

# WHY IS THE SQUARE ROOT OF 2 IRRATIONAL?

FRANCESCO MACRÌ 

ABSTRACT. ...Based on a classical mathematical proof of the ancient Greeks, this article summarizes what irrational numbers are. ...

## 1. IS IT POSSIBLE TO SPECIFY ALL POINTS ON THE NUMBER LINE EXACTLY BY DRAWING THEM?

...history Hippasus...

The Greek proof that there is no rational number whose square equals 2 is one of the great intellectual achievements of humanity and it should be experienced by every educated person [1, p. 4].

## 2. ROOT? SQUARE ROOT?

Before we deal with the question, what *irrational* numbers are, it is crucial to understand what a root and what a square root of a number are. Our exploration starts with an imaginative number line, where the integer numbers, or simply the *integers* and the *rational numbers* are located. As you probably know, *integers* are the numbers

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

that continue infinitely in both directions of the number line: On left side of the number zero there are the *negative integers* and on the right side of zero there are the *positive integers*. Likewise, also the *rational numbers* continue infinitely in both directions of our imaginative number line: they are numbers that are expressed as a ratio or a quotient of two integers, for example

$$\dots, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \dots$$

Please note, that whereas the top number of the fraction, that is the numerator, can be 0, because 0 divided by 1 as in the example above is still 0, the bottom number of the fraction, namely the denominator, cannot have the value 0, because if we assume that the quotient of such a fraction would be  $q = \frac{a}{0}$ , then by basic algebra we would get  $q \cdot 0 = a$ , and hence, by the

fact that multiplying any number with 0 gives always 0, we get  $0 = a$ . But this would also mean that in theory we could insert as  $q$  any number, which would lead to  $1 \cdot 0 = 0 \cdot 0, 2 \cdot 0 = 0 \cdot 0, 3 \cdot 0 = 0 \cdot 0, \dots$ . If we would now cancel out 0 on the left-hand side (LHS) and on the right-hand side (RHS), we would get obviously wrong results such as  $1 = 0, 2 = 0, 3 = 0, \dots$ . This is one of the reasons why it is not allowed to have a denominator with the value 0.

If we now multiply an integer  $x$  by itself  $n$  times (with  $n$  being a positive integer), we get the product  $y$ , which in this case is also an integer. The resulting equation can be expressed as

$$(1) \quad \underbrace{(x \cdot x \cdot x \cdots x)}_{n \text{ times}} = y,$$

or more conveniently, as

$$(2) \quad x^n = y.$$

Now a *square* root is a special case of this rule, namely when  $n$  equals 2

$$(3) \quad \underbrace{y \cdot y}_{2 \text{ times}} = y^2 = x$$

**Theorem 1.** The square root of 2 (i. e.  $\sqrt{2} = 2^{1/2}$ ) is irrational.

Main Idea behind the Proof: Try to express the square root as a fraction. You will see that this leads to a contradiction.

*Proof.* We suppose that  $2^{1/2}$  is rational. By definition, a number belongs to the set of the rational numbers  $\mathbb{Q}$ , if it can be expressed as a ratio of two integers  $\frac{p}{q}$ , where the numerator  $p$  can be any integer and the denominator  $q$  must be a non-zero integer.

If  $2^{1/2}$  is rational, then it can be expressed as the ratio of two integers  $p$  and  $q$ , where  $p$  and  $q$  have no common factor other than 1:

$$(4) \quad 2^{1/2} = \frac{p}{q}.$$

Now we square both sides of the equation (4), which on the left-hand side (LHS) gives:

$$(5) \quad \left(2^{1/2}\right)^2 = 2^{2/2} = 2 =,$$

and on the right-hand side (RHS) it gives:

$$(6) \quad = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}.$$

That is:

$$(7) \quad 2 = \frac{p^2}{q^2},$$

which by basic algebra can be rearranged into:

$$(8) \quad p^2 = 2q^2.$$

Now, by definition of an even integer, that has the form  $(2 \cdot \text{an integer})$ , the RHS, i. e.  $2q^2$  is an even integer. It follows that also the LHS, that is  $p^2$  must be even. This leads us to the question of whether  $p$  is also even.

**Proposition 1.** For every integer  $s$ , if  $s^2$  is even then  $s$  is even.

*Proof.* Suppose  $s$  is any odd integer. Then, by the definition of an odd integer,  $s = 2k + 1$  for some integer  $k$ . By substitution and basic algebra, we get:

$$(9) \quad s^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Due to the closure under addition and multiplication that holds within the set of integers  $\mathbb{Z}$ ,  $L = 2k^2 + 2k$  is an integer. Hence, also  $s^2 = 2 \cdot L + 1$  is an integer, and by the definition of an odd integer,  $s^2$  is odd.

Now we know, that also  $p$  must be even. And by definition of an even integer, we also deduce that:

$$(10) \quad p = 2r \quad \text{for some integer } r.$$

Now, by substitution, we insert into equation (8) what we got in equation (10), and we see that:

$$(11) \quad p^2 = (2r)^2 = 4r^2 = 2q^2.$$

By dividing both,  $4r^2$  and  $2q^2$  by 2, we get:

$$(12) \quad 2r^2 = q^2.$$

As we can see, by the definition of an even integer,  $q^2$  is even, and by proposition (1) also  $q$  is even. But earlier, we deduced from (8) that  $p$  is even. And this would mean, that both  $p$  and  $q$  are even, and that they have the common factor 2. But this contradicts the supposition, that  $p$  and  $q$  do not have a common factor, other than 1. Hence, the supposition is false and the theorem, which states that the square root of 2 cannot be expressed as a ratio of two integers, is true, which means that it is irrational.  $\square$

<sup>1</sup>S. J. Axler, *Algebra & trigonometry: with student solutions manual* (Wiley, Hoboken, N.J, 2012).