> The Greek proof that there is no rational number whose square equals 2 is one of the great intellectual achievements of humanity and it should be experienced by every educated person [1, p. 4].
>
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THE SQUARE ROOT OF 2 IS IRRATIONAL

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Abstract. This article presents a very famous proof that the square root of 2 cannot be expressed by a rational number.

The main idea behind the proof: Try to express the square root of 2 as a fraction without common factors, i.e. as a rational number. If this succeeds, it would mean that the square root of 2 is rational. But this is simply not possible and ultimately leads to a contradiction.

1. Historical Notes

The ancient Greeks, the Pythagoreans, studied prime numbers, progressions, and those ratios and proportions, but in contrast to our current understanding, a ratio of two whole numbers was not a fraction, i. e. a distinct kind of number with respect to the whole numbers [2, p. 32]. The own discovery of the role of whole numbers in musical harmony inspired Pythagoreans to seek whole-number patterns everywhere [3, p. 11]. If quantities could have been measured by a common unit using whole numbers, they had a common measure and where called *com-mensurable* [2, p. 32]. The discovery of ratios that were not measurable in this way, i. e. that where *in-commensurable*, is attrubuted to Hippasus of Metapontum [2, p. 32], and was a turning point in Greek mathematics [4, p. 1]. Furthermore, it has affected mathematics and philisophy from the time of the Greeks to the present day [5, p. 59] and it is assumed that it marked the origin of what these days is considered the Greek contribution to rigorous procedure in mathematics [5, p. 59], [4, p. 1]. The starting point was the Pythagorean theorem, that was discovered indipendently in several ancient cultures [4, p. 3]. There is evidence [3, p. 4] that the Babylonians (1800 BC), the Chinese (between

200 and 220 BC) and Indian (between 500 and 200 BC) mathematicians were interested in triangles whose sides where whole-number triples that - denoted in modern way - satisfy the equation $a^2 + b^2 = c^2$ [4, pp. 3–4] such as for instance the following ones, that today we call *Pythagorean triples* [3, p. 4], e. g. $\langle 3, 4, 5\rangle, \langle 5, 12, 13\rangle, \langle 8, 15, 17\rangle$:

$$3^2 + 4^2 = 5^2 = 9 + 16 = 25,$$
$$5^2 + 12^2 = 13^2 = 25 + 144 = 169,$$
$$8^2 + 15^2 = 17^2 = 64 + 225 = 289.$$

But it is assumed, that only the Pythagoreans were interested in a special case that eventually led to the discovery of the *incommensurable* ratios: Given that two sides of the right-angled triangle are $a = b$; the crucial question - again, denoted in the modern way - must have been whether there are whole numbers a and c that satisfy the equation:

$$c^2 = 2a^2, \tag{1}$$

fulfilling *commensurability* [4, pp. 6–7], [5, p. 58].

2. Theorem and Proof

Theorem 1. The square root of 2 is irrational.

Proof. We suppose that the square root of 2 is rational. By definition, a number belongs to the set of the rational numbers $\mathbb{Q}$, if it can be expressed as a ratio of two integers $\frac{p}{q}$, where the numerator p can be any integer and the denominator q must be a non-zero integer.

If the square root of 2 is rational, then it can be expressed as the ratio of two integers p and q, where p and q have no common factor other than 1:

$$\sqrt{2} = 2^{1/2} = \frac{p}{q}. \tag{2}$$

Now we square both sides of the equation (2). On the left-hand side (LHS) it gives:

$$\left(2^{1/2}\right)^2 = 2^{2/2} = 2 =, \tag{3}$$

whereas on the right-hand side (RHS) it gives:

$$= \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}. \tag{4}$$

That is:

$$2 = \frac{p^2}{q^2}, \tag{5}$$

which by basic algebra can be rearranged into:

$$p^2 = 2q^2. \tag{6}$$

Now, by definition of an even integer, that has the form $(2 \cdot \text{an integer})$, the RHS, i. e. $2q^2$ is an even integer. It follows that also the LHS, that is p^2 must be even. This leads us to the question of whether p is also even.

Proposition 1. For every integer s, if s^2 is even then s is even.

Proof. Suppose s is any odd integer. Then, by the definition of an odd integer, $s = 2k + 1$ for some integer k. By substitution and basic algebra, we get:

$$s^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \tag{7}$$

If we add or multiply integers toghether, the resulting sum or product will still be in the same set $\mathbb{Z}$, i. e. it is an integer. This is referred to as the closure property of addition and multiplication which holds within the set of integers. Due to these properties, the expression $2k^2 + 2k$ must be an integer. We will denote it as $L = 2k^2 + 2k$. Hence, also $s^2 = 2 \cdot L + 1$ is an integer, and by the definition of an odd integer, s^2 is odd. □

Now we know, that also p must be even. And by definition of an even integer, we also deduce that:

$$p = 2r \qquad \text{for some integer } r. \tag{8}$$

Now, by substitution, we insert into equation (6) what we got in equation (8), and we see that:

$$p^2 = (2r)^2 = 4r^2 = 2q^2. \tag{9}$$

By dividing both, $4r^2$ and $2q^2$ by 2, we get:

$$2r^2 = q^2. \tag{10}$$

As we can see, by the definition of an even integer, q^2 is even, and by proposition (1) also q is even. But earlier, we deduced from (6) that p is even. And this would mean, that both p and q are even, and that they have the common factor 2. But this contradicts the supposition, that p and q do not have a common factor, other than 1. Hence, the supposition is false and the theorem, which states that the square root of 2 cannot be expressed as a ratio of two integers, is true, which means that it is irrational. □

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