Lecture Notes

Geometric Measure Theory

Course held by

Prof. Giovanni Alberti

Notes written by

Francesco Paolo Maiale

Department of Mathematics

Pisa University

September 25, 2017

Disclaimer

These notes came out of the *Geometric Measure Theory* course, held by Professor Giovanni Alberti in the second semester of the academic year 2016/2017.

They include all the topics that were discussed in class; I added some remarks, simple proof, etc.. for my convenience.

I have used them to study for the exam; hence they have been reviewed thoroughly. Unfortunately, there may still be many mistakes and oversights; to report them, send me an email to frank094 (at) hotmail (dot) it.

Contents

Chapter 1

Introduction

In this chapter, we introduce the main topics of the course and give a brief overview of what we will see and what we will be able to prove by the end of the course.

1.1 Plateau's Problem

The primary goal and the motivating example of this course is the **Plateau's problem**, that is, the problem to find the d-dimensional surface Σ of the minimal area with prescribed $(d-1)$-dimensional boundary Γ.

By the end, we will be able to prove that a solution indeed exists, but we will not find it explicitly since it is a NP (hard) numerical problem.

As of now, the problem is not well defined. In fact, the notions of *surface*, *area*, and *boundary* make sense in the smooth setting but, as the examples below show, we need to work in a less regular setting.

More precisely, requiring the surface to be smooth is not enough for modeling reasons (e.g., dip a wire frame into a soap solution, form a soap film, and look for the minimal surface whose boundary is the wire frame), and also for existence reasons.

Example 1.1. Here we give a list of Plateau's problems with prescribed boundary conditions, and we write down the correct solutions, without proving anything.

(a) Let us identify $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$ and, if $d = 2$, let us consider the smooth boundary given by

$$\Gamma_1 := \left(S^1 \times \{0\}\right) \cup \left(\{0\} \times S^1\right).$$

Surprisingly, every minimizing sequence of smooth surfaces converges to a surface which is not smooth at all. Indeed, the solution of the problem is

$$\Sigma_1 := \left(D^2 \times \{0\}\right) \cup \left(\{0\} \times D^2\right).$$

The surface Σ_1 is clearly singular at the origin, but the singularity may be removed (by factorizing it into two nonsingular surfaces).

(b) Let us identify $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$ and, if $d = 2$, let us consider the smooth boundary given by

$$\Gamma_2 := \left\{(z^2, z^3) \,:\, z \in S^1\right\}.$$

The solution to the Plateau's problem is

$$\Sigma_2 := \left\{(z^2, z^3) \,:\, z \in D^2\right\},$$

which is a non-smooth surface, whose singularity cannot be removed (since the polynomial $z_1^3 = z_2^2$ cannot be factorized).

(c) Let us identify $\mathbb{R}^8 \cong \mathbb{R}^4 \times \mathbb{R}^4$ and, if $d = 7$, let us consider the smooth boundary given by

$$\Gamma_3 := S^3 \times S^3.$$

The minimal surface of prescribed boundary Γ_3 is

$$\Sigma_3 := \left\{(x_1,\, x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 \;:\; |x_1| = |x_2| \le 1\right\}.$$

To conclude this introductive chapter, we give a brief overview of the main approaches (studied in this course) to the Plateau's problem, as d ranges between 1 and ∞.

1.2 Geodesics problem ($d = 1$)

The geodesics problem (that is, find the shortest curve connecting two points) is, surprisingly, still an open in the non-Riemannian setting. However, in the Riemannian setting, the geodesics problem is completely solved.

Indeed, if we consider the curves parametrized by paths, the *length* is a well-defined notion, and the associated functional is lower semi-continuous and coercive; hence the compactness is easy to prove.

There are many possible approaches to the geodesics problem, e.g., the Steiner approach and the set theoretical approach, which we describe briefly in the remainder of the section.

Steiner Problem. It is also called networks approach, and it is used to prove the existence of the geodesics and find the explicit expression for it. The reader may consult [1] for a detailed dissertation on the topic.

Set Theoretical Approach. The main idea is to find a closed and connected set Σ of minimum *length*, containing a given finite set Γ. As we shall see later in the course, in this case the length is a well defined concept: the *Hausdorff distance*.

In fact, if X is a suitable space (metric, endowed with Hausdorff distance, etc...), then the class defined by

$$\mathcal{X} := \{K \subseteq X \;:\; K \text{ compact and connected}\}$$

is compact and, by Gotab theorem[1], $\mathcal{H}^1$ is lower semi-continuous on X.

1.3 "Surface" Problem ($d > 1$)

3. The Plateau's problem is much harder when $d = 2$, but there are still many approaches possible some of which relying, in a certain sense, on the work already done in the geodesics case.

Set Theoretical Approach. This approach is highly nontrivial. For example, one may ask what does it mean that a compact set Σ spans a boundary Γ? Moreover, there is another problem one should deal with: the 2-dimensional Hausdorff measure $\mathcal{H}^2$ is, generally, not lower semi-continuous. The reader may consult [9] for a complete treatise of the topic.

Remark 1.1. Suppose that $d = 2$, $n = 3$ and that Σ is a surface with boundary Γ. If γ is another closed curve, linked to Γ (by a nonzero linking number), then $\gamma \cap \Sigma \neq \emptyset$.

[1] [4] Let $\mathscr{C}$ be an infinite collection of non-empty compact sets all lying in a bounded portion B of $\mathbb{R}^n$. Then there exists a sequence $\{E_j\}$ of distinct sets of $\mathfrak{C}$ convergent in the Hausdorff metric to a non-empty compact set E.

Parametric Approach. This method is essentially due to Douglas [3]. The main idea is the following: since a parametrization $\phi : D^2 \to \mathbb{R}^n$ defines surfaces in $\mathbb{R}^n$, the area functional is well-defined and given by the formula

$$A(\phi) := \int_{D^2} \left| \frac{\partial \phi}{\partial s_1} \wedge \frac{\partial \phi}{\partial s_2} \right| \mathrm{d}s_1 \, \mathrm{d}s_2.$$

On the other hand, the existence through lower semi-continuity and the compactness are a delicate matter, since coercivity is not an easy property to obtain (the integrand is similar to a determinant).

There is a trick which is similar to the one we can use to find geodesics in the differential geometry setting. More precisely, we consider the functional

$$E(\phi) := \frac{1}{2} \int_{D^2} |\nabla \phi|^2 \, \mathrm{d}s_1 \, \mathrm{d}s_2.$$

If we find a minimal point ϕ for E, then ϕ will be a **conformal parametrized** minimum for A. This trick, on the other hand, heavily depends on a nontrivial theorem: every such Σ admits a conformal re-parametrization.

The lack of conformal parametrization, though, is what stop us from extending the same trick to dimension d strictly bigger than 2.

Higher Dimension. If the codimension of Σ is equal to 1 (that is, $n = d+1$), then finite perimeter sets generalize the notion of open $(d+1)$-dimensional sets with smooth boundary in $\mathbb{R}^n$.

The class of finite perimeter sets has excellent compactness properties and a notion of area lower semi-continuous.

This approach is called ”weak” surfaces approach, and it is essentially due to Caccioppoli [2] and De Giorgi [7]. A different approach, working for any d and n, referred to as *integral currents*, was introduced by Federer and Flaming in their joint paper [6].

Chapter 2

Measure Theory

In this chapter, the chief goal is to give a brief introduction to the theory of measure and acquire the fundamental notions needed for the remainder of the course. The reader with an adequate background, may immediately jump to the next section, and use this one as a reference.

In the first part, we introduce the only three classes of measures we shall be concerned about in this course, and then we study the weak-$*$ topology and some of the fundamental properties of the weak-$*$ convergence.

In the second part, we introduce the notion of *outer measure*, and we show how it can be used to construct a measure that belongs to one of the classes mentioned above.

In the final part, we construct the Hausdorff d-dimensional measure, denoted by $\mathcal{H}^d$, and we prove some of the main properties (e.g., the relation with Lipschitz functions). The last section is devoted to the computation of the Hausdorff dimension of a Cantor-type set.

2.1 Definitions and Elementary Properties

Introduction. In this course, we will essentially be only concerned with measures that belong to one of the following classes:

(1) Positive, σ-additive measures, defined on the Borel σ-algebra of a reasonable space X, e.g. we might assume X metric, locally compact and separable.[1]

(2) Positive, σ-subadditive measures, defined on the set of all parts $P(X)$. These are generally called *outer measures*.

(3) Real-valued and vector-valued (bounded) σ-additive measures, defined on the Borel σ-algebra.

Remark 2.1. The Borel σ-algebra $\mathcal{B}(X)$ is not closed under the action of continuous maps.

Proof. Let $X = [0, 1] \subset \mathbb{R}$, and let us consider a space-filling curve $g : [0, 1] \longrightarrow [0, 1]^2$ and the projection $\pi : [0, 1]^2 \longrightarrow [0, 1]$; the reader may easily check that the map $f := \pi \circ g : [0, 1] \longrightarrow [0, 1]$ is a continuous function.

Thus, given a Borel set $S \subset [0, 1]^2$ with $\pi(S)$ not Borel, it is easy to prove that the image of the Borel set $g^{-1}(S)$ under f is not Borel. □

[1] We are not very specific here since the reader may try, whenever possible, to derive, as an exercise, the minimal assumption on X for an assertion to be true.

In this course, we will often introduce a suitable[2] outer measure defined on X and, by taking the restriction to the Borel σ-algebra (see Theorem 2.14), we automatically obtain a measure which belongs to the class **(1)**.

Remark 2.2. If X is a separable Banach space[3], then the class of measures **(3)** is the dual space of a particular subspace of $C^0(X)$. In the next section, we will also prove that, if the space of measures is endowed with the weak-$*$ topology, then this class has good compactness properties (that is, the Banach-Alaoglu theorem holds true).

Definition 2.1 (Vector-valued Measure)**.** Let $\mathcal{B}(X)$ be a Borel σ-algebra on X, and let F be a normed space. A function $\mu : X \longrightarrow F$ is a σ-additive F*-valued* measure if

$$\sum_{n\in\mathbb{N}} \mu\left(E_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right), \tag{2.1}$$

for any countable disjoint family of Borel sets $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(X)$.

Remark 2.3. The identity (2.1) does not only imply the σ-additivity of the measure, but it also gives us a stronger property.

More precisely, the sum on the left-hand side does not depend on the order of the indexes, and thus, if F is a finite-dimensional space, then it is enough to infer that the sum converges absolutely.

Representation Theorem. In this final brief paragraph, we sketch the proof of the *Riesz representation theorem*, according to which the class of measures **(3)** is the dual of a suitable subset of the space of all continuous functions.

Definition 2.2 (Total Variation)**.** Let μ be a measure. The *total variation* of a set $E \subseteq X$ is the supremum of the measure over the possible countable partitions, that is,

$$|\mu|\,(E) := \sup\left\{ \sum_{n=0}^{+\infty} |\mu(E_n)| \;\middle|\; \{E_n\}_{n\in\mathbb{N}} \text{ countable partition of } E \right\}. \tag{2.2}$$

Clearly, the total variation is a positive bounded measure, defined on the Borel σ-algebra $\mathcal{B}(X)$. The **mass** of μ is defined by setting

$$\|\mu\| := |\mu|\,(X). \tag{2.3}$$

The formula (2.3) defines a norm on the space of all the F-valued σ-Borel measures, which is also complete. Moreover, it is easy to see that μ is absolutely continuous with respect to its total variation $|\mu|$, and hence by **Radon-Nikodym Theorem** there exists a Borel function $f : X \longrightarrow F$ such that

$$\mu = f \cdot |\mu| \rightsquigarrow \mu(E) := \int_E f(x)\,\mathrm{d}\,|\mu|\,(x),$$

with the additional property that its norm is almost everywhere equal to one, that is, $|f(x)|_F = 1$ for $|\mu|$-almost every $x \in X$.

Consequently, vector-valued measures may be identified with the product between a positive measure λ and a function λ-summable, that is, there exists $f \in L^1(\lambda)$ such that

$$\mu = f \cdot \lambda.$$

[2]See Section 2.4 for a more precise definition of what suitable means here.
[3]It is not necessary to require X to be a separable Banach space, but it is more than enough for our purposes.

This identification is particularly useful when we need to integrate a function $g : X \longrightarrow F$ with respect to a vector valued measure. Indeed, it turns out that

$$\int_X g(x)\,\mathrm{d}\mu(x) = \int_X g(x)\cdot f(x)\,\mathrm{d}|\mu|(x),$$

where $\cdot$ is a suitable notion of product[4] between f and g.

Notation. Let μ be an $\mathbb{R}^n$-valued measure defined on X. We denote by Λ_μ the functional given by

$$g \longmapsto \int_X g\,\mathrm{d}\mu = \int_X g(x)\cdot f(x)\,\mathrm{d}\,|\mu|\,(x), \tag{2.4}$$

where $\cdot$ is the scalar product in $\mathbb{R}^n$.

The map (2.4) is well-defined at every $|\mu|$-summable function $g : X \longrightarrow \mathbb{R}^n$ function; thus, the functional is well-defined at every continuous function $h : X \longrightarrow \mathbb{R}^n$ which is infinitesimal at ∞ (i.e., equal to zero in the one-point compactification of $\mathbb{R}^n$.)

In particular, Λ_μ is well-defined on $C_0\,(X;\;\mathbb{R}^n)$ - or, equivalently, on $C\,(X;\;\mathbb{R}^n)$ if X is compact -, which is a separable Banach spaces with respect to the supremum norm. The functional

$$\Lambda_\mu : C_0\,(X;\;\mathbb{R}^n) \longrightarrow \mathbb{R}$$

is linear and bounded (i.e., continuous). More precisely, from (2.4) it follows that

$$|\Lambda_\mu(g)| \le \int_X |g(x)|\,\mathrm{d}\,|\mu|\,(x) \le \|g\|_{C_0(X;\,\mathbb{R}^n)}\,\|\mu\|,$$

which, in turn, implies that $\|\Lambda_\mu\| \le \|\mu\|$. If we set $g := f$, then it turns out that $\|g\| = \|f\| = 1$ and that the equality holds true, i.e.

$$\|\Lambda_\mu\| = \|\mu\|.$$

Theorem 2.3 (Riesz)**.** *Let $\mathcal{M}\,(X;\;\mathbb{R}^n)$ be the space of all the $\mathbb{R}^n$-valued measures. The map*

$$\mathcal{M}\,(X;\;\mathbb{R}^n) \longrightarrow \left(C_0\,(X;\;\mathbb{R}^n)\right)^*, \qquad \mu \longmapsto \Lambda_\mu$$

is an isometry. Moreover, if X is a compact space, then

$$\mathcal{M}\,(X;\;\mathbb{R}^n) \longrightarrow \left(C\,(X;\;\mathbb{R}^n)\right)^*, \qquad \mu \longmapsto \Lambda_\mu$$

is also an isometry.

Theorem 2.4 (Riesz)**.** *Let F be a finite-dimensional normed space. The map*

$$\mathcal{M}\,(X;\;F^*) \longrightarrow \left(C_0(X;\;F^*)\right)^*, \qquad \mu \longmapsto \Lambda_\mu$$

is an isometry. Moreover, if F is a separable Banach space, then

$$\mathcal{M}\,(X;\;F^*) \longrightarrow \left(C_0(X;\;F^*)\right)^*, \qquad \mu \longmapsto \Lambda_\mu$$

is also an isometry, provided that F^* is good enough.

[4] For example it could be a scalar product, or an external product.

2.2 Weak-$*$ Topology on Measures Space

In this section, we endow the space of measures $\mathcal{M}(X;\,\mathbb{R}^n)$ with the weak-$*$ topology since the associated notion of convergence is particularly pleasant.

Definition 2.5 (Measures convergence). A sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R}^n)$ of $\mathbb{R}^n$-valued measures converges weakly-$*$ to a measure $\mu \in \mathcal{M}(X;\,\mathbb{R}^n)$ if and only if

$$\lim_{n\to\infty} \int_X g(x)\,\mathrm{d}\mu_n(x) = \int_X g(x)\,\mathrm{d}\mu(x), \qquad \forall g \in C_0^0(X;\,\mathbb{R}^n)\,. \tag{2.5}$$

Remark 2.4. If X is a compact space, then $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R}^n)$ converges weakly-$*$ to a measure $\mu \in \mathcal{M}(X;\,\mathbb{R}^n)$ if and only if

$$\lim_{n\to\infty} \int_X g(x)\,\mathrm{d}\mu_n(x) = \int_X g(x)\,\mathrm{d}\mu(x), \qquad \forall g \in C^0(X;\,\mathbb{R}^n)\,. \tag{2.6}$$

Remark 2.5. The notion of strong convergence in the space $\mathcal{M}(X;\,\mathbb{R}^n)$ is, unfortunately, "*too strong*", and, in general, it is not interesting at all. For this reason, from now on we say that μ_n converges (in the sense of measures) to μ if (2.5) is satisfied.

Remark 2.6.

(1) The **Banach-Alaoglu Theorem**, in the particular case of $\mathcal{M}(X;\,\mathbb{R}^n)$, may be stated in the following, particularly simple, way.

Theorem 2.6 (Banach-Alaoglu). *Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R}^n)$ be a sequence with uniformly bounded masses, that is, there exists $C > 0$ such that*

$$\|\mu_n\| \le C < \infty \qquad \forall\, n \in \mathbb{N}.$$

Then, up to subsequences, it converges in the sense of measures to an element $\mu \in \mathcal{M}(X;\,\mathbb{R}^n)$.

(2) Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R}^n)$ be a sequence of measures with uniformly bounded masses. Then μ_n converges to $\mu \in \mathcal{M}(X;\,\mathbb{R}^n)$ if and only if

$$\lim_{n\to\infty} \int_X g(x)\,\mathrm{d}\mu_n(x) = \int_X g(x)\,\mathrm{d}\mu(x), \qquad \forall g \in D,$$

where D is a dense subset of $C_0(X;\,\mathbb{R}^n)$, e.g. the space of compactly supported functions.

(3) The "local version of the theory" works in a very similar way. For example, if we assume that $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R}^n)$ is a sequence of measures with locally bounded masses, that is, for any $K \subseteq X$ compact there exists $C(K) > 0$ such that

$$\|\mu_n\|_K \le C(k) < \infty, \qquad \forall n \in \mathbb{N},$$

then we can infer that μ_n converges in the sense of measures.

Proposition 2.7. *Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R})$ be a sequence of real-valued positive measures, and assume that it converges to $\mu \in \mathcal{M}(X;\,\mathbb{R})$.*

(a) *For any open subset $A \subseteq X$, it turns out that*

$$\liminf_{n\to+\infty} \mu_n(A) \ge \mu(A).$$

(b) *For any compact subset $K \subseteq X$, it turns out that*

$$\limsup_{n\to+\infty} \mu_n(K) \le \mu(K).$$

(c) *For any relatively compact $E \subseteq X$ such that $\mu(\partial E) = 0$, it turns out that*

$$\lim_{n\to+\infty} \mu_n(E) = \mu(E).$$

Proof.

(a) The first assertion is equivalent to the fact that the functional

$$\Phi_A : \mathcal{M}(X;\ \mathbb{R}) \longrightarrow \mathbb{R}, \qquad \lambda \longmapsto \int_X \chi_A \,\mathrm{d}\lambda$$

is (weakly-$*$) lower semi-continuous for every $A \in \mathcal{B}(\mathbb{R})$. The characteristic function of an open set can be approximated by an increasing sequence of continuous function, e.g.,

$$f_n^{(A)}(x) := \begin{cases} 0 & \text{if } x \notin A \\ \min\left\{1,\ \sup_{r>0}\{n\cdot r \mid B(x,\ r) \subseteq A\}\right\} & \text{if } x \in A. \end{cases}$$

The supremum of weakly-$*$ lower semi-continuous function[5] is weakly-$*$ lower semi-continuous; thus from the relation

$$\chi_A(x) = \sup_{n\in\mathbb{N}} f_n(x),$$

it follows easily that χ_A is lower semi-continuous. Therefore, if we apply the standard Fatou Lemma, it turns out that

$$\liminf_{n\to+\infty} \mu_n(A) \ge \mu(A), \qquad \forall\, A \in \mathcal{B}(\mathbb{R}).$$

(b) This assertion follows immediately from the previous one by taking the complement. On the other hand, we can mimic the proof above and obtain the result directly by proving that

$$\Phi_K : \mathcal{M}(X;\ \mathbb{R}) \longrightarrow \mathbb{R}, \qquad \lambda \longmapsto \int_X \chi_K \,\mathrm{d}\lambda$$

is weakly-$*$ upper semi-continuous for every $K \subseteq X$ compact. The characteristic function of a compact set can be approximated by a decreasing sequence of continuous function, e.g.

$$g_n(x) := \begin{cases} 1 & \text{if } x \in K \\ 1 - \min\left\{0,\ \sup_{r>0}\{n\cdot r \mid B(x,\ r) \subseteq K^c\}\right\} & \text{if } K^c. \end{cases}$$

The infimum of a collection of (weakly-$*$) upper semi-continuous functions is (weakly-$*$) upper semi-continuous; thus from the equality

$$\chi_K(x) = \inf_{n\in\mathbb{N}} g_n(x),$$

it follows that χ_K is upper semi-continuous. Therefore, if we apply the reverse inequality of the Fatou Lemma, it turns out that

$$\limsup_{n\to+\infty} \mu_n(K) \le \mu(K).$$

[5]The sequence $(f_n)_{n\in\mathbb{N}}$ is made up of continuous function, but the supremum will lose the upper semi-continuity.

(c) Let $E \subseteq X$ be a relatively compact set such that $\mu(\partial E) = 0$. The interior part $\operatorname{Int} E$ has the same measure of E and it is open; thus it follows from **(a)** that

$$\liminf_{n\to+\infty} \mu_n(E) = \liminf_{n\to+\infty} \mu_n(\operatorname{Int} E) =\geq \mu(\operatorname{Int} E) = \mu(E).$$

In a similar fashion, since E is relatively compact, the closure $\overline{E}$ is compact and has the same measure of E; thus it follows from **(b)** that

$$\limsup_{n\to+\infty} \mu_n\left(\overline{E}\right) = \limsup_{n\to+\infty} \mu_n(E) \leq \mu(E) = \mu\left(\overline{E}\right),$$

and this proves the sought identity, i.e., $\lim_{n\to+\infty} \mu_n(E) = \mu(E)$.

□

Proposition 2.8. *Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\,\mathbb{R})$ be a sequence of real-valued positive measures, and assume that it converges to $\mu \in \mathcal{M}(X;\,\mathbb{R})$.*

(a) *If X is compact, then*

$$\lim_{n\to\infty} \|\mu_n\| = \|\mu\|.$$

(b) *If X is locally compact, then*

$$\liminf_{n\to\infty} \|\mu_n\| \geq \|\mu\|.$$

In particular, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures, converging to some μ, such that the mass $\|\mu_n\|$ does not converge to $\|\mu\|$.

Proof.

(a) The norm is weakly-$*$ lower semi-continuous; hence it suffices to prove that

$$\mu_n \to \mu \text{ and } X \text{ compact} \implies \limsup_{n\to\infty} \|\mu_n\| \leq \|\mu\|.$$

By assumption $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of positive measures, which means that the mass is simply given by $\|\mu_n\| = \mu_n(X)$, that is,

$$\|\mu_n\| = \int_X \mathrm{d}\mu_n, \qquad \forall\, n \in \mathbb{N}.$$

By compactness of X, the convergence in sense of measures is equivalent to (2.6), and thus

$$\|\mu_n\| = \int_X \mathrm{d}\mu_n \to \int_X \mathrm{d}\mu = \|\mu\|.$$

(b) It is enough to provide a counterexample to the convergence. Let $X = \mathbb{R}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}$ such that $x_n \to +\infty$ as $n \to +\infty$. If we define

$$\mu_n := \delta_{x_n},$$

where δ_y is the delta of Dirac at y, then it is straightforward to prove that $\mu_n \to 0$ and $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$, that is,

$$\liminf_{n\to+\infty} \|\mu_n\| = 1 > \|\mu\| = 0.$$

$\square$

Definition 2.9 (Tightness)**.** Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\ \mathbb{R})$ be a sequence of real-valued positive measures. The sequence is *tight* if, for every $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset X$ such that

$$\mu_n(X \setminus K_\epsilon) \leq \epsilon, \qquad \forall\, n \in \mathbb{N}.$$

Lemma 2.10. *Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\ \mathbb{R})$ be a sequence of real-valued positive measures, converging to an element $\mu \in \mathcal{M}(X;\ \mathbb{R})$. Then the sequence is tight if and only if the mass converges, that is,*

$$\|\mu_n\| \xrightarrow{n\to+\infty} \|\mu\|. \tag{2.7}$$

Proof. If the sequence of the masses converges to $\|\mu\|$, then the tightness of $(\mu_n)_{n\in\mathbb{N}}$ follows from the definition (namely, the tail of the masses sequence is as small as we want.)

Vice versa, suppose that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is tight. The norm $\|\cdot\|$ is weakly-$*$ lower semi-continuous; hence it suffices to prove the opposite inequality, that is,

$$\limsup_{n\to\infty} \|\mu_n\| \leq \|\mu\|.$$

Let $\epsilon > 0$ and let $K_\epsilon \subset X$ be the compact subset satisfying (2.7). If we consider the decomposition of the ambient space $X = K_\epsilon \cup K_\epsilon^c$, then it turns out that

$$\mu_n(X) = \int_{K_\epsilon} \mathrm{d}\mu_n + \int_{X\setminus K_\epsilon} \mathrm{d}\mu_n \leq \int_X \chi_{K_\epsilon}(x)\, \mathrm{d}\mu_n(x) + \epsilon.$$

Finally, we take the limit as $n \to +\infty$, and we notice that from **(b)** of Proposition 2.7 it follows that that

$$\limsup_{n\to+\infty} \mu_n(X) \leq \limsup_{n\to+\infty} \int_X \chi_{K_\epsilon}\, \mathrm{d}\mu_n + \epsilon \leq \mu(K_\epsilon) + \epsilon,$$

which is enough to conclude the proof since $\epsilon > 0$ may be taken arbitrarily small. $\square$

"Strong" Convergence. In this final paragraph, we introduce a stronger notion of convergence, which turns out to be the right replacement for the convergence in norm for the space of all measures.

Definition 2.11 (Convergence in Variation)**.** Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X;\ \mathbb{R})$ be a sequence of real-valued measures. The sequence *converges in variation* to some $\mu \in \mathcal{M}(X;\ \mathbb{R})$ if and only if

$$\mu_n \longrightarrow \mu \qquad \text{and} \qquad \|\mu_n\| \xrightarrow{n\to+\infty} \|\mu\|. \tag{2.8}$$

Remark 2.7. The convergence in variation induces a finer topology than the weak-$*$ one, and it is the right replacement for the norm convergence.

Remark 2.8. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of positive measures converging in variation. The assertions of Proposition 2.7 hold true, provided that we modify them accordingly with the new definition:

($\mathbf{b_1}$) For any closed subset $C \subseteq X$, it turns out that

$$\limsup_{n\to+\infty} \mu_n(C) \leq \mu(C).$$

($\mathbf{c_1}$) For any $E \subseteq X$ such that $\mu(\partial E) = 0$, it turns out that

$$\lim_{n\to+\infty} \mu_n(E) = \mu(E).$$

Remark 2.9. If $\mu_n \to \mu$ is a converging sequence of $\mathbb{R}^n$-valued measures (not necessarily positive), then there is a subsequence $(n_k)_{k\in\mathbb{N}} \subset (n)_{n\in\mathbb{N}}$ such that

$$|\mu_{n_k}| \to \lambda,$$

where λ is a positive measure satisfying $\lambda \geq |\mu|$. Moreover, given a relatively compact set $E \subseteq X$ with $\lambda(E) = 0$, the reader may prove, as an exercise, that

$$\mu_n(E) \to \mu(E).$$

Exercise 2.1. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$-valued measures converging in variation to an element μ. Prove that:

(1) The sequence of positive measures $(|\mu_n|)_{n\in\mathbb{N}}$ weakly-$*$ converges to $|\mu|$.

(2) For every subset $E \subseteq X$ such that $|\mu|(E) = 0$, it turns out that $\mu_n(E) \to \mu(E)$.

Exercise 2.2. Let X be a suitable ambient space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X such that $x_n \to x$ as $n \to +\infty$, and set $\mu_n := \delta_{x_n} - \delta_x$. Prove that

$$|\mu_n| = \delta_{x_n} \qquad \text{and} \qquad |\mu_n| \to |\mu| = 0.$$

2.3 Outer Measures

In this section, we introduce the notion of *outer measure*, and we set the ground for the main result of this chapter: the Carathéodory construction of a measure defined on the Borel σ-algebra $\mathcal{B}(X)$.

Definition 2.12 (Outer Measure)**.** Let X be a set. An *outer measure* on X is a set function $\mu : P(X) \longrightarrow [0, \infty]$, where $\mathcal{P}(X)$ denotes the power set, such that

(a) $\mu(\varnothing) = 0$;

(b) μ is monotone, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$;

(c) μ is σ-additive, i.e. $\{A_n\}_{n\in\mathbb{N}} \subset P(X) \implies \mu(\cup_n A_n) \leq \sum_n \mu(A_n)$.

Example 2.1. A simple example of an outer measure, which is defined on every set X, is given by

$$\mu(E) := \begin{cases} n & \text{if } |E| = n, \\ +\infty & \text{if } |E| \geq \aleph_0. \end{cases}$$

Example 2.2. Let $X := \mathbb{R}$. The outer measure from which the Lebesgue measure derives is defined by

$$\mu(E) := \inf \left\{ \sum_{n\in\mathbb{N}} |I_n| \;\middle|\; E \subset \bigcap_{n\in\mathbb{N}} I_n \right\},$$

where the I_n are, for example, open intervals.

Definition 2.13 (Carathéodory measurable)**.** Let μ be an outer measure defined on X. A subset $A \subseteq X$ is *Carathéodory μ-measurable* if and only if

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c), \qquad \forall\, E \subseteq X. \tag{2.9}$$

Theorem 2.14 (Carathéodory)**.** *Let $\mathcal{M}$ be the class of Carathéodory μ-measurable sets in X. Then $\mathcal{M}$ is a σ-algebra and the restriction of μ to $\mathcal{M}$, denoted by $\mu \llcorner \mathcal{M}$, is σ-additive.*

Proof. To ease the notation for the reader we divide the proof into three steps.

Step 0. The relation (2.9) is symmetric with respect to the complement; hence $A \in \mathcal{M}$ immediately implies $A^c \in \mathcal{M}$, and vice versa. In particular, the empty set $\varnothing$ belongs to $\mathcal{M}$.

Step 1. Let $A_1, A_2 \in \mathcal{M}$. For every $E \subseteq X$ it turns out that

$$\begin{aligned}\mu(E) &= \mu(E \cap A_1) + \mu(E \cap A_1^c) = \\ &= \mu(E \cap A_1 \cap A_2) + \mu(E \cap A_1 \cap A_2^c) + \mu(E \cap A_1^c) \geq \\ &\geq \mu\left(E \cap (A_1 \cap A_2)\right) + \mu\left(E \cap (A_1 \cap A_2)^c\right),\end{aligned}$$

where the last inequality follows from the monotonicity property of μ. The opposite inequality is always true as a consequence of the subadditivity of the measure; hence we can infer that

$$A_1, A_2 \in \mathcal{M} \implies A_1 \cap A_2 \in \mathcal{M}.$$

Step 2. Let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of elements of $\mathcal{M}$. Then

$$\begin{aligned}\mu(E) &= \mu(E \cap A_1) + \mu(E \cap A_1^c) = \\ &= \mu(E \cap A_1) + \mu(E \cap A_1^c \cap A_2) + \mu(E \cap A_1^c \cap A_2^c) = \\ &= \mu(E \cap A_1) + \mu(E \cap A_2) + \mu(E \cap A_1^c \cap A_2^c) = \\ &= \sum_{n=1}^{m} \left[\mu(E \cap A_n)\right] + \mu\left(E \cap \left(\bigcup_{n=1}^{m} A_n\right)^c\right).\end{aligned}$$

If we take the supremum, the identity above yields to

$$\mu(E) \geq \sum_{n=1}^{+\infty} \left[\mu(E \cap A_n)\right] + \mu\left(E \cap \left(\bigcup_{n=1}^{+\infty} A_n\right)^c\right),$$

and thus, using the σ-subadditivity of the outer measure μ, we obtain the nontrivial inequality

$$\mu(E) \geq \mu\left(E \cap \left(\bigcup_{n=1}^{+\infty} A_n\right)\right) + \mu\left(E \cap \left(\bigcup_{n=1}^{+\infty} A_n\right)^c\right),$$

that is, the countable union of disjoint elements of $\mathcal{M}$ still belongs to $\mathcal{M}$.

Step 3. In a similar fashion, one can prove that $\mu \llcorner \mathcal{M}$ is σ-additive. In fact, one can take a family of disjoint elements $\{A_n\}_{n\in\mathbb{N}}$ and take $E := \bigcup_{n\in\mathbb{N}} A_n$. $\square$

This theorem, despite the simple proof, is incredibly powerful. Indeed, it is relatively easy to find an outer measure on X and derive a σ-additive measure, while it is much harder to define it directly.

Remark 2.10. The Carathéodory result, on the other hand, gives no information whatsoever on the σ-algebra $\mathcal{M}$. For example, if we consider the outer measure

$$\mu(E) := \begin{cases} 0 & \text{if } E = \varnothing \\ 1 & \text{if } E \neq \varnothing, \end{cases}$$

then it is easy to prove that the σ-algebra associated via Theorem 2.14) is the trivial one, that is,

$$\mathcal{M} = \{\varnothing, X\},$$

and thus it is not an interesting object to deal with.

The next theorem will give us an easy-to-check criterion for an outer measure to be "*interesting*," in the sense that the σ-algebra $\mathcal{M}$ contains (at least) the Borel σ-algebra.

Theorem 2.15. *Let X be a metric space and let μ be an outer measure defined over X. If*

$$\operatorname{dist}(A_1, A_2) > 0 \implies \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2),$$

then $\mathcal{M}$ contains the Borel algebra.

Proof. First, we notice that it is enough to prove that closed sets belong to $\mathcal{M}$. We divide the argument into two steps: a particular, fairly straightforward, case and the general case.

Step 1. Let X be a Cantor-type set (in particular, it is totally disconnected). The identity

$$\mu(E) = \mu(E \cap I) + \mu(E \cap I^c), \qquad \forall\, E \subseteq X,$$

holds for any interval I contained in X. By construction, the interval I and the complement I^c are distant, and this concludes the proof since the intervals generate the Borel σ-algebra of X.

Step 2. Let $C \subseteq X$ be a closed subset, and let us consider the sequence given by

$$A_n := \left\{ x \in X \;\middle|\; d(x, C) \geq \frac{1}{n} \right\}.$$

The sequence $(A_n)_{n \in \mathbb{N}}$ is increasing, and its limit is the complement of C, that is,

$$A_n \uparrow (X \setminus C) = C^c.$$

We may always assume, without loss of generality, that $\mu(E) < +\infty$. It follows from the monotonicity of μ that

$$\mu(E) \geq \mu(E \cap C) + \mu(E \cap A_n), \qquad \forall\, n \in \mathbb{N},$$

and hence, by taking the limit as $n \to +\infty$, an application of Lemma 2.16 proves that

$$\mu\left(E \cap A_n\right) \to \mu\left(E \cap C^c\right),$$

which is the nontrivial inequality. $\square$

Lemma 2.16. *Let $A_n \uparrow A$ be an increasingly converging sequence of subsets, and assume that*

$$\operatorname{dist}(A_n, A_{n+1}^c) > 0 \qquad \forall\, n \in \mathbb{N}. \tag{2.10}$$

If the measure of the limit is finite ($\mu(A) < +\infty$), then $\mu(A_n) \uparrow \mu(A)$.

Proof. Let $a \in \mathbb{R}$ be the real number such that $\mu(A_n) \uparrow a$ (it exists, as a consequence of the monotonicity of the sequence). The reader can easily prove that

$$a \leq \mu(A).$$

To prove the opposite inequality, we consider the sequence of sets defined by induction as

$$\begin{cases} B_1 := A_1, \\ B_n := A_n \setminus B_{n-1}. \end{cases}$$

By construction, for every $k \in \mathbb{N}$ it turns out that

$$\mu(A_k) \geq \mu(A) - \sum_{n=k+1}^{+\infty} \mu(B_n),$$

and thus, if we can prove that the sum $\sum_{n\in\mathbb{N}} \mu(B_n)$ is finite, then we can take the limit as $k \to +\infty$ and find that

$$a = \liminf_{k\to+\infty} \mu(A_k) \geq \mu(A).$$

The assumption (2.10) implies that B_n and B_{n+2} are distant for every $n \in \mathbb{N}$; therefore we can decompose the sum as

$$\sum_{n\in\mathbb{N}} \mu(B_n) = \sum_{k=0}^{+\infty} \mu(B_{2k}) + \sum_{k=0}^{+\infty} \mu(B_{2k+1}).$$

The measure μ satisfies the assumption of Theorem 2.15; hence

$$\sum_{k\in\mathbb{N}} \mu(B_{2k}) = \mu\left(\bigcup_{k=0}^{+\infty} B_{2k}\right) \leq \mu(A).$$

In conclusion, the assumption $\mu(A) < \infty$ implies that

$$\sum_{n\in\mathbb{N}} \mu(B_n) \leq 2\mu(A) < \infty,$$

which is exactly the estimate we needed. □

Lemma 2.17. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of outer measures. If $\mu_n \nearrow \mu$ or $\mu_n \searrow \mu$, then also μ is an outer measure.*

Proof. We may assume that $(\mu_n)_{n\in\mathbb{N}}$ is an increasing sequence of outer measures converging to some μ, that is, $\mu_n \nearrow \mu$. The opposite case is obtained in a similar way.

Step 1. The equality $\mu(\varnothing) = 0$ is trivial. If $A \subset B$, then

$$\mu_n(A) \leq \mu_n(B) \implies \mu(A) := \sup_{n\in\mathbb{N}} \mu_n(A) \leq \sup_{n\in\mathbb{N}} \mu_n(B) =: \mu(B).$$

Step 2. Let $\{A_k\}_{k\in\mathbb{N}}$ be any countable collection of subsets of X. Then

$$\mu_n\left(\bigcup_{k\in\mathbb{N}} A_k\right) \leq \sum_{k\in\mathbb{N}} \mu_n(A_k) \qquad \forall\, n \in \mathbb{N},$$

and it follows from Fatou's lemma that

$$\begin{aligned}
\mu\left(\bigcup_{k\in\mathbb{N}} A_k\right) &= \limsup_{n\to+\infty} \mu_n\left(\bigcup_{k\in\mathbb{N}} A_k\right) \leq \\
&\leq \limsup_{n\to+\infty} \sum_{k\in\mathbb{N}} \mu_n(A_k) \leq \\
&\leq \sum_{k\in\mathbb{N}} \limsup_{n\to+\infty} \mu_n(A_k) = \\
&= \sum_{k\in\mathbb{N}} \mu(A_k).
\end{aligned}$$

□

Proposition 2.18. *Let X be a compact space, and let μ be a positive finite measure, defined on the Borel σ-algebra. For every $E \subseteq X$ Borel it turns out that*

$$\begin{aligned}
\mu(E) &= \inf\left\{\mu(A) \mid A \supseteq E,\ A \textit{ is open}\right\} = \\
&= \inf\left\{\mu(K) \mid K \subseteq E,\ K \textit{ compact}\right\}.
\end{aligned}$$

The first identity is usually referred to as outer-regularity of a measure; the second one inner-regularity of a measure.

Sketch of the Proof. First, we notice that the space X is compact, and thus the inner-regularity follows from the outer-regularity by passing to the complement. If we define the measure

$$\nu(E) := \inf\left\{\mu(A) \mid A \supseteq E,\ A \text{ is open}\right\},$$

then it is easy to prove that this is an outer measure (thus outer-regular by definition), which is additive on distant sets.

It follows from Theorem 2.15 that ν is a positive measure on the Borel σ-algebra. Finally, since $\mu(A) = \nu(A)$ for every open subset A, a straightforward application of the *Monotone Class Theorem*[6] allows us to infer that $\mu \equiv \nu$ on the whole σ-algebra of Borel. □

Remark 2.11. Let X and Y be spaces satisfying suitable assumption (e.g., metric and locally compact.) If E is a Borel subset of X, and $f : E \longrightarrow Y$ is a continuous function, then $f(E)$ is universally measurable[7], but it is, in general, not Borel.

2.4 Carathéodory Construction

Let X be a metric space, let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets, and let $\rho : \mathcal{F} \longrightarrow [0, +\infty]$ be the associated **gauge** function. We also assume that

(a) $\varnothing \in \mathcal{F}$;

(b) $\rho(\varnothing) = 0$.

[6] A monotone class is a collection of sets which is closed under countable monotone union and intersection.

[7] **Definition.** The set $f(E) \subset Y$ is universally measurable if and only if, for every finite positive measure μ on Y, there are two Borel sets $A \subseteq f(E) \subseteq B$ such that $\mu(A) = \mu(B)$.

For every positive real number $\delta \in (0, +\infty]$ and every $E \in \mathcal{P}(X)$, we define

$$\Psi_\delta(E) := \inf \left\{ \sum_{n\in\mathbb{N}} \rho(F_n) \;\middle|\; \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F},\ \operatorname{diam}(F_n) < \delta,\ \bigcup_{n\in\mathbb{N}} F_n \supseteq E \right\}, \tag{2.11}$$

where the diameter of a set F is defined by setting

$$\operatorname{diam}(F) := \sup \{ d(x, x') \mid x, x' \in F \}.$$

It may not be clear at this moment, but it is crucial to take the infimum only over countable covers in the definition (2.11). We assume that

$$\inf \varnothing = +\infty,$$

in such a way that the function

$$(0, +\infty] \ni \delta \longmapsto \Psi_\delta$$

is (weakly) decreasing. In particular, for any $E \in \mathcal{P}(X)$ we define

$$\Psi(E) := \lim_{\delta\to 0^+} \Psi_\delta(E) = \sup_{\delta>0} \Psi_\delta(E). \tag{2.12}$$

Lemma 2.19.

(a) *For any $\delta > 0$, the set function Ψ_δ is an outer measure and it is additive on sets which are distant more than δ in $\mathcal{F}$, that is, for every $A_1, A_2 \in \mathcal{F}$ such that $d(A_1, A_2) > \delta$, it turns out that*

$$\Psi_\delta(A_1 \cup A_2) = \Psi_\delta(A_1) + \Psi_\delta(A_2).$$

(b) *The limit set function Ψ is an outer measure and it is additive on distant set of $\mathcal{F}$, that is, for every $A_1, A_2 \in \mathcal{F}$ such that $d(A_1, A_2) > 0$, it turns out that*

$$\Psi(A_1 \cup A_2) = \Psi(A_1) + \Psi(A_2).$$

In particular, the outer measure Ψ satisfies the assumption of Theorem 2.15.

Proof.

(a) The reader may prove that Ψ_δ is an outer measure, as a relatively straightforward check of the characterizing properties; here we only show that it is additive on distant sets.

Let $\{E_n\}_{n\in\mathbb{N}}$ be an admissible countable cover of $A_1 \cup A_2$, with $\operatorname{diam}(E_n) < \delta$. By assumption

$$d(A_1, A_2) > \delta,$$

therefore it is immediate to check that, for any $n \in \mathbb{N}$, either $E_n \cap A_1 \neq \varnothing$ or $E_n \cap A_2 \neq \varnothing$. For $i = 1, 2$, let us set

$$I_i := \{ n \in \mathbb{N} \ :: E_n \cap A_i \neq \varnothing \},$$

and consider the countable covers $\mathscr{E} := \{E_n\}_{n\in I_1}$ and $\mathscr{F} := \{E_n\}_{n\in I_2}$ of A_1 and A_2 respectively. Then the nontrivial inequality follows easily:

$$\Psi_\delta(A_1) + \Psi_\delta(A_2) \leq \sum_{n\in I_1} \rho(E_n) + \sum_{n\in I_2} \rho(E_n) =$$

$$= \sum_{n\in\mathbb{N}} \rho(E_n) = \Psi_\delta(A_1 \cup A_2).$$

(b) This assertion is an immediate consequence of **(a)** and Lemma 2.17.

□

Corollary 2.20. *The limit set function Ψ is a σ-additive measure defined on the σ-algebra of Borel.*

Example 2.3 (Lebesgue Measure)**.** Let $X = \mathbb{R}^d$, and let us consider the collection of all *rectangles*, that is,

$$\mathcal{F} = \{I_1 \times \cdots \times I_d \mid I_i \subset \mathbb{R} \text{ interval}\}.$$

The gauge function associated to this collection is given by

$$\rho (I_1 \times \cdots \times I_d) = \prod_{i=1}^{d} |I_i|,$$

where $|I|$ denotes the length of the interval $I \subset \mathbb{R}$. For every $\delta \in (0, +\infty]$, it is easy to prove that

$$\Psi_\delta = \Psi = \mathcal{L}^d,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Moreover, the limit function coincides with the gauge function on rectangles set, i.e.,

$$\Psi (I_1 \times \cdots \times I_d) = \rho (I_1 \times \cdots \times I_d).$$

Exercise 2.3. Let $X = \mathbb{Q}$ and $\mathcal{F}$ be the family of 1-dimensional rectangles, that is,

$$\mathcal{F} = \{I \mid I \subset \mathbb{R} \text{ interval}\}.$$

Prove that

$$\rho (I) = |I| \implies \Psi_\delta \equiv 0.$$

Hint: the set of all rational numbers $\mathbb{Q}$ is totally disconnected countable space, and thus we can take countable covers made up of points, whose length - in the sense of ρ - is zero.)

2.5 Hausdorff d-dimensional Measure $\mathcal{H}^d$

Let X be a metric space, let $d \in [0, \infty)$ be any positive real number, let $\mathcal{F} = \mathcal{P}(X)$ be the family of all subsets, and let

$$\rho (E) = (\mathrm{diam}(E))^d$$

be the gauge function. The Hausdorff outer measure is defined by formula (2.11), but there is also a multiplicative constant (depending on d only), that is,

$$\mathcal{H}^d_\delta(E) = c_d \Psi_\delta(E).$$

The Hausdorff measure is the limit as $\delta \to 0^+$, that is, it is defined by formula (2.12) up to the same multiplicative constant:

$$\mathcal{H}^d(E) = c_d \Psi(E).$$

The constant c_d, for integer values of d, is defined as follows:

$$c_d := \frac{\alpha_d}{2^d},$$

where α_d is the measure of the d-dimensional unitary ball, that is,

$$\alpha_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}.$$

Lemma 2.21 (Properties of the Hausdorff Measure)**.**

(a) *If $d = 0$, then $\mathcal{H}^d$ is the counting measure.*

(b) *If $X = \mathbb{R}^n$, then*

$$\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E), \qquad \forall\, \lambda > 0,\ \forall\, E \subseteq \mathbb{R}^n.$$

(c) *If $f : E \subseteq X \longrightarrow Y$ is an isometry, then*

$$\mathcal{H}^d(f(E)) = \mathcal{H}^d(E).$$

(d) *If $f : E \subseteq X \longrightarrow Y$ is an L-Lipschitz map, then*

$$\mathcal{H}^d(f(E)) \leq L^d \mathcal{H}^d(E).$$

(e) *Let $X := \mathbb{R}^n$, and let Ψ_δ be the outer measure associated to the rectangles gauge function. For every $\delta \in (0, +\infty]$ and every $E \subseteq \mathbb{R}^n$, it turns out that*

$$\mathcal{H}^d_\delta(E) = \Psi_\delta(E).$$

Moreover, the Hausdorff d-dimensional measure and the Lebesgue d-dimensional measure coincide on $\mathbb{R}^n$, that is,

$$\mathcal{H}^d \equiv \mathcal{L}^d.$$

Proof.

(a) Let E be a finite set with $n := |E|$. Since E is discrete, there exists a real number $\epsilon > 0$ such that any countable cover of E made up by sets of diameter less than ϵ, must be of cardinality at least n.

On the other hand, there is a trivial covering by n nonempty sets of diameter less than ϵ (i.e., small neighborhoods of the n points); hence $\mathcal{H}^0(E) = n$. Employing the monotonicity of the outer measures, this proves the claim.

(b) First, we observe that any subset $A \subset \mathbb{R}^n$ has the following property: for every positive constant $\lambda > 0$, it turns out that

$$\operatorname{diam}(\lambda A) = \lambda \cdot \operatorname{diam}(A).$$

Let $E \subseteq \mathbb{R}^n$ be given, and let $\{E_n\}_{n\in\mathbb{N}}$ be an optimal cover[8] for E of diameter $\delta > 0$. It follows easily that $\{\lambda E_n\}_{n\in\mathbb{N}}$ is a countable cover for the rescaling λE, of diameter $\lambda\delta$. Hence

$$\mathcal{H}^d_{\lambda\delta}(\lambda E) \leq c_d \sum_{n\in\mathbb{N}} (\operatorname{diam}(\lambda E_n))^d =$$

$$= c_d \lambda^d \sum_{n\in\mathbb{N}} (\operatorname{diam}(E_n))^d = \lambda^d \mathcal{H}^d_\delta(E),$$

and, by taking the limit as δ approaches 0^+, we infer that

$$\mathcal{H}^d(\lambda E) \leq \lambda^d \mathcal{H}^d(E).$$

[8] A cover realizing the infimum in definition (2.11).

In a similar fashion, let $\{F_n\}_{n\in\mathbb{N}}$ be an optimal cover for λE of diameter $\delta > 0$. It follows easily that $\{\lambda^{-1}E_n\}_{n\in\mathbb{N}}$ is a countable cover for E, of diameter $\lambda^{-1}\delta$. Hence

$$\mathcal{H}^d_{\lambda^{-1}\delta}(E) \leq c_d \sum_{n\in\mathbb{N}} (\operatorname{diam}(E_n))^d =$$

$$= c_d \lambda^{-d} \sum_{n\in\mathbb{N}} (\operatorname{diam}(\lambda E_n))^d = \lambda^d \mathcal{H}^d_\delta(\lambda E),$$

and, by taking the limit as δ approaches 0^+, we infer that

$$\mathcal{H}^d(\lambda E) \geq \frac{1}{\lambda^d}\mathcal{H}^d(E).$$

(c) First, we notice that for every isometry $f : X \longrightarrow Y$ and every subset $E \subseteq X$, it turns out that

$$d(x, x') = d(f(x), f(x')) \implies \operatorname{diam}(E) = \operatorname{diam}(f(E)).$$

The argument shown in **(b)** applies here without any significant change, provided that we take into account the property above.

(d) Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable cover of E with diameter less than or equal to $\delta > 0$. The map f is L-Lipschitz, and thus $\{f(E_n)\}_{n\in\mathbb{N}}$ is a countable cover of $f(E)$ with diameter less than or equal to $L\delta$. Therefore

$$\mathcal{H}^d_{L\delta}(f(E)) \leq c_d \sum_{n\in\mathbb{N}} (\operatorname{diam}(f(E_n)))^d \leq$$

$$\leq c_d L^d \sum_{n\in\mathbb{N}} (\operatorname{diam}(E_n))^d = L^d \mathcal{H}^d_\delta(E),$$

and the thesis follows by taking the limit as δ approaches 0^+.

(e) This assertion is rather nontrivial, and we give a sketch of the proof after we introduce the Steiner symmetrization and the isoperimetrical inequality (see Theorem 2.27.)

□

Lemma 2.22. *Let E be a Borel set contained in a d-dimensional surface $\Sigma \subset \mathbb{R}^n$ of class C^1. Then the d-dimensional Hausdorff measure of E is the d-dimensional volume, that is,*

$$\mathcal{H}^d(E) = \operatorname{vol}_d(E).$$

Proof. The proof presented here is **not** rigorous, but the reader may try to expand it, fill the gaps and fix the imprecision as a particularly useful exercise.

Step 1. Fix $\delta > 0$ and cover the surface Σ with a countable collection $\mathcal{U} := \{U_n\}_{n\in\mathbb{N}}$ of sets such that, for every $n \in \mathbb{N}$, there is an almost-isometry

$$f_n : U_n \xrightarrow{\sim} A_n \subseteq \mathbb{R}^d,$$

where A_n is a flat subset of $\mathbb{R}^d$. More precisely, there exists $\delta > 0$ such that

$$(1-\delta)\cdot d(x, x') \leq d(f_n(x), f_n(x')) \leq (1+\delta)\cdot d(x, x'),$$

where d is the distance of the metric space X.

Step 2. The set E may be written as the disjoint union of a collection of subsets $\mathcal{E} := \{E_n\}_{n\in\mathbb{N}}$ satisfying the inclusion $E_n \subseteq U_n$ for every $n \in \mathbb{N}$, that is,

$$E = \bigsqcup_{n\in\mathbb{N}} E_n.$$

The d-dimensional volume of E is given by the sum of the d-dimensional volumes of the E_ns, i.e.,

$$\mathrm{vol}_d(E) = \sum_{n\in\mathbb{N}} \mathrm{vol}_d(E_n),$$

which is equal, taking the preimages via the almost-isometries, to

$$\mathrm{vol}_d(E) = \left(\sum_{n\in\mathbb{N}} \mathcal{L}^d\left(f_n^{-1}(A_n) \cap E_n\right)\right) \cdot (1 + \mathcal{O}(\delta)).$$

The function f_n is $(1+\delta)$-Lipschitz; hence

$$\mathcal{H}^d(E) = \sum_{n\in\mathbb{N}} \mathcal{H}^d(E_n) = \left(\sum_{n\in\mathbb{N}} \mathcal{H}^d\left(f_n(E_n)\right)\right) \cdot (1 + \mathcal{O}(\delta)),$$

and this concludes the proof since the right-hand side coincides with $\mathrm{vol}_d(E)$. □

Remark 2.12. The Hausdorff measure $\mathcal{H}^d$ does not change if we replace the family $\mathcal{F}$ with the following alternatives:

(1) Closed Sets. The diameter of a set A coincides with the diameter of its closure $\overline{A}$.

(2) Open Sets. Given a set A, for every $\epsilon > 0$ one can find an open set B_ϵ such that

$$A \subseteq B_\epsilon \qquad \text{and} \qquad \mathrm{diam}(B_\epsilon) \leq \mathrm{diam}(A) + \epsilon.$$

(3) Convex Sets. In the particular case $X = \mathbb{R}^n$, the reader may prove that the diameter of the convex hull of A is equal to the diameter of A.

On the other hand, one cannot replace the family $\mathcal{F}$ with, e.g., the family of balls[9].

Hausdorff Spherical Measure. Let X be a metric space, let $d \in [0, \infty)$ be any positive real number, let $\mathcal{F}_s$ be the family of all the balls, and let

$$\rho(E) = (\mathrm{diam}(E))^d$$

be the gauge function. The spherical Hausdorff outer measure is defined by formula (2.11), but there is also a multiplicative constant (depending on d only), that is,

$$\mathcal{H}^d_{\delta,\, s}(E) = c_d \Psi_{\delta,\, s}(E).$$

The spherical Hausdorff measure is the limit as $\delta \to 0^+$, that is, it is defined by formula (2.12) up to the same multiplicative constant:

$$\mathcal{H}^d_s(E) = c_d \Psi(E).$$

Lemma 2.23. *There is a constant $C > 0$ such that*

$$\mathcal{H}^d(E) \leq \mathcal{H}^d_s(E) \leq C \cdot \mathcal{H}^d(E), \qquad \forall\, E \subseteq X.$$

Proof. Let $E \subseteq X$ be a subset of X, and let $\{E_n\}_{n\in\mathbb{N}}$ be a countable covering of E with sets whose diameter is strictly less than δ.

[9]The reader may prove this assertion formally, but the rough idea behind it is evident: in general, a r-diameter set is **not** contained in a r-diameter ball (see, e.g., an equilateral triangle).

Step 1. Let $x_n \in E_n$ be a sequence of points, and let us consider the family of balls defined by

$$B_n := B\left(x_n,\, \mathrm{diam}(E_n)\right).$$

By construction, one can easily check that

$$E_n \subset B_n \qquad \text{and} \qquad \mathrm{diam}\left(B_n\right) = 2\mathrm{diam}(E_n),$$

and thus $\{B_n\}_{n\in\mathbb{N}}$ is a covering of E, made up of balls whose diameter does not exceed 2δ.

Step 2. A straightforward application of the definitions proves the inequality

$$\mathcal{H}^d_{2\delta,\, s}(E) \leq \sum_{n\geq 0} c_d \left(\mathrm{diam}(B_n)\right)^d = 2^d c_d \sum_{n\geq 0} \left(\mathrm{diam}(E_n)\right)^d.$$

If we take the infimum over all such families $\{E_n\}_{n\in\mathbb{N}}$, then we get the thesis with $C = 2^d$, that is,

$$\mathcal{H}^d(E) \leq \mathcal{H}^d_s(E) \leq 2^d \cdot \mathcal{H}^d(E), \qquad \forall\, E \subseteq X.$$

One easily notices that 2^d is not the sharp constant, and it can be improved up to

$$C(d) := \left(\frac{2n}{n+1}\right)^{\frac{d}{2}},$$

but we will not furnish a proof of this result in the course. □

Lemma 2.24. *Let E be a Borel set contained in a d-dimensional surface $\Sigma \subset \mathbb{R}^n$ of class C^1. Then*

$$\mathcal{H}^d(E) = \mathrm{vol}_d(E) = \mathcal{H}^d_s(E).$$

Proof. The argument is along the lines of the one used in Lemma 2.22. □

Lemma 2.25. *Let $d < d'$ be two real numbers, and let $E \subseteq X$ be any subset.*

(1) If $\mathcal{H}^d(E)$ is finite, then $\mathcal{H}^{d'}(E) = 0$.

(2) If $\mathcal{H}^{d'}(E) > 0$, then $\mathcal{H}^d(E) = +\infty$.

In particular, for a fixed set E, the function $d \longmapsto \mathcal{H}^d(E)$ is decreasing, and it attains a finite nonzero value at most once.

Proof.

(1) Let $\{E_n\}_{n\in\mathbb{N}}$ be a covering of E, whose diameter does not exceed a fixed $\delta > 0$. By definition, it turns out that

$$\mathcal{H}^{d'}_\delta(E) \leq \sum_{n\geq 0} \left(\mathrm{diam}(E_n)\right)^{d'} \leq \delta^{d'-d} \cdot \sum_{n\geq 0} \left(\mathrm{diam}(E_n)\right)^d, \tag{2.13}$$

which, in turn, implies that

$$\mathcal{H}^{d'}_\delta(E) \leq \delta^{d'-d} \cdot \mathcal{H}^d_\delta(E).$$

In conclusion, we notice that $d' - d$ is strictly greater than 0, and hence the thesis follows by taking the limit as $\delta \to 0^+$.

(2) The inequality (2.13) may be rewritten as

$$\mathcal{H}^d_\delta(E) \geq \delta^{d-d'} \cdot \mathcal{H}^{d'}_\delta(E),$$

and thus we conclude as in **(1)** since $d - d'$ is strictly less than 0.

$\square$

Definition 2.26 (Hausdorff Dimension)**.** Let $E \subseteq X$. The Hausdorff dimension of E, denoted by $\dim_{\mathcal{H}} E$, is the unique real number such that

$$\begin{cases} \mathcal{H}^d(E) = 0 & d > \dim_{\mathcal{H}} E \\ \mathcal{H}^d(E) = +\infty & d < \dim_{\mathcal{H}} E. \end{cases}$$

Remark 2.13 (Basic Properties)**.**

(a) If $\underline{d} = \dim_{\mathcal{H}} E$ is the Hausdorff dimension of E, then $\mathcal{H}^{\underline{d}}(E)$ might be either zero or infinite (i.e., it is not necessarily finite).

(b) The Hausdorff dimension of a countable union is equal to the supremum of the Hausdorff dimensions, that is,

$$\dim_{\mathcal{H}} \bigcup_{n\in\mathbb{N}} E_n = \sup_{n\in\mathbb{N}} \{\dim_{\mathcal{H}} E_n\}.$$

(c) Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ and assume that $\dim_{\mathcal{H}} E_n \nearrow d$. The dimension of the union is equal to d, that is,

$$\dim_{\mathcal{H}} \bigcup_{n\in\mathbb{N}} E_n = d,$$

and the Hausdorff measure of the union is zero:

$$\mathcal{H}^d\left(\bigcup_{n\in\mathbb{N}} E_n\right) = 0.$$

Exercise 2.4. Let $E \subseteq X$. The following properties are equivalent:

(a) The Hausdorff measure of E is zero, i.e. $\mathcal{H}^d(E) = 0$.

(b) For every $\epsilon > 0$ there is a countable cover $\{E_n\}_{n\in\mathbb{N}}$, whose diameters satisfy the inequality

$$\sum_{n\geq 0} (\operatorname{diam}(E_n))^d \leq \epsilon.$$

(c) The ∞-Hausdorff measure of E is zero, i.e. $\mathcal{H}^d_\infty(E) = 0$.

Theorem 2.27 ($\mathcal{H}^d \equiv \mathcal{L}^d$)**.** *Let $X := \mathbb{R}^n$. For every $\delta > 0$ and every $E \subseteq X$, it turns out that*

$$\mathcal{H}^d(E) = \mathcal{H}^d_\delta(E) = \mathcal{L}^d(E).$$

Before we can give the proof of this result, we need to introduce two fundamental tools (the isoperimetrical property of balls and the Steiner symmetrization), which are also extremely important per se.

Theorem 2.28 (Isoperimetrical Property of Balls)**.** *Let $B(r)$ be the* ***closed*** *ball centered at the origin of radius r in $\mathbb{R}^d$. Among all closed sets with given diameter in $\mathbb{R}^d$, the ball $B(r)$ has maximum Lebesgue measure, that is,*

$$c_d\left(\operatorname{diam}(B_r)\right)^d = \mathcal{L}^d\left(B(r)\right) \geq \mathcal{L}^d(E), \qquad \forall\, E \subseteq X \;:\; \operatorname{diam}(E) = 2r.$$

Sketch. The proof of the isoperimetrical property is a direct consequence of the Steiner symmetrization (the reader may consult [5, pp 195–198] for a rigorous argument.)

Symmetrization Construction. Let $E \subseteq \mathbb{R}^d$ be a bounded closed subset and let V be an affine hyperplane in $\mathbb{R}^d$. For every point $x \in V$, we may consider the 1-dimensional subspace e_x, orthogonal to V, and replace the intersection $e_x \cap E$ by a closed segment, centered at x_0, with the same length[10]. From now on, we denote by $\widetilde{E}$ the symmetrization of E.

Symmetrization Properties. Let $E \subseteq \mathbb{R}^d$ be a bounded closed subset and let V be an affine hyperplane in $\mathbb{R}^d$. The reader may prove that the Lebesgue measure does not change, that is,

$$\mathcal{L}^d(E) = \mathcal{L}^d\left(\widetilde{E}\right),$$

while the diameter decreases, that is,

$$\operatorname{diam}(E) \geq \operatorname{diam}\left(\widetilde{E}\right).$$

Moreover, one could easily prove that the set $\widetilde{E}$ is closed.

Proof ($d = 2$). In this particular case the proof is fairly straightforward (it suffices to look at Figure 2.1), but the idea for $d > 2$ is exactly the same.

Let E be any subset of $\mathbb{R}^2$, and consider an orthogonal basis $\{e_1, e_2\}$ of the real plane. If we denote by $\widetilde{E}$ the Steiner symmetrization of E with respect to e_1 first, and e_2 after, then it is easy to prove that $\widetilde{E}$ symmetric with respect to the origin, and it is thus contained in a ball of diameter $\operatorname{diam}\left(\widetilde{E}\right)$. □

Proof of Theorem 2.27 . Recall that the normalization constant of the Hausdorff measure is given by

$$c_d = \frac{\alpha_d}{2^d},$$

and thus, for every ball $B_{x,\,r} := B(x,\,r) \subseteq \mathbb{R}^d$, it turns out that

$$\operatorname{vol}_d(B_{x,\,r}) = c_d\left(\operatorname{diam}(B_{x,\,r})\right)^d.$$

Step 1. First, we notice that it suffices to prove that

$$\mathcal{H}^d_\infty(E) \geq \mathcal{L}^d(E), \qquad \forall\, E \subseteq \mathbb{R}^d,$$

in order to obtain the first inequality, that is,

$$\mathcal{H}^d(E) \geq \mathcal{L}^d(E), \qquad \forall\, E \subseteq \mathbb{R}^d.$$

[10] Hausdorff measure.

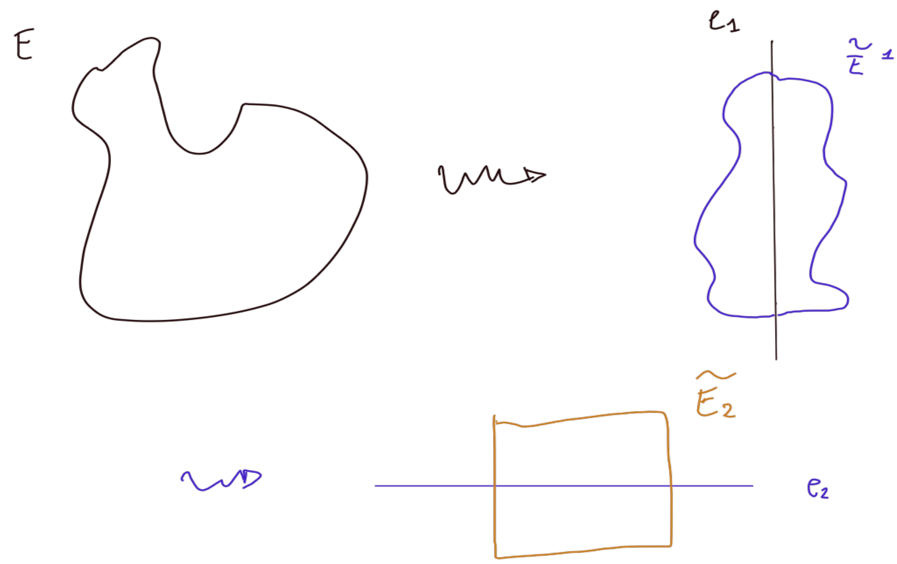

Figure 2.1: Idea of the proof in dimension 2

Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable cover of E. It follows from the isoperimetrical property of balls (see Theorem 2.28) that

$$c_d\,(\mathrm{diam}(E_n))^d \geq \mathcal{L}^d(E_n),$$

and thus

$$c_d \sum_{n\in\mathbb{N}} (\mathrm{diam}(E_n))^d = \sum_{n\in\mathbb{N}} c_d\,(\mathrm{diam}(E_n))^d \geq \mathcal{L}^d(E).$$

By taking the infimum of the left-hand side, we conclude that

$$\mathcal{H}^d_\infty(E) \geq \mathcal{L}^d(E).$$

Step 2. The opposite inequality follows if we are able to prove that

$$\mathcal{H}^d_\delta(E) \leq \mathcal{L}^d(E), \qquad \forall\, E \subseteq \mathbb{R}^d,\ \ \forall\, \delta > 0.$$

Let $\{E_n\}$ be a countable optimal cover for computing the Lebesgue measure[11] of E. The optimal cover cannot be made up of cubes or rectangles[12], but by Vitali's Covering Lemma 3.1 it follows that it can be chosen among all covers made up of (closed) balls. Moreover, for every fixed $\epsilon > 0$, there exists a collection of balls $\{B_n\}_{n\in\mathbb{N}}$ such that

$$\epsilon + \mathcal{L}^d(E) \geq \sum_{n\in\mathbb{N}} \mathcal{L}^d(B_n) = \sum_{n\in\mathbb{N}} c_d\,(\mathrm{diam}(B_n))^d \geq \mathcal{H}^d_\delta(E),$$

and this concludes the proof by arbitrariness of $\delta > 0$. □

[11] In particular, the cover does not need to be optimal for computing the Hausdorff measure.

[12] The reader may prove, as an exercise, that covers made up of cubes or rectangles yield to the sought inequality up to a constant strictly bigger than one.

2.6 Hausdorff Dimension of Cantor-Type Sets

In this brief section, we compute the Hausdorff dimension of the standard Cantor set $\mathcal{C}$, and we generalize the process to more elaborate sets.

Lemma 2.29. *Let $\mathcal{C}$ be the standard Cantor set. Its Hausdorff dimension is given by*

$$\underline{d} := \dim_{\mathcal{H}}(\mathcal{C}) = \frac{\ln 2}{\ln 3},$$

and the Hausdorff measure is finite, that is,

$$0 < \mathcal{H}^{\underline{d}}(\mathcal{C}) < +\infty.$$

Proof. In this proof for simplicity purposes we assume that the renormalization constant c_d is equal to 1, and we prove that the Hausdorff measure of $\mathcal{C}$ is bounded as follows:

$$\frac{1}{2} \leq \mathcal{H}^{\underline{d}}(\mathcal{C}) \leq 1.$$

Upper bound. The bound from above is, as usual, relatively easy to prove: it is enough to find a cover for the Cantor set $\mathcal{C}$ satisfying the bound. By definition

$$\mathcal{C} = \bigcap_{k=0}^{\infty} \left(\bigcup_{i=1}^{2^k} I_{k,\,i} \right),$$

where the diameter of each interval is given by

$$|I_{k,\,i}| = 3^{-k}.$$

Fix $\delta > 0$, and consider the minimal integer $k \in \mathbb{N}$ such that $3^{-k} < \delta$. Clearly

$$\mathcal{C} \subset \bigcup_{i=1}^{2^k} I_{k,\,i},$$

and this implies, by σ-subadditivity, that

$$\mathcal{H}^{\underline{d}}_{\delta}(\mathcal{C}) \leq \sum_{i=1}^{2^k} \mathcal{H}^{\underline{d}}_{\delta}(I_{k,\,i}) \leq 2^k \frac{1}{3^{\underline{d}k}} = 1.$$

Lower bound. First, we notice that

$$\mathcal{H}^{\underline{d}}_{\delta}(\mathcal{C}) \geq \mathcal{H}^{\underline{d}}_{\infty}(\mathcal{C}),$$

and thus it is enough to prove the lower bound for the ∞ measure, that is,

$$\mathcal{H}^{\underline{d}}_{\delta}(\mathcal{C}) \geq \mathcal{H}^{\underline{d}}_{\infty}(\mathcal{C}) \overset{?}{\geq} \frac{1}{2}.$$

Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable cover of $\mathcal{C}$, and assume that each E_n is open and convex. Moreover, by compactness, we may assume that there are only finitely many. Fix $n \in \mathbb{N}$ and take the smallest interval $I_{k,\,i} := I_n$ that contains E_n. The reader may prove by herself that

$$|I_n| \leq 3\,|E_n|\,,$$

as a consequence of the fact that, if I_n splits into I'_n and I''_n, then E_n must intersect both[13] (otherwise I_n would not be minimal, as required.) As a consequence of our claim, it turns out that

$$\sum_{n\in\mathbb{N}} (\mathrm{diam}\,(E_n))^d \geq 3^{-d} \sum_{n\in\mathbb{N}} (\mathrm{diam}\,(I_n))^d =$$

$$= 3^{-d} = \frac{1}{2}.$$

The equality is left as an exercise for the reader. The rough idea behind it is to throw away the repeated intervals I_n in the previous sum; it is not hard to prove that, in this way, we end up with finitely many. ☐

Remark 2.14. Let $\mathcal{C}$ be a Cantor set and let $\underline{d}$ be its Hausdorff dimension. The lower bound is not sharp, and one could prove that

$$\mathcal{H}^{\underline{d}}(\mathcal{C}) = 1.$$

Exercise 2.5 (Cantor-Type Set)**.** Fix $0 < \lambda < 1$, and let us consider the following partition of the unitary interval:

$$I_\lambda = \left[0,\, \frac{\lambda}{2}\right] \sqcup \left[\frac{\lambda}{2},\, 1 - \frac{\lambda}{2}\right] \sqcup \left[1 - \frac{\lambda}{2},\, 1\right].$$

Let $\mathcal{C}_\lambda$ be the generalized Cantor set, that is

$$\mathcal{C}_\lambda = \bigcap_{k=0}^{\infty} \left(\bigcup_{i=1}^{2^k} I_{k,\,i}^{(\lambda)} \right),$$

where $|I_{k,\,i}^{(\lambda)}| = \left(\frac{\lambda}{2}\right)^k$. Prove that the Hausdorff dimension of $\mathcal{C}_\lambda$ is given by the solution of the equation

$$2\left(\frac{\lambda}{2}\right)^{d_\lambda} = 1,$$

and, actually, it turns out that

$$0 < \mathcal{H}^{d_\lambda}(\mathcal{C}_\lambda) \leq 1.$$

Remark 2.15. The Hausdorff dimension of $\mathcal{C}_\lambda$ is explicitly given by

$$d_\lambda = \frac{\log 2}{\log 2 - \log \lambda},$$

and thus we get any Hausdorff dimension in $(0,\, 1)$ as λ ranges in $[0,\, 1]$.

[13]**N.B.** There could be points of E_n lying between I'_n and I''_n, but, since they do not belong to the Cantor set, we can ignore them (with some care).

Chapter 3

Covering Theorem

In this chapter, we investigate several covering theorems, that is, the possibility, given a family $\mathcal{F}$ of balls covering E, to extract a subfamily $\mathcal{F}'$ which covers E and behaves "better."

More precisely, let X be a metric space and μ a measure define on $\mathcal{B}(X)$; we would like to extract a subfamily $\mathcal{F}'$ satisfying one of the following properties:

(a) The collection $\mathcal{F}'$ is a disjoint covering of E. It is, clearly, the best possibility we could be hoping for, but, unfortunately, it is false[1] even in $\mathbb{R}^2$.

(b) The collection $\mathcal{F}'$ is a covering of E made up of balls that do not overlap "too much" (i.e., the measure of the overlapping portion is arbitrarily small.)

(c) The collection $\mathcal{F}'$ is a disjoint covering of μ-almost all of E, that is, the portion of E that is not covered has measure zero.

(d) The collection $\mathcal{F}'$ is a covering of E satisfying the inequality

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \epsilon.$$

In this chapter, we are mainly concerned with two classes of covering theorems, depending on the ambient space:

(1) Covering theorems that works on *every* metric space X, provided that the measure μ satisfies some extra assumptions.

(2) Covering theorems that works only on $X := \mathbb{R}^n$, with no requirement on the measure μ.

3.1 Vitali Covering Theorem

In this section, to ease the notation we denote by $B(x, r)$ the **closed** ball of center x and radius r, and by $\widehat{B}(x, r)$ the closed ball of center x and radius $5r$.

Lemma 3.1 (Vitali Covering Lemma). *Let X be a metric space, let $\mathcal{F} := \{B(x_i, r_i)\}_{i \in I}$ be a family of closed balls with uniformly bounded radii that covers a set $E \subseteq X$. Then there exists a disjoint subfamily $\mathcal{F}'$ such that the rescaling*

$$\widehat{\mathcal{F}}' := \{\widehat{B}(x_i, r_i) \mid B(x_i, r_i) \in \mathcal{F}'\}$$

[1] **Exercise.** Prove that the unitary square $[0, 1]^2$ in $\mathbb{R}^2$ cannot be covered by disjoint balls.

is a covering of E.

Remark 3.1. The uniform bound on the radii of the family $\mathcal{F}$ is fundamental, and it is impossible to drop it. Indeed, there is a simple counterexample in the real plane $\mathbb{R}^2$. The family

$$\mathcal{F} := \{B(n, n) \mid n \in \mathbb{N}\},$$

is an increasing covering of the upper-half plane E. Therefore, any disjoint subfamily is necessarily formed by a single ball, and hence it cannot cover E for any $n \in \mathbb{N}$ (see Figure 3.1.)

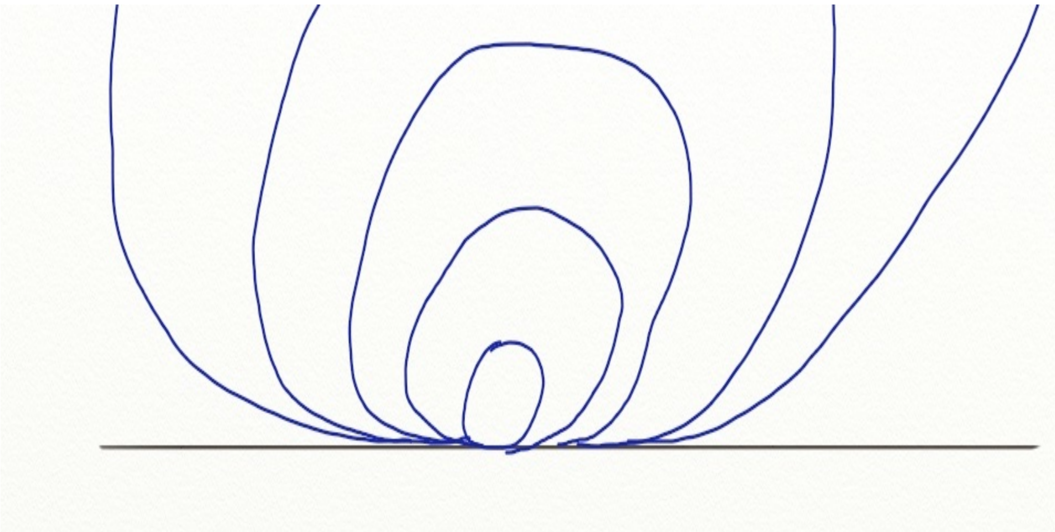

Figure 3.1: Necessity of the uniform boundedness on radii

Proof. We first prove the result by assuming some additional properties, and then we reduce the general case to it with a simple trick.

Step 1. Assume that the family $\mathcal{F} = \{B(x_n, r_n)\}_{n\in\mathbb{N}}$ is a countable cover of E, and assume also that the sequence of the radii is weakly decreasing, i.e. $r_1 \geq r_2 \geq \dots$. To construct the disjoint subfamily, we proceed as follows:

(1) The first ball B_1 automatically belongs to $\mathcal{F}'$.

(2) The second ball B_2 either intersect B_1 or it does not. If the latter holds true, then we add B_2 to the family $\mathcal{F}'$; if the former alternative holds true, we solely throw it away.

(3) Iterate the process for every $n \geq 3$ - taking into account all of the intersections with the balls that were previously added to the subfamily $\mathcal{F}'$.

At this point, we only need to prove that the rescaled collection of balls $\widehat{\mathcal{F}}'$ is a covering of E. Clearly, we may equivalently show that every ball B_n is a subset of an element in $\widehat{\mathcal{F}}'$. In particular, for every $n \in \mathbb{N}$ there are only two possible outcomes:

(i) The ball B_n already belongs to $\mathcal{F}'$, and thus there is nothing to prove.

(ii) The ball B_n intersects the ball B_m for some $m < n$. The distance between the centers x_n and x_m is less than or equal to the sum of the radii, which means that

$$d(x_m, x_n) \leq r_m + r_n \leq 2r_m \implies B(x_n, r_n) \subset B(x_m, 3r_m) \subset \widehat{B}_m.$$

Step 2. In the general case, we cannot assume that the family is countable or that the radii give an increasing sequence, and thus everything depends on the boundedness assumption. More precisely, there exists a real number $R > 0$ such that

$$B(x_i, r_i) \in \mathcal{F} \implies r_i \leq R.$$

Let us consider the following partition

$$\mathcal{F}_n := \left\{ B \in \mathcal{F} \;\middle|\; \frac{R}{2^{n+1}} < r \leq \frac{R}{2^n} \right\}, \qquad \forall n \in \mathbb{N},$$

in such a way that we can extract a subfamily from each $\mathcal{F}_n$ as follows:

(i) Extract a maximal (with respect to the inclusion) disjoint subfamily $\mathcal{F}_0'$ from $\mathcal{F}_0$.

(ii) Extract a maximal disjoint subfamily $\mathcal{F}_1'$ from $\mathcal{F}_1$, satisfying the additional requirement that it does not intersect any element of $\mathcal{F}_0'$.

(iii) Iterate the process for $n \geq 3$ - taking into account all of the intersections with the subfamilies already chosen in the previous steps.

Let $\mathcal{F}' = \cup_{n\in\mathbb{N}} \mathcal{F}_n'$: we need to show that the rescaled family $\widehat{\mathcal{F}}'$ is a covering of E. Clearly, we may equivalently show that every ball B_n is a subset of an element in $\widehat{\mathcal{F}}'$. In particular, for every $B(x, r) \in \mathcal{F}_N$ there are only two possible outcomes:

(i) The ball $B(x, r)$ already belongs to $\mathcal{F}_M'$ for some $M \in \mathbb{N}$, and thus there is nothing to prove.

(ii) The ball $B(x, r)$ intersects the ball $B(y, s)$, with $B(y, s) \in \mathcal{F}_M'$ for some $M \leq N$. It follows that $r \leq 2s$, and therefore the distance between the centers is less than or equal to the sum of the radii, which means that

$$d(x, y) \leq r + s \leq 3s \implies B(x, r) \subseteq \widehat{B}(y, s).$$

□

Theorem 3.2 (Vitali Covering Theorem). *Let X be a metric space, let μ be a doubling*[2] *locally finite measure*[3]*, and let $E \subseteq X$ be a Borel set. Then, for every $\epsilon > 0$ and every $\mathcal{F}$ fine cover*[4]*. of E made up of closed balls, there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$ which covers μ-almost all of E, that is,*

$$\mu\left(E \setminus \bigcup_{B\in\mathcal{F}'} B\right) = 0,$$

and the mass does not exceed $\mu(E)$ "too much", that is,

$$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E) + \epsilon. \tag{3.1}$$

Remark 3.2. The statement of the Vitali covering theorem makes sense, but there is a topological issue which may not be apparent: The disjoint subfamily needs to be countable, and thus we need some additional assumptions on X (locally compact, separable, etc.)

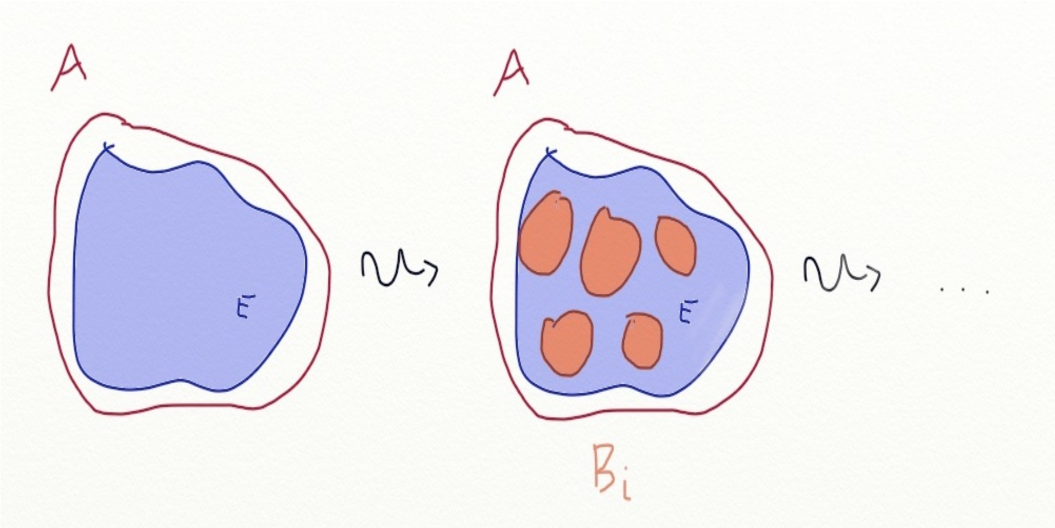

Figure 3.2: Idea of the proof

Proof. Assume, without loss of generality, that $\mu(E) < \infty$, and fix an open neighborhood A_0 of E satisfying the additional property

$$\mu\left(A_0 \setminus E\right) \leq \epsilon \wedge \frac{\mu(E)}{2C^3}, \tag{3.2}$$

where C is the doubling constant.

Step 1. Let us consider the collection of balls

$$\mathcal{F}_0 := \{B(x, r) \in \mathcal{F} \mid B(x, r) \subseteq A_0\}.$$

By construction, this is still a cover of E, and thus, by Lemma 3.1, it follows that there exists a disjoint subfamily of balls $\mathcal{F}_0'$ such that the rescaling $\widehat{\mathcal{F}}_0'$ covers E.

Step 2. We apply the doubling property of μ three times (since $2^2 < 5 < 2^3$), and it is easy to see that we can find a lower bound for the measure of $\mathcal{F}_0'$:

$$\mu(E) \leq \sum_{B \in \mathcal{F}_0'} \mu\left(\widehat{B}\right) \leq C^3 \sum_{B \in \mathcal{F}_0'} \mu(B) \implies \sum_{B \in \mathcal{F}_0'} \mu(B) \geq \frac{\mu(E)}{C^3}.$$

[2]**Definition.** A measure μ is doubling if and only if there exists a positive constant $C > 0$ such that

$$\mu\left(B(x, 2r)\right) \leq C\, \mu\left(B(x, r)\right).$$

[3]**Definition.** A measure μ is locally finite if and only if for every $x \in X$ there exists a neighborhood $U_x \ni x$ such that $\mu(U_x) < +\infty$.

[4]**Definition.** Let $\mathcal{F}$ be a family of closed balls that covers E. We say that $\mathcal{F}$ is fine if and only if each $x \in E$ is the center of a closed ball in $\mathcal{F}$ with arbitrarily small radius.

In a similar fashion[5], we obtain the following estimate[6]:

$$C^3 \sum_{B\in\mathcal{F}_0'} \mu(B) \le C^3 \left(\mu\left(\bigcup \mathcal{F}_0' \cap E\right) + \mu\left(\bigcup \mathcal{F}_0' \setminus E\right)\right) \le$$

$$\le C^3 \mu\left(\bigcup \mathcal{F}_0' \cap E\right) + \frac{\mu(E)}{2},$$

from which it follows that

$$\frac{\mu(E)}{2\,C^3} \le \mu\left(\left(\bigcup \mathcal{F}_0'\right) \cap E\right).$$

As an immediate consequence, the measure of the portion of E that has not been covered yet by $\mathcal{F}_0'$ may be estimated as follows:

$$\mu\left(E \setminus \bigcup \mathcal{F}_0'\right) \le \left(1 - \frac{1}{2C^3}\right)\mu(E). \tag{3.3}$$

Step 3. Let us take a suitable **finite** subfamily $\mathcal{F}_0''$, and let $E_1 := E \setminus \bigcup \mathcal{F}_0''$. The inequality (3.3) proves that

$$\mu(E_1) \le \left(1 - \frac{1}{3C^3}\right)\mu(E).$$

The process can now be iterated in the following way: Let A_1 be an open neighborhood of E_1 satisfying the additional requirement[7] that $E_1 \subset A_1 \subset (\bigcup \mathcal{F}_0'')^c$, and notice that everything works out smoothly as in the previous steps. In particular, it turns out that

$$\mu(E_k) \le \left(1 - \frac{1}{3C^3}\right)^k \mu(E),$$

and thus the inevitable candidate is given by

$$\mathcal{F}' := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n''.$$

The reader may check, as an exercise, that the following properties hold (and conclude the proof):

(1) The family $\mathcal{F}'$ is disjoint.

(2) The family $\mathcal{F}'$ is a covering of μ-almost all of E. Hint: prove that

$$E \setminus \bigcup_{n\in\mathbb{N}} \mathcal{F}_n'' = \bigcap_{n\in\mathbb{N}} E_n.$$

(3) The measure is arbitrarily near to the measure of E, that is,

$$\sum_{B\in\mathcal{F}'} \mu(B) \le \mu(E) + \epsilon.$$

[5] *Hint.* Decompose E as the disjoint union of the interior part ($\cap$) and the exterior part ($\setminus$). Then, apply the subadditivity of the measure and the assumption (3.2) on the measure of the open set A_0.

[6] **Notation.** The union of all the elements in a collection is denoted by

$$\bigcup \mathcal{F} := \bigcup_{B\in\mathcal{F}} B.$$

[7] Here we use the finiteness of the subfamily $\mathcal{F}_0''$ to ensure that the complement is an open set.

□

Lemma 3.3. *Let X be a metric space, and let μ be a doubling locally finite measure. Let $E \subseteq X$ be a Borel subset, and let $\mathcal{F}$ be a fine cover of E. Then there exist a finite disjoint subfamily $\mathcal{F}''$ and a positive real number $\delta > 0$ such that*

$$\mu\left(E \setminus \bigcup \mathcal{F}''\right) \leq \delta \cdot \mu(E).$$

Proof. The statement follows easily from the proof of the Vitali's Covering Theorem 3.2. □

Lemma 3.4. *Let X be a metric space, and let μ be a doubling locally finite measure. Let $E_0 \subset X$ be a Borel null-set, let $\epsilon > 0$ be a positive real number, and let $\mathcal{F}$ be a fine cover of E. Then there exist a subfamily $\mathcal{F}''$, which covers E_0, such that*

$$\sum_{B \in \mathcal{F}''} \mu(B) \leq \epsilon.$$

Proof. Let A_0 be an open neighborhood of E_0 satisfying the inequality

$$\mu(A_0) \leq \frac{\epsilon}{C^3}.$$

Let us consider the family of balls

$$\mathcal{G} := \left\{ B_i := B(x_i, r_i) \;\middle|\; B_i \subset A_0 \text{ and } \widehat{B}_i \in \mathcal{F} \right\}.$$

The reader may check, as an exercise, that $\mathcal{G}$ is a fine cover of E_0. By Vitali's Covering Lemma 3.1 we can find a disjoint subfamily $\mathcal{G}'$ such that $\widehat{\mathcal{G}}'$ covers E_0; hence, we set

$$\mathcal{F}'' := \widehat{\mathcal{G}}'.$$

By construction $\mathcal{F}'' \subset \mathcal{F}$, and it is also a covering for E_0 such that

$$\sum_{B \in \mathcal{G}'} \mu\left(\widehat{B}\right) \leq C^3 \sum_{B \in \mathcal{G}'} \mu(B) \leq C^3 \mu(A_0) \leq \epsilon,$$

which is exactly what we wanted to prove. □

Remark 3.3. The cover $\mathcal{F}$ need not be fine in the statement of Lemma 3.4, but, since this is the only result in this section whose assumptions can be weakened, we will just ignore it.

Theorem 3.5. *Let X be a metric space, and let μ be a doubling locally finite measure. Let $E \subseteq X$ be a Borel subset, let $\epsilon > 0$ be a positive real number, and let $\mathcal{F}$ be a fine cover of E. Then there exists a subfamily $\mathcal{F}'$, which covers E, such that*

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \epsilon,$$

that is we can cover E completely with balls, whose measure does not exceed too much $\mu(E)$, by dropping the disjoint requirement.

Proof. First, we notice that the Vitali's Covering Theorem 3.2 allows us to restrict out attention to null-set (i.e., $\mu(E) = 0$.) Then, we apply Lemma 3.4 to conclude. □

3.2 Besicovitch Covering Theorem

In this section, we investigate a different type of covering theorem, which is, actually, very similar to the Vitali's result.

The originality of the Besicovitch covering theorem is that it works without further assumption on the measure μ, provided that X is the Euclidean space $\mathbb{R}^n$.

Lemma 3.6 (Besicovitch Covering Lemma)**.** *Let $X := \mathbb{R}^n$, and let*

$$\mathcal{F} := \{B(x_i, r_i)\}_{i \in I}$$

be a family of closed balls with uniformly bounded radii, which is also a fine cover of a Borel set $E \subset \mathbb{R}^n$. Then there exist a natural number $N := N(n)$, which depends only on the dimension n, and $\mathcal{F}_1, \dots, \mathcal{F}_{N+1} \subset \mathcal{F}$ disjoint subfamilies such that

$$\bigcup_{i=1}^{N+1} \mathcal{F}_i \supseteq E.$$

Before we work out the details of the proof of this result, we need to state a technical lemma that allows us to infer easily that the natural number $N(n)$ depends solely on the dimension of the ambient space.

Lemma 3.7. *There exists a constant $N := N(n) \in \mathbb{N}$, depending only on the dimension of the ambient space, such that for any finite collection of closed balls $B_0, \dots, B_p \subset \mathbb{R}^n$ satisfying*

- ***(a)*** *$B_0 \cap B_i \neq \varnothing$ for every $i = 1, \dots, p$;*
- ***(b)*** *the 0th ball has the smaller radius, that is $r_0 \leq r_i$, for any $i = 1, \dots, p$;*
- ***(c)*** *the center x_i of the ball B_i does not belong to the ball B_j for every $i \neq j \in \{1, \dots, p\}$;*

*it turns out that $p \leq N(n)$. Moreover, the conclusion does not change if we replace the assumption **(b)** with a slightly different one, that is*

- ***(b)'*** *$r_0 \leq 2\, r_i$, for any $i = 1, \dots, p$.*

Proof. See [10, pp. 99–101]. □

Proof of Lemma 3.6. We first prove the result by assuming some additional properties, and then we reduce the general case to it with a simple trick.

Step 1. Assume that the family $\mathcal{F} = \{B(x_n, r_n)\}_{n \in \mathbb{N}}$ is a countable fine cover of E, and assume also that the sequence of the radii is weakly decreasing, i.e. $r_1 \geq r_2 \geq \dots$. To construct the disjoint subfamilies, we proceed as follows:

- **(1)** Select any index $j \in \{1, \dots, N+1\}$ and add the ball $B(x_0, r_0)$ to the subfamily $\mathcal{F}_j$.
- **(2)** Assume that $n-1$ balls have already been placed in $\mathcal{F}_1, \dots, \mathcal{F}_{N+1}$, or thrown away. Let $\{i_1, \dots, i_k\} \subset \{1, \dots, n-1\}$ be the subset of the indices of the kept balls.
- **(3)** We consider the n-th ball, and we notice that there are only two possible outcomes:
  - **(i)** If x_n belongs to $\bigcup_{j=1}^{k} B_{i_j}$, then we throw the ball B_n away.

(ii) If x_n does not belong to $\bigcup_{j=1}^{k} B_{i_j}$, then there is an index $i \in \{1, \ldots, N+1\}$ such that B_n can be added to $\mathcal{F}_i$, and that subfamily is still disjoint.

We argue by contradiction. If such an index i does not exist, then, for any $j \in \{1, \ldots, N+1\}$, there exists a ball $B_j \in \mathcal{F}_j$ such that $B_j \cap B_n \neq \varnothing$. By Lemma 3.7 we immediately derive the contradiction $N+1 \leq N$, which is exactly what we needed to conclude the first part of the proof.

Step 2. In the general case, we cannot assume that the family is countable or that the radii give an increasing sequence, and thus everything depends on the boundedness assumption. More precisely, there exists a real number $R > 0$ such that

$$B(x_i, r_i) \in \mathcal{F} \implies r_i \leq R. \tag{3.4}$$

Let us put a well order on $\mathcal{F} := \{B_\alpha\}_{\alpha<\omega}$ for some ordinal ω satisfying the additional property

$$\alpha' < \alpha \implies r_\alpha < 2\, r_{\alpha'},$$

which is always possible as a result of (3.4). The construction is no different than the previous one, but we do need to use the transfinite induction. In particular, we proceed as follows:

(i) Select any index $j \in \{1, \ldots, N+1\}$ and add the ball $B(x_0, r_0)$ to the subfamily $\mathcal{F}_j$.

(ii) Assume that α' balls have already been placed in $\mathcal{F}_1, \ldots, \mathcal{F}_{N+1}$, or thrown away. Let $\tau \subset \alpha'$ be the subset of the indices of the kept balls.

(iii) Let $\alpha > \alpha'$. There are two possible outcomes:

  (i) If x_α belongs to $\bigcup_{j\in\tau} B_j$, then we throw the ball B_α away.

  (ii) If x_α does not belong to $\bigcup_{j\in\tau} B_j$, then there is an index $i \in \{1, \ldots, N+1\}$ such that B_α can be added to $\mathcal{F}_i$, and that subfamily is still disjoint.

□

Lemma 3.8. *Let $X = \mathbb{R}^n$, and let μ be a locally finite measure. Let $E \subseteq X$ be a Borel subset, and let $\mathcal{F}$ be a fine cover of E. Then there exist a finite disjoint subfamily $\mathcal{F}''$ and a positive real number $\delta > 0$ such that*

$$\mu\left(E \setminus \bigcup \mathcal{F}''\right) \leq \delta \cdot \mu(E).$$

Proof. By Besicovitch Covering Lemma 3.6 there are $N := N(n)$ disjoint subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_N \subset \mathcal{F}$, whose union covers E, that is

$$E \subset \bigcup_{i=1}^{N} \mathcal{F}_i.$$

For every $i \in \{1, \ldots, N\}$ we set

$$E_i := E \setminus \bigcup \mathcal{F}_i,$$

and the reader may readily check that there exists an index $i \in \{1, \ldots, N\}$ such that

$$\mu(E_i) \leq \frac{\mu(E)}{N}.$$

The thesis follows immediately by taking to a proper finite subfamily of $\mathcal{F}_i$. □

Remark 3.4. It is not necessary for $\mathcal{F}$ to be a fine cover, but it is enough to have each point of E be the center of such a ball.

Lemma 3.9. *Let $X := \mathbb{R}^n$, and let μ be a locally finite measure. Let $E_0 \subset X$ be a Borel null-set, let $\epsilon > 0$ be any positive number, and let $\mathcal{F}$ be a fine cover of E. Then there exist a subfamily $\mathcal{F}''$, which covers E_0, such that*

$$\sum_{B \in \mathcal{F}''} \mu(B) \leq \epsilon.$$

Proof. Let A_0 be an open neighborhood of E_0 satisfying the inequality

$$\mu(A_0) \leq \frac{\epsilon}{N}.$$

Let us consider the family of balls

$$\mathcal{G} := \{B_i := B(x_i, r_i) \mid B_i \subseteq A_0\}.$$

The reader may check, as an exercise, that $\mathcal{G}$ is a fine cover of E_0. By Besicovitch Covering Lemma 3.6 we can find disjoint subfamilies $\mathcal{G}_1, \ldots, \mathcal{G}_N$ covering E_0. Set

$$\mathcal{F}'' := \bigcup_{i=1}^{N} \mathcal{G}_i,$$

and notice that it is also a covering of E_0 satisfying the additional property

$$\sum_{B \in \mathcal{F}''} \mu(B) = \sum_{i=1}^{N} \left[\sum_{B \in \mathcal{G}_i} \mu(B) \right] \leq N \cdot \mu(A_0) \leq \epsilon.$$

Indeed, the sum of the measures of the balls inside any subfamily $\mathcal{G}_i$ (as a consequence of the disjointness) needs to be equal or less than the measure of A_0. □

Theorem 3.10 (Besicovitch Covering Theorem)**.** *Let $X := \mathbb{R}^n$ and let μ be a locally finite measure. Let $E \subseteq X$ be a Borel subset, let $\epsilon > 0$ be any positive real number and let $\mathcal{F}$ be a fine cover of E. Then there exists a disjoint subfamily $\mathcal{F}'$, which covers μ-almost all of E, satisfying the inequality*

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \epsilon.$$

Proof. This is a direct consequence[8] of the Besicovitch Covering Lemma 3.6 and Lemma 3.8. □

[8] The argument is very similar to the one used in Vitali's lemma. The reader may try to fill-in the details as an exercise.

Chapter 4

Density of Measures

In this brief chapter, we investigate the notion of *density* associated with a measure μ. In particular, we first study the upper density related to the d-dimensional Hausdorff measure, and then we show that some of its essential properties may be generalized to other measures satisfying certain assumptions.

4.1 Density of Doubling Locally Finite Measures

In this section, we prove that the density of a doubling[1] locally finite measure μ is a well-defined quantity.

Definition 4.1 (Density)**.** Let $E \subseteq X$ be a Borel subset of a metric space. The *density* of a measure μ of the set E is given, at the point $x \in X$, by the following limit:

$$\Theta(\mu,\, E,\, x) := \lim_{r \to 0} \frac{\mu\left(E \cap B(x,\, r)\right)}{\mu\left(B(x,\, r)\right)}.$$

Theorem 4.2. *Let μ be a (doubling) locally finite measure defined on a metric space X, and let $E \subseteq X$ be a Borel subset. Then the density is either 1 or 0, that is,*

$$\Theta(\mu,\, E,\, x) := \lim_{r \to 0} \frac{\mu\left(E \cap B(x,\, r)\right)}{\mu\left(B(x,\, r)\right)} = \begin{cases} 1 & \text{for } \mu\text{-almost every } x \in E, \\ 0 & \text{for } \mu\text{-almost every } x \notin E. \end{cases}$$

Remark 4.1. The theorem holds true even if μ is not a doubling measure. Indeed, one can require μ to be *asymptotically* doubling, that is

$$\lim_{r \to 0} \frac{\mu\left(B(x,\, 2r)\right)}{\mu\left(B(x,\, r)\right)} < +\infty$$

for μ-almost every $x \in X$.

Proof. Here we only prove that the density is 1 for μ-almost every $x \in E$. The reader may either prove the other case with a similar argument or obtain it automatically by taking the complement.

[1] This assumption is necessary if X is a general metric space, but we can drop it if $X = \mathbb{R}^n$.

Step 1. Fix $\lambda \in (0, 1)$, and let us consider the set

$$E_\lambda := \left\{ x \in E \;\middle|\; \liminf_{r \to 0} \frac{\mu\left(E \cap B(x, r)\right)}{\mu\left(B(x, r)\right)} < \lambda \right\}.$$

It is easy to see that we may equivalently prove that measure[2] of E_λ is zero for any λ, that is

$$\mu(E_\lambda) = 0, \qquad \forall\, \lambda \in (0, 1).$$

Step 2. Let $\mathcal{F}$ be the family of all the closed balls $B(x, r)$ with center $x \in E_\lambda$ and radius satisfying the inequality

$$\frac{\mu\left(E \cap B(x, r)\right)}{\mu\left(B(x, r)\right)} \leq \lambda,$$

which is possible because the inferior limit of the ratio is estimated by λ in E_λ. Then $\mathcal{F}$ is a fine cover of E_λ, and thus by Vitali's Covering Theorem 3.2 there exists a disjoint subfamily $\mathcal{F}'$, covering μ-almost all of E_λ, such that

$$\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E_\lambda) + \epsilon.$$

An easy estimate for the measure of E_λ follows:

$$\mu(E_\lambda) \leq \sum_{B \in \mathcal{F}'} \mu\left(E \cap B\right) \leq \lambda \sum_{B \in \mathcal{F}'} \mu\left(B\right) \leq \lambda(\mu(E_\lambda) + \epsilon).$$

In conclusion, if we take the limit as $\epsilon \to 0^+$, then we obtain the estimate

$$\mu(E_\lambda) \leq \lambda \mu(E_\lambda),$$

and this the sought contradiction since we assumed λ to be strictly less than 1. $\square$

4.2 Upper Density of the Hausdorff Measure

In this section, we focus our efforts on the upper density of the d-dimensional Hausdorff measure since we can find a lower bound and an upper bound explicitly.

Definition 4.3 (Upper Density)**.** Let E be a Borel subset of a metric space X. The *upper d-dimensional density* (with respect to $\mathcal{H}^d$) of E at the point x is defined by setting

$$\Theta^*_d(E, x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^d\left(E \cap B(x, r)\right)}{\alpha_d r^d}, \tag{4.1}$$

where $\alpha_d 2^{-d}$ is the renormalization constant introduced the definition of the Hausdorff measure, and $B(x, r)$ denotes the closed ball of center x and radius r.

Theorem 4.4. *Let E be a Borel subset of a metric space X, and assume that E is locally $\mathcal{H}^d$-finite. Then the following properties hold true:*

(a) *The upper density of external points (outer density) is almost everywhere zero, that is,*

$$\Theta^*_d(E, x) = 0, \qquad \text{for } \mathcal{H}^d\text{-almost every } x \notin E.$$

[2] Here, we would need to check carefully that the set E_λ is actually Borel! The intuitive idea behind this goes as follows: The density ratio is increasing with respect to r, and thus it is a Borel function. The inferior limit for $r \to 0$, on the other hand, can be computed using rational sequences for the radii; thus it is Borel, and so is the set E_λ. The reader may try to fill in the details here as an exercise.

(b) *The upper density is bounded from below, that is,*

$$\frac{1}{2^d} \leq \Theta^*_d(E,\, x), \qquad \textit{for } \mathcal{H}^d\textit{-almost every } x \in E.$$

(c) *If $X \cong \mathbb{R}^n$, then the upper density is bounded from above by 1, that is,*

$$\Theta^*_d(E,\, x) \leq 1, \qquad \textit{for } \mathcal{H}^d\textit{-almost every } x \in E.$$

(d) *If X is a generic metric space, then the upper density is bounded from above by a slightly different constant, that is,*

$$\Theta^*_d(E,\, x) \leq 5^d, \qquad \textit{for } \mathcal{H}^d\textit{-almost every } x \in E.$$

Remark 4.2. The reason we are only concerned with upper density is the following: There exists $E \subseteq \mathbb{R}$ with finite and positive Hausdorff measure such that the lower density is 0 for $\mathcal{H}^d$-almost every $x \in E$.

On the other hand, there is a dual concept of the Hausdorff dimension, called the **packing dimension**. Similar statements hold for this kind of measure, but the lower density replaces the upper one (see, e.g., [8, pp. 81–86].)

Proof.

(a) Fix $m > 0$ and let us consider the set

$$E_m := \{x \notin E \mid \Theta^*_d(E,\, x) > m\}\,.$$

The thesis follows easily if we can prove that the d-dimensional Hausdorff measure of E_m is equal to zero for any fixed m.

Let $\mu := \mathcal{H}^d \llcorner E$ the restriction to E of the d-dimensional Hausdorff measure, and notice that μ is locally finite by construction. Let A be an open neighborhood of E_m, and let us consider the family of balls

$$\mathcal{F} = \left\{ \overline{B(x,\, r)} \;\middle|\; \overline{B(x,\, r)} \subset A,\, x \in E_m \text{ and } \mathcal{H}^d\left(E \cap \overline{B(x,\, r)}\right) > m \cdot \alpha_d r^d \right\}.$$

The reader may easily prove that $\mathcal{F}$ is a fine cover of E_m. Thus, by Vitali's Covering Lemma 3.1 it follows that there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $\widehat{\mathcal{F}}'$ is also a covering of E_m. A straightforward computation proves that

$$\mu(A) \geq \sum_{B \in \mathcal{F}'} \mu(B) \geq m \cdot \alpha_d \sum_{B \in \mathcal{F}'} r(B)^d =$$

$$= \frac{m}{5^d} \frac{\alpha_d}{2^d} \sum_{B \in \mathcal{F}'} \left(\operatorname{diam}(\widehat{B})\right)^d \geq$$

$$\geq \frac{m}{5^d} \mathcal{H}^d_\infty(E_m),$$

since $E_m \subseteq \cup_{B \in \mathcal{F}'} \widehat{B}$. By definition, the set E_m is disjoint from E, and thus $\mu(E_m) = 0$ (as μ is the restriction to E), and, by regularity (of the measure μ), we can always choose A in such a way that $\mu(A)$ is as small as we want. It turns out that

$$\epsilon \geq \frac{m}{5^d} \mathcal{H}^d_\infty(E_m), \quad \forall\, \epsilon > 0 \implies \mathcal{H}^d(E_m) = 0$$

since $\mathcal{H}^d(E) = 0$ is equivalent to $\mathcal{H}^d_\infty(E) = 0$.

(b) We divide the proof into three steps: we prove it in a very particular case, and then we reduce it to the general case using a simple trick.

Fix $m < 2^{-d}$ and let us consider the set

$$E_m := \{x \in E \mid \Theta^*_d(E, x) < m\}.$$

Let $\mu := \mathcal{H}^d \llcorner E$ the restriction to E of the d-dimensional Hausdorff measure, and notice that μ is locally finite by construction. By assumption, for every $x \in E_m$ there is a real number $r_0(x) := r_0 > 0$ such that

$$\mathcal{H}^d \left(E \cap \overline{B(x, r_0)} \right) < m \cdot \alpha_d r_0^d. \tag{4.2}$$

Step 1. Assume that[3] the inequality above holds for every $r > 0$, uniformly with respect to x. Let $\mathcal{F} := \{F_i \mid i \in I\}$ be a covering of E_m made up of balls, and consider the family of balls $\mathcal{F}' := \{B_i \mid i \in I\}$ defined in the following way:

1) The radius of B_i is equal to the radius of F_i for any $i \in I$.

2) The center of B_i is a point of E_m for any $i \in I$.

3) The collection of balls $\mathcal{F}'$ covers E_m.

We can easily estimate the Hausdorff measure of E_m as follows:

$$2^d \mathcal{H}^d(E_m) \geq \alpha_d \sum_{F_i \in \mathcal{F}} \mu(F_i) \geq \alpha_d \sum_{B_i \in I} (\operatorname{diam}(B_i))^d \geq \frac{1}{m} \sum_{i \in I} \mathcal{H}^d (E \cap B_i),$$

from which it follows that

$$2^d \mathcal{H}^d(E_m) \geq \frac{1}{m} \mathcal{H}^d(E_m) \implies \mathcal{H}^d(E_m) = 0$$

since $m < 2^{-d}$ by assumption.

Step 2. Assume now that the inequality (4.2) does not hold uniformly for every $r > 0$, but there is a uniform constant $r_0 > 0$ such that (4.2) holds true for every $x \in E$ and every $r \leq r_0$. In a similar fashion, we can estimate the outer measure $\mathcal{H}^d_\delta$ for $\delta < r_0/2$ - up to an arbitrarily small error $\epsilon > 0$ -, that is, for every $\epsilon > 0$ we obtain the inequality

$$2^d \mathcal{H}^d(E_m) + \epsilon \geq \frac{1}{m} \mathcal{H}^d_\delta(E_m),$$

which immediately implies

$$\mathcal{H}^d(E_m) = 0,$$

since $m < 2^{-d}$ by assumption.

Step 3. Finally, assume that the inequality (4.2) is not uniform with respect to x, that is, for any $x \in E_m$, it holds for $r_0 := r_0(x) > 0$. We consider the set

$$E_{m, r_0} := \left\{ x \in E \;\middle|\; \mathcal{H}^d \left(E \cap \overline{B(x, r)} \right) < m \cdot \alpha_d r^d, \ \forall\, r \leq r_0 \right\}$$

and we prove, using the previous step, that this set has measure zero for any choice of $r_0 > 0$ and $m < 2^{-d}$.

(c) We first give a rough idea of the proof of this assertion and then we formalize it correctly.

[3] We will get rid of this assumption in the next step using a simple trick.

Step 0. Fix $m > 1$ and let us consider the set

$$E_m := \{x \in E \mid \Theta^*_d(E, x) > m\}.$$

Let $\mu := \mathcal{H}^d \llcorner E$ the restriction to E of the d-dimensional Hausdorff measure, and notice that μ is locally finite by construction. By assumption, there are infinitely many balls such that

$$\mu\left(\overline{B(x, r)}\right) = \mathcal{H}^d\left(\overline{B(x, r)} \cap E\right) > m \cdot \alpha_d r^d,$$

and thus

$$\mu(E_m) \simeq \sum_i \mu(B_i) \geq m \cdot \frac{\alpha_d}{2^d} \sum_i (2r_i)^d \geq m \cdot \mathcal{H}^d(E_m).$$

In conclusion, since μ is the restriction of $\mathcal{H}^d$ to E, it suffices to notice that $E_m \subset E$ to infer that

$$\mathcal{H}^d(E_m) = \mu(E_m) \geq m \cdot \mathcal{H}^d(E_m),$$

which is absurd since $m > 1$.

Step 1. Fix $\delta > 0$ and let us consider the collection of balls

$$\mathcal{F} := \left\{ \overline{B(x, r)} \;\middle|\; x \in E_m,\, r \leq \frac{\delta}{2} \text{ and } \mathcal{H}^d\left(\overline{B(x, r)} \cap E\right) > m \cdot \alpha_d r^d \right\}.$$

It is easy to prove that $\mathcal{F}$ is a fine cover of E_m. By Besicovitch Covering Lemma 3.6, for every $\epsilon > 0$ we can extract a subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}'$ is a covering of E_m and

$$\mu(E_m) + \epsilon \geq \sum_{B \in \mathcal{F}'} \mu(B).$$

A straightforward computation proves that

$$\begin{aligned}
\mu(E_m) + \epsilon \geq \sum_{B \in \mathcal{F}'} \mu(B) \geq& \\
&\geq m \sum_{B \in \mathcal{F}'} \alpha_d r(B)^d = \\
&= m \cdot \frac{\alpha_d}{2^d} \sum_{B \in \mathcal{F}'} (2r(B))^d \geq \\
&\geq m \mathcal{H}^d_\delta(E_m).
\end{aligned}$$

Therefore, if we take the limit as δ and ϵ goes to 0^+, then it turns out that

$$\mathcal{H}^d(E_m) + \epsilon = \mu(E_m) + \epsilon \geq m\, \mathcal{H}^d_\delta(E_m) \implies \mathcal{H}^d(E_m) \geq m\, \mathcal{H}^d(E_m),$$

which is absurd since $m > 1$.

(d) Let $\mu := \mathcal{H}^d \llcorner E$ the restriction to E of the d-dimensional Hausdorff measure, and notice that μ is locally finite by construction. Unfortunately, the measure μ does not have the doubling property (or the asymptotic one); therefore we need to rely on a different covering theorem.

Fix $m > 5^d$, fix $\delta > 0$, let A be an open neighborhood of E_m, and let us consider the family of closed balls

$$\mathcal{F} := \left\{ \overline{B(x,\, r)} \subset A \;\middle|\; x \in E_m,\, r \le \delta \text{ and } \mathcal{H}^d\left(\overline{B(x,\, r)} \cap E\right) > m \cdot \alpha_d r^d \right\}.$$

The cover $\mathcal{F}$ is fine, and thus there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $\widehat{\mathcal{F}}'$ covers E_m. Then

$$\begin{aligned} \mu(A) \ge \sum_{B \in \mathcal{F}'} \mu(B) \ge & \\ & \ge m \sum_{B \in \mathcal{F}'} \alpha_d\, r(B)^d = \\ & = \frac{m}{5^d} \frac{\alpha_d}{2^d} \sum_{B \in \mathcal{F}'} \left(2r\left(\widehat{B}\right)\right)^d \ge \\ & \ge \frac{m}{5^d} \mathcal{H}^d_{\delta'}(E_m), \end{aligned}$$

where $\delta' = 10\delta$. We conclude the proof by noticing that, since μ is a regular measure (both inner and outer regular), we have

$$\mathcal{H}^d(E_m) = \mu(E_m) = \inf_{A \supset E_m} \mu(A)$$

and hence, by taking the limit as $\delta,\, \epsilon \to 0^+$, it turns out that

$$\mathcal{H}^d(E_m) \ge \frac{m}{5^d} \mathcal{H}^d_{\delta'}(E_m) \implies \mathcal{H}^d(E_m) = 0.$$

□

Remark 4.3. In the previous results, we used the covering theorems assuming either that $X = \mathbb{R}^n$ or μ doubling, but for the applications we need them to be true for a larger class of measures. More precisely, the two statements

A1) If $E \subset X$ is a Borel subset and $\mathcal{F}$ is a fine cover of E, then for any $\epsilon > 0$ there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$, covering E μ-almost everywhere, such that

$$\sum_{B \in \mathcal{F}'} \mu(B) \le \mu(E) + \epsilon.$$

A2) If $E \subset X$ is a Borel subset and $\mathcal{F}$ is a fine cover of E, then for any $\epsilon > 0$ there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$, covering E, such that

$$\sum_{B \in \mathcal{F}'} \mu(B) \le \mu(E) + \epsilon.$$

hold true even for a measure $\mu = \mu' \llcorner F$, where μ' is doubling and F is arbitrary (actually, it is enough to ask $\mu \ll \mu' \llcorner F$.)

4.3 Upper d-Dimensional Density

Support. Let μ be a measure defined on a metric space X. The *support* of μ is the smallest closed subset $F \subset X$ such that the complement F^c is a null set, that is,

$$\operatorname{spt}(\mu) = \inf \{ F \subset X \mid F \text{ is closed and } \mu(F^c) = 0 \}.$$

The measure μ is *supported* on a Borel set $E \subset X$ if the complement of E is a null set for μ. In particular, the set E contains the support of μ, that is,

$$\operatorname{spt}(\mu) \subseteq E.$$

Definition 4.5. Let μ and λ be two measures defined on the same metric space X. We say that μ is *orthogonal* to λ, and we denote it by $\mu \perp \lambda$, if and only if they are supported on disjoint Borel sets.

Remark 4.4. The support of a measure μ is a well-defined notion, but it is not the optimal one concerning the orthogonality. Indeed, the reader may prove, as an exercise, that there exists a measure μ orthogonal to a measure λ such that

$$\operatorname{spt}(\mu) \cap \operatorname{spt}(\lambda) \neq \varnothing.$$

Hint. Consider the Lebesgue measure and the Dirac measure on the real line.

Lemma 4.6. *Let λ and μ be locally finite measures defined on a metric space X. Assume that λ is orthogonal to μ and λ satisfies the assumption* ***A2)***. *Then the Radon-Nikodym derivative is zero, that is,*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(x) := \lim_{r \to 0} \frac{\lambda\left(\overline{B(x,\, r)}\right)}{\mu\left(\overline{B(x,\, r)}\right)} = 0,$$

for μ-almost every $x \in X$.

Proof. Fix $m > 0$, let $F \subset X$ be a Borel set satisfying

$$\lambda(F^c) = 0 \qquad \text{and} \qquad \mu(F) = 0,$$

and let

$$E_m := \left\{ x \notin F \;\middle|\; \limsup_{r \to 0^+} \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(x) > m \right\}.$$

Since F is a μ-null set, we may equivalently prove that $\mu(E_m) = 0$ for every fixed $m > 0$. Consider the family of closed balls

$$\mathcal{F} := \left\{ \overline{B(x,\, r)} \;\middle|\; x \in E_m,\, \lambda\left(\overline{B(x,\, r)}\right) \geq m\mu\left(\overline{B(x,\, r)}\right) \right\}.$$

The reader may check that $\mathcal{F}$ is a fine cover of E_m; hence, by assumption **A2)** it follows that for every $\epsilon > 0$ there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$ such that

$$E_m \subseteq \bigcup_{B \in \mathcal{F}'} B \qquad \text{and} \qquad \sum_{B \in \mathcal{F}'} \lambda(B) \leq \lambda(E_m) + \epsilon.$$

The complement of F is a λ-null set, and thus we finally infer that

$$\underbrace{\lambda(E_m)}_{=0} + \epsilon \geq m \sum_{B \in \mathcal{F}'} \mu(B) \geq m\mu(E_m) \implies \mu(E_m) = 0.$$

□

Corollary 4.7. *Let λ and μ be locally finite measures defined on a metric space X. Assume that λ is orthogonal to μ and λ satisfies the assumption **A2)**. Then the Radon-Nikodym derivative is infinite, that is,*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = +\infty,$$

for μ-almost every $x \in X$.

Theorem 4.8. *Let μ be locally finite measures defined on a metric space X satisfying the assumption **A2)**, and let $f \in L^p_{\mathrm{loc}}(X, \mu)$ be a function. Then*

$$\fint_{\overline{B(x,\,r)}} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) \xrightarrow{r \to 0} 0,$$

for μ-almost every $x \in X$.

Proof. Fix $\epsilon > 0$. By Lusin's Theorem[4] we can always find a continuous function $\widetilde{f} : X \longrightarrow \mathbb{R}$ and a subset $E \subset X$ such that

$$f\big|_E = \widetilde{f}\big|_E \qquad \text{and} \qquad \mu\,(X \setminus E) < \epsilon.$$

The complement of E can be taken arbitrarily small; thus it suffices to prove the theorem for μ-almost every $x \in E$. In particular, given $x \in E$ it turns out that

$$\fint_{\overline{B(x,\,r)}} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) = \fint_{\overline{B(x,\,r)} \cap E} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) + \fint_{\overline{B(x,\,r)} \setminus E} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) =$$

$$= \frac{1}{\mu\left(\overline{B_r}\right)} \left[\int_{\overline{B_r} \cap E} \left| \widetilde{f}(y) - \widetilde{f}(x) \right|^p \mathrm{d}\mu(y) + \int_{\overline{B_r} \setminus E} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) \right] \overset{(*)}{\leq}$$

$$\overset{(*)}{\leq} \left[\omega\left(\widetilde{f}\right) \right]^p + \frac{2^{p-1}}{\mu\left(\overline{B_r}\right)} \int_{\overline{B(x,\,r)} \setminus E} \left[|f(x)|^p + |f(y)|^p \right] \mathrm{d}\mu(y),$$

where $\omega\left(\widetilde{f}\right)$ denotes the oscillation of $\widetilde{f}$, and the inequality $(*)$ follows from the following trivial fact: If $p \geq 1$, then for any real numbers a and b it turns out that

$$|a - b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Let us consider the measures

$$\lambda_1 := \left[|f|^p \, \mathbb{1}_{X \setminus E} \right] \cdot \mu \qquad \text{and} \qquad \lambda_2 := \mathbb{1}_{X \setminus E} \cdot \mu,$$

and let $\widetilde{\mu}$ be the restriction of μ to the set E. Then

$$\fint_{\overline{B(x,\,r)}} |f(y) - f(x)|^p \,\mathrm{d}\mu(y) \leq \left[\omega\left(\widetilde{f}\right) \right]^p + 2^{p-1} \frac{\lambda_1\left(\overline{B(x,\,r)}\right)}{\widetilde{\mu}\left(\overline{B(x,\,r)}\right)} + 2^{p-1} \, |f(x)|^p \, \frac{\lambda_2\left(\overline{B(x,\,r)}\right)}{\widetilde{\mu}\left(\overline{B(x,\,r)}\right)},$$

from which it easily follows that:

[4]**Lusin's Theorem:** Let (X, Σ, μ) be a Radon measure space and let Y be a second-countable topological space. If $f : X \to Y$ is a measurable function and $\epsilon > 0$, then for any $A \in \Sigma$ there is a closed set E with $\mu(A \setminus E) < \epsilon$ such that $f\big|_E$ is continuous.

1) The oscillation $\omega\left(\widetilde{f}\right)$ goes to 0 as $r \to 0^+$ for **every** $x \in E$, as a consequence of the continuity of $\widetilde{f}$.

2) The measures λ_1 and λ_2 are both orthogonal to $\widetilde{\mu}$; hence both the ratios go to 0 as $r \to 0^+$, for $\widetilde{\mu}$-almost every $x \in X$, which means for μ-almost every x in E.

$\square$

Definition 4.9. The function $f : X \longrightarrow \mathbb{R}$ is L^p*-approximately continuous* if and only if

$$\fint_{B(x,\,r)} |f(y) - f(x)|^p \, \mathrm{d}\mu(y) \xrightarrow{r \to 0^+} 0$$

for μ-almost every $x \in X$.

Corollary 4.10. *Let μ be locally finite measures defined on a metric space X satisfying the assumption **A2)**, and let $f \in L^p_{\mathrm{loc}}(X,\, \mu)$ be a function. Then*

$$\fint_{B(x,\,r)} f(y) \, \mathrm{d}\mu(y) \xrightarrow{r \to 0^+} f(x),$$

for μ-almost every $x \in X$.

Proof. By Jensen's inequality[5], it turns out that

$$L^q_{\mathrm{loc}}(\Omega) \subset L^p_{\mathrm{loc}}(\Omega), \qquad \forall\, q < p,$$

and thus we can apply Theorem 4.8 with $p = 1$. $\square$

Theorem 4.11. *Let μ and λ be locally finite measures satisfying the assumption **A2)**. Assume that there is a decomposition*

$$\lambda = f \cdot \mu + \lambda_s,$$

where λ_s denotes the singular part of the measure λ. Then the density of λ with respect to μ can be computed pointwise, and it is equal to

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(x) = f(x),$$

for μ-almost every $x \in X$.

Proof. By Lemma 4.6 it turns out that

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\mu}(x) = 0$$

for μ-almost every $x \in X$. On the other hand, by Theorem 4.8 it follows that

$$\frac{\mathrm{d}\lambda_{ac}}{\mathrm{d}\mu}(x) = f(x),$$

where λ_{ac} denotes the part of λ which is absolutely continuous with respect to μ, that is,

$$\lambda_{ac} = f \cdot \mu.$$

$\square$

[5]Let $(\Omega,\, \mathcal{G},\, \mu)$ be a probability space. If $g : \Omega \longrightarrow \mathbb{R}$ is a μ-summable function, and if $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, then

$$\varphi\left(\int_\Omega g(x)\, \mathrm{d}\mu(x)\right) \le \int_\Omega \varphi \circ g(x)\, \mathrm{d}\mu(x).$$

Upper Density. Let μ be a locally finite measure defined on a metric space X, and let $d > 0$ be a positive real number. The *d-dimensional upper density* at the point x with respect to μ is defined by setting

$$\Theta_d^*(\mu,\, x) := \limsup_{r \to 0^+} \frac{\mu\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d}, \tag{4.3}$$

where $\frac{\alpha_d}{2^d}$ is the renormalization constant used in the definition of the Hausdorff measure.

Theorem 4.12. *Let μ be a locally finite measure defined on a metric space X, and let $d > 0$ be a positive real number. The following properties are equivalent:*

(1) *The upper density is finite and nonzero, that is,*

$$0 < \Theta_d^*(\mu,\, x) < +\infty,$$

for μ-almost every $x \in X$.

(2) *There is a locally summable function $f \in L^1_{\text{loc}}\left(X,\, \mathcal{H}^d\right)$ such that*

$$\mu = f \cdot \mathcal{H}^d.$$

The proof of this theorem is rather involved. Hence, we split it into different propositions and lemmas and, at the end of the section, we get much more than the statement above.

Proposition 4.13. *Let μ be a locally finite measure on X and let $d > 0$. Suppose that there is $f \in L^1_{\text{loc}}\left(\mathcal{H}^d\right)$ such that $\mu = f \cdot \mathcal{H}^d$. Then*

$$2^{-d}\, f(x) \le \Theta_d^*(\mu,\, x) \le 5^d\, f(x) \quad \textit{for } \mathcal{H}^d\textit{-almost every } x \in X.$$

In particular the upper-density of μ is finite and nonzero ($0 < \Theta_d^*(\mu,\, x) < +\infty$) for μ-almost any $x \in X$.

Proof. We first prove the result in some particular cases, and then we generalize it using a simple trick.

Step 1. Assume that $X = \mathbb{R}^n$ and assume that there are constants $m < M \in \mathbb{R}$ such that

$$0 < m < f(x) \le M < +\infty \quad \text{for } \mathcal{H}^d\text{-almost every } x \in X \text{ s.t. } f(x) \neq 0.$$

Set $E := \{x \in X \mid f(x) \neq 0\}$ and consider the measure $\lambda := \mathbb{1}_E \cdot \mathcal{H}^d$. Clearly λ is locally finite, and thus we can easily compute the density ratio, that is,

$$\begin{aligned}
\frac{\mu\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d} &= \frac{1}{\alpha_d r^d} \int_{\overline{B(x,\, r)}} f(y)\, \mathrm{d}\mathcal{H}^d(y) = \\
&= \frac{1}{\alpha_d r^d} \int_{\overline{B(x,\, r)}} f(y)\, \mathrm{d}\lambda(y) = \\
&= \frac{\lambda\left(\overline{B(x,\, r)}\right)}{\alpha_d\, r^d} \fint_{\overline{B(x,\, r)}} f(y)\, \mathrm{d}\lambda(y).
\end{aligned}$$

Since λ is equal to the restriction to E of the d-dimensional Hausdorff measure, it follows from Theorem 4.4 that

$$\limsup_{r \to 0} \frac{\lambda\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d} \in \left[\frac{1}{2^d},\, 5^d\right]$$

for $\mathcal{H}^d$-almost every $x \in E$. On the other hand, the Lebesgue result (see Corollary 4.10) implies that

$$\fint_{B(x,\, r)} f(y)\, \mathrm{d}\lambda(y) \xrightarrow{r \to 0} f(x),$$

for $\mathcal{H}^d$-almost every $x \in E$, which is exactly what we wanted to prove.

Step 2. Let X be any reasonable[6] space and assume that there are constants $m < M \in \mathbb{R}$ such that

$$0 < m < f(x) \leq M < +\infty$$

for $\mathcal{H}^d$-almost every $x \in X$ such that $f(x) \neq 0$. Let f_1 and f_2 be finite sums of step functions in such a way that

$$f_1(x) \leq f(x) \leq f_2(x).$$

We claim that

$$2^{-d} f_1(x) \leq \Theta^*_d(\mu,\, x) \leq 5^d f_2(x)$$

for $\mathcal{H}^d$-almost every $x \in X$ such that $f(x) \neq 0$.

Step 2.1. Let $\alpha_1,\, \dots,\, \alpha_m \in \mathbb{R}$ and $A_1,\, \dots,\, A_m \subset X$, and assume that

$$f_1(x) = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{E_i},$$

where $E_i = E \cap A_i$ for every $i = 1,\, \dots,\, m$. The following inequality of measures is tirvial

$$\mu \geq \sum_{i=1}^{m} \alpha_i\, \mathbb{1}_{E_i} \cdot \mathcal{H}^d,$$

and hence

$$\Theta^*_d(E_i,\, x) \geq \alpha_i 2^{-d} \overset{(x \in E_i)}{=} f_1(x)\, 2^{-d}$$

for $\mathcal{H}^d$-almost every $x \in E_i$. The union of the E_is is almost all of E, and thus it turns out that

$$2^{-d} f_1(x) \leq \Theta^*_d(\mu,\, x)$$

for $\mathcal{H}^d$-almost every $x \in E$.

[6] Here we do not specify what kind of assumptions are needed on X, but any space that "looks like" $\mathbb{R}^n$ will do. The reader may try, as an exercise, to find the minimal assumptions that can make the argument of the second step work.

Step 2.2. Let us consider the function

$$f_2(x) = \sum_{i=1}^{m} \beta_i \mathbb{1}_{E_i},$$

where the family $\{E_i := A_i \cap E\}_{i=1,\dots,m}$ is disjoint. Since $E = \bigsqcup_{i=1,\dots,m} E_i$, it turns out that $\Theta^*_d(\mu, x) = 0$ for $\mathcal{H}^d$-almost every $x \notin E$. If $x \in E_j$, then

$$\Theta^*_d(\mu, x) \leq \sum_{i=1}^{m} \beta_i \Theta^*_d(E_i, x) = \beta_j \Theta^*_d(E_j, x) \leq 5^d f_2(x)$$

for $\mathcal{H}^d$-almost every $x \in E_j$, as a consequence of Theorem 4.4.

Step 3. Assume $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{H}^d)$. Fix $m > 1$, set

$$E_m := \left\{ x \in E \;\middle|\; \frac{1}{m} \leq f(x) \leq m \right\},$$

and let $\mu = \mu_m + \widetilde{\mu_m}$ be the Radon-Nikodym decomposition of μ associated to E_m. More precisely, we consider the decomposition

$$\mu_m = f\, \mathbb{1}_{E_m} \cdot \mathcal{H}^d \qquad \text{and} \qquad \widetilde{\mu_m} = f\, \mathbb{1}_{E \setminus E_m} \cdot \mathcal{H}^d,$$

in such a way that

$$\frac{\mu\left(\overline{B(x, r)}\right)}{\mu_m\left(\overline{B(x, r)}\right)} \xrightarrow{r \to 0} 1 \quad \text{for } \mu_m\text{-almost every } x \in E.$$

Since $\widetilde{\mu_m}$ is singular, it turns out that

$$\Theta^*_d(\mu, x) = \Theta^*_d(\mu_m, x) \leq 5^d f(x)$$

for μ_m-almost every $x \in E$, i.e., for $\mathcal{H}^d$-almost every $x \in E_m$ The union of the E_ms covers almost all of E; thus

$$\Theta^*_d(\mu, x) \leq 5^d f(x)$$

for μ-almost every $x \in E$. If, on the other hand, $x \notin E$, then we can prove[7] that the d-dimensional upper density is equal to 0 for $\mathcal{H}^d$-almost every $x \notin E$.

Step 4. Let X be a reasonable metric space and let f be a function in $L^1_{\text{loc}}(X, \mathcal{H}^d)$. This step follows from the previous ones since for a locally summable function, for every $\epsilon > 0$, there are constants $m, M > 0$ and $E_{m,M} \subseteq X$ such that

$$0 < m < f(x) \leq M < +\infty$$

for $\mathcal{H}^d$-almost every $x \in E$ such that $f(x) \neq 0$ and $\mathcal{H}^d(E_{m,M}) < \epsilon$. □

Lemma 4.14. *Let μ be a locally finite measure with finite d-dimensional upper density, that is,*

$$\Theta^*_d(\mu, x) < +\infty$$

for μ-almost every $x \in X$. Then μ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^d$, that is,

$$\mathcal{H}^d(E) = 0 \implies \mu(E) = 0.$$

[7] The reader may prove this assertion in a similar fashion to Theorem 4.4.

Sketch of the Proof. By assumption, if $r > 0$ is small enough, it turns out that

$$\frac{\mu\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d} \leq m.$$

Let E be a $\mathcal{H}^d$-null subset of X, that is $\mathcal{H}^d(E) = 0$. For any $m,\, \rho > 0$ let us consider the collection of sets

$$E_{m,\,\rho} = \left\{ x \in E \;\middle|\; \frac{\mu\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d} \leq m \text{ for every } r < \rho \right\}.$$

The countable union over the rational numbers covers almost every point of E, that is,

$$\bigcup_{(m,\,\rho)\in\mathbb{Q}^2} E_{m,\,\rho} = \mu\text{-almost all of } E,$$

since there may be $A \subset X$ null set such that

$$x \in A \implies \Theta^*_d(\mu,\, x) = +\infty.$$

It is clearly enough to prove that the measure of $E_{m,\,\rho}$ is equal to 0 for every fixed couple $(m,\, \rho) \in \mathbb{Q}^2$.

By assumption $\mathcal{H}^d(E_{m,\,\rho}) = 0$, and thus $\mathcal{H}^d_\delta(E_{m,\,\rho}) = 0$. In particular, for every $\epsilon > 0$ there exists a covering $\{E_i\}_{i\in I_{m,\,\rho}}$ for $E_{m,\,\rho}$ satisfying the additional property

$$\operatorname{diam}(E_i) < \delta \quad \text{and} \quad \sum_i \left(\operatorname{diam}(E_i)\right)^d \leq \epsilon.$$

Let us consider the closed ball $B_i := \overline{B(x_i,\, r_i)}$, centered at $x_i \in E_i \cap E$ with radius $r_i = \operatorname{diam}(E_i)$, for every $i \in I_{m,\,\rho}$. The collection of balls $\{B_i\}_{i\in I_{m,\,\rho}}$ covers $E_{m,\,\rho}$, and thus

$$\mu(E_{m,\,\rho}) \leq \sum_{i\in I_{m,\,\rho}} \mu(B_i) \leq m\alpha_d \sum_{i\in I_{m,\,\rho}} r_i^d \leq m\alpha_d \sum_{i\in I_{m,\,\rho}} \left(\operatorname{diam}(E_i)\right)^d \leq m\alpha_d \cdot \epsilon,$$

which is exactly what we wanted to prove. □

Lemma 4.15. *Let μ be a locally finite measure defined on a metric space X. Fix $m > 0$ and let*

$$E_m := \{x \in E \mid \Theta^*_d(\mu,\, x) > m\}.$$

The d-dimensional Hausdorff measure of E_m is bounded from above, that is,

$$\mathcal{H}^d(E_m) \leq \frac{5^d}{m}\mu(E_m).$$

Proof. Let A be an open neighborhood of E_m, and let us consider the collection of closed balls

$$\mathcal{F} := \left\{ \overline{B(x,\, r)} \;\middle|\; \overline{B(x,\, r)} \subseteq A \text{ and } \frac{\mu\left(\overline{B(x,\, r)}\right)}{\alpha_d r^d} > m \right\}.$$

The reader may prove that $\mathcal{F}$ is a fine cover of E_m; thus by Vitali's covering Lemma 3.1 there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $\widehat{\mathcal{F}'}$ covers E_m. It follows that

$$\begin{aligned}\mu(A) \geq \sum_{B\in\mathcal{F}'} \mu(B) \geq& \\ &\geq m \sum_{B\in\mathcal{F}'} \alpha_d r^d = \\ &= \frac{m}{5^d}\frac{\alpha_d}{2^d} \sum_{B\in\mathcal{F}'} (10r)^d \geq \\ &\geq \frac{m\alpha_d}{10^d} \sum_{B\in\mathcal{F}'} \left(\operatorname{diam}(\hat{B})\right)^d \geq \frac{m}{5^d}\mathcal{H}^d_\infty(E_m).\end{aligned}$$

In particular, to estimate $\mathcal{H}^d_\delta$ it is enough to consider a cover made up of balls whose diameter is less than $\delta/10$. If we take the infimum with respect to A and the supremum with respect to $\delta > 0$, then it turns out that

$$\mathcal{H}^d(E_m) \leq \frac{5^d}{m}\mu(E_m).$$

□

Corollary 4.16. *Let μ be a locally finite measure defined on a metric space X and assume that*

$$\Theta^*_d(\mu, x) > 0$$

for μ-almost every $x \in X$. Then μ is supported on the set

$$E := \{x \in X \mid \Theta^*_d(\mu, x) > 0\},$$

which is σ-finite with respect to $\mathcal{H}^d$.

Proof. It follows immediately from the previous theorem since

$$E = \bigcup_{n\in\mathbb{N}} E_{\frac{1}{n}}$$

and we have, for every $n \in \mathbb{N}$, the estimate

$$\mathcal{H}^d(E_{\frac{1}{n}}) \leq \frac{5^d}{m}\mu(E_{\frac{1}{n}}) < +\infty.$$

□

Proposition 4.17. *Let μ be a locally finite measure defined on a metric space X and assume that*

$$0 \leq \Theta^*_d(\mu, x) < +\infty$$

for μ-almost every $x \in X$. Then

$$\mu = f \cdot \mathcal{H}^d$$

for some locally summable function $f \in L^1_{\text{loc}}(\mathcal{H}^d)$.

Proof. By Lemma 4.14, the measure μ is absolutely continuous with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$, while the previous corollary implies that μ is supported on the set E defined above. Therefore

$$\mu \ll \lambda := \mathbb{1}_E \cdot \mathcal{H}^d,$$

and λ is σ-finite; thus the Radon-Nikodym theorem implies that $\mu = f \cdot \lambda$. □

Remark 4.5. Notice that the Radon-Nikodym theorem cannot be applied directly to $\mathcal{H}^d$ since the Hausdorff measure is not σ-finite (which is a necessary assumption for the result to hold true.)

4.4 Applications

In this final brief section, we investigate some of the main applications of the theory developed so far (especially to Cantor sets.)

Corollary 4.18. *Let E be a Borel set contained in a metric space X, and let $d \geq 0$ be a positive real number. Assume that there exists a finite measure μ, defined on X, such that:*

(a) *The d-dimensional upper density is finite, that is, $\Theta_d^*(\mu, x) < +\infty$ for μ-almost every $x \in E$.*

(b) *The set E is not null with respect to μ, that is, $\mu(E) > 0$.*

Then the d-dimensional Hausdorff measure of E is strictly bigger than zero.

Proof. By Theorem 4.12, the measure $\lambda := \mathbb{1}_E \cdot \mu$ is absolutely continuous with respect to $\mathcal{H}^d$; therefore the Hausdorff measure of the set E cannot be zero (otherwise also $\mu(E)$ would be equal to zero). □

Cantor Set. We have proved in the previous chapter that

$$d := \frac{\log 2}{\log 3},$$

is the Hausdorff dimension of the Cantor set $\mathcal{C}$, and recall that $\mathcal{H}^d(\mathcal{C}) > 0$.

Remark 4.6. The Cantor set is defined by setting

$$\mathcal{C} := \bigcap_{i=0}^{+\infty} C_i,$$

where $C_0 = [0,\, 1]$, $C_1 = [0,\, 1/3] \cup [2/3,\, 1]$ and C_k is the disjoint union of 2^k intervals, whose length is equal to 3^{-k}.

Proposition 4.19. *Let*

$$\Phi(I_{i,\,j}) = \Phi(I_{i+1,\,k}) + \Phi(I_{i+1,\,k+1})$$

be a set function (that preserves the mass). Then Φ can be uniquely extended to a finite measure ν on $\mathbb{R}$, which is supported on the Cantor set $\mathcal{C}$.

Proof. Let $\mathcal{F} = \{I_{i,\,j}\}_{i,\,j \in \mathbb{N}} \cup \varnothing$ and let μ be the outer measure on $\mathcal{C}$ given by the Carathéodory construction. We claim that the following properties are satisfied:

(a) For every $i,\, j \in \mathbb{N}$ it turns out that $\mu(I_{i,\,j}) = \Phi(I_{i,\,j})$.

(b) The outer measure μ is additive on distant sets (and thus the restriction to the Borel σ-algebra is a σ-additive measure.)

As a consequence of the second claim, the restriction of μ to the Borel σ-algebra, denoted by ν, is exactly the sought measure. The reader may prove, as an exercise, that the measure defined in this way is **unique**.

(a) Let us consider the outer measure

$$\mu\left(I_{i,\,j}\right) := \inf\left\{\Phi(E_k) \mid E_k \in \mathcal{F} \text{ and } I_{i,\,j} \subseteq E_k\right\}.$$

Notice that $(E_k)_{k\in\mathbb{N}}$ is an open covering of $\mathcal{C}$ and, for every $i,\, j \in \mathbb{N}$, the interval $I_{i,\,j}$ is compact. It follows that there exists a finite subfamily $E_{i_1}, \ldots, E_{i_k}$ such that

$$\mu\left(I_{i,\,j}\right) = \inf_{\ell=1,\,\ldots,\,k} \Phi(E_{i_\ell}).$$

This proves that $\mu(I_{i,\,j}) = \Phi(I_{i,\,j})$ for every $i,\, j$ since Φ is additive on distant sets[8].

(b) We consider the outer measure

$$\mu_\delta\left(I_{i,\,j}\right) := \inf\left\{\Phi(E_k) \mid E_k \in \mathcal{F},\, \mathrm{diam}(E_k) \leq \delta \text{ and } I_{i,\,j} \subseteq E_k\right\}.$$

Then μ_δ coincides with μ on a suitable class of sets, and it is larger on where the distance is bigger than δ. This concludes the proof.

□

Canonical Measure on $\mathcal{C}$. Let μ be the measure given by Proposition 4.19 associated to the set function

$$\Phi(I_{i,\,j}) = 2^{-i}.$$

It is enough to prove that $\Theta^*_d(\mu,\, x) < +\infty$ for μ-almost every $x \in \mathcal{C}$. First, we notice that for every $x \in \mathcal{C}$, it turns out that

$$\widetilde{\Theta^*_d}(\mu,\, x) := \lim_{\mathrm{diam}(I_{i,\,j})\to 0,\, x\in I_{i,\,j}} \frac{\mu(I_{i,\,j})}{\left(\mathrm{diam}(I_{i,\,j})\right)^d} = \frac{2^{-i}}{3^{-id}} = 1.$$

Then

$$\Theta^*_d(\mu,\, x) \in \left[c_1\widetilde{\Theta^*_d}(\mu,\, x),\, c_2\widetilde{\Theta^*_d}(\mu,\, x)\right],$$

and this is exactly what we wanted to prove. Indeed, the closed ball $\overline{B(x,\, r)}$ satisfies the inequality $3^{-i-1} < r \leq 3^{-i}$, and thus $\overline{B(x,\, r)} \cap \mathcal{C} \subseteq I_{i,\,j}$ for some $j \in \mathbb{N}$.

[8] It follows easily from the definition of Φ and of $\mathcal{C}$.

Chapter 5

Self-Similar Sets

In this brief chapter, we investigate the notion of Hutchinson's *fractals* (or self-similar sets), and we prove that, under suitable assumptions, given a certain number of similarities, there exists a unique compact self-similar set with a precise Hausdorff dimension d.

Definition 5.1 (Self-Similar)**.** A subset $E \subseteq \mathbb{R}^n$ is *self-similar* if there exist a finite collection $\phi_1, \ldots, \phi_N$ of similarities

$$\phi_i(x) := x_i + \lambda_i \cdot R_i x, \quad \text{where } R_i \in O(n) \text{ and } \lambda_i \in (0, 1),$$

such that

$$E = \bigcup_{i=1}^{N} \phi_i(E).$$

Remark 5.1. If the $\phi_i(E)$ are (essentially) disjoint[1], then we expect the Hausdorff dimension of E to be equal to the unique solution of the equation

$$\sum_{i=1}^{N} \lambda_i^d = 1. \tag{5.1}$$

It is important to stress that the Hausdorff dimension should be equal to d, but, a priori, there is no guarantee that the d-dimensional Hausdorff measure of E is finite. We now state two lemmas that explain what we mean by *should be equal to.*

Lemma 5.2. *The equation* (5.1) *admits a unique solution $d \geq 0$, provided that the λ_is are positive and strictly less than one.*

Lemma 5.3. *If the $\phi_i(E)$ are (essentially) disjoint and if there exists $d \geq 0$ such that $0 < \mathcal{H}^d(E) < +\infty$, then d is the unique solution of* (5.1).

Proof. This is a straightforward application of the properties of the Hausdorff measure:

$$E = \bigsqcup_{i=1}^{N} \phi_i(E) \implies \mathcal{H}^d(E) = \sum_{i=1}^{N} \mathcal{H}^d\left(\phi_i(E)\right) = \left(\sum_{i=1}^{N} \lambda_i^d\right) \mathcal{H}^d(E),$$

that is, the factor $\left(\sum_{i=1}^{N} \lambda_i^d\right)$ needs to be equal to one for the identity to hold. □

[1] More precisely, we say that $\phi_i(E)$ and $\phi_j(E)$ are *essentially* disjoint if and only if $\mathcal{L}^n\left(\phi_i(E) \cap \phi_j(E)\right) = 0$.

Hutchinson Construction of Self-Similar Sets. Let be given $\phi_1, \ldots, \phi_N$ similarities with scaling factors $\lambda_i \in (0, 1)$. For the next theorem to be true, we need to introduce the *open set condition.*

The open set condition (OSC). The finite family of similarities $\phi_1, \ldots, \phi_N$ satisfies the open set condition if and only if there exists a nonempty open set $V \subset \mathbb{R}^n$ such that

$$\bigcup_{i=1}^{N} \phi_i(V) \subseteq V \qquad \text{and} \qquad \phi_i(V) \cap \phi_j(V) = \varnothing \text{ for } i \neq j.$$

Theorem 5.4 (Hutchinson)**.** *Let be given a finite collection $\phi_1, \ldots, \phi_N$ of similarities with scaling factors $\lambda_i \in (0, 1)$ satisfying the (OSC) condition. Then there exists a unique self-similar compact set $C \subseteq \mathbb{R}^n$, that is,*

$$C = \bigcup_{i=1}^{N} \phi_i(C).$$

The Hausdorff dimension of C is the unique solution d of the equation (5.1), and the d-dimensional Hausdorff measure of C is finite and nonzero, that is,

$$0 < \mathcal{H}^d(C) < +\infty.$$

The proof of this theorem is hard, and we need extra care to deal properly with the open set condition. For this reason, we replace it with a stronger condition:

$$\phi_i\left(\overline{V}\right) \cap \phi_j\left(\overline{V}\right) = \varnothing, \qquad \forall\, i \neq j.$$

More precisely, we assume that the images of V via the similarities are distant.

Proof. Let $X := \overline{V}$, and let $\mathcal{F} := \{F \subseteq X \mid F \text{ is closed.}\}$ be the family of all the closed (and thus compact) subsets of X.

Step 1. The Hausdorff distance induces a structure of metric space on $\mathcal{F}$. Indeed, if we denote by $\mathcal{U}_r(A)$ an open neighborhood of A with diameter r, then one can prove that

$$d_{\mathcal{H}}(C_1, C_2) := \inf\left\{r > 0 \mid C_1 \subseteq \mathcal{U}_r(C_2) \text{ and } C_2 \subseteq \mathcal{U}_r(C_1)\right\},$$

is a metric. Moreover, the following implications hold[2]:

(a) If X is complete, then $(\mathcal{F}, d_{\mathcal{H}})$ is complete.

(b) If X is compact, then $(\mathcal{F}, d_{\mathcal{H}})$ is compact.

Step 2. Let us consider the following operator

$$\Phi : \mathcal{F} \longrightarrow \mathcal{F}, \qquad F \longmapsto \bigcup_{i=1}^{N} \phi_i(F).$$

between two Banach spaces, and let $\lambda_{max} := \max_{i=1,\ldots,N} \lambda_i \in (0, 1)$. By definition, it turns out that

$$d_{\mathcal{H}}\left(\Phi(F_1), \Phi(F_2)\right) \leq \lambda_{max} \cdot d_{\mathcal{H}}(F_1, F_2),$$

[2] These two statements are not related to the content of this course; hence both are left as an exercise for the reader.

which means that the operator Φ is a contraction. The Banach fixed point theorem[3] proves that there exists a unique fixed point C which is given by

$$\lim_{j \to +\infty} \Phi^j(F),$$

for every $F \in \mathcal{F}$. In particular, one can choose X as a starting point for the sequence.

Step 3. First, we observe that

$$X \supset \Phi(X) \supset \Phi^2(X) \supset \cdots \implies \lim_{j \to +\infty} \Phi^j(X) = \bigcap_{j \in \mathbb{N}} \Phi^j(X) = C.$$

For any $j \in \mathbb{N}$ and any j-tuple of indices $(i_1, \ldots, i_j) \in \{1, \ldots, N\}^j$, we denote by $X_{i_1, \ldots, i_j}$ the iterated image of X, that is,

$$X_{i_1, \ldots, i_j} := \phi_{i_j} \circ \cdots \circ \phi_{i_1}(X).$$

This notation is particularly useful since we can now express C as an infinite intersection of finite unions (a fairly straightforward generalization of the construction of the Cantor set), that is,

$$\bigcap_{j \in \mathbb{N}} \left(\bigcup_{1 \le i_1, \ldots, i_j \le N} X_{i_1, \ldots, i_j} \right) = C. \tag{5.2}$$

Step 4. In this brief step, we want to find an **upper bound** on the d-dimensional Hausdorff measure of C and, more precisely, we prove that

$$\mathcal{H}^d(C) \le \frac{\alpha_d}{2^d} \left(\operatorname{diam}(X)\right)^d.$$

Fix $\delta > 0$, choose j so that $\lambda_{max}^j \cdot \operatorname{diam}(X) < \delta$, and observe that

$$\operatorname{diam}\left(X_{i_1, \ldots, i_j}\right) \le \lambda_{max}^j \cdot \operatorname{diam}(X) \quad \text{and} \quad \operatorname{diam}\left(X_{i_1, \ldots, i_j}\right) = \lambda_{i_1} \ldots \lambda_{i_j} \cdot \operatorname{diam}(X).$$

As a consequence of the identity (5.2), it follows that the family $\left\{X_{i_1, \ldots, i_j}\right\}$ is a covering of C with diameter less than δ as $(i_1, \ldots, i_j)$ ranges in the set of all the j-tuples; hence

$$\begin{aligned} \mathcal{H}^d_\delta(C) &\le \frac{\alpha_d}{2^d} \sum_{1 \le i_1, \ldots, i_j \le N} \left(\operatorname{diam}\left(X_{i_1, \ldots, i_j}\right)\right)^d \le \\ &\le \frac{\alpha_d}{2^d} \left(\operatorname{diam}(X)\right)^d \sum_{1 \le i_1, \ldots, i_j \le N} \left(\lambda_{i_1} \ldots \lambda_{i_j}\right)^d. \end{aligned}$$

The right-hand side is, up to a constant, equal to $(\operatorname{diam}(X))^d$ since

$$\sum_{1 \le i_1, \ldots, i_j \le N} \left(\lambda_{i_1} \ldots \lambda_{i_j}\right)^d = \left(\sum_{i=1}^N \lambda_i^d\right)^j = 1.$$

In particular, notice that the existence of the fixed point C and the upper bound on the d-dimensional Hausdorff measure rely on the completeness of X only: all the other assumptions are necessary for the lower bound!

[3] **Theorem.** Let B be a Banach space, and let $T : B \longrightarrow B$ be a contraction. Then there exists a unique $b \in B$ such that $T(b) = b$, and for every $x \in B$ it turns out that b is the limit point of the sequence $\{x_n := T^n x\}_{n \in \mathbb{N}}$.

Step 5. The lower bound is rather delicate, and the rough idea behind it is to construct a probability measure μ on C such that

$$\Theta_d^*(\mu, x) < +\infty, \qquad \forall\, x \in C.$$

More precisely, we construct μ as the weak limit of a sequence of probability measures $(\mu_k)_{k\in\mathbb{N}}$ as follows. Let $x \in C$ be a point, and let $\mu_0 := \delta_x$ be the Dirac measure centered at that point. We define the measure

$$\mu_1 := \lambda_1^d \delta_{\phi_1(x)} + \cdots + \lambda_N^d \delta_{\phi_N(x)},$$

where the renormalization constants need to be chosen in this way since the problem is asymmetrical[4] as the weight is not uniformly distributed.

Step 5.1. Let us denote by $x_{i_1,\dots,i_j}$ the image via the collection $\phi_1, \dots, \phi_N$ of x, that is,

$$x_{i_1,\dots,i_j} := \phi_{i_j} \circ \cdots \circ \phi_{i_1}(x).$$

The jth element of the sequence is thus given by

$$\mu_j := \sum_{1 \le i_1,\dots,i_j \le N} \left(\lambda_{i_1} \dots \lambda_{i_j}\right)^d \delta_{x_{i_1,\dots,i_j}}.$$

The set function μ_j is a probability measure defined on C for every $j \in \mathbb{N}$; hence there exists a subsequence[5] μ_{j_k} that converges to a probability measure μ. We now claim that for every $j \in \mathbb{N}$ and for every j-tuple of indices $(i_1, \dots, i_j) \in \{1, \dots, N\}^j$, it turns out that

$$\mu\left(X_{i_1,\dots,i_j}\right) = \lambda_{i_1}^d \dots \lambda_{i_j}^d.$$

It is easy to see that for any $k \ge j$, we have

$$\mu_k\left(X_{i_1,\dots,i_j}\right) = \lambda_{i_1}^d \dots \lambda_{i_j}^d,$$

and hence our claim follows immediately if we can pass this identity to the limit as $k \to +\infty$.

Here we use the strong replacement for the open set condition: For every j-tuple, the set $X_{i_1,\dots,i_j}$ is both open and closed in C, and thus we can take the limit for $k \to +\infty$.

Step 5.2. Fix $x \in C$. Then x is uniquely identified by a sequence of indices $(i_j)_{j\in\mathbb{N}}$ in such a way that $x \in X_{i_1,\dots,i_j}$ for every $j \in \mathbb{N}$. Moreover, we have the identity

$$\frac{\mu\left(X_{i_1,\dots,i_j}\right)}{\left[\operatorname{diam}(X_{i_1,\dots,i_j})\right]^d} = \frac{1}{\operatorname{diam}(X)^d}, \tag{5.3}$$

since $\operatorname{diam}(X_{i_1,\dots,i_j}) = \lambda_{i_1} \dots \lambda_{i_j} \cdot \operatorname{diam}(X)$ by construction.

It remains to prove that (5.3) is enough to infer that the d-dimensional upper density of μ is finite. Fix $x \in C$ and fix $0 < r < d_{min}$, where

$$d_{min} := \inf_{i,j=1,\dots,N} d\left(\phi_i(x), \phi_j(x)\right).$$

[4]The reader may check that a different choice of renormalization constants yields to a different result (e.g., with $1/N$).

[5]The whole sequence converges to μ, but we do not use this fact in the proof. On the other hand, we will use this property in the next chapters; thus the reader may try to prove it by herself.

There exists a natural number $j \in \mathbb{N}$ such that

$$d_{min} \cdot \lambda_{i_1} \dots \lambda_{i_{j+1}} < r \leq d_{min} \cdot \lambda_{i_1} \dots \lambda_{i_j},$$

and therefore

$$\overline{B(x,\, r)} \cap C \subseteq X_{i_1,\, \dots,\, i_j}.$$

It follows that

$$\mu\left(\overline{B(x,\, r)}\right) \leq \mu\left(X_{i_1,\, \dots,\, i_j}\right),$$

and this is enough to estimate the density ratio, that is,

$$\frac{\mu\left(\overline{B(x,\, r)}\right)}{r^d} \leq \frac{\mu\left(X_{i_1,\, \dots,\, i_j}\right)}{\left(d_{min} \cdot \lambda_{i_1} \dots \lambda_{i_j}\right)^d} = \left(\frac{1}{d_{min} \cdot \mathrm{diam}(X)}\right)^d < +\infty.$$

Step 5.3 Here we rephrase the argument presented in [8]. We want to prove that

$$\Theta^*_d(\mu,\, x) < +\infty, \qquad \forall\, x \in C$$

is enough to infer that $\mathcal{H}^d(C) > 0$ strictly. Suppose that

$$B_r \subseteq X \subseteq B_R,$$

and let $\rho > 0$ be a fixed real number. Denote by $\mathscr{S}$ the space of all finite sequences $(i_1,\, \dots,\, i_j)$ such that the following inequality holds:

$$\lambda_{min} \cdot \rho \leq \lambda_{i_1} \dots \lambda_{i_j} \leq \rho. \tag{5.4}$$

It follows that the collection

$$\mathscr{X} := \left\{X_{i_1,\, \dots,\, i_j} \;:\; (i_1,\, \dots,\, i_j) \in \mathscr{S}\right\}$$

is disjoint, and thus every $X_{i_1,\, \dots,\, i_j} \in \mathscr{X}$ contains a ball of radius $r \cdot \lambda_{i_1} \dots \lambda_{i_j}$ and it is contained in a ball of radius $R \cdot \lambda_{i_1} \dots \lambda_{i_j}$. By (5.4) we infer that

$$B_{r \cdot \lambda_{min} \cdot \rho} \subseteq X_{i_1,\, \dots,\, i_j} \subseteq B_{R \cdot \rho}.$$

In particular, every ball of radius ρ intersects q sets of the collection $\mathscr{X}$, where

$$q := \left(\frac{1 + 2R}{\lambda_{min} r}\right)^n.$$

Moreover, by definition the support of $\mu_{i_1,\, \dots,\, i_j}$ is contained in $\overline{X_{i_1,\, \dots,\, i_j}}$ for every $j \geq 1$, and therefore

$$\mu = \sum_{(i_1,\, \dots,\, i_j) \in \mathscr{S}} \left(\lambda_{i_1} \dots \lambda_{i_j}\right)^d \mu_{i_1,\, \dots,\, i_j}.$$

For every ball B_ρ of radius ρ such that $B_\rho \cap \overline{X_{i_1,\, \dots,\, i_j}} \neq \varnothing$, it turns out that

$$\mu(B_\rho) \leq \sum_{(i_1,\, \dots,\, i_j) \in \mathscr{S}} \left(\lambda_{i_1} \dots \lambda_{i_j}\right)^d \mu_{i_1,\, \dots,\, i_j}(\mathbb{R}^n),$$

which means that

$$\mu(B_\rho) \leq q\rho^d = \frac{q|B_\rho|^d}{2^d}$$

whenever $|B_\rho| < |X|$. Let $\{U_i\}_{i\in I}$ be a countable cover of C, and notice that

$$C \subseteq \bigcup_{i\in I} B_i \quad \text{where } |B_i| \leq 2|U_i|,$$

from which it follows that

$$1 = \mu(X) \leq \sum_{i\in I} \mu(B_i) \leq q \sum_{i\in I} |U_i|^d \simeq q\mathcal{H}^d(E),$$

and thus $\mathcal{H}^d(E) \geq q^{-1} > 0$, which is what we wanted to prove. □

Chapter 6

Other measures and dimensions

In this chapter, we first introduce the *integralgeometric* measure, and then we investigate the Haar invariant k-dimensional measure.

In the second half, we show how the Haar measure may be used to define an invariant measure on the Grassmannian manifold $G(n, m)$, which will be extremely useful to study rectifiable sets.

6.1 Geometric Integral Measure

In this first section, we propose an alternative k-dimensional measure to the Hausdorff one and, at the same time, we exhibit a motivational example for introducing the notion of *invariant measures.*

Definition (1-dimensional). Let $E \subseteq \mathbb{R}^n$ be a subset, and fix a projection

$$P_L : \mathbb{R}^n \longrightarrow L$$

onto a 1-dimensional linear subspace (i.e., a line). We may easily define (see Figure 6.1) a measure, which is invariant under translation but not under rotations, as follows:

$$\int_{x \in L} \# \left(P_L^{-1}(x) \cap E \right) \, \mathrm{d}\mathcal{H}^1(x), \tag{6.1}$$

where $\#(\cdot)$ denotes the cardinality function.

In order to define a rotation-invariant measure, the simplest idea that comes in mind is to consider the average value of (6.1), as L ranges in the set of all the 1-dimensional subspaces of $\mathbb{R}^n$. More precisely, we "define" the 1-dimensional integralgeometric measure as follows:

$$\mathcal{I}^1(E) := \fint_{L \in G(n, 1)} \left[\int_{x \in L} \# \left(P_L^{-1}(x) \cap E \right) \, \mathrm{d}\mathcal{H}^1(x) \right]. \tag{6.2}$$

The measure (6.2) is not well-defined, since we do not know which measure we need to use to integrate over all $L \in G(n, 1)$; we will come back to this issue later.

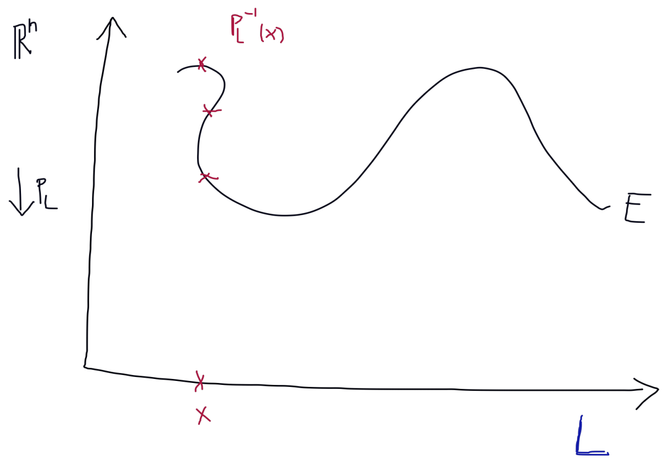

Figure 6.1: One-dimensional translation-invariant measure.

Definition (k-dimensional). The same construction can be easily generalized to a k-dimensional measure. Indeed, if we denote by $G(n, k)$ the Grassmannian manifold (i.e, the set of all k-dimensional subspaces of $\mathbb{R}^n$) then, it turns out that

$$\mathcal{I}^k(E) := c_{k,n} \fint_{V \in G(n,k)} \left[\int_{x \in V} \# \left(P_V^{-1}(x) \cap E \right) \, \mathrm{d}\mathcal{H}^k(x) \right] \tag{6.3}$$

is a measure, which is invariant under translations and rotations. The reader may prove that, if the renormalization constant $c_{k,n}$ is chosen properly, then

$$\mathcal{I}^k(E) = \mathcal{L}^k(E) = \mathcal{H}^k(E),$$

provided that E is contained in a k-hyperplane of $\mathbb{R}^n$.

Definition Issue. The k-dimensional integralgeometric measure (6.3) is not well-defined (as we have already mentioned in the 1-dimensional case.) Indeed, we are taking the average integral over all the elements of the Grassmannian manifold (n, k), but we have not introduced yet a measure on that space that is also invariant.

This issue is the main reason why we are so interested in developing (at least partially) the theory of *invariant measures.* At the end of the chapter, we will be able to prove the existence of an invariant measure $\gamma_{n,k}$ on the Grassmannian manifold (n, k).

Basic Properties. To conclude this introduction we give a list of relevant and compelling facts about the k-dimensional geometric integral measure and leave them as exercises for the reader.

Lemma 6.1. *Let E be a subset of a h-dimensional surface Σ. Assume that $h > k$ strictly, and that the h-dimensional volume of Σ is positive. Then $\mathcal{I}^k(E) = +\infty$.*

Remark 6.1. The k-dimensional Hausdorff measure is, in general, different from the k-dimensional geometric integral measure.

More precisely, it may happen that for some subset $E \subset \mathbb{R}^n$, the Hausdorff dimension is $\dim_{\mathcal{H}}(E) > k$, but $\mathcal{I}^k(E) = 0$.

Example 6.1 (Cantor Set)**.** Let us consider the set $\mathcal{C}_2 \subset \mathbb{R}^2$ given by the product of two Cantor-type set with scaling factor equal to $1/4$ (see Figure 6.1.)

In particular, we have the similarities $\Phi_1, \dots, \Phi_4$, all with scaling factor equal to $1/4$, in such a way that $\mathcal{C}_2$ is a self-similar fractal set in the sense of Hutchinson, that is,

$$\mathcal{C}_2 = \bigcup_{i=1}^{4} \Phi_i(\mathcal{C}_2).$$

By Theorem 5.4, the Hausdorff dimension of $\mathcal{C}_2$ is the unique solution to the equation

$$4\left(\frac{1}{4}\right)^d = 1 \implies d = 1,$$

but the reader may prove, as an exercise, that $\mathcal{I}^1(\mathcal{C}_2) = 0$.

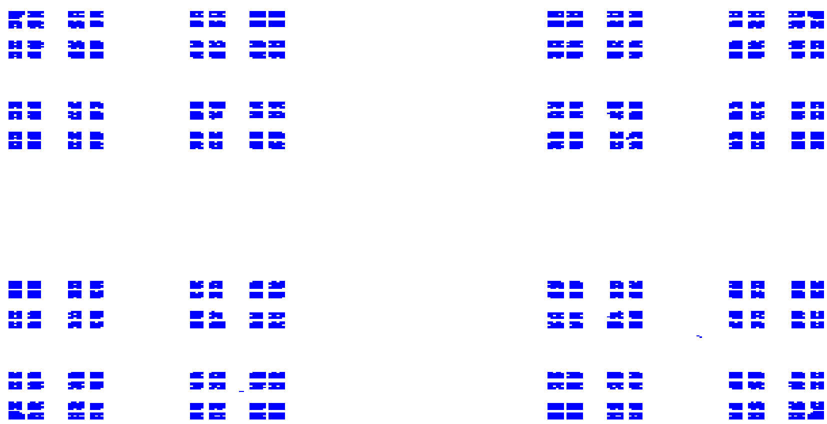

Figure 6.2: Cantor Square

6.2 Invariant Measures on Topological Groups

In this section, we examine the assumptions needed for the existence and uniqueness of an invariant measure defined on a topological group $\mathcal{G}$.

Definition 6.2 (Push-Forward)**.** Let μ be a positive measure on X, and let $f : X \longrightarrow Y$ be a Borel function. The *push-forward* measure of μ via f is defined by setting

$$f_{\#}\mu(E) := \mu\left(f^{-1}(E)\right), \qquad \forall\, E \in \mathcal{B}(Y).$$

Lemma 6.3. *Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be measurable spaces, and let μ be a positive measure on X. Then the push-forward $f_{\#}\mu$ is a well-defined measure on the the Borel σ-algebra of Y.*

Topological Groups. Let $\mathcal{G}$ be a topological group. For any $y \in \mathcal{G}$, we denote by τ_y the left-multiplication $(x \mapsto y \cdot x)$ and by τ_y^* the right-multiplication $(x \mapsto x \cdot y)$.

Definition 6.4 (Invariant Measure)**.** Let μ be a measure defined on a topological group $\mathcal{G}$. The measure μ is left-invariant on $\mathcal{G}$ if and only if

$$(\tau_y)_{\#}\,\mu = \mu \qquad \forall\, y \in \mathcal{G}.$$

In a similar fashion, the measure μ is right-invariant if and only if

$$\left(\tau_y^*\right)_{\#}\,\mu = \mu \qquad \forall\, y \in \mathcal{G},$$

and, clearly, μ is *invariant* if and only if μ is both left-invariant and right-invariant.

We are now ready to state the existence and uniqueness results. We will not prove the second theorem in this course, but we will give, at the end of the section, a complete sketch of the proof of the first result (which is the only one we shall be using in this course).

Theorem 6.5. *Let $\mathcal{G}$ be a compact group. Then there exists a unique invariant probability measure on $\mathcal{G}$, called Haar measure.*

Theorem 6.6. *Let $\mathcal{G}$ be a locally compact and separable group. Then there exists a locally finite invariant measure on $\mathcal{G}$, which is unique up to a multiplicative constant.*

Grassmannian Manifold. The Grassmannian $G(n,\, k)$ is, in general, not a group. Therefore, we cannot apply to it the theorem stated above, but there is another way around since, as we shall see soon, we can identify $G(n,\, k)$ with a quotient $\mathcal{G}/_{\Gamma}$, where Γ satisfies certain properties.

Let $\mathcal{G}$ be a topological group acting on a set X, and let $\tau : \mathcal{G} \times X \longrightarrow X$ be the left action map, that is,

$$\tau(g,\, x) = g \cdot x.$$

In order to be coherent with the previous paragraph, we denote by $\tau_g(x)$ the element $\tau(g,\, x)$ so that

$$\tau_{g_1 \cdot g_2} = \tau_{g_1}\tau_{g_2}.$$

A measure μ defined on X is *invariant* under the action of $\mathcal{G}$ if and only if

$$(\tau_g)_{\#}\,\mu = \mu$$

for every $g \in \mathcal{G}$. However, in this general setting, the existence (let alone the uniqueness) of an invariant measure is not guaranteed and, actually, we can exhibit an easy counterexample assuming both $\mathcal{G}$ and X compact.

Example 6.2. Let $X = \mathbb{P}^1(\mathbb{R}) \cong \mathbb{R} \cup \{0\}$ and let $\mathcal{G}$ be the group of all the projective transformation of X. Since the translation group is contained in $\mathcal{G}$, the only possible invariant measure is the 1-dimensional Lebesgue measure. On the other hand, the Lebesgue measure is not invariant under homothety, and hence there are no invariant measures.

Theorem 6.7. *Let $\mathcal{G}$ be a topological group acting on a set X. An invariant measure (not necessarily unique) exists if one of the following conditions is satisfied:*

(1) *The group $\mathcal{G}$ is abelian (or, more generally, it satisfies the Weyl condition[1].)*

[1] We do not need to deal with this delicate condition, but the interested reader may find more information here.

(2) The set X is isomorphic to the quotient ${}^{\mathcal{G}}\!/\!_{H}$, where H is a closed subspace[2] *of $\mathcal{G}$.*

Remark 6.2. If μ is a left-invariant measure on $\mathcal{G}$ and $\pi : \mathcal{G} \longrightarrow {}^{\mathcal{G}}\!/\!_{H}$ is a projection onto a closed subset, then the push-forward $\pi_{\#}\mu$ is also an invariant measure.

The non-oriented Grassmannian manifold $G(n, k)$ is diffeomorphic to a certain quotient, so that, by Theorem 6.7, an invariant measure $\gamma_{n,k}$ exists. In particular, we will finally be able to show that the integralgeometric k-dimensional measure (6.3) is well-defined.

Lemma 6.8. *Let $O(m)$ denote the group of the orthogonal $m \times m$ matrices. Then there is a diffeomorphism*

$$G(n, k) \cong {}^{O(n)}\!/\!_{(O(k) \times O(n-k))},$$

where $O(k) \times O(n-k)$ is the space of orthogonal matrices made up of a $k \times k$ orthogonal block and a $(n-k) \times (n-k)$ orthogonal block.

The reader may work out the details of the proof by themselves, but the intuitive idea behind it is simple: Given $W \in G(n, k)$ element of the Grassmannian manifold, consider an orthonormal basis $w_1, \ldots, w_k$ for it. Complete it to an orthonormal basis $w_1, \ldots, w_n$ of $\mathbb{R}^n$ and, at this point, define an equivalence class naturally (that is, up to change of orthonormal basis for W and the complement of W separately.)

In conclusion, as promised, we sketch the proof of Theorem 6.5 in the special case of $\mathcal{G}$ Lie group, and we give a complete proof (of the existence, at least) in the case of $\mathcal{G}$ commutative).

Proof 1. Let $\mathcal{G}$ be a k-dimensional Lie group. The idea is to define a left-invariant k-form ω, that is, a k-form such that the pull-back according to τ_y is ω itself. Then it suffices to check that

$$\mu(E) := \int_E \omega$$

is the sought invariant measure, and also that it is unique. □

Proof. Let $\mathcal{G}$ be a commutative group, and let $\mathcal{P}$ be the space of probability measures defined on $\mathcal{G}$. For any $g \in \mathcal{G}$, set

$$\mathcal{P}_g := \left\{ \mu \in \mathcal{P} \;\middle|\; (\tau_g)_{\#}\, \mu = \mu \right\}$$

be the subset of $\mathcal{P}$ containing all the g-invariant probability measures defined on X.

Step 1. We want to prove that, for every $g \in \mathcal{G}$, the subset $\mathcal{P}_g$ is nonempty. Fix $\mu_0 \in \mathcal{P}$ and let us consider, for every $n \in \mathbb{N}$, the probability measure defined by setting

$$\mu_n := \frac{\mu_0 + (\tau_g)_{\#}\, \mu_0 + \cdots + (\tau_{g^n})_{\#}\, \mu_0}{n+1} \in \mathcal{P},$$

where g^n denotes the product of n copies of g.

By compactness there exists a subsequence μ_{n_k} weakly-$*$ converging to a measure μ_∞. We now claim that μ_∞ is a τ_g-invariant probability measure. Indeed, by definition of μ_n, it follows that

$$(\tau_g)_{\#}\, \mu_{n_k} \to \mu_\infty \implies (\tau_g)_{\#}\, \mu_\infty = \mu_\infty.$$

[2] Notice that H does not need to be a normal subgroup of $\mathcal{G}$.

Step 2. We want to prove that the intersection of all the $\mathcal{P}_g$ is nonempty, which is clearly enough to infer the existence of an invariant measure.

Let g, $h \in \mathcal{G}$ be two elements, let $\mu_0 \in \mathcal{P}_g$ be an invariant measure, and let μ_∞ be the weakly-$*$ limit of the sequence

$$\mu_n := \frac{\mu_0 + (\tau_h)_\# \mu_0 + \cdots + (\tau_{h^n})_\# \mu_0}{n+1} \in \mathcal{P}.$$

The set $\mathcal{P}_g$ is weakly-$*$ closed; therefore $\mu_\infty \in \mathcal{P}_g \cap \mathcal{P}_h$. By induction we can prove that the family $\{\mathcal{P}_g\}_{g \in \mathcal{G}}$ has the finite intersection property, and thus, by compactness of $\mathcal{G}$, it immediately follows that

$$\bigcap_{g \in \mathcal{G}} \mathcal{P}_g \neq \varnothing.$$

Step 3. We want to prove that the intersection above contains only one element. In order to do that, we define the convolution product of two measures by setting

$$\mu_1 * \mu_2(E) := (\mu_1 \times \mu_2)\left(\{(x_1, x_2) \mid x_1 + x_2 \in E\}\right).$$

The reader may prove that the convolution is commutative, and also that

$$\mu_1 * \mu_2 = \mu_1,$$

if μ_1 is an invariant measure.

This is enough to infer that the invariant measure is unique. Indeed, if λ, $\mu \in \cap_{g \in \mathcal{G}} \mathcal{P}_g$ are two invariant measures, then the properties above of the convolution implies that

$$\mu = \mu * \lambda = \lambda * \mu = \lambda \implies \mu = \lambda.$$

$\square$

Chapter 7

Lipschitz Functions

In the first half of the chapter, we present some of the basic properties of Lipschitz maps and their relations to Hausdorff measures.

In the second half, we derive the so-called *area formula* for Lipschitz maps and we show the close relations with the *coarea formula*.

7.1 Definitions and Main Properties

In this section, we introduce Lipschitz function and Lipschitz maps (between metric spaces), and we state some of the main properties (which explains why they are such a good replacement in geometric measure theory for C^1 functions.)

Definition 7.1 (Lipschitz)**.** Let X and Y be metric spaces. A map $f : X \longrightarrow Y$ is *Lipschitz* if there exists a positive constant $L > 0$ such that

$$d_Y\left(f(x),\, f(y)\right) \leq L \cdot d_X\left(x,\, y\right)$$

for every $x,\, y \in X$. The Lipschitz constant of f is the optimal one, that is,

$$\mathrm{Lip}(f) := \inf\left\{L > 0 \mid d_Y\left(f(x),\, f(y)\right) \leq L \cdot d_X\left(x,\, y\right) \text{ for every } x,\, y \in X\right\}.$$

Compactness. We now present an Ascoli-Arzelà result, that is, a compactness criteria for Lipschitz maps.

Theorem 7.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n : X \longrightarrow Y$ between compact metric spaces, and assume that it is uniformly Lipschitz, that is, each f_n has the same Lipschitz constant.*

Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ converging uniformly to a Lipschitz map $f_\infty : X \longrightarrow Y$ with the same Lipschitz constant.

This result follows immediately from an application of the typical Ascoli-Arzelà theorem for metric spaces, but the reader may notice that the assumptions are not optimal. Indeed, one may only assume that X is locally compact and Y is a pre-compact space.

Extension Property. Lipschitz functions $f : X \longrightarrow \mathbb{R}$ have an excellent extension property, which preserves the Lipschitz constant so that, even if it is defined on a subset $E \subset X$, we can always talk about f as a function defined on X.

Lemma 7.3 (Mc Shame)**.** *Let $f : E \subset X \longrightarrow \mathbb{R}$ be a Lipschitz function. There exists $F : X \longrightarrow \mathbb{R}$ such that*

$$F\big|_E = f \qquad \textit{and} \qquad \mathrm{Lip}(F) = \mathrm{Lip}(f).$$

Proof. Let $L := \mathrm{Lip}(f)$, and set

$$F(x) := \inf\left\{ f(y) + L \cdot d_X(x,\, y) \mid y \in E \right\}. \tag{7.1}$$

The reader may prove by themselves that F is an extension of f, and also that F is a L-Lipschitz map (since it is a lower envelope.) □

The extension is, in general, not unique. Indeed, we can easily define a different extension by setting

$$\widetilde{F}(x) := \sup\left\{ f(y) - L \cdot d_X(x,\, y) \mid y \in E \right\}. \tag{7.2}$$

The reader may check that (7.1) and (7.2) define, in general, two different extension of f. The extension property is also true for $Y := \mathbb{R}^m$-valued functions, but the argument above does not preserve the Lipschitz constant in general.

Lemma 7.4. *Let $f : E \subset X \longrightarrow \mathbb{R}^m$ be a Lipschitz function. There exists $F : X \longrightarrow \mathbb{R}^m$ such that*

$$F\big|_E = f \qquad \textit{and} \qquad \mathrm{Lip}(F) \le \sqrt{m} \cdot \mathrm{Lip}(f).$$

Proof. Let $L := \mathrm{Lip}(f)$. We may extend each component by setting

$$F_i(x) := \inf\left\{ f_i(y) + L \cdot d_X(x,\, y) \mid y \in E \right\}. \tag{7.3}$$

The reader may prove by themselves that $F = (F_1,\, \dots,\, F_m)$ is an extension of f satisfying the properties mentioned above. □

If X and Y are metric spaces, then there is no guarantee that such an extension exists. On the other hand, if they are Hilbert spaces, then there is a highly nontrivial theorem that proves the existence of an extension that preserves the Lipschitz constant.

Theorem 7.5 (Kirszbraun)**.** *Let $f : E \subseteq X \longrightarrow Y$ be a Lipschitz map between Hilbert spaces. Then there exists $F : X \longrightarrow Y$ such that*

$$F\big|_E = f \qquad \textit{and} \qquad \mathrm{Lip}(F) = \mathrm{Lip}(f).$$

7.2 Differentiability of Lipschitz Functions

In this section, we investigate the Lusin property of Lipschitz maps (from $\mathbb{R}^n$ to $\mathbb{R}^m$) with C^1 maps and, in particular, we prove that Lipschitz maps are $\mathcal{L}^n$-almost everywhere differentiable.

Theorem 7.6 (Lusin Property)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz map. Then, for every $\epsilon > 0$ there exists a function $g_\epsilon \in C^1(\mathbb{R}^n;\, \mathbb{R})$ satisfying the following properties:*

(1) The two maps coincide up to a set of measure at most ϵ, that is,

$$\mathcal{L}^n\left(\left\{ x \in \mathbb{R}^n \mid f(x) \neq g_\epsilon(x) \right\}\right) \le \epsilon.$$

(2) The Lipschitz constant is bounded from above, that is,

$$\mathrm{Lip}(g_\epsilon) \le \mathrm{Lip}(f).$$

We will not prove this theorem entirely as we will only sketch the proof of a "local" statement. On the other hand, at the end of the section we briefly show how to obtain the global statement via a *partition of unity* (but this is a highly nontrivial result!)

Regularity. First, we prove that Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}$ (and thus to $\mathbb{R}^m$) are almost everywhere differentiable with respect to the Lebesgue measure.

Theorem 7.7 (Rademacher)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz map. Then f is $\mathcal{L}^n$-almost everywhere differentiable.*

Remark 7.1. The extension property (i.e., Mc Shame Lemma 7.3) automatically gives us the Rademacher theorem for Lipschitz maps defined on a subset $E \subset \mathbb{R}^n$.

Remark 7.2. The statement of the Rademacher theorem presented here is not the most general possible. Indeed, the proof we are about to give works for Lipschitz maps with values in *any finite-dimensional normed space.*

On the other hand, for infinite-dimensional Banach spaces, the statement is usually false. There is a particular class of Banach spaces for which the theorem holds, usually referred in the literature as *Banach spaces with the Lusin property*).

Lemma 7.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz map. Then f belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, that is, the distributional gradient Df is essentially bounded.*

Sketch of the Proof. Given a locally summable $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and given a direction $v \in \mathbb{R}^n$, it is a well-known fact that the distributional directional derivative is given by

$$\frac{\partial f}{\partial v} = \lim_{h \to 0} \frac{f - \tau_{hv} f}{h}.$$

Since f is a Lipschitz map, we easily infer that

$$\sup_{x \in \mathbb{R}^n} \left| \frac{f(x) - \tau_{hv} f(x)}{h} \right| \leq \mathrm{Lip}(f)|v|,$$

that is, the directional distributional derivative belongs to $L^\infty(\mathbb{R}^n)$, and so also the distributional gradient. □

Remark 7.3. For every $p \in [1, +\infty)$ it turns out that

$$W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n).$$

Theorem 7.9. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous Sobolev function, that is $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p \in (n, +\infty)$. Then f is $\mathcal{L}^n$-almost everywhere differentiable, and the pointwise gradient agrees with the distributional gradient for $\mathcal{L}^n$-almost every $x \in \mathbb{R}^n$.*

Proof. Fix a ball $B := B(\underline{x}, r)$.

Step 1. We claim that

$$\omega(f, B) \leq C(n) \cdot r \left(\fint_B |\nabla f(x)|^p \, \mathrm{d}x \right)^{1/p}, \tag{7.4}$$

where $C(n)$ is a universal constant[1] and ω is oscillation function, that is,

$$\omega(f, B) := \sup \{ |f(x) - f(\underline{x})| \mid |x - \underline{x}| < r \}.$$

[1] We say that a constant is universal if it depends only on the dimension of the ambient space

Step 2. In order to prove (7.4), we notice that we can always reduce to the case $B = B\,(0,\,1)$ by composing with the transformation

$$x \longmapsto \frac{x - \underline{x}}{r}.$$

The Poincaré inequality gives the inequality[2]

$$\left| \int_{B(0,\,1)} f(x)\,\mathrm{d}x - \fint_{B(0,\,1)} f(x)\,\mathrm{d}x \right| \le c(n)\|\nabla f\|_{L^p(B)},$$

from which we infer that

$$\Phi(f) = \|\nabla f\|_{L^p(B(0,\,1))} + \left| \fint_{B(0,\,1)} f(x)\,\mathrm{d}x \right|$$

is equivalent to the usual $W^{1,\,p}(B(0,\,1))$ norm. Therefore, we apply Sobolev embedding theorem and obtain the inequality

$$\|f\|_{\infty,\,B(0,\,1)} \le c_1(n) \left[\|\nabla f\|_{L^p(B(0,\,1))} + \left| \fint_{B(0,\,1)} f(x)\,\mathrm{d}x \right| \right].$$

In conclusion, it turns out that

$$\omega\,(f,\,B) \le 2\,\|f\|_\infty \le 2\,c_1(n) \left[\|\nabla\,f\|_{L^p(B(0,\,1))} + \left| \fint_{B(0,\,1)} f(x)\,\mathrm{d}x \right| \right],$$

and, by replacing f with $f - \mathrm{av}(f)$, we infer that (7.4) holds true since the left-hand side does not change if we add a constant to the function.

Step 3. For every $h \in \mathbb{R}^n$ it turns out that

$$|f(\underline{x} + h) - f(\underline{x})| \le C(n)|h| \left(\fint_B |\nabla f(x)|^p\ \mathrm{d}x \right)^{1/p}.$$

Therefore, if L is a linear application, it turns out that

$$|f(\underline{x} + h) - f(\underline{x}) - Lh| \le C(n)|h| \left(\fint_B |\nabla f(x) - L|^p\ \mathrm{d}x \right)^{1/p},$$

and this concludes the proof. Indeed, if $\underline{x}$ is a point of L^p-approximate continuity of $\nabla f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$, and if we take $L := \nabla f(\underline{x})$, then it turns out that

$$|f(\underline{x} + h) - f(\underline{x}) - \nabla f(\underline{x})h| \le C(n)|h|\mathcal{O}(1) = C(n)\,\mathcal{O}(\|h\|).$$

□

The Rademacher theorem for Lipschitz maps follows easily combining the lemma and the theorem above. We are finally ready to state the local version of the Lusin property theorem and to give, at least, the main ideas behind it.

[2]The constant depends only on the dimension n since the domain is the unitary ball!

Theorem 7.10. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $\mathcal{L}^n$-almost everywhere differentiable map, and let Ω be an open bounded subset of $\mathbb{R}^n$. For every $\epsilon > 0$ there exist a compact subset $K_\epsilon \subset \Omega$ and a function $g_\epsilon \in C^1(\mathbb{R}^n;\, \mathbb{R})$ satisfying the following properties:*

(1) *The two maps coincide up to a set of measure at most ϵ, that is,*

$$f\big|_{K_\epsilon} = g_\epsilon\big|_{K_\epsilon} \qquad \textit{and} \qquad \mathcal{L}^n\left(\Omega \setminus K_\epsilon\right) \leq \epsilon.$$

(2) *The Lipschitz constant is bounded from above, that is,*

$$\mathrm{Lip}(g_\epsilon) \leq \mathrm{Lip}(f) + \epsilon$$

provided that f is Lipschitz.

Proof. Let

$$D := \{x \in \mathbb{R}^n \mid f \text{ is differentiable at } x\}$$

be the set of points where f is differentiable. Then, for any $x \in D$, it turns out that

$$\omega(x,\, r) := \sup_{|h|<r} \left| \frac{f(x+h) - \nabla f(x)h}{h} \right| \to 0$$

decreasingly as $r \to 0^+$. For every $\epsilon > 0$ we can find a compact set $K_\epsilon \subset \Omega$ such that

(a) $\mathcal{L}^n\left(\Omega \setminus K_\epsilon\right) \leq \epsilon$;

(b) ∇f is continuous at each point of K_ϵ (Lusin);

(c) $\omega_r(x) \searrow 0$ as $r \to 0^+$, uniformly with respect to $x \in K_\epsilon$.

The reader may check these assertions as an exercise[3]. Let $A = \mathbb{R}^n \setminus K_\epsilon$ be the complement of K_ϵ in Ω; for any integer $i \in \mathbb{Z}$ we consider the set

$$A_i := \left\{ x \in A \;\middle|\; \frac{1}{2^{i+1}} < d(x,\, K_\epsilon) < \frac{1}{2^{i-1}} \right\}.$$

Clearly $\{A_i\}_{i\in\mathbb{Z}}$ is a covering of A; hence there exists a smooth partition of unity $\{\sigma_i\}_{i\in\mathbb{Z}}$ subordinated to it, with the additional property

$$|\nabla \sigma_i| \leq 2^{i+3}.$$

Let ρ be a regularizing kernel with support contained in the unitary ball, and assume also that $\int \rho = 1$ and $\int x\rho = 0$. If we let $\rho_i := \rho_{\frac{1}{2^{i+1}}}$ be the rescaling, then we can set

$$g(x) := \begin{cases} f(x) & \text{if } x \in K_\epsilon, \\ \displaystyle\sum_{i\in\mathbb{Z}} (f(x)\sigma_i(x)) * \rho_i(x) & \text{if } x \notin K_\epsilon. \end{cases}$$

To conclude the proof the reader may check that

(a) $f \equiv g$ on K_ϵ;

(b) g is smooth on A;

(c) g is differentiable at every $x \in K_\epsilon$ and $\nabla g = \nabla f$ on A.

$\square$

[3]**Hint.** The second assertion follows from the Lusin theorem, while the third one follows from the Egorov theorem.

7.3 Area Formula for Lipschitz Maps

The main result of this section is the following theorem.

Theorem 7.11. *Let Σ be a d-dimensional surface of class C^1, and let $f : \Omega \longrightarrow \Sigma$ be a Lipschitz function defined on an open subset $\Omega \subseteq \mathbb{R}^d$. Then for every $E \subseteq \Omega$ Borel it turns out that*

$$\int_\Sigma m_{f,E}(y)\,\mathrm{d}\mathcal{H}^d(y) = \int_E Jf(x)\,\mathrm{d}\mathcal{L}^d(x), \tag{7.5}$$

where $m_{f,E}(y) := \#\left(f^{-1}(y) \cap E\right)$ and $Jf(x)$ denotes the Jacobian[4] *of f at x.*

Remark 7.4. If f is an injective Lipschitz function, the area formula (7.5) reduces to an expression we are more familiar with (the substitution of variables in integral calculus):

$$\mathcal{H}^d\left(f(E)\right) = \int_E Jf(x)\,\mathrm{d}\mathcal{H}^d(x). \tag{7.6}$$

Remark 7.5. The codomain Σ does not need to be a d-dimensional surface, but it is enough to require Σ Riemannian manifold since a scalar product is all we need to define orthonormal basis.

Remark 7.6 (Jacobian, I)**.** If Σ is a d-dimensional surface embedded in $\mathbb{R}^n$ for some n and f is differentiable at x, then the Jacobian of f at x can be easily computed using the standard formula

$$Jf(x) = \sqrt{\det\left(\nabla_t f(x)\right)^T\left(\nabla_t f(x)\right)}, \tag{7.7}$$

where ∇_t denotes *tangent gradient*. More precisely, if we look at f as a function from Ω to $\mathbb{R}^n$, then the differential at x is a linear map

$$\mathrm{d}f_x : T_x\Omega \cong \mathbb{R}^d \longrightarrow \mathbb{R}^n.$$

By choosing orthonormal bases of both $\mathbb{R}^d$ and $\mathbb{R}^n$, we obtain a matrix that represents the linear map $\mathrm{d}f_x$, which is also the matrix representing $\nabla_t f(x)$.

Proof. We notice that the adjoint map $(\mathrm{d}f_x)^*$ sends $T_{f(x)}\Sigma \cong \mathbb{R}^n$ to $\mathbb{R}^d$ since the scalar product canonically identifies a vector space with its dual. The linear map

$$(\mathrm{d}f_x)^* \circ \mathrm{d}f_x : T_x\Sigma \longrightarrow T_x\Sigma$$

is represented by the square matrix $(\nabla_t f(x))^T\,(\nabla_t f(x))$ by construction, and thus we infer that the formula (7.7) holds.

Remark 7.7. The main result of this section (7.5) is right as it is stated, but the reader may notice that the proof we present completely ignores the measurability issues[5].

A weaker statement, which can be proved as an exercise, with no measurability issue to check, is given by the following theorem.

Theorem 7.12. *Let $f : \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a Lipschitz function defined on an open subset $\Omega \subseteq \mathbb{R}^d$. Then for every $F \subset \mathbb{R}^n$ Borel set it turns out that*

$$\int_F \#\left(f^{-1}(y)\right)\,\mathrm{d}\mathcal{H}^d(y) = \int_{f^{-1}(F)} Jf(x)\,\mathrm{d}x. \tag{7.8}$$

[4] The Jacobian, in this abstract setting, will be defined below in Definition 7.14.
[5] A nice exercise would be to clean up the proof and make it rigorous.

A stronger (more general) statement, which requires a lot of work to be proved rigorously, is given by the following theorem.

Theorem 7.13. *Let Σ be a d-dimensional surface of class C^1, and let $f : \Omega \longrightarrow \Sigma$ be a Lipschitz function defined on an open subset $\Omega \subseteq \mathbb{R}^d$.For every positive Borel function $h : \Omega \longrightarrow [0, +\infty]$, it turns out that*

$$\int_\Sigma [\sum_{s \in f(x)} h(s)] \, \mathrm{d}\mathcal{H}^d(x) = \int_\Omega h(x) Jf(x) \, \mathrm{d}x. \tag{7.9}$$

Area Formula. In this final paragraph, the main goal is to sketch the proof of (7.5). The first step is, as promised, to give meaning to the Jacobian of a Lipschitz function.

Definition 7.14 (Jacobian)**.** Let $f : \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a function differentiable at x, and $d \leq n$. The d-dimensional Jacobian of f at x, denoted by $J_d f(x)$, is defined by setting

$$J_d f(x) = \sup \left\{ \frac{\mathcal{H}^d \left(\nabla_t f(x)(P) \right)}{\mathcal{H}^d[P]} \;\middle|\; \begin{array}{c} P \text{ is a } d\text{-dimensional} \\ \text{parallelepiped contained in } \mathbb{R}^d \end{array} \right\}.$$

We are now ready to prove the area formula (7.5). We shall see soon that it is an immediate consequence of two technical lemmas.

Lemma 7.15. *Let $A \subseteq \Omega$ be an open set such that the restriction $f\big|_A$ is a diffeomorphism, that is, injective and with maximal rank:*

$$\operatorname{rank}\left(\mathrm{d}f(s)\right) = d.$$

Then the push-forward measure

$$\mu := f_\# \left(\mathbb{1}_A(x) Jf(x) \cdot \mathcal{L}^d \right)$$

is equal to the measure

$$\mathbb{1}_{f(A)} \cdot \mathcal{H}^d.$$

In particular, given the subset $F = f(A) \subseteq \Sigma$ as in (7.8), it turns out that

$$\mathcal{H}^d\left(F\right) = \int_A Jf(s) \, \mathrm{d}s,$$

that is, the area formula (7.5) holds if $F = f(E)$ for every E Borel.

Lemma 7.16. *Let $E \subseteq \Omega$ be a Borel set such that $Jf(s) = 0$ for every $s \in E$ (i.e., the rank of the differential map is not maximal). Then*

$$\mathcal{H}^d\left(f(E)\right) = 0$$

and the area formula (7.5) holds for every such E.

Suppose, for now, that both lemmas hold true. Then, for every Borel set $E \subset \Omega$, it is enough to split it into countably many pieces E_i satisfying the following properties:

i) The restriction $f\big|_{E_i}$ is a diffeomorphism for every $i \in I$.

ii) The complement set

$$E \setminus \bigcup_{i \in I} E_i$$

is the set of all points $s \in E$ such that the differential of f at s has rank strictly less than d.

Proof of Lemma 7.15. First, we notice that the thesis follows easily if we can prove that the Radon-Nikodym density

$$\frac{\mathrm{d}\mu}{\mathrm{d}\left(\mathbb{1}_{f(A)} \cdot \mathcal{H}^d\right)}(s) = 1 \tag{7.10}$$

for **every** $s \in f(A)$.

Step 0. Suppose that (7.10) holds. The measure μ is supported, by definition, in the set $f(A)$ and it has no singular part[6]; hence

$$f_{\#}\left(\mathbb{1}_A \, Jf \cdot \mathcal{L}^d\right) = \mathbb{1}_{f(A)} \cdot \mathcal{H}^d.$$

Step 1. Recall that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\left(\mathbb{1}_{f(A)} \cdot \mathcal{H}^d\right)}(s) = 1 \iff \frac{\mu\left(B(s,\, r)\right)}{\mathcal{H}^d\left((B(s,\, r)) \cap \Sigma\right)} \xrightarrow{r \to 0^+} 1,$$

and thus it is enough to find an asymptotic estimate of both the numerator and the denominator.

Step 2. If $\pi_s : \mathbb{R}^n \longrightarrow T_s\Sigma$ is the orthogonal projection onto $T_s\Sigma$, then it turns out that

$$(1 - o(1)) \cdot |x - y| \leq |\pi(x) - \pi(y)| \leq |x - y|$$

that is, the projection is an almost-isometry. It follows that

$$\mathcal{H}^d\left(B(s,\, r) \cap \Sigma\right) \sim \mathcal{H}^d\left(\pi\left(B(s,\, r) \cap \Sigma\right)\right) \sim \alpha_d r^d.$$

Step 3. In a similar fashion, given a ball $B_r := B(0,\, r) \subseteq T_s\Sigma$, one can consider the ellipsoid[7]

$$E_r := \mathrm{d}f^{-1}\left(B_r\right).$$

The image of E_r according to f has measure

$$\mu\left(f(s + E_r)\right) \sim \mu\left(B(s,\, r)\right),$$

and, on the other hand, it is easy to show that

$$\mu\left(f(s + E_r)\right) = \int_{s + E_r} Jf(x)\, \mathrm{d}x = Jf(s)\, |s + E_r| = Jf(s)\, \frac{|B_r|}{Jf(s)} = |B_r|.$$

In particular, it turns out that

$$\mu\left(B(s,\, r)\right) \sim \mu\left(f(s + E_r)\right) \sim |B_r| = \alpha_d r^d,$$

which is exactly what we wanted to prove. □

Proof of Lemma 7.16. By assumption, for every point $s \in E$ and positive real number $r > 0$ small enough, it turns out that

$$f\left(B(s,\, r)\right) \subseteq B^{k<d}\left(f(s),\, \mathcal{O}(r)\right) \times B\left(f(s),\, o(r)\right),$$

[6] Indeed, if μ has a singular part, then there exists a point $s \in f(A)$ such that (7.10) is $+\infty$, against our assumption.
[7] The preimage of a ball via the differential is an ellipsoid because the rank is maximal.

where $\mathcal{O}(-)$ and $o(-)$ are, respectively, the Landau's symbols. It follows that

$$\mathcal{H}^d\left(f(B(s,\, r))\right) \simeq \mathcal{O}(r)^k \cdot o(r)^{d-k} \sim o(r^d),$$

and this is enough to conclude since we can cover the ellipsoid with a countable union of balls satisfying the estimate above, that is,

$$\sum_i \operatorname{diam}\left((B_i)\right)^d \sim \left(\frac{\mathcal{O}(r)}{o(r)}\right)^k \cdot o(r)^d \sim o(r^d),$$

which is exactly what we wanted to prove. $\square$

7.4 Coarea Formula for Lipschitz Maps

Coarea. Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a map of class C^1, and assume that $m < n$. If $h : \Omega \longrightarrow [0,\, +\infty]$ is a positive function, then the coarea formula is given by

$$\int_{\mathbb{R}^n} \left[\int_{f^{-1}(y)\cap\Omega} h(x)\,\mathrm{d}\mathcal{H}^{n-m}(x) \right] \mathrm{d}y = \int_\Omega h(x) J(f)(x)\,\mathrm{d}x, \tag{7.11}$$

where

$$J(f)(x) = \sqrt{\det\left((\nabla f(x))\, (\nabla f(x))^t \right)} \in M(n \times n,\, \mathbb{R})$$

is a suitable notion of Jacobian. The proof of formula (7.11) is simple, and it is left to the reader to fill in the details in what follows.

Sketch of the Proof.

Particular Case. If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the projection over the first coordinate, then the formula reduces to the Fubini-Tonelli theorem.

General Case. In the general case, the main idea is to split the set Ω as

$$\Omega = \Omega_s \sqcup \Omega_n,$$

where Ω_s is the set of all singular points (i.e., all the $x \in \Omega$ such that the Jacobian at x has non-maximal rank), and Ω_n is the set of all points such that the Jacobian is a matrix of maximal rank.

The formula (7.11) clearly holds at every point of the first kind since the Jacobian is not invertible (i.e., $J(f)(x) = 0$), while the set $f^{-1}(x) \cap \Omega$ is $\mathcal{H}^{n-m}$-null as expected.

The formula for regular points, on the other hand, can be easily deduced from the particular case discussed above ($f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ projection) via a simple change of variables.

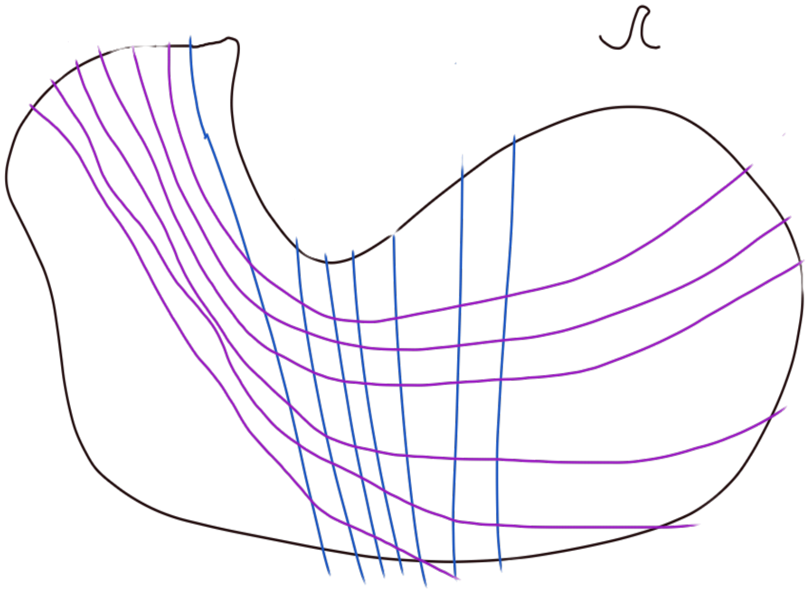

Figure 7.1: Idea of the coarea formula: integrate first along the blue lines, and then integrate the result along the violet lines

Chapter 8

Rectifiable Sets

In this chapter, we shall be mainly concerned with the notions of d-dimensional *rectifiable* and *purely unrectifiable* sets in metric spaces.

In the first section, we furnish the definitions of rectifiable (and purely unrectifiable) set through Lipschitz maps, and we state some of the main criteria in $\mathbb{R}^n$.

In the second section, we work towards a (suitably weak) definition of the tangent bundle for rectifiable sets, and, in the next section, we end up proving that any Borel set E with an approximate tangent cone and d-dimensional lower density bounded from below is d-rectifiable.

In the last section, we prove the area formula for rectifiable sets using the characterization as the union of the graphs of Lipschitz maps.

8.1 Introduction and Elementary Properties

Definition 8.1 (Rectifiable Set)**.** Let X be a metric space. A Borel set $E \subseteq X$ is a d-dimensional *rectifiable set* if and only if there exists a collection of Borel sets $\{E_i\}_{i\in\mathbb{N}}$ such that

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

satisfying the following properties:

1) E_0 is a $\mathcal{H}^d$-null set, and
2) $E_i \subset f_i(\mathbb{R}^d)$, where $f_i : \mathbb{R}^d \longrightarrow X$ is a Lipschitz function for every $i \geq 1$.

Remark 8.1. Let X be a metric space, and let $E \subset X$ be a d-dimensional rectifiable set. Then the Hausdorff dimension of E is less than or equal to d, i.e.,

$$\dim_{\mathcal{H}} E \leq d$$

Proof. Let $\{E_i\}_{i\in\mathbb{N}}$ be the collection given by the definition of d-rectifiable set. The assumption that E_0 is a $\mathcal{H}^d$-null set immediately implies that

$$\dim_{\mathcal{H}} E_0 \leq d.$$

On the other hand, by Lemma 2.21 it follows that a Lipschitz map does not increase the Hausdorff dimension, that is,

$$\dim_{\mathcal{H}} \mathbb{R}^d = d \implies \dim_{\mathcal{H}} f_i(\mathbb{R}^d) \leq d,$$

which is enough to infer (as a consequence of Remark 2.13) that

$$E = \bigcup_{i \in \mathbb{N}} E_i \implies \dim_{\mathcal{H}} E = \sup_{i \in \mathbb{N}} \dim_{\mathcal{H}} E_i \leq d.$$

□

Exercise 8.1. Prove that $\mathcal{H}^d(E) = 0$ does not imply that E is contained in a countable union of Lipschitz images of $\mathbb{R}^d$, that is,

$$E \not\subseteq \bigcup_{n \in \mathbb{N}} f_n(\mathbb{R}^d).$$

Exercise 8.2. Prove that there exists a Borel set $E_0 \subseteq \mathbb{R}^2$ such that $\mathcal{H}^d(E_0) = 0$ which cannot be covered by a rectifiable curve (hint: self-similar sets).

Proposition 8.2 (Criteria of Rectifiability)**.** *Let $X := \mathbb{R}^n$, and let $E \subset X$ be a Borel set. The following assertions are equivalent:*

(a) *The set E is d-rectifiable.*

(b) *There exists a collection of Borel sets $\{E_i\}_{i \in \mathbb{N}}$ such that*

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

satisfying the following properties:

1) *E_0 is a $\mathcal{H}^d$-null set.*

2) *$E_i \subset f_i(A_i)$, where $A_i \subseteq \mathbb{R}^d$ is an open subset and $f_i : A_i \longrightarrow \mathbb{R}^n$ is a differentiable function.*

(c) *There exists a collection of Borel sets $\{E_i\}_{i \in \mathbb{N}}$ such that*

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

satisfying the following properties:

1) *E_0 is a $\mathcal{H}^d$-null set.*

2) *$E_i \subset f_i(A_i)$, where $A_i \subseteq \mathbb{R}^d$ is an open subset and $f_i : A_i \longrightarrow \mathbb{R}^n$ is a diffeomorphism.*

(d) *There exists a collection of Borel sets $\{E_i\}_{i \in \mathbb{N}}$ such that*

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

satisfying the following properties:

1) *E_0 is a $\mathcal{H}^d$-null set.*

2) *$E_i \subset \Sigma_i$, where Σ_i is a d-dimensional surface of class C^1 contained in $\mathbb{R}^n$.*

Proof. By definition (differentiable maps are Lipschitz) **(b)** $\implies$ **(a)**, while the opposite implications is a straightforward consequence of the following result.

Lemma 8.3. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a Lipschitz map. Then there is a collection $\{f_i\}_{i\in\mathbb{N}}$ of functions of class C^1 and a $\mathcal{H}^d$-null Borel set A_0 such that*

$$f(\mathbb{R}^d) \subseteq \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^d) \cup A_0.$$

Proof. By Theorem 7.6, Lipschitz maps have the Lusin property in the class of differentiable maps C^1; hence, for every $i \in \mathbb{N}$, there exists a differentiable map $f_i : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ such that

$$|E_i| := \left|\left\{x \in \mathbb{R}^d \mid f_i(x) \neq f(x)\right\}\right| \leq \frac{1}{i}.$$

The reader may easily prove that, by construction, it turns out that

$$f(\mathbb{R}^d) \subset \left(\bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^d)\right) \cup \left(f\left(\bigcap_{i\in\mathbb{N}} E_i\right)\right).$$

In particular, the image of the intersection is a $\mathcal{H}^d$-null set since

$$\mathcal{L}^d\left(\bigcap_{i\in\mathbb{N}} E_i\right) = 0 \implies \mathcal{H}^d\left(\bigcap_{i\in\mathbb{N}} E_i\right) = 0,$$

and f is a Lipschitz map (see Lemma 2.21). $\square$

Clearly **(c)** $\implies$ **(b)** (since diffeomorphisms are also differentiable maps); thus we only need to prove the opposite implication.
Suppose **(b)** holds, and let $\{E_i\}_{i\in\mathbb{N}}$ be the collection of Borel sets satisfying the definition, i.e,,

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

where E_0 is a $\mathcal{H}^d$-null set and $E_i \subset f_i(A_i)$ for differentiable maps $f_i : A_i \longrightarrow \mathbb{R}^n$. For every $i \geq 1$ we consider a partition of A_i

$$A_i = A_i^{max} \sqcup A_i^{min},$$

where

$$A_i^{max} := \{x \in A_i \mid \mathrm{d}f_x \text{ has maximal rank}\}$$

and

$$A_i^{min} := \{x \in A_i \mid \mathrm{d}f_x \text{ has NOT maximal rank}\}.$$

Recall that A_i^{min} is a $\mathcal{H}^d$-null set for every differentiable map f_i. Hence, we can consider the collection of Borel sets given by

$$\widetilde{E_0} := E_0 \cup \left(\bigcup_{i\geq 1} f\left(A_i^{min}\right)\right) \qquad \text{and} \qquad \widetilde{E_i} := f_i\left(A_i^{max}\right)$$

together with the diffeomorphisms $f_i\big|_{A_i^{max}}$.

In conclusion, it remains to prove the equivalence **(c)** $\implies$ **(d)**. The "only if" part is immediate from the definitions, while the "if" part is left to the reader as a simple exercise (in the same spirit of the implications we have already proved here). ☐

Remark 8.2. If $X := \mathbb{R}^n$, we just proved that Lipschitz maps may be replaced by C^1 maps in the definition of d-rectifiable set. On the other hand, it is not possible to replace C^1 by a more regular space $C^{k\geq 2}$ since C^1 does not have the Lusin property in C^2.

Definition 8.4 (Purely Unrectifiable)**.** Let X be a metric space. A Borel set $E \subseteq X$ is a d-dimensional *unrectifiable set* if and only if

$$\mathcal{H}^d\left(E \cap f\left(\mathbb{R}^d\right)\right) = 0 \quad \text{for every Lipschitz map } f : \mathbb{R}^d \longrightarrow X.$$

Proposition 8.5 (Criteria of Unrectifiability)**.** *Let $X := \mathbb{R}^n$, and let $E \subset X$ be a Borel set. The following assertions are equivalent:*

- ***(a)*** *E is purely d-unrectifiable.*
- ***(b)*** *For every C^1 map $f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ it turns out that*

$$\mathcal{H}^d\left(E \cap f\left(\mathbb{R}^d\right)\right) = 0.$$

- ***(c)*** *For every diffeomorphism $f : A \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}^n$ defined on an open subset A, it turns out that*

$$\mathcal{H}^d\left(E \cap f\left(A\right)\right) = 0.$$

- ***(d)*** *For every d-dimensional surface $\Sigma \subseteq \mathbb{R}^n$ of class C^1 it turns out that*

$$\mathcal{H}^d\left(E \cap \Sigma\right) = 0.$$

Notation. From now on, we will use purely unrectifiable and 1-purely unrectifiable as synonyms.

Proposition 8.6. *Let $X = \mathbb{R}^2$. For every $d \in (0,\, 2]$ there exists a compact set $K_d \subset \mathbb{R}^2$ such that $\dim_{\mathcal{H}} K = d$ and K purely unrectifiable.*

Proof. Here we present the proof of the assertion for $d \in (0,\, 2)$. The reader may extend the construction to dimension 2 as an exercise.

Step 1. First, we claim that if $K = K_1 \times K_2$ is a product of compact subsets of $\mathbb{R}$, then

$$\mathcal{L}^1\left(K_1\right) = \mathcal{L}^1\left(K_2\right) = 0 \implies K \text{ purely unrectifiable.}$$

Indeed, let $\mathcal{C}$ be a curve of class C^1 (a submanifold of $\mathbb{R}^2$). Since $\mathcal{C}$ is locally the graph of a function, we may assume, without loss of generality, that

$$\mathcal{C} = \left\{ (x_1,\, g(x_1)) \;\middle|\; g \in C^1\left(\mathbb{R};\, \mathbb{R}\right) \right\}.$$

The 1-dimensional Hausdorff measure of the intersection $K \cap \mathcal{C}$ may be computed by parts, and thus we can use the assumption on the Lebesgue measure to obtain

$$\mathcal{H}^1\left(K \cap \mathcal{C}\right) \leq \mathcal{H}^1\left(\left(K_1 \times \mathbb{R}\right) \cap \mathcal{C}\right) = \int_{K_1} \sqrt{1 + \dot{g}^2}\, \mathrm{d}t = 0.$$

Therefore, we conclude that K is purely 1-unrectifiable as a consequence of Proposition 8.5.

Step 2. If $K_1 = K_2$ are self-similar fractals (in Hutchinson sense) of dimension d', then the product $K = K_1 \times K_2$ has Hausdorff dimension $2d'$.

Indeed, the reader may prove that for any two separable metric spaces X and Y with Y totally bounded that

$$\dim_{\mathcal{H}} X + \dim_{\mathcal{H}} Y \le \dim_{\mathcal{H}} (X \times Y) \le \dim_{\mathcal{H}} X + \dim_{\mathcal{B}} Y,$$

where $\dim_{\mathcal{B}} Y$ denotes the upper box counting dimension of Y (see, e.g. [8]). In particular, if Y has equal Hausdorff and upper box counting dimension (which holds if Y is a compact interval) then

$$\dim_{\mathcal{H}} X + \dim_{\mathcal{H}} Y = \dim_{\mathcal{H}} (X \times Y)\,.$$

Step 3. In conclusion, in Section 2.6 we have proved that there is a Cantor-type set of every dimension $d \in (0,\,1)$. Hence, for every $d \in (0,\,1)$, the product $\mathcal{C}_d \times \mathcal{C}_d$ is a purely unrectifiable set of dimension $2d \in (0,\,2)$, which is exactly what we wanted to exhibit. $\square$

Proposition 8.7. *Let $E \subset X$ be a Borel set with finite Hausdorff measure $\mathcal{H}^d(E) < +\infty$. Then E is the union of a d-rectifiable part and a purely d-unrectifiable part, that is,*

$$E = E_r \cup E_{pu}.$$

Proof. Let us consider the family

$$\mathcal{F} = \{F \subset E \mid F \text{ is a } d\text{-rectifiable subset}\}\,.$$

Let $m := \sup\left\{\mathcal{H}^d(F) \mid F \in \mathcal{F}\right\}$, and let $(F_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be a maximizing sequence, that is,

$$\mathcal{H}^d(F_n) \nearrow m.$$

If we set

$$E_r := \bigcup_{n\in\mathbb{N}} F_n,$$

then it turns out that $E_r \in \mathcal{F}$ (being rectifiable is closed under countable unions), and the Hausdorff dimension of E_r is exactly m. In conclusion, we set

$$E_{pu} := E \setminus E_r,$$

and we prove[1], by contradiction, that E_{pu} is purely d-unrectifiable.

Indeed, suppose that $E' := E \setminus E_r$ is not purely d-unrectifiable. Let us consider the family

$$\mathcal{F}' = \{F \subset E' \mid F \text{ is a } d\text{-rectifiable subset}\}\,,$$

which is not empty by assumption. Let $m' := \sup\{\dim_{\mathcal{H}} F \mid F \in \mathcal{F}'\}$, and let $(F_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be a maximizing sequence, that is,

$$\mathcal{H}^d(F_n) \nearrow m'.$$

If we set

$$E'_r := \bigcup_{n\in\mathbb{N}} F_n,$$

then it turns out that $E_r \in \mathcal{F}'$, and the Hausdorff dimension of E'_r is exactly m'. In particular, it turns out that

$$\widetilde{E_r} := E_r \cup E'_r$$

is a d-rectifiable set with Hausdorff dimension $\ge m$; thus we have derived a contradiction with the maximality of E_r. $\square$

[1] The reader may prove this assertion as an exercise.

Characterization of d-rectifiable sets in $\mathbb{R}^n$. In this final paragraph, we state two fundamental criteria for rectifiable sets (but we do not give the proof of them).

Theorem 8.8 (Marstrand-Preiss)**.** *Let $E \subseteq \mathbb{R}^n$ be a Borel set. If $\mathcal{H}^d(E) < +\infty$ and*

$$\lim_{r\to 0} \frac{\mathcal{H}^d\left(E \cap B(x,\, r)\right)}{\alpha_d r^d} \neq 0,\, \infty$$

exists for $\mathcal{H}^d$-almost every $x \in E$, then d is an integer, and E is a d-rectifiable set.

Remark 8.3. If $E \subseteq \mathbb{R}^n$ is a d-rectifiable set with positive Hausdorff measure $\mathcal{H}^d(E) > 0$, then for almost every $V \in G(n,\, d)$ it turns out that

$$\mathcal{H}^d\left(P_V(E)\right) > 0.$$

Theorem 8.9 (Besicovitch-Federer)**.** *Let $E \subseteq \mathbb{R}^n$ be a Borel set. If E is purely d-unrectifiable, then*

$$\mathcal{H}^d\left(P_V(E)\right) = 0$$

for $\gamma_{n,\,d}$-almost every $V \in G(n,\, d)$.

Proof. This result is the main topic of my seminary. I will try to upload some written notes about it after the exam. □

8.2 Tangent Bundle

In this section, we want to introduce a weaker notion of the tangent bundle for rectifiable sets. From now on, we will denote by E a d-rectifiable set in $\mathbb{R}^n$ unless stated otherwise.

Proposition 8.10. *Let Σ and Σ' be d-dimensional surfaces of class C^1. Then the tangent planes are equal at almost every point in the intersection $x \in \Sigma \cap \Sigma'$, that is,*

$$T_x\,\Sigma = T_x\,\Sigma'.$$

The proof of this assertion is a mere consequence of the following Lemma since the set of all points $x \in \Sigma \cap \Sigma'$ such that Σ is transversal to Σ' (i.e., where the tangent planes do not coincide) is negligible for the Hausdorff/Lebesgue measure.

Lemma 8.11. *Let $f,\, \widetilde{f} : \mathbb{R}^d \longrightarrow \mathbb{R}$ be functions of class $C^1(\mathbb{R}^d)$. Then*

$$\nabla f(x) = \nabla \widetilde{f}(x)$$

for $\mathcal{L}^d$-almost every point $x \in \mathbb{R}^d$ such that $f(x) = \widetilde{f}(x)$.

Proof. We may always assume without loss of generality that $\widetilde{f} \equiv 0$. Let K denote the set of the zeros of f, and let $x \in K$ be a density point, that is,

$$\lim_{r\to 0} \frac{\mathcal{H}^d\left(K \cap B(x,\, r)\right)}{\alpha_d\, r^d} = 1.$$

Suppose that $\nabla f(x) \geq \epsilon > 0$. By continuity, there exist a positive constant $\delta_i > 0$ and a point $y_i \in \mathbb{R}^n$ such that $f(y_i) \neq 0$ and $y_i \in B(x,\, \delta_i)$. In particular, by choosing a suitable sequence $\delta_i \searrow 0$, it turns out that

$$B\left(y_i,\, \frac{\delta_i}{2}\right) \cap K = \varnothing.$$

We are now ready to derive a contradiction. The Hausdorff density of K at x cannot be equal to one because there is a sequence of balls shrinking down to x that does not intersect K. More precisely, it turns out that

$$\lim_{i\to+\infty}\frac{\mathcal{H}^d\left(K^c\cap B(y_i,\,r_i)\right)}{\alpha_d\,r_i^d}>0\implies\lim_{r\to 0}\frac{\mathcal{H}^d\left(K\cap B(x,\,r)\right)}{\alpha_d\,r^d}<1.$$

□

Definition 8.12 (Tangent Bundle)**.** Let E be a Borel d-rectifiable set. A map T from E to the Grassmannian manifold $G(n,\,d)$ that sends x to $T(x)$ is a *weak tangent bundle* for the set E if and only if for every Σ d-dimensional surface of class C^1 it turns out that

$$T_x\,\Sigma=T(x)$$

for $\mathcal{H}^d$-almost every $x\in\Sigma\cap E$.

Proposition 8.13. *A d-rectifiable Borel set $E\subset\mathbb{R}^n$ admits a "unique" - up to $\mathcal{H}^d$ negligible sets - weak tangent bundle.*

Proof. By Proposition 8.2 there exists a collection of Borel subsets $\{E_i\}_{i\in\mathbb{N}}$ such that

$$E=\bigcup_{i=0}^{+\infty}E_i,$$

satisfying the following properties:

(1) The set E_0 is $\mathcal{H}^d$-null.

(2) For every $i\geq 1$ there is a d-dimensional surface $\Sigma_i\subset\mathbb{R}^n$ of class C^1 such that $E_i\subset\Sigma_i$.

Therefore, we set

$$T(x):=\begin{cases}T_x\,\Sigma_1 & x\in E_1,\\ T_x\,\Sigma_2 & x\in E_2\setminus E_1,\\ \dots,\end{cases}$$

and, for every point

$$x\notin\bigcup_{i\geq 1}E_i,$$

we set $T(x)$ to be equal to any d-dimensional vector space. The reader may easily check that T is the sought map since it is unique up to the negligible set $\left(\bigcup_{i\geq 1}E_i\right)^c$. □

Approximate Tangent. In this brief paragraph, we introduce an even weaker notion called *approximate tangent plane*. From now on, we shall assume that E is a d-rectifiable Borel set, which is locally $\mathcal{H}^d$-finite, that is for every $p\in E$ there is an open neighborhood $U_p\ni p$ such that

$$\mathcal{H}^d(U_p)<+\infty.$$

Definition 8.14 (Cone)**.** Let α be a fixed angle, let $x\in\mathbb{R}^n$ be a point and let V be a d-dimensional plane in $\mathbb{R}^n$. The cone of angle α around V centered at x is defined by setting

$$\mathcal{C}\left(x,\,V,\,\alpha\right):=\left\{x'\in\mathbb{R}^n\;:\;|x'-x|\cdot\sin(\alpha)\geq d(x-x',\,V)\right\}.$$

Definition 8.15 (Strong Tangent Plane)**.** Let $V \in G(n, d)$ be a d-dimensional plane. If E is a Borel set and $x \in E$ a point, then V is a *strong tangent plane* to E at x if and only if for every $\alpha > 0$ there exists a positive radius $r_0 > 0$ such that

$$E \cap B(x, r) \subseteq \mathcal{C}(x, V, \alpha) \quad \text{for every } r \leq r_0.$$

Definition 8.16 (Approximate Tangent Plane)**.** Let $V \in G(n, d)$ be a d-dimensional plane. If E is a Borel set and $x \in E$ a point, then V is an *approximate tangent plane* to E at x if and only if for every $\alpha > 0$ it turns out that

$$\mathcal{H}^d\left((E \cap B(x, r)) \setminus \mathcal{C}(x, V, \alpha)\right) = o(r^d), \tag{8.1}$$

and

$$\mathcal{H}^d\left((E \cap B(x, r)) \cap \mathcal{C}(x, V, \alpha)\right) \sim \alpha_d r^d. \tag{8.2}$$

Theorem 8.17. *Let $E \subseteq \mathbb{R}^n$ be a Borel set. If E is a d-rectifiable $\mathcal{H}^d$-locally finite set, then the weak tangent bundle $T(x)$ is the approximate tangent plane to E at x for $\mathcal{H}^d$-almost every $x \in E$.*

Proof. We divide the argument into different steps to highlight what we are doing here.

Step 1. First, observe that the thesis is equivalent to the following assertion: *For every $i \geq 1$ the vector space $T_x \Sigma_i$ is the approximate tangent plane to E at x for $\mathcal{H}^d$-almost every $x \in E \cap \Sigma_i$.*

Step 2. Let us define the measures

$$\lambda := \mathbb{1}_{\Sigma_i} \cdot \mathcal{H}^d,$$

$$\mu' := \mathbb{1}_{\Sigma_i \setminus E} \cdot \mathcal{H}^d,$$

$$\mu'' := \mathbb{1}_{E \setminus \Sigma_i} \cdot \mathcal{H}^d.$$

Fix $\alpha > 0$ and $x \in \Sigma_i$, and let $V := T_x \Sigma_i$ be the d-dimensional tangent plane. By construction, it turns out that

$$\begin{aligned} \mathcal{H}^d\left((E \cap B(x, r)) \cap \mathcal{C}(x, V, \alpha)\right) = \lambda\left(B(x, r) \cap \mathcal{C}(x, V, \alpha)\right) - \mu'\left(B(x, r) \cap \mathcal{C}(x, V, \alpha)\right) + \\ \cdots + \mu''\left(B(x, r) \cap \mathcal{C}(x, V, \alpha)\right) \end{aligned} \tag{8.3}$$

since the restriction $\mathcal{H}^d \llcorner E$ is equal to $\lambda - \mu' + \mu''$.

Step 3. By definition, as r goes to 0

$$\lambda\left(B(x, r) \cap \mathcal{C}(x, V, \alpha)\right) \sim \mathcal{H}^d\left(E \cap B(x, r)\right),$$

and thus

$$\lambda\left(B(x, r) \cap \mathcal{C}(x, V, \alpha)\right) \sim \alpha_d r^d \quad \text{for } r \text{ sufficiently small.}$$

Step 4. We now claim that

$$\mu' \left(B(x,\, r) \cap \mathcal{C}\left(x,\, V,\, \alpha\right)\right) = o(r^d)$$

for $\mathcal{H}^d$-almost every $x \in E \cap \Sigma_i$. Indeed, we observe that

$$\mu' \left(B(x,\, r) \cap \mathcal{C}\left(x,\, V,\, \alpha\right)\right) \leq \mathcal{H}^d \left(B(x,\, r) \cap (\Sigma_i \setminus E)\right),$$

and thus it suffices to show[2] that the density of $\Sigma_i \setminus E$ is 0 at almost every point of E with respect to the measure λ (which is, by the way, rather intuitive). Indeed, if

$$\Theta\left(\Sigma_i \setminus E,\, \lambda,\, x\right) = \frac{\mathcal{H}^d \left(B(x,\, r) \cap (\Sigma_i \setminus E)\right)}{\mathcal{H}^d \left(\Sigma_i \cap B(x,\, r)\right)} = 0,$$

then the numerator must be an element of the class $o(r^d)$ since the denominator is $\sim \alpha_d r^d$.

Step 5. In a similar fashion, we claim that

$$\mu'' \left(B(x,\, r) \cap \mathcal{C}\left(x,\, V,\, \alpha\right)\right) = o(r^d)$$

for $\mathcal{H}^d$-almost every $x \in E \cap \Sigma_i$. This is an easy consequence of the inequality

$$\mu'' \left(B(x,\, r) \cap \mathcal{C}\left(x,\, V,\, \alpha\right)\right) \leq \mu'' \left(B(x,\, r)\right).$$

Indeed, the right-hand side is an element of the class $o(r^d)$ since μ'' is orthogonal to λ by construction, and thus the density with respect to λ is zero for λ-almost every x. More precisely,

$$\frac{\mathrm{d}\mu''}{\mathrm{d}\lambda}(x) = \lim_{r \to 0} \frac{\mathcal{H}^d \left(B(x,\, r)\right)}{\lambda \left(B(x,\, r)\right)} = 0,$$

and the denominator is $\sim \alpha_d r^d$ as in the previous step.

Step 6. In the same spirit, the reader may prove that

$$\mathcal{H}^d \left((E \cap B(x,\, r)) \setminus \mathcal{C}\left(x,\, V,\, \alpha\right)\right) = o(r^d)$$

using the same decomposition introduced in (8.3). ☐

Exercise 8.3. Let Σ be a line in $\mathbb{R}^2$, and let

$$E := \Sigma \cup \left(\bigcup_{n \in \mathbb{N}} \partial B(x_n,\, r_n) \right),$$

where $\{x_n\}_{n \in \mathbb{N}}$ is a dense countable collection of points in $\mathbb{R}^2$ and $(r_n)_{n \in \mathbb{N}}$ is a summable family of positive real numbers. Prove that:

1) The set E is d-rectifiable.

2) The set E is locally $\mathcal{H}^1$-finite.

3) For almost every $x \in \Sigma$ it turns out that

$$\mathcal{H}^1 \left((E \setminus \Sigma) \cap B(x,\, r)\right) = o(r).$$

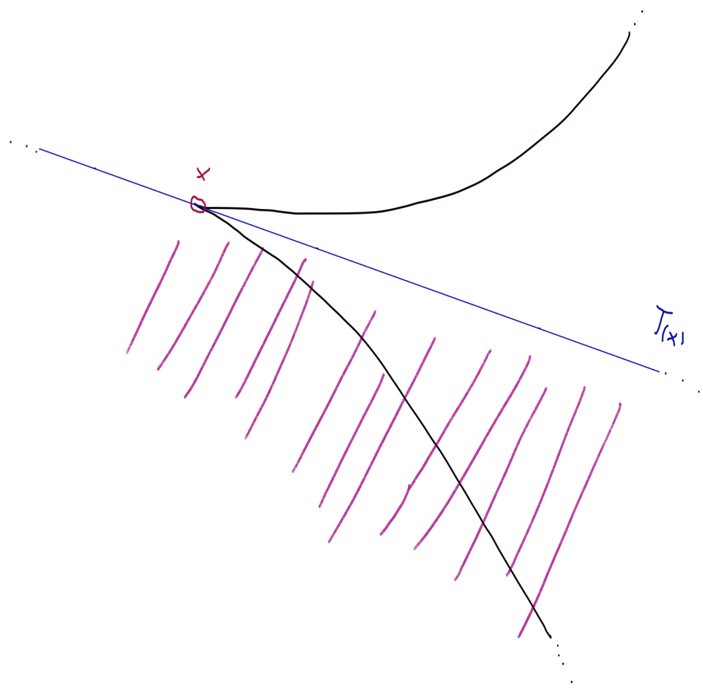

Figure 8.1: The mass is distributed only on one side of the approximate tangent plane.

Remark 8.4. According to the statement of Theorem 8.17, it might happen that the mass is distributed only on one side of the tangent plane (see Figure 8.1).

To prove that the statement can be improved (i.e., the situation described above cannot happen), we need to introduce a different definition of *approximate tangent plane*, which is easier to deal with, and then prove that it is stronger than Definition 8.16.

Definition 8.18 (Approximate Tangent Plane)**.** Let $\psi_{x,\,r} : B(x,\,r) \longrightarrow B(0,\,1)$ be the magnifying glass map, that is,

$$\psi_{x,\,r}(x') := \frac{x'-x}{r},$$

and let $E_{x,\,r}$ be the image of E via $\psi_{x,\,r}$. An element V of the Grassmannian manifold $G(n,\,d)$ is an *approximate tangent plane* to the set E at the point x if and only if

$$\mathbb{1}_{E_{x,\,r}} \cdot \mathcal{H}^d \rightharpoonup \mathbb{1}_V \cdot \mathcal{H}^d$$

locally weakly-$*$, that is,

$$\int_{E_{x,\,r}} \varphi(t)\,\mathrm{d}\mathcal{H}^d(t) \xrightarrow{r\to 0} \int_V \varphi(t)\,\mathrm{d}\mathcal{H}^d(t), \qquad \forall\,\varphi \in C^0_c(\mathbb{R}^d).$$

Remark 8.5. If V is an approximate tangent plane to E at x in the sense of Definition 8.18, then V is an approximate tangent plane to E at x in the sense of Definition 8.16.

[2]This assertion is left as a simple exercise for the reader.

Proof. Let us consider the following measures:

$$\mu_{x,\,r} := \mathbb{1}_{E_{x,\,r}} \cdot \mathcal{H}^d \qquad \text{and} \qquad \mu_x := \mathbb{1}_V \cdot \mathcal{H}^d.$$

The d-dimensional Hausdorff measure of $\partial B(0,\,1) \cap V$ is equal to zero since $B(0,\,1)\cap V$ is a d-dimensional set (which means that the boundary has dimension $d-1$); hence

$$\mu_{x,\,r} \rightharpoonup \mu_x \implies \mu_{x,\,r}\left(B(0,\,1)\right) \xrightarrow{r\to 0} \mu_x\left(B(0,\,1)\right),$$

as a consequence of Proposition 2.7. By definition, we have $\mu_x\left(B(0,\,1)\right) = \alpha_d$ and

$$\mu_{x,\,r}\left(B(0,\,1)\right) = \mathcal{H}^d\left(E_{x,\,r}\cap B(0,\,1)\right) = \frac{1}{r^d}\,\mathcal{H}^d\left(E \cap B(x,\,r)\right),$$

which implies a condition weaker than **(b)**, that is

$$\mathcal{H}^d\left(E\cap B(x,\,r)\right) \sim \alpha_d r^d \quad \text{as } r \to 0^+.$$

In a similar fashion, it turns out that

$$\mu_{x,\,r}\left(B(0,\,1)\cap \mathcal{C}(0,\,V,\,\alpha)\right) \xrightarrow{r\to 0} \mu\left(B(0,\,1)\cap\mathcal{C}(0,\,V,\,\alpha)\right) = \alpha_d,$$

from which it follows that

$$\mathcal{H}^d\left(E\cap B(x,\,r)\cap\mathcal{C}(0,\,V,\,\alpha)\right) \sim \alpha_d r^d \quad \text{as } r\to 0^+,$$

that is, the condition **(b)** holds. The reader may prove, in a similar way, that the condition **(a)** also holds true. □

Theorem 8.19. *Let $E \subseteq \mathbb{R}^n$ be a Borel set. If E is d-rectifiable and locally $\mathcal{H}^d$-finite, then $T(x)$ is the approximate tangent plane (Definition 8.18) to E at x for $\mathcal{H}^d$-almost every $x \in E$.*

Sketch of the Proof. We divide the argument into different steps to ease the notation.

Step 1. First, observe that the thesis is equivalent to the following assertion: *For every $i \geq 1$ the vector space $T_x\Sigma_i$ is the approximate tangent plane to E at x for $\mathcal{H}^d$-almost every $x \in E\cap\Sigma_i$.*

Step 2. Fix $i \geq 1$ and let $\Sigma_{x,\,r}$ be the image of Σ_i via $\psi_{x,\,r}$. We consider the measures

$$\lambda_{x,\,r} := \mathbb{1}_{\Sigma_{x,\,r}} \cdot \mathcal{H}^d,$$

$$\mu'_{x,\,r} := \mathbb{1}_{\Sigma_{x,\,r}\setminus E_{x,\,r}} \cdot \mathcal{H}^d,$$

$$\mu''_{x,\,r} := \mathbb{1}_{E_{x,\,r}\setminus \Sigma_{x,\,r}} \cdot \mathcal{H}^d,$$

in such a way that the following decomposition holds:

$$\mu_{x,\,r} = \lambda_{x,\,r} - \mu'_{x,\,r} + \mu''_{x,\,r}.$$

Step 3. The reader may prove that

$$\lambda_{x,\,r} \rightharpoonup \mathbb{1}_V \cdot \mathcal{H}^d,$$

locally weakly-$*$, using the fact that the projection $\pi_{x,\,r} : \sigma_{x,\,r} \longrightarrow V$ is an almost-isometry.

Step 4. In a similar way to what we have done in the proof of Theorem 8.17, one can check that

$$\mu'_{x,\,r} \rightharpoonup 0 \iff \mu'_{x,\,r}\,(B_R) \xrightarrow{r\to 0} 0 \quad \text{for any fixed radius } R > 0$$

as a consequence of Proposition 2.7. We now observe that

$$\mu'_{x,\,r}\,(B(0,\,1)) = \mathcal{H}^d\,(B(0,\,1)\cap(\Sigma_{x,\,r}\setminus E_{x,\,r})) = \frac{1}{r^d}\mathcal{H}^d\,(B(x,\,r)\cap(\Sigma_i\setminus E)) \xrightarrow{r\to 0} 0$$

since $\mathcal{H}^d\,(B(x,\,r)\cap(\Sigma_i\setminus E))$ is an element of the class $o(r^d)$, as proved in Theorem 8.17.

Step 5. Similarly, by Proposition 2.7 it follows that

$$\mu''_{x,\,r} \rightharpoonup 0 \iff \mu''_{x,\,r}\,(B_R) \xrightarrow{r\to 0} 0 \quad \text{for any fixed radius } R > 0.$$

We now observe that

$$\mu''_{x,\,r}\,(B(0,\,1)) = \mathcal{H}^d\,(B(0,\,1)\cap(E_{x,\,r}\setminus\Sigma_{x,\,r})) = \frac{1}{r^d}\mathcal{H}^d\,(B(x,\,r)\cap(E\setminus\Sigma_i))\,,$$

thus it suffices to prove that the right-hand side goes to zero as $r\to 0^+$. Indeed, the limit of the ratio is the Radon-Nikodym density, which is zero as a consequence of the fact that the two measures are orthogonal:

$$\frac{1}{r^d}\mathcal{H}^d\,(B(x,\,r)\cap(E\setminus\Sigma_i)) \xrightarrow{r\to 0} \frac{\mathrm{d}\left(\mathbb{1}_{E\setminus\Sigma}\cdot\mathcal{H}^d\right)}{\mathrm{d}\left(\mathbb{1}_{\Sigma}\cdot\mathcal{H}^d\right)} = 0.$$

□

8.3 Rectifiability Criteria

The primary goal of this section is to relate d-rectifiable set with properties of the approximate tangent plane. More precisely, in this section we denote by $V\in G(n,\,d)$ a d-dimensional plane, by $E\subseteq\mathbb{R}^n$ a Borel set, and we fix a point $x\in\mathbb{R}^n$.

Definition 8.20 (Tangent Cone)**.** Fix $\alpha\in(0,\,\pi/2)$. The cone $\mathcal{C}(x,\,V,\,\alpha)$ is *tangent* to a set E at x if there exists a positive constant $r_0>0$ such that

$$B(x,\,r)\cap E\subseteq\mathcal{C}(x,\,V,\,\alpha), \qquad \forall\, r\le r_0.$$

Theorem 8.21. *Assume that E admits a tangent cone at every point $x\in E$. Then there are (at most) countably many Lipschitz functions $f_i:\mathbb{R}^d\longrightarrow\mathbb{R}^n$ such that*

$$E\subseteq\bigcup_{i\ge 1} f_i(\mathbb{R}^d).$$

In particular, the set E is d-rectifiable.

Proof. We first prove a particular case, and then we reduce the general case to it using a standard argument.

Step 1. Let us assume[3] that:

a) The d-dimensional plane V, the angle α and the radius r_0 do not depend on the point $x \in E$.

b) The linear space V is a straight d-dimensional plane.

We divide the ambient space in strips of thickness $r_0 \sin \alpha$, that is,

$$\mathbb{R}^n = \bigcup_{i \in \mathbb{Z}} S_i, \quad \text{with } |S_i| = r_0 \sin \alpha.$$

The set E is thus given by the union

$$E = \bigcup_{i \in \mathbb{Z}} (E \cap S_i),$$

which means that it is enough to prove that the intersection of E with each strip S_i is contained in the graph of a Lipschitz map, that is,

$$E \cap S_i \subset \Gamma(g_i),$$

where $g_i : V \longrightarrow V^\perp$ is Lipschitz[4]. By assumption, it turns out that (see Figure 8.2) the intersection of E with each strip is contained in the cone centered at any $x \in E \cap S_i$, that is,

$$E \cap S_i \subset \mathcal{C}(x, V, \alpha) \qquad \forall x \in E \cap S_i.$$

Therefore, if we denote by (x, y) the coordinates associated with the decomposition $V \oplus V^\perp = \mathbb{R}^n$, then one can prove that

$$|y' - y| \leq \tan(\alpha) \cdot |x' - x|,$$

which means that the y coordinate furnishes a Lipschitz map (see Figure 8.3).

Step 2. We are now ready to get rid of the extra assumptions, one by one.

(i) Assume that α and r_0 do not depend on the point $x \in E$, and fix an angle $\alpha' > \alpha$.

The cones of the form $\mathcal{C}(0, V, \alpha)$, as V ranges in the set of all admissible d-dimensional planes, are contained in a finite family $C(0, V_i, \alpha')$ of slightly ampler cones. We can reduce to the first step by splitting E in the finite union associated to the family above, that is,

$$E = \bigcup_{i=1}^{N} E_i,$$

with V_i fixed d-dimensional straight plane for each $i = 1, \ldots, N$.

(ii) Assume that the radius r_0 does not depend on the point $x \in E$.

For every $\alpha \in \mathbb{Q}$ there exists $\alpha'(\alpha) > \alpha$ such that the cones of the form $\mathcal{C}(0, V, \alpha)$, as V ranges in the set of all admissible d-dimensional planes, are contained in a given finite family $C(0, V_i, \alpha'(\alpha))$ of slightly ampler cones. We can reduce to the first step by splitting E in the (at most) countable union

$$E = \bigcup_{\alpha \in \mathbb{Q}} \left[\bigcup_{i=1}^{N_{\alpha'(\alpha)}} E_i \right].$$

[3] We will get rid of these extra assumptions later. It is important to understand that the argument presented in the first step only proves a particular case.

[4] The graph of the function g_i is a subset of $\mathbb{R}^n$ since g_i sends V to its orthogonal, and $V \oplus V^\perp = \mathbb{R}^n$.

(iii) Finally, if no extra assumption holds, then it suffices to consider the collection

$$E_r := \{x \in E \mid r_0(x) < r\}$$

and split E as follows

$$E = \bigcup_{r,\,\alpha \in \mathbb{Q}} \left[\bigcup_{i=1}^{N_{r,\,\alpha'(\alpha)}} E_{r,\,i} \right].$$

□

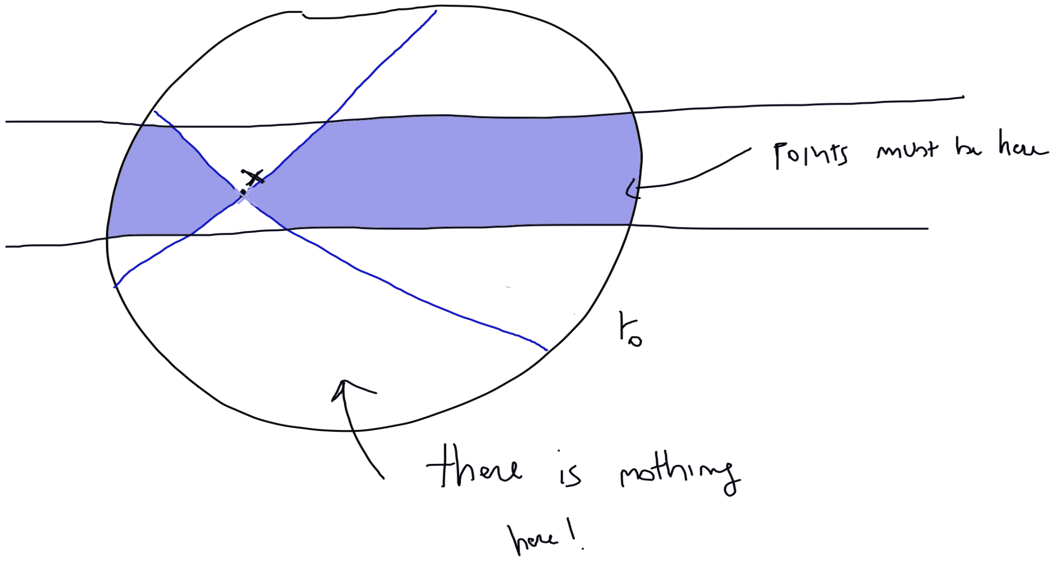

Figure 8.2: Sketch of the first step.

Definition 8.22 (Approximate Tangent Cone)**.** Fix $\alpha \in (0,\,\pi/2)$. The cone $\mathcal{C}(x,\,V,\,\alpha)$ is a *approximately tangent* to a set E at x if and only if

$$\mathcal{H}^d\left((B(x,\,r) \cap E) \setminus \mathcal{C}(x,\,V,\,\alpha)\right) = o(r^d).$$

Theorem 8.23. *Assume that E admits an approximate tangent cone at every point $x \in E$, and assume also that the lower density is bounded from below, that is,*

$$\Theta_*(E,\,x) > 0 \qquad \forall\, x \in E.$$

Then there are (at most) countably many Lipschitz functions $f_i : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ such that

$$E \subseteq \bigcup_{i \geq 1} f_i(\mathbb{R}^d).$$

In particular, the set E is d-rectifiable.

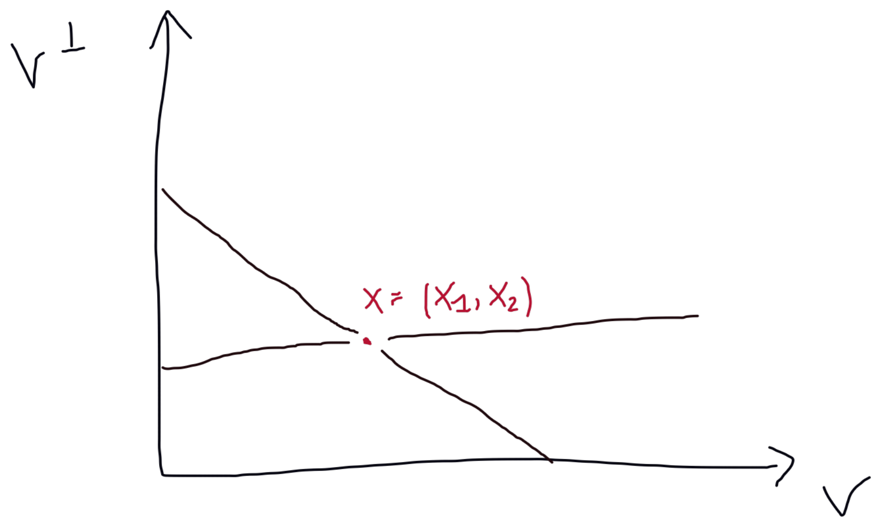

Figure 8.3: The coordinates give a Lipschitz map.

Corollary 8.24. *If E admits an approximate tangent cone at $\mathcal{H}^d$-almost every point $x \in E$, and the lower density is bounded from below, i.e.,*

$$\Theta_*(E, x) > 0 \qquad \textit{for } \mathcal{H}^d\textit{-almost every point } x \in E,$$

then the set E is d-rectifiable.

Proof of Theorem 8.23. We first prove a particular case, and then we reduce the general case to it using a standard argument.

Step 1.1. Let us assume that:

a) The d-dimensional plane V and the angle α are the same for every points $x \in E$.

b) There exist $r_0 > 0$ and $\delta > 0$ such that for every $x \in E$ it turns out that

$$\mathcal{H}^d \left(E \cap B(x, r) \right) \geq \delta \cdot r^d, \tag{8.4}$$

for every $r \leq r_0$.

First, we notice that the existence of an approximate tangent plane implies that for every $r \leq r_0$ it turns out that

$$\mathcal{H}^d \left(\left(E \cap B(x, r) \right) \setminus \mathcal{C} \left(x, V, \alpha \right) \right) \leq \delta' \cdot r^d, \tag{8.5}$$

where δ' is a positive constant that can be chosen arbitrarily. To prove the particular case, we first need to show that there exists a radius $\overline{r} > 0$ such that

$$E \cap \left(V \times B \left(x_0, \overline{r} \right) \right) \subset \Gamma(g),$$

where $x_0 \in V^\perp$ and $\Gamma(g)$ is the graph of a Lipschitz function $g : V \longrightarrow V^\perp$.

Step 1.2. Fix an angle $\alpha' \in (\alpha, \pi/2)$, and fix a point $x \in E$. We claim that

$$E \cap B\left(x_0, \frac{r_0}{2}\right) \subseteq \mathcal{C}\left(x, V, \alpha'\right).$$

We argue by contradiction. Suppose that there exists a point

$$x' \in \left(E \cap B\left(x_0, \frac{r_0}{2}\right)\right) \setminus \mathcal{C}\left(x, V, \alpha'\right),$$

and let $r, r' > 0$ be positive real numbers such that

$$B(x', r') \subset B(x, r) \qquad \text{and} \qquad B(x', r') \cap \mathcal{C}(x, V, \alpha) = \varnothing.$$

More precisely, let us consider the following radii:

$$r := 2\,|x - x'| \qquad \text{and} \qquad r' := |x - x'|\,\sin(\alpha' - \alpha).$$

The assumptions (8.4) and (8.5) proves that

$$\delta' \cdot r^d \geq \mathcal{H}^d\left((E \cap B(x, r)) \setminus \mathcal{C}\left(x, V, \alpha\right)\right) \geq \mathcal{H}^d\left(E \cap B(x', r')\right) \geq \delta \cdot (r')^d,$$

from which we can derive a contradiction by fixing δ' such that

$$\delta' < \delta \cdot \left(\frac{\sin(\alpha' - \alpha)}{2}\right)^d.$$

Step 1.3. In this final step, the goal is to clean the proof of the particular case by fixing the values of δ' and α' in such a way that the contradiction above holds. In particular, let us consider

$$\alpha' = \frac{2\,\alpha + \pi}{4},$$

and

$$\delta' = \frac{\delta}{2} \cdot \left(\frac{1}{2}\sin\left(\frac{\pi - 2\,\alpha}{4}\right)\right)^d.$$

In the previous step, we proved that

$$E \cap B\left(x_0, \frac{r_0}{2}\right) \subseteq \mathcal{C}\left(x, V, \alpha'\right),$$

and this is enough to infer that

$$E \cap \left(V \times B\left(x_0, \overline{r}\right)\right) \subseteq \Gamma(g),$$

where $g : V \longrightarrow V^{\perp}$ is a Lipschitz function of constant $\tan(\alpha')$, provided that the thickness of the cylinder strips $2\overline{r}$ is less or equal than $(r_0/2) \cdot \sin(\alpha')$, that is,

$$\overline{r} \leq \frac{r_0}{4} \sin\left(\frac{\pi + 2\alpha}{4}\right).$$

Step 2. In the general case, it is enough to consider the countable covering of E defined by setting

$$E_n := \left\{ x \in E \;\middle|\; (8.4) \text{ and } (8.5) \text{ holds for } r = \delta = \frac{1}{n} \right\}.$$

□

Corollary 8.25. *If E admits a tangent plane at every point $x \in E$, then the set E is d-rectifiable.*

8.4 Area Formula for Rectifiable Sets

Summary. Let Σ be a d-dimensional surface of class C^1, and let $f : \Omega \longrightarrow \Sigma$ be a Lipschitz function defined on an open subset $\Omega \subseteq \mathbb{R}^d$. In Section 7.3 we proved that for every Borel set $E \subseteq \Omega$ it turns out that

$$\int_{y\in\Sigma} m_{f,\,E}(y)\,\mathrm{d}\mathcal{H}^d(y) = \int_E Jf(x)\,\mathrm{d}\mathcal{L}^d(x), \tag{8.6}$$

where $m_{f,\,E}(y) := \#\left(f^{-1}(y)\cap E\right)$ and $Jf(x)$ is the Jacobian of f at x.

The formula presented here may be extend with little efforts to a more general setting. More precisely, let Σ' be a d-dimensional surface of class C^1, and let $f : \Sigma' \longrightarrow \Sigma$ be a Lipschitz function. The (area) formula

$$\int_\Sigma \left[\sum_{s\in f^{-1}(x)} h(s)\right] \mathrm{d}\mathcal{H}^d(x) = \int_{\Sigma'} h(x)Jf(x)\,\mathrm{d}\mathcal{L}^d \tag{8.7}$$

holds, and the proof is very similar[5] to the one presented in Section 7.3.

Rectifiable Sets. Let $E \subset \mathbb{R}^n$ be a d-rectifiable set, and let $f : E \longrightarrow f(E)$ be a Lipschitz function[6]. In this paragraph, we want to prove that for every Borel set $F \subseteq f(E)$ it turns out that

$$\int_{y\in F} m_{f,\,E}(y)\,\mathrm{d}\mathcal{H}^d(y) = \int_{f^{-1}(F)} Jf(x)\,\mathrm{d}\mathcal{L}^d(x), \tag{8.8}$$

where $Jf(x)$ denotes the Jacobian of f at x, that is

$$Jf(x) := \sqrt{\det\left(\nabla_t f(x)\right)^T\left(\nabla_t f(x)\right)}. \tag{8.9}$$

Definition 8.26 (Tangent Differential)**.** The function $f : E \longrightarrow X \subseteq \mathbb{R}^m$ is tangentially differentiable at $\mathcal{H}^d$-almost every $x \in E$ if and only if there exists a linear map $L : T_xE \longrightarrow \mathbb{R}^m$ such that

$$f(x') = f(x) + L\left(\pi(x - x')\right) + o(|x - x'|)$$

for every $x' \in E$, where π is the projection onto the tangent space T_xE.

Remark 8.6. If we consider a Lipschitz extension $\widetilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ of f, then f is tangentially differentiable at $x \in E$ if and only if there exists a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\widetilde{f}(x + h) = f(x) + Lh + o(|h|) \qquad \forall\, h \in T_xE.$$

Proof of Area Formula. Let $\{E_i\}_{i\geq 0}$ be given by the equivalent definition of rectifiable set (see Proposition 8.2), that is

$$E = \bigcup_{i=0}^{+\infty} E_i,$$

and the collection satisfies the following properties:

(a) The d-dimensional Hausdorff measure of E_0 is zero.

[5] The reader may try to fill in the details as an exercise. Indeed, it is enough to take a compatible atlas for the surface Σ' and use the previous result (8.6) on each chart $\varphi : U \xrightarrow{\sim} V$. To conclude, the reader may use a suitable partition of unity and prove that the formula glues as expected.

[6] The codomain is not important since the image of E via f is always a d-rectifiable set.

(b) The set E_i is contained in a surface of class C^1, denoted by Σ_i, for every $i \geq 1$.

(c) The restriction $f\big|_{E_i}$ may be extended to a differentiable function $F_i : \Sigma_i \longrightarrow \mathbb{R}^m$.

(d) The set E_i is equal to the disjoint union of E_i' and E_i'', where E_i' is the set of points $x \in E_i$ where the rank of $\mathrm{d}f_x$ is not maximal, and E_i'' is the set of points where it is maximal.

For every $i \geq 1$, the function $F_i : \Sigma_i \longrightarrow \Sigma_i'$ is differentiable, and thus the area formula holds for every $x \in F_i^{-1}(\Sigma_i') \cap E_i''$:

$$\int_{\Sigma_i'} \left[\sum_{s \in F_i^{-1}(x) \cap S_i''} h(s) \right] \mathrm{d}\mathcal{H}^d(x) = \int_{S_i''} h(x) Jf(x) \, \mathrm{d}\mathcal{L}^d(x). \tag{8.10}$$

In a similar fashion, the area formula also holds for every $x \in F_i^{-1}(\Sigma_i') \cap E_i'$ since the Jacobian is zero at every point, and the image of a $\mathcal{H}^d$-null set via a differentiable function is also $\mathcal{H}^d$-null.

Remark 8.7. Recall that, if two Lipschitz functions agree on a subset, then they have the same tangential gradient at almost every x in the intersection (see Lemma 8.11).

The area formula introduced in this final section can be slightly improved. Indeed, we do not need to consider a Lipschitz function f, but it is enough to require the following properties:

(i) The function f is differentiable almost everywhere.

(ii) The function f sends $\mathcal{H}^d$-null sets into $\mathcal{H}^d$-null sets.

Exercise 8.4. Prove that for every $d \in (0, +\infty)$ there exists a set E such that $\mathcal{H}^d(E)$ is finite and nonzero, but the lower density is zero:

$$\Theta_*(E, x) = 0 \qquad \text{for } \mathcal{H}^d\text{-almost every } x \in E.$$

Chapter 9

Bounded Variation Functions

In this brief chapter, we introduce the notion of *bounded variation* functions, and we investigate some of the main properties (extension, approximation, compactness, trace, etc.)

Notation. In this chapter, we denote by $|\cdot|$ the Lebesgue measure $\mathcal{L}^n(\cdot)$, and we denote by $\mathrm{d}x$ the differential $\lceil \mathcal{L}^n(x)$, unless it is necessary to indicate the dimension.

9.1 Definition and Elementary Properties

Definition 9.1 (Bounded Variation)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A function $u : \Omega \longrightarrow \mathbb{R}$ is of bounded variation on Ω, and we denote it by $u \in \mathrm{BV}(\Omega)$, if and only if the following properties hold:

(i) The function u belongs to $L^1(\Omega)$.

(ii) There exists a vector-valued measure $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{M}(\Omega; \mathbb{R}^n)$ such that

$$\int_\Omega \frac{\partial \varphi}{\partial x_i}(x) u(x) \,\mathrm{d}x = - \int_\Omega \varphi(x) \,\mathrm{d}\mu_i(x), \qquad \forall \varphi \in C_c^\infty(\Omega) \tag{9.1}$$

for every $i = 1, \dots, n$.

Remark 9.1. Let $u \in L^1(\Omega)$ be a summable function, and let us denote by Du the weak derivative (in the sense of distributions). Then the condition (9.1) is equivalent to requiring that u is weakly differentiable and Du is $\mathbb{R}^n$-valued measure.

Remark 9.2. In a similar fashion, the condition (9.1) is equivalent to the existence of a positive real-valued measure λ and a vector-valued function $\tau : \Omega \longrightarrow \mathbb{R}^n$ such that

$$\int_\Omega \frac{\partial \varphi}{\partial x_i}(x) u(x) \,\mathrm{d}x = - \int_\Omega \varphi(x) \tau_i(x) \,\mathrm{d}\lambda(x), \qquad \forall \varphi \in C_c^\infty(\Omega) \tag{9.2}$$

for every $i = 1, \dots, n$.

There is an equivalent definition of the set $\mathrm{BV}(\Omega)$ that relies more on the vector-valued structure of $\mathbb{R}^n$ (that is, it is not equivalent in a generic metric space.)

Definition 9.2 (Bounded Variation)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A function $u : \Omega \longrightarrow \mathbb{R}$ is of bounded variation on Ω, and we denote it by $u \in \mathrm{BV}(\Omega)$, if and only if the following properties hold:

(i) The function u belongs to $L^1(\Omega)$.

(ii) There exists a vector-valued measure $\mu \in \mathcal{M}(\Omega;\, \mathbb{R}^n)$ such that

$$\int_\Omega \operatorname{div}(\varphi)(x) u(x)\, \mathrm{d}x = -\int_\Omega \varphi(x)\, \mathrm{d}\mu(x), \qquad \forall\, \varphi \in C_c^\infty(\Omega) \tag{9.3}$$

for every $i = 1, \ldots, n$.

Remark 9.3. In a similar fashion, the condition (9.3) is equivalent to the existence of a positive real-valued measure λ and a vector-valued function $\tau : \Omega \longrightarrow \mathbb{R}^n$ such that

$$\int_\Omega \operatorname{div}(\varphi)(x) u(x)\, \mathrm{d}x = -\int_\Omega \varphi(x) \cdot \tau(x)\, \mathrm{d}\lambda(x), \qquad \forall\, \varphi \in C_c^\infty(\Omega) \tag{9.4}$$

for every $i = 1, \ldots, n$, where $\cdot$ denotes the scalar product between two $\mathbb{R}^n$ vectors.

Exercise 9.1. Let $\Omega \subseteq \mathbb{R}^n$. A function u satisfies (9.1) if and only if it satisfies (9.3), i.e., the given definitions are actually equivalent.

Remark 9.4. The space $\mathrm{BV}(\Omega)$, endowed with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega), \tag{9.5}$$

is a non-separable Banach space.

Proof. The space $\big(\mathrm{BV}(\Omega),\, \|\cdot\|_{\mathrm{BV}(\Omega)}\big)$ is clearly a Banach space, as it can be checked quickly by the reader; thus it is enough to prove that it is not separable.

We prove that $\mathrm{BV}([0,\,1])$ is not separable since the general case follows from a similar argument. For every $\alpha \in (0,\,1)$ let us consider the characteristic function

$$\mathbb{1}_\alpha := \mathbb{1}_{[\alpha,\,1]}.$$

For every choice $\alpha \neq \beta \in [0,\,1]$, it turns out that

$$\|\mathbb{1}_\alpha - \mathbb{1}_\beta\|_{\mathrm{BV}([0,\,1])} = 2 + |\alpha - \beta|,$$

and hence we can consider the family of balls

$$B_\alpha := \left\{ \psi \in \mathrm{BV}([0,\,1]) \;\middle|\; \|\psi - \mathbb{1}_\alpha\|_{\mathrm{BV}([0,\,1])} \leq 1 \right\}.$$

This collection of balls is indexed by the interval $[0,\,1]$, which means that it is a family with the cardinality of the continuum. Therefore we can infer that $\mathrm{BV}([0,\,1])$ is not separable since each dense subset must intersect every ball B_α in at least one point, which means that any dense subset has a cardinality bigger or equal than the continuum. □

Remark 9.5. The measure μ given by (9.1) is unique (since the weak derivative in the sense of distribution is unique, up to set of measure zero). We shall denote it by Du through the entire chapter since the symbol ∇u is reserved for the classical/pointwise gradient.

Lemma 9.3. *Let $u \in L^1(\Omega)$ be a summable function. Then u belongs to $\mathrm{BV}(\Omega)$ if and only if the induced linear functional, defined by*

$$\Lambda_u : C_c^\infty(\Omega) \longrightarrow \mathbb{R}, \qquad \varphi \longmapsto \int_\Omega \operatorname{div}(\varphi)(x) u(x)\, \mathrm{d}x,$$

is bounded with respect to the uniform norm $\|\cdot\|_\infty$.

Proof.

" $\implies$ " Assume that $u \in \mathrm{BV}(\Omega)$. The absolute value of the functional can be easily estimated using the identity (9.3) as follows:

$$\left| \int_\Omega \mathrm{div}\,(\varphi)\,(x)\,u(x)\,\mathrm{d}x \right| = \left| - \int_\Omega \varphi(x)\,\mathrm{d}\mu(x) \right| \leq \|\varphi\|_\infty |\mu|(\Omega),$$

where $|\mu|(\Omega)$ denotes the total variation. In particular, the linear functional Λ_u is bounded with respect to the uniform norm, and thus it can be extended up to the closure of the domain, that is,

$$\Lambda_u : \overline{C_c^\infty(\Omega)} = C_0^0(\Omega) \longrightarrow \mathbb{R},$$

where $C_0^0(\Omega)$ denotes the set of all continuous functions on Ω that are infinitesimal on the boundary $\partial\Omega$. Notice that the identity (9.3) holds for every function $\varphi \in C_0^1(\Omega)$ since the divergence operator is not defined on the set of all continuous functions.

" $\impliedby$ " Vice versa, assume that $u \in L^1(\Omega)$ and Λ_u is a bounded functional with respect to the uniform norm. From the Riesz Representation Theorem 2.3 one can find a measure $\mu \in \mathcal{M}\,(\Omega;\,\mathbb{R}^n)$ such that

$$\int_\Omega \mathrm{div}(\varphi)(x)\,u(x)\,\mathrm{d}x =: \Lambda_u(\varphi) = \int_\Omega \varphi(x)\,\mathrm{d}\mu(x).$$

In particular, the function u satisfies the property (9.3), that is u is weakly differentiable (in the sense of distributions) and its weak derivative is equal to $-\mathrm{d}\mu$, accordingly with the definition of bounded variation functions. □

Example 9.1.

1) The Sobolev space $W^{1,1}(\Omega)$ is contained in $\mathrm{BV}(\Omega)$ for every bounded set $\Omega \subset \mathbb{R}^n$.

Indeed, let $f \in W^{1,1}(\Omega)$ be a Sobolev function, and denote by $g \in L^1(\Omega)$ its weak derivative. It suffices to show that the distribution associated with g is a vector-valued measure in $\mathcal{M}\,(\Omega;\,\mathbb{R}^n)$ satisfying (9.3) holds. By definition of weak derivative

$$\int_\Omega \mathrm{div}\,(\varphi)\,(x)u(x)\,\mathrm{d}\mathcal{L}^n(x) = -\int_\Omega \varphi(x)g(x)\,\mathrm{d}\mathcal{L}^n(x), \qquad \forall\,\varphi \in C_c^\infty(\Omega),$$

thus it is enough to set $\mu := g \cdot \mathcal{L}^n$, which is well defined because g is a summable function.

2) If $E \subseteq \Omega$ is an open bounded set with differentiable (C^1 is enough) boundary, then the characteristic function $\mathbb{1}_E$ belongs to $\mathrm{BV}\,(\mathbb{R}^n)$.

The distributional derivative is zero on both $\mathrm{Int}(E)$ and $\Omega \setminus \overline{E}$; hence it is a measure supported on the boundary ∂E. More precisely, one can check that

$$D\mathbb{1}_E = \nu_i\,\mathbb{1}_{\partial E} \cdot \mathcal{H}^{n-1},$$

where ν_i denotes the inner normal vector. Indeed, for any $\varphi \in C_c^\infty\,(\mathbb{R}^n;\,\mathbb{R}^n)$ it turns out that

$$\int_{\mathbb{R}^n} \mathrm{div}(\varphi)(x)\mathbb{1}_E(x)\,\mathrm{d}x = \int_E \mathrm{div}(\varphi)(x)\,\mathrm{d}x = -\int_{\partial E} \varphi(x)\nu_i(x)\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

where the red identity follows from the divergence theorem.

Remark 9.6. The characteristic $\mathbb{1}_E$ introduced above is the first example of a function of bounded variation on $\mathbb{R}^n$ which is not also Sobolev function. Moreover, it turns out that

$$\sup_{\|\varphi\| \leq 1} |\Lambda_{\mathbb{1}_E}(\varphi)| \leq \mathcal{H}^{n-1}(\partial E).$$

We will see (in the next chapter) that the inequality above is, for every smooth set E, an equality:

$$\|\Lambda_{\mathbb{1}_E}(\varphi)\|_* = \mathcal{H}^{n-1}(\partial E).$$

9.2 Alternative Definitions: Bounded Variation on the Real Line

In this brief section, we want to discuss a different definition of the space $\mathrm{BV}([a, b]; \mathbb{R})$, which is still used in some fields of mathematics (as partial differential equations evolution theory).

Definition 9.4 (Bounded Variation, II)**.** A function u is of bounded variation on $[a, b] \subseteq \mathbb{R}$ if and only if the *total variation* is finite, i.e.

$$T_v(u) := \sup_{P \in \mathcal{P}} \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)| < +\infty, \tag{9.6}$$

where $\mathcal{P}$ is the set of all finite partitions of the interval $[a, b]$.

Definition 9.5 (Essential Variation)**.** A function u is of bounded essential variation on $[a, b]$, and we denote it by $u \in \mathrm{BV}_{\mathrm{ess}}([a, b])$, if and only if

$$T_{v_{ess}}(u) := \sup_{P \in \mathcal{P}_{ess}} \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)| < +\infty, \tag{9.7}$$

where $\mathcal{P}_{ess}$ is the set of all finite partitions of the interval $[a, b]$ with extremal points x_i such that u is L^1-approximate continuous at x_i for every i.

Remark 9.7. The essential total variation (9.7) is closely related to the total variation introduced in (9.6) since, as one can easily check, it turns out that

$$T_{v_{ess}}(u) = \inf_{w(x) \,=\, u(x) \text{ a.e.}} T_v(w).$$

The next result shows that the definition of bounded variation introduced in this section is related to the one we will be dealing with in this course through the finiteness of the essential variation.

Theorem 9.6. *A function u belongs to $\mathrm{BV}((a, b))$ (in the sense of definition (9.1)) if and only if $u \in L^1((a, b))$ and $T_{v_{ess}}(u) < +\infty$.*

Lemma 9.7. *Let $u \in \mathrm{BV}((a, b))$. There exists a constant*[1] *$c \in \mathbb{R}$ such that*

$$u(x) = c + \int_a^x \mathrm{d}\mu(x) = c + \mu((a, x)) \tag{9.8}$$

for $\mathcal{L}^1$-almost every $x \in (a, b)$.

Proof. It is enough to notice that the identity (9.8) is equivalent to the following identity:

$$\left(u(x) - \int_a^x \mathrm{d}\mu(x) \right)' = 0. \tag{9.9}$$

□

Proof of Theorem 9.6.

[1] In the proof of Theorem 9.6 is not necessary to know the exact value of c, but the reader may try to prove that it is equal to the trace.

" $\Longrightarrow$ " Assume that $u \in \mathrm{BV}((a,\, b))$. By (9.8) there exists a constant $c \in \mathbb{R}$ such that

$$u(x) = c + \mu\left((a,\, x)\right).$$

The function u is summable by definition; thus it is enough to prove that the total essential variation is finite. The explicit expression for u immediately yields to

$$T_{v_{ess}}(u) := \sup_{P \in \mathcal{P}_{ess}} \sum_{i=0}^{n-1} \left|\mu\left((a,\, x_{i+1})\right) - \mu\left((a,\, x_i)\right)\right| = \sup_{P \in \mathcal{P}_{ess}} \sum_{i=0}^{n-1} \left|\mu\left((x_i,\, x_{i+1})\right)\right|.$$

In particular, we notice that the total essential variation is bounded by the mass of μ, and thus it is finite by assumption:

$$T_{v_{ess}}(u) = \sup_{P \in \mathcal{P}_{ess}} \sum_{i=0}^{n-1} \left|\mu\left((x_i,\, x_{i+1})\right)\right| \leq |Du|\left((a,\, b)\right) < +\infty.$$

" $\Longleftarrow$ " This assertion is left as an exercise for the reader.

□

9.3 Functional Properties

In this final section, we investigate some of the main functional properties of bounded variation functions (extension operator, trace operator, approximation results, etc.) which will be extensively used when we introduce the notion of finite perimeter set.

Definition 9.8 (Lipschitz Boundary)**.** A domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary is an open subset, which is locally the graph of a Lipschitz map with respect to some choice of orthogonal bases.

Theorem 9.9 (Extension)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary $\partial\Omega$. Then there exists*[2] *an extension operator*

$$E : \mathrm{BV}(\Omega) \longrightarrow \mathrm{BV}(\mathbb{R}^n).$$

Theorem 9.10 (Approximation, I)**.** *The immersion*

$$C^\infty\left(\mathbb{R}^n\right) \hookrightarrow \mathrm{BV}\left(\mathbb{R}^n\right)$$

is dense, that is, for every $u \in \mathrm{BV}\left(\mathbb{R}^n\right)$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty\left(\mathbb{R}^n\right)$ such that

$$\begin{cases} u_n \longrightarrow u & \text{in } L^1(\mathbb{R}^n), \\ \nabla u_n \cdot \mathcal{L}^n \rightharpoonup Du & \text{weakly (in the sense of distributions)}, \\ \|\nabla u_n\|_{L^1(\mathbb{R}^n)} \longrightarrow \|Du\|. \end{cases}$$

Proof.

Step 1. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a mollifier and, for any $\epsilon > 0$, consider the scaling

$$\rho_\epsilon(x) := \frac{1}{\epsilon^n}\rho\left(\frac{x}{\epsilon}\right).$$

The function $u_\epsilon := \rho_\epsilon * u$ belongs to $C_c^\infty(\mathbb{R}^n)$ and a standard approximation argument is enough to infer that

$$u_\epsilon \to u \quad \text{in } L^1(\mathbb{R}^n).$$

[2] The operator E is, in general, not unique in any sense.

Step 2. The key point of the proof is the following inequality (from which the last two properties will follow immediately):

$$\|\rho_\epsilon * Du\|_{L^1(\mathbb{R}^n)} = \|\nabla u_\epsilon\|_{L^1(\mathbb{R}^n)} \overset{?}{\leq} \|Du\|. \tag{9.10}$$

If we give this inequality for granted, then we can easily conclude that the family of functions $(\nabla u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $L^1(\mathbb{R}^n)$. In particular, $\left(\nabla u_\epsilon \cdot \mathcal{L}^d\right)_{\epsilon>0}$ is a uniformly bounded sequence of measures, and therefore (by an Ascoli-Arzelà compactness theorem) it converges in sense of measures to a vector-valued measure $\mu \in \mathcal{M}(\mathbb{R}^n;\, \mathbb{R}^n)$. For every $\varphi \in C_c^\infty(\mathbb{R}^n)$ it turns out that

$$\int_{\mathbb{R}^n} \nabla u_\epsilon(x)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} u_\epsilon(x)\mathrm{div}(\varphi)(x)\,\mathrm{d}x,$$

and the left-hand side converges to

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}\mu(x),$$

while the right-hand side converges to

$$-\int_{\mathbb{R}^n} u(x)\mathrm{div}(\varphi)(x)\,\mathrm{d}x$$

as $\epsilon \to 0^+$. In particular, we obtain the identity

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}\mu(x) = -\int_{\mathbb{R}^n} u(x)\mathrm{div}(\varphi)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}(Du)(x), \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^n),$$

which implies that $\mu = Du$ (e.g., using the fundamental lemma in Calculus of Variations).

Step 3. The third property follows easily from the fact that $\|\cdot\|_{L^1(\mathbb{R}^n)}$ is a lower semi-continuous function. Indeed, we have that

$$\liminf_{\epsilon\to 0^+} \|\nabla u_\epsilon\|_{L^1(\mathbb{R}^n)} \geq \|Du\|,$$

which gives the sought convergence of the masses, as the opposite inequality holds by (9.10). □

Theorem 9.11 (Approximation, II). *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary $\partial\Omega$. The immersion*

$$C^\infty\left(\overline{\Omega}\right) \hookrightarrow \mathrm{BV}\left(\Omega\right)$$

is dense, that is, for every $u \in \mathrm{BV}\left(\Omega\right)$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty\left(\Omega\right)$ such that

$$\begin{cases} u_n \longrightarrow u & \text{in } L^1(\Omega), \\ \nabla u_n \cdot \mathcal{L}^n \rightharpoonup Du & \text{in the weak sense of distributions,} \\ \|\nabla u_n\|_{L^1(\mathbb{R}^n)} \longrightarrow |Du|\left(\Omega\right). \end{cases}$$

Proof. The key idea is to extend u to $\mathbb{R}^n$ via an extension operator

$$E : \mathrm{BV}(\Omega) \longrightarrow \mathrm{BV}(\mathbb{R}^n),$$

which exists as a consequence of Theorem 9.9, and apply the first approximation result to the function $E(u)$ (see Theorem 9.10).

We need to be careful though: It is in general false that every extension operator E can be used here since, e.g., the sequence $\widetilde{u}_\epsilon$, which approximates $E(u)$, does not satisfy the third property (see Figure 9.1). More precisely, the norm is lower semi-continuous and hence

$$\liminf_{\epsilon\to 0^+} \|\nabla u_\epsilon\|_{L^1(\mathbb{R}^n)} \geq \|Du\|,$$

but the opposite inequality needs not to hold. Indeed, we need to ask E to be an extension operator satisfying the following key property:

$$|DE(u)|(\partial\Omega) = 0. \tag{9.11}$$

Such an extension operator exists, but it needs to be chosen carefully (we will not discuss this further, but the reader may try to do it as an exercise). $\square$

Remark 9.8. The immersion

$$C^\infty\left(\overline{\Omega}\right) \hookrightarrow \mathrm{BV}(\Omega)$$

is not dense with respect to the norm (9.5).

Exercise 9.2. Let Ω be a bounded set with Lipschitz boundary $\partial\Omega$, and let $u \in \mathrm{BV}(\Omega)$. Prove that the following is an extension operator

$$E : \mathrm{BV}(\Omega) \longrightarrow \mathrm{BV}(\mathbb{R}^n), \qquad u \longmapsto \widetilde{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega, \end{cases}$$

and prove also that it is not the right one for the proof of Theorem 9.11.

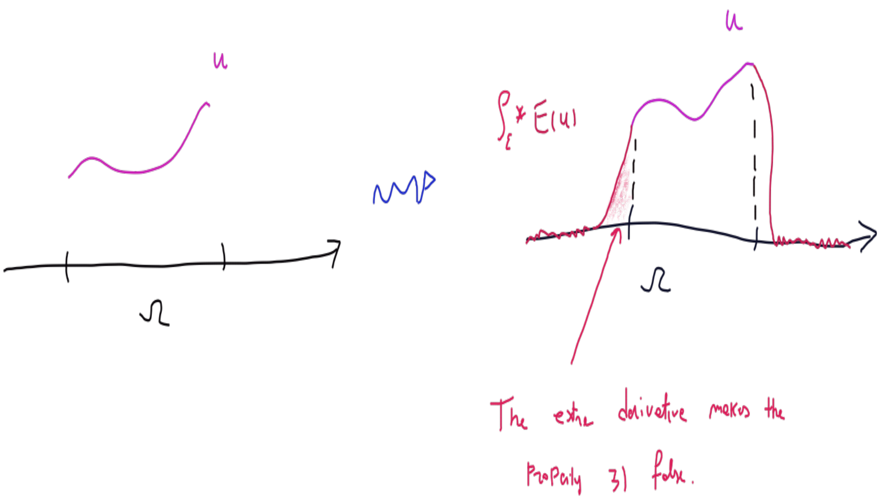

Figure 9.1: What could go wrong in the proof?

Remark 9.9. Recall that the Sobolev critical exponent in $\mathbb{R}^n$ is defined by setting

$$1^* := \frac{n}{n-1}.$$

Theorem 9.12 (Sobolev Embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary $\partial\Omega$. The immersion*

$$\mathrm{BV}(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous for every $1 \leq p \leq 1^*$ and compact for every $1 \leq p < 1^*$.

Characterization of $\mathrm{BV}(\mathbb{R}^n)$**.** In this brief paragraph, we want to state and prove a useful characterization of $\mathrm{BV}(\mathbb{R}^n)$, which is exactly the one that does not work for $L^1(\mathbb{R}^n)$.

Proposition 9.13. *Let $u \in L^1(\mathbb{R}^n)$. The following assertions are equivalent:*

(1) The function u belongs to $\mathrm{BV}(\mathbb{R}^n)$.

(2) There is a constant $C > 0$ such that

$$\|\tau_h u - u\|_{L^1(\mathbb{R}^n)} \leq C|h|.$$

Proof.

" $\Longrightarrow$ " Assume that $u \in C^\infty(\mathbb{R}^n)$ in such a way that

$$\frac{u(x+he_i)-u(x)}{h} \xrightarrow{h\to 0^+} \frac{\partial u}{\partial x_i}.$$

First, we notice that it suffices to prove the thesis for an orthonormal basis of directions, e.g. $\{e_1, \ldots, e_n\}$. A simple computation yields to

$$\|\tau_i u - u\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u(x+he_i)-u(x)| \, \mathrm{d}x \leq$$

$$\leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)} |h|,$$

which is enough to conclude using the density of $C^\infty(\mathbb{R}^n)$ in $\mathrm{BV}(\mathbb{R}^n)$.

" $\Longleftarrow$ " Vice versa, it suffices to prove that for every weakly differentiable summable function u it turns out that

$$\frac{u(x+he_i)-u(x)}{h} \xrightarrow{h\to 0^+} D_i u$$

in the sense of distributions. □

Theorem 9.14. *Let Ω be either $\mathbb{R}^n$ or a bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. If $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ is a bounded sequence, then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$u_{n_k} \to u \in \mathrm{BV}(\Omega) \qquad \text{in } L^1(\Omega),$$

$$Du_{n_k} \rightharpoonup Du \qquad \text{weakly-}* \text{ (in the sense of measures).}$$

Proof. The sequence of measures $(Du_n)_{n\in\mathbb{N}}$ is bounded by assumption; hence, up to subsequences, it converges in the weak sense of measures to a measure μ in $\mathcal{M}(\Omega;\, \mathbb{R}^n)$. By Sobolev Embedding Theorem 9.12 the immersion

$$\mathrm{BV}(\Omega) \hookrightarrow L^1(\Omega)$$

is compact, and thus the sequence $(u_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ is relatively compact, which means that - up to subsequences - it converges strongly (with respect to the L^1 norm) to an element u. To conclude the proof, it remains to prove that

$$\mu = Du.$$

The same argument used in the proof of Theorem 9.10 works here; therefore it is left to the reader to fill in the details. □

Trace Operator. In the last paragraph of this section, we investigate the trace operator, and we state the main theorem (which is somewhat equivalent to the one valid for Sobolev spaces), and we briefly explain why in the finite perimeter setting, we cannot introduce boundary condition using the trace operator.

Theorem 9.15 (Trace Operator)**.** *The trace operator, defined by*

$$T : C^1\left(\overline{\Omega}\right) \longrightarrow L^1\left(\partial\Omega\right), \qquad u \longmapsto u\big|_{\partial\Omega},$$

is bounded with respect to the $\|\cdot\|_{\mathrm{BV}(\Omega)}$ *norm, and it can be uniquely extended to a linear operator*

$$T : \mathrm{BV}(\Omega) \longrightarrow L^1\left(\partial\Omega\right).$$

Remark 9.10. Recall that the space $C^1\left(\overline{\Omega}\right)$ is dense in $\mathrm{BV}(\Omega)$ in the weak sense (see Theorem 9.10). This embedding is the main difference with the Sobolev spaces setting we mentioned above and, as we will see later, it is also the reason why the trace operator is not used to assign boundary conditions.

Remark 9.11. Recall that the trace operator defined on $W^{1,p}(\Omega)$ can be used to assign boundary conditions as follows. Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ be a minimizing sequence for a functional $F(u) : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, and assume that

$$Tu_n = g.$$

If $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, then the *Sobolev trace theorem* implies that $Tu = g$, which is exactly what we would like to happen when trying to find a solution for a minimum problem.

On the other hand, if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{BV}(\Omega)$, then - up to subsequences - u_n converges to u strongly in $L^1(\Omega)$. However, in general, it is not true that Tu_n converges (in any reasonable sense) to Tu (as we shall prove in the following examples).

Example 9.2. Let $\Omega := (0,\, 1)$. Consider the sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}\left((0,\, 1)\right)$ that is defined as it is portrayed in Figure 9.2. Clearly, for every $n \in \mathbb{N}$, it turns out that

$$Tu_n(0) = 0 \qquad \text{and} \qquad Tu_n(1) = 1,$$

but u_n converges pointwise to $u(x) \equiv 1$ on Ω, which means that a boundary condition is not preserved under the limit $(Tu(0) = 1)$. The reader may check by herself that

$$u_n \rightharpoonup 1 \qquad \text{in } \mathrm{BV}(\Omega),$$

that is,

$$\begin{cases} u_n \to u & \text{in } L^1(\Omega), \\ \|u_n\|_{\mathrm{BV}(\Omega)} \leq M & \forall\, n \in \mathbb{N}. \end{cases}$$

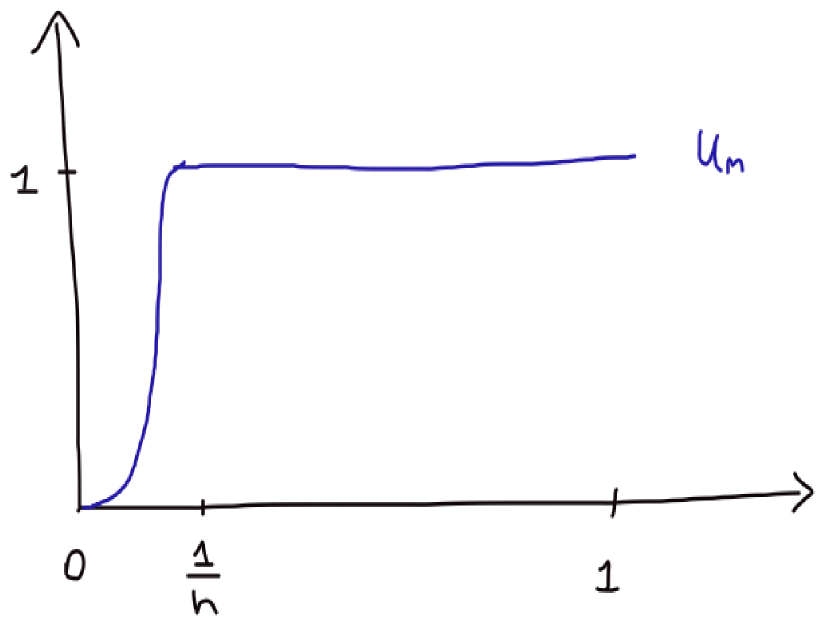

Figure 9.2: First Counterexample

Example 9.3. Let $\Omega \subseteq \mathbb{R}^n$, and consider the sequence of subsets $(E_k)_{k\in\mathbb{N}} \subset \mathcal{P}(\Omega)$ defined as it is portrayed in Figure 9.3. Moreover, assume that for any $n \in \mathbb{N}$

(a) E_k is smooth,

(b) $\overline{E_k} \subset \Omega$, and

(c) the measure $\mathcal{H}^{n-1}(\partial E_k)$ is of the same order of $\mathcal{H}^{n-1}(\partial\Omega)$.

The sequence given by the characteristic functions $u_k := \mathbb{1}_{E_k} \in \mathrm{BV}(\Omega)$ is uniformly bounded since the following properties hold:

(a) $\|u_k\|_{L^1(\Omega)} \leq \mathcal{H}^n(\Omega)$;

(b) $\|Du_k\| = \mathcal{H}^{n-1}(\partial E_k) \to \mathcal{H}^{n-1}(\partial\Omega)$ as n goes to $+\infty$.

Clearly, the limit of the sequence u_k is the characteristic function of Ω, that is, the constant function $1 \in \mathrm{BV}(\Omega)$. The reader may easily prove that

$$Tu_n = 0 \qquad \text{but} \qquad Tu = 1,$$

which means that Tu_n does not converge in any reasonable sense to Tu.

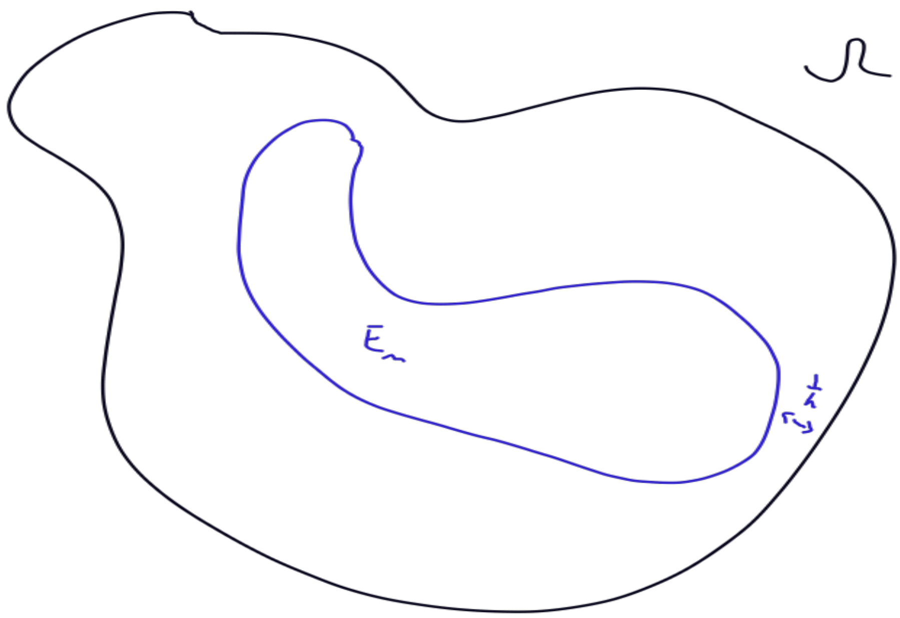

Figure 9.3: Second Counterexample

Chapter 10

Finite Perimeter Sets

In this final chapter, we introduce the notion of *finite perimeter set*, and we also present a variational setting that allows us to prove the existence of a solution to the Plateau's problem (for suitable boundary conditions) and the capillarity problem.

In the second half of the chapter, we introduce the essential and the reduced boundary, and we prove the *Structure Theorem* due to De Giorgi and Federer.

10.1 Main Definitions and Elementary Properties

In this section, we present the definition of finite perimeter set, and we exploit the theory of bounded variation functions introduced in the previous chapter to derive approximation and compactness results.

Definition 10.1 (Finite Perimeter Set)**.** Let $E \subseteq \mathbb{R}^n$. We say that E has finite perimeter if and only if the characteristic function has bounded variation, that is,

$$\mathbb{1}_E \in \mathrm{BV}\left(\mathbb{R}^n\right).$$

Recall that the functional associated to $\mathbb{1}_E$ is given by

$$\Lambda_{\mathbb{1}_E}(\varphi) := \int_E \mathrm{div}(\varphi)(x)\,\mathrm{d}x,$$

and it is bounded with respect to the uniform norm. The perimeter of E is defined by the operator norm of the functional $\Lambda_{\mathbb{1}_E}(\varphi)$, that is,

$$\mathrm{Per}(E) := \left\|\Lambda_{\mathbb{1}_E}\right\|_* .$$

Definition 10.2 (Relative F.P.S.)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $E \subseteq \Omega$ be a subset. We say that E has finite perimeter in Ω if and only if

$$\mathbb{1}_E \in \mathrm{BV}\left(\Omega\right).$$

The perimeter of E in Ω is defined by the operator norm of the functional associated to $\mathbb{1}_E$, that is,

$$\mathrm{Per}_\Omega(E) = \left\|\Lambda_{\mathbb{1}_E}\right\|_* ,$$

where $\Lambda_{\mathbb{1}_E}$ denotes the restriction of the functional to the space $C_c^\infty(\Omega)$.

Remark 10.1. Let $E \subseteq \Omega \subseteq \mathbb{R}^n$ be a smooth subset (e.g., assume that the boundary of E is at least of class C^1). The reader may prove that the perimeter of E in Ω is simply given by the following formula:

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{n-1}\left(\partial E \cap \Omega\right). \tag{10.1}$$

Hint. A similar argument to the one used in the Example 9.1 works here. In particular, we notice that the derivative of the characteristic function is given by

$$D\left(\mathbb{1}_E\right) = \nu_i \mathbb{1}_{\partial E \cap \Omega} \cdot \mathcal{H}^{n-1}.$$

Remark 10.2. It is important to notice that, in general, the formula

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{n-1}\left(\partial E\right)$$

is not true, not even if E is smooth. The intuitive idea is clear: Since Ω is an open set, the boundary of E may overlap with the boundary of Ω as it happens in Figure 10.1.

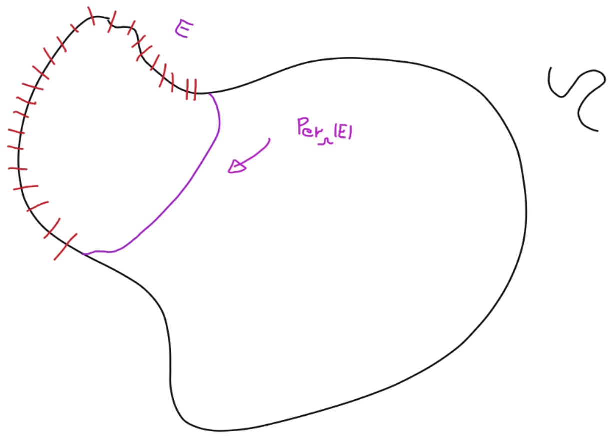

Figure 10.1: The boundary of the set E partially overlaps with the boundary of Ω. Therefore only the magenta part of ∂E contributes to the perimeter of E in Ω.

Compactness Results. In this paragraph, we state two compactness results for finite perimeter sets, both of which derive from the functional properties of bounded variations functions.

Notation. Let Ω be an open set, and let $E,\, F \subseteq \Omega$ be two subsets. The distance between E and F is the n-Lebesgue measure of the symmetric difference, that is,

$$d(E,\, F) := \|\mathbb{1}_E - \mathbb{1}_F\|_{L^1(\Omega)} = \mathcal{L}^n\left(E \triangle F\right).$$

Theorem 10.3 (Compactness, I). *Let $(E_k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ be a uniformly bounded sequence of finite perimeter sets, that is,*

$$\mathrm{Per}(E_k) < +\infty \qquad \textit{and} \qquad E_k \subseteq \Omega' \subset\subset \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open subset. Then there exist a finite perimeter set $E \subseteq \Omega$ and an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$d(E_k,\, E) \xrightarrow{k\to+\infty} 0.$$

Moreover, the perimeter is a lower semi-continuous functional, that is

$$\liminf_{k\to+\infty} (\mathrm{Per}(E_k)) \geq \mathrm{Per}(E).$$

Remark 10.3. The assumption "$(E_k)_{k\in\mathbb{N}}$ is a uniformly bounded sequence" is crucial here, otherwise one can exhibit the easy counterexample

$$E_k := B(k,\, \frac{1}{2}),$$

which is a collection given by integer translations of a ball centered at zero with radius one half.

Theorem 10.4 (Compactness, II)**.** *Let $(E_k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ be a uniformly bounded sequence of finite perimeter sets, that is,*

$$\mathcal{L}^n\left(E_k\right),\, \mathrm{Per}(E_k) \leq c < +\infty.$$

Then there exist a finite perimeter set $E \subseteq \Omega$ and an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$|(E_k \triangle E) \cap B(x,\, r)| \xrightarrow{k\to+\infty} 0, \qquad \forall\, B(x,\, r) \subseteq \mathbb{R}^n.$$

Moreover, the perimeter is a lower semi-continuous functional, that is

$$\liminf_{k\to+\infty} (\mathrm{Per}(E_k)) \geq \mathrm{Per}(E).$$

10.2 Approximation Theorem via Coarea Formula

The primary goal of this section is to prove that any finite perimeter set can be approximated via a sequence of smooth (e.g. C^∞ boundary) sets. In order to prove this assertion, we will need to use the *coarea* formula presented in Section 7.4.

Theorem 10.5. *Let $E \subseteq \mathbb{R}^n$ be a finite perimeter set. Then there exists a sequence of smooth (boundary of class C^∞) sets $(E_k)_{k\in\mathbb{N}}$ such that*

$$\begin{cases} E_k \xrightarrow{d} E, \\ \mathcal{H}^{n-1}\left(\partial E_k\right) = \mathrm{Per}(E_k) \xrightarrow{k\to+\infty} \mathrm{Per}(E), \end{cases}$$

where d denotes the symmetric distance introduced in the previous section.

Proof. Let ρ be a mollifier kernel. Denote by ρ_ϵ the ϵ-rescaling, which is defined in the usual way:

$$\rho_\epsilon(x) := \frac{1}{\epsilon^n}\,\rho\left(\frac{x}{\epsilon}\right).$$

Let $u_\epsilon := \rho_\epsilon * \mathbb{1}_E$ be the convolution. For every $s_\epsilon \in [0,\, 1]$ and $\epsilon > 0$ the set

$$E_{s_\epsilon,\,\epsilon} := \{x \in \Omega \mid u_\epsilon(x) > s_\epsilon\}$$

is open and smooth, provided that we pick an s_ϵ such that it is not a critical value (which is always possible by Sard's Lemma).

Step 1. We claim that, if $\delta \le s_\epsilon \le 1-\delta$, then

$$u_\epsilon \xrightarrow{L^1(\Omega)} u \implies E_{s_\epsilon,\,\epsilon} \xrightarrow{d} E. \tag{10.2}$$

Indeed, a straightforward application of the Fubini-Tonelli's theorem proves that

$$\|u_\epsilon - u\|_{L^1(\Omega)} = \int_0^1 d\left(E,\, E_{s,\,\epsilon}\right)\,\mathrm{d}s.$$

On the other hand, for every $s' < s$ it turns out that $E_{s',\,\epsilon} \supseteq E_{s,\,\epsilon}$; hence we can easily estimate the L^1-norm of the difference as follows:

$$\|u_\epsilon - u\|_{L^1(\Omega)} \ge \int_{s_\epsilon}^1 |E \setminus E_{s_\epsilon,\,\epsilon}|\,\mathrm{d}s \ge (1-s_\epsilon)\cdot |E \setminus E_{s_\epsilon,\,\epsilon}|\,,$$

where $\setminus$ denotes the usual difference between two sets.

In particular, if we take s_ϵ strictly less than 1, then $\|u_\epsilon - u\|_{L^1(\Omega)} \to 0$ implies $|E \setminus E_{s_\epsilon,\,\epsilon}| \to 0$, which is exactly what we wanted to prove.

Step 2. The function u_ϵ is smooth (by convolution). Hence we can apply the *coarea formula* (7.11) with $h \equiv 1$ and $f = u_\epsilon$ to obtain the following relation:

$$\int_{\mathbb{R}^n} |\nabla u_\epsilon(x)|\,\mathrm{d}x = \int_0^1 \mathcal{H}^{n-1}\left(u_\epsilon^{-1}(s)\right)\,\mathrm{d}s,$$

where $u_\epsilon^{-1}(s) = \partial E_{s,\,\epsilon}$ if s is a regular value[1]. One can easily prove that

$$\frac{1}{1-2\delta}\,\mathrm{Per}(E) \ge \frac{1}{1-2\delta}\int_0^1 \mathrm{Per}(E_{s,\,\epsilon})\,\mathrm{d}s \ge -\!\!\!\!\!\!\int_\delta^{1-\delta} \mathrm{Per}(E_{s,\,\epsilon})\,\mathrm{d}s,$$

for $\delta > 0$ small enough; hence there exists a non-null (with respect to the Lebesgue measure) set of regular values in the interval $[\delta,\,1-\delta]$ such that

$$\frac{1}{1-2\delta}\,\mathrm{Per}(E) \ge \mathrm{Per}(E_{s_\epsilon,\,\epsilon}) \implies \limsup_{\epsilon\to 0^+} \mathrm{Per}(E_{s_\epsilon,\,\epsilon}) \le \frac{1}{1-2\delta}\,\mathrm{Per}(E)$$

for every regular value $s_\epsilon \in [\delta,\,1-\delta]$. Recall that the perimeter is a lower semi-continuous functional; hence we only need to refine the estimate and rule δ out of the equation. $\square$

Exercise 10.1. State the approximation theorem in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

10.3 Existence Results for Plateau's Problem

Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth subset of $\mathbb{R}^n$. Assume that Γ is the boundary of a regular hypersurface $\mathcal{G} \subset \partial\Omega$, that is,

$$\Gamma = \partial\mathcal{G}.$$

The Plateau's problem consists of finding the minimal hypersurface Σ contained in Ω such that its boundary is given by Γ, that is, satisfying the boundary condition $\partial\Sigma = \Gamma$.

[1] In particular, the formula holds with $u_\epsilon^{-1}(s) = \partial E_{s,\,\epsilon}$ for $\mathcal{L}^n$-almost every $s \in [0,\,1]$ since the set of singular values is negligible by Sard's Lemma.

Finite Perimeter Setting. Let E be a subset of Ω satisfying the following condition:

$$\partial E \cap \partial \Omega = \mathcal{G}.$$

If Σ is a regular hypersurface given by the intersection $\partial E \cap \Omega$, then the following relations holds:

$$\operatorname{Per}(E) = \mathcal{H}^{n-1}\left(\partial E \cap \partial \Omega\right) + \mathcal{H}^{n-1}(\Sigma). \tag{10.3}$$

This identity allows us to infer that minimizing the functional $\mathcal{H}^{n-1}(\Sigma)$ is equivalent to minimizing the functional $\operatorname{Per}(E)$ among all finite perimeter sets E such that

$$T\left(\mathbb{1}_E\right) = \mathbb{1}_{\mathcal{G}}.$$

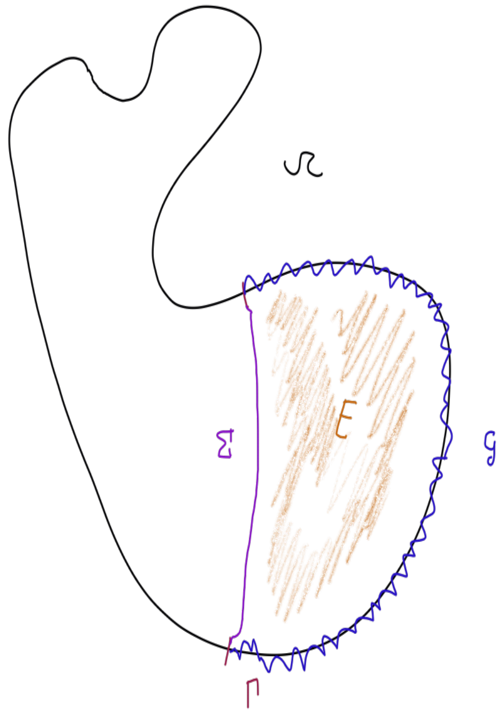

Figure 10.2: Plateau's Problem in the F.P.S. formulation.

Remark 10.4. In the previous sections, we proved that the perimeter is a nice functional, which is also semi-lower continuous. Hence the equivalent formulation in the f.p.s. setting should be enough to demonstrate the existence of a solution. But, as the problem is formulated now, many things could go wrong:

1) Assume that one can find a solution of the minimum problem (10.3). How can one recover a surface Σ?

2) The minimum of the functional (10.3) actually exists?

3) If $(E_n)_{n\in\mathbb{N}}$ is a minimizing sequence, then the trace $T(\mathbb{1}_{E_n})$ may not converge to $\mathbb{1}_{\mathcal{G}}$, as proved in the previous chapter.

4) Topological issues. The hypersurface $\mathcal{G}$ with boundary Γ may not even exist.

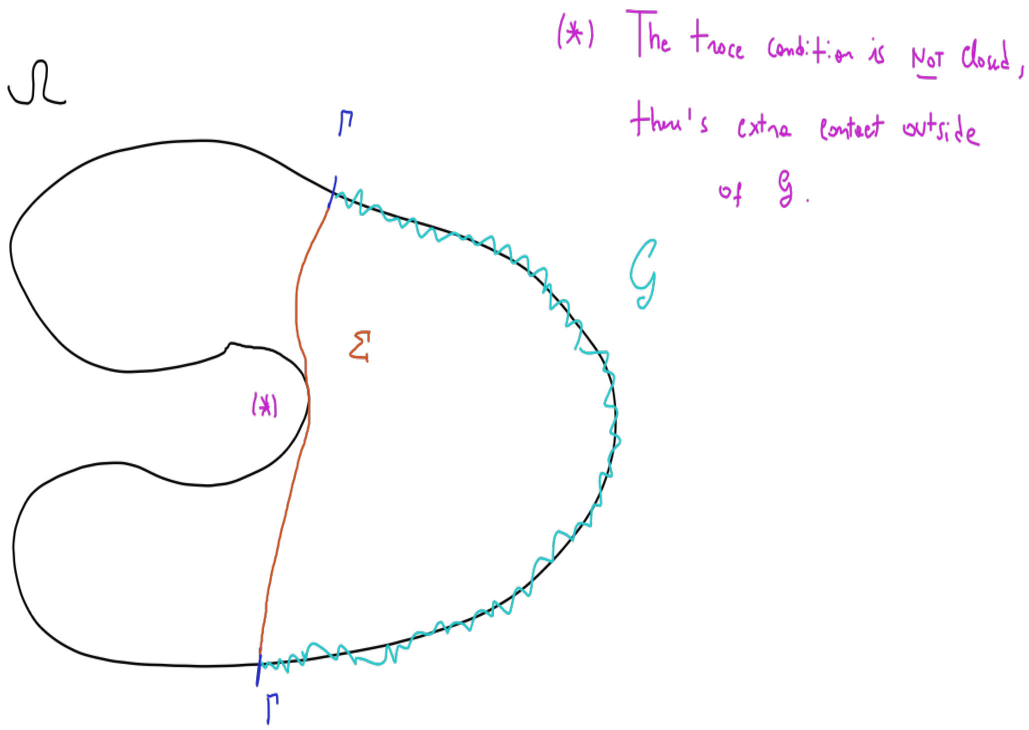

Figure 10.3: In this Plateau's problem, the minimizing surface Σ makes an extra contact with the boundary of Ω, which means that $T(\mathbb{1}_{E_n})$ does not converge to $\mathbb{1}_{\mathcal{G}}$.

Alternative Formulation. The Plateau's problem needs to be reformulated differently (in the f.p.s. setting) to avoid the issues listed above (except the topological ones, which are way more delicate to deal with).

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth subset of $\mathbb{R}^n$. Assume that Γ is the boundary of a regular hypersurface $\mathcal{G} \subset \partial\Omega$, that is,

$$\Gamma = \partial\mathcal{G}.$$

Let E_0 be a fixed subset contained in the complement of Ω, that is, $E_0 \subset \mathbb{R}^n \setminus \overline{\Omega}$. The Plateau's problem, again, constist of finding the minimal hypersurface Σ contained in Ω such that

$$\partial\Sigma = \Gamma.$$

In the finite perimeter setting, the idea is to look for the minimum point(s) of the functional

$$\mathrm{Per}(E) - \mathcal{H}^{n-1}\left(\partial E_0 \setminus \overline{\Omega}\right) \tag{10.4}$$

among all finite perimeter sets E satisfying the following condition:

$$E \setminus \Omega = E_0 \qquad \text{up to } \mathcal{L}^n\text{-null sets.}$$

Plateau's problem. In this brief paragraph, we summarize the classical formulation of the Plateau's problem and we reinterpret it into the finite perimeter set framework.

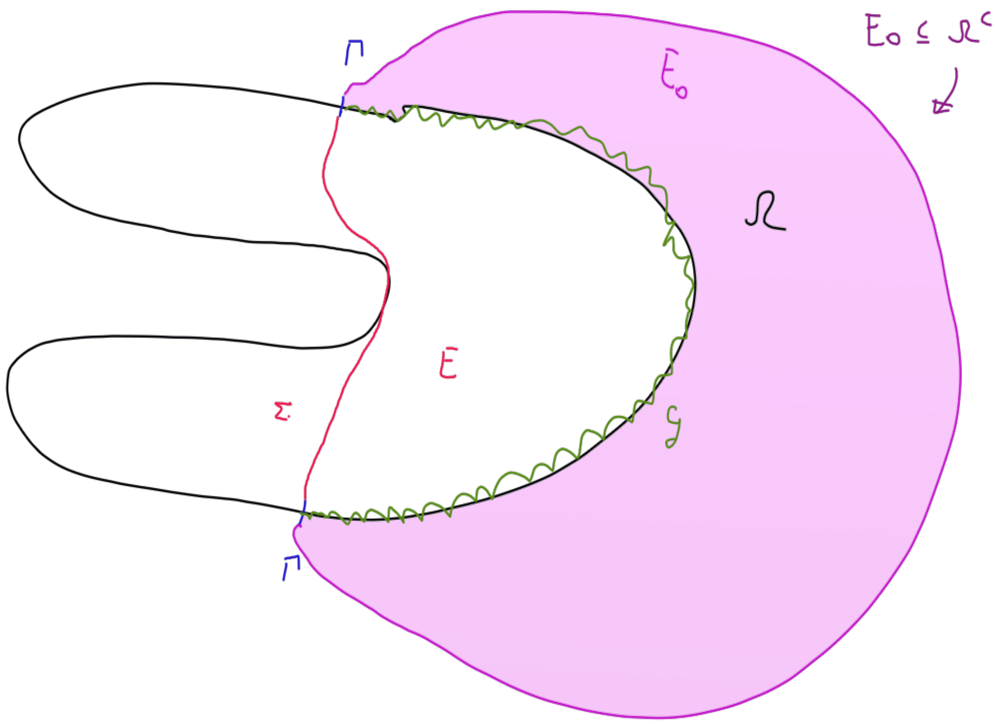

Figure 10.4: In this picture, one can see how the trace issues are no longer a treat using this alternative formulation of the Plateau's problem.

C1) Given Ω be a bounded smooth subset of $\mathbb{R}^n$.

C2) Given Γ be a $(n-2)$-dimensional surface contained in $\partial\Omega$.

C3) Goal: Find a $(n-1)$-dimensional surface Σ such that $\partial\Sigma = \Gamma$ and Σ minimizes the area functional $\mathcal{H}^{n-1}(\Sigma)$.

In the finite perimeter set framework, the Plateau's problem can be formulated in the following way:

F1) Given Ω be a bounded smooth subset of $\mathbb{R}^n$.

F2) Given $\mathcal{G} \subset \partial\Omega$ such that there exists a finite perimeter set E_0 in $\mathbb{R}^n$ contained in the complement of Ω, that is,

$$E_0 \subseteq \mathbb{R}^n \setminus \overline{\Omega} \qquad \text{and} \qquad T(\mathbb{1}_{E_0}) = \mathbb{1}_{\mathcal{G}}$$

up to $\mathcal{L}^n$-null sets.

F3) Goal: Find a finite perimeter set E in $\mathbb{R}^n$ such that $E \setminus \overline{\Omega}$ is - up to $\mathcal{L}^n$-null sets - equal to E_0, such that E is a minimum point of the perimeter functional.

In conclusion, a natural question arises: For which $\mathcal{G}$ there exists E_0 satisfying the property **F2)**? Surprisingly, it is enough to ask that $\mathcal{G}$ is any Borel set as a consequence of the following theorems.

Theorem 10.6. *The trace operator*

$$T : C^1\left(\overline{\Omega}\right) \longrightarrow L^1\left(\partial\Omega\right), \qquad u \longmapsto u\big|_{\partial\Omega}$$

is surjective.

Remark 10.5. In general, the trace operator introduced above does not admit a linear right inverse.

Theorem 10.7. *Let u be a characteristic function in $L^1(\partial\Omega)$. Then there exists a finite perimeter set $E \subseteq \mathbb{R}^n$ such that*

$$T(\mathbb{1}_E) = u.$$

In particular, the only problems one needs to deal with in Plateau's problem are of topological nature (see, e.g., Figure 10.5).

Figure 10.5: The surface Γ is not the boundary of any hypersurface $\mathcal{G}$ contained in the torus Ω.

Conclusion. To conclude this chapter, we sketch the framework and the solution of a Plateau's problem. The reader may refer to Figure 10.6 for a better understanding of what is going on, but the main idea is to take Γ equal to the disjoint union of two circumferences (lying on the planes π_1 and π_2 respectively), and $\mathcal{G}$ equal to the disjoint union of the two closed disks.

The set Ω is convex, but once again the condition on the trace is **not** closed. Therefore this example shows how important is the introduction of a different formulation where the operator T does not appear.

The solution of the minimum problem depends on the distance d between the planes π_1 and π_2, and it is either a catenoid or a disjoint union of disks.

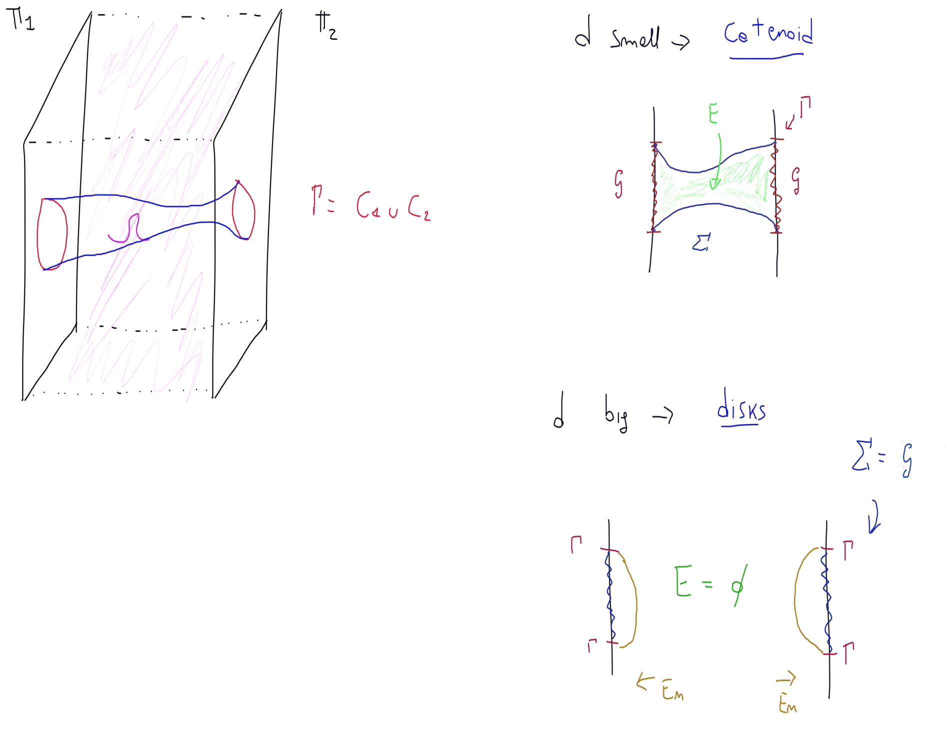

Figure 10.6: Example of Plateau's problem

10.4 Capillarity Problem

Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$. In this section, we shall discuss the capillarity problem equilibrium when the container (Ω) is fixed and the volume of the liquid is fixed. More precisely, the drop of liquid (denoted by E), contained in Ω, is at equilibrium if and only if it is a critical point (local minima) of the capillarity energy functional

$$\mathscr{F}(E) := \mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma \mathcal{H}^{n-1}(\partial E \cap \partial\Omega), \tag{10.5}$$

where $\sigma \in \mathbb{R}$ is a constant, subject to the constraint that the volume is fixed, e.g. $|E| = m$.

In the Appendix, we shall prove that, if we set the first variation of the functional $\mathscr{F}$ equal to zero, then we find that the mean curvature H is constant. More precisely, it is equal to the Lagrange multiplier associated with the constraint on[2] Σ^f, and $\theta := \theta_Y$ is constant on Γ and equal to the so-called *Young angle*, i.e. the solution to the following trigonometric equation:

$$\cos(\theta_Y) = -\sigma.$$

Remark 10.6. Suppose that the drop of liquid is subject to the action of the gravitational force. Then one needs to slightly modify the functional $\mathscr{F}$ to take into account the gravity in the following way:

$$\mathscr{G}(E) := \mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma \mathcal{H}^{n-1}(\partial E \cap \partial\Omega) + \int_E \vec{g} \cdot x. \tag{10.6}$$

[2] We denote by Σ^f the free portion of the surface, that is, the portion of the surface that does not lie on $\partial\Omega$.

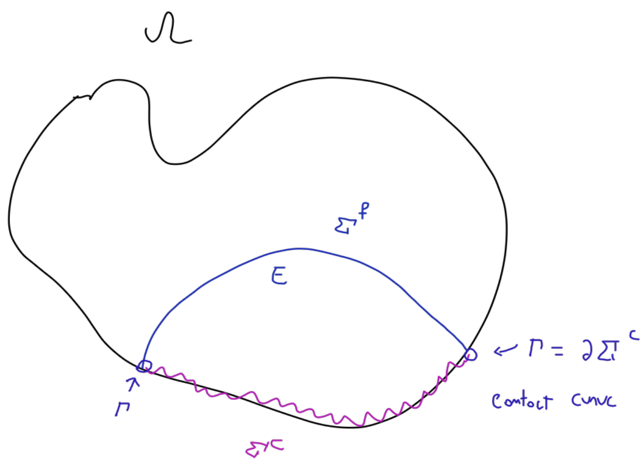

Figure 10.7: Capillarity Problem

The reader may compute the first variation of the functional $\mathscr{G}$ and set it equal to zero. It is not hard to check that the mean curvature H is not constant anymore.

Finite Perimeter Formulation. In the f.p.s. framework, we can translate the capillarity problem associated to the functional $\mathscr{F}$ (subject to the volume constraint) as follows:

F1) Given Ω bounded smooth subset of $\mathbb{R}^n$.

F2) Goal: Find a finite perimeter set $E \subseteq \Omega$ in $\mathbb{R}^n$, with fixed volume $|E| = m$, which minimizes the functional

$$\mathscr{F}(E) = \mathrm{Per}_{\Omega}(E) + \sigma \mathcal{H}^{n-1}(\Sigma^c), \tag{10.7}$$

where $\mathbb{1}_{\Sigma^c} = T(\mathbb{1}_E)$ denotes the portion of the surface Σ that lies on the boundary of Ω.

Proposition 10.8. *Suppose that $\sigma \notin [-1, 1]$. Then the minimization problem (10.7), with fixed volume, is ill-posed.*

Remark 10.7. More precisely, we shall prove that the functional (10.7) is not lower semi-continuous with respect to the distance d whenever $|\sigma| > 1$.

Proof. Let us consider the sequence of subsets $(E_{1/n})_{n\in\mathbb{N}}$ portrayed in Figure 10.8, and suppose that $E_{1/n}$ is smooth for every $n \in \mathbb{N}$. If $|\sigma| > 1$, then one can prove that the functional $\mathscr{F}$ is not lower semi-continuous using this sequence, that is,

$$\mathscr{F}(E_{1/n}) = \left|\Sigma^f\right| + |\Sigma^c| + o(1) \xrightarrow{\epsilon\to 0^+} \left|\Sigma^f\right| \pm |\Sigma^c| < \mathscr{F}(E).$$

In other words, the relaxation of the functional $\mathscr{F}$ is given by

$$\widetilde{\mathscr{F}}(E) = \begin{cases} \left|\Sigma^f\right| + |\Sigma^c|, & \text{if } \sigma > 1, \\ \left|\Sigma^f\right| - |\Sigma^c| & \text{if } \sigma < -1, \end{cases}$$

and this is enough to prove the claim above since

$$\sigma > 1 \implies \widetilde{\mathscr{F}}(E) < \mathscr{F}(E) = \left|\Sigma^f\right| + \sigma \left|\Sigma^c\right|,$$

and similarly

$$\sigma < -1 \implies \widetilde{\mathscr{F}}(E) > \mathscr{F}(E) = \left|\Sigma^f\right| + \sigma \left|\Sigma^c\right|.$$

□

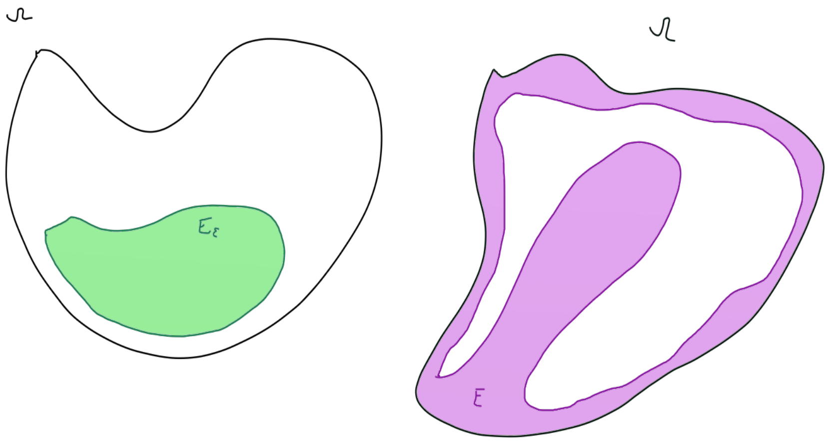

Figure 10.8: Left: Sequence when $\sigma > 1$. Right: Sequence when $\sigma < -1$.

Claim. If $\sigma \in [-1, 1]$, then the functional (10.7) is lower semi-continuous.

The proof of this claim cannot be tackled with the tools introduced in this section since we need to use more sophisticated methods. We will come back to this result in the next chapter.

10.5 Finite Perimeter Sets: Structure Theorem

The primary goal of this section is to state and prove the structure theorem for finite perimeter set, due to both De Giorgi and Federer (it appeared in different papers, but we shall merge them to obtain a very reasonable result.)

Lemma 10.9. *Let $E \subset \mathbb{R}^n$ be a bounded subset of $\mathbb{R}^n$ such that ∂E is a Lipschitz boundary. If ν_e is the external normal to E, then*

$$D(\mathbb{1}_E) = -\nu_e \, \mathbb{1}_{\partial E} \cdot \mathcal{H}^{n-1}$$

and, in particular, it turns out that

$$\operatorname{Per}(E) = \mathcal{H}^{n-1}(\partial E). \tag{10.8}$$

Proof. The identity follows immediately from the divergence theorem for Lipschitz boundaries. □

Example 10.1. In general, the $\mathcal{H}^{n-1}$ measure of the boundary ∂E does not coincide with the perimeter of the set $\mathrm{Per}(E)$. For example, if E is any $\mathcal{L}^n$-null set, then it is easy to prove that

$$\mathrm{Per}(E) = \mathrm{Per}(\varnothing) = 0.$$

On the other hand, the interior of E is empty and hence

$$\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}\left(\overline{E}\right),$$

and this last quantity is, in general, strictly positive.

There is a result which links the notion of the perimeter, and the Hausdorff measure of the boundary for every Borel set E Borel, but the proof is rather involved. Here we state the result and sketch the proof of a slightly weaker statement.

Theorem 10.10. *For every $E \subset \mathbb{R}^n$ Borel, it turns out that*

$$\mathrm{Per}(E) \leq \mathcal{H}^{n-1}(\partial E). \tag{10.9}$$

Theorem 10.11. *There is a constant $c > 1$ such that*

$$\mathrm{Per}(E) \leq c \cdot \mathcal{H}^{n-1}(\partial E) \tag{10.10}$$

for every $E \subset \mathbb{R}^n$ Borel.

Sketch of the Proof. We can assume without loss of generality that E is a closed and bounded Borel set with finite Hausdorff measure of the boundary, that is,

$$\mathcal{H}^{n-1}(\partial E) < +\infty.$$

Step 1. Let $\delta > 0$ be a positive number, and let $\{B_i^\delta\}_{i\in\mathbb{N}}$ be a collection of open balls of uniformly bounded radii $r_i \leq \delta$, which is an optimal cover of the boundary ∂E with respect to the spherical Hausdorff measure. By compactness of ∂E, it turns out that

$$\partial E \subseteq \bigcup_{i=1}^{N(\delta)} B_i^\delta,$$

where $N(\delta)$ is a positive integer. In particular, we have the estimate

$$\sum_{i=1}^{N(\delta)} (2r_i)^{n-1} \leq \mathcal{H}_s^{n-1}(\partial E) \leq C \cdot \mathcal{H}^{n-1}(\partial E),$$

as a consequence of the fact that the Hausdorff measure is equivalent to the Hausdorff spherical measure. Let

$$E_\delta := E \cup \bigcup_{i=1}^{N(\delta)} B_i^\delta,$$

and notice that $E_\delta \xrightarrow{d} E$ as $\delta \to 0^+$, that is,

$$|E_\delta \triangle E| = |E_\delta \setminus E| \xrightarrow{\delta\to 0^+} 0,$$

since we assumed E to be a closed set.

Step 2. Suppose that the balls intersect transversally. Then the boundary ∂E_δ is Lipschitz, which means that the perimeter formula (10.8) holds. Moreover, the perimeter functional is a lower semi-continuous function, and therefore

$$\mathrm{Per}(E) \leq \liminf_{\delta \to 0^+} \mathrm{Per}(E_\delta) = \liminf_{\delta \to 0^+} \mathcal{H}^{n-1}(\partial E_\delta).$$

On the other hand, the boundary of E_δ is covered by the finite family of δ-balls, which means that

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E_\delta) \leq \mathcal{H}^{n-1}\left(\bigcup_{i=1}^{N} \partial B_i^\delta\right) \leq \\ \leq \sum_{i=1}^{N} \mathcal{H}^{n-1}\left(\partial B_i^\delta\right) \leq \\ \leq C' \sum_{i=1}^{N} (2r_i)^{n-1} \leq c \cdot \mathcal{H}^{n-1}(\partial E),\end{aligned}$$

where $c := C \cdot C'$ is the sought constant. □

Exercise 10.2. Let $E \subset \mathbb{R}$ be a finite perimeter set. Prove that E is, up to $\mathcal{L}^1$-null sets, the union of finitely many intervals.

Lemma 10.12. *Let $E \subseteq \mathbb{R}$ be a finite perimeter set. There exists a representative of E, also denoted by E, such that formula (10.8) holds.*

Proof. The function $\mathbb{1}_E$ belongs to $\mathrm{BV}(\mathbb{R})$ by assumption. Therefore, by Lemma 9.7 there exists a constant $c \in \mathbb{R}$ such that

$$\mathbb{1}_E(x) = c + \mu([-\infty, x]),$$

where μ denotes the weak derivative $D(\mathbb{1}_E)$. The reader may prove that, if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$, then this is a continuous representative of $\mathbb{1}_E$ that satisfies the formula (10.8). □

Proposition 10.13. *Let $E \subset \mathbb{R}^n$ and $n \geq 2$. Then there exists a Borel set E such that no representative of E satisfies the perimeter formula (10.8), that is, the inequality is strict for every representative E' of E.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ be a dense sequence of points, and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying the condition

$$\sum_{n\in\mathbb{N}} n \cdot r_n < +\infty. \tag{10.11}$$

Step 1. Let

$$L := \sum_{n\in\mathbb{N}} r_n < +\infty$$

and set $B_n := B(x_n, r_n)$. We define inductively a sequence of finite perimeter sets given by the symmetric differences, that is,

$$\begin{cases} E_0 = B_0, \\ E_n = E_{n-1} \triangle B_n. \end{cases}$$

We claim that $(\mathbb{1}_{E_n})_{n\in\mathbb{N}} \subset \mathrm{BV}(\mathbb{R}^2)$ is a Cauchy sequence with respect to the $\|\cdot\|_{\mathrm{BV}}$-norm. If the claim holds, then we can infer that

$$\mathbb{1}_{E_n} \xrightarrow{L^1} \mathbb{1}_E,$$

and also that

$$D(\mathbb{1}_E) = \pm\,\nu\,\mathbb{1}_\Sigma \cdot \mathcal{H}^1,$$

where

$$\Sigma = \bigcup_{n\in\mathbb{N}} \partial B(x_n,\, r_n)$$

is a 1-rectifiable surface. In fact, it is enough to notice that

$$\partial E_n = \bigcup_{i=0}^{n} \partial B_i \implies \mathcal{H}^1\left(\partial E_n\right) \leq 2\pi L,$$

and

$$D\mathbb{1}_{E_n} = \pm\,\nu_n\,\mathbb{1}_{\partial E_n} \cdot \mathcal{H}^1.$$

Step 2. We now prove the claim. By definition of symmetric difference it turns out that

$$\mathbb{1}_{E_n} = \mathbb{1}_{B_0} + \left(\mathbb{1}_{B_1\setminus E_0} - \mathbb{1}_{B_1\cap E_0}\right) + \cdots =: \sum_{i=0}^{n} u_i,$$

and therefore it is equivalent to show that

$$\sum_{i=0}^{+\infty} \|u_i\|_{\mathrm{BV}(\mathbb{R}^n)} < +\infty.$$

The L^1-norm of u_i may be roughly estimated by the area of the ball B_i, which means that

$$\|u_i\|_{L^1(\mathbb{R}^n)} = \pi \cdot r_i^2.$$

In a similar way, the total variation of the derivative may be estimated by

$$\|Du_i\| = 2\pi \cdot r_i + 2\mathcal{H}^1\left(\partial E_{i-1} \cap B_i\right) \leq 2\pi(i+1)r_i,$$

and therefore the assumption (10.11) proves the claim.

In particular, the support of the measure $D\mathbb{1}_E$ is the whole plane $\mathbb{R}^2$, and hence every representative of E has boundary containing the support of $\mathbb{1}_E$, that is,

$$\text{"}\partial\widetilde{E} = \mathbb{R}^2.\text{"}$$

We finally infer that no representative $\widetilde{E}$ of E has $\mathcal{H}^1$-finite boundary. $\square$

Main Result. In this final paragraph, we are finally ready to state and prove the so-called *structure theorem* due to De Giorgi and Federer. The proof is quite involved. Therefore we will need a lot of work to get through it.

Definition 10.14 (Essential Boundary)**.** Let E be a Borel set in $\mathbb{R}^n$. The *essential boundary* of E in $\mathbb{R}^n$ is defined by

$$\partial_* E := \left\{x \in \partial E \mid \Theta_n\left(E,\, x\right) \neq 0,\, 1 \text{ or } \Theta_n\left(E,\, x\right) \text{ does not exist}\right\},$$

where Θ_n denotes the n-dimensional Hausdorff density.

Remark 10.8. The essential boundary is, in general, strictly contained in the boundary. For example, if ∂E is locally the graph of a function f that admits a cuspid p, then $p \in \partial E \setminus \partial_* E$.

Definition 10.15 (Normal Vector Field)**.** If Σ is a $(n-1)$-rectifiable set, then $\eta : \Sigma \longrightarrow \mathbb{R}^n$ is a *normal vector field* if $|\eta(x)| = 1$ for every $x \in \Sigma$ and

$$\eta(x) \perp \operatorname{Tan}(\Sigma, x) \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } x \in \Sigma.$$

Theorem 10.16 (De Giorgi-Federer)**.** *Let E be a finite perimeter set in $\mathbb{R}^n$ (or in Ω). Then the following assertions hold:*

(1) The essential boundary $\partial_* E$ is $\mathcal{H}^{n-1}$-finite.

(2) The essential boundary $\partial_* E$ is $(n-1)$-rectifiable.

(3) There exists a normal vector field η which is "inner normal" in the following sense:

$$\mathcal{L}^n\left(\left(E \triangle (x + M^+_{\eta(x)})\right) \cap B_{x,\,r}\right) \ll r^n \quad \text{as } r \to 0^+ \tag{10.12}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \partial_* E$.

(4) The derivative of the characteristic function of E is given by

$$D(\mathbb{1}_E) = \mathbb{1}_{\partial_* E}\, \eta \cdot \mathcal{H}^{n-1} \tag{10.13}$$

Remark 10.9. The assertion (10.12) is completely equivalent to the following one:

$$\frac{1}{r}(E - x) \xrightarrow{L^1_{loc}(\mathbb{R}^n)}_{r \to 0^+} M^+_{\eta(x)}. \tag{10.14}$$

Corollary 10.17. *Under the assumptions of the Theorem 10.16 it turns out that*

$$\Theta_n(E, x) = \frac{1}{2}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \partial_* E$.

Corollary 10.18. *Under the assumptions of the Theorem 10.16 it turns out that*

$$\operatorname{Per}(E) = \mathcal{H}^{n-1}\left(\partial_* E\right) < \infty.$$

In a different paper, Federer proved that also a sort of converse of this theorem holds true, but we will only state it here since the proof is far beyond the reach of this course.

Theorem 10.19 (Federer)**.** *Let $E \subseteq \mathbb{R}^n$ be a Borel set such that $\mathcal{H}^{n-1}(\partial_* E) < +\infty$. Then E is a finite perimeter set.*

Exercise 10.3. Let $E \subseteq \mathbb{R}^n$ be a set with empty essential boundary, that is $\partial_* E = \varnothing$. Prove that either $|E| = 0$ or $|\mathbb{R} \setminus E| = 0$.

Definition 10.20 (Reduced Boundary)**.** Let E be a finite perimeter set in $\mathbb{R}^n$. The *reduced boundary* of E in $\mathbb{R}^n$, denoted by $\partial^* E$, is the set of all $x \in \mathbb{R}^n$ such that the Radon-Nikodym density

$$\eta(x) := \frac{\mathrm{d}(D\mathbb{1}_E)}{\mathrm{d}|D\mathbb{1}_E|}(x)$$

exists and has norm equal to one.

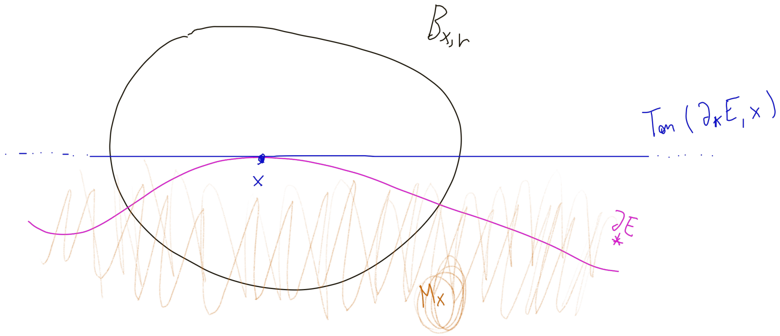

Figure 10.9: Intuitive idea of the statement of the De Giorgi-Federer structure theorem. The picture is slightly misleading: the vector space $M_\eta(x)^+$ is the whole space "under" the tangent to $\partial_* E$ at x.

Remark 10.10. Equivalently, if we denote by $\tau \cdot \mu$ the weak derivative of $\mathbb{1}_E$, then $\eta(x)$ is the L^1-approximate limit, with $|\tau| = 1$, of τ at $\mathcal{H}^{n-1}$-almost every x, that is,

$$\fint_{B(x,r)} |\tau(y) - \eta(x)| \, \mathrm{d}\mu(y) \xrightarrow{r \to 0^+} 0.$$

Example 10.2. In general, the reduced boundary is a proper subset of the boundary. For example, if E is a rhombus, then the four vertexes are not in the reduced boundary $\partial^* E$.

Theorem 10.21 (De Giorgi)**.** *Let $E \subseteq \mathbb{R}^n$ be a finite perimeter set in $\mathbb{R}^n$, and set $D(\mathbb{1}_E) := \tau \cdot \mu$. Then the following assertions hold true:*

(1) *The Radon-Nikodym density $\eta(x)$ coincides μ-almost everywhere with $\tau(x)$. Moreover, the measure μ is supported in the reduced boundary, that is,*

$$\mu\left(\mathbb{R}^n \setminus \partial^* E\right) = 0. \tag{10.15}$$

(2) *The reduced boundary $\partial^* E$ is $(n-1)$-rectifiable.*

(3) *The measure μ is the restriction of the $(n-1)$-dimensional Hausdorff measure to the reduced boundary, that is,*

$$\mu = \mathcal{H}^{n-1} \llcorner \partial^* E. \tag{10.16}$$

(4) *The Radon-Nikodym density $\eta(x)$ is normal to the reduced boundary at x for every $x \in \partial^* E$, that is,*

$$\eta(x)^\perp = \mathrm{Tan}_*\left(\partial^* E, x\right)$$

where $\mathrm{Tan}_*\left(-, \cdot\right)$ denotes the approximate tangent plane.

(5) The normal vector field η satisfies the following property:

$$\mathcal{L}^n\left(\left(E \triangle (x + M^+_{\eta(x)})\right) \cap B_{x,\,r}\right) \ll r^n \quad \text{as } r \to 0^+ \tag{10.17}$$

for every $x \in \partial^* E$. Equivalently,

$$\frac{1}{r}(E - x) \xrightarrow{L^1_{loc}(\mathbb{R}^n)}_{r \to 0^+} M^+_{\eta(x)}. \tag{10.18}$$

Corollary 10.22. *Under the assumptions of the Theorem 10.21 it turns out that*

$$\Theta_n(E,\, x) = \frac{1}{2}$$

for every $x \in \partial^* E$.

The proof of De Giorgi's Theorem 10.21 will follow reasonably quickly from a sequence of many technical lemmas that we will patently state and prove here.

In order to ease the statements of the following results, we will fix a point $x \in \partial^* E$ and denote it by 0 unless otherwise stated.

Lemma 10.23. *There exists an universal constant $c := c(n) > 0$ such that*

$$\mu(B_r) \le c \cdot r^{n-1}. \tag{10.19}$$

In particular, it turns out that there exists an universal constant $c' := c'(n) > 0$ such that

$$\mu(B_r) \le c' \cdot \mathcal{H}^{n-1}\left(\partial B_r \cap E\right). \tag{10.20}$$

Proof. Set $E_r := E \cap B_r$.

Step 1. We claim that, for almost every $r > 0$, the weak derivative of the characteristic function satisfies an additive formula:

$$D(\mathbb{1}_{E_r}) = \mathbb{1}_{B_r} \cdot D(\mathbb{1}_E) + \mathbb{1}_E \cdot D(\mathbb{1}_{B_r}). \tag{10.21}$$

We notice that the integral of $\eta(0)$ with respect to the measure $D(\mathbb{1}_{E_r})$ is given by

$$\int_{\mathbb{R}^n} \eta(0)\, \mathrm{d}D(\mathbb{1}_{E_r}) = 0,$$

since $\eta(0)$ is constant. On the other hand, it follows from (10.21) that

$$\int_{\mathbb{R}^n} \eta(0)\, \mathrm{d}D(\mathbb{1}_{E_r}) = \int_{B_r} \eta(0)\eta(x)\, \mathrm{d}\mu(x) + \int_E \eta(0) \mathrm{d}D(\mathbb{1}_{B_r})(x) =$$

$$= \int_{B_r} \eta(0)\eta(x)\, \mathrm{d}\mu(x) + \int_{\partial B_r \cap E} \eta(0)\nu_{inn}(x)\, \mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\nu_{inn}(x)$ denotes the inner normal vector to the sphere at x. In particular, it turns out that

$$\int_{B_r} \eta(0)\eta(x)\, \mathrm{d}\mu(x) = -\int_{\partial B_r \cap E} \eta(0)\nu_{inn}(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) \le$$

$$\le \mathcal{H}^{n-1}(\partial B_r) \sim \alpha_n r^{n-1},$$

where the inequality follows from the fact that both ν_{inn} and η are vectors of norm less or equal than 1.

In conclusion, we notice that for $r \to 0^+$ the left-hand side of the inequality above is asymptotically equivalent to $\mu(B_r)$, which means that there exists a $r_0 > 0$ small enough such that

$$\frac{1}{2}\,\mu(B_r) \leq \int_{B_r} \eta(0)\eta(x)\,\mathrm{d}\mu(x) \leq \mathcal{H}^{n-1}(\partial B_r) \sim c(n)\cdot r^{n-1},$$

for almost every $0 < r < r_0$.

Step 2. We now sketch the proof of the claim (10.21). More precisely, we show that the formula holds for every f.p.s. E and almost every $r > 0$ - even when 0 does not belong to the reduced boundary -.

Step 2.1. First, we notice that (10.21) holds for every smooth f.p.s. E that is also transversal to the boundary of the ball B_r. Indeed, one can easily prove that

$$\partial E_r = \left(\partial B_r \cap \overline{E}\right) \cup \left(\overline{B_r} \cap \partial E\right).$$

Step 2.2. Let E^n be an approximation of E by smooth sets, that is

$$E^n \xrightarrow{L^1_{loc}} E.$$

Choose $r > 0$ in such a way that ∂E^n_r is transversal to the sphere ∂B_r for every $n \in \mathbb{N}$. The previous step proves that E^n satisfies (10.21) for every $n \in \mathbb{N}$, which means that we can pass the identity to the limit[3]. $\square$

Lemma 10.24. *There are two universal constants $c := c(n)$, $c' := c'(n) > 0$ such that for every $r > 0$ small enough*

$$c \cdot r^n \leq |E \cap B_r| \leq c' \cdot r^n, \tag{10.22}$$

and

$$c \cdot r^n \leq |E^c \cap B_r| \leq c' \cdot r^n. \tag{10.23}$$

Remark 10.11. It is easy to prove that the complement of a f.p.s. is also a f.p.s. with the same weak derivative up to null sets. In particular, the inequality (10.22) implies the inequality (10.23) by passing to the complement.

Proof. Set $E_r := E \cap B_r$.

Step 1. We claim that the following *isoperimetrical inequality* holds: For every $r > 0$ there exists an universal constant $c := c(n) > 0$ such that

$$v(r)^{\frac{n-1}{n}} \leq c \cdot \mathrm{Per}(E_r), \tag{10.24}$$

where $v(r)$ denotes the Lebesgue measure of E_r. If the inequality holds, then it is easy to see that

$$\begin{aligned} v(r)^{\frac{n-1}{n}} &\leq c_1 \cdot \|D(\mathbb{1}_{E_r})\| \leq \\ &\leq c_2 \left(\mathrm{Per}_{B_r}(E) + \mathcal{H}^{n-1}\left(E \cap \partial B_r\right)\right) \leq \\ &\leq c_3 \cdot \mathcal{H}^{n-1}\left(E \cap \partial B_r\right) = c_3 \cdot \dot{v}(r), \end{aligned}$$

[3] We will not give any detail, but one needs to be careful to the notion of convergence that comes into play here.

since

$$\operatorname{Per}_{B_r}(E) = |D(\mathbb{1}_E)|(B_r) = \mu(B_r),$$

and clearly $\mu(B_r)$ and $\mathcal{H}^{n-1}(E \cap \partial B_r)$ are comparable quantities as a consequence of the previous result.

Step 2. The symbol $\dot{v}$ denotes the classic derivative of v since a Lipschitz function is almost everywhere differentiable (by Rademacher Theorem 7.7). The reader may prove, as an exercise, that the formula used above holds:

$$\dot{v}(r) = \mathcal{H}^{n-1}(E \cap \partial B_r) \quad \text{for almost every } r > 0. \tag{10.25}$$

Step 3. In conclusion, we notice that $v(r)$ is an almost everywhere differentiable function which satisfies the following inequality:

$$v(r)^{\frac{n-1}{n}} \leq c_3 \cdot \dot{v}(r).$$

A straightforward computation yields to

$$v(r) \geq c(n) \cdot r^n,$$

which is exactly the nontrivial part of (10.22). The trivial one, on the other hand, follows immediately from the worst estimate possible:

$$v(r) \leq |B_r| \sim c'(n) \cdot r^n.$$

Step 4. It remains to prove the the claimed isoperimetrical inequality (10.24); surprisingly, it follows easily from a far more general theorem.

Precisely, one can prove that there exists a universal constant $c := c(n)$ such that for any given F bounded f.p.s., it turns out that

$$|F|^{\frac{n-1}{n}} \leq c(n) \cdot \operatorname{Per}(F).$$

Recall that the immersion

$$\mathrm{BV}(\Omega) \hookrightarrow L^{1^*}(\Omega),$$

where $1^* = \frac{n}{n-1}$, is continuous (and compact). Therefore, there exists a universal constant $c' := c'(n) > 0$ such that

$$\|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq c' \cdot \|u\|_{\mathrm{BV}(\Omega)}.$$

The reader may prove that one can always replace the $\mathrm{BV}(\Omega)$-norm with the total variation $\|Du\|$ if u is, for example, compactly supported. The isoperimetrical inequality follows immediately by taking $u := \mathbb{1}_F$ for every **bounded** f.p.s. F. ☐

Lemma 10.25. *Let $E_r := \frac{1}{r}(E - x) = \frac{1}{r}E$. Then*

$$E_r \xrightarrow{L^1_{loc}} M^+_{\eta(0)}, \tag{10.26}$$

or, equivalently, it turns out that

$$\left|(E \triangle M^+_{\eta(0)}) \cap B_r\right| \ll r^n \quad \text{as } r \to 0^+. \tag{10.27}$$

Proof. Denote by η_r and μ_r the rescaling of η and μ respectively, in such a way that

$$D(\mathbb{1}_{E_r}) = \eta_r \cdot \mu_r.$$

Step 1. By Lemma 10.23 it turns out that

$$\mu_r(B_1) = \frac{\mu(B_r)}{r^{n-1}} \leq c(n) \implies \mu_r(B_R) \leq c(n) \cdot R^{n-1} \tag{10.28}$$

for every $R > 0$. Similarly, by Lemma 10.24 it turns out that

$$c_1(n) \leq |E_r \cap B_1| \leq c_1'(n),$$

from which it follows that

$$c_1(n) \cdot R^n \leq |E_r \cap B_R| \leq c_1'(n) \cdot R^n \tag{10.29}$$

for every $R > 0$. The compactness property of the f.p.s. (see Theorem 10.3), together with the inequality (10.28), imply that, up to subsequences,

$$\mathbb{1}_{E_r} \xrightarrow{L^1_{loc}} \mathbb{1}_{E_0}$$

$$D\left(\mathbb{1}_{E_r}\right) \xrightarrow{\text{locally}} D\left(\mathbb{1}_{E_0}\right)$$

for some f.p.s. E_0. Since 0 belongs to the reduced boundary $\partial^* E$, we have that

$$\fint_{B_r} |\eta(x) - \eta(0)| \, \mathrm{d}\mu(x) \xrightarrow{r \to 0^+} 0,$$

which means that for every $R > 0$

$$\fint_{B_R} |\eta_r(x) - \eta(0)| \, \mathrm{d}\mu_r(x) \xrightarrow{r \to 0^+} 0.$$

We may assume without loss of generality that $\eta(0) = e_1$, where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. It turns out that

$$D\left(\mathbb{1}_{E_0}\right) = \eta(0) \cdot \mu_0,$$

from which we infer that $D^i\left(\mathbb{1}_{E_0}\right)$ is zero for every $i = 2, \ldots, n$.

Step 2. By Lemma 10.26 it turns out that $\mathbb{1}_{E_0}$ has a representative which depends only on the first variable x_1, and thus we can infer that

$$E_0 = x_0 + M^+_{\eta(0)}$$

for some $x_0 \in \mathbb{R}^n$. It remains to prove that x_0 is necessarily equal to zero.

We argue by contradiction: suppose that x_0 is not 0. Then one can find a radius $R > 0$ small enough such that the Lebesgue measure of the intersection $E_0 \cap B_R$ is zero, in contradiction with the inequality (10.29). Therefore

$$E_0 = M^+_{\eta(0)} \qquad \text{and} \qquad E_r \xrightarrow{L^1_{loc}} E_0 = M^+_{\eta(0)},$$

which is exactly what we wanted to prove. □

Lemma 10.26. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ be a function such that*

$$D^i f \equiv 0 \qquad \forall\, i = 2, \ldots, n.$$

Then there exists $\widetilde{f} \in L^1_{loc}(\mathbb{R})$ such that

$$\widetilde{f}(x_1) = f(x_1, \ldots, x_n) \quad \textit{for } \mathcal{L}^n\textit{-almost every } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Lemma 10.27. *Let $E_r := \frac{1}{r}(E - x) = \frac{1}{r}E$. Then*

$$D(\mathbb{1}_{E_r}) \xrightarrow{\textit{locally}} D(\mathbb{1}_{M^+_{\eta(0)}}) = \eta(0)\, \mathbb{1}_{\eta(0)^\perp} \cdot \mathcal{H}^{n-1}. \tag{10.30}$$

Lemma 10.28. *For every $r > 0$ it turns out that*

$$\mu(B_r) \sim \alpha_{n-1} r^{n-1}. \tag{10.31}$$

Proof. We divide the argument into two steps to ease the notation.

Step 1. First, we deduce from Lemma 10.27 that

$$|D(\mathbb{1}_{E_r})| \xrightarrow{\text{locally}} \left| D(\mathbb{1}_{M^+_{\eta(0)}}) \right|. \tag{10.32}$$

Indeed, if $\eta(0) = e_1$, then

$$⨍_{B_R} |\eta_r(x) - \eta(0)| \, \mathrm{d}\mu_r(x) \xrightarrow{r \to 0^+} 0, \qquad \forall\, R > 0$$

implies that

$$\left| \frac{\partial\, \mathbb{1}_{E_r}}{\partial x_1} \right| \xrightarrow{r \to 0^+} \left| \frac{\partial\, \mathbb{1}_{E_0}}{\partial x_1} \right|,$$

which is exactly what we wanted to prove since $E_0 = M^+_{\eta(0)}$.

Step 2. From (10.32) it follows that[4]

$$\frac{\mu(B_r)}{r^{n-1}} = \mu_r(B_1) \longrightarrow \mathcal{H}^{n-1}(\eta(0)^\perp \cap B_1) = \alpha_{n-1}$$

for almost every $r > 0$. To conclude the proof, we need the following auxiliary lemma.

Lemma 10.29. *Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of vector measures on X. Assume that $\lambda_n := \tau_n \cdot \mu_n$ converges in the sense of measures (weakly-$*$) to a measue $\lambda := \tau \cdot \mu$, and assume that there exists a constant vector v such that*

$$\int_X |\tau_n - v|(x) \, \mathrm{d}\mu_n(x) \xrightarrow{n \to +\infty} 0.$$

Then μ_n converges to μ and $\lambda = v \cdot \mu$ almost everywhere.

□

We have finally introduced all the technical tools to prove the statements of the De Giorgi's theorem presented above.

Proof of Theorem 10.21.

(1) The first statement about the density and the support of μ is an immediate consequence of the argument used in the next point.

[4] Here the reader needs to be careful. The convergence is not true in general, but one can prove that it is enough to ask that $\mathcal{H}^{n-1}(\eta(0)^\perp \cap B_1) = 0$.

(2) We sketch two different proofs of the fact that the reduced boundary is $(n-1)$-rectifiable. The first proof is due to De Giorgi, while the second one relies more on the sequence of technical lemmas above.

(a) For every $r_0 > 0$ and $\delta_0 > 0$ we consider

$$S_{r_0,\,\delta_0} := \left\{ x \in \partial^* E \;\middle|\; \left|(E \triangle (x + M^+_{\eta(x)})) \cap B(x,\, r)\right| \leq \delta_0 r^n \text{ for every } r \leq r_0. \right\}.$$

The idea is to prove that each $S_{r_0,\,\delta_0}$ is $(n-1)$-rectifiable. In fact, we know that the countable union of d-rectifiable set is also d-rectifiable, and hence the thesis follows from the fact that

$$\partial^* E = \bigcup_{(n,\, m) \in \mathbb{N}^2} S_{\frac{1}{n},\, m}.$$

(b) The support of the measure μ is contained in the reduced boundary $\partial^* E$; hence by Lemma 10.28 we can infer that

$$\mu\,(B(x,\, r)) \sim \alpha_{n-1} r^{n-1} \quad \text{for } \mu\text{-almost every } x \in \partial^* E.$$

We claim that there exists $F \subseteq \operatorname{spt}(\mu)$ such that

$$c_1\, \mathbb{1}_F \cdot \mathcal{H}^{n-1} \leq \mu \leq c_2\, \mathbb{1}_F \cdot \mathcal{H}^{n-1} \qquad \text{and} \qquad \mu\,(\partial^* E \setminus F) = 0.$$

Indeed, it is enough to notice that $\mu\,(E \setminus \partial^* E) = 0$ as the support of μ is contained in the reduced boundary. Moreover, the $(n-1)$-dimensional upper density is zero at almost every point of F, that is,

$$\Theta^*(\mu,\, x) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } x \in F,$$

and this implies that

$$\begin{cases} \mathcal{H}^{n-1}\,(\partial^* E \setminus F) = 0, \\ \mathcal{H}^{n-1}\,(\partial^* E \triangle F) = 0. \end{cases}$$

The measure μ is thus equivalent to the restriction of the $\mathcal{H}^{n-1}$ measure to the reduced boundary, that is, there are two constants $c_1,\, c_2 > 0$ such that

$$c_1'\, \mathbb{1}_{\partial^* E} \cdot \mathcal{H}^{n-1} \leq \mu \leq c_2'\, \mathbb{1}_{\partial^* E} \cdot \mathcal{H}^{n-1} \implies \mu \sim \mathcal{H}^{n-1} \llcorner \partial^* E.$$

We deduce that the blowup of μ at x is given by

$$\mathbb{1}_{\eta(x)^\perp} \cdot \mathcal{H}^{n-1},$$

which means that $\eta(x)^\perp$ is the approximate tangent plane of $\partial^* E$ at x.
Moreover, the lower density is bounded from below; therefore, by Corollary 8.24, we can finally infer that $\partial^* E$ is $(n-1)$-rectifiable.

(3) In the previous point we have proved that μ is equivalent to $\mathbb{1}_{\partial^* E} \cdot \mathcal{H}^{n-1}$, which means that there exists a function g such that

$$\mu = \mathbb{1}_{\partial^* E}\, g \cdot \mathcal{H}^{n-1}.$$

Therefore it is enough to show that g is equal to 1 $\mathcal{H}^{n-1}$-almost everywhere. The $(n-1)$-dimensional upper density of the Hausdorff measure is given by

$$\Theta^*_{n-1}(\partial^* E,\, x) = 1 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } x \in \partial^* E,$$

while the $(n-1)$-dimensional upper density of μ is given by

$$\Theta^*_{n-1}(\mu, x) = 1 \quad \text{for } \mu\text{-almost every } x \in \partial^* E.$$

In particular, it follows that $g = 1$ almost everywhere (with respect to either μ or $\mathcal{H}^{n-1}$).

(4) This assertion also follows immediately from the second proof of **(2)**.

(5) This assertion simply summarize the content of Lemma 10.25.

□

We are finally ready to prove the De Giorgi-Federer Theorem 10.16. The reduced boundary is a subset of the essential boundary, therefore everything follows from the previous result if we are able to prove that

$$\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0.$$

Proposition 10.30 (Isoperimetrical Inequality)**.** *Let $E \subseteq \mathbb{R}^n$ be a finite perimeter set in $B(x, r)$. Then there exists an universal constant $c := c(n) > 0$ such that*[5]

$$(|E| \wedge |B(x, r) \setminus E|)^{\frac{n-1}{n}} \le c \cdot \mathrm{Per}_{B(x,r)}(E). \tag{10.33}$$

Proof. The Sobolev Embedding Theorem implies that the immersion

$$\mathrm{BV}(B(x, r)) \hookrightarrow L^{1^*}(B(x, r)),$$

where $1^* = \frac{n}{n-1}$, is continuous. Therefore, there exists a universal constant $c := c(n) > 0$ (which does not depend on the radius and the center of the ball) such that

$$\|u\|_{L^{\frac{n}{n-1}}(B(x,r))} \le \|u\|_{\mathrm{BV}(B(x,r))}.$$

We replace the $\mathrm{BV}(B(x, r))$ norm with the equivalent one

$$\|Du\| + \left| \fint_{B(x,r)} u \right|.$$

It turns out that

$$\|u - \fint_{B(x,r)} u\|_{L^{\frac{n}{n-1}}(B(x,r))} \le c(n)\, \|Du\|.$$

By symmetry, we may assume without loss of generality that $|E| < |B(x, r) \setminus E|$. Then it is easy to prove that

$$\mathbb{1}_E(x) - \frac{|E|}{|B(x, r)|} > \frac{1}{2}, \qquad \text{if } x \in E,$$

and the thesis follows from the inequality above since

$$c(n)\, \|Du\| \ge \|u - \fint_B u\|_{L^{\frac{n}{n-1}}(B(x,r))} \ge \frac{1}{2}|E|^{\frac{n-1}{n}}.$$

□

[5]The inequality presented here is **not** sharp! The result can be refined, but it is not necessary for our purposes.

Lemma 10.31. *Under the assumptions of the Theorem 10.16, it turns out that*

$$\mu\left(B(x,\, r)\right) \ll r^{n-1} \implies \Theta_n^*(E,\, x) \in \{0,\, 1\}.$$

In particular, the point x does not belong to the essential boundary $\partial_* E$.

Proof. Let $x \in \mathbb{R}^n$ be a point such that $\Theta_n(E,\, x) \neq 0,\, 1$ (i.e., x belongs to the essential boundary). Then there exists $\delta > 0$ and a sequence $(r_n)_{n\in\mathbb{N}}$ converging to 0 such that either

$$\exists\, (r_{n_k})_{k\in\mathbb{N}} \,:\, r_k \xrightarrow{k\to+\infty} 0 \quad \text{and} \quad \frac{|E \cap B(x,\, r_{n_k})|}{r_{n_k}^n} \geq \delta,$$

or

$$\exists\, (r_{n_k})_{k\in\mathbb{N}} \,:\, r_k \xrightarrow{k\to+\infty} 0 \quad \text{and} \quad \frac{|E \cap B(x,\, r_{n_k})|}{r_{n_k}^n} \leq \alpha_n - \delta,$$

Either way the function

$$\mathbb{R} \ni r \longmapsto \frac{|E \cap B(x,\, r)|}{r^n}$$

is continuous; hence there exists $\lambda' < \lambda$ real numbers such that

$$\lambda' \leq \frac{|E \cap B(x,\, r_{n_k})|}{r_{n_k}^n} < \lambda$$

for $r_{n_k} \xrightarrow{k\to+\infty} 0$. On the other hand, the isoperimetrical inequality (10.33) implies that

$$|E \cap B(x,\, r) \setminus E|^{\frac{n-1}{n}} \leq c \cdot \mathrm{Per}_{B(x,\, r)}(E) = \mu\left(B(x,\, r)\right),$$

from which we derive a contradiction with the assumption $\mu(B(x,\, r)) \ll r^{n-1}$ by noticing that

$$\lambda' r^{n-1} \leq |E \cap B(x,\, r) \setminus E|^{\frac{n-1}{n}} \leq c \cdot \mathrm{Per}_{B(x,\, r)}(E) = \mu\left(B(x,\, r)\right)$$

for every $r > 0$ small enough. □

Lemma 10.32. *Under the assumptions of the Theorem 10.16, it turns out that*

$$\mathcal{H}^{n-1}\left(\partial_* E \setminus \partial^* E\right) = 0.$$

Proof. Equivalently, we may prove that for $\mathcal{H}^{n-1}$-almost every point $x \notin \partial^* E$, it turns out that x does not belong to $\partial_* E$ as well. By Lemma 10.31, it suffices to prove that

$$\mu\left(B(x,\, r)\right) \ll r^{n-1},$$

but this follows easily from the fact that

$$\mu = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

□

10.6 Back to the Capillarity Problem

We are finally ready to prove the existence of a solution to the capillarity problem for any $\sigma \in [-1,\, 1]$ using the new notion of essential boundary.

Framework. Fix $\Omega \subset \mathbb{R}^n$ regular bounded open set, and fix a volume

$$0 < m < |\Omega|.$$

The capillarity energy is given by

$$\mathscr{F}(E) := \mathcal{H}^{n-1}(\Sigma^f) + \sigma\, \mathcal{H}^{n-1}(\Sigma^c), \tag{10.34}$$

where we define

$$\begin{cases} \Sigma^f := \partial_* E \cap \Omega, \\ \Sigma^c := \partial_* E \cap \partial\Omega. \end{cases}$$

Theorem 10.33. *The functional (10.34) is lower semi-continuous on the set of all finite perimeter set. Moreover, there exists (at least) a minimum point E for $\mathscr{F}$ for every $\sigma \in [-1, 1]$.*

Proof. We split the proof into two cases: positive and negative σ.

Case "$1 \geq \sigma \geq 0$": The functional (10.34) can be equivalently written as

$$\begin{aligned} \mathscr{F}(E) &= \sigma\, \mathcal{H}^{n-1}(\partial_* E) + (1-\sigma)\, \mathcal{H}^{n-1}\left(\Sigma^f\right) = \\ &= \sigma \cdot \operatorname{Per}(E) + (1-\sigma) \cdot \operatorname{Per}_\Omega(E). \end{aligned} \tag{10.35}$$

In particular, the functional (10.35) is equal to the convex sum of two lower semi-continuous functionals, which means that it is also lower semi-continuous.

The functional $\mathscr{F}$ is also coercive. Indeed, let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ be a uniformly bounded sequence, that is, there exists a constant $M \in \mathbb{R}$ such that

$$\mathscr{F}(E_n) \leq M, \qquad \forall\, n \in \mathbb{N}.$$

Then, up to subsequences, there exists a finite perimeter set E such that $E_n \xrightarrow{\mathcal{X}} E$. If σ is strictly greater than 0, then the perimeters are equibounded

$$\sigma \cdot \operatorname{Per}(E_n) \leq \mathscr{F}(E_n) \leq M,$$

which means that we can apply a compactness theorem for f.p.s. and conclude that the functional is coercive. On the other hand, if $\sigma = 0$, then the coerciveness follows immediately from the following estimate:

$$\operatorname{Per}(E_n) \leq \mathcal{H}^{n-1}(\partial\Omega) + \operatorname{Per}_\Omega(E_n) \leq \mathcal{H}^{n-1}(\partial\Omega) + M < +\infty.$$

Case "$-1 \geq \sigma \leq 0$": The idea is to pass everything to the complement, that is, we consider as a variable for the functional (10.34) the set $E^c := \Omega \setminus E$ instead of E. In particular, we notice that there are nice relations between the surfaces, that is,

$$\begin{cases} \Sigma^f_c = \Sigma^f, \\ \Sigma^c_c = \partial\Omega \setminus \Sigma^c. \end{cases}$$

It turns out that

$$\begin{aligned} \mathscr{F}(E) &= \sigma\, \mathcal{H}^{n-1}(\partial\Omega) - \sigma\, \mathcal{H}^{n-1}(\partial\Omega) + \mathscr{F}(E) = \\ &= \sigma\, \mathcal{H}^{n-1}(\partial\Omega) + \mathcal{H}^{n-1}(\Sigma^f_c) + |\sigma|\, \mathcal{H}^{n-1}(\Sigma^c_c) = \\ &= \mathscr{F}_{|\sigma|}(E^c) + \sigma\, \mathcal{H}^{n-1}(\partial\Omega). \end{aligned} \tag{10.36}$$

Since $\sigma\, \mathcal{H}^{n-1}(\partial\Omega)$ is a constant, we can immediately reduce to the previous case $1 \geq |\sigma| \geq 0$. $\square$

Remark 10.12. The restriction of (10.34) to any smooth subset Ω in $\mathcal{X}$ agrees with the classical capillarity energy presented in the previous section, and the relaxation of $\mathscr{F}$ to $\mathcal{X}$ is given by $\mathscr{F}$ itself. Therefore every minimizing sequence, made up of smooth sets, converges to the minimum of $\mathscr{F}$ on $\mathcal{X}$.

Remark 10.13. Up to dimension 8, the inner regularity theory of the capillarity problem works just fine; the boundary regularity, on the other hand, is quite messy.

Remark 10.14. A similar problem arises when dealing with Plateau's type problems. More precisely, the surface

$$\Sigma = (\partial_* E_{min}) \cap \overline{\Omega}$$

is closed (in Ω), and it is also analytic up to dimension 7. As we shall prove at the end of the course, starting from dimension 8, one can find a counterexample to this statement (actually, it is possible to find a minimal surface such that it is analytic outside of a closed singular set of dimension $n-8$).

Chapter 11

Appendix

11.1 First Variation of a Functional

Framework. Let Σ be a d-rectifiable and $\mathcal{H}^d$-finite set in $\mathbb{R}^n$, and let $(\Phi_t)_{t\in\mathbb{R}}$ be a one parameter family of diffeomorphisms of $\mathbb{R}^n$ such that $\Phi_0 = \mathrm{id}_{\mathbb{R}^n}$. Let

$$v(x) := \frac{\mathrm{d}\Phi_t(x)}{\mathrm{d}t}\Big|_{t=0},$$

and suppose that $\Sigma_t := \Phi_t(\Sigma)$ is the collection of all the competitors (of Σ) to a certain minimum problem.

Lemma 11.1. *Let $x \in \Sigma$ be a point, and let $\{e_1, \dots, e_d\}$ be an orthonormal basis of the tangent space* $\mathrm{Tan}(\Sigma, x)$*. Then it turns out that*

$$\frac{\mathrm{d}\mathcal{H}^d(\Sigma_t)}{\mathrm{d}t}\Big|_{t=0} = \int_\Sigma \mathrm{div}_T(v)(x)\,\mathrm{d}\mathcal{H}^d(x), \tag{11.1}$$

where $\mathrm{div}_T(-)$ *denotes the divergence along the tangent plane, that is,*

$$\mathrm{div}_T(v)(x) = \sum_{i=1}^d \frac{\partial\left(\langle v,\, e_i\rangle\right)}{\partial e_i}(x).$$

Proof. The map Φ_t is a diffeomorphism for every t; hence we can apply the area formula (7.6) to Σ_t, and obtain the following identity:

$$\mathcal{H}^d\left(\Sigma_t\right) = \int_\Sigma J_T(\Phi_t)(x)\,\mathrm{d}\mathcal{H}^d(x), \tag{11.2}$$

where J_T denotes the tangential Jacobian, that is,

$$J_T(\Phi_t) = \sqrt{\det\left((\nabla_T\Phi_t)^t(\nabla_T\Phi_t)\right)}.$$

Fix $x \in \Sigma$. By Taylor's expansion theorem one can prove that

$$\begin{cases} \Phi_t(x) = x + t\,v(x) + \mathcal{O}_x(t), \\ \mathrm{d}\Phi_t(x) = \mathrm{Id} + t\,\mathrm{d}v(x) + \mathcal{O}_x(t). \end{cases}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ such that $\{e_1, \dots, e_d\}$ is an orthonormal basis of the tangent space $\mathrm{Tan}(\Sigma, x)$. Then we can write the relations above as follows:

$$\nabla_T \Phi_t = \left[\begin{array}{c|c} \mathrm{Id}_{d\times d} & \underline{0} \\ \underline{0} & \underline{0} \end{array}\right] + t \begin{pmatrix} \nabla_T v \\ \underline{0} \end{pmatrix} + \mathcal{O}_x(t).$$

If we refer to the first matrix on the right-hand side by J, then it is easy to prove that

$$(\nabla_T \Phi_t)^t (\nabla_T \Phi_t)(x) = \mathrm{Id}_{d\times d}(x) + t \left(J^t \, \nabla_T v(x) + (\nabla_T v(x))^t \, J \right) + \mathcal{O}_x(t),$$

from which it follows that

$$\begin{aligned} J_T(\Phi_t)(x) &= \sqrt{\det\left((\nabla_T \Phi_t)^t (\nabla_T \Phi_t) \right)} \simeq \\ &\simeq \sqrt{1 + t \cdot \mathrm{tr} \left(J^t \, \nabla_T v(x) + (\nabla_T v(x))^t \, J \right) + \mathcal{O}_x(t)} = \\ &= \sqrt{1 + 2t \, \mathrm{div}_T(v)(x) + \mathcal{O}_x(t)} \simeq \\ &\simeq 1 + t \, \mathrm{div}_T(v)(x) + \mathcal{O}_x(t), \end{aligned}$$

where the first approximation $\simeq$ follows from the Taylor's expansion

$$\det(\mathrm{Id}_{n\times n} + tA) \simeq 1 + t \, \mathrm{Tr}(A) + o(t) \quad \text{as } t \to 0,$$

and the last approximation follows from the Taylor's expansion of $\sqrt{1+x}$. In particular, it turns out that

$$\begin{aligned} \int_\Sigma J_T(\Phi_t)(x) \, \mathrm{d}\mathcal{H}^d(x) &= \int_\Sigma \left(1 + t \, \mathrm{div}_T(v)(x) + \mathcal{O}_x(t) \right) \, \mathrm{d}\mathcal{H}^d(x) = \\ &= \mathcal{H}^d(\Sigma) + t \int_\Sigma \mathrm{div}_T(v)(x) \, \mathrm{d}\mathcal{H}^d(x) + o(t). \end{aligned}$$

Notice that the remainder needs a reasonable assumption to behave properly: For example, we can require either Σ bounded or Φ_t is the identity outside of a compact set K for every $t \in \mathbb{R}$.

In conclusion, it is enough to take the derivative with respect to the time of (11.2), and evaluate it at $t = 0$, to obtain exactly the sought formula (11.1). □

Theorem 11.2 (Divergence Theorem)**.** *Let Σ be a compact $(d-1)$-dimensional surface with a boundary $\partial\Sigma$ of class C^2. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}^{d-1}(\Sigma_t) \Big|_{t=0} = - \int_\Sigma \vec{H}(x) v(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) + \int_{\partial\Sigma} \eta_{\partial\Sigma}(x) v(x) \, \mathrm{d}\mathcal{H}^{d-2}(x), \tag{11.3}$$

where $\vec{H}(x)$ is the mean curvature vector of Σ at x and $\eta_{\partial\Sigma}$ is the outward normal of $\partial\Sigma$ at x.

Lemma 11.3. *Let $\Sigma_0 \subset \mathbb{R}^n$ be a hypersurface which minimizes the area (the $(n-1)$-dimensional volume) among all Σ such that*

$$\Sigma \triangle \Sigma_0 \subset\subset \Omega,$$

where Ω is a fixed open set in $\mathbb{R}^n$. Then

$$\vec{H}_{\Sigma_0}(x) = 0, \qquad \forall \, x \in \Omega \cap \Sigma_0.$$

Proof. Let v be a compactly supported smooth vector field on Ω. Then there exists a family $(\Phi_t)_{t\in I}$ of diffeomorphisms with v as initial speed[1] and $\Phi_0 = \mathrm{id}_{\mathbb{R}^n}$. It turns out that the hypersurfaces $(\Sigma_t)_{t\in I} := (\Phi_t(\Sigma_0))_{t\in I}$ are all competitors for the area problem; hence the function

$$t \longmapsto \mathcal{H}^{n-1}(\Sigma_t)$$

has a (local) minimum at $t = 0$. From the divergence formula (11.3) we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{n-1}(\Sigma_t)\,\Big|_{t=0} = 0 \implies \int_\Sigma \vec{H}_{\Sigma_0}(x)v(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = 0,$$

since the boundary term is equal to zero. Therefore, a straightforward application of the fundamental lemma in calculus of variation allows us to infer that

$$\vec{H}_{\Sigma_0}(x) = 0, \qquad \forall\, x \in \Omega \cap \Sigma_0.$$

□

Remark 11.1. If Σ is a finite perimeter set, then a similar computation shows that

$$\int_{\partial_*\Sigma} \mathrm{div}_T(v) = 0,$$

which means that we cannot choose any smooth vector field v when dealing with it.

We can summarize the results obtained in this section with a slightly different version of the divergence formula (11.3), which holds for every smooth vector field v.

Proposition 11.4. *Let Σ be a compact $(n-1)$-dimensional (hyper)surface with a boundary $\partial\Sigma$ of class C^2. Then*

$$\int_\Sigma \mathrm{div}_T(v)(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = -\int_\Sigma \vec{H}(x)v(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{\partial\Sigma} \eta_{\partial\Sigma}(xv(x)\,\mathrm{d}\mathcal{H}^{n-2}(x) \tag{11.4}$$

for every[2] $v \in C^\infty(\mathbb{R}^n)$.

Proof. We can decompose the vector field as follows:

$$v = v_N \cdot \eta_{\partial\Sigma} + v_T,$$

where v_N is the normal component, and v_T is the tangential component. Then one can easily check that the following identity holds:

$$\mathrm{div}_T(v) = \underbrace{\mathrm{div}_T(v_N)}_{=0} \cdot \eta_{\partial\Sigma} + v_N \cdot \mathrm{div}_T(\eta_{\partial\Sigma}) + \mathrm{div}_T(v_T).$$

We notice that $\mathrm{div}_T(\eta_{\partial\Sigma})$ is the trace of the second fundamental form, which means that

$$\mathrm{div}_T(\eta_{\partial\Sigma}) = |\vec{H}|(x) =: H(x).$$

The usual divergence theorem for a smooth vector field gives us the identity

$$\int_\Sigma \mathrm{div}_T(v)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\partial\Sigma} v(x)\eta_{\partial\Sigma}(x)\,\mathrm{d}\mathcal{H}^{n-2}(x).$$

[1] For example, the reader may consider the family of diffeomorphisms $\Phi_t(x) := x + t\,v(x)$, which is well-defined in a small neighborhood of 0.

[2] Here we can take any smooth vector field v because Σ is compact by assumption. If not, then v needs to be a compactly supported vector field.

If we plug the first relation (for the divergence) into the second one (divergence theorem for smooth functions), then it turns out that

$$\int_{\Sigma} \operatorname{div}_T(v)(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) = -\int_{\Sigma} \vec{H}(x) v(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) + \int_{\partial\Sigma} \eta_{\partial\Sigma}(x) v(x)\, \mathrm{d}\mathcal{H}^{n-2}(x),$$

which is exactly what we wanted to prove. □

Remark 11.2. Let $\Sigma \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional surface with a boundary $\partial\Sigma$ of class C^2. If Σ minimizes the $(n-1)$-dimensional volume among all $\widetilde{\Sigma}$ such that

$$\left(\widetilde{\Sigma} \triangle \Sigma\right) \cap \partial\Sigma = \varnothing,$$

then the mean curvature $\vec{H}_\Sigma$ is identically equal to 0.

We can state and prove a similar result for finite perimeter sets, but first we need a technical result concerning the closure of $\mathrm{BV}(\Omega)$ with respect to the action of a diffeomorphism.

Lemma 11.5. *Let Ω be a fixed open set in $\mathbb{R}^n$, and let Φ be a diffeomorphism of Ω. If $u \in \mathrm{BV}(\Omega)$, then the composition $u \circ \Phi^{-1}$ also belongs to $\mathrm{BV}(\Omega)$.*

Lemma 11.6. *Let Ω be a fixed open set in $\mathbb{R}^n$, and let E be a finite perimeter set which minimizes the perimeter inside Ω among all $\widetilde{E}$ f.p.s. in Ω such that*

$$E \triangle \widetilde{E} \subset\subset \Omega$$

up to null sets. Then it turns out that

$$\int_{\partial_* E} \operatorname{div}_T(v)\, \mathrm{d}\mathcal{H}^{n-1} = 0, \qquad \forall\, v \in C_c^\infty(\Omega).$$

Proof. Fix v compactly supported smooth vector field on Ω. Then there exists a family $(\Phi_t)_{t\in I}$ of diffeomorphisms with v as initial speed[3] and $\Phi_0 = \mathrm{id}_{\mathbb{R}^n}$. Since $\Phi_t(x) = x$ for every x which does not belong to the support of v, Lemma 11.6 proves that $(E_t)_{t\in I} := (\Phi_t(E_0))_{t\in I}$ is a collection of finite perimeter set such that

$$\partial_* E_t = \Phi_t(\partial_* E).$$

In particular, the f.p.s. E_t are all competitors for the perimeter problem; hence the function

$$t \longmapsto \mathrm{Per}_\Omega(E_t)$$

admits a (local) minimum at $t = 0$. The thesis follows from a simple identity which is left to the reader:

$$\frac{\mathrm{d}\left(\mathrm{Per}_\Omega(E_t)\right)}{\mathrm{d}t}\Big|_{t=0} = \int_{\partial_* E} \operatorname{div}_T(v)\, \mathrm{d}\mathcal{H}^{n-1}.$$

□

Definition 11.7 (Weak Mean Curvature)**.** Let Σ be a d-rectifiable set in $\mathbb{R}^n$. We say that Σ has mean curvature in the weak sense if and only if there exists $\vec{g} \in L^p\left(\mathbb{R}^n, \mathcal{H}^d \llcorner \Sigma\right)$ such that

$$\int_{\Sigma} \operatorname{div}_T(v)(x)\, \mathrm{d}\mathcal{H}^d(x) = \int_{\Sigma} \vec{g}(x) v(x)\, \mathrm{d}\mathcal{H}^d(x), \qquad \forall\, v \in C_c^\infty(\mathbb{R}^n).$$

[3]For example, the reader may consider the family of diffeomorphisms $\Phi_t(x) := x + t\,v(x)$, which is well-defined in a small neighborhood of 0.

Remark 11.3. Let Σ be a compact $(n-1)$-dimensional (hyper)surface with a boundary $\partial\Sigma$ of class C^2. If Σ admits a weak mean curvature $\vec{g}$, then the following properties hold:

(i) The weak mean curvature $\vec{g}$ coincides with the mean curvature $\vec{H}$ almost everywhere.

(ii) The boundary of Σ is empty[4].

Definition 11.8 (k-Varifold)**.** An *integral k-dimensional varifold* is a couple (Σ, θ), where Σ is a k-rectifiable set in $\mathbb{R}^n$, and $\theta : \Sigma \longrightarrow \mathbb{N}$ is a function in $L^1\left(\mathbb{R}^n, \mathcal{H}^k \llcorner \Sigma\right)$.

Definition 11.9 (Varifold W.M.C.)**.** Let (Σ, θ) be a k-dimensional varifold in $\mathbb{R}^n$. We say that Σ has mean curvature in the weak sense if and only if there exists $\vec{g} \in L^p\left(\mathbb{R}^n, \mathcal{H}^d \llcorner \Sigma\right)$ such that

$$\int_\Sigma \theta(x)\mathrm{div}_T(v)(x)\,\mathrm{d}\mathcal{H}^k(x) = \int_\Sigma \theta(x)\vec{g}(x)v(x)\,\mathrm{d}\mathcal{H}^k(x), \qquad \forall\, v \in C_c^\infty(\mathbb{R}^n).$$

Exercise 11.1. Let Σ be a compact $(n-1)$-dimensional (hyper)surface with a boundary $\partial\Sigma$ of class C^2, and let $\theta \in \mathrm{BV}(\Sigma;\ \mathbb{N})$. Suppose that

$$\int_\Sigma \theta(x)\,\mathrm{div}(v)(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma v(x)\,\mathrm{d}\theta(x)$$

for every tangent vector field $v \in C_c^\infty(\Sigma)$. Prove that:

(i) The weak mean curvature $\vec{g}$ coincides with the mean curvature $\vec{H}$ almost everywhere.

(ii) The boundary of Σ is empty.

(iii) The function θ is locally constant.

Exercise 11.2. Let Σ be a compact $(n-1)$-dimensional (hyper)surface with a boundary $\partial\Sigma$ of class C^2, and let $\theta \in C^\infty(\Sigma;\ \mathbb{R})$. Suppose that

$$\int_\Sigma \theta(x)\,\mathrm{div}(v)(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma v(x)\,\mathrm{d}\theta(x)$$

for every tangent vector field $v \in C_c^\infty(\Sigma)$. Prove or disprove the following formula:

$$\vec{g} = \vec{H} - \frac{\nabla_T\theta}{\theta}.$$

11.2 Regularity in Capillarity Problems

Framework. Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$. In this section, we shall discuss the capillarity problem equilibrium when the container (Ω) is fixed and the volume of the liquid is fixed. More precisely, the drop of liquid (denoted by E), contained in Ω, is at equilibrium if and only if it is a critical point (local minima) of the capillarity energy functional

$$\mathscr{F}(E) := \mathcal{H}^{n-1}\left(\partial E \cap \Omega\right) + \sigma\mathcal{H}^{n-1}\left(\partial E \cap \partial\Omega\right), \tag{11.5}$$

where $\sigma \in \mathbb{R}$ is a constant, subject to the constraint that the volume is fixed, e.g. $|E| = m$.

[4] This follows fairly easily from the integration by parts formula.

Mean Curvature. Assume that E is a f.p.s. which minimizes the capillarity energy (11.5) and satisfies the constraint on the volume. If E is regular enough, then one can prove that

$$\begin{cases} H_f \text{ is constant on the free portion of the surface } \Sigma^f, \\ \theta_Y \text{ is constant on the boundary } \Gamma. \end{cases}$$

More precisely, the free mean curvature H_f is the Lagrange multiplier associated to the volume constraint, while θ_Y is the solution of the trigonometric equation

$$\cos\theta_Y = -\sigma. \tag{11.6}$$

The first assertion is easy to prove. Let $(\Phi_t)_{t\in I}$ be a family of diffeomorphisms fixing the boundary, and choose a smooth vector field v in such a way that the volume constraint is not violated, that is,

$$\text{the constraint is preserved} \iff \int_{\Sigma^f} v\cdot\eta = 0,$$

which means that the first variation of the volume is zero. As a result of the technical tools introduced in the previous section, it turns out that if we set the first variation equal to zero, then we obtain

$$\int_{\Sigma^f} H(v\cdot\eta) = 0, \qquad \forall\, v\in C_c^\infty \ : \ \int_{\Sigma^f} v\cdot\eta = 0.$$

The Du Bois-Reymond's fundamental lemma in calculus of variation allows us to infer that

$$H\big|_{\Sigma^f} = c \implies H_f = c,$$

that is, the free mean curvature is constant. The proof of (11.6) is very similar, but we need to consider a family of transformations which moves the boundary.

Remark 11.4. If E is not regular enough, then one might translate the problem in the setting of k-varifolds, and prove that the weak mean curvature is constant on Σ^f. Unfortunately, it is not easy at all to give a meaning to θ, and recover something like formula (11.6).

11.3 Regularity in the F.P.S. Setting

Introduction. Let $\Omega\subset\mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$, and let E be a finite perimeter set in Ω minimizing the perimeter $\mathrm{Per}_\Omega(-)$ among all f.p.s. $\widetilde{E}$ such that

$$E \triangle \widetilde{E} \subset\subset \Omega.$$

Theorem 11.10. *The essential boundary $\partial_* E$ is closed in Ω, and there is a representative of E in its equivalence class which satisfies the following properties:*

(a) *The topological boundary ∂E coincides with the essential boundary $\partial_* E$.*

(b) *The perimeter formula is satisfies, that is,*

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{n-1}(\partial E).$$

Moreover, the representative above can be uniquely characterized as follows:

$$E = \{x\in E \mid \Theta(E,x) = 1\}.$$

Lemma 11.11. *There exists a universal constant $\delta_0 := \delta_0(n) > 0$ such that for every $(x_0, r_0) \in \Omega \times \mathbb{R}$ satisfying $B(x_0, r_0) \subset \Omega$, it turns out that*

$$\frac{|E \cap B(x_0, r_0)|}{\alpha_n r_0^n} \leq \delta_0 \implies \left| E \cap B\left(x_0, \frac{r_0}{2}\right) \right| = 0,$$

$$\frac{|E \cap B(x_0, r_0)|}{\alpha_n r_0^n} \geq 1 - \delta_0 \implies \left| E \cap B\left(x_0, \frac{r_0}{2}\right) \right| = |B(x_0, r_0)|,$$

almost everywhere.

Truncation Argument. Fix $(x_0, r_0) \in \Omega \times \mathbb{R}$. For every $r \in [\frac{r_0}{2}, r_0]$, set

$$E_r := E \setminus B(x_0, r).$$

Step 1. The set E minimizes the perimeter; hence

$$\mathrm{Per}_\Omega(E_r) \geq \mathrm{Per}_\Omega(E).$$

The usual isoperimetrical inequality yields to[5]

$$\begin{aligned} c(n)\, |E \cap B(x_0, r)|^{1-\frac{1}{n}} &\leq \mathrm{Per}_{B(x_0, r)}(E) = \\ &= \mathcal{H}^{n-1}\left(\partial_* E \cap B(x_0, r)\right) \leq \\ &\leq \mathcal{H}^{n-1}\left(E \cap \partial B(x_0, r)\right) \end{aligned}$$

for almost every $r \in [\frac{r_0}{2}, r_0]$. Recall that

$$D(\mathbb{1}_{E_r}) = \mathbb{1}_{\Omega \setminus B(x_0, r)}\, D(\mathbb{1}_E) + \nu\, \mathbb{1}_{E \cap \partial B(x_0, r)} \cdot \mathcal{H}^{n-1},$$

from which it follows that

$$\partial_* E_r = \left(\partial_* E \setminus \overline{B(x_0, r)}\right) \cup \left(E \cap \partial B(x_0, r)\right)$$

for almost every admissible r.

Step 2. Let $v(r) := |E \cap B(x_0, r)|$ be the n-dimensional Lebesgue measure. Then[6]

$$\dot{v}(r) = \mathcal{H}^{n-1}\left(E \cap \partial B(x_0, r)\right) \qquad \text{for a.e. admissible } r.$$

It follows from the isoperimetrical inequality that

$$c(n)\, v(r)^{1-\frac{1}{n}} \leq \dot{v}(r),$$

from which it follows that[7]

$$v\left(r_0\right)^{\frac{1}{n}} - v\left(\frac{r_0}{2}\right)^{\frac{1}{n}} \geq \frac{c(n)}{n} \frac{r_0}{2}.$$

[5] The last inequality needs to be justified. The reader should check very carefully the details since a lot of issues could arise here.

[6] Here the reader needs to be extremely careful. A priori v is simply differentiable in the weak sense of distributions, while in the computation above we need to deal with the classical derivative of v.

[7] To solve the differential inequality one needs to divide by v. If $v(r) \neq 0$ for a.e. admissible r, then everything is fine. On the other hand, if $v(r) = 0$ in the middle of the interval, then one needs to check that v is strictly increasing, and thus $\{r \,:\, v(r) = 0\}$ is a null set.

If we rearrange the inequality, then we obtain the following estimate:

$$v\left(\frac{r_0}{2}\right)^{\frac{1}{n}} \le \left[\alpha_n^{\frac{1}{n}} \left(\frac{v(r_0)}{\alpha_n r_0^n}\right)^{\frac{1}{n}} - \frac{c(n)}{2n}\right] r_0.$$

In particular, the thesis follows by choosing the universal constant δ_0 in such a way that $v(r_0/2)$ is either 0 or equal to the volume of the ball, that is,

$$\delta_0 = \frac{1}{\alpha_n}\left(\frac{c(n)}{2n}\right)^n.$$

□

Proof of Theorem 11.10. Everything is obvious except the fact that $\partial_* E$ is a closed set; we shall show that the complement is open.

Case 1. Fix a point x_0 with density $\Theta(E, x_0)$ equal to 0. Then there exists $r_0 > 0$ small enough such that

$$\frac{|E \cap B(x_0, r_0)|}{\alpha_n r_0^n} \le \delta_0,$$

and thus by Lemma 11.11 it turns out that

$$\left|E \cap B\left(x_0, \frac{r_0}{2}\right)\right| = 0.$$

In particular, it turns out that

$$\Theta(E, x) = 0 \qquad \forall x \in E \cap B\left(x_0, \frac{r_0}{2}\right).$$

Case 2. Fix a point x_0 with density $\Theta(E, x_0)$ equal to 1. Then there exists $r_0 > 0$ small enough such that

$$\frac{|E \cap B(x_0, r_0)|}{\alpha_n r_0^n} \ge 1 - \delta_0,$$

and thus by Lemma 11.11 it turns out that

$$\left|E \cap B\left(x_0, \frac{r_0}{2}\right)\right| = \left|B\left(x_0, \frac{r_0}{2}\right)\right|.$$

In particular

$$\Theta(E, x) = 1 \qquad \forall x \in E \cap B\left(x_0, \frac{r_0}{2}\right),$$

which means that the complement of $\partial_* E$ is open (i.e., $\partial_* E$ is closed). □

Exercise 11.3. What happens if the f.p.s. E minimizes the perimeter $\mathrm{Per}_\Omega(-)$ among all f.p.s. $\widetilde{E}$ such that

$$E \triangle \widetilde{E} \subset\subset \Omega,$$

subject to a volume constraint? Is $\partial_* E$ still closed?

11.4 Structure of Finite Length Continua in $\mathbb{R}^n$

The main goal of this section is to prove that a continua (= compact and connected) set K in $\mathbb{R}^n$ with finite length is the image of a surjective path γ such that

$$\text{Length}(\gamma) \leq 2 \cdot \mathcal{H}^1(K) < +\infty.$$

Definition 11.12. Let $I := [a, b]$. A continuous path $\gamma : I \longrightarrow \mathbb{R}^n$ of finite length has constant speed c if and only if

$$\text{Length}(\gamma, J) < c \cdot \text{Length}(\gamma, I)$$

for every $J \subset I$.

Remark 11.5 (Arc Length)**.** Let $\gamma : I \longrightarrow \mathbb{R}^n$ be a continuous path of finite length L. If we set

$$\sigma(s) := \inf \{ t \in [a, b] \mid \text{Length}\,(\gamma, [a, t]) \geq Ls \},$$

then the riparametrization $\gamma \circ \sigma : [0, 1] \longrightarrow \mathbb{R}^n$ is a continuous path with constant speed L.

Remark 11.6 (Tangent Plane)**.** Let K be a continua set in $\mathbb{R}^n$. Then the tangent plane $\text{Tan}(K, x)$ is the vector space in the classical sense for $\mathcal{H}^1$-almost every x.

Lemma 11.13. *Let K be a continua set in $\mathbb{R}^n$ with finite length. Then K is connected by arcs and, more precisely, it is connected by injective paths of finite length.*

Sketch of the Proof. The main idea is to obtain the injective path γ as the uniform limit (with respect to the Hausdorff distance) of a sequence of finite length paths γ^δ satisfying certain properties.

Step 1. Fix $x_0, x \in K$. For every $\delta > 0$ we may consider the (almost) shortest δ-chain of points $x_0^\delta, \dots, x_n^\delta \in K$ and times $0 =: t_0^\delta \leq \cdots \leq t_n^\delta := 1$, where n depends on δ, such that the following properties hold:

i) The first point is $x_0 := x_0^\delta$, and the final point is $x := x_n^\delta$ for every $\delta > 0$.

ii) The distance between two consecutive points is less than δ, that is,

$$|x_i^\delta - x_{i-1}^\delta| < \delta,$$

and the total length is less than the length of K, that is,

$$\sum_{i=1}^{n} |x_i^\delta - x_{i-1}^\delta| < \mathcal{H}^1(K).$$

iii) For every $i = 1, \dots, n$ it turns out that γ^δ reaches the point x_i^δ at the time t_i^δ.

iv) The Lipschitz constant of γ^δ is less or equal than 4 times its length.

The reader may prove that the curve γ can be obtained as the uniform limit of the sequence $(\gamma^\delta)_{\delta > 0}$. $\square$

Theorem 11.14. *Let K be a continua set in $\mathbb{R}^n$ with finite length. Then there exists a surjective path $\gamma : [0, 1] \longrightarrow K$ such that*

$$\text{Length}(\gamma) \leq 2 \cdot \mathcal{H}^1(K) < +\infty.$$

In particular, every continua $\mathcal{H}^1$-finite set is rectifiable.

Proof. The assertion is an immediate consequence of Lemma 11.13. ☐

Remark 11.7 (Degree)**.** The degree of γ at x, which will be denoted by $\deg(\gamma, x)$, is well-defined for $\mathcal{H}^1$-almost every x. In particular, for every vector field $v \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, it turns out that

$$\int v(\gamma_i(s))\, \dot{\gamma}_i(s)\, \mathrm{d}s = 0$$

for every $i \in \mathbb{N}$, which means that

$$\int v(\gamma(s))\, \dot{\gamma}(s)\, \mathrm{d}s = 0, \qquad \forall\, v \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

Since $v(\gamma_n)$ converges uniformly to $v(\gamma)$ and $\dot{\gamma}_n$ converges to $\dot{\gamma}$ in $L^\infty_{w^*}$ (i.e., in the weak-$*$ topology of L^∞), then the area formula implies that

$$0 = \int v(\gamma(s)) \frac{\dot{\gamma}(s)}{|\dot{\gamma}(s)|}\, J(\gamma)(s)\mathrm{d}s = \int_K v(x) \sum_{s \in \gamma^{-1}(x)} \tau(s) \cdot \deg(\gamma, x)\, \mathrm{d}\mathcal{H}^1(x),$$

where

$$\tau(s) := \frac{\dot{\gamma}(s)}{|\dot{\gamma}(s)|}$$

indicates the orientation of the curve. In particular, it turns out that

$$\tau(x) \cdot \deg(\gamma, x) = 0$$

for $\mathcal{H}^1$-almost every x, which means that the cardinality of the set $\gamma^{-1}(x)$ is even for $\mathcal{H}^1$-almost every x.

11.5 Lower Semi-Continuity: Golab Theorem

In this brief section, we use the structure theorem of a continua set to prove the famous Golab compactness result.

Theorem 11.15 (Golab)**.** *Let $(K_i)_{i \in \mathbb{N}}$ be a sequence of continua sets in $\mathbb{R}^n$. If K_i converges to a continua set K with respect to the Hausdorff distance, then*

$$\liminf_{i \to +\infty} \mathcal{H}^1(K_i) \geq \mathcal{H}^1(K).$$

Proof. By Theorem 11.14 for every $i \in \mathbb{N}$ there exists a surjective path γ_i of finite length that parametrizes the corresponding continua set K_i. Up to subsequences, it turns out that γ_i converges to some path γ, which parametrizes K.

The preimage $\gamma^{-1}(x)$ is nonempty, and its cardinality is even (see Remark 11.7). Therefore, one can easily infer that

$$\left|\gamma^{-1}(x)\right| \geq 2 \quad \text{at } \mathcal{H}^1\text{-almost every } x \in K,$$

which means that

$$\mathcal{H}^1(K) \leq \frac{1}{2} \cdot \mathrm{Length}(\gamma).$$

The length is a semi-lower continuous functional

$$\mathrm{Length}(\gamma) \leq \liminf_{i \to +\infty} \mathrm{Length}(\gamma_i),$$

and by Theorem 11.14 it turns out that

$$\operatorname{Length}(\gamma) \le \liminf_{i \to +\infty} \operatorname{Length}(\gamma_i) \le 2 \liminf_{i \to +\infty} \mathcal{H}^1(K_i).$$

If we combine the two inequalities, then we obtain the thesis:

$$\mathcal{H}^1(K) \le \frac{1}{2} \cdot \operatorname{Length}(\gamma) \le \liminf_{i \to +\infty} \mathcal{H}^1(K_i).$$

□

11.6 Simons' Cone

In this final section, we show that the minimality of the functional $\mathcal{H}^{n-1}(- \cap K)$ may be achieved via a calibration. We shall apply this result to prove the minimality of the Simons' Cone, which is a singular surface, in dimension $n = 8$.

Theorem 11.16 (Minimality through Calibration)**.** *Let Σ_0 be a compact $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$ with boundary $\partial\Sigma_0$ oriented by a continuous unit normal η_0. Assume that there exists a calibration for Σ_0, that is, a vector field v, defined on the space $\mathbb{R}^n$, with the following properties:*

- ***(i)*** *The vector field v coincides with η_0 on Σ_0.*
- ***(ii)*** *The norm of v is less than or equal to 1.*
- ***(iii)*** *The divergence of v is identically zero.*

Then Σ_0 is the hypersurface minimizing the functional $\mathcal{H}^{n-1}(-)$ among all the $(n-1)$-dimensional hypersurfaces Σ with the same boundary, oriented in the same way.

Proof. A straightforward computation shows that

$$\begin{aligned}\mathcal{H}^{n-1}(\Sigma_0) &= \int_{\Sigma_0} v \cdot \eta_0 \, \mathrm{d}\mathcal{H}^{n-1} = \\ &= \int_{\Sigma} v \cdot \eta_0 \, \mathrm{d}\mathcal{H}^{n-1} \le \\ &\le \mathcal{H}^{n-1}(\Sigma),\end{aligned}$$

where the blue equality follows from **(i)**, and the red follows from **(ii)**, **(iii)**, and the divergence theorem. □

Theorem 11.17 (Minimality through Calibration, II)**.** *Let Σ_0 be a complete $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$ without boundary. Assume that there exist a Borel set E and a calibration for Σ_0, that is, a vector field v, defined on the space $\mathbb{R}^n$, with the following properties:*

- ***(i)*** *The vector field v coincides with η_0 on Σ_0.*
- ***(ii)*** *The norm of v is less than or equal to 1.*
- ***(iii)*** *The divergence of v is less than or equal to zero in E, and greater than or equal to zero in the complement $\mathbb{R}^n \setminus E$.*
- ***(iv)*** *The surface Σ_0 is the boundary of E, and η_0 is the outwards normal.*

Then Σ_0 is the hypersurface minimizing the functional $\mathcal{H}^{n-1}(- \cap K)$ among all the $(n-1)$-dimensional hypersurfaces Σ which agrees with Σ outside of a compact set K.

Simons' Cone. The surface

$$\Sigma_S := \left\{ (x,\, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \;|\; |x| = |y| \right\}$$

admits a calibration v. In particular, the Simons' cone is the minimal surface (with respect to the $(n-1)$-dimensional Hausdorff measure) in dimension $n = 8$, and it is easy to prove that the origin $(0,\, 0)$ is a singular point (as stated in the final remark of the previous chapter). We will not present a proof here, but the calibration is given by

$$v(x,\, y) := \frac{\nabla f(x,\, y)}{f(x,\, y)}, \quad \text{where } f(x,\, y) = \frac{|x|^4 - |y|^4}{4},$$

and the Borel set E is given by

$$E := \left\{ (x,\, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \;|\; |x| < |y| \right\}.$$

Index

Bibliography

[1] Alexei A. Tuzhilin Alexander O. Ivanov. *Minimal NetworksThe Steiner Problem and Its Generalizations.* CRC Press, 1994.

[2] Renato Caccioppoli. Sulla quadratura delle superfici piane e curve. *Atti Accad. Naz. Lincei Rnd. Cl. Sci. Fis. Mat. Natur.*, 6:142–146, 1927.

[3] Jesse Douglas. Solution of the problem of plateau. *Transactions of the American Mathematical Society*, 33(1):263–321, 1931.

[4] K. J. Falconer. *The Geometry of Fractal Sets.* Cambridge University, New York, NY, USA, 1986.

[5] Hebert Federer. *Geometric Measure Theory.* Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 1 edition, 1996.

[6] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Annals of Mathematics*, 72(3):458–520, 1960.

[7] Ennio De Giorgi. Definizione ed espressione analitica del perimetro di un insieme. *Atti Accad. Naz. Lincei Rnd. Cl. Sci. Fis. Mat. Natur.*, 14:29–41, 1953.

[8] Pertti Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambriidge studies in advanced mathematics. Cambridge University, 1995.

[9] E. R. Reifenberg. Solution of the plateau problem form-dimensional surfaces of varying topological type. *Acta Mathematica*, 104(1):1–92, 1960.

[10] Harold R. Parks Steven G. Krantz. *Geometric Integration Theory.* Cornerstones, 2008.