> The Greek proof that there is no rational number whose square equals 2 is one of the great intellectual achievements of humanity and it should be experienced by every educated person [1, p. 4].
>
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WHY IS THE SQUARE ROOT OF 2 IRRATIONAL?

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Abstract. ...

1. Is it possible to specify all points on the number line exactly by drawing them?

...history Hippasus...

Note 1: text for right-hand side of pages, it is set justified.

2. Root? Square root?

Before we deal with the question, what *irrational* numbers are, it is crucial to understand what a root and what a square root of a number are. Our exploration starts with an imaginative number line, where the integer numbers, or simply the *integers* and the *rational numbers* are located. As you probably know, *integers* are the numbers

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

that continue infinitely in both directions of the number line: On left side of the number zero there are the *negative integers* and on the right side of zero there are the *positive integers*. Likewise, also the *rational numbers* continue infinitely in both directions of our imaginative number line: they are numbers that are expressed as a ratio or a quotient of two integers, for example

$$\dots, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \dots.$$

Please note, that whereas the top number of the fraction, that is the numerator, can be 0, because 0 divided by 1 as in the example above is still

0, the bottom number of the fraction, namely the denominator, cannot have the value 0, because if we assume that the quotient of such a fraction would be $q = \frac{a}{0}$, then by basic algebra we would get $q \cdot 0 = a$, and hence, by the fact that multiplying any number with 0 gives always 0, we get $0 = a$. But this would also mean that in theory we could insert as q any number, which would lead to $1 \cdot 0 = 0 \cdot 0, 2 \cdot 0 = 0 \cdot 0, 3 \cdot 0 = 0 \cdot 0, \ldots$. If we would now cancel out 0 on the left-hand side (LHS) and on the right-hand side (RHS), we would get obviously wrong results such as $1 = 0, 2 = 0, 3 = 0, \ldots$. This is one of the reasons why it is not allowed to have a denominator with the value 0.

If we now multiply an integer x by itself n times (with n being a positive integer), we get the product y, which in this case is also an integer. The resulting equation can be expressed as

$$\underbrace{(x \cdot x \cdot x \cdot \cdots \cdot x)}_{n\ times} = y, \tag{1}$$

or more conveniently, as

$$x^n = y. \tag{2}$$

Now *squaring* is a special case of this rule, namely when n equals 2

$$\underbrace{y \cdot y}_{2\ times} = y^2 = x \tag{3}$$

Theorem 1. The square root of 2 (i. e. $\sqrt{2} = 2^{1/2}$) is irrational.

Main Idea behind the Proof: Try to express the square root as a fraction. You will see that this leads to a contradiction.

Proof. We suppose that $2^{1/2}$ is rational. By definition, a number belongs to the set of the rational numbers $\mathbb{Q}$, if it can be expressed as a ratio of two integers $\frac{p}{q}$, where the numerator p can be any integer and the denominator q must be a non-zero integer.

If $2^{1/2}$ is rational, then it can be expressed as the ratio of two integers p and q, where p and q have no common factor other than 1:

$$2^{1/2} = \frac{p}{q}. \tag{4}$$

Now we square both sides of the equation (4), which on the left-hand side (LHS) gives:

$$\left(2^{1/2}\right)^2 = 2^{2/2} = 2 =, \tag{5}$$

and on the right-hand side (RHS) it gives:

$$= \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}. \tag{6}$$

That is:

$$2 = \frac{p^2}{q^2}, \tag{7}$$

which by basic algebra can be rearranged into:

$$p^2 = 2q^2. \tag{8}$$

Now, by definition of an even integer, that has the form $(2 \cdot \text{an integer})$, the RHS, i. e. $2q^2$ is an even integer. It follows that also the LHS, that is p^2 must be even. This leads us to the question of whether p is also even.

Proposition 1. For every integer s, if s^2 is even then s is even.

Proof. Suppose s is any odd integer. Then, by the definition of an odd integer, $s = 2k + 1$ for some integer k. By substitution and basic algebra, we get:

$$s^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \tag{9}$$

Due to the closure under addition and multiplication that holds within the set of integers $\mathbb{Z}$, $L = 2k^2 + 2k$ is an integer. Hence, also $s^2 = 2 \cdot L + 1$ is an integer, and by the definition of an odd integer, s^2 is odd.

Now we know, that also p must be even. And by definition of an even integer, we also deduce that:

$$p = 2r \quad \text{for some integer } r. \tag{10}$$

Now, by substitution, we insert into equation (8) what we got in equation (10), and we see that:

$$p^2 = (2r)^2 = 4r^2 = 2q^2. \tag{11}$$

By dividing both, $4r^2$ and $2q^2$ by 2, we get:

$$2r^2 = q^2. \tag{12}$$

As we can see, by the definition of an even integer, q^2 is even, and by proposition (1) also q is even. But earlier, we deduced from (8) that p is even. And this would mean, that both p and q are even, and that they have the common factor 2. But this contradicts the supposition, that p and q do not have a common factor, other than 1. Hence, the supposition is false and the theorem, which states that the square root of 2 cannot be expressed as a ratio of two integers, is true, which means that it is irrational. □

References

[1] S. J. Axler, *Algebra & trigonometry: with student solutions manual* (Wiley, Hoboken, N.J, 2012).