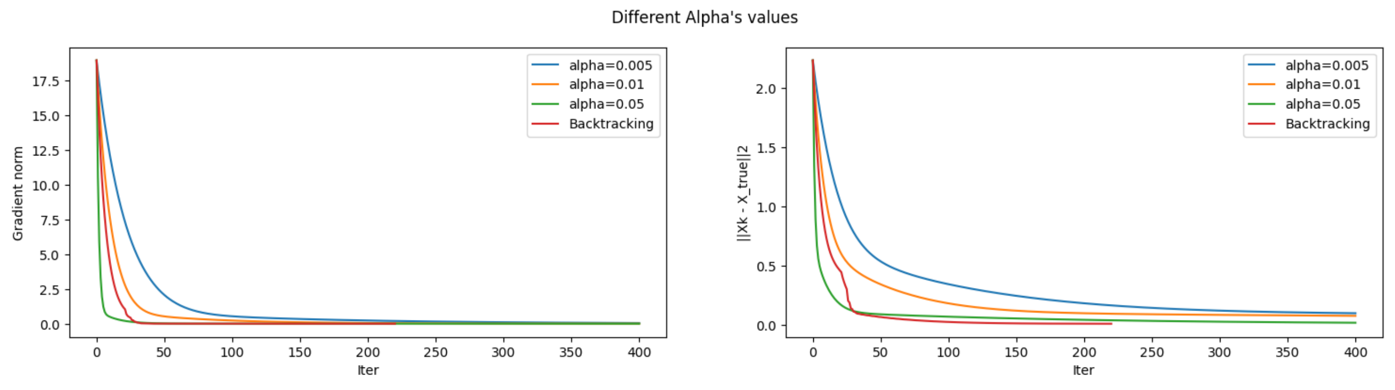
```
In [ ]: showContour(f, grad_f, x_true,x0 = np.zeros((input_size,)),k_max = 200,tol_f = 1e-5,tol_
```



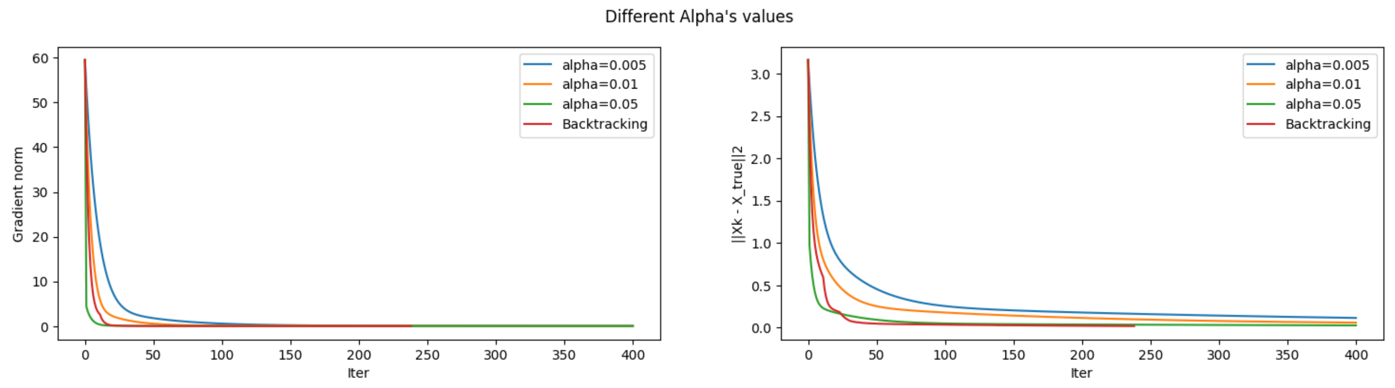
function 3

```
In [ ]: n_values = [5,10,15]
        for n in n_values:
          print(f"Vandermonde matrix N value equal to {n}")
          f, grad_f, x_true, input_size = function3(n)
          testVaryingAlpha(f, grad_f, input_size, x_true,k_max = 400, try_alpha=[0.005, 0.01, 0.
          print('\n')
```

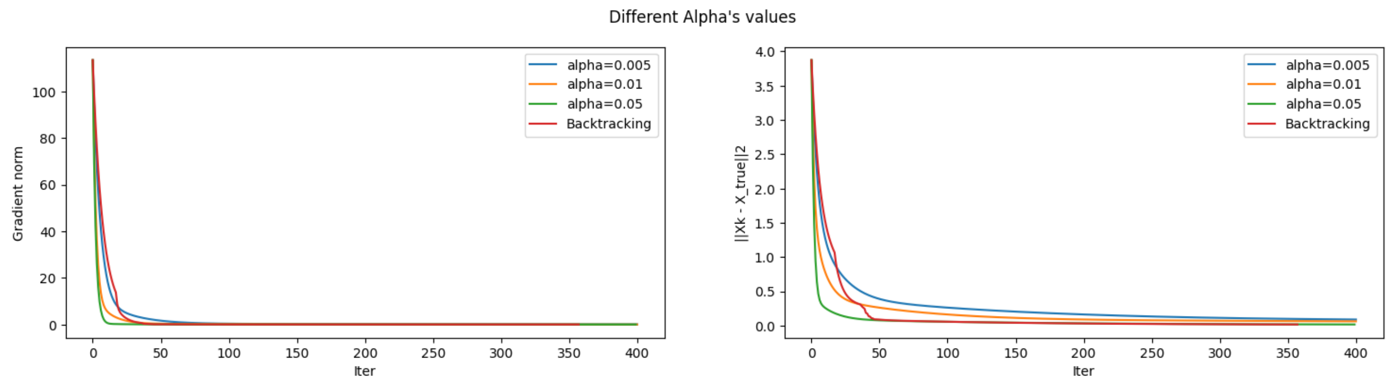
```
Vandermonde matrix N value equal to 5
```



```
Vandermonde matrix N value equal to 10
```



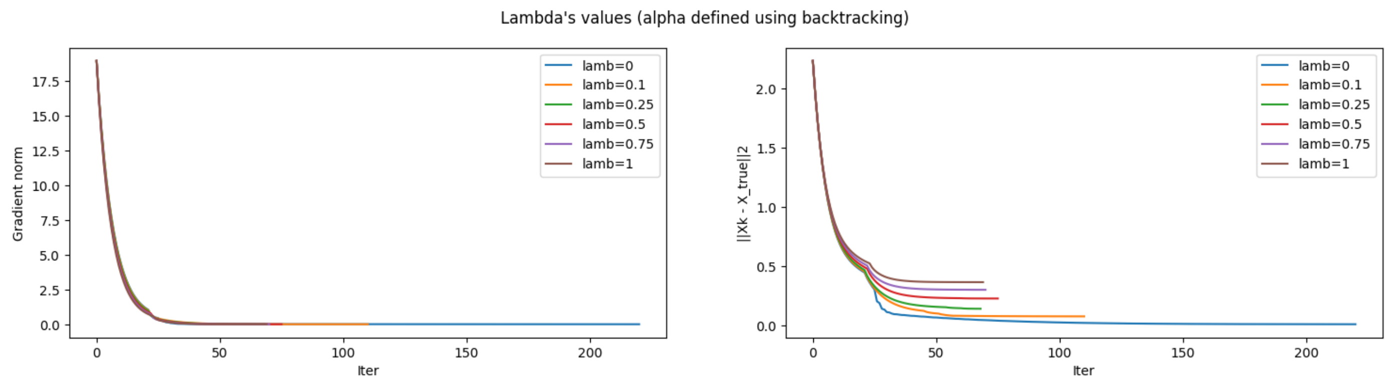
```
Vandermonde matrix N value equal to 15
```

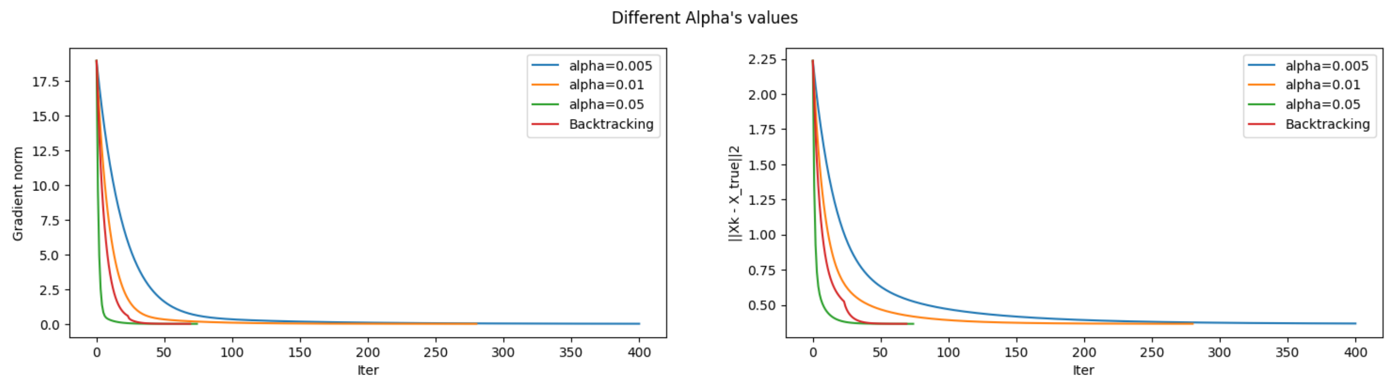


function 4

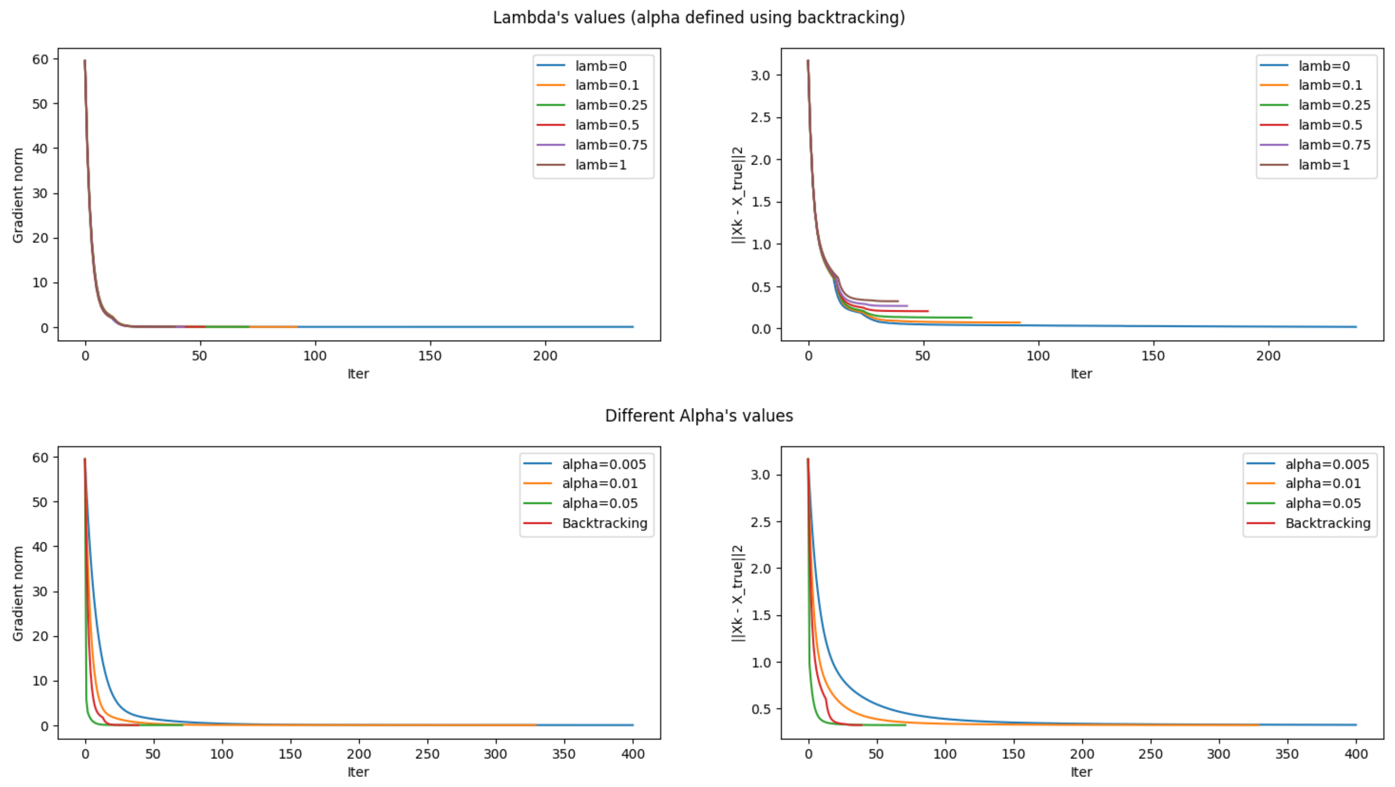
```
In [ ]: n_values = [5,10,15]
        for n in n_values:
          print(f"Vandermonde matrix N value equal to {n}, Lambda equal to 1 for different value
          f, grad_f, x_true, input_size = function4(n, lamb=1)
          testVaryingLambda(function4, input_size, x_true, n=n, k_max = 400)
          testVaryingAlpha(f, grad_f, input_size, x_true, try_alpha=[0.005, 0.01, 0.05, None], k
          print('\n')
```

```
Vandermonde matrix N value equal to 5, Lambda equal to 1 for different values of alpha
```

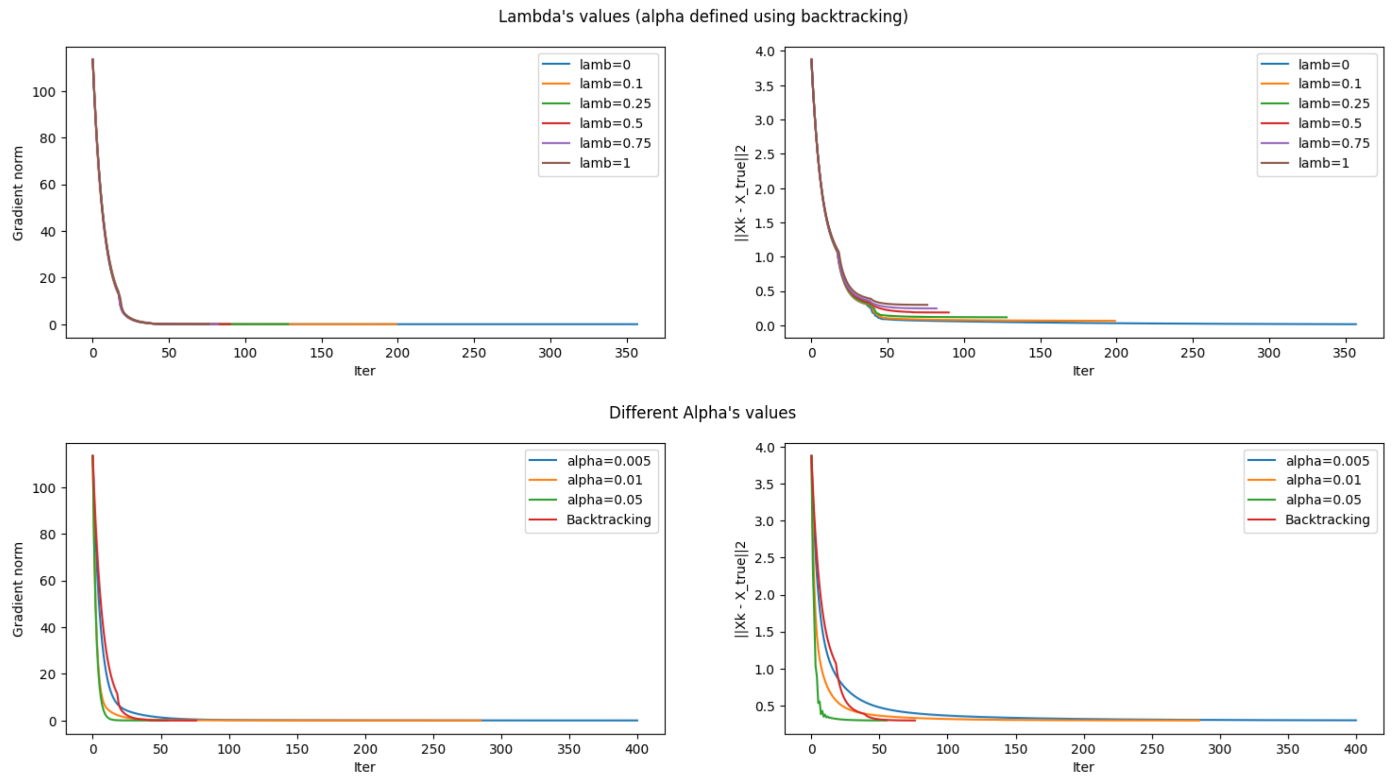




```
Vandermonde matrix N value equal to 10, Lambda equal to 1 for different values of alpha
```



```
Vandermonde matrix N value equal to 15, Lambda equal to 1 for different values of alpha
```



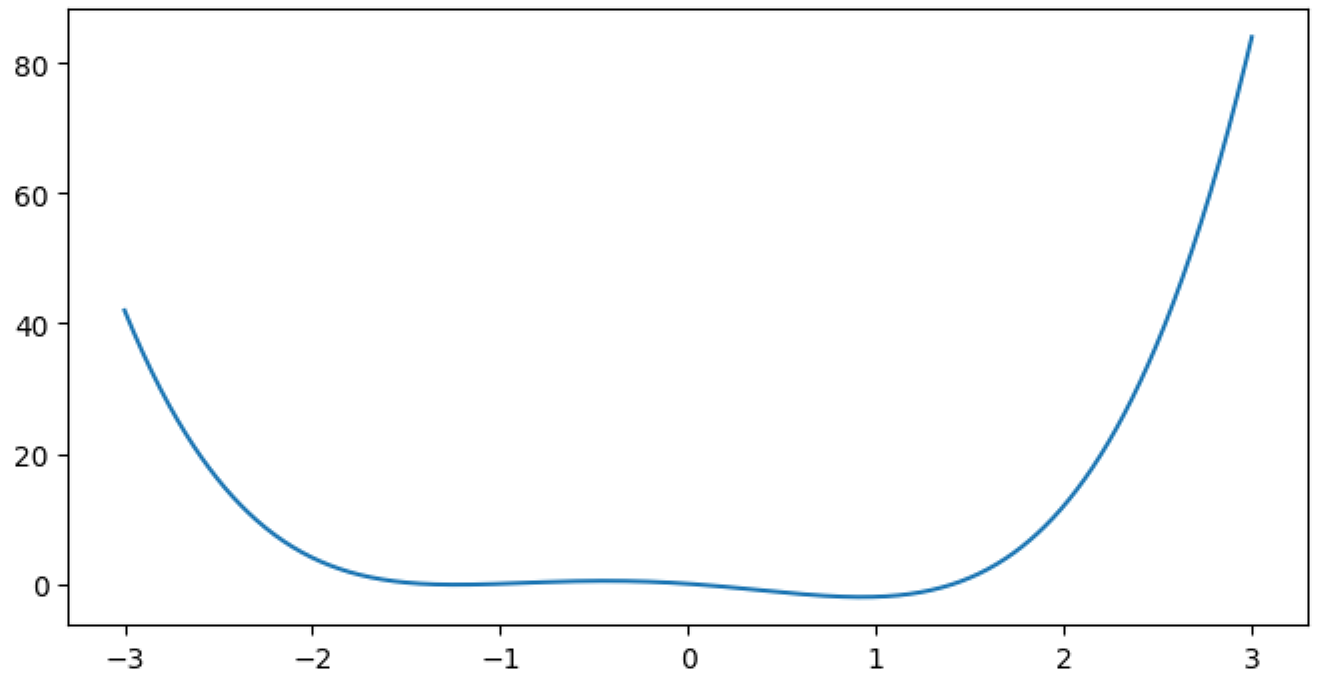
Out[ ]:
```
'\nwhen lamda is equal to 1, we obtain that the error is equal approximately to 0.5, it
means that we are in the middle between the minimization of lambda*x (=0) and the reachi
ng of the optimal solution (=1)\n'
```

when lamda is equal to 1, we obtain that the error is equal approximately to 0.5, it means that we are in the middle between the minimization of lambda*x (=0) and the reaching of the optimal solution (=1)

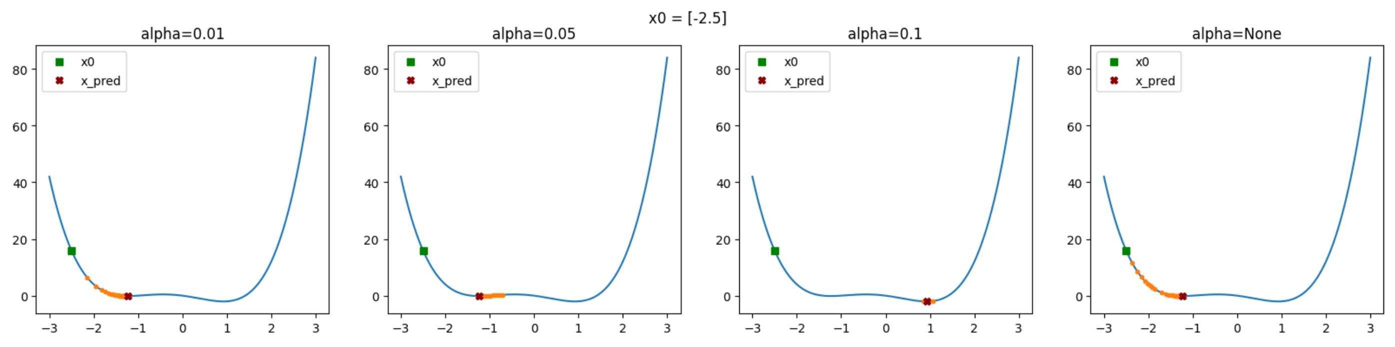
function 5

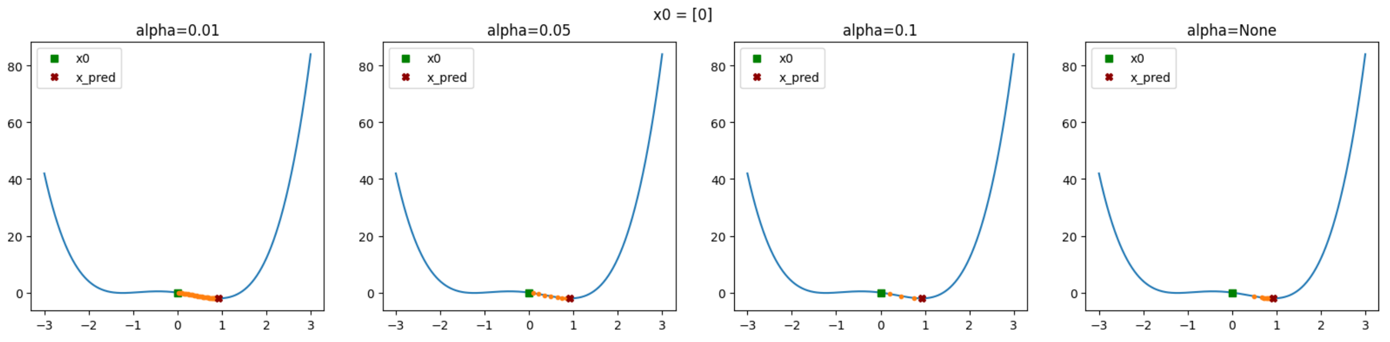
In [ ]:
```python
def plotSteps(fn, grad_fn, x0, try_alpha=[0.01, 0.05, 0.1, None]):
    plt.figure(figsize=(20, 4))
    plt.suptitle(f"x0 = {x0}")
    for i, alpha in enumerate(try_alpha):
        history_x, curr_k, history_f, history_grad, history_err = gd(fn, grad_fn, x0, 50
        plt.subplot(1, len(try_alpha), i+1)
        plt.title(f"alpha={alpha}")
        plt.plot(np.linspace(-3, 3, 1000), fn(np.linspace(-3, 3, 1000)))
        plt.plot(history_x[0], fn(np.array(history_x[0])), "s", color="green", label="x0
        plt.plot(history_x[1:-1], fn(np.array(history_x[1:-1])), ".")
        plt.plot(history_x[-1], fn(history_x[-1]), "X", color="darkred", label="x_pred")
        plt.legend()
    plt.show()
```

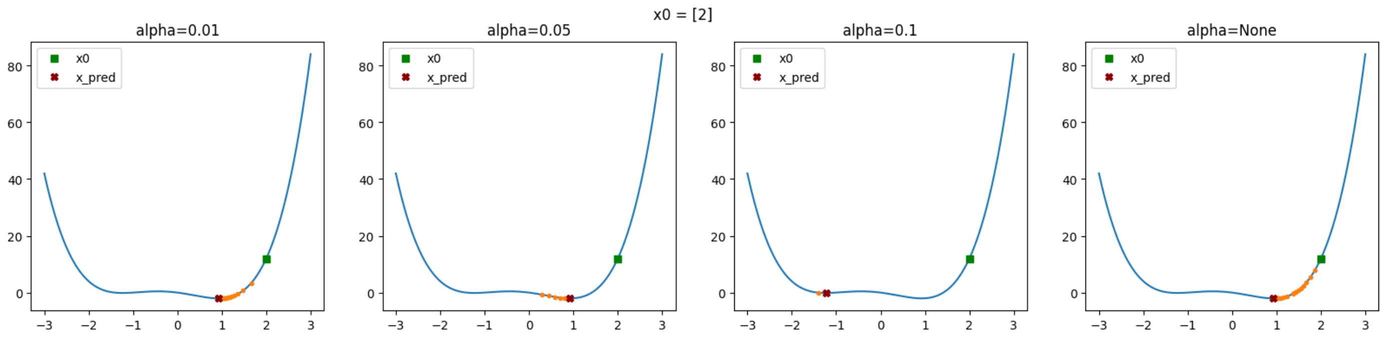
In [ ]:
```
f, grad_f, x_true, in_size = function5()
x_axis = np.linspace(-3, 3, 1000)
y_axis = f(x_axis)
plt.figure(figsize=(8, 4))
plt.plot(x_axis, y_axis)
plt.show()
```



In [ ]:
```
plotSteps(f, grad_f, np.array([-2.5]))
plotSteps(f, grad_f, np.array([0]))
plotSteps(f, grad_f, np.array([2]))
#note: steps could happens that are very small with an alpha value huge, this is obviosl
```







Optimization via Stochastic Gradient Descent

To test the script above, consider the MNIST dataset we used in the previous laboratories, and do the following:

1. From the dataset, select only two digits. It would be great to let the user input the two digits to select.
2. Do the same operation of the previous homework to obtain the training and test set from (X, Y), selecting the Ntrain you prefer.
3. Implement a logistic regression classificator as described in the corresponding post on my website.

In [ ]:

```python
from google.colab import drive
drive.mount('/content/drive')
import sys
sys.path.append('/content/drive/MyDrive/smm/homeworks/')
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import pandas as pd
import math
from tqdm import tqdm
from itertools import combinations
from utils.optimization import gd
from utils.PCAClassifier import PCAClassifier
from utils.SVDClassifier import SVDClassifier

np.random.seed(42)
```

```
Mounted at /content/drive
```

In [ ]:

```python
def train_test_split(X, Y, train_size, random_seed=42):
    idxs = np.arange(0, X.shape[1])
    np.random.default_rng(random_seed).shuffle(idxs)
    X_train = X[:, idxs[:train_size]]
    Y_train = Y[idxs[:train_size]]
    X_test = X[:, idxs[train_size:]]
    Y_test = Y[idxs[train_size:]]
    return X_train, Y_train, X_test, Y_test
```

```
def filterDigits(X, Y, digits):
    select_mask = np.isin(Y, digits)
    return X[:, select_mask], Y[select_mask]

#we deal with a dataset where dimensions are along the rows, and the samples along the c
# adding the bias: i define a new array (1 dim) or a new matrix (2 dim) with only ones,
def addCoefficient(A):
    if A.ndim == 1:
        out = np.ones((A.shape[0]+1,))
        out[1:] = A #we have just added one dimension equal to 1
    else:
        out = np.ones((A.shape[0]+1, A.shape[1]))
        out[1:, :] = A
    return out

def createDataset(X, Y, digits, train_ratio = 0.75):
    X, Y = filterDigits(X, Y, digits)
    # all the label became 0 or 1 in according to the digit
    Y[Y == digits[0]] = 0
    Y[Y == digits[1]] = 1
    return train_test_split(X, Y, int(train_ratio*X.shape[1]))
```

In [ ]:
```
from scipy.io import loadmat
from google.colab import drive
drive.mount('/content/drive')
file_path = '/content/drive/MyDrive/smm/homeworks/homework2/data/MNIST.mat'
data = loadmat(file_path)

full_X = data["X"]
full_Y = np.squeeze(data["I"], axis=0)

print(f"Images shape: {full_X.shape}")
print(f"Labels shape: {full_Y.shape}")
```

```
Drive already mounted at /content/drive; to attempt to forcibly remount, call drive.moun
t("/content/drive", force_remount=True).
Images shape: (256, 1707)
Labels shape: (1707,)
```

In [ ]:
```
#logistic regressor
def sigmoid(x): #activation function
    return 1 / (1 + np.exp(-x))

def f(w, x):
    return sigmoid(x.T @ w) # classifier

def mse(fn, w, X, y):
    return (1/2) * np.linalg.norm(fn(w, X) - y, 2)**2

def mse_grad(fn, w, X, y):
    N = X.shape[1]
    fn_X = fn(w, X)
    return ( X @ (fn_X * (1-fn_X) * (fn_X - y)) )

def loss(w, X, Y):
    N = X.shape[1]
    return (1/N) * mse(f, w, X, Y) #loss function = mean of mse of classifier prediction

def grad_loss(w, X, Y):
    N = X.shape[1]
    return (1/N) * mse_grad(f, w, X, Y) #mean of the derivatives of the loss function
```

In [ ]:
```
def sgd(loss, grad_loss, w0, data, batch_size, n_epochs, lr, random_seed=42):
    X, y = data
    data_size = X.shape[1] #dimension on the rows
    curr_w = w0
    history_w = [w0]
    history_loss = [loss(w0, X, y)]
    history_grad = [grad_loss(w0, X, y)]
    history_err = [np.linalg.norm(history_grad[-1], 2)]
    rng = np.random.default_rng(random_seed)

    for _ in range(n_epochs):
        idxs = np.arange(0, data_size)
        rng.shuffle(idxs) #for each epoch, i shuffle the data index to generate differen

        for i in range(math.ceil(data_size / batch_size)): # for each batch (n batch = n
            batch_idxs = idxs[i*batch_size : (i+1)*batch_size] #current batch (for examp
            batch_X = X[:, batch_idxs] #define the batch
            batch_y = y[batch_idxs]

            curr_w = curr_w - lr*grad_loss(curr_w, batch_X, batch_y)

        history_w.append(curr_w)
        history_loss.append(loss(curr_w, X, y))
        history_grad.append(grad_loss(curr_w, X, y))
        history_err.append(np.linalg.norm(history_grad[-1], 2) )

    return history_w, history_loss, history_grad, history_err
```

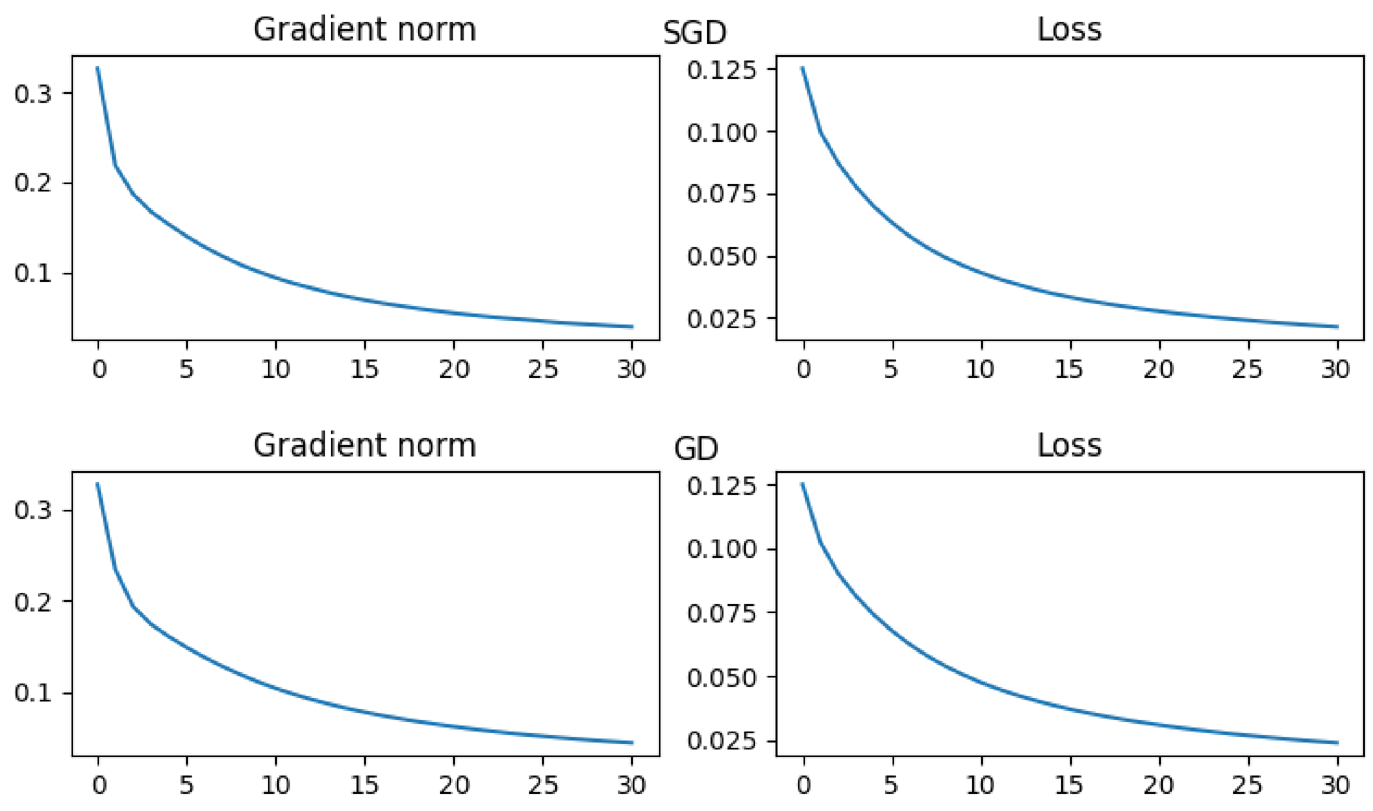
. Test the logistic regression classificator for different digits and different training set dimensions. The training procedure will end up with a set of optimal parameters w ∗

. Compare w∗ when computed with Gradient Descent and Stochastic Gradient Descent, for different digits and different training set dimensions.

In [ ]:
```
def binary_logistic_regression(X_train, Y_train, algorithm, batch_size=64, epochs=30, lr
    X_train = addCoefficient(X_train) #adding the bias
    w0 = np.zeros(X_train.shape[0])
    if algorithm == "sgd":
        history_w, history_loss, history_grad, history_err = sgd(loss=loss,grad_loss=gra
            #batch_size=batch_size, n_epochs=epochs,lr=lr)
            batch_size=64, n_epochs=epochs,lr=lr)
        plt.figure(figsize=(9, 2))
        plt.suptitle("SGD")
        plt.subplot(1, 2, 1)
        plt.plot(range(len(history_err)), history_err)
        plt.title("Gradient norm")
        plt.subplot(1, 2, 2)
        plt.plot(range(len(history_loss)), history_loss)
        plt.title("Loss")
        plt.show()
    else:
        history_w, curr_k, history_loss, history_grad, history_err = gd(loss=loss,grad_l
            w0=w0,data=(X_train, Y_train),k_max=epochs,tol_loss=tol_loss,tol_w=tol_w, al
        )
        plt.figure(figsize=(9, 2))
        plt.suptitle("GD")
        plt.subplot(1, 2, 1)
        plt.plot(range(len(history_err)), history_err)
        plt.title("Gradient norm")
        plt.subplot(1, 2, 2)
        plt.plot(range(len(history_loss)), history_loss)
        plt.title("Loss")
        plt.show()
    return history_w[-1]

# we assign 1 or 0 in order allow the evaluation
def predict_binary_logistic(X, w, threshold=0.5):
    X = addCoefficient(X) # we add the coeff. 1 for the bias
    if X.ndim == 1:
        return 1 if f(w, X) >= threshold else 0
    else:
        return np.array([1 if f(w, X[:, i]) >= threshold else 0 for i in range(X.shape[1

def evaluate_binary_logistic(model_w, X_test, Y_test):
    correct = 0
    for i in range(X_test.shape[1]):
        if predict_binary_logistic(X_test[:, i], model_w) == Y_test[i]:
            correct += 1
    return correct / X_test.shape[1]
```

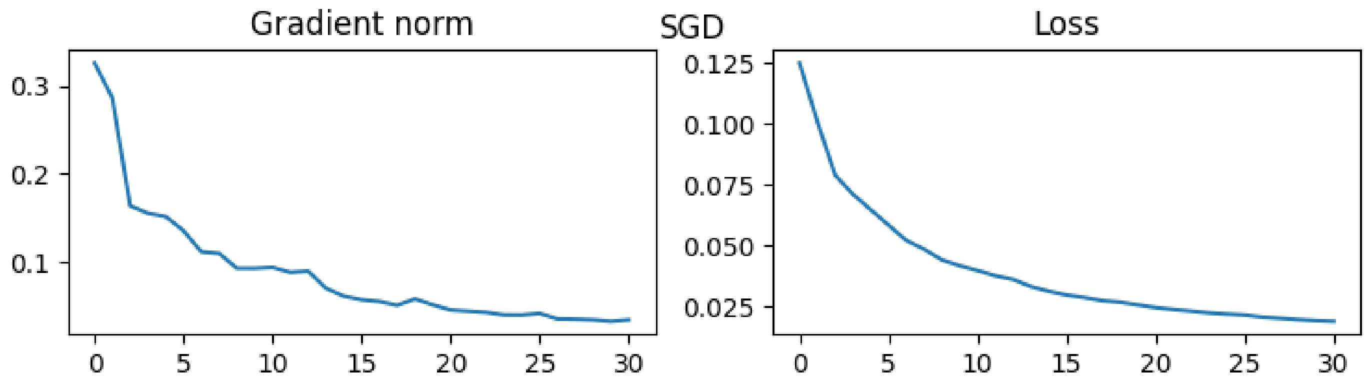
In [ ]:

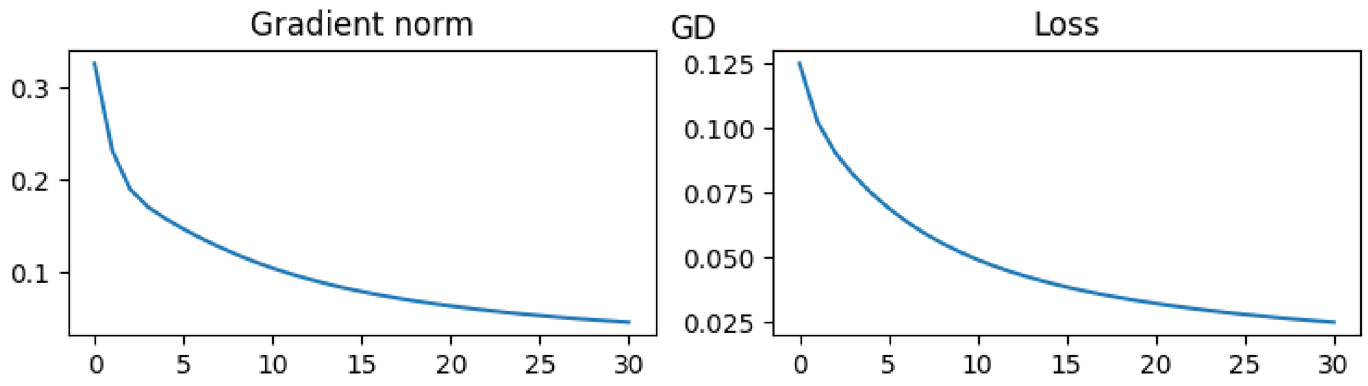
```
def binary_digit_evaluation(digit1, digit2, train_ratio):
    X_train, Y_train, X_test, Y_test = createDataset(full_X, full_Y, [digit1, digit2], t
    model_w_sgd = binary_logistic_regression(X_train, Y_train, algorithm="sgd")
    model_w_gd = binary_logistic_regression(X_train, Y_train, algorithm="gd")
    accuracy_sgd = evaluate_binary_logistic(model_w_sgd, X_test, Y_test)
    accuracy_gd = evaluate_binary_logistic(model_w_gd, X_test, Y_test)
    accuracy_sgd_train= evaluate_binary_logistic(model_w_sgd, X_train, Y_train)
    accuracy_gd_train= evaluate_binary_logistic(model_w_gd, X_train, Y_train)
    norm_diff = np.linalg.norm(model_w_sgd - model_w_gd)
    print(f'Xtrain = {X_train.shape[1]}, Xtest = {X_test.shape[1]}')
    print(f"Digits: ({digit1}, {digit2}), Train Ratio: {train_ratio}, ||w_sgd - w_gd||_2
          f"SGD Acc Test: {accuracy_sgd:.4f}, GD Acc Test: {accuracy_gd:.4f}")

import random
num_pairs = 3
train_ratios = [0.5, 0.6, 0.7]
for _ in range(num_pairs):
    print('\n\n\n')
    digit1, digit2 = random.sample(range(10), 2)
    for train_ratio in train_ratios:
        binary_digit_evaluation(digit1, digit2, train_ratio)
```



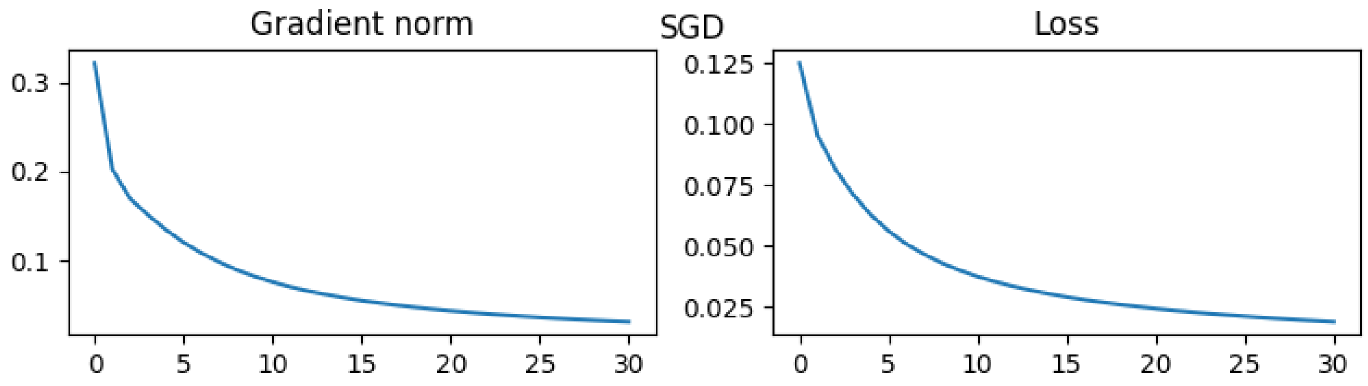
```
Xtrain = 162, Xtest = 162
```

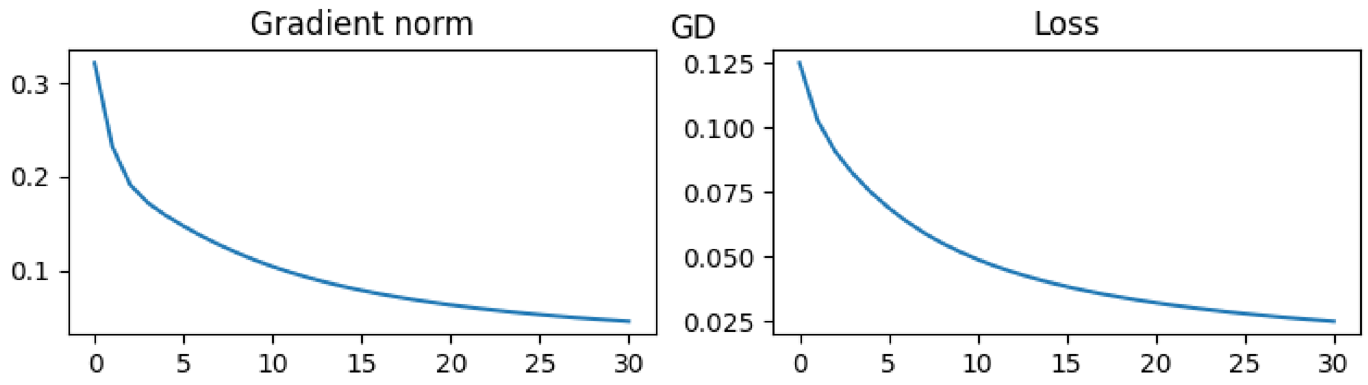
```
Digits: (4, 2), Train Ratio: 0.5, ||w_sgd - w_gd||_2 = 0.0596, SGD Acc Train: 0.9877, GD
Acc Train: 0.9877, SGD Acc Test: 0.9630, GD Acc Test: 0.9568
```



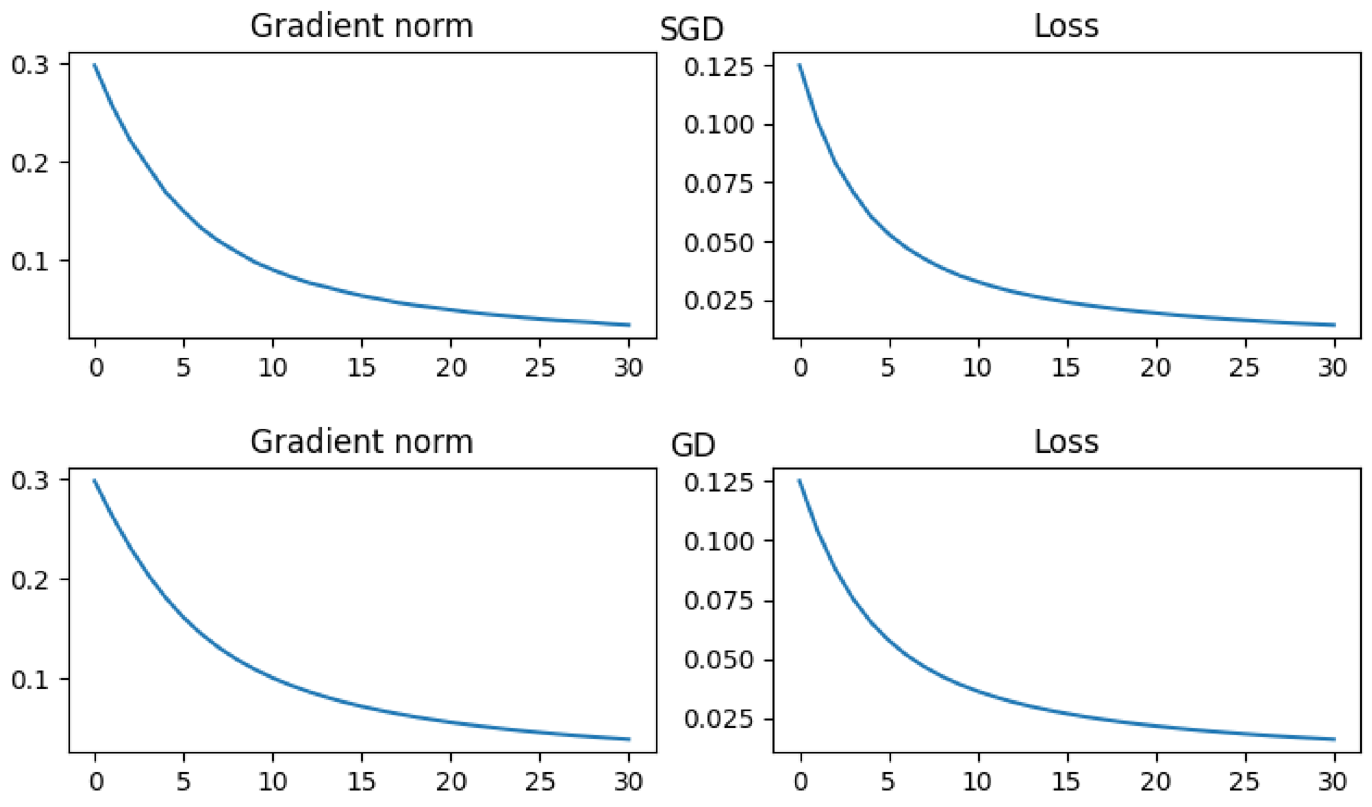


```
Xtrain = 194, Xtest = 130
Digits: (4, 2), Train Ratio: 0.6, ||w_sgd - w_gd||_2 = 0.1813, SGD Acc Train: 0.9897, GD
Acc Train: 0.9794, SGD Acc Test: 0.9462, GD Acc Test: 0.9462
```

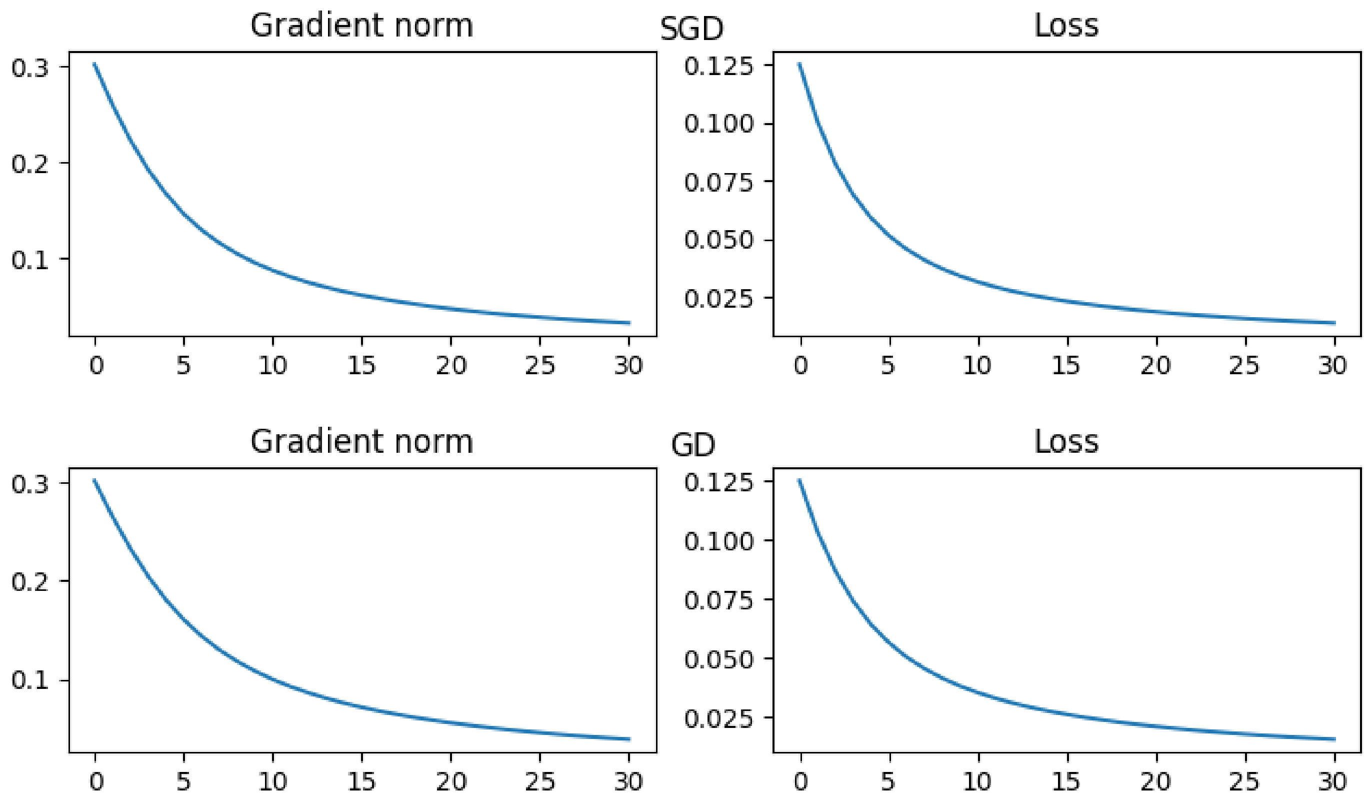




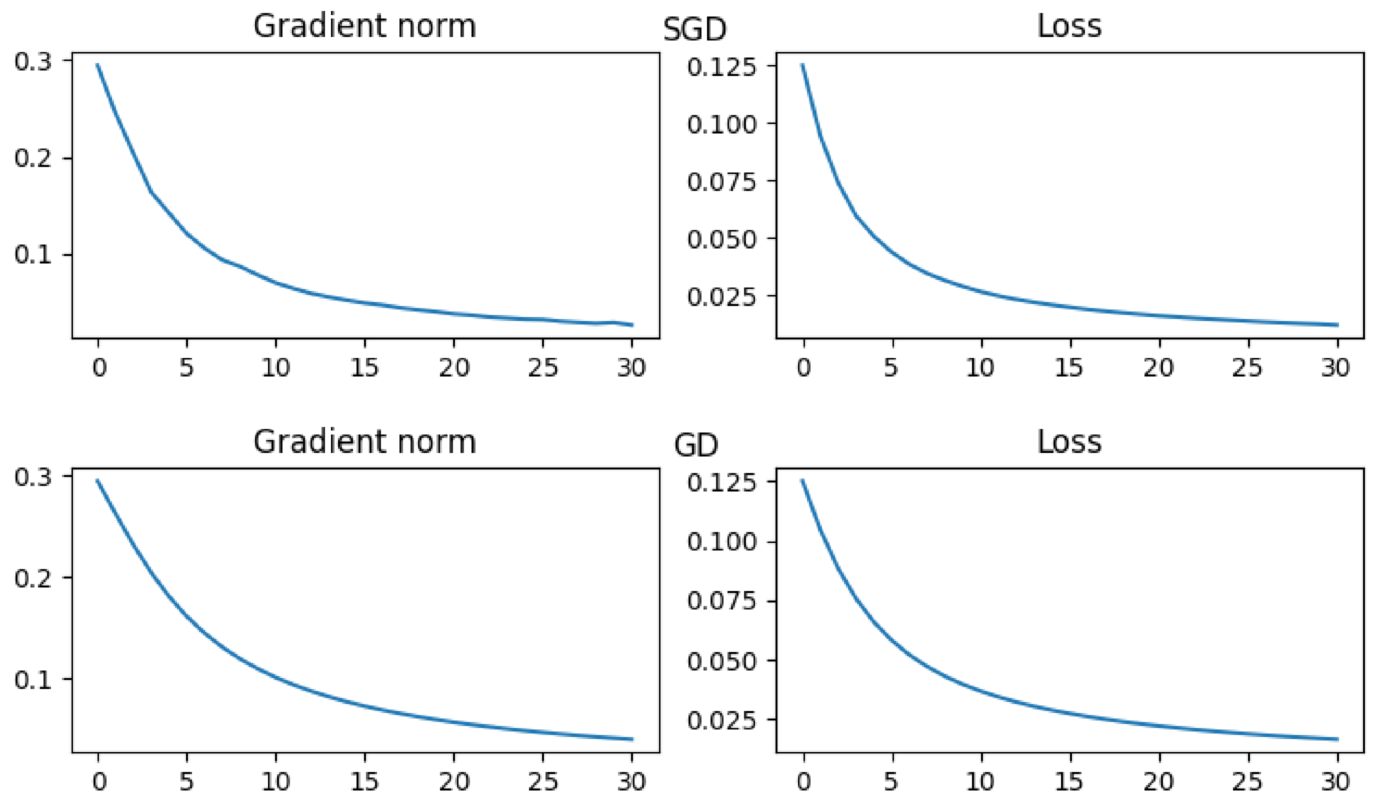
```
Xtrain = 226, Xtest = 98
Digits: (4, 2), Train Ratio: 0.7, ||w_sgd - w_gd||_2 = 0.1653, SGD Acc Train: 0.9912, GD
Acc Train: 0.9779, SGD Acc Test: 0.9490, GD Acc Test: 0.9388
```



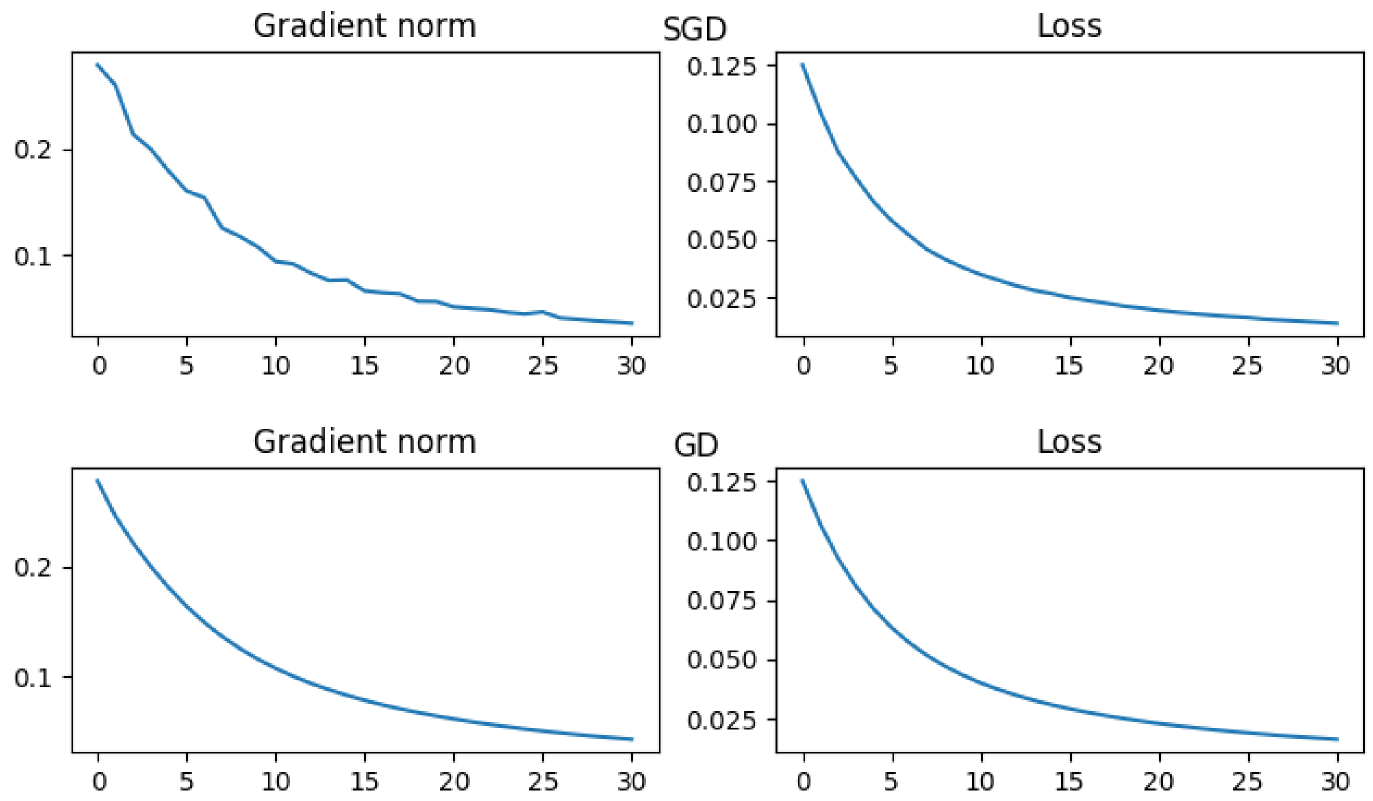
```
Xtrain = 141, Xtest = 141
Digits: (3, 6), Train Ratio: 0.5, ||w_sgd - w_gd||_2 = 0.0562, SGD Acc Train: 0.9858, GD
Acc Train: 0.9858, SGD Acc Test: 0.9929, GD Acc Test: 0.9929
```



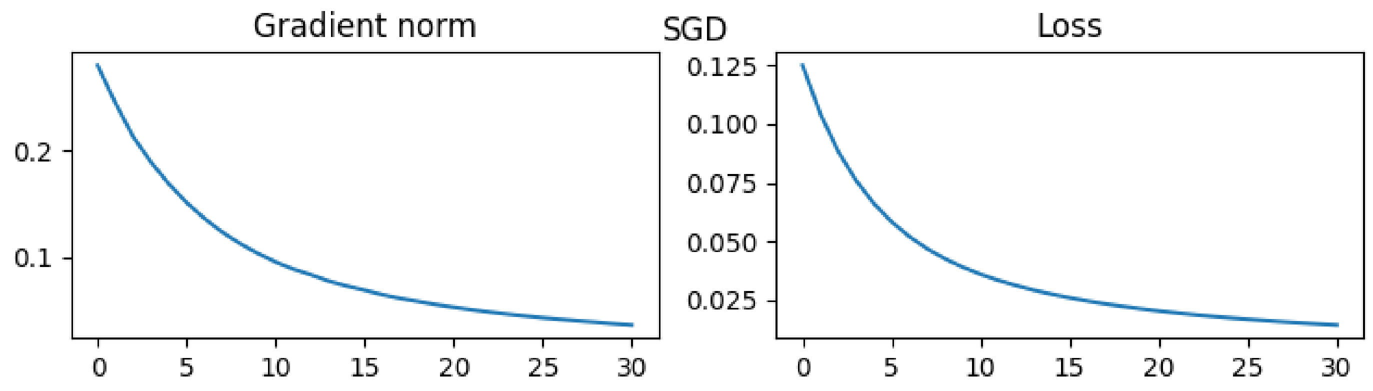
```
Xtrain = 169, Xtest = 113
Digits: (3, 6), Train Ratio: 0.6, ||w_sgd - w_gd||_2 = 0.0499, SGD Acc Train: 0.9882, GD
Acc Train: 0.9882, SGD Acc Test: 0.9912, GD Acc Test: 0.9912
```

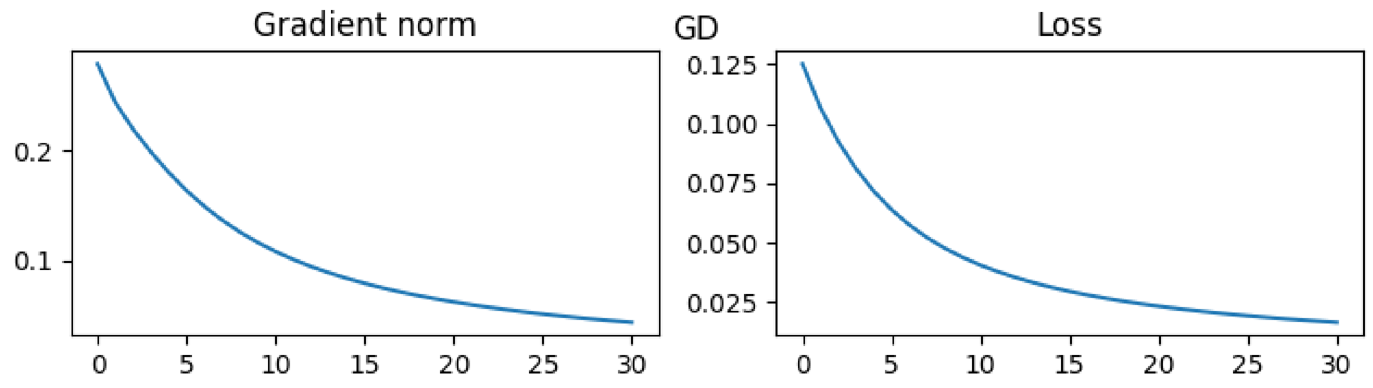


```
Xtrain = 197, Xtest = 85
Digits: (3, 6), Train Ratio: 0.7, ||w_sgd - w_gd||_2 = 0.1462, SGD Acc Train: 0.9898, GD
Acc Train: 0.9898, SGD Acc Test: 1.0000, GD Acc Test: 1.0000
```

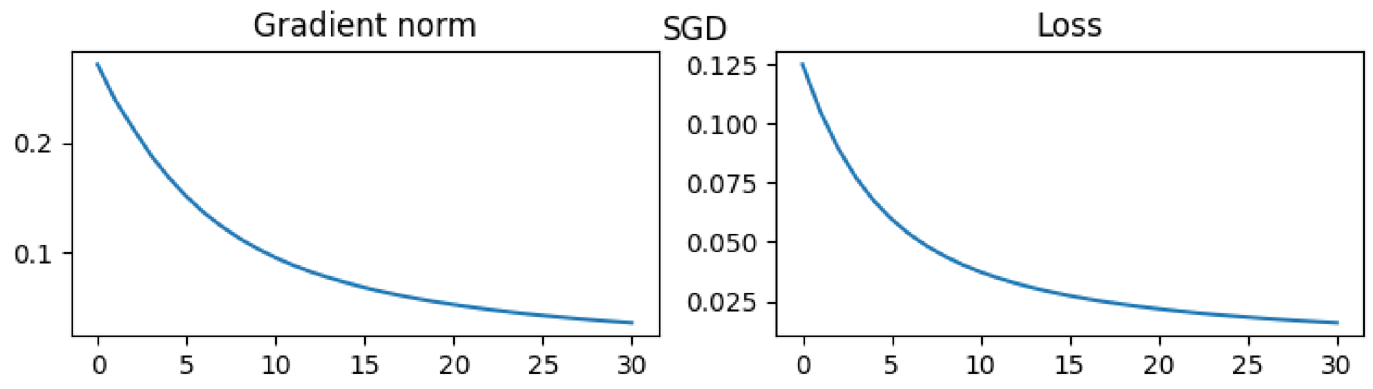


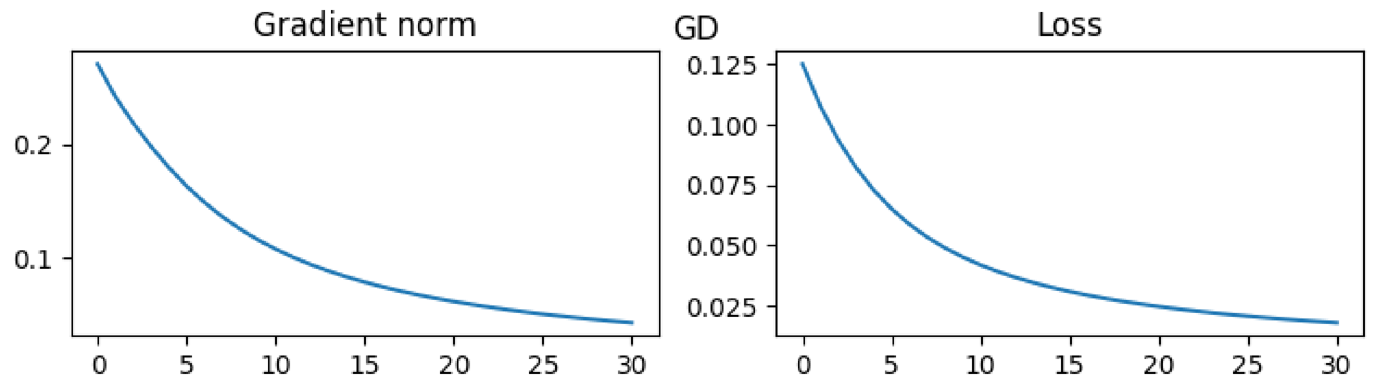
```
Xtrain = 131, Xtest = 132
Digits: (3, 9), Train Ratio: 0.5, ||w_sgd - w_gd||_2 = 0.0779, SGD Acc Train: 1.0000, GD
Acc Train: 1.0000, SGD Acc Test: 0.9697, GD Acc Test: 0.9697
```





```
Xtrain = 157, Xtest = 106
Digits: (3, 9), Train Ratio: 0.6, ||w_sgd - w_gd||_2 = 0.0568, SGD Acc Train: 1.0000, GD
Acc Train: 1.0000, SGD Acc Test: 0.9623, GD Acc Test: 0.9623
```





```
Xtrain = 184, Xtest = 79
Digits: (3, 9), Train Ratio: 0.7, ||w_sgd - w_gd||_2 = 0.0551, SGD Acc Train: 0.9946, GD
Acc Train: 0.9946, SGD Acc Test: 0.9620, GD Acc Test: 0.9620
```