Lecture Notes

Institutional Analysis

Course held by

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Disclaimer

These notes came out of the *Functional Analysis/Sobolev Spaces* course, held by Professor Pietro Majer in the second semester of the academic year 2016/2017.

They include all the topics that were discussed in class; I added some remarks, simple proof, etc.. for my convenience.

I have used them to study for the exam; hence they have been reviewed thoroughly. Unfortunately, there may still be many mistakes and oversights; to report them, send me an email at **francescopaolo (dot) maiale (at) gmail (dot) com**.

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Part I

Functional Analysis

Chapter 1

Hilbert Spaces

In the first half of this chapter, we investigate the main properties of *Hilbert spaces*, e.g., the projection onto a closed convex subspace, the characterization of the orthogonal projection onto a linear subspace, etc. In the second part, we introduce the notion of *topological dual* of a Hilbert space, and we prove the fundamental representation theorem due to Riesz. We also present a simple characterization of Hilbert basis in terms of Fourier sums, and briefly investigate the cardinality of such bases. In the appendix, we prove the Radon-Nikodym theorem, we describe the structure of particular subgroups of Hilbert spaces entirely, and we also find an orthogonal family in $L^2([0, \infty), \mathrm{d}\mu)$, known as *Laguerre polynomials*.

1.1 Pre-Hilbert Spaces

In this section, we introduce the notion of *pre-Hilbert space*, that is, a vector space V endowed with a real scalar product $(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R}$ or a Hermitian product $(\cdot, \cdot) : V \times V \longrightarrow \mathbb{C}$.

1.1.1 Real Scalar Product

Recall that $(V, +, \cdot)$ is a <u>real</u> vector space if V is a vector space whose field of scalars is $\mathbb{R}$.

Definition 1.1 (Symmetric Form)**.** A map $B : V \times V \longrightarrow \mathbb{R}$ is a *symmetric form* if

$$B(x, y) = B(y, x) \quad \text{for every } x, y \in V.$$

Definition 1.2 (Bilinear Form)**.** A symmetric map $B : V \times V \longrightarrow \mathbb{R}$ is a *bilinear form* if

$$B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z) \quad \text{for every } x, y, z \in V \text{ and } \alpha, \beta \in \mathbb{R}.$$

Definition 1.3 (Positive Form)**.** A map $B : V \times V \longrightarrow \mathbb{R}$ is a *positive form* if the associated quadratic form is positive, that is,

$$Q(x) := B(x, x) \geq 0 \quad \text{for every } x \in V.$$

Example 1.1. The space

$$V = C^0([0, 1]; \mathbb{R})$$

is a real vector space. The reader may check as an exercise that a bilinear, symmetric and positive form defined on V is given by

$$B(f,\, g) := \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

Lemma 1.4 (Cauchy-Schwarz)**.** *Let $B : V \times V \to \mathbb{R}$ be a bilinear, symmetric and positive form defined over V. For every $x,\, y \in V$ it turns out that*

$$|B(x,\, y)|^2 \leq |B(x,\, x)|\, |B(y,\, y)|\,. \tag{1.1}$$

First Proof. The quadratic form Q associated to B is positive; hence

$$Q\left(B(x,\, x)\, y - B(x,\, y)\, x\right) \geq 0,$$

for every $x,\, y \in V$. It follows from the definition of Q that

$$B(x,\, x)^2 B\,(y,\, y) - 2B(x,\, y)^2 B(x,\, x) + B(x,\, y)^2 B(x,\, x) \geq 0,$$

from which we infer that

$$B(x,\, x)^2 B\,(y,\, y) - B(x,\, y)^2 B(x,\, x) \geq 0.$$

It is easy to see that there are only two possibilities we need to study separately:

1) If $B(x,\, x) = B(y,\, y) = 0$, then

$$B(x+y,\, x+y) \geq 0 \implies B(x,\, y) \geq 0,$$

$$B(x-y,\, x-y) \geq 0 \implies -B(x,\, y) \geq 0,$$

and thus the inequality (1.1) is trivially satisfied.

2) If $B(x,\, x) \neq 0$, then we can divide the inequality above by $B(x,\, x)$ and obtain the Cauchy-Schwarz inequality, that is,

$$B(x,\, x)B\,(y,\, y) - B(x,\, y)^2 \geq 0.$$

The opposite case ($B(y,\, y) \neq 0$) follows in the same way replacing x with y (and vice versa).

□

Second Proof. Let us consider the second-order polynomial

$$\mathbb{R} \ni t \longmapsto Q(x+ty) = B(y,\, y)t^2 + 2B(x,\, y)t + B(x,\, x) \in \mathbb{R}.$$

The bilinear form B is positive, that is, $B(x+ty,\, x+ty) \geq 0$ for every $t \in \mathbb{R}$. A standard argument for second-order polynomials proves that the discriminant Δ needs to be negative at every $t \in \mathbb{R}$, which yields to the sought inequality

$$\Delta = 4B(x,\, y)^2 - 4B(x,\, x)B(y,\, y) \leq 0.$$

□

Remark 1.1. Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ be a linear map, and assume that $L(e_1) = x$ and $L(e_2) = y$. Then

$$A^\intercal A \in \mathrm{M}\,(2 \times 2,\, \mathbb{R}) \qquad \text{and} \qquad \det A^\intercal A = \sum_{1 \le i < j \le n} (x_i y_j - x_j y_i)^2\,,$$

i.e., it is the sum of the squared determinant of all the 2×2 minors of A.

Definition 1.5 (Scalar product)**.** A map $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$ is a *scalar product* if the following properties are satisfied:

(a) $(\cdot,\, \cdot)$ is a bilinear, symmetric and positive form;

(b) $(\cdot,\, \cdot)$ is a positive-definite form, that is,

$$(x,\, x) \ge 0 \text{ for every } x \in V \quad \text{and} \quad (x,\, x) = 0 \iff x = 0.$$

Lemma 1.6 (Cauchy-Schwarz)**.** *Let $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$ be a scalar product defined over V. For every $x,\, y \in V$ it turns out that*

$$|(x,\, y)|^2 \le |(x,\, x)|\, |(y,\, y)|\,, \tag{1.2}$$

and the equality holds if and only if there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$.

Proposition 1.7. *Let $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$ be a scalar product defined over V. The map $\| \cdot \| : V \longrightarrow \mathbb{R}$ defined by setting*

$$\|x\| := \sqrt{(x,\, x)}, \qquad \forall\, x \in V, \tag{1.3}$$

is a norm, i.e., the following properties are satisfied:

(1) $\|x\| \ge 0$ *for every* $x \in V$;

(2) $\|x\| = 0$ *if and only if* $x = 0$;

(3) $\|\lambda x\| = |\lambda| \|x\|$ *for every* $x \in X$ *and for every* $\lambda \in \mathbb{R}$;

(4) for every $x,\, y \in X$ the triangle inequality holds true, i.e.

$$\|x + y\| \le \|x\| + \|y\|. \tag{1.4}$$

Moreover, the norm $\| \cdot \|$ induces a distance, invariant under translations, defined as follows:

$$d(x,\, y) := \|x - y\|, \qquad \forall\, x,\, y \in V. \tag{1.5}$$

Proposition 1.8 (Parallelogram Law)**.** *Let V be a real vector space.*

(a) If $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$ is a scalar product over V, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \qquad \forall\, x,\, y \in V. \tag{1.6}$$

(b) If $\| \cdot \| : V \longrightarrow \mathbb{R}$ is a norm satisfying the parallelogram law (1.6) at every couple of points $x,\, y \in V$, then there exists a scalar product $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$ such that

$$\| \cdot \| := \sqrt{(\cdot,\, \cdot)}.$$

Proof.

(a) By definition of $\|\cdot\|$ we have

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= (x+y,\, x+y)+(x-y,\, x-y) = \\ &= \|x\|^2+2(x,\, y)+\|y\|^2+\|x\|^2-2(x,\, y)+\|y\|^2 = \\ &= 2\|x\|^2+2\|y\|^2,\end{aligned}$$

which is exactly what we wanted to prove.

(b) We define a scalar product using the polarization identity, that is,

$$(x,\, y) := \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right) \quad \text{for every } x,\, y \in V.$$

The reader may check as an exercise that $(\cdot,\, \cdot)$ is a scalar product, and also that $\|x\| = \sqrt{(\cdot,\, \cdot)}$ for any $x \in V$.

□

1.1.2 Hermitian Product

Let V be a complex vector space, that is, the field of scalars of V is given by $\mathbb{C}$.

The definitions introduced in the previous section need to be slightly modified to adapt to the different structure of $\mathbb{C}$.

In particular, we introduce the notion of *hermitian product*, and we prove that the Cauchy-Schwarz inequality still holds - as a consequence of some properties of the complex numbers.

Definition 1.9 (Semilinear Form)**.** A map $B : V \times V \longrightarrow \mathbb{C}$ is a *right-semilinear form* if

$$B(x,\, \alpha y+\beta z) = \overline{\alpha}B(x,\, y)+\overline{\beta}B(x,\, z) \quad \text{for every } x,\, y,\, z \in V \text{ and } \alpha,\, \beta \in \mathbb{C}.$$

Definition 1.10 (Hermitian product)**.** A map $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{C}$ is a *hermitian product* if the following properties are satisfied:

(a) The form $(\cdot,\, \cdot)$ is sesquilinear, that is, it is linear on the left and semilinear on the right.

(b) The form $(\cdot,\, \cdot)$ is hermitian, that is,

$$(y,\, x) = \overline{(x,\, y)} \quad \text{for every } x,\, y, \in V.$$

(c) The form $(\cdot,\, \cdot)$ is positive-definite, that is,

$$(x,\, x) \geq 0 \text{ for every } x \in V \quad \text{and} \quad (x,\, x) = 0 \iff x = 0.$$

Remark 1.2. If V is a complex vector space and $(\cdot,\, \cdot)_{\mathbb{C}}$ is a Hermitian product defined over V, we can define a real scalar product by setting

$$(\cdot,\, \cdot)_{\mathbb{R}} := \mathfrak{Re}\,(\cdot,\, \cdot)_{\mathbb{C}} \quad \text{or} \quad (\cdot,\, \cdot)_{\mathbb{R}} := \mathfrak{Im}\,(\cdot,\, \cdot)_{\mathbb{C}}.$$

These scalar product are, a priori, distinct. The reader may prove that both satisfy the axioms of a scalar product, and provide an example showing that they might be very different.

Lemma 1.11 (Cauchy-Schwarz)**.** *Let $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{C}$ be a scalar product defined over V. For every $x,\, y \in V$ it turns out that*

$$|(x,\, y)|^2 \leq |(x,\, x)|\, |(y,\, y)|\,, \tag{1.7}$$

and the equality holds if and only if there exists $\lambda \in \mathbb{C}$ such that $x = \lambda y$.

Proof. For every $x,\, y \in V$ it turns out that

$$0 \leq \|x - \lambda y\|^2 \implies 0 \leq \|x\|^2 + \lambda\overline{\lambda}\|y\|^2 - \lambda(y,\, x) - \overline{\lambda}(x,\, y). \tag{1.8}$$

Assume that both x and y are nonzero vectors (otherwise the inequality is trivial), and take

$$\lambda := \frac{\overline{(y,\, x)}}{\|y\|^2}.$$

If we plug the value of λ into (1.8), we obtain the inequality

$$0 \leq \|x\|^2 - \frac{|(y,\, x)|}{\|y\|^2},$$

which yields to the Cauchy-Schwarz inequality multiplying everything by $\|y\|^2$. □

1.2 Hilbert Spaces

In this section, we introduce the notion of *Hilbert space*, and we investigate the properties of the projection onto a closed convex subset.

Definition 1.12 (Hilbert Space)**.** A real vector space H, endowed with a scalar product $(\cdot,\, \cdot) : V \times V \longrightarrow \mathbb{R}$, is a (real) Hilbert space if it is complete with respect to the metric

$$d(x,\, y) := \|x - y\|.$$

In a similar fashion, a complex vector space H, endowed with a Hermitian product $(\cdot,\, \cdot) :$ $V \times V \longrightarrow \mathbb{R}$, is a complex Hilbert space if it is complete with respect to the metric above.

1.2.1 Projection Onto a Convex Subspace

Recall that a set C is convex if and only if the segment between any two points $x,\, y \in C$ lies entirely in C, that is,

$$\lambda x + (1 - \lambda)y \in C \quad \text{for every } \lambda \in [0,\, ,\ 1].$$

Theorem 1.13 (Convex Projection)**.** *Let H be a Hilbert space and assume that $C \neq \varnothing$ is a closed convex subset of H. Then there exists a unique point of minimal norm in C, that is,*

$$\exists_1\, x \in C \ : \ \|x\| = \inf_{y\in C} \|y\|.$$

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset C$ be a minimizing sequence, i.e.,

$$\|x_n\| \xrightarrow{n\to+\infty} \inf_{y\in C} \|y\| =: d.$$

Step 1. For any p, $q \in \mathbb{N}$, the parallelogram law (1.6) gives us the identity

$$\|x_p - x_q\|^2 = 2\left(\|x_p\|^2 + \|x_q\|^2\right) - 4\left\|\frac{x_q + x_q}{2}\right\|^2,$$

and the convexity of C implies that

$$\frac{x_p + x_q}{2} \in C.$$

By definition of infimum $\left\|\frac{x_p+x_q}{2}\right\| \geq d$; hence the inequality above yields to

$$\|x_p - x_q\|^2 \leq 2\left(\|x_p\|^2 + \|x_q\|^2\right) - 4\,d = o(1), \qquad \text{as } p,\, q \to +\infty,$$

which means that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space H. Therefore x_n converges to x in H and, since C is closed, we finally infer that $x \in C$.

Step 2. The uniqueness of $x \in C$ follows from a standard argument. Let $(x_n)_{n\in\mathbb{N}} \subset C$ and $(x'_n)_{n\in\mathbb{N}} \subset C$ be two minimizing sequences. Then

$$y_k := \begin{cases} x_n & \text{if } k = 2n, \\ x'_n & \text{if } k = 2n+1, \end{cases}$$

is also a minimizing sequences, which converges to some element $x \in C$. Since both $(x_n)_{n\in\mathbb{N}} \subset C$ and $(x'_n)_{n\in\mathbb{N}} \subset C$ are subsequences of a converging sequence, the limit points must be the same. ☐

Lemma 1.14. *Let H be a Hilbert space, and let $C \neq \varnothing$ be a closed convex subset. The map*

$$P : H \longrightarrow \mathbb{C}, \qquad x \longmapsto P(x) = \textit{point of minimal distance from } x \textit{ in } C$$

is well-defined. In particular, the map P is a metric projection (i.e., $P^2 = P$), and it is 1-Lipschitz.

Proof. The map P is well-defined as a consequence of Theorem 1.13, since it sends $x \in H$ to the unique point $P(x) \in C$ satisfying

$$\|x - P(x)\| = \inf_{y\in C} \|x - y\|.$$

Moreover, the map P is a metric projection because it is the identity on C by definition, and its range is C, which means that $P(P(x)) = P(x)$ for every $x \in H$.

Hence, it remains to prove that P is Lipschitz-continuous of Lipschitz constant 1. We claim that given a point $x \in H$, the scalar product of the projection satisfies the inequality

$$(x - P(x),\, y - P(x)) \leq 0 \quad \text{for every } y \in C. \tag{1.9}$$

If (1.9) holds, then given any two points $x, y \in H$, we have

$$(x - P(x),\, z - P(x)) \leq 0 \qquad \forall\, z \in C,$$

$$(y - P(y),\, w - P(y)) \leq 0 \qquad \forall\, w \in C.$$

If we set $z = P(y)$ and $w = P(x)$, then the sum of the above inequalities yields to

$$\|P(x) - P(y)\|^2 \leq (x - y,\, P(x) - P(y))\,,$$

and hence we conclude using the Cauchy-Schwarz inequality (1.1). To prove the claim (1.9), we simply notice that by convexity

$$\lambda y + (1 - \lambda) P(x) \in C,$$

which means that

$$\begin{aligned}\|x - P(x)\|^2 &\leq \|x - [\lambda y + (1-\lambda)P(x)]\,\|^2 = \\ &= \|x - P(x) + \lambda\,[y - P(x)]\,\|^2 = \\ &= \|x - P(x)\|^2 + \lambda^2 \|y - P(x)\|^2 - 2\lambda (x - P(x),\, y - P(x)).\end{aligned}$$

Therefore

$$\lambda \|y - P(x)\|^2 \geq 2(x - P(x),\, y - P(x)),$$

and this yields to the desired result by taking the limit for $\lambda \to 0$. The reader may prove that, actually, the claim (1.9) characterizes entirely the projection onto a convex set. ☐

Proposition 1.15. *Let $E \subset \mathbb{R}^n$ be a subset satisfying the following property:*

$$\text{"}\forall\, x \in \mathbb{R}^n \text{ there exists a unique point } P(x) \in E \,:\, \|x - P(x)\| = \inf_{y \in E} \|x - y\|.\text{"}$$

Then E is a nonempty convex and closed set.

Lemma 1.16. *Let H be a Hilbert space, and let $E \subset H$ be a nonempty bounded subset. If*

$$r_E := \inf \left\{ R > 0 \;\middle|\; \exists\, x \in H \,:\, E \subset \overline{B_R(x)} \right\},$$

then there exists a unique $x_E \in H$ such that $E \subset \overline{B_{r_E}(x_E)}$.

Remark 1.3. Let $x, y \in H$ be any two points of H, and let $r > 0$ be any real number. Then there is an obvious inclusion of balls

$$\overline{B\,(x,\, r)} \cap \overline{B\,(y,\, r)} \subseteq \overline{B\left(\frac{x+y}{2},\, \sqrt{r^2 - \left\|\frac{x-y}{2}\right\|^2}\right)}. \tag{1.10}$$

Hint. This inclusion is clearly translation-invariant. In particular, one may assume that $x = -y$, apply the parallelogram law (1.6) and conclude as follows:

$$\|z - x\| \leq r,\ \|z + x\| \leq r \implies \|z\|^2 \leq r^2 - \|x\|^2.$$

Proof. Let $(r_n, x_n)_{n\in\mathbb{N}}$ be a minimizing sequence, that is

$$r_n \xrightarrow{n\to+\infty} r_E,$$

and notice that, for every $n \in \mathbb{N}$, we have $E \subset \overline{B_{r_n}(x_n)}$.

Step 1. First, we want to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ admits a limit $x_E \in H$, that is, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. However this is a simple consequence of the inclusion (1.10) mentioned above, since for every $p, q \in \mathbb{N}$ it turns out that

$$E \subseteq \overline{B\left(x_p, \max\{r_p, r_q\}\right)}, \quad E \subseteq \overline{B\left(x_q, \max\{r_p, r_q\}\right)},$$

and thus

$$E \subseteq \overline{B\left(x_p, \max\{r_p, r_q\}\right)} \cap \overline{B\left(x_q, \max\{r_p, r_q\}\right)} \subseteq$$

$$\subseteq \overline{B\left(\frac{x_p + x_q}{2}, \sqrt{\max\{r_p^2, r_q^2\} - \left\|\frac{x_p - x_q}{2}\right\|^2}\right)}.$$

Consequently,

$$\max\{r_p^2, r_q^2\} - \left\|\frac{x_p - x_q}{2}\right\|^2 \geq r_E^2 \implies \max\{r_p^2, r_q^2\} - r_E^2 \geq \left\|\frac{x_p - x_q}{2}\right\|^2,$$

which means that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., there exists $x_E \in H$ such that $x_n \to x_E \in H$.

Step 2. For every $n \in \mathbb{N}$ we have the inclusion

$$E \subseteq \overline{B(x_n, r_n)} \subseteq \overline{B\left(x_E, r_n + \|x - x_E\|\right)},$$

hence it suffices to take the limit as $n \to +\infty$ to derive the result. □

1.2.2 Projection onto a Closed Linear Subspace

In this section, we prove that any closed linear subspace F of a Hilbert space H admits a projection map $P : H \longrightarrow F$ that may be characterized via the notion of *orthogonality.*

Definition 1.17 (Orthogonal)**.** The orthogonal of a subspace $F \subset H$ of a Hilbert space, denoted by $F^\perp$, is defined via the scalar product as

$$F^\perp := \{x \in H \mid (x, v) = 0 \text{ for every } v \in F\}.$$

Theorem 1.18. *Let H be a Hilbert space and let $F \subseteq H$ be a linear and closed subspace.*

(a) *For any $x \in H$ there exists a unique point $P(x) \in F$ such that*

$$\|x - P(x)\| = \inf_{y\in F} \|x - y\|.$$

(b) *The map P is uniquely determined by the scalar product, that is,*

$$P(x) = y \iff (x - y,\, z) = 0 \qquad \forall z \in F.$$

Namely, the point $P(x) \in F$ is the projection of $x \in H$ if and only if $P(x) - x$ belongs to the orthogonal space $F^{\perp}$.

(c) *The map $P : H \longrightarrow F$ is a linear continuous projection (i.e., $P = P^2$).*

(d) *The kernel of P is the orthogonal space $F^{\perp}$, and the rank/image of P is the space F itself. Moreover, the Hilbert space H may be decomposed as the topological direct sum*[1] *of F and $F^{\perp}$.*

(e) *The orthogonal of the orthogonal is the space itself, that is,*

$$\left(F^{\perp}\right)^{\perp} = F.$$

Proof. The first assertion is an immediate consequence of Theorem 1.13. Indeed, for any $x \in H$ the projection $P(x)$ is the point of minimal norm in the affine, linear and closed subspace $F + x$.

(b) It suffices to prove that

$$P(x) = y \iff x - y \in F^{\perp}.$$

" $\Longrightarrow$ " : Let $y = P(x)$. By **(a)** it follows that $\|x - y\| = \inf_{u \in F} \|x - u\|$; hence, for every $t > 0$ and for every $v \in F$, we have the inequality

$$\|x - y\|^2 \leq \|(x - y) - tv\|^2.$$

We expand the right-hand side, and also divide both sides by t (which is positive); it is easy to see that

$$2(x - y,\, v) \leq t\|v\|^2.$$

Sending t to 0^+ (i.e., to 0 from right) yields to

$$(x - y,\, v) \leq 0, \qquad \forall\, v \in F. \tag{1.11}$$

Finally, since F is a linear subspace, we may always replace v with $-v$ in the inequality (1.11); thus, we can, at last, infer that

$$(x - y,\, v) = 0, \qquad \forall\, v \in F,$$

that is, $x - y \in F^{\perp}$.

[1] A topological direct sum is a direct sum $H = B \oplus C$, with the additional requirement that the map $(x,\, y) \in B \times C \mapsto x + y \in H$ is linear, bijective and an homeomorphism.

" $\Longleftarrow$ " : Conversely, assume that $x - y \in F^{\perp}$ and $y \in F$. For every $u \in F$, the vector $y - u$ still belongs to F (linear subspace); hence the orthogonality condition implies that

$$\|x-u\|^2 = \|(x-y)+(y-u)\|^2 = \|x-y\|^2 + \|y-u\|^2 \geq \|x-y\|^2,$$

which is exactly what we wanted to prove.

(c) The map P is obviously a projection since $P^2 = P$. The linearity follows easily from **(b)**; indeed, it is enough to prove that

$$(\alpha x + \beta y) - (\alpha P(x) + \beta P(y)) \in F^{\perp}.$$

For any $v \in F$ it turns out that

$$(\alpha\,(x - P(x)) + \beta\,(y - P(y))\,,\, v) = \alpha\,(x - P(x),\, v) + \beta\,(y - P(y),\, v) = 0$$

since both $x - P(x)$ and $y - P(y)$ belongs to $F^{\perp}$. In conclusion, we prove that P is a 1-Lipschitz map, that is,

$$\|Px - Py\| \leq \|x - y\|.$$

The characterization **(b)** proves that for any $x \in H$ it turns out that $x - Px \in F^{\perp}$ and $Px \in F$. Consequently, the scalar product $(x - Px,\, Px)$ is equal to 0, and thus

$$\|x\|^2 = \|Px + (x - Px)\|^2 = \|Px\|^2 + \|(\mathrm{id}_H - P)x\|^2 \geq \|Px\|^2.$$

In particular, for any $x \in H$ we have the inequality $\|x\| \geq \|Px\|$, and this is - by linearity of P - enough to infer that P is a 1-Lipschitz map:

$$Px - Py = P(x-y) \implies \|Px - Py\| \leq \|x - y\|.$$

(d) The assertion **(b)** proves that $Px = 0$ if and only if $x \in F^{\perp}$, and hence the kernel is $F^{\perp}$. In a similar fashion, one can check that the range of P is F.

General Fact. Let X be a topological vector space, and let $P : X \longrightarrow X$ be a linear and continuous projection. One may always decompose X as the direct sum of $\operatorname{Ker} P$ and $\operatorname{Ran} P$ in such a way that the restriction of the map

$$\begin{aligned} \operatorname{Ker} P \times \operatorname{Ran} P &\longrightarrow X \\ (v,\, w) &\longmapsto v + w \end{aligned}$$

is a homeomorphism. The converse assertion is also true: If there is a topological decomposition $X = V \oplus W$, then there exists a linear and continuous projection $P : X \longrightarrow V$.

(e) Here the assumption F closed comes into play: the orthogonal of a subspace A is always closed, and hence it is needed for the equality to hold.

The inclusion $F \subseteq \left(F^{\perp}\right)^{\perp}$ is always true, and it is an immediate consequence of the definition of $^{\perp}$. On the other hand, if $x \in \left(F^{\perp}\right)^{\perp}$, then $(x,\, x - Px) = 0$ and

$$\|x - Px\|^2 = 0 \implies x = Px \implies x \in F.$$

$\square$

1.3 Dual Space of a Hilbert Space

In this section, we investigate the basic properties of the *topological dual* of a Hilbert space, and we prove the fundamental representation result known as *Riesz theorem.*

Definition 1.19 (Topological Dual)**.** Let $(E, \|\cdot\|_E)$ be a normed space over a field $\mathbb{K}$. The *topological dua*l of E, denoted by E^*, is the space of all the continuous linear forms, i.e.,

$$E^* := \{\varphi : E \longrightarrow \mathbb{K} \mid \varphi \text{ linear and continuous}\}.$$

The norm $\|\cdot\|_E$ induces a norm on the dual E^*, which is given by

$$\|\varphi\|_{E^*} := \sup_{\|x\|_E \leq 1} |\varphi(x)| = \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|_E}. \tag{1.12}$$

One can easily check that (1.12) is the uniform norm restricted to the ball $B_E(0, 1)$, that is,

$$\|\varphi(x)\|_{E^*} = \left\| \phi \left|_{B_E(0,\,1)} \right. \right\|_\infty.$$

Proposition 1.20. *Let $(E, \|\cdot\|)$ be a normed space over a field $\mathbb{K}$. A linear form $\varphi : E \longrightarrow \mathbb{K}$ is continuous if and only if it is bounded.*

Proof. This equivalence holds in a far more general setting: it is enough to require both X and Y are topological vector spaces (see Section 4.1 for more details.)

" $\Longleftarrow$ " : Assume that $\varphi : E \longrightarrow \mathbb{K}$ is a bounded functional, that is, there exists a positive constant $C > 0$ such that

$$|\varphi(x)| \leq C\|x\|, \qquad \forall\, x \in E.$$

By linearity, it turns out that φ is a C-Lipschitz function, i.e.

$$|\varphi(x) - \varphi(y)| = |\varphi(x - y)| \leq C\|x - y\|,$$

and this implies also that φ is continuous (as Lipschitz-continuity is stronger than continuity).

" $\Longrightarrow$ " : Assume that $\varphi : E \longrightarrow K$ is continuous at $x = 0$. The ϵ-δ definition, which works only for metric spaces, implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x\| \leq \delta \implies |\varphi(x) - \varphi(0)| \leq \epsilon,$$

and thus, by linearity, it follows that

$$\|x\| \leq \delta \implies |\varphi(x)| \leq \epsilon.$$

In conclusion, we observe that for every $x \in E$ we have

$$\varphi(x) = \frac{\|x\|}{\delta} \varphi \left(\delta \frac{x}{\|x\|} \right),$$

from which it follows that

$$|\varphi(x)| \leq \frac{\epsilon}{\delta} \|x\|.$$

□

Lemma 1.21. *Let* $(E, \|\cdot\|_E)$ *be a* $\mathbb{K}$*-normed space. Then the topological dual* E^* *is a Banach space (=complete), provided that* $\mathbb{K}$ *is complete.*

Proof. The topological dual is the space of linear continuous forms; hence

$$E^* = \mathcal{L}(E, \mathbb{K}).$$

By Theorem 2.10 the dual E^* is complete since the codomain $\mathbb{K}$ is complete by assumption. In particular, the dual of a real (or complex) normed space is always a Banach space. □

Theorem 1.22 (Riesz-Fréchet)**.** *Let* H *be a Hilbert space, and let* $\varphi \in H^*$ *be a given continuous linear functional. Then there exists a unique element* $x_\varphi \in H$ *such that*

$$\langle \varphi, u \rangle = (x_\varphi, u) \quad \text{for every } u \in H,$$

where $\langle \cdot, \cdot \rangle$ *is the duality product* $\langle H^*, H \rangle$*, and* $(\cdot, \cdot)$ *is the scalar product in* H*. Moreover, the isomorphism*

$$H^* \ni \varphi \longmapsto x_\varphi \in H$$

is also an isometry, that is,

$$\|x_\varphi\|_H = \|\varphi\|_{H^*}.$$

Proof. The idea is to define a map that associated a continuous linear functional to any element $x \in H$, and then prove that it is an invertible isometry.

Step 1. More precisely, for any $x \in H$ we define the linear continuous functional $\varphi_x : H \longrightarrow \mathbb{K}$ as follows:

$$\varphi_x(v) := (v, x).$$

Notice that φ_x is also linear in a complex Hilbert space, since the Hermitian product is left-linear and right-semilinear, and v is on the left on purpose. By the Cauchy-Schwarz inequality (1.1) it turns out that

$$|\varphi_x(v)| \leq \|x\|\|v\| \implies \|\varphi_x\|_{H^*} \leq \|x\|,$$

and, actually, the dual norm $\|\varphi_x\|_{H^*}$ is exactly equal to $\|x\|$, as the reader may check by herself computing the functional at $v := x$. The mapping

$$\Phi : H \longrightarrow H^*, \qquad \Phi(x) := \varphi_x$$

is a linear (or anti-linear, in the complex case) isometric inclusion, thus we are only left to prove that Φ is surjective.

Step 2. Let $\psi : H \longrightarrow \mathbb{R}$ be any linear and continuous nonzero functional. The kernel $\operatorname{Ker}\psi$ is not the whole space H, and thus the orthogonal is nonempty:

$$(\operatorname{Ker}\psi)^\perp \neq \varnothing.$$

Let $u \in (\operatorname{Ker}\psi)^\perp$ be a nonzero element, whose norm is equal to 1, and let us consider the vector

$$v := \langle \psi, x \rangle\, u - \langle \psi, u \rangle\, x \in \operatorname{Ker}\psi,$$

as x ranges in H. By orthogonality, it turns out that

$$(\langle\psi,\, x\rangle\, u - \langle\psi,\, u\rangle\, x,\, u) = 0,$$

from which it follows that

$$\langle\psi,\, x\rangle\, \|u\|^2 - \langle\psi,\, u\rangle\, (x,\, u) = 0 \implies \langle\psi,\, x\rangle = \left(x,\, \overline{\langle\psi,\, u\rangle}\, u\right),$$

which means that the mapping Φ is surjective:

$$\psi = \Phi\left(\overline{\langle\psi,\, u\rangle}\, u\right).$$

□

1.4 Hilbert Sums and Orthonormal Bases

In this section, we introduce the notion of orthonormal system in a Hilbert space, and characterize maximal orthogonal systems in terms of the Perseval formula.

Definition 1.23 (Summable Collection)**.** Let X be a topological vector space. A collection of elements $\{x_\lambda\}_{\lambda\in\Lambda} \subset X$ is *summable*, with sum S, if, for every $U \subset X$ neighborhood of the origin, there exists a finite subset $A \subset \Lambda$ such that

$$\forall\, B \;:\; A \subset B \subset \Lambda \text{ and } |B| < \infty \implies S - \sum_{\lambda\in B} x_\lambda \in U.$$

Definition 1.24 (Orthonormal System)**.** Let H be a Hilbert space. A collection of elements $\{e_\lambda\}_{\lambda\in\Lambda} \subset H$ is an *orthonormal system* for H if, for every $\mu,\, \nu \in \Lambda$, it turns out that

$$(e_\mu,\, e_\nu) = \delta_{\mu,\,\nu}.$$

Proposition 1.25. *Let H be a Hilbert space and let $\{e_\lambda\}_{\lambda\in\Lambda} \subset X$ be an orthonormal system. The following properties are equivalent:*

(a) *The system $\{e_\lambda\}_{\lambda\in\Lambda}$ is maximal, that is,*

$$(e_\lambda,\, x) = 0 \quad \forall\, \lambda \in \Lambda \implies x = 0.$$

(b) *The system $\{e_\lambda\}_{\lambda\in\Lambda}$ is complete, that is,*

$$\overline{\text{Span}\,\langle e_\lambda \;:\; \lambda \in \Lambda\rangle} = H.$$

(c) *Every element $x \in H$ is equal to the sum of its "Fourier coefficients", that is,*

$$x = \sum_{\lambda\in\Lambda} x_\lambda e_\lambda,$$

where $x_\lambda := (x,\, e_\lambda)$.

(d) *The squared norm of x is equal to the ℓ^2-norm of its "Fourier coefficients", that is,*

$$\|x\|^2 = \sum_{\lambda\in\Lambda} |x_\lambda|^2 = \|\mathbf{x}_\lambda\|_{\ell^2(\Lambda)}. \tag{1.13}$$

(e) *For every $x,\, y \in H$, the following identity holds true:*

$$(x,\, y) = \sum_{\lambda \in \Lambda} x_\lambda y_\lambda.$$

Proof. The argument presented here is rather simple, and there are no important ideas behind; we divide it into five steps, to ease the notation for the reader.

Step 1. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is not a complete orthonormal system, that is, the linear span is not dense in H, that is

$$\overline{\mathrm{Span}\,\langle e_\lambda \;:\; \lambda \in \Lambda\rangle} \neq H,$$

and let us prove that $\{e_\lambda\}_{\lambda\in\Lambda}$ cannot be a maximal system. The orthogonal is nonempty

$$\left(\overline{\mathrm{Span}\,\langle e_\lambda \;:\; \lambda \in \Lambda\rangle}\right)^{\perp} \neq \varnothing$$

since the closure of the linear span is a closed linear subspace of H, and the point **(e)** of the Theorem 1.18 asserts that

$$\left(F^{\perp}\right)^{\perp} = F.$$

In particular, there exists $x \neq 0$ in the orthogonal of the linear span of the family $\{e_\lambda\}_{\lambda\in\Lambda}$, that is, an element of H such that

$$(x,\, e_\lambda) = 0, \qquad \forall\, \lambda \in \Lambda,$$

which means that the collection $\{e_\lambda\}_{\lambda\in\Lambda} \cup \{x/\|x\|\}$ is strictly bigger than $\{e_\lambda\}_{\lambda\in\Lambda}$, and hence $\{e_\lambda\}_{\lambda\in\Lambda}$ is not maximal.

Step 2. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a complete orthonormal system. For any finite subset $B \subset \Lambda$ we consider the linear span

$$H_B := \mathrm{Span}\,\langle e_\lambda \;:\; \lambda \in B\rangle\,,$$

which is a finite-dimensional subspace of H, linearly homeomorphic to $\mathbb{C}^n$ (or $\mathbb{R}^n$, if H is a real vector space), for some $n \in \mathbb{N}$.

In particular, the subspace H_B is closed and complete with respect to the induced metric; it follows from Theorem 1.18 that there exists a projection $P_B : H \longrightarrow H_B$, which is defined by

$$P_B(x) := \sum_{\lambda \in B} x_\lambda e_\lambda \in H_B. \tag{1.14}$$

Indeed, one can easily check that this is the (unique) orthogonal projection onto a linear subspace since it satisfies the orthogonality criterion given in the theorem above. More precisely, it is enough to prove that (1.14) satisfies the following property:

$$\lambda \in B \implies (x - P_B(x),\, e_\lambda) = (x,\, e_\lambda) - x_\lambda = 0.$$

Consequently, for any finite subset $B \subset \Lambda$, there is a well-defined distance function

$$d(x,\, H_B) := \left\| x - \sum_{\lambda \in B} x_\lambda\, e_\lambda \right\| = \|x - P_B(x)\|,$$

and the assumption $\{e_\lambda\}_{\lambda\in\Lambda}$ complete implies that

$$\inf_{B\subset\Lambda} d(x,\, H_B) = 0$$

or, equivalently, that for any $\epsilon > 0$ and any $x \in H$ there exist $B_0 \subset \Lambda$ finite such that

$$\left\| x - \sum_{\lambda\in B_0} x_\lambda\, e_\lambda \right\| < \epsilon.$$

Moreover, for any finite subset C such that $B_0 \subset C \subset \Lambda$, the inequality above is still true, that is,

$$\left\| x - \sum_{\lambda\in C} x_\lambda\, e_\lambda \right\| < \epsilon$$

since the distance decreases as the finite subset of Λ gets bigger (recall that the projection gives the point of minimal norm). In conclusion, from the simple identity

$$d(x,\, H_B)^2 = \left\| x - \sum_{\lambda\in B} x_\lambda\, e_\lambda \right\|^2$$

it follows that

$$\{e_\lambda\}_{\lambda\in\Lambda} \text{ is complete} \iff (1.13) \iff x = \sum_{\lambda\in\Lambda} x_\lambda e_\lambda.$$

Step 3. Let $x,\, y \in H$. The Perseval identity (1.13) implies that

$$\|x+ty\|^2 = \sum_{\lambda\in\Lambda} |(x+ty,\, e_\lambda)|^2,$$

from which it follows that

$$\bar{t}(x,\, y) + t(y,\, x) = 2t \sum_{\lambda\in\Lambda} x_\lambda y_\lambda.$$

In conclusion, it is enough to choose $t := (x,\, y)$ for the identity above to become the so-called generalized Parseval formula, that is,

$$(x,\, y) = \sum_{\lambda\in\Lambda} x_\lambda y_\lambda.$$

Step 4. We argue by contradiction. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a non-maximal orthonormal collection. In particular, there exists $x \neq 0$ such that

$$(x,\, e_\lambda) = 0, \qquad \forall\, \lambda \in \Lambda.$$

The norm $\|x\|$ is different from zero, but from the Perseval identity (1.13) we obtain

$$\|x\|^2 = \sum_{\lambda\in\Lambda} |x_\lambda|^2 = 0 \implies \|x\| = 0,$$

and this is the sought contradiction. $\square$

Proposition 1.26. *Let H be a Hilbert space and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal system. Then there exists a complete orthonormal system $\{e'_\lambda\}_{\lambda\in\Lambda'}$, which is not necessarily unique, extending $\{e_\lambda\}_{\lambda\in\Lambda}$.*

Definition 1.27 (Hilbert Basis)**.** An orthonormal system $\{e_\lambda\}_{\lambda\in\Lambda}$ in a Hilbert space H is a *Hilbert basis* if it is maximal.

Theorem 1.28. *Let H be a Hilbert space, and let $\{e_\lambda\}_{\lambda\in\Lambda}$ and $\{f_\mu\}_{\mu\in\Delta}$ be two Hilbert bases. The cardinality is the same, that is,*

$$|\Lambda| = |\Delta|\,,$$

and it is usually called Hilbert dimension of H in the literature.

Proof.

First Case. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a finite-cardinality basis, then the thesis is trivially true since H is either isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ for some $n \geq 0$.

Second Case. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an infinite basis of H, that is, we have $|\Lambda| \geq \aleph_0$. We consider the rational linear span

$$D := \mathrm{Span}_{\mathbb{Q}} \langle e_\lambda \;:\; \lambda \in \Lambda\rangle\,,$$

and it is immediate to see that by definition D is dense in H and has the same cardinality, i.e.

$$|\Lambda| = |D|\,.$$

The family of open balls

$$\{B(f_\mu,\, 1/2)\}_{\mu\in\Delta}$$

is necessarily disjoint, and hence, by density, one can find at least one element of D in any such ball. It follows that $|\Lambda| \geq |\Delta|$ and this gives us the thesis since the argument is symmetric with respect to Λ and Δ. ☐

Remark 1.4. Let $(\mathscr{X},\, \mathcal{A},\, \mu)$ be a measure space, with μ a positive σ-additive measure. We define the usual p-summable functions space as

$$L^p\left(\mathscr{X},\, \mathcal{A},\, \mu\right) := \left\{ f : \mathscr{X} \longrightarrow \mathbb{C} \;\middle|\; f \text{ measurable and } \int_{\mathscr{X}} |f|^p \,\mathrm{d}\mu < \infty \right\}.$$

It is easy to prove that this is a Banach space for any $p \in [1,\, +\infty]$, and also that it is a Hilbert space if and only if $p = 2$. In that case the scalar product is given by

$$(f,\, g) := \int_{\mathscr{X}} f(x)\overline{g}(x)\,\mathrm{d}\mu(x).$$

If Λ is a set, we can consider the discrete measure (cardinality) and let $\mathcal{A} = P(\Lambda)$. In this special case $L^p(\Lambda) := \ell_p(\Lambda)$, which is the set of all the p-summable sequences, that is,

$$\ell_p\left(\Lambda\right) := \left\{ \mathbf{x} : \Lambda \longrightarrow \mathbb{C} \;\middle|\; \sum_{\lambda\in\Lambda} |\mathbf{x}(\lambda)|^p < \infty \right\}.$$

Corollary 1.29. *Let H be a Hilbert space. There exists a set Λ of cardinality equal to the Hilbert dimension* $\dim_{\mathcal{H}} H$ *such that the map*

$$\ell_2(\Lambda) \ni \mathbf{x} \longmapsto \sum_{\lambda \in \Lambda} \mathbf{x}(\lambda) e_\lambda \in H$$

is an isomorphism.

In particular, any infinite-dimensional Hilbert space H is separable if and only if $H \cong \ell^2(\mathbb{N})$; thus there exists, up to isomorphism, only one separable Hilbert space.

1.5 Appendix

In this section, we briefly introduce some additional notions that are either interesting exercises or useful in different courses (e.g., probability and geometric measure theory).

1.5.1 Radon-Nikodym Theorem

Definition 1.30 (Absolute Continuity)**.** A measure ν is *absolutely continuous* with respect to a measure μ, and we denote it by $\nu \ll \mu$, if and only if every

$$\mu(A) = 0 \implies \nu(A) = 0.$$

Theorem 1.31 (Radon-Nikodym)**.** *Let μ and ν be bounded and positive measures defined on a σ-algebra $\mathcal{F}$ of a set $\mathscr{X}$, and assume that $\nu \ll \mu$. Then there exists a unique*[2] *summable positive function $f \in L^1(\mathscr{X}, \mu)^+$ such that $fu \in L^1(\mathscr{X}, \mu)$ and*

$$\int_{\mathscr{X}} u(x) \, \mathrm{d}\nu(x) = \int_{\mathscr{X}} f(x) u(x) \, \mathrm{d}\mu(x), \qquad \forall\, u \in L^1(\nu).$$

Moreover, the same statement holds true if we replace $L^1(\mathscr{X}, \mu)$ with $L^2(\mathscr{X}, \mu)$, and the function f is unique, provided that it is μ-almost everywhere 0 in any ν-null set.

Proof. The measure $\mu + \nu$ is finite since μ and ν are both finite measures. It follows that there is a chain of inclusions

$$L^2(\mu + \nu) \hookrightarrow L^1(\mu + \nu) \hookrightarrow L^1(\nu),$$

where the last one is due to the fact that $\nu \ll \mu$. Indeed, if we consider $v \in L^1(\mu + \nu)$, then

$$\int_{\mathscr{X}} v(x) \, \mathrm{d}\nu(x) < +\infty$$

since $\mu(A) = 0$ implies $\nu(A) = 0$, and thus $(\mu + \nu)(A) = 0$, which means that $\mu + \nu \ll \mu$. In a similar fashion, there is another chain of maps

$$L^2(\mu + \nu) \hookrightarrow L^1(\mu + \nu) \xrightarrow{\varphi} L^1(\mu) \xrightarrow{f \mapsto \int_{\mathscr{X}} f \, \mathrm{d}\mu} \mathbb{R},$$

[2] Here the reader needs to be careful. The space $L^1(\mu)$ is used in these notes either as $L^1(\mu)$ or $L^1(\mu)/\sim$, where $f \sim g$ if and only if $f(x) = g(x)$ almost everywhere. In this case, the uniqueness is intended as equivalence class.

but, in general, φ is **not** an inclusion. In particular, the functional

$$L^2(\mu+\nu) \ni f \mapsto \int_{\mathscr{X}} f(x)\,\mathrm{d}\mu(x) \in \mathbb{R}$$

is bounded and linear, and thus by Riesz Theorem 1.22 there exists a unique element $g \in L^2(\mu+\nu)$ such that

$$\int_{\mathscr{X}} u(x)\,\mathrm{d}\mu(x) = \int_{\mathscr{X}} u(x)g(x)\,\mathrm{d}(\mu+\nu)(x),$$

or, equivalently, such that

$$\int_{\mathscr{X}} [1-g(x)]\,u(x)\,\mathrm{d}\mu(x) = \int_{\mathscr{X}} u(x)g(x)\,\mathrm{d}\nu(x).$$

Let $u(x) := \chi_{g\le 0}(x)$ be the characteristic function of the set $\{x \in X \mid g(x) \le 0\}$. Then

$$\int_{\mathscr{X}} [1-g(x)]\,u(x)\,\mathrm{d}\mu(x) \ge \int_{\mathscr{X}} u(x)\,\mathrm{d}\mu(x) \ge 0,$$

$$\int_{\mathscr{X}} u(x)g(x)\,\mathrm{d}\nu(x) \le 0,$$

and therefore $\{x \in X \mid g(x) \le 0\}$ is a μ-null set and, in particular, a ν-null set. Equivalently, if we let $u_\epsilon(x) := \chi_{g\ge 1+\epsilon}(x)$, then

$$\int_{\mathscr{X}} [1-g(x)]\,u_\epsilon(x)\,\mathrm{d}\mu(x) \le 0,$$

$$\int_{\mathscr{X}} u(x)g(x)\,\mathrm{d}\nu(x) \ge \int_{\mathscr{X}} u(x)\,\mathrm{d}\nu(x) \ge 0,$$

implies that $\{g \ge 1+\epsilon\}$ is a μ-null set for any $\epsilon > 0$. In particular, $\{x \in X \mid g(x) > 1\}$ is a μ-null set, and thus a ν-null set, that is,

$$0 < g(x) \le 1, \qquad \text{for } \nu\text{-almost every } x \in X.$$

For any $n \in \mathbb{N}$ and for any $u \in L^1(\nu)^+$, the function

$$\left(\frac{u(x)}{g(x)} \wedge n\right),$$

where $\wedge$ denotes the minimum, belongs to L^∞; thus

$$\int_{\mathscr{X}} [1-g(x)]\left(\frac{u(x)}{g(x)} \wedge n\right)\,\mathrm{d}\mu(x) = \int_{\mathscr{X}} \left(\frac{u(x)}{g(x)} \wedge n\right) g(x)\,\mathrm{d}\nu(x),$$

and applying the Beppo-Levi property we obtain

$$\int_{\mathscr{X}} f(x)u(x)\,\mathrm{d}\mu(x) = \int_{\mathscr{X}} u(x), \mathrm{d}\nu(x),$$

where

$$f(x) := \frac{g(x)}{1-g(x)}.$$

In conclusion, let $u \in L^1(\nu)$ and decompose it as the sum of the positive part and the negative part, i.e. $u = u^+ - u^-$, and apply the argument above to both addendum. $\square$

1.5.2 Subgroups of a Hilbert Spaces

Let $V \subseteq \mathbb{R}^n$ be an additive, closed and connected subgroup of $\mathbb{R}^n$. The first goal of this section is to prove that V is a linear subspace of $\mathbb{R}^n$.

Definition 1.32 (Discrete)**.** An additive subgroup $\mathcal{G}$ of $\mathbb{R}^n$ is *discrete* if it is generated by $m \leq n$ linearly independent vectors.

It follows that every additive subgroup $V \subseteq \mathbb{R}^n$ that is not the linear span of m linearly independent vectors is dense and, since it is also closed, equal to $\mathbb{R}^n$. Therefore, the only nontrivial V are the following ones:

(1) V is a point, i.e., $V = \{(0, \dots, 0)\}$.

(2) V is a line, i.e., $V = \operatorname{Span}\langle e_1\rangle$;

(3) V is a plane, i.e., $V = \operatorname{Span}\langle e_1, e_2\rangle$;

(4) V is a k-hyperplane, i.e., $V = \operatorname{Span}\langle e_1, \dots, e_k\rangle$

(5) V is a hyperplane, i.e., $V = \operatorname{Span}\langle e_1, \dots, e_{n-1}\rangle$;

(6) $V = \mathbb{R}^n$.

It is straightforward to prove that these are all linear subspaces of $\mathbb{R}^n$ and, as an immediate consequence of the criterion mentioned above, they are the only ones additive, closed and connected.

Hilbert Spaces. The primary goal is to prove an analogous result that characterizes, in a certain sense, the additive closed connected subgroups $\mathcal{G}$ of a Hilbert space H.

Lemma 1.33 (Generalized Parallelogram Identity)**.** *Let H be a Hilbert space, and let $x_1, \dots, x_n$ be elements of H. Then*

$$\sum_{\epsilon \in \{0,1\}^n} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^2 = 2^n \sum_{j=1}^n \|x_j\|^2. \tag{1.15}$$

Lemma 1.34. *Let $\mathcal{G} \subseteq H$ an additive subgroup of a Hilbert space H. Suppose that $\mathcal{G}$ is a lattice generated by a finite number of elements with norms $\|g_i\| \leq 1$ for $i = 1, \dots, n$. Then $\mathcal{G}$ is a $\sqrt{n}$-net in his linear span, that is,*

$$\mathcal{G} \text{ is a } \sqrt{n}\text{-net in } \operatorname{Span}_{\mathbb{R}}\langle g_1, \dots, g_n\rangle.$$

Proof.

Step 1. We claim that all the elements of the rescaling $2^{-1} \cdot \mathcal{G}$ are distant, at most, $\sqrt{n}/2$ from any other element of $\mathcal{G}$. In fact, given $x \in 2^{-1} \cdot \mathcal{G}$ we may always assume - up to translations - that it is given by a reduced sum

$$x = \frac{1}{2} \sum_{i=1}^k g_i \quad \text{for some } k \leq n. \tag{1.16}$$

More precisely, the point x is the center of the fundamental parallelogram associated to the generators of the lattice with nonzero coefficients in (1.16). Hence (by the parallelogram identity) the distance between x and any vertexes $y \in \mathcal{G}$ is less than or equal to $\sqrt{k}/2$, which can be estimated by $\sqrt{n}/2$.

Step 2. The idea is to iterate this process and conclude with a density argument. For every $m \in \mathbb{N}$, the subgroup $2^{-m-1} \cdot \mathcal{G}$ is distant at most $\frac{\sqrt{n}}{2^{m+1}}$ from $2^{-m} \cdot \mathcal{G}$. Thus, any element of the union

$$\bigcup_{m \in \mathbb{N}} 2^{-m} \cdot \mathcal{G} \tag{1.17}$$

is, at most, distant $\sqrt{n}$ from any element of $\mathcal{G}$ since

$$\sum_{m=1}^{+\infty} 2^{-m} = 1.$$

The proof is now complete since the union (1.17) is a dense set in H, as the reader may check as an easy exercise. □

We are finally ready to state the main result of this section. Namely, a sufficient condition for an additive closed connected subgroup $\mathcal{G} \subset H$ of a Hilbert space to be linear, is that $\mathcal{G}$ is arc-wise connected by α-Hölder arcs for $\alpha > 1/2$. The reader should notice that it is not a necessary condition by any means.

Theorem 1.35. *Let $\mathcal{G} \subseteq H = \ell^2(\mathbb{N};\ \mathbb{R})$ be an additive, closed and connected subgroup of H.*

(a) *There exists a nonlinear subspace $\mathcal{G}$ that is arc-wise connected by $1/2$-Hölder arcs.*

(b) *If $\mathcal{G}$ is arc-wise connected by α-Hölder arcs, and $\alpha > 1/2$, then $\mathcal{G}$ is a linear subspace.*

Proof.

(a) Let $H := L^2([0,\, 1])$ and let $\mathcal{G} := L^2([0,\, 1];\ \mathbb{Z})$. It is an easy exercise to show that $\mathcal{G}$ is an additive, closed subgroup of H. Given $f \in \mathcal{G}$, the arc defined by

$$\gamma(t) := f\chi_{[0,\, t]} = \begin{cases} f(x) & x \in [0,\, t] \\ 0 & x \in [t,\, 1] \end{cases}$$

takes values in $\mathcal{G}$, and its extremal points are $\gamma(0) \equiv 0$ and $\gamma(1) \equiv f$. Then

$$\|\gamma(t) - \gamma(s)\|^2_{L^2([0,\, 1])} \leq \int_s^t |f(x)|^2 \, \mathrm{d}x,$$

but we cannot conclude that γ is $1/2$-Hölder because f is not - a priori - in $L^4([0,\, 1])$. We consider a reparametrization

$$\sigma : [0,\, 1] \longrightarrow [0,\, 1 + \|f\|^2_{L^2([0,\, 1])}]$$

defined explicitly by setting

$$\sigma(t) := t + \int_0^t |f(x)|^2 \, dx.$$

It is easy to prove that σ is continuous, increasing (strictly) and bijective; thus

$$\|\gamma(t) - \gamma(s)\|_2 \leq |\sigma(t) - \sigma(s)|^{1/2},$$

which means that the composition $\gamma \circ \sigma^{-1}$ is a 1/2-Hölder path, and yet $\mathcal{G}$ is nonlinear.

(b) Suppose that $\mathcal{G}$ is arc-wise connected by α-Hölder arcs for some $\alpha > 1/2$. It follows from Lemma 1.34 that the lattice generated by n vectors $g_1, \dots, g_n \in H$, with norms $\|g_k\| \leq r$, is a $r\sqrt{n}$-net in their linear span. Therefore, if $\gamma : [0, 1] \longrightarrow G$ is an α-Hölder path, for any $n \in \mathbb{N}$, the n elements

$$g_{k,n} := \gamma\left(\frac{k+1}{n}\right) - \gamma\left(\frac{k}{n}\right) \in \mathcal{G}, \quad k = 0, \dots, n-1,$$

form a $Cn^{1/2-\alpha}$-net in their linear span. Since $\mathcal{G}$ is closed this implies that it is a cone, and hence a linear subspace.

$\square$

1.5.3 Laguerre Polynomials

Let $\mathcal{A}$ be an alphabet. Given a finite word formed by $(n_1, a_1), \dots, (n_k, a_k) \in \mathbb{N} \times \mathcal{A}$, a natural question would be how many anagrams of this word can we find, and how many[3] of them have no fixed points? Recall that, if $n := n_1 + \cdots + n_k$, the number of anagrams is given by

$$G(n_1, \dots, n_k) := \frac{n!}{n_1! \, n_2! \, \dots \, n_k!},$$

and thus we can immediately infer that $F(n_1, \dots, n_k) \leq G(n_1, \dots, n_k)$.

2-words. The first nontrivial case is given by an alphabet made up of two letters. Clearly, if $n_1 < n_2$ or $n_2 < n_1$ every permutation (=anagram) of the word has at least a fixed point, therefore we can assume that $n_1 = n_2$. In this case, the problem admits one and only one solution (replace each a_1 with a_2, and vice versa). In particular, we have a orthonormal basis, that is,

$$F(n, m) = \delta_{n,m}.$$

General Case. If k is arbitrary, the question becomes much harder, and there are two possible approaches; here we give an idea of the first method, and we investigate the second one entirely.

[3] We denote by $F(n_1, \dots, n_k)$ the number of anagrams with no fixed points.

First Approach. Let $k \geq 1$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and let $D(\alpha) := F(\alpha_1, \ldots, \alpha_k)$. Then

$$\sum_{\alpha \in \mathbb{N}^k} D(\alpha) x^\alpha = \frac{1}{1 - \sum_S (|S| - 1) \prod_{i \in S} x_i},$$

where S ranges over all nonempty subsets of $\{1, \ldots, k\}$, and $x^\alpha := x_1^{\alpha_1} \ldots x_k^{\alpha_k}$.

Second Approach. Let Λ denote an arbitrary alphabet, and suppose that the words have length n. Consider a set of indices $I = \{1, 2, \ldots, n\}$, and let A be the set of all the permutations, whose cardinality is thus given by

$$|A| = \frac{n!}{n_1! \, n_2! \, \ldots \, n_k!}, \quad \text{where } n_1 + \cdots + n_k = n.$$

For any $J \subseteq I$, consider the set of all permutations fixing J

$$A_J := \{\sigma \in \mathfrak{S}_n \mid J \subseteq \mathrm{Fix}(\sigma)\},$$

and notice that the cardinality is given by

$$|A_J| = \frac{|I \setminus J|!}{\prod_{\lambda \in \Lambda} |I_\lambda \setminus J_\lambda|!},$$

where $I_\lambda = \sigma^{-1}(\lambda)$ and $J_\lambda = \sigma^{-1}(\lambda) \cap J$. Since

$$A_J \cap A_K = A_{J \cup K},$$

the inclusion-exclusion principle gives us a formula for $A_\varnothing$ (the set of permutations with no fixed points), i.e.

$$|A_\varnothing| = |A| - \sum_{i \in I} |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots = \sum_{J \subseteq I} (-1)^{|J|} \, |A_J|.$$

Using the explicit formula for the cardinality of A_J and the obvious equivalence

$$\mathcal{P}(I) \longleftrightarrow \prod_{\lambda \in \Lambda} \mathcal{P}(I_\lambda),$$

we find that, if we set $K_\lambda := I_\lambda \setminus J_\lambda$, then

$$|A_\varnothing| = \sum_{J \in \prod_\lambda \mathcal{P}(I_\lambda)} (-1)^{\sum_\lambda |J_\lambda|} \frac{(\sum_\lambda |I_\lambda \setminus J_\lambda|)!}{\prod_\lambda (|I_\lambda \setminus J_\lambda|!)} =$$

$$= \sum_{K \in \prod_\lambda \mathcal{P}(I_\lambda)} (-1)^{\sum_\lambda (|I_\lambda| - |K_\lambda|)} \frac{(\sum_\lambda |K_\lambda|)!}{\prod_\lambda (|K_\lambda|!)}.$$

At this point, we want to find a way to "implement" the following identity:

$$\prod_{\lambda \in \Lambda} \sum_{j \in X_\lambda} c(\lambda, j) = \sum_{\Phi \in \prod_\lambda X_\lambda} \prod_{\lambda \in \Lambda} c(\lambda, \Phi(j)),$$

where $c(\cdot, -)$ is any function defined on $\Lambda \times \left(\prod_{\lambda\in\Lambda} X_\lambda\right)$. By Euler formula, it turns out that

$$m! = \int_0^{+\infty} x^m \mathrm{e}^{-x}\,\mathrm{d}x,$$

and hence

$$|A_\varnothing| = \sum_{K\in\prod_\lambda \mathcal{P}(I_\lambda)} \left[\prod_{\lambda\in\Lambda} \frac{(-1)^{|I_\lambda|-|K_\lambda|}}{|K_\lambda|!}\right] \int_0^\infty x^{\sum_\lambda |K_\lambda|}\,\mathrm{e}^{-x}\,\mathrm{d}x =$$

$$= \int_0^\infty \left[\sum_{K\in\mathcal{P}(I_\lambda)} \prod_\lambda \frac{(-1)^{|I_\lambda|-|K_\lambda|}}{|K_\lambda|!}\, x^{\sum_\lambda |K_\lambda|}\right] \mathrm{e}^{-x}\,\mathrm{d}x.$$

If we set $\mathrm{d}\mu := e^{-x}\,\mathrm{d}x$, then

$$|A_\varnothing| = \int_0^\infty \left[\prod_\lambda \sum_{K\in\mathcal{P}(I_\lambda)} \frac{(-1)^{|I_\lambda|-|K_\lambda|}}{|K_\lambda|!}\, x^{\sum_\lambda |K_\lambda|}\right] \mathrm{d}\mu(x) =$$

$$= \int_0^\infty \left[\prod_\lambda P_{n_\lambda}(x)\right] \mathrm{d}\mu(x) =,$$

where P_m is the polynomial defined by

$$P_m(x) := \sum_{k=0}^m (-1)^{m-k}\binom{m}{k}\frac{x^k}{k!}.$$

Since $F(m,\, n) = \delta_{m,\,n}$, then

$$\ldots = \int_0^\infty P_n(x)\cdot P_m(x)\,\mathrm{d}\mu(x) = 0, \qquad \forall\, n \neq m,$$

thus $\{P_n\}_{n\in\mathbb{N}}$ is an orthogonal family in $L^2\left([0,\,\infty),\,\mathrm{d}\mu\right)$ and they are called **Laguerre polynomials**.

1.6 Exercises

Exercise 1.1. The locus of the zeros of a quadratic form $Q(x) = B(x,\, x)$, which is usually denoted by $\mathcal{Z}(B)$, is a linear subspace of V.

Exercise 1.2. For any $x,\, y \in \mathbb{R}^n$ the Cauchy-Schwarz inequality holds, i.e.

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right) - \left(\sum_{i=1}^n x_i\, y_i\right)^2 \geq 0.$$

Moreover, the following equality is satisfied

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right) - \left(\sum_{i=1}^n x_i\, y_i\right)^2 = \sum_{1\leq i<j\leq n} \left[x_i\, y_j - x_j\, y_i\right]^2,$$

Exercise 1.3 (Open Problem[4]). Let H be a Hilbert space, and let $C \subset H$ a subset together

[4] This problem is likely still open in some sense, but I am not certain about this.

with a projection $P : H \longrightarrow C$ such that P is continuous and $\|P(x) - x\| = \inf_{y \in C} \|x - y\|$. Then C is convex.

Exercise 1.4. Let μ be a measure over $I \subseteq \mathbb{R}$ (e.g. $\mathbb{R}$, $\mathbb{R}^+$, a bounded interval, etc..). When the polynomial in $\mathbb{C}[x]$ are dense in $L^2(I, \mu)$?

In affirmative cases, prove that the Gram-Schmidt algorithm produces a orthonormal basis of polynomial such that P_n has degree n.

Exercise 1.5. Let H be a Hilbert space and assume that $E \subset H$ is bounded and nonempty. In the setting of Lemma 1.16, the following properties holds true:

(a) If E is also convex, then x_E belongs to E.

(b) Both x_E and r_E depend continuously (actually, 1-Lipschitz) on the sets (with respect to the distance $\mathcal{H}$), that is

$$\|x_E - x_{E'}\| \leq d_{\mathcal{H}}(E, E'), \qquad |r_E - r_{E'}| \leq d_{\mathcal{H}}(E, E').$$

Exercise 1.6. Let H be a Hilbert space, and assume that $E \subset H$ is bounded, closed and nonempty. The Hausdorff distance is given by

$$d_{\mathcal{H}}(E, E') = \|d_E - d_{E'}\|_\infty,$$

where d_E is the usual distance. In a Banach space, since we cannot assume that the distance is a bounded function, we can define the Hausdorff distance as

$$d_{\mathcal{H}}(E, E') = \|(d_E - \|x\|) - (d_{E'} - \|x\|)\|_\infty.$$

In particular, we defined a map

$$E \in B(H) := \{E \subseteq H \ : \ E \neq \varnothing, \ E \text{ bounded and closed}\} \mapsto d_E - \|x\| \in C_b(H; \mathbb{R})$$

and $B(H)$ is a complete metric space with the Hausdorff distance. Prove that

$$d_{\mathcal{H}}(E, E') \leq r \iff \begin{cases} E \subseteq r \cdot B(0, 1) + E' \\ E' \subseteq r \cdot B(0, 1) + E. \end{cases}$$

Exercise 1.7. Let $H := L^2([0, 1])$. Find the projection over the following convex subspaces:

(a) $C = \overline{B(0, 1)}$;

(b) $C = \{f \in H \ : \ f \geq 0 \text{ a.e.}\}$;

(c) $C = \overline{B((0, 1); L^\infty)}$.

Chapter 2

Banach Spaces

In this chapter, we generalize the concept of Hilbert space when a scalar product inducing the norm does not exist. More precisely, we introduce the notion of *Banach space*, and describe the main properties (e.g., completeness) of the space of all linear operators between two Banach spaces

2.1 Definitions and Elementary Properties

Definition 2.1 (Norm)**.** Let X be a (real or) complex vector space. A *norm* is an application $\|\cdot\| : X \longrightarrow \mathbb{R}_+$ satisfying the following properties:

(a) The norm is always positive and equal to 0 only for $x = 0$, that is,

$$\|x\| \geq 0 \text{ for every } x \in X \quad \text{and} \quad \|x\| = 0 \iff x = 0.$$

(b) The norm is positively homogeneous, that is,

$$\|\lambda x\| = |\lambda| \|x\| \quad \text{for all } \lambda \in \mathbb{C} \text{ and } x \in X.$$

(c) The norm satisfies the triangular inequality, that is,

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for every } x,\, y \in X. \tag{2.1}$$

Remark 2.1. Any normed space $(E, \|\cdot\|_E)$ is also a metric space. Indeed, it is straightforward to check that the mapping $\rho : X \times X \longrightarrow \mathbb{R}$, defined by

$$\rho(x, y) := \|x - y\|,$$

satisfies the conditions **(a)**-**(c)** introduced in the previous definition. The opposite implication is false, that is, a metric space is generally not a normed space.

Proof. The first assertion is evident. To find a metric space which is not a normed space, the reader may consider a **bounded** metric d on a topological space X. □

Definition 2.2 (Banach Space)**.** A normed space $(B, \|\cdot\|_B)$ is a *Banach space* if it is complete with respect to the induced distance ρ.

Definition 2.3 (Seminorm)**.** Let X be a complex vector space. A *seminorm* is a mapping $p : X \longrightarrow \mathbb{R}$ satisfying the following properties:

(a) *Positive*, that is, $p(x) \geq 0$ for all $x \in X$.

(b) *Positively homogeneous*, that is $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{C}$ and $x \in X$.

(c) Satisfies the *triangular inequality*, that is

$$p(x+y) \leq p(x) + p(y) \quad \text{for every } x,\, y \in X. \tag{2.2}$$

Remark 2.2. Let (X, p_X) be a seminormed space. The map $\rho_s : X \times X \longrightarrow \mathbb{R}$, defined by setting

$$\rho_s(x, y) := p_X(x-y),$$

is a *pseudometric* on X, that is, a positive function satisfying all the properties of a metric, except

$$x = y \iff \rho_s(x,\, y) = 0.$$

In general, if (X, p_s) is a seminormed space, then the topology induced on (X, ρ_s) is not Hausdorff, that is, the points are not separable since $\rho_s(x,\, y) = 0$ does **not** imply $x = y$.

Example 2.1. The reader may check that $L^2([0,\, 1])$, endowed with the seminorm

$$p_2(f) = \int_0^1 |f(x)|^2 \, \mathrm{d}x,$$

is not a Hausdorff space. The reader may jump to Exercise 3.2, where we investigate this property more in-depth.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a normed space. The following are equivalent:*

(a) The space X is complete with respect to $\|\cdot\|$.

(b) Every absolutely converging series

$$\sum_{n \in \mathbb{N}} \|x_n\| < \infty$$

is convergent, that is,

$$\exists x \in X \ : \ \lim_{n \to +\infty} \left\| \sum_{k=0}^{n} x_k - x \right\| = 0.$$

(c) Every absolutely convergent geometric[1] *series*

$$\sum_{n \in \mathbb{N}} \|x_n\| < \infty$$

is convergent, that is,

$$\exists x \in X \ : \ \lim_{n \to +\infty} \left\| \sum_{k=0}^{n} x_k - x \right\| = 0.$$

[1]**Definition.** Let $(X, \|\cdot\|)$ be a normed space. The series $\sum_n x_n$ is geometric if $\|x_n\| \leq 2^{-n}$ for every $n \in \mathbb{N}$.

Remark 2.3. Let $(X,\, d)$ be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence converging to some $x \in X$, then $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof. The triangular inequality implies that

$$d(x_n,\, x) \leq d(x_n,\, x_{n_k}) + d(x_{n_k},\, x),$$

for every $n \in \mathbb{N}$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists a natural number $N_1 \in \mathbb{N}$ such that

$$d(x_n,\, x_m) \leq \epsilon, \qquad \forall\, n,\, m > N_1.$$

In a similar fashion, the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to x, and hence there exists a natural number $N_2 \in \mathbb{N}$ such that

$$d(x_{n_k},\, \underline{x}) \leq \epsilon, \qquad \forall\, n_k > N_2.$$

In conclusion, if we let $N := \max\{N_1,\, N_2\}$, then it turns out that

$$d(x_n,\, x) \leq 2\,\epsilon, \qquad \forall n\, > N,$$

that is, $x_n \to x$ as n goes to $+\infty$. □

Proof of Theorem 2.4. The assertion "**(b)** $\implies$ **(c)**" follows directly from the definitions; therefore we only show the remaining two implications here.

"**(a)** $\implies$ **(b)**" Suppose that X is complete, and let $(x_n)_{n\in\mathbb{N}}$ be an absolutely convergent sequence, that is,

$$\sum_{n\in\mathbb{N}} \|x_n\| < \infty.$$

If we denote by $(S_k)_{k\in\mathbb{N}}$ the sequence of the partial sums, i.e.

$$S_k := \sum_{n=0}^{k} x_n,$$

then the thesis is equivalent to the existence of $S \in X$ such that

$$S_k \xrightarrow{k\to+\infty} S.$$

By completeness, it is enough to prove that $(S_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X. But this follows easily from the following estimate:

$$\|S_j - S_k\| = \left\| \sum_{n=k}^{j} x_n \right\| \leq \sum_{n=k}^{j} \|x_n\| = o(1), \qquad \text{for } k,\, j \to +\infty.$$

"**(c)** $\implies$ **(a)**" Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. A standard argument implies that there always exists a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that

$$\left\| x_{n_{k+1}} - x_{n_k} \right\| \leq 2^{-k}.$$

Let $(y_k)_{k\in\mathbb{N}} := \left(x_{n_{k+1}} - x_{n_k}\right)_{k\in\mathbb{N}}$. Clearly, the sum

$$S_k := \sum_{k\in\mathbb{N}} y_k$$

is an absolutely convergent geometric series, and thus by assumption it converges to a certain element $S \in X$. On the other hand, we have the identity

$$x_{n_{k+1}} = x_{n_0} + S_k,$$

and therefore there exists $\underline{x} = x_{n_0} + S \in X$ such that

$$x_{n_k} \xrightarrow{k\to+\infty} \underline{x}.$$

In particular, we proved that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ always admits a converging subsequence. By Remark 2.3 we finally infer that $(x_n)_{n\in\mathbb{N}}$ converges to $\underline{x}$ as well, which means that X is complete. $\square$

Proposition 2.5 (Bounded Functions)**.** *Let S be a set, and let $(X, \|\cdot\|_X)$ be a Banach space. The space of all the bounded functions with values in X, denoted by*

$$\left(\mathcal{B}\left(S;\ X\right),\ \|\cdot\|_{\infty,\,S}\right),$$

is complete with respect to the uniform norm.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{B}\left(S;\ X\right)$ be a sequence such that

$$\sum_{k\in\mathbb{N}} \|f_k\|_{\infty,\,S} < \infty.$$

For every $s \in S$, it turns out that

$$\sum_{k\in\mathbb{N}} \|f_k(s)\|_X \le \sum_{k\in\mathbb{N}} \|f_k\|_{\infty,\,S} < \infty,$$

and thus (by completeness) the series $\sum_{k\in\mathbb{N}} f_k(s)$ converges in X. For every $s \in S$ it turns out that

$$\left\|\sum_{k\in\mathbb{N}} f_k(s)\right\|_X \le \sum_{k\in\mathbb{N}} \|f_k(s)\|_X < \infty,$$

and hence the map

$$F : S \longrightarrow X, \qquad s \longmapsto \sum_{k=0}^{+\infty} f_k(s)$$

is a bounded function. In conclusion, we notice that the convergence is uniform since

$$\left\| F - \sum_{k=0}^{N} f_k \right\|_{\infty,\,S} = \left\| \sum_{k=N+1}^{+\infty} f_k \right\|_{\infty,\,S} \le \sum_{k\ge N+1} \|f_k\|_{\infty,\,S} = o(1)$$

for $N \to +\infty$, and this is enough to conclude the proof. $\square$

Remark 2.4. Let E be a topological space. The inclusion

$$\left(C_b^0\,(E;\;X)\,,\,\|\cdot\|_\infty\right) \subset \left(\mathcal{B}\,(E;\;X)\,,\,\|\cdot\|_\infty\right)$$

is closed. Therefore, the space $C_b^0\,(E;\;X)$ is complete as a corollary of Proposition 2.5.

Proposition 2.6. *Let X be a topological space, and let μ be any reasonable measure on a σ-algebra $\mathcal{F}$ of X. The space of all μ-summable functions, denoted by $L^1(X;\;\mu)$, is complete.*

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset L^1(X;\;\mu)$ be a sequence such that

$$\sum_{k\in\mathbb{N}} \|f_k\|_{L^1(X;\;\mu)} < \infty,$$

and let us define

$$g_n(x) := \sum_{k=0}^{n} |f_k(x)|, \qquad \forall\, n \in \mathbb{N}.$$

Clearly $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive and measurable functions; thus the Beppo-Levi[2] property implies that

$$\sum_{k=0}^{+\infty} \|f_k\|_{L^1(X;\;\mu)} = \lim_{n\to+\infty} \int_X g_n(x)\,\mathrm{d}\mu(x) = \int_X \left(\sum_{k=0}^{+\infty} |f_k(x)|\right) \mathrm{d}\mu(x),$$

that is, the series $\sum_k |f_k(x)|$ converges μ-almost everywhere. But $\mathbb{R}$ is a complete space; hence the mapping

$$F : X \longrightarrow \mathbb{R}, \qquad x \longmapsto F(x) := \sum_{k=0}^{+\infty} f_k(x)$$

is well-defined and, as the reader may easily check, also an element of $L^1(X;\;\mu)$ since

$$|F(x)| \le \sum_{k=0}^{+\infty} |f_k(x)| \in L^1(X;\;\mu).$$

In conclusion, we observe that

$$\left\| F - \sum_{k=0}^{N} f_k \right\|_{L^1(X;\;\mu)} \le \sum_{k\ge N+1} \|f_k\|_{L^1(X;\;\mu)} = o(1),$$

for $N \to +\infty$, to infer that the convergence is uniform. $\square$

Remark 2.5. In a similar fashion, one can prove that $L^p(X;\;\mu)$ is complete for every $p \ge 1$ and any reasonable measure μ.

---

[2]Let $(X,\,\Sigma,\,\mu)$ be a measure space, and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of non-decreasing (point-wise), positive and measurable functions. Then the punctual limit $f(x) := \lim_k f_k(x)$ is measurable and

$$\int f_k\,\mathrm{d}\mu \xrightarrow{k\to+\infty} \int f\,\mathrm{d}\mu.$$

Step 1. Let $(f_k)_{k\in\mathbb{N}} \subset L^p(X;\, \mu)$ be a sequence such that

$$\sum_{k\in\mathbb{N}} \|f_k\|_p < \infty,$$

and assume that $\mu(X) < \infty$ and $1 \leq p < +\infty$. The inclusion

$$L^p(X;\, \mu) \hookrightarrow L^1(X;\, \mu)$$

is continuous, and thus $\sum_k f_k$ is also a absolutely converging series in $L^1(X;\, \mu)$. It follows from Proposition 2.6 that it converges in $L^1(X;\, \mu)$ to an element $f \in L^1(X;\, \mu)$ and, up to a subsequences, it converges to f μ-almost everywhere.

Step 2. For every $n \in \mathbb{N}$, it turns out that

$$|f_m(x) - f_n(x)| \xrightarrow{m\to+\infty} |f(x) - f_n(x)|\,,$$

and thus - by Fatou's Lemma[3] - we find that

$$\|f - f_n\|^p_{L^p(X;\, \mu)} \leq \liminf_{m\to+\infty} \|f_m - f_n\|^p_{L^p(X;\, \mu)} = o(1),$$

for $n \to +\infty$, and this is enough to conclude.

Step 3. If X is a σ-finite set, then we can find a μ-almost everywhere converging subsequence and use the same argument in a exhaustion by compact sets (e.g., diagonal procedure).

If X is not σ-finite, it is enough to notice that the support of a $L^p(X;\, \mu)$ function needs to be σ-finite and that the countable union of σ-finite sets is also σ-finite.

2.2 Linear Operators between Banach Spaces

We denote the space of all the linear bounded operators $L : X \longrightarrow Y$ by $\mathcal{L}(X,\, Y)$. Coherently with the case $Y = \mathbb{C}$, there is a naturally induced norm, i.e.

$$\|L\|_{\mathcal{L}(X,\, Y)} := \|L\|_{\infty,\, \overline{B_X}} = \sup_{\|x\|_X \leq 1} \|Lx\|_Y\,,$$

where $\overline{B_X}$ is the closed ball of X, centered at 0 with radius 1.

Definition 2.7 (BL Operator)**.** Let $L \in \mathcal{L}(X,\, Y)$ be a linear operator. We say that L is *bounded* if

$$\|L\|_{\mathcal{L}(X,\, Y)} < +\infty,$$

that is, if the set $L\left(\overline{B_X}\right)$ is bounded as a subset of Y.

[3] Let $(X,\, \Sigma,\, \mu)$ be a measure space, and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of positive and measurable functions. If $f(x) := \liminf_k f_k(x)$, then

$$\int_X f \, \mathrm{d}\mu \leq \liminf_{k\to+\infty} \int_X f_k \, \mathrm{d}\mu.$$

Lemma 2.8. *Let $L \in \mathcal{L}(X, Y)$ be a linear operator. The following properties are all equivalent:*

- ***(a)*** *L is Lipschitz.*
- ***(b)*** *L is continuous.*
- ***(c)*** *L is continuous at $x = 0$.*
- ***(d)*** *L is locally bounded, that is there exists a neighborhood U of the origin whose image $L(U)$ is bounded in Y.*

Proof. The only nontrivial implication is (**d**) $\implies$ (**a**). First, we notice that for a locally bounded linear operator L there are positive constants r, $s > 0$ such that

$$L\left(\overline{B_X(0,\, r)}\right) \subseteq \overline{B_Y(0,\, s)}. \tag{2.3}$$

Let $x \in X$ be a point. The inclusion (2.3) implies that

$$Lx = \frac{\|x\|}{r} L\left(\frac{rx}{\|x\|}\right) = \frac{\|x\|}{r} y \quad \text{for some } y \text{ such that } \|y\| \leq s,$$

from which it follows that

$$\|Lx\| \leq \frac{s}{r}\|x\|.$$

In particular, the operator L is Lipschitz, and the Lipschitz constant is exactly equal to the operator norm $\|L\|_{\mathcal{L}(X,\, Y)}$ introduced above. □

Proposition 2.9. *Let L_1, $L_2 \in \mathcal{L}(X, Y)$ be linear bounded operators, and let $\lambda \in \mathbb{K}$ be an element of the scalar field. The following properties hold true:*

- ***(a)*** *$L_1 + L_2 \in \mathcal{L}(X, Y)$.*
- ***(b)*** *$\lambda L_1 \in \mathcal{L}(X, Y)$.*
- ***(c)*** *If L_1, $L_2 \in \mathcal{L}(X, X)$, then both $L_1 L_2$ $(=L_1 \circ L_2)$ and $L_2 L_1$ $(=L_2 \circ L_1)$ belong to $\mathcal{L}(X, X)$.*

Theorem 2.10. *The space $\mathcal{L}(X, Y)$ is Banach, provided that Y is a Banach space.*

We now present two similar arguments to prove that the space $\mathcal{L}(X, Y)$ of linear and continuous function is complete. We will see that the completeness of Y plays a fundamental role in both.

Proof 1. The main idea behind this argument is to employ the completeness of the space $\mathcal{B}(S, Y)$ with respect to the uniform norm, choosing S to be the unit closed ball of X.

Step 1. The embedding

$$\big(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)}\big) \hookrightarrow \big(\mathcal{B}(\overline{B_X}, Y), \|\cdot\|_\infty\big),$$

defined by sending L to its restriction to the unit ball $L\,|_{\overline{B_X}}$, is continuous since

$$\|L\,|_{\overline{B_X}}\|_\infty = \|L\|_{\mathcal{L}(X,Y)}.$$

In particular, it is an *isometric* embedding. Therefore, if we can prove that

$$\mathcal{L}(X, Y) \subseteq C_b^0(X, Y)$$

is a closed inclusion (even with respect to the pointwise convergence), then we can infer that $\mathcal{L}(X, Y)$ is complete as a result of Remark 2.4.

Step 2. Let $(L_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X, Y)$ be a sequence of linear and continuous operators converging to a bounded continuous application $L \in C_b^0(X, Y)$, that is,

$$L_n(x) \xrightarrow{n\to+\infty} L(x) \quad \text{for all } x \in X.$$

The sequence given by the restrictions to the closed unit ball $\left(L_n\,|_{\overline{B_X}}\right)_{n\in\mathbb{N}}$ converges uniformly to the restriction $L\,|_{\overline{B_X}}$, which is bounded by Lemma 2.8.

Step 3. It follows that the operator L is linear (by definition) and with bounded restriction to the unit closed ball. On the other hand, the operator norm is equal to the uniform norm of the restriction, and hence $L \in \mathcal{L}(X, Y)$, that is,

$$\imath : \mathcal{L}(X, Y) \hookrightarrow C_b^0(X, Y)$$

is a closed inclusion, and this concludes. □

The previous proof is simple, but it exploits the completeness of another function space; here we give a slightly different proof of Theorem 2.10, which shows more explicitly what happens.

Proof 2. The space of all the linear continuous operators $\mathcal{L}(X, Y)$ is a normed vector space, as it follows immediately from Proposition 2.9.

Step 1. Let $(L_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X, Y)$ be a Cauchy sequence, that is, $\|L_n - L_m\| \to 0$ as $n, m \to +\infty$. The thesis is equivalent to the existence of a bounded linear operator $L \in \mathcal{L}(X, Y)$ such that

$$\|L_n - L\| \xrightarrow{n\to+\infty} 0.$$

Indeed, for every $x \in X$, it turns out that

$$\|(L_n - L_m)x\|_Y \leq \|L_n - L_m\|\|x\|_X \xrightarrow{n,\, m\to+\infty} 0,$$

and thus the sequence $(L_n x)_{n\in\mathbb{N}} \subset Y$ is a Cauchy sequence in a complete space. In particular, there exists an element $y \in Y$ such that

$$\|L_n x - y\|_Y \xrightarrow{n\to+\infty} 0.$$

Step 2. Let us define L as the pointwise limit of the sequence L_n, that is,

$$Lx := y = \lim_{n \to +\infty} L_n x.$$

The operator $L : X \longrightarrow Y$ is clearly linear. Furthermore, by Lemma 2.8 it suffices to show that L is bounded to infer that L is continuous. Using the fact that $(L_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that $\sup_{n\in\mathbb{N}} \|L_n\| < \infty$, and thus

$$\|Lx\|_Y = \lim_{n \to +\infty} \|L_n x\|_Y \leq K\|x\|_X, \quad \text{for all } x \in X,$$

that is, $L \in \mathcal{L}(X, Y)$. □

Chapter 3

The Hahn-Banach Theorems

In this chapter, we are concerned with the various statements of the Hahn-Banach theorem. In the first half, we prove the analytic form: *Every functional defined on a linear subspace of a vector space may be extended (not uniquely, in general) to a linear functional, provided that there exists a particular function p.*

In the second half of the chapter, we prove the geometric weak (strong) form: *Two subsets A, $B \subset X$, satisfying certain assumptions, of a (locally convex) topological vector space X may be (strictly) separated via a closed hyperplane.*

3.1 The Analytic Form of the Hahn-Banach Theorem

Theorem 3.1 (Hahn-Banach, analytic form)**.** *Let X be a real vector space and let $p : X \longrightarrow \mathbb{R}$ be a sublinear function, that is, a function satisfying the following properties:*

1) *$p(\lambda x) = \lambda p(x)$ for every $x \in X$ and every positive $\lambda \in \mathbb{R}$.*

2) *$p(x + y) \leq p(x) + p(y)$ for every x, $y \in X$.*

Let $G \subset X$ be a linear subspace of X, and let $g : G \longrightarrow \mathbb{R}$ be a linear functional such that

$$g(x) \leq p(x), \qquad \forall\, x \in G.$$

Then there exists a linear functional $f : X \longrightarrow \mathbb{R}$, which extends g, satisfying the following property:

$$f(x) \leq p(x) \qquad \forall\, x \in X.$$

Proof. Let us consider the family

$$\mathcal{F} := \left\{ h : Y \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{c} Y \text{ is a linear subspace of } X, \\ h \text{ is linear, } G \subseteq Y,\ h \text{ extends } g, \\ \text{and } h(x) \leq p(x) \text{ for every } x \in Y \end{array} \right\}.$$

A standard argument proves that

$$h_1 \le h_2 \iff Y_1 \subset Y_2 \text{ and } h_2 \text{ extends } h_1$$

is a partial order relation on $\mathcal{F}$. The reader may easily check that the family $\mathcal{F}$ is nonempty (since $(g, Y) \in \mathcal{F}$) and that it is *inductive*. Indeed, let $\{h_i\}_{i \in I} \subset P$ be a totally ordered chain. If we set

$$Y = \bigcup_{i \in I} Y_i, \qquad h(x) = h_i(x) \quad \text{if } x \in Y_i \text{ for some } i,$$

then it is easy to prove that (h, Y) is an upper bound for the chain. By Zorn's lemma, there is a maximal element $(f, Y) \in \mathcal{F}$: If we can prove that Y is equal to X, then the functional $f : X \longrightarrow \mathbb{R}$ will be the sought one.

We argue by contradiction. Suppose that there exists an element $x_0 \in X \setminus Y$, and let us consider the linear space $Z := Y \oplus \mathbb{R}x_0$. For every $(x, t) \in Y \times \mathbb{R}$, we also define

$$h(x + tx_0) = f(x) + t\alpha,$$

where the constant $\alpha \in \mathbb{R}$ $(=h(0, x_0))$ needs to be chosen in such a way that $(h, Z) \in \mathcal{F}$. It suffices to take any $\alpha \in \mathbb{R}$ among the real numbers satisfying the inequality

$$\sup_{y \in Y} [f(y) - p(y - x_0)] \le \alpha \le \inf_{x \in Y} [p(x + x_0) - f(x)]\,. \tag{3.1}$$

It is easy to prove that such an α must exist. Indeed, the subadditivity of p yields to

$$f(u + v) = f(u) + f(v) \le p(u + v + x_0 - x_0) \le p(u - x_0) + p(v + x_0) \quad \text{for all } u,\, v \in Y,$$

which, in turn, implies

$$f(u) - p(u - x_0) \le -f(v) + p(v + x_0) \quad \text{for every } u,\, v \in Y.$$

It follows from the completeness of $\mathbb{R}$ that the set defined by the left-hand side and the set defined by the right-hand side are necessarily separated by an element, and hence such an α exists. It follows from (3.1) that for every $x \in Y$ we have

$$\begin{cases} f(x) + \alpha \le p(x + x_0) \\ \\ f(x) - \alpha \le p(x - x_0), \end{cases}$$

and, multiplying both by $t > 0$, if we set $y = tx$, then we obtain

$$\begin{cases} f(y) + t\alpha \le p(y + tx_0) \\ \\ f(y) - t\alpha \le p(y - tx_0). \end{cases}$$

The second inequality, for $t < 0$, can be easily rewritten as

$$f(y) - (-t)\alpha \le p(y - (-t)x_0) \iff f(y) + t\alpha \le p(y + tx_0) \quad \text{for } t < 0,$$

which yields to

$$f(x) + t\alpha \le p(x + tx_0) \quad \text{for all } t \in \mathbb{R}.$$

In particular, the map h extends f and it satisfies the inequality $h \le p$; hence $(h, Z) \in \mathcal{F}$, which is in contradiction with the maximality of f. $\square$

Notation. Let $f \in X^*$ and $x \in X$ be given. From now on, we denote by $\langle f, x\rangle$ the value of f at x, i.e. $f(x)$. In the literature, the mapping $\langle \cdot, \cdot\rangle$ is called the *duality* (X^*, X) *scalar product.*

Corollary 3.2. *Let $G \subset X$ be a linear subspace, and let $D := \overline{B_X(0, 1)}$. If $g : G \longrightarrow \mathbb{R}$ is a continuous linear functional, then there exists $f \in X^*$ that extends g and such that*

$$\|f\|_{X^*} = \sup_{x \in G \cap D} |g(x)| = \|g\|_{G^*}.$$

Proof. Let us consider the function $p(x) = \|g\|_{G^*}\|x\|$. The reader may check by herself that p is positively homogeneous and subadditive, that is, for every $x, y \in X$ and $\lambda \in \mathbb{R}^+$ it turns out that

$$p(\lambda x) = \lambda p(x) = \lambda p(x),$$

$$p(x + y) = \|g\|_{G^*}\|x + y\|_X \le \|g\|_{G^*}(\|x\| + \|y\|).$$

By definition, for every $x \in G$ the inequality $g(x) \le p(x)$ holds; thus by the Hahn-Banach Theorem 3.1 there exists a functional $f : X \longrightarrow \mathbb{R}$ extending g. To prove that the norm is preserved, it suffices to notice that

$$\|g\|_{G^*} \le \|f\|_{X^*} \le \|p\|_{X^*} = \|g\|_{G^*}.$$

□

Remark 3.1 (Complex)**.** Let X be a complex normed space, and let $f : G \subset X \longrightarrow \mathbb{C}$ be a continuous linear functional, defined on a linear subspace G. Then

$$\mathfrak{Re}\, f : G \longrightarrow \mathbb{R}$$

is a continuous $\mathbb{R}$-linear functional, which is still defined on G since X is a real normed space as well. By the Hahn-Banach Theorem 3.1 there exists a functional $g : X \longrightarrow \mathbb{R}$, extending $\mathfrak{Re}\, f$, such that

$$\langle g, x\rangle \le \|f\|_{G^*}\|x\|.$$

The functional $\widetilde{f} : X \longrightarrow \mathbb{C}$ defined by setting

$$\left\langle \widetilde{f}, x \right\rangle := \langle g, x\rangle - \imath \langle g, \imath x\rangle$$

is $\mathbb{C}$-linear, and, clearly, it is an extension of f. Moreover, one can check that

$$\left\langle \mathfrak{Re}\, \widetilde{f}, x \right\rangle = \langle g, x\rangle \le \|f\|_{G^*}\|x\| \implies$$

$$\implies \mathfrak{Re}\left(\theta \left\langle \widetilde{f}, x \right\rangle\right) \le \|f\|_{G^*} \|x\|,$$

for any $\theta \in \mathbb{C}$ of unitary norm; hence we can finally infer that

$$\left\langle \widetilde{f}, x \right\rangle \le \|f\|_{G^*}\|x\|, \qquad \forall\, x \in X.$$

Example 3.1 (Extension: norm-preserving)**.** Let $(X, \|\cdot\|)$ be a normed space, and let $x_0 \in X$ be a point. If we let $G = \mathbb{R}x_0$, then the linear continuous functional $g : G \longrightarrow \mathbb{R}$ defined by setting

$$g(x_0) = \|x_0\|^2,$$

may be extended, via Hahn-Banach, to a continuous linear functional $f \in X^*$ with the additional properties

$$\|f\|_{X^*} = \|x_0\|, \qquad \langle f, x_0\rangle = \|x_0\|^2.$$

Example 3.2 (Extension: unitary)**.** Let $(X, \|\cdot\|)$ be a normed space and let $x_0 \in X$ be a point. If we let $G = \mathbb{R}x_0$, then the linear continuous functional $g : G \longrightarrow \mathbb{R}$ defined by setting

$$g(x_0) := \|x_0\|,$$

may be extended, via Hahn-Banach, to a continuous linear functional $x_0^* \in X^*$ with the additional properties

$$\|x_0^*\|_{X^*} = 1, \qquad \langle x_0^*, x_0\rangle = \|x_0\|.$$

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a normed space, and let $x_0 \in X$ be a point. Then*

$$\|x_0\|_X := \sup_{\|f\|_{X^*}\leq 1} |\langle f, x_0\rangle| = \max_{\|f\|_{X^*}\leq 1} |\langle f, x_0\rangle| .$$

Proof. One inequality is trivial since

$$\sup_{\|f\|_{X^*}\leq 1} |\langle f, x\rangle| \leq \|x\| \quad \text{for all } x \in X.$$

On the other hand, in Example 3.2 we have proved the existence of a linear functional $x_0^* \in X^*$ with unitary operator norm ($\|x_0^*\|_{X^*} = 1$) and satisfying $\langle x_0^*, x_0\rangle = \|x_0\|$; thus the equality follows immediately by choosing $f = x_0^*$. □

3.2 The Geometric Forms of the Hahn-Banach Theorem

In this section, we set the ground for the statements of the geometric forms of the Hahn-Banach theorem; namely, we briefly introduce the notion of (locally convex) topological vector space, which is approached more in-depth in Section 4.1, and we investigate the separation properties.

Definition 3.4 (Minkowski Functional)**.** Let X be a vector space, and let $C \subset X$ be a subset containing the origin. The *Minkowski functional* is defined as follows:

$$p_C(x) := \inf\left\{t \in \mathbb{R}^+ \mid x \in t\cdot C\right\} \in [0, +\infty].$$

Lemma 3.5. *Let $C \subset X$ be a subset of a real vector space X containing the origin, and let p_C be the associated Minkowski functional. Then the following properties hold:*

(a) *The image of X via p_C is bounded (i.e., $p(x) < +\infty$ for every $x \in X$) if and only if C is absorbing*[1]*.*

(b) *If C is convex and absorbing, then the functional p_C is positively homogeneous and subadditive.*

(c) *If C is convex absorbing and balanced, the functional p_C is a seminorm.*

(d) *If C is convex and absorbing, then following inclusions hold:*

$$\{p_C < 1\} \subseteq C \subseteq \{p_C \leq 1\}.$$

[1] **Definition.** Let $C \subset X$ be a subset of a vector space X. We say that C is absorbing if for every $x \in X$ there exists $t \in \mathbb{R}^+$ such that $x \in t\cdot C$.

Proof. We refer the reader to Section 4.1 for a better overview of topological vector spaces, included the importance of seminorms via Minkowsi functionals.

(a) If C is absorbing, then for any $x \in X$ there is a positive constant $r_x > 0$ such that

$$x \in r_x \cdot C,$$

and this implies that $p_C(x) \leq r_x < \infty$. The vice versa is obvious.

(b) First, we want to prove that for all $\lambda \in \mathbb{R}^+$

$$p_C(\lambda x) = \lambda p_C(x).$$

There are two possible way to show this equality. The first one follows from taking the limit as $\epsilon \to 0^+$ of the trivial chain of inequalities below:

$$\frac{p_C(\lambda x) + \epsilon}{\lambda} \geq p_C(x) \geq \frac{p_C(\lambda x) - \epsilon}{\lambda}.$$

Alternatively, notice that

$$\begin{aligned} p_C(\lambda x) &= \inf\{t > 0 \,:\, \lambda x \in t \cdot C\} = \\ &= \inf\{t > 0 \,:\, x \in \frac{t}{\lambda} \cdot C\} = \\ &= \lambda \inf\{\frac{t}{\lambda} > 0 \,:\, x \in \frac{t}{\lambda} \cdot C\} = \\ &= \lambda p_C(x). \end{aligned}$$

We now prove the subadditivity of p_C. For every $\epsilon > 0$ and every $\lambda \in (0, 1)$, it turns out that

$$\lambda \frac{x}{p_C(x) + \epsilon} + (1 - \lambda) \frac{y}{p_C(y) + \epsilon} \in C,$$

since C is convex. If we choose

$$\lambda := \frac{p_C(x) + \epsilon}{p_C(x) + p_C(y) + 2\epsilon} \implies 1 - \lambda = \frac{p_C(y) + \epsilon}{p_C(x) + p_C(y) + 2\epsilon},$$

then it turns out that

$$\frac{x + y}{p_C(x) + p_C(y) + 2\epsilon} \in C.$$

In particular, by definition we have the inequality

$$p_C(x) + p_C(y) + 2\epsilon \geq p_C(x + y),$$

and hence, if we take the limit as $\epsilon \to 0^+$, we can infer that p_C is subadditive.

(c) Recall that, if C is a balanced set, then $C = -C$. In particular, it turns out that for all $\lambda < 0$ we have

$$\begin{aligned} p_C(\lambda x) &= \inf\{t > 0 \,:\, \lambda x \in t \cdot (C)\} = \\ &= \inf\{t > 0 \,:\, (-\lambda) x \in t \cdot (-C)\} = \\ &= p_{-C}(-\lambda x) = -\lambda p_{-C}(x) = |\lambda| p_C(x), \end{aligned}$$

which means that p_C is homogeneous (and not only positively homogeneous).

(d) One of the inclusions is trivial, i.e.,

$$\{x \in X \mid p_C(x) < 1\} \subseteq C.$$

On the other hand, notice that for a convex absorbing set C, the family $\{t \mid x/t \in C\}$ is either given by

$$[p_C(x), +\infty) \quad \text{or} \quad (p_C(x), +\infty).$$

Therefore, if $p_C(x) > 1$, both intervals do not contain the value $t = 1$, which means that x cannot belong to C.

□

Theorem 3.6 (Geometric Hahn-Banach). *Let X be a topological vector space, and let $A \subset X$ and $B \subset X$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume that A is open. Then there exist $f \in X^*$ and $\gamma \in \mathbb{R}$ such that*

$$f(a) < \gamma \leq f(b) \quad \textit{for all } a \in A \textit{ and } b \in B.$$

More precisely, there exists a closed hyperplane H that separates A and B.

Proof. The difference[2] $A - B$ is open since it is arbitrary union of open sets

$$A - B = \bigcup_{b \in B} A - \{b\},$$

and it is also convex as a consequence of the identity

$$t(A - B) + (1 - t)(A - B) = (tA + (1 - t)A) + (tB + (1 - t)B).$$

By assumption A and B do not intersect. Thus, the origin is not an element of the difference $A - B$, and, in particular, given any $x_0 \in A - B$, the translation $C := A - B - x_0$ is a convex neighborhood of 0 endowed with a Minkowski functional p_C.

By Lemma 3.5, it follows that $p_C(-x_0) \geq 1$ (since $-x_0 \notin C$) and, by Hahn-Banach Theorem 3.1 there exists a linear functional $f : X \longrightarrow \mathbb{R}$ such that

$$\langle f, -x_0 \rangle = 1 \quad \text{and} \quad f(x) \leq p_C(x) \quad \text{for all} x \in X.$$

For every $a \in A$ and every $b \in B$, the difference $a - b - x_0$ belongs to C, and thus

$$f(a) - f(b) - f(x_0) \leq p_C(a - b - x_0) \leq 1 \implies f(a) - f(b) \leq 0.$$

In particular, there exists $\gamma \in \mathbb{R}$ such that

$$f(a) \leq \gamma \leq f(b) \quad \text{for all } a \in A \text{ and all } b \in B.$$

We now claim that f is an open mapping. If the claim holds true, then the image of A via f is an open subset of $\mathbb{R}$, and therefore f does not admit a maximum on $f(A)$, that is,

$$f(a) < \gamma \leq f(b) \quad \text{for all } a \in A \text{ and all } b \in B.$$

In conclusion, notice that the functional f is continuous because it is linear and bounded on a neighborhood of 0 (see Lemma 2.8), and thus the hyperplane is closed. More precisely, if we consider the neighborhood of the origin $U := C \cap (-C)$, then for all $x \in U$, it turns out that

$$f(x) \leq 1 \quad \text{and} \quad f(-x) \leq 1.$$

[2] The difference $X - Y$ is the collection of all the elements of the form $x - y$ for some $x \in X$ and $y \in Y$.

Claim. We prove here that any nonconstant linear function $f : X \longrightarrow \mathbb{R}$ is an open mapping, that is, the image of all open sets $V \subset X$ is open in $\mathbb{R}$.

Proof. Let $x^* \in X$ be a point such that $f(x^*) = 1$, and let $V \subseteq X$ be an open set. Fix a point $y \in f(V)$ and a point $x \in V$ such that $f(x) = y$. The continuity of the scalar product proves that

$$\exists\, \delta > 0 \,:\, \forall\, |r| < \delta \implies x + rx^* \in V,$$

which means that

$$f(x + rx^*) = y + r \in f(V).$$

In particular, the one-dimensional open ball of center y and radius δ is contained in the image $f(V) \subset \mathbb{R}$, and this is enough to infer that $f(V)$ is open. □

Lemma 3.7. *Let X be a topological vector space, and let $f : X \longrightarrow \mathbb{R}$ be a nonzero linear functional. The following properties are equivalent:*

- ***(a)*** *The functional f is continuous.*
- ***(b)*** *The kernel $\operatorname{Ker} f$ is closed as a subset of X.*
- ***(c)*** *The kernel $\operatorname{Ker} f$ is not dense in X.*
- ***(d)*** *The functional f is bounded in a neighborhood of the origin.*

Proof.

- **(a)** Assume that f is continuous. The singlet $\{0\} \in \mathbb{R}$ is closed in $\mathbb{R}$; thus the preimage $f^{-1}(0) = \operatorname{Ker} f$ is closed in X.
- **(b)** Assume that $\operatorname{Ker} f$ is closed. By assumption f is not identically null; hence $\operatorname{Ker} f \neq X$, which means that it cannot be dense.
- **(c)** Assume that $\operatorname{Ker} f$ is not dense in X, that is, its complement has nonempty interior. By definition, there exist a point $x \in X$ and a neighborhood V of the origin such that

  $$(x + V) \cap \operatorname{Ker} f = \varnothing.$$

  In particular, the linearity of f is enough to infer that

  $$0 \notin f(x + V) \implies f(y) \neq -f(x), \qquad \forall y \in V.$$

  In Lemma 4.2 we prove that one can always find a $V' \subset V$ balanced neighborhood of the origin; hence we may assume without loss of generality that V is balanced. The reader may prove as an exercise that the image $f(V)$ is also balanced.

  A balanced set in $\mathbb{R}$ is either bounded, in which case we are done, or equal to the whole real line $\mathbb{R}$, which clearly contradicts the requirements above.
- **(d)** Assume that $f(V)$ is bounded for some V neighborhood of the origin, that is, there exists $M > 0$ constant such that $|f(x)| \leq M$ for any $x \in V$.

  Let $\epsilon > 0$ and set $W := \left(\frac{\epsilon}{M}\right) \cdot V$. Then for all $y \in W$ it turns out that

  $$|f(y)| \leq \frac{\epsilon}{M} \sup_{x \in V} |f(x)| \leq \epsilon,$$

  which proves that f is continuous at zero (and hence everywhere).

□

Lemma 3.8. *Let X be a topological vector space. Suppose that $C \subset X$ is a closed set and $K \subset X$ is a compact set such that $K \cap C = \varnothing$. Then there exists a neighborhood V of 0 such that*

$$(K+V) \cap (C+V) = \varnothing.$$

Proof. For any $x \in K$ there is an open neighborhood W_x of 0 such that

$$(x + W_x) \cap C = \varnothing.$$

By continuity of the vector sum it follows that, for any $x \in K$, there exists an open neighborhood V_x of 0 such that $V_x = -V_x$ and $V_x + V_x + V_x \subseteq W_x$. In particular,

$$(x + V_x + V_x + V_x) \cap C = \varnothing \implies (x + V_x + V_x) \cap (C + V_x) = \varnothing,$$

and $\{x + V_x\}_{x \in K}$ is an open cover of K; thus there exists a finite subcover $\{x_i + V_i\}_{i=1,\dots,k}$ of K. If we define

$$V := \bigcap_{i=1}^{k} V_i,$$

then it is easy to prove that V is an open neighborhood of 0, $V = -V$ and $V + V + V \subseteq W_{x_i}$. Finally, we notice that there is an inclusion

$$(K+V) \subset \bigcup_{i=1}^{k} (x_i + V_i + V) \subset \bigcup_{i=1}^{k} (x_i + V_i + V_i),$$

and no term in the last union intersects $C + V$ since we chose V_i to ensure this property; thus we infer that

$$(K+V) \cap (C+V) = \varnothing,$$

which is exactly what we wanted to prove. □

Theorem 3.9 (Hahn-Banach, second geometric form)**.** *Let X be a locally convex topological vector space, and let $K \subset X$ and $C \subset X$ be two nonempty convex subsets such that $K \cap C = \varnothing$. Assume that K is compact and C is closed. Then there exist $f \in X^*$ and $\alpha < \beta \in \mathbb{R}$ such that*

$$f(x) \le \alpha < \beta \le f(y) \quad \textit{for all } x \in K \textit{ and all } y \in C.$$

More precisely, there exists a closed hyperplane H that strictly separates K and C.

Proof. By Lemma 3.8 we can always find an open and convex neighborhood V of the origin, separating K and C, that is,

$$(K + V - V) \cap C = \varnothing.$$

The subset $A := K + V$ is open and convex; thus by the Hahn-Banach Theorem 3.6 it follows that there are $f \in X^*$ and $\beta \in \mathbb{R}$ such that

$$f(x) < \beta \le f(y) \quad \text{for all } x \in A \text{ and all } y \in C,$$

or, equivalently,

$$f(x) < \beta \le f(y) \quad \text{for all } x \in K \text{ and all } y \in C.$$

The functional f is continuous and the set K is compact; hence f has a maximum on K. If we denote it by α, then the thesis follows easily:

$$f(x) \le \alpha < \beta \le f(y) \qquad \forall\, x \in K, \forall\, y \in C.$$

□

3.3 Appendix

In this appendix, we first introduce the notions of *bidual* space and *adjoint operator*, and then we characterize the dual of subspaces and quotients of Banach spaces via adjoint immersion/quotient.

In the second half, we investigate two possible completions of a metric space X. More precisely, we prove that there exists a complete metric space $\widetilde{X}$ and an isometric embedding $j : X \hookrightarrow \widetilde{X}$ with dense image.

In the final part, we fully exploit the tools previously introduced to find the dual spaces of $c_0(\mathbb{N};\ \mathbb{C})$, $\ell_1(\mathbb{N};\ \mathbb{C})$ and $\ell_\infty(\mathbb{N};\ \mathbb{C})$, and we show the relation between them. Moreover, we use the characterization of the dual space of a quotient to prove that c_0 is **not** the dual of a Banach space.

3.3.1 The Bidual Space X^{**} and the Adjoint Operator

Let $(X,\ \|\cdot\|_X)$ be a complex Banach space. The topological bidual space, denoted by X^{**}, is the set of all the linear and continuous functionals from X^* to $\mathbb{C}$, i.e.

$$X^{**} := \{\varphi : X^* \longrightarrow \mathbb{C} \mid \varphi \text{ linear and continuous}\}.$$

The norm $\|\cdot\|_{X^*}$ induces a norm on the dual X^{**}, which is given by

$$\|\varphi\|_{X^{**}} := \sup_{\|f\|_{X^*}\le 1} |\varphi(f)| = \sup_{f\neq 0} \frac{|\varphi(f)|}{\|f\|_{X^*}}. \tag{3.2}$$

There is a linear inclusion from X to its bidual X^{**}, which is given by

$$(X,\ \|\cdot\|_X) \ni x \longmapsto j_x \in (X^{**},\ \|\cdot\|_{X^{**}}),$$

where $j_x : X^* \longrightarrow \mathbb{C}$ is the valuation at the point x, that is,

$$\langle j_x,\ f\rangle := \langle f,\ x\rangle.$$

This inclusion is also an **isometry** since (3.2) implies that

$$\|j_x\|_{X^{**}} = \sup_{\|f\|_{X^*}\le 1} |\langle j_x,\ f\rangle| = \sup_{\|f\|_{X^*}\le 1} |\langle f,\ x\rangle| = \|x\|_X.$$

Remark 3.2. If X is not a Banach space (i.e., if X is a topological vector space), then we can still find an isometric linear inclusion.

In particular, since the dual space is always complete, the map $x \longmapsto j_x$ defined above gives us a way to include isometrically a non-complete metric space into a complete metric space (see Subsection 3.3.2 for more details).

Definition 3.10 (Dual Operator)**.** Let $T \in \mathcal{L}(X,\ Y)$ be a linear continuous operator between two Banach spaces. The *dual operator* is the element $T^* \in \mathcal{L}(Y^*,\ X^*)$ defined by

$$T(f) := f \circ T.$$

Remark 3.3. For any $x \in X$ and any $f \in Y^*$, it turns out that

$$\langle T^* f, x \rangle = \langle f, Tx \rangle,$$

where the left-hand side is the duality scalar product (X^*, X), and the right-hand side is the duality scalar product (Y^*, Y). From the Hahn-Banach Theorem 3.1 it follows that

$$\begin{aligned} \|T^*\|_{\mathcal{L}(Y^*, X^*)} &= \sup_{\|f\|_{Y^*} \le 1} |T^* f| = \\ &= \sup_{\|f\|_{Y^*} \le 1} \sup_{\|x\| \le 1} |\langle T^* f, x \rangle| = \\ &= \sup_{\|x\| \le 1} |\langle T, x \rangle| = \|T\|_{\mathcal{L}(X, Y)}. \end{aligned}$$

Dual of Subspaces and Quotients. Let $(X, \|\cdot\|_X)$ be a Banach space and let $\imath_Y : Y \hookrightarrow X$ be the natural inclusion of a closed linear subspace (endowed with the induced norm).

Theorem 3.11. *In this setting, there exists an isometry between the dual of Y and a suitable quotient of the dual of X, that is,*

$$Y^* \xrightarrow{\sim} {X^*}\big/_{Y^\perp}, \qquad \textit{where} \quad Y^\perp := \left\{ f \in X^* \;\middle|\; f\big|_Y \equiv 0 \right\} \subset X^*.$$

Proof. First, we observe that the dual operator $\imath_Y^* : X^* \longrightarrow Y^*$ is nothing more than the restriction map, that is,

$$\imath_Y^*(f) = f\big|_Y.$$

In particular, the orthogonal space of Y is also equal to the kernel of the dual inclusion, that is,

$$Y^\perp = \operatorname{Ker} \imath_Y^*.$$

The operator $\imath_Y^*$ is clearly linear, and it passes to the quotient. Indeed, if $f \in Y^\perp$ is any element of the orthogonal, then

$$\imath_Y^*(f) = f\big|_Y \equiv 0,$$

and thus there exists a well-defined operator $\imath_Y^* : {X^*}\big/_{Y^\perp} \longrightarrow Y^*$.

The operator $\imath_Y^*$ is injective by definition; hence we only need to prove that it is surjective, that is, for any $g \in Y^*$ there exists $f \in X^*$ such that $f\big|_Y \equiv g$.

But, given $g \in Y^*$, the extension $f \in X^*$ exists as a straightforward application of the Hahn-Banach Theorem 3.1. Finally, it is also an isometry since

$$\|f + Y^\perp\|_{X^*} = \inf_{h \in Y^\perp} \|f + h\|_{X^*} = \|f\|_{Y^*}.$$

$\square$

Remark 3.4. The subspace Y has the property of "unique extension preserving the norm" if and only if $Y^\perp$ has a unique point of minimal distance.

Let $(X, \|\cdot\|_X)$ be a Banach space and let $\imath_Y : Y \hookrightarrow X$ be the natural inclusion of a closed linear subspace (endowed with the induced norm).

Theorem 3.12. *In this setting, there exists an isometry between the dual of the quotient* $X/_Y$ *and the orthogonal of* Y*, that is,*

$$\left(X/_Y\right)^* \cong Y^\perp.$$

Proof. Let $\pi_Y : X \twoheadrightarrow X/_Y$ be the projection map. The dual map is given by

$$\pi_Y^* : \left(X/_Y\right)^* \longrightarrow X^*, \qquad f \longmapsto f \circ \pi,$$

and clearly its image is equal to $Y^\perp$ since

$$f \circ \pi_Y(y) = f(0) = 0, \qquad \forall\, y \in Y.$$

Therefore π_Y^* is a map onto $Y^\perp$ and it is also injective, that is,

$$f \circ \pi \equiv 0 \implies f \equiv 0$$

as a consequence of the Hahn-Banach Theorem 3.1. In conclusion, the reader may prove that it is an isometry as a consequence of the fact that the projection π preserves the balls:

$$\pi\left(B_X(0,\,1)\right) = B_{X/_Y}(0,\,1).$$

□

Example 3.3. Let $C^1([a,\,b])$ be the space of continuously differentiable functions, equipped with the norm $\|\cdot\|_\infty + \|\partial_x \cdot\|_\infty$, then there exists an inclusion

$$C^1([a,\,b]) \hookrightarrow C^0([a,\,b]) \times C^0([a,\,b]).$$

The norm in the product space is given by $\|(\cdot,\,*)\| := \|\cdot\|_\infty + \|*\|_\infty$, and $C^1([a,\,b])$ is actually a closed subspace (Borel).

In particular, the dual of $C^1([a,\,b])$ may be represented through the dual of $C^0([a,\,b])$. The reader may jump to Exercise 3.4 for a more detailed explanation of what happens.

3.3.2 Completion of Metric Spaces

The main result of this subsection is that every metric space can be densely embedded in a complete metric space. Here we sketch the construction of two completion: the Cauchy sequence one, and the Fréchet-Kuratowski one.

Theorem 3.13. *Let* $(X,\,d)$ *be a metric space. There exists a complete metric space* $\left(\widetilde{X},\,\widetilde{d}\right)$ *and an isometric embedding* $j : X \hookrightarrow \widetilde{X}$ *with dense image.*

Cauchy Sequences Method. Let us consider the set of all the Cauchy sequences of X, i.e.,

$$\mathcal{C} := \left\{(x_n)_{n\in\mathbb{N}} \subset X \mid (x_n)_n \text{ is a Cauchy sequence}\right\}.$$

We can always equip the set $\mathcal{C}$ with a semi-distance, which is defined by setting

$$\delta\left((x_n)_{n\in\mathbb{N}},\,(y_n)_{n\in\mathbb{N}}\right) := \lim_{n\to+\infty} d(x_n,\,y_n).$$

The map δ is well-defined (namely, the limit exists and is finite). Indeed, for any $\epsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that

$$\|x_n - x_N\| \leq \epsilon \qquad \text{and} \qquad \|y_n - y_N\| \leq \epsilon.$$

It follows that

$$d(x_n,\,y_n) \leq d(x_n,\,x_N) + d(x_N,\,y_N) + d(y_N,\,y_n) \leq 2\,\epsilon + d(x_N,\,y_N),$$

that is, $\left(d(x_n,\,y_n) - d(x_N,\,y_N)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $\mathbb{R}$, and thus it converges to a finite limit. We define the completion as follows:

$$\widetilde{X} = {}^{\mathcal{C}}\!/\!_{\sim}, \qquad \widetilde{d} = \delta\,|_{{}^{\mathcal{C}}\!/\!_{\sim}},$$

where $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if and only if their semi-distance is equal to 0. The inclusion is clearly given by sending $x \in X$ to the constant sequence, i.e.,

$$j : X \ni x \longmapsto (x)_{n\in\mathbb{N}} \in \widetilde{X}.$$

We now prove that the image of j is dense in $\widetilde{X}$. Indeed, for any $(x_n)_{n\in\mathbb{N}} \in \widetilde{X}$ and for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n,\,x_m) \leq \epsilon$ for any $n,\,m \geq N$. In particular, we can find a constant sequence arbitrarily near to $(x_n)_{n\in\mathbb{N}}$, which is given by x_{N+k} for some $k \in \mathbb{N}$.

Lemma 3.14. *Let $(X,\,d)$ be a metric space and $D \subset X$ a dense subset. If every Cauchy sequence in D converges in X, then X is complete.*

We do not discuss this result, but we give it for granted to show that $\widetilde{X}$ is a complete metric space. In particular, it is enough to prove that every Cauchy sequence in $j(X)$ converges in $\widetilde{X}$.

Let $(\widetilde{y_n})_{n\in\mathbb{N}}$ be a Cauchy sequence in $j(X)$, and assume that $\widetilde{y_n}$ is represented by the constant (Cauchy) sequence $(y_n,\,y_n,\,\dots)$ for every $n \in \mathbb{N}$. Since j is an isometry, it turns out that

$$d(y_n,\,y_m) = \widetilde{d}\,(\widetilde{y_n},\,\widetilde{y_m})\,, \qquad \forall\, m,\,n \in \mathbb{N}.$$

Thus $(y_1,\,y_2,\,\dots)$ is a Cauchy sequence in X and, if we let $\widetilde{y} = [(y_1,\,y_2,\,\dots)]$ be its equivalence class, then we can prove that $\widetilde{y_n} \to \widetilde{y}$. Indeed, for any $\epsilon > 0$ there is a natural number $N \in \mathbb{N}$ such that

$$d(y_n,\,y_m) < \frac{\epsilon}{2}, \qquad \forall\, m,\,n \geq N.$$

Hence, for every $k \geq N$, it turns out that

$$\widetilde{d}\,(\widetilde{y_k},\,\widetilde{y}) = \lim_{n\to+\infty} d(y_k,\,y_n) \leq \frac{\epsilon}{2},$$

and this concludes the proof of the completeness of $\widetilde{X}$.

Fréchet-Kuratowski Method. Let $x_0 \in X$ be a point, and let us consider the map

$$X \ni x \longmapsto \Phi_x, \qquad \Phi_x(y) := d(y,\, x) - d(y,\, x_0).$$

The functional $\Phi_x : X \longrightarrow \mathbb{R}$ is bounded

$$|\Phi_x(y)| \le |d(y,\, x) - d(y,\, x_0)| \le d(x,\, x_0),$$

and thus the operator norm of Φ_x satisfies the following inequality:

$$\|\Phi_x\| \le d(x,\, x_0).$$

In particular, the mapping above $x \longmapsto \Phi_x$ send points of X to bounded maps from X to $\mathbb{R}$, that is,

$$j : X \ni x \longmapsto \Phi_x \in \mathcal{B}(X;\, \mathbb{R})$$

and the latter is a complete space, as we have already proved in the Example 2.5.

In conclusion, the reader may try to prove that the embedding is isometric and also densely defined as an exercise using the lemmas provided below.

Lemma 3.15. *In the setting above, it turns out that*

$$\|\Phi_x - \Phi_{x'}\|_\infty = d(x,\, x'), \qquad \forall\, x,\, x' \in X.$$

Proof. The first inequality is rather obvious:

$$\sup_{y \in X} |\Phi_x(y) - \Phi_{x'}(y)| = \sup_y |d(x,\, y) - d(x',\, y)| \le d(x,\, x').$$

On the other hand, if we choose $y = x$, then we obtain the opposite one; hence we infer that

$$|\Phi_x(x) - \Phi_{x'}(x)| = d(x',\, x).$$

□

Lemma 3.16. *If $(X,\, d_X)$ is a metric space and $(Y,\, d_Y)$ is a complete metric space, then every uniformly continuous mapping*

$$f : D \subseteq X \longrightarrow Y$$

defined on a dense subset $D \subset X$, can be extended to a uniformly continuous mapping $F : X \longrightarrow Y$ that preserves the continuity module of f.

Sketch of the Proof. In order to prove this Lemma, we need to use some basic results:

(a) If $F : X \to Y$ is a uniformly continuous map between two metric spaces and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

(b) Let X and Y be metric spaces, $S \subseteq X$ and $f : S \to Y$ be uniformly continuous. If two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S converge to the same limit in X and if the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges in Y, then $\lim_n f(x_n) = \lim_n f(y_n)$.

Back to the theorem, we can define the extension as follows

$$F(x) = \lim_{k\to+\infty} f(d_k),$$

where $(d_k)_{k\in\mathbb{N}}$ is any sequence such that $(d_k)_k \subset D$ and $d_k \to x$ as k approaches $+\infty$. The reader can easily prove that F is well-defined, uniform continuous, and unique.

$\square$

Corollary 3.17. *The completion of a normed space* $(X, \|\cdot\|)$ *is a Banach space (i.e., it is complete).*

3.3.3 Dual and Bidual of c_0

In this brief section, we study the dual and the bidual of $c_0(\mathbb{N};\ \mathbb{C})$, and we answer to a natural question: is c_0 the dual of a Banach space B?

Theorem 3.18. *Let* $c_0(\mathbb{N};\ \mathbb{C}) := \left\{x : \mathbb{N} \longrightarrow \mathbb{C} \;\middle|\; x(n) \xrightarrow{n\to+\infty} 0\right\}$ *be the space of infinitesimal sequences equipped with the uniform norm.*

(a) *The dual is the space of all the summable sequences, that is,*

$$c_0^*(\mathbb{N};\ \mathbb{C}) = \ell_1(\mathbb{N};\ \mathbb{C}).$$

(b) *The bidual is the space of all the essentially bounded sequences, that is,*

$$(\ell_1(\mathbb{N};\ \mathbb{C}))^* = \ell_\infty(\mathbb{N};\ \mathbb{C}).$$

(c) *The dual of* $\ell_\infty(\mathbb{N};\ \mathbb{C}))^*$ *is the space of all the additive finite measures on* $\mathbb{N}$*, that is,*

$$\mathrm{ba}(\mathbb{N}) := \{\mu \in \mathcal{M}(\mathbb{N};\ \mathbb{R}) \mid \mu \text{ additive, finite and eventually negative}\}.$$

Proof.

(a) We want to prove that the dual of c_0 is isometric to ℓ_1. Let

$$\Phi : \ell_1 \longrightarrow c_0^*, \qquad y \longmapsto \Phi_y \;:\; \Phi_y(x) = \sum_{k\in\mathbb{N}} x_k y_k.$$

It is straightforward to prove that Φ is a well-defined map, that is,

$$\sum_{k\in\mathbb{N}} |x_k y_k| \le \|x\|_\infty \|y\|_1 < +\infty.$$

In particular, for any $y \in Y$, the image Φ_y is a linear and bounded (thus continuous) functional, and we can easily estimate the norm by

$$\|\Phi_y\|_{c_0^*} \le \|y\|_1.$$

We want to prove that Ψ is an isometry, i.e. $\|\Phi_y\|_{c_0^*} = \|y\|_1$; hence we only need to prove that the equality holds true at one point. Let

$$u_n = \sum_{k=0}^{n} \operatorname{sgn}(y_k) e_k \in c_0.$$

It follows from the definition that

$$\|\Phi_y\|_{c_0^*} = \sup_{\|x\| \le 1} |\langle \Phi_y, x\rangle| \ge \langle \Phi_y, u_n\rangle = \sum_{k=0}^{n} |y_n|$$

for any $n \in \mathbb{N}$; hence, by passing to the limit, we can infer that

$$\|\Phi_y\|_{c_0^*} \ge \|y\|_1 \implies \|\Phi_y\|_{c_0^*} = \|y\|_1.$$

It remains to prove that Φ is **surjective**. Let $f \in c_0^*$ be any linear and continuous functional and define $y_k := \langle f, e_k\rangle$ for any $k \in \mathbb{N}$. We can easily prove that $y = (y_k)_{k\in\mathbb{N}}$ belongs to ℓ_1 since

$$\sum_{k=0}^{n} |y_k| = \sum_{k=0}^{n} |\langle f, e_k\rangle| = \langle f, u_n\rangle \le \|f\|_{c_0^*} < +\infty.$$

Finally, notice that $\langle \Phi_y, e_k\rangle = \langle f, e_k\rangle$ for any $k \in \mathbb{N}$, that is, f and Φ_y coincide on a dense subset with respect to the norm $\|\cdot\|_{c_0}$ (the span of $\{e_k\}_{k\in\mathbb{N}}$) and, by continuity, they coincide in the whole space.

(b) We want to prove that the dual of ℓ_1 is isometric to ℓ_∞. Let

$$\Psi : \ell_\infty \longrightarrow \ell_1^*, \qquad y \longmapsto \Psi_y \ : \ \Psi_y(x) = \sum_{k\in\mathbb{N}} x_k y_k.$$

It is straightforward to prove that it is well-defined, that is,

$$\sum_{k\in\mathbb{N}} |x_k y_k| \le \|y\|_\infty \|x\|_1 < +\infty.$$

In particular, for any $y \in Y$, the image Ψ_y is a linear and bounded (thus continuous) functional, and the dual norm can be easily estimated by

$$\|\Psi_y\|_{\ell_1^*} \le \|y\|_\infty.$$

We want to prove that Ψ is an **isometry**, i.e. $\|\Psi_y\|_{\ell_1^*} = \|y\|_\infty$; hence we only need to prove that the equality holds true in one point. It follows from the definition that

$$\|\Psi_y\|_{\ell_1^*} = \sup_{\|x\| \le 1} |\langle \Psi_y, x\rangle| \ge |\langle \Psi_y, e_k\rangle| = |y_k|$$

for any $k \in \mathbb{N}$, and thus

$$\|\Psi_y\|_{\ell_1^*} \ge \|y\|_\infty \implies \|\Phi_y\|_{\ell_1^*} = \|y\|_\infty.$$

It remains to prove that Ψ is **surjective**. Let $f \in \ell_1^*$ be any linear and continuous functional and define $y_k := \langle f, e_k\rangle$ for any $k \in \mathbb{N}$. We can easily prove that $y = (y_k)_{k\in\mathbb{N}}$ belongs to ℓ_∞ since

$$|y_k| \le \|f\|_{\ell_1^*} \implies \|y\|_\infty \le \|f\|_{\ell_1^*} < +\infty.$$

Finally, notice that $\langle \Psi_y, e_k\rangle = \langle f, e_k\rangle$ for any $k \in \mathbb{N}$, that is, f and Ψ_y coincide on a dense subset with respect to the norm $\|\cdot\|_{\ell_1}$ (the span of $\{e_k\}_{k\in\mathbb{N}}$) and by continuity they coincide in the whole space.

Extra. The inclusion $c_0 \hookrightarrow c_0^{**}$ is exactly the set inclusion, and the following diagram is commutative

$$\begin{tikzcd}
c_0 \arrow[r, hook] \arrow[dd, leftrightarrow, "\mathrm{id}"'] & c_0^{**} \arrow[dd, leftrightarrow] \arrow[dr, "\Phi^*"] & \\
 & & \ell_1^* \\
c_0 \arrow[r, hook] & \ell_\infty \arrow[ur, "\Psi"] &
\end{tikzcd}$$

(c) We want to prove that the dual of ℓ_∞ is the space of the measures $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ additive and finite, i.e.,

$$\mathrm{ba}(\mathbb{N}) := \{\mu \in \mathcal{M}(\mathbb{N};\ \mathbb{R}) \mid \mu \text{ additive, finite and eventually negative}\}.$$

Clearly $\mathrm{ba}(\mathbb{N})$ is a Banach space with the total variation norm, that is,

$$\|\mu\|_{\mathrm{ba}} := \sup_{A \subset \mathbb{N}} [\mu(A) - \mu(\mathbb{N} \setminus A)].$$

Let $\mu \in \mathrm{ba}(\mathbb{N})$ and let $f \in \ell_\infty$ be a function of finite rank (i.e. a simple function), e.g.,

$$f(x) = \sum_{k=1}^{r} c_k \chi_{E_k}(x),$$

where $c_k \in \mathbb{R}$ (or $\mathbb{C}$) and $E_k \subseteq \mathbb{N}$. If we set

$$\int_{\mathbb{N}} f(x)\, \mathrm{d}\mu(x) := \sum_{k=1}^{r} c_k\, \mu(E_k),$$

then it is easy to prove that the integral is well-defined, that is, it does not depend on the particular representation of f. The linear subspace

$$F := \{f \in \ell_\infty(\mathbb{N};\ \mathbb{C}) \mid f(\mathbb{N}) \text{ is finite}\}$$

is *dense* with respect to the ℓ_∞ norm; thus the map

$$\mu \longmapsto \int_{\mathbb{N}} f(x)\, \mathrm{d}\mu(x)$$

may be extended to the whole space because it is continuous. Indeed, it is easy to prove that the map is bounded on F since

$$\left| \int_{\mathbb{N}} f(x)\, \mathrm{d}\mu(x) \right| \le \|f\|_\infty\, \|\mu\|_{\mathrm{ba}},$$

and thus there is a linear and continuous extension

$$I : \mathrm{ba}(\mathbb{N}) \longrightarrow \ell_\infty^*(\mathbb{N};\ \mathbb{C}), \qquad \mu \longmapsto I_\mu\ :\ I_\mu(f) := \int_{\mathbb{N}} f(x)\, \mathrm{d}\mu(x).$$

It is straightforward to prove that I is an **isometry**. Indeed, the inequality above holds true also when passing to the limit, and thus

$$\|I_\mu\|_{\ell_\infty^*} \le \|\mu\|_{\mathrm{ba}}.$$

On the other hand

$$\|\mu\|_{\mathrm{ba}} = \sup_{A\subset\mathbb{N}} [\mu(A) - \mu(\mathbb{N}\setminus A)] = \sup_{A\subset\mathbb{N}} \int_{\mathbb{N}} \left(\chi_A - \chi_{\mathbb{N}\setminus A}\right) \mathrm{d}\mu(x) \le \|I_\mu\|_{\ell_\infty^*},$$

and hence it is an isometry. Finally, to prove that it is **surjective** notice that given $\alpha \in \ell_\infty^*$, we can define a measure μ_α by setting

$$\mu_\alpha(A) = \langle \alpha,\, \chi_A\rangle. \tag{3.3}$$

Clearly (3.3) is a finite measure (since α is finite in the ℓ_∞ measure) and it is additive (by the linearity of α).

In conclusion, notice that the equality $\alpha = I(\mu_\alpha)$ holds on the dense subset F; thus they coincide on the whole space and the thesis is proved. Observe that the immersion

$$\ell_1 \hookrightarrow \ell_1^{**} \cong \mathrm{ba}\,(\mathbb{N})$$

gives us the identification $\ell_1 \rightsquigarrow \mathrm{ba}_\sigma$, that is, the subset of the σ-additive finite measure on $\mathbb{N}$.

□

Theorem 3.19. *The space c_0 is not the dual of any Banach space X, that is, $c_0 \neq X^*$ for any X Banach.*

We are quite not ready to prove this result. We need to state and give a proof of a technical lemma which will allow us to derive a contradiction when we assume that a Banach space X such that $X^* = c_0$ exists.

Lemma 3.20. *Any infinite-dimensional closed subspace X of ℓ_1 contains a subspace Y linearly homeomorphic to ℓ_1.*

Proof. We divide the proof into three steps.

Step 0. Let $f \in \ell_1^*$ be any linear continuous functional. Since X is an infinite-dimensional closed subspace, the intersection $\mathrm{Ker}\, f \cap X$ is also infinite-dimensional and closed.

We can find, inductively, a sequence $\mathbf{u}^n \in X$ with the first n components zero ($u_0 = u_1 = \cdots = u_n = 0$) and unitary norm ($\|u\| = 1$). Furthermore, given a unitary sequence $\mathbf{u} \in X$, we can find a natural number $N \in \mathbb{N}$ such that

$$\sum_{k>N} |u_k| < \frac{1}{4}.$$

Step 1. The preliminary remark allows us to find a sequence of sequences $(\mathbf{u}^n)_{n\in\mathbb{N}} \subset X$ and an increasing sequence of natural numbers $(k_n) \subset \mathbb{N}$ satisfying the following properties:

(1) The norm of each elements is unitary, that is, $\|\mathbf{u}^n\|_{\ell_1} = 1$ for all $n \in \mathbb{N}$.

(2) The mass of the sequence $\mathbf{u}^n$ is focused on the first k_n components, that is,

$$\|\mathbf{u}^n\big|_{[0,\,k_n]}\|_{\ell_1} \geq 3/4 \quad \text{for all } n \in \mathbb{N}.$$

(3) The $(n+1)$th elements is zero on the interval where the mass of the previous one is focused, that is,

$$\mathbf{u}^{n+1}\big|_{[0,\,k_n]} \equiv 0 \quad \text{for any } n \in \mathbb{N}.$$

Let us consider the operator $L : \ell_1 \longrightarrow X$ defined by setting

$$L(\lambda) := \sum_{n\in\mathbb{N}} \lambda_n \mathbf{u}_n,$$

which is well-defined as a consequence of the assumption X closed.

Step 2. First, notice that

$$\|L(\lambda)\|_{\ell_1} \leq \|\lambda\|_{\ell_1},$$

and thus L is a linear and bounded (=continuous) operator. If we set $I_n := (k_{n-1},\, k_n]$, then it follows immediately that

$$\|L(\lambda)\|_{\ell_1} = \left\| \sum_{n\in\mathbb{N}} \lambda_n \mathbf{u}_n \chi_{I_n} + \sum_{n\in\mathbb{N}} \lambda_n \mathbf{u}_n \chi_{X\setminus I_n} \right\|_{\ell_1} \geq$$

$$\geq \left\| \sum_{n\in\mathbb{N}} \lambda_n \mathbf{u}_n \chi_{I_n} \right\|_{\ell_1} - \sum_{n\in\mathbb{N}} \lambda_n \|\mathbf{u}_n \chi_{I_n^c}\|_{\ell_1}.$$

On the other hand, the family of intervals I_n is disjoint. Therefore the above inequality yields to

$$\|L(\lambda)\|_{\ell_1} \geq \sum_{n\in\mathbb{N}} \|\lambda_n \mathbf{u}_n \chi_{I_n}\|_{\ell_1} - \frac{1}{4}\|\lambda\|_{\ell_1} \geq \frac{1}{2}\|L\|_{\ell_1},$$

which means that, for every $\lambda \in \ell_1$, the inequality

$$\frac{1}{2}\|\lambda\|_{\ell_1} \leq \|L(\lambda)\|_{\ell_1} \leq \|\lambda\|_{\ell_1},$$

holds, and

$$L : \ell_1 \longrightarrow L(\ell_1) \subset X$$

induces an invertible operator. In conclusion, it suffices to set $Y := L(\ell_1)$ and take the homeomorphism $L : \ell_1 \longrightarrow Y$. $\square$

Proof of Theorem 3.19. We argue by contradiction. Suppose that there exists a Banach space X such that $X^* = c_0$. Then (see Theorem 3.18), it turns out that

$$X \hookrightarrow X^{**} \cong \ell_1 \implies X \subseteq \ell_1,$$

and therefore we can apply Lemma 3.20. In particular, there exists $Y \subset X$ such that $Y \cong \ell_1$. On the other hand, in Section 3.3.1 we were able to find a way to express the dual of a linear closed subspace, and therefore

$$\ell_\infty \cong Y^* = {}^{X^*}\!/_{Y^\perp} = {}^{c_0}\!/_{Y^\perp},$$

which implies that ℓ_∞ is a quotient of c_0. This is the sought contradiction: the space c_0 is separable, but ℓ_∞ is not separable, and thus it cannot be a quotient of c_0. $\square$

Remark 3.5. Let X be a reflexive Banach space. Then the bidual inclusion is surjective (i.e. $X \cong X^{**}$), and this gives us a small chain of duals:

$$X \longrightarrow X^* \longrightarrow X \longmapsto X^* \longrightarrow \dots$$

On the other hand, there is a theorem asserting that X^* reflexive implies X reflexive. The implications of this statement are fascinating. Indeed, it is always possible to start from a non-reflexive Banach space and obtain a infinite chain of non-surjective inclusions $X \subset X^* \subset \dots$.

3.4 Exercises

Exercise 3.1. Prove the completeness of the following spaces:

(a) $\mathcal{L}^p(X, \mathrm{d}\mu)$, for any $1 \leq p \leq \infty$.

(b) $\left(C_b^0(X), \|\cdot\|_\infty\right)$, i.e. the bounded function on a topological space X with the uniform norm.

(c) Let X be a Banach space and let N be a closed linear(vector) subspace. Then X/N is complete with the quotient norm, that is

$$\|x + N\| \stackrel{\text{def}}{=} \inf_{y \in N} \|x + y\|.$$

(d) Let X be an Hilbert space and let N be a closed linear(vector) subspace. Then the quotient is the orthogonal $N^\perp$, the quotient norm is equal to the orthogonal norm and $\pi : X \to X/N$ is an open map.

Solution.

(a) The reader may refer to Example 2.5 for the proof in the case $1 \leq p < +\infty$.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in L^∞. Given an integer $k \geq 1$ there is an integer N_k such that

$$\|f_n - f_m\|_{L^\infty} \leq \frac{1}{k},$$

for any $m, n \geq N_k$. Hence there is a null set E_k such that

$$|f_n(x) - f_m(x)| \leq \frac{1}{k}, \qquad \forall\, x \in X \setminus E_k,\ \forall\, m, n \geq N_k.$$

Then we let $E := \cup_{k\in\mathbb{N}} E_k$, so that E is a null set, and we see that for any $x \in X \setminus E$ it turns out that the sequence $f_n(x)$ is Cauchy in $\mathbb{R}$. By completeness

$$f_n(x) \xrightarrow{n\to+\infty} f(x), \qquad \forall\, x \in X \setminus E.$$

In particular, taking the limit $m \to +\infty$ on the inequality above, yields to

$$|f_n(x) - f(x)| \leq \frac{1}{k}, \qquad \forall\, x \in X \setminus E_k,\ \forall\, n \geq N_k.$$

We conclude that $f \in L^\infty(X)$ and $\|f - f_n\|_{L^\infty} \leq \frac{1}{k}$ for all $n \geq N_k$, thus f_n converges to f also (strongly) in $L^\infty(X)$. Note that this implies that, up to a subsequence, the convergence is also *point-wise*.

(b) The reader may refer to Example 2.5 for a more general situation concerning bounded functions.

(c) For each $x \in X$ let us denote $\hat{x} := x + N \in {}^{X}\!/_{N}$. Let $(\hat{x}_n)_{n\in\mathbb{N}}$ be any sequence converging uniformly (i.e. $\sum_n \|\hat{x}_n\|$ converges). From the definition of $\|\cdot\|$, it follows that, for each $n \in \mathbb{N}$, there exists $x_n \in \hat{x}_n$ such that $\|x_n\|_X \leq 2\,\|\hat{x}_n\|$. Therefore

$$\sum_{n=1}^{+\infty} \|x_n\|_X \leq 2 \sum_{n=1}^{+\infty} \|\hat{x}_n\| < \infty,$$

then $(x_n)_{n\in\mathbb{N}}$ converges uniformly as well. Since X is Banach, by Theorem 2.4 the sum converges to some vector $x \in X$. Then, from the definition of the norm in ${}^{X}\!/_{N}$, it follows that $\sum_n \hat{x}_n$ converges to $\hat{x}$ in ${}^{X}\!/_{N}$ and, applying again Theorem 2.4, we infer that the quotient space is complete.

(d) We define the projection

$$p : {}^{X}\!/_{N} \to N^\perp, \qquad x + N \mapsto p(x+N) = \pi(x),$$

where π is the projection over $N^\perp$. The mapping is well defined since

$$x+n_1 = x+n_2 \implies p(x+n_1) = \pi(x+n_1) = \pi(x)+\pi(n_1) = \pi(x)+\pi(n_2) = p(x+n_2),$$

i.e. it is independent from the particular representative of the coset.

The projection is **injective**. If $p([x]) = 0$, then $[x] = x + N \subset \mathrm{Ker}(\pi)$ which is equal to N (see Theorem 1.18), and thus $[x] = 0$ (since $x \in N$).

The projection is **surjective**. If $y \in N^\perp$, then $[y] = y + N$ has image via p exactly equal to y.

Finally, we prove that the spaces are isometric, that is

$$\|x + N\| = \inf_{y\in N} \|x + y\| = \|\pi\, x\| = \|\pi(x + N)\|.$$

□

Exercise 3.2. Let (X, p) be a seminormed space and let N be a closed linear(vector) subspace. Then ${}^{X}\!/_{N}$ is normed with $\|\cdot\| := p\,\big|_{X\,/\,N}$.

Solution. Suppose that N is closed, and that $\|x + N\| = 0$. Then, by definition of the norm, it turns out that

$$\inf_{y\in N} \|x - y\| = 0.$$

Hence there exist a minimizing sequence $(y_n)_{n\in N}$ such that $\|x - y_n\| \to n$ as $n \to +\infty$. But N is closed, thus x belongs to N. Finally, we have that

$$x + N = N = 0 + N \implies [x] = 0$$

as a class of equivalence in ${}^{X}\!\diagup_{N}$. □

Exercise 3.3. Let $(X,\, p)$ be a semi-normed space. Then ${}^{X}\!\diagup_{\{\bar{0}\}}$ is normed with norm p.

Solution. Corollary of Exercise 3.2. □

Exercise 3.4. Let $X_1,\, \ldots,\, X_n$ be normed spaces. Represent the dual of $X_1 \times \cdot \times X_n$, in terms of X_i^* as i ranges in $\{1,\, \ldots,\, n\}$.

Solution. Let $\|\cdot\|_{X_i}$ be the norm on X_i for each i. On the product space define the norm

$$\|(x_1,\, \ldots,\, x_n)\|_X = \sum_{i=1}^{n} \|x_i\|_{X_i},$$

and consider the following mapping

$$F : \prod_{i=1}^{n} X_i^* \to \left(\prod_{i=1}^{n} X_i\right)^*, \qquad (f_1,\, \ldots,\, f_n) \longmapsto f := (f_1,\, \ldots,\, f_n).$$

This map is clearly bijective (since it admits an inverse) and linear, thus it is an isomorphism. □

Exercise 3.5. Let $0 < p < 1$. Prove that $L^p\left([0,\, 1]\right)$ is a topological vector space, metric and complete. Prove also that the only open convex subset is the whole space itself, i.e. the only linear continuous form is $f \equiv 0$.

Exercise 3.6. Let $(f_n)_{n\in\mathbb{N}} \subset \ell_1$ be a sequence. Then it converges weakly to f if and only if it converges to f in norm.

Chapter 4

Banach-Steinhaus and Open Mapping Theorem

In the first half of this chapter, we formally introduce the notion of *topological vector space*, and we investigate some of its fundamental properties (e.g., separability, special neighborhoods of the origin, and metrizability).

The next section is entirely devoted to state and prove the uniform boundedness theorem (due to Banach and Steinhaus) for families of operators between topological vector spaces.

In the final section, we exploit the boundedness principle to derive two fundamental results in functional analysis: the open mapping theorem, and the closed graph theorem (both widely used).

4.1 Topological Vector Spaces

Definition 4.1 (Topological Vector Space)**.** A *topological vector space* is a $\mathbb{K}$ vector space X endowed with a compatible topology τ, that is, a topology such that the operations

$$+ : X \times X \longrightarrow X \qquad \text{and} \qquad \cdot : \mathbb{K} \times X \longrightarrow X$$

are both τ-continuous.

Remark 4.1. Let (X, τ) be a topological vector space. The topology is translation-invariant, which means that τ can be completely determined by a local basis of neighborhoods of the origin.

Lemma 4.2. *Let (X, τ) be a topological vector space. Then the following properties hold:*

(1) Every neighborhood of the origin is absorbing.

(2) Every neighborhood U of the origin contains a neighborhood of the origin V such that

$$V + V \subseteq U \quad \textit{and} \quad V = -V.$$

(3) Every neighborhood U of the origin contains a balanced[1] *neighborhood of the origin $V \subseteq U$.*

Proof. The assertion **(1)** is rather obvious since the product is continuous and

$$x \in X \rightsquigarrow \lim_{t \to 0^+} tx = 0 \implies tx \in U \quad \text{for all } t \in [0, \tau].$$

(2) Recall that, in a topological vector space (X, τ), the sum is τ-continuous. Consequently, there are V_1 and V_2 neighborhoods of 0 such that

$$(x, y) \in V_1 \times V_2 \implies x + y \in U.$$

In particular, we have the inclusion $V_1, V_2 \subseteq U$ and, since the family of the neighborhoods of a point is closed under intersection, we also have that $W := V_1 \cap V_2$ is a neighborhood of the origin. In conclusion, if we set $V := W \cap (-W)$, then it is easy to prove that that V is the sought neighborhood of the origin, that is,

$$V + V \subseteq U \quad \text{and} \quad V = -V.$$

(3) Recall that, in a topological vector space, the product is τ-continuous. In particular, there is a positive real number $\delta > 0$ and a neighborhood W of 0 such that

$$\alpha \cdot W \subseteq U, \qquad \forall \alpha \, : \, |\alpha| < \delta.$$

Let us denote by V the union of the scalings $\alpha \cdot W$, i.e.

$$V := \bigcup_{|\alpha| < \delta} \alpha \cdot W.$$

It is easy to see that V is a neighborhood of the origin, and also that $V \subseteq U$. In conclusion, to prove that V is balanced, it suffices to take $|\beta| \leq 1$ and compute the product:

$$\beta \cdot V = \bigcup_{|\alpha| < \delta} (\beta \alpha) \cdot W,$$

but $|\beta\alpha| < \delta$ for any $|\alpha| < \delta$, which is enough to infer that V is balanced.

□

Lemma 4.3. *Let X be a vector space, and let $\mathcal{U}$ be a collection of subsets satisfying the following properties:*

(1) The origin belongs to all $U \in \mathcal{U}$.

(2) For all $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$.

(3) Every set $U \in \mathcal{U}$ is absorbing.

[1]**Definition.** A subset $C \subset X$ of a vector space is balanced if and only if $\theta \cdot C \subseteq C$ for every $|\theta| \leq 1$.

(4) *For all $U \in \mathcal{U}$ one can find $V \in \mathcal{U}$ such that*

$$V + V \subseteq U \quad \textit{and} \quad V = -V.$$

(5) *For all $U \in \mathcal{U}$ one can find a balanced $V \in \mathcal{U}$ such that $V \subseteq U$.*

Then $\mathcal{B}_x := \{x + \mathcal{U}\}$ is a local basis of neighborhoods for all $x \in X$. The generated topology τ on X is compatible with the vector space operations, and it turns out that

$$A \in \tau \iff \forall\, x \in A,\, \exists\, U \in \mathcal{U} \,:\, x \in U \subset A.$$

4.1.1 Locally Convex Spaces

In this brief subsection, we investigate some of the fundamental properties of locally convex topologies (since most of the spaces we will deal with are locally convex.)

Definition 4.4 (Locally Convex)**.** Let (X, τ) be a topological vector space. We say that τ is *locally convex* if there exists a local basis of convex neighborhoods of 0.

Remark 4.2. Equivalently, a topological vector space X is locally convex if and only if there exists a family $\mathcal{U}$ of subsets such that:

(1) The origin belongs to all $U \in \mathcal{U}$.

(2) For all U, $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$.

(3) Every set $U \in \mathcal{U}$ is absorbing.

(4) For all $U \in \mathcal{U}$ one can find $V \in \mathcal{U}$ such that

$$V + V \subseteq U \quad \text{and} \quad V = -V.$$

(5) For all $U \in \mathcal{U}$ one can find a convex balanced set $V \in \mathcal{U}$ such that $V \subseteq U$.

Example 4.1. Here we give a short list of locally convex spaces, and we also furnish a brief explanation of what the notions of convergence are.

1) The space of all continuous functions on the real line $C^0(\mathbb{R}; \mathbb{R})$ equipped with the family of seminorms $P_K := \| \cdot \|_{\infty,\, K}$, where $K \subset \mathbb{R}$ is a compact subset. The notion of convergence is the uniform convergence on compact subsets.

2) The space of all differentiable functions on a bounded subset $C^k(\Omega)$ equipped with the family of seminorms $P_{\alpha,\, K} := \|D^\alpha \cdot \|_{\infty,\, K}$, where $K \subset \Omega$ is a compact subset. The notion of convergence is the uniform convergence of **all** derivatives on compact subsets.

3) The space of distributions $\mathcal{D}(\Omega)$.

4) The space of all sequences $\mathbb{R}^{\mathbb{N}}$ (or its subset made up of compactly supported sequences) is a locally convex space with the finer topology that makes the inclusions continuous).

Theorem 4.5 (Seminorm Characterization). *Let X be a vector space, and let $\mathcal{P} = \{p_\alpha\}_{\alpha \in I}$ be a family of seminorms defined on X. The following properties hold true:*

(a) *The ball B_{p_α} is absorbing, convex and balanced for every $\alpha \in I$.*

(b) *The family of neighborhoods*

$$\mathcal{U} := \left\{ \bigcap_{\alpha \in S \subset I} B_{p_\alpha} \mid S \text{ finite subset of indices of } I \right\}$$

*satisfies the properties **(1)**-**(5)** described above, and hence it defines a locally convex topology τ on X.*

(c) *The extended family of seminorms, i.e.*

$$\widetilde{\mathcal{P}} := \left\{ \widetilde{p}_S := \max_{\alpha \in S} p_\alpha \mid S \text{ finite subset of indices of } I \right\}$$

induces a family of balls

$$\widetilde{\mathcal{U}} := \{ B_{\widetilde{p}_S} \mid S \text{ finite subset of indices of } I \}$$

*which defines the same locally convex topology τ of **(b)**.*

(d) *Let $\mathcal{U}$ be a local basis of the origin made up of absorbing, convex and balanced sets, and let σ be the locally convex topology induced on X. Then the family of the Minkowski functionals*

$$\mathcal{P} := \{ p_U \mid U \in \mathcal{U} \}$$

is a family of seminorms, which generates the same topology σ.

Remark 4.3. Let X be a topological vector space, and let $\mathcal{U}$ be a local basis of neighborhoods of the origin. The following topological properties hold:

(a) The space X is T_1 if and only if X is T_0.

(b) The space X is T_1 if and only if $\{0\}$ is closed if and only if $\bigcap_{U \in \mathcal{U}} U = \{0\}$ or, equivalently, if and only if $p(x) = 0$ for any $p \in \mathcal{P}$ implies $x = 0$.

(c) The space X is T_0 if and only if X is T_2 (i.e., Hausdorff).

(d) Assume that (X, τ) is a locally convex T_0 space, and assume also that there exists a countable separated family of seminorms generating τ. Then X is metrizable.

Proof.

(a) Any topological group which is T_0 is automatically T_1. The converse is always true.

(b) Recall that a topological space is T_1 if and only if each singlet $\{x\}$ is closed. But if $\{0\}$ is closed in a topological vector space, then every $\{x\}$ is closed as x ranges in X.

(c) Fix $x, y \in X$. The singlet $\{x\}$ is always compact, while the singlet $\{y\}$ is closed because of the T_1 assumption. Therefore, the existence of open neighborhoods separating x and y follows immediately from Lemma 3.8.

(d) Let us set

$$d(x, y) := \max_{k \in \mathbb{N}} \left[2^{-k} \frac{p_k(x-y)}{1+p_k(x-y)} \right]. \tag{4.1}$$

The reader may check by herself that (4.1) is a distance on X. For example, it is easy to see that

$$d(x, y) = 0 \implies p_k(x-y) = 0 \quad \text{for all } k \in \mathbb{N},$$

and this is enough to infer that x is necessarily equal to y, as a consequence of the fact that the family of seminorms is separated.

Furthermore, the function d is translation-invariant. Thus we only need to prove that the collection of open balls

$$B_r(0) := \{x \in X \mid d(x, 0) < r\}$$

is a local basis of neighborhoods of the origin, which induces the very same topology τ.

Step 1. The condition $d(x, 0) < r$ easily implies that

$$\left(2^{-k} - r\right) p_k(x) < r,$$

and this inequality is automatically satisfied for all $k > \log_2(1/r) =: r^*$. The finite number of remaining indices satisfy the inequality

$$p_k(x) < \frac{r}{2^{-k} - r} =: r_k,$$

and thus B_r contains the intersection of a finite number of $\mathcal{P}$-balls, that is,

$$\bigcap_{k=0}^{\lfloor r^* \rfloor} r_k \cdot B_{p_k} \subseteq B_r(0).$$

In particular, the metric ball $B_r(0)$ contains a τ-neighborhood of the origin.

Step 2. Vice versa, given a τ-neighborhood V of the origin, it follows from the definition of local basis that one can always find positive real numbers $r_j > 0$ such that

$$\bigcap_{j=0}^{m} r_j \cdot B_{p_j} \subseteq V.$$

If we take r satisfying the inequality

$$2r < \max\left\{2^{-j} \cdot r_j \mid j = 1, \ldots, m\right\},$$

then any $x \in B_r(0)$ satisfies the inequality

$$d(x, 0) < r < \frac{2^{-j} \cdot r_j}{2} \implies p_j(x) < r_j \quad \text{for all } j = 1, \ldots, m.$$

Consequently, the metric ball $B_r(0)$ is contained in the finite intersection of the balls $r_j \cdot B_{p_j}$, and therefore it is also contained in V.

□

Theorem 4.6. *A* T_0 *topological vector space* X *is second-countable if and only if it is metrizable.*

Hint. Take a countable basis of neighborhoods of the origin such that each element is balanced and monotone (with respect to the inclusion) $U_{n+1} + U_{n+1} \subseteq U_n$. Then define

$$\rho(x) = \inf \left\{ \sum_{i=1}^{n} 2^{-k_i} \mid k_i \in \mathbb{N},\ x \in U_{k_1} + \cdots + U_{k_n} \right\}$$

and prove that $\rho(x - y) = d(x, y)$ is a metric on X.

The proof of this theorem is rather technical and I will not write it down, but it can be found e.g. in **W. Rudin, Functional Analysis** (Theorem 1.24). □

4.2 The Uniform Boundedness Principle

Let X be a topological vector space. In this section, we shall denote by $\mathcal{U}_p(X)$ a local basis of neighborhoods around $p \in X$.

Definition 4.7 (Bounded)**.** A set $E \subset X$ is *bounded* if and only if for every $U \in \mathcal{U}_0(X)$ there exists $r > 0$ such that $E \subseteq r \cdot U$.

Remark 4.4. If X is a metric space, a set is bounded in the sense above if and only if it is bounded with respect to the metric.

Lemma 4.8. *Let* X *be a topological vector space and let* $S \subset X$ *be any subset. The closure of* S *is given by the (possibly) infinite intersections of the* $\mathcal{U}$*-translations:*

$$\overline{S} = \bigcap_{U \in \mathcal{U}_0(X)} S + U.$$

In particular, any point $x \in X$ admits a local basis of *closed neighborhoods*. Indeed, it is straightforward to prove that

$$\{V + x \mid V \in \mathcal{U}_0(X)\}$$

is a local basis of neighborhoods of x, and hence

$$\left\{\overline{V} + x \mid V \in \mathcal{U}_0(X)\right\}$$

is also a local basis.

Lemma 4.9. *Let* X *be a topological vector space, and let* $E \subset X$ *be a bounded subset. Then the closure* $\overline{E}$ *is also bounded.*

Proof. By definition, for every $U \in \mathcal{U}_0(X)$ there exists a positive real number $r > 0$ such that $E \subset r \cdot U$. Hence, by taking the closure we obtain the chain of inclusions

$$\overline{E} \subseteq r \cdot \overline{U} \subset (r + \epsilon) \cdot U,$$

for some $\epsilon > 0$, as a consequence of Lemma 4.8. □

Lemma 4.10. *Let X and Y be topological vector spaces. If $L \in \mathcal{L}(X, Y)$ is a linear and continuous operator, then it is bounded. The converse is true if, e.g., X is locally bounded.*

Theorem 4.11. *A topological vector space X is normable if and only if it is both locally convex and locally bounded.*

We now present a simple criterion to show whether or not a set is bounded looking at the behavior of all the sequences.

Lemma 4.12. *Let X be topological vector space and let $E \subset X$. Then E is bounded if and only if for every sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ converging to 0 and every sequence $(x_n)_{n\in\mathbb{N}} \subset E$, it turns out that*

$$\alpha_n x_n = o(1) \quad \text{for } n \to +\infty.$$

Proof. Suppose that E is a bounded subset of X. Fix $U \in \mathcal{U}_0(X)$ neighborhood of the origin in X, and let $r > 0$ be the positive real number such that $E \subset r \cdot U$. It follows that

$$x_n \in r \cdot U \implies \frac{x_n}{r} \in U \quad \text{for } n \in \mathbb{N}.$$

Therefore, we can always choose $N \in \mathbb{N}$ in such a way that $\alpha_N < 1/r$, that is,

$$\alpha_n x_n \in U \quad \text{for all } n \geq N.$$

The opposite implication is trivial (e.g., one could consider $x_n := x$ to be the constant sequence). □

Corollary 4.13. *If E is relatively compact (or relatively sequentially compact), then E is bounded.*

Remark 4.5. Let $(X, \|\cdot\|)$ be a normed space. Then:

(1) A subset $E \subset X$ is strongly bounded if and only if $\sup_{x\in E} \|x\| < +\infty$.

(2) A subset $E \subset X$ is weakly bounded if and only if for all $f \in X^*$ we have $\sup_{x\in E}\langle f, x\rangle < +\infty$.

(3) A subset $E \subset X^*$ is weakly-$*$ bounded if and only if for all $u \in X$ we have $\sup_{f\in E}\langle f, u\rangle < +\infty$.

Notation. Let X and Y be topological vector spaces, and let Γ be a family (eventually uncountable) of linear and continuous applications from X to Y. For every set $S \subset X$, we denote by $\Gamma(S)$ the union of the images, that is

$$\Gamma(S) := \bigcup_{T\in\Gamma} T(S),$$

and, for every $R \subset Y$, we denote by $\Gamma^{-1}(R)$ the intersection of the preimages, that is

$$\Gamma^{-1}(R) := \bigcap_{T\in\Gamma} T^{-1}(R).$$

In particular, the inclusion $\Gamma(S) \subset R$ is a compact way to express that

$$T(S) \subset R \quad \text{for every } T \in \Gamma,$$

and, similarly, it is also equivalent to the inclusion

$$S \subset T^{-1}(R) \quad \text{for every } T \in \Gamma.$$

Definition 4.14 (Equicontinuous)**.** Let X and Y be topological vector spaces. A family $\Gamma \subset \mathcal{L}(X, Y)$ of linear and continuous applications is *equicontinuous* if and only if

$$\forall V \in \mathcal{U}_0(Y),\, \exists U \in \mathcal{U}_0(X)\,:\, \Gamma(U) \subset V.$$

Remark 4.6. If X and Y are metric spaces, this notion is completely equivalent to the equicontinuity in the sense of ϵ-δ.

Remark 4.7. If X and Y are normed spaces, a family Γ is equicontinuous if and only if Γ is equibounded in $\mathcal{L}(X, Y)$ with respect to the operator norm.

Definition 4.15 (Meager Set)**.** Let (X, τ) be a topological space, and let $S \subset X$. We say that S is a *meager* (or *first-category*) set if and only if there exists a countable cover made up of nowhere dense subsets of X, that is,

$$S = \bigcup_{n\in\mathbb{N}} X_n, \qquad \operatorname{Int}\overline{X_n} = \varnothing.$$

Furthermore, we say that S is a *second-category* set if it is not a first-category set.

Theorem 4.16 (Baire)**.** *Let X be either a complete metric space or a locally compact topological space. Then each open nonempty subset of X is a second-category set.*

Theorem 4.17 (Banach-Steinhaus)**.** *Let X and Y be topological vector spaces and let $\Gamma \subset \mathcal{L}(X, Y)$ be a collection (eventually uncountable) of linear and continuous applications. If*

$$E := \{x \in X \mid \Gamma(\{x\}) \text{ is bounded}\}$$

is a second-category set (i.e., Γ is pointwise bounded in a second-category set), then Γ is a equicontinuous family.

Proof. Fix $U \in \mathcal{U}_0(Y)$ neighborhood of the origin in Y. Recall that we can always find a neighborhood $V \in \mathcal{U}_0(Y)$ closed, balanced and satisfying the the inclusion $V + V \subset U$.

By assumption, for all $x \in E$ there exists a positive natural number $m(x) \in \mathbb{N}$ such that $\Gamma(x) \subset m(x) \cdot V$ (or, equivalently, $x \in m(x) \cdot \Gamma^{-1}(V)$). In particular, we have that

$$E \subseteq \bigcup_{n\in\mathbb{N}} n \cdot \Gamma^{-1}(V) \quad \text{and} \quad \Gamma^{-1}(V) \text{ closed.}$$

Since E is a second-category set, there exists $m \in \mathbb{N}$ such that $m \cdot \Gamma^{-1}(V)$ has nonempty internal part (and, hence, the same applies to $n \cdot \Gamma^{-1}(V)$ for every $n \in \mathbb{N}$). In particular, it turns out that

$$\mathfrak{V} := \Gamma^{-1}(V) - \Gamma^{-1}(V) \in \mathcal{U}_0(X),$$

and by linearity of Γ we also have that

$$\Gamma(\mathfrak{V}) = \Gamma(\Gamma^{-1}(V)) - \Gamma(\Gamma^{-1}(V)) = V - V \subseteq U.$$

Therefore, the set $\mathfrak{V}$ is a neighborhood of the origin in X whose image is contained in U, and this is exactly what we wanted to prove. ☐

Remark 4.8. The theorem is false if Γ is a family of linear (but not necessarily continuous) operators from X to Y. The easiest counterexample is the following: Consider a Banach space X, and let $Y := \mathbb{R}$ and $\Gamma := \{L\}$, for any L discontinuous linear functional.

Applications. In this brief paragraph, we investigate some basic results which may be obtained through a simple application of the uniform boundedness principle.

Theorem 4.18. *Let X be a complete metric space, and let Y and Z be topological vector spaces. If*

$$B : X \times Y \longrightarrow Z$$

is a bilinear application separately sequentially continuous, then B is jointly sequentially continuous (i.e., seq. continuous w.r.t. the couple).

Proof. We first prove a particular case, and then we generalize it with a simple algebraic trick.

Step 1. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence converging to 0, and let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a converging sequence. The linear mapping

$$B(\cdot,\, y_n) : X \longrightarrow Z$$

is clearly continuous for every $n \in \mathbb{N}$, and it is pointwise bounded. Indeed, for $u \in X$ fixed, the subset

$$\{B(u,\, y_n)\}_{n\in\mathbb{N}} \subset Z$$

is bounded in Z, since converging sequences are bounded. The Baire Theorem 4.16 holds in the complete metric space X, and hence the Banach-Steinhaus Theorem 4.17 implies that for every $U \in \mathcal{U}_0(Z)$ there exists $V \in \mathcal{U}_0(X)$ such that

$$B\left(V,\, y_n\right) \subset U \quad \text{for every } n \in \mathbb{N}.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ converges to 0; thus, $x_n \in V$ definitively, and this means that $B(x_n,\, y_n)$ belongs to U for n sufficiently large, that is, $B(x_n,\, y_n) \to 0$.

Step 2. If $x_n \to x \in X$, then the thesis follows from the previous step if one notices that the difference $(x_n - x)_{n\in\mathbb{N}}$ converges to 0. Then

$$B(x_n,\, y_n) = B(x_n - x,\, y_n) + B(x,\, y_n) \xrightarrow{n\to+\infty} 0 + B(x,\, y) = B(x,\, y).$$

□

Corollary 4.19. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,\, Y)$ be a family of linear and continuous operators between two Banach spaces, and assume that the sequence is pointwise bounded, that is,*

$$\sup_{n\in\mathbb{N}} \|T_n(x)\| < +\infty \quad \textit{for all } x \in X.$$

Then T_n is equibounded,that is, there exists a positive constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{L}(X,\,Y)} \leq C < +\infty.$$

Remark 4.9. Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,\, Y)$ be a family of linear and continuous operators between two Banach spaces, and suppose that

$$T_n(x) \xrightarrow{n\to+\infty} T(x) \quad \text{w.r.t. the strong convergence in } X.$$

Then $T_n \to T \in \mathcal{L}(X,\, Y)$ in the operator norm, and the convergence is uniform on any compact set.

Proof. The sequence $(T_n)_{n\in\mathbb{N}}$ is pointwise bounded. Therefore, the previous corollary asserts that it is also equibounded, that is, there exists a positive constant c such that

$$\sup_{n\in\mathbb{N}} \|T_n\| = c < \infty.$$

It follows that

$$\|T_n x\| \le c\|x\| \quad \text{for all } x \in X,$$

which means that T_n is also an equi-Lipschitz continuous sequence.

The Ascoli-Arzelà theorem allows us to extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that T_{n_k} converges, uniformly on compact sets, to a linear operator T. Furthermore, the assumption

$$\|T_n x\| \le c\|x\| \quad \text{for all } x \in X,$$

is preserved under the limit for $n \to +\infty$, and therefore T is a continuous operator, which means that

$$T_{n_k} \xrightarrow{k\to+\infty} T \in \mathcal{L}(X, Y),$$

uniformly on compact sets. □

4.3 The Open Mapping Theorem

The statement and the proof of the open mapping theorem are both taken **verbatim** from the Brezis book [1].

Theorem 4.20 (Open Mapping)**.** *Let X and Y be two Banach spaces, and let T be a continuous linear operator from X to Y that is surjective (=onto). Then there exists a constant $c > 0$ such that*

$$T\left(B_X(0, 1)\right) \supset B_Y(0, c).$$

Remark 4.10. The conclusion of the open mapping theorem is equivalent to saying that T is an open map. Indeed, let us suppose that U is open in X and let us prove that $T(U)$ is open in Y.

Fix any point $y_0 \in T(U)$, so that $y_0 = T x_0$ for some $x_0 \in U$. Let $r > 0$ be such that $B(x_0, r) \subset U$, i.e., $x_0 + B(0, r) \subset U$. It follows that

$$y_0 + T\left(B(0, r)\right) \subset T(U).$$

Then by open mapping theorem it follows that

$$T\left(B(0, r)\right) \supset B(0, rc)$$

and therefore

$$B(y_0, rc) \subset T(U).$$

Proof. We split the argument into two steps.

Step 1. Assume that T is a linear surjective operator from X onto Y. Then there exists a constant $c > 0$ such that

$$\overline{T\,(B_X(0,\,1))} \supset B_Y(0,\,2c).$$

Set $X_n = n \cdot \overline{T\,(B_X(0,\,1))}$. Since T is surjective, we clearly have

$$Y = \bigcup_{n\in\mathbb{N}} X_n,$$

and by Baire category theorem there exists some N such that $\mathrm{Int}X_N \neq \varnothing$. It follows that the same holds true (by homeomorphisms) for any other X_n and thus

$$\mathrm{Int}\left(\overline{T\,(B_X(0,\,1))}\right) \neq \varnothing.$$

Pick $c > 0$ and $y_0 \in Y$ such that

$$B_Y(y_0,\,4c) \subset \overline{T\,(B_X(0,\,1))}.$$

In particular, $y_0 \in \overline{T\,(B_X(0,\,1))}$, and by symmetry,

$$-y_0 \in \overline{T\,(B_X(0,\,1))}.$$

Therefore

$$B_Y(0,\,4c) \subset \overline{T\,(B_X(0,\,1))} + \overline{T\,(B_X(0,\,1))}.$$

On the other hand, since $\overline{T\,(B_X(0,\,1))}$ is convex, we have

$$\overline{T\,(B_X(0,\,1))} + \overline{T\,(B_X(0,\,1))} = 2 \cdot \overline{T\,(B_X(0,\,1))},$$

and the first claim follows.

Step 2. Assume that T is continuous linear operator from X into Y that satisfies the first claim. Then we have

$$T\,(B_X(0,\,1)) \supset B_Y(0,\,c).$$

Choose any $y \in Y$ with $\|y\| < c$. The goal is to find some $x \in X$ such that

$$\|x\| < 1 \qquad \text{and} \qquad Tx = y.$$

We know that

$$\forall \epsilon > 0,\ \exists\, z \in X \text{ such that } \|z\| < \frac{1}{2} \text{ and } \|y - Tz\| < \epsilon.$$

Choosing $\epsilon = c/2$, we find some $z_1 \in X$ such that

$$\|z_1\| < \frac{1}{2} \qquad \text{and} \qquad \|y - Tz_1\| < \frac{c}{2}.$$

Proceeding similarly, by induction we obtain a sequence $(z_n)_{n\in\mathbb{N}} \subset X$ such that

$$\|z_n\| < \frac{1}{2^n} \qquad \text{and} \qquad \|y - T(z_1 + \cdots + z_n)\| < \frac{c}{2^n}, \quad \text{for any } n \in \mathbb{N}.$$

It follows that the sequence $x_n = z_1 + \cdots + z_n$ is a Cauchy sequence. Let x be its limit (since X is complete) with, clearly, $\|x\| < 1$ and $y = Tx$ (since T is continuous). $\square$

Corollary 4.21. *Let X and Y be Banach spaces, and let T be a bijective continuous and linear operator from X to Y. Then T^{-1} is a continuous operator from Y to X.*

Proof. The Open Mapping Theorem 4.20, together with the assumption that T is injective, is enough to infer that

$$x \in X \ : \ \|Tx\|_Y < c \implies \|x\|_X < 1.$$

By homogeneity, it turns out that

$$\|x\|_X \leq \frac{1}{c} \|Tx\|_Y, \qquad \forall\, x \in X,$$

and hence T^{-1} is continuous. □

Corollary 4.22. *Let X be a vector space provided with two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that X is Banach with respect to both norms, and assume that there exists a constant $C \geq 0$ such that*

$$\|x\|_2 \leq C\, \|x\|_1, \qquad \forall\, x \in X.$$

Then the two norms are ***equivalent****, i.e., there is a constant $c > 0$ such that*

$$\|x\|_1 \leq c\, \|x\|_2, \qquad \forall\, x \in X.$$

Proof. It suffices to apply Corollary 4.21 with

$$X := (X, \|\cdot\|_1), \qquad Y := (X, \|\cdot\|_2) \quad \text{and} \quad T = \mathrm{id}_X.$$

□

Remark 4.11. The assumption that the two norms are comparable is critical. Indeed, if $(X, \|\cdot\|)$ is an infinite-dimensional Banach space, then one can always find a linear form $\ell : X \longrightarrow \mathbb{R}$ which is not continuous. Therefore, we may define a non-continuous operator $A : X \longrightarrow X$ as follows. Pick a point $x_0 \in X$ such that $Tx_0 = 1$, and set

$$A(x) := x - 2x_0\, \langle \ell, x \rangle.$$

We can easily define a norm

$$\|x\|_A := \|Ax\|_X,$$

which makes A an isometry of X (and thus continuous), if seen as an operator between $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_A)$.

The reader may check as an exercise that the corollary above fails since $(X, \|\cdot\|_A)$ is also a Banach space, but $\|\cdot\|_A$ and $\|\cdot\|$ are not equivalent norms.

Theorem 4.23 (Closed Graph)**.** *Let X and Y be two Banach spaces, and let $T : X \longrightarrow Y$ be a linear operator. Then T is continuous if and only if the graph of T, denoted by $\Gamma(T)$, is closed in the Cartesian product $X \times Y$.*

Proof. First, we notice that the graph

$$\Gamma(T) := \{(x, Tx) \mid x \in X\}$$

is the kernel of the map

$$X \times Y \ni (x, y) \longmapsto Tx - y.$$

Step 1. If T is continuous, then the map $(x, y) \longmapsto Tx - y$ is also continuous. Therefore, the preimage of the singlet $\{0\}$, which is closed, is exactly equal to the graph $\Gamma(T)$.

Step 2. Vice versa, let us consider on X the two norms

$$\|x\|_1 := \|x\|_X + \|Tx\|_Y \qquad \text{and} \qquad \|x\|_2 := \|x\|_X.$$

It is easy to check, using the assumption that $G(T)$ is closed, that X is a Banach space for the norm $\|\cdot\|_1$. On the other hand, by assumption, X is also a Banach space for $\|\cdot\|_2$ and clearly

$$\|x\|_1 \geq \|x\|_2.$$

It follows from Corollary 4.22 that the two norms are equivalent; hence there exists a constant $c > 0$ such that $\|x\|_1 \leq c\,\|x\|_2$, from which it follows that

$$\|Tx\|_Y \leq c'\,\|x\|_X,$$

i.e. T is continuous. $\square$

Chapter 5

Banach-Alaoglu and Closed Rank Theorem

In this chapter, we exploit the results we have proved so far to define a locally convex topology on a Banach space X (which is weaker than the topology induced by the norm.)

In a similar fashion, we show that the dual space X^* may be endowed with three (eventually) different topologies: the one induces by the dual norm, the weak one, and the weak-$*$ one.

In the second half of the chapter, we investigate some of the main properties of the weak-$*$ topology, e.g. the Banach-Alaouglu theorem (which asserts that the ball is weakly-$*$ compact.)

5.1 Initial Topology

Let $\mathscr{X}$ be a vector space, and let $\mathscr{F} \subset \mathscr{X}'$ be a subfamily of the **algebraic dual**[1]. Notice that every linear functional induces a seminorm, defined by

$$|\varphi|(x) := |\varphi(x)|,$$

and therefore the collection

$$\mathcal{P} := \{|\varphi| \ : \ \varphi \in \mathscr{F}\}$$

generates a locally convex topology on $\mathscr{X}$. More precisely, let

$$\mathcal{B}_\varphi^r := \{x \in X \mid |\varphi(x)| < r\}$$

denote the ball associated to $\varphi \in \mathcal{P}$. The enlarged family, which is defined by

$$\widetilde{\mathcal{P}} := \left\{ \max_{i \in S} |\varphi_i| \mid S \subset \mathscr{F} \text{ finite subset} \right\},$$

[1]**Definition.** Let X be a vector space. The algebraic dual of X, denoted by X', is the set of all the linear functional $\varphi : X \longrightarrow \mathbb{R}$ - since a priori there is no topology on X.

has the following property: *Every ball associated to an element of* $\widetilde{\mathcal{P}}$ *is equal to a finite intersection of balls associated to elements of* $\mathcal{P}$.

The family of seminorms $\mathcal{P}$ induces on $\mathscr{X}$ a locally convex topology τ, which is the coarsest topology that makes any $\varphi \in \mathcal{F}$ continuous.

Example 5.1. Let us consider a collection $(X_i, \tau_i)_{i\in I}$ of topological vector spaces, and let us denote by X the product of the X_i's, that is,

$$X := \prod_{i\in I} X_i.$$

Our goal is to define a topology τ on X that makes it a topological vector space. The natural way to do it is to require that a map $f : Y \longrightarrow X$ is continuous if and only if the components $p_i \circ f : Y \longrightarrow X_i$ are τ_i-continuous for every $i \in I$, where

$$p_i : X := \prod_{i\in I} X_i \longrightarrow X_i$$

denotes the ith canonical projection. Furthermore, one can easily prove that

$$(X_i, \tau_i) \text{ is locally convex for all } i \in I \implies (X, \tau) \text{ is locally convex.}$$

For example, both the sum and the scalar product on X are τ-continuous operations since the composition with each projection p_i is continuous.

Back to the general case, we observe that the *initial topology* τ, generated by the family $\mathcal{P}$, is defined in such a way that it is the **coarsest** topology making every $\varphi \in \mathscr{F}$ continuous.

Proposition 5.1. *Let* $\mathscr{X}$ *be a vector space, let* $\mathscr{F} \subset \mathscr{X}'$ *be a subfamily of the algebraic dual, and let* τ *be the initial topology relative to* $\mathscr{F}$. *Then continuous linear forms of* $(\mathscr{X}, \tau)$ *are precisely the elements of* $\mathscr{F}$ *and all of their linear combinations, that is,*

$$\mathscr{X}^*_\tau = \operatorname{Span}\langle f \,:\, f \in \mathscr{F}\rangle.$$

The proof of this result is an immediate consequence of the following technical lemma.

Lemma 5.2. *Let* X *be a vector space, and let* $f, f_1, \dots, f_n \in X'$. *Then the following properties are equivalent:*

(a) *There are real numbers* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *such that*

$$f = \sum_{i=1}^n \lambda_i f_i.$$

(b) *There exists a positive constant* $C > 0$ *such that*

$$|f(x)| \le C \max_{i=1,\dots,n} |f_i(x)| \quad \text{for all } x \in X.$$

(c) *The kernel of* f *contains the intersection of the kernels, that is,*

$$\ker f \supseteq \bigcap_{i=1}^n \ker f_i.$$

Proof. The unique nontrivial implication is (**c**) $\Longrightarrow$ (**a**). In the linear algebra course we have proved that there exists a unique $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the diagram below is commutative

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "f"'] & \mathbb{R}^n \arrow[d, dotted, "\lambda"] \\ & \mathbb{R} \end{tikzcd}$$

where $F(x) := (f_1(x), \ldots, f_n(x))$.

On the other hand, the mapping $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ is clearly linear and satisfying the identity $\lambda \circ F(x) = f(x)$. It follows that f is a linear combination of the components of F, that is,

$$f(x) = \sum_{i=1}^{n} \lambda_i f_i(x).$$

$\square$

In particular, if f is continuous with respect to the initial topology τ, then f is bounded in a neighborhood of the origin of the form

$$\bigcap_{i=1}^{n} \{x \in X \mid |f_i(x)| < \epsilon_i\}$$

Therefore, it follows from Lemma 5.2 that f is a linear combination of a finite number of elements $f_i \in \mathscr{F}$, precisely $f_1, \ldots, f_n$, that is,

$$\mathscr{X}_\tau^* = \operatorname{Span}\langle f \ : \ f \in \mathscr{F}\rangle.$$

5.2 Elementary Properties of the Weak Topologies

Let $(X, \|\cdot\|)$ be a Banach space, and let $\mathscr{F} := X^*$ be the topological dual, that is, the set of all the linear forms continuous with respect to the strong convergence τ_s.

The initial topology associated to the family $\mathscr{F}$ is called *weak topology* of X, and it is usually denoted by τ_w. A natural question arises: *The weak topology τ_w and the strong topology τ_s are actually different?*

Answer. The reader may readily check that the dual spaces are the same, that is,

$$X_{\tau_w}^* = X_{\tau_s}^*,$$

and also that both topologies are separated. Luckily, this is not enough for them to coincide and, actually, if X is an infinite-dimensional Banach space, the strong topology and the weak topology are always different, as a consequence of the following result.

Lemma 5.3. *Let U be a neighborhood of the origin in τ_w. Then there exists a finite-codimensional subspace $H \subset X$ contained in U.*

Proof. A subbasis of the neighborhoods of the origin in the weak topology $\sigma(X, X^*)$ is given by the sets of the form

$$W_{f,\epsilon} = \{x \in X \mid |\langle f, x\rangle| < \epsilon\} \quad \text{for } f \in X^* \text{ and } \epsilon > 0.$$

In particular, given $U \in \mathcal{U}_0(X)$ neighborhood of the origin, one can always find $f_1, \ldots, f_N \in X^*$ and $\epsilon_1, \ldots, \epsilon_N > 0$ such that

$$U \supseteq \bigcap_{i=1}^{N} W_{f_i, \epsilon_i}.$$

Furthermore, we may always assume that the finite intersection is made up of **linearly independent** elements. The reader can easily check that the finite-dimensional subspace

$$H = \{x \in X \mid f_i(x) = 0 \text{ for all } i = 1, \ldots, N\}$$

is contained in U, and it is thus the sought subspace. □

Weak-$*$ Topology. Let $(X^*, \|\cdot\|_*)$ be the topological dual of a Banach space X, endowed with the operator norm. The valuations $j_x : X^* \longrightarrow \mathbb{R}$ form a linear subspace of the algebraic bidual X'', which may be identified with X in the obvious way.

Definition 5.4. The initial topology on X^* associated to the family

$$\mathscr{F} := \{j_x : X^* \longrightarrow \mathbb{R} \mid x \in X\}$$

is called *weak-$*$ topology*, and it is usually denoted by τ_w^*.

In particular, on the dual space X^* one can define three remarkable topologies satisfying the following chain of inclusions:

$$\tau_w^* \subseteq \tau_w \subseteq \tau_s.$$

Notation. In the literature, the initial topology on a vector space X associated to a family $\mathscr{F}$ is usually denoted by $\sigma(X, \mathscr{F})$. Therefore, from now on, we denote by $\sigma(X, X^*)$ the weak topology on X, by $\sigma(X^*, X^{**})$ the weak topology on X^*, and by $\sigma(X^*, X)$ is the weak-$*$ topology on X^*.

Definition 5.5 (Separated)**.** Let X be a vector space, and let $\mathscr{F}$ be a subfamily of the algebraic dual. The initial topology $\sigma(X, \mathscr{F})$ is *separated* if and only if for every $x \in X$, $x \neq 0$, there is an element $f \in \mathscr{F}$ such that $\langle f, x\rangle \neq 0$.

Remark 5.1. Equivalently, the initial topology $\sigma(X, \mathscr{F})$ is separated if and only if for every $x \neq 0$ there are a neighborhood U of the origin and a positive real number $c > 0$ such that

$$x \notin c \cdot U.$$

Lemma 5.6. *Let X be a Banach space. The weak topology $\sigma(X, X^*)$ and the weak-$*$ topology $\sigma(X^*, X)$ are separated.*

Proof. Let $x \in X \setminus \{0\}$. It follows from Example 3.2 that one can always find a continuous linear functional $f : X \longrightarrow \mathbb{R}$ such that

$$\|f\|_{X^*} = 1 \quad \text{and} \quad \langle f, x\rangle = \|x\| \neq 0.$$

On the other hand, a functional $f \in X^*$ is nonzero if and only if there exists $x \in X$ such that $\langle f, x\rangle \neq 0$. Therefore, the valuation j_x is different from zero in f, and this concludes the proof. □

Lemma 5.7. *Let X be a Banach space.*

(a) *A subset $E \subset X$ is bounded in norm if and only if it is weakly bounded.*

(b) *A subset $E \subset X^*$ is bounded in norm if and only if it is weakly-$*$ bounded.*

Proof. One implication is clear in both the assertions: if E is bounded w.r.t. the strong topology, then it is bounded in the weak (or weak-$*$) topology since there are less neighborhoods of 0.

(a) Assume that $E \subset X$ is weakly bounded. For every $f \in X^*$, the image $f(E)$ is bounded and hence there exists a positive constant $c_f > 0$ such that

$$|\langle f, x\rangle| \leq c_f \quad \text{for all } x \in E.$$

If we identify E with the subspace of the bidual $\{j_x \mid x \in E\}$, then the estimate above can be rewritten in terms of the valuation:

$$|j_x(f)| \leq c_f \quad \text{for all } x \in E \text{ and } f \in X^*.$$

Then E is pointwise bounded as a subset of the bidual X^{**}, and hence the Banach-Steinhaus Theorem 4.17 implies that E is uniformly bounded as a subset of the bidual.

On the other hand, X is a Banach space and so is X^{**}. Consequently, the set E is bounded w.r.t. the norm in X^{**}, and therefore also w.r.t. the norm in X.

(b) In this case the Banach-Steinhaus Theorem 4.17 may be applied directly to the family E since, by assumption, we have

$$\sup_{f \in E} |\langle f, x\rangle| < C \quad \text{for all } x \in X.$$

□

Remark 5.2. If X is a locally convex topological vector space, the first assertion is still true while the second is, generally, not true.

Lemma 5.8. *Let X be a Banach space.*

(a) *If $C \subset X$ is a convex set, then C is closed in norm if and only if C is weakly closed.*

(b) *If $C \subset X^*$ is a convex set, then C is closed in norm if it is weakly-$*$ closed.*

Proof. First, notice that C closed in the weak (or weak-$*$) topology, always implies C closed in norm.

(a) Suppose that C is a convex closed set w.r.t. the strong topology. We shall now prove that the complement $A := C^c$ is weakly open. Let $x_0 \notin C$. By Hahn-Banach Theorem 3.9 there exist a linear continuous functional $f \in X^*$ and a real number $\alpha \in \mathbb{R}$ such that

$$\langle f, x_0\rangle < \alpha < \langle f, x\rangle \quad \text{for all } x \in C.$$

It follows that $V := \{y \in X \mid \langle f, y\rangle < \alpha\}$ is an open neighborhood of x_0, strictly contained in A, and this is enough to infer that A is weakly open.

(b) It follows easily from the definitions, but it is interesting to see a counterexample for the opposite implication. For example, a non-reflexive Banach space X is closed in norm (as a subset of the bidual X^{**}), but it is weakly-$*$ dense and thus it cannot be weakly-$*$ closed.

A beautiful, concrete example may be found at this Math Stackexchange post.

□

5.3 The Banach-Alaoglu Theorem

In this section, we set the ground to state and prove a major result concerning the weak-$*$ topology. Namely, the closed ball is weakly-$*$ compact.

Let $\mathscr{X}$ be any set. The space of all the functions from $\mathscr{X}$ to $\mathbb{C}$, denoted by $\mathbb{C}^{\mathscr{X}}$, is a separated locally convex topological vector space, endowed with the product topology.

More precisely, one can easily prove that the product topology coincides with the initial topology associated to the family of the projections

$$\mathscr{F} := \left\{ \pi_x : \mathbb{C}^{\mathscr{X}} \longrightarrow \mathbb{C}_x \mid \mathbb{C}_x \cong \mathbb{C},\, x \in \mathscr{X} \right\},$$

and it is thus the coarsest that makes them continuous.

Remark 5.3. There is a natural inclusion $\mathscr{X}^* \subset \mathbb{C}^{\mathscr{X}}$ and, surprisingly, the subspace topology coincides with the weak-$*$ topology $\sigma\left(\mathscr{X}^*,\, \mathscr{X}\right)$.

The assertion of the previous remark is, actually, a consequence of a more general fact. Let $\mathscr{Y} \subset \mathscr{X}$, and endow $\mathscr{X}$ with the initial topology relative to the family

$$\mathscr{F} := \{f_\alpha : \mathscr{X} \longrightarrow \mathscr{Z}_\alpha\}_{\alpha \in \Gamma},$$

for some set of indices Γ. Then the subspace topology induces on $\mathscr{Y}$ is the initial topology relative to the restricted family

$$\mathcal{F}\big|_{\mathscr{Y}} := \{f_\alpha : \mathscr{X} \cap \mathscr{Y} \longrightarrow \mathscr{Z}_\alpha\}_{\alpha \in \Gamma}.$$

Remark 5.4. The algebraic dual of $\mathscr{X}$ is closed w.r.t. the inclusion $\mathscr{X}' \subset \mathbb{C}^{\mathscr{X}}$. Indeed, using the definitions, it is always possible to write it as the arbitrary intersection of closed sets:

$$\mathscr{X}' = \{f : \mathscr{X} \longrightarrow \mathbb{C} \mid \langle f,\, \alpha x + \beta y\rangle = \alpha\langle f,\, x\rangle + \beta\langle f,\, y\rangle \text{ for all } \alpha,\, \beta \in \mathbb{C} \text{ and } x,\, y \in \mathscr{X}\} =$$

$$= \bigcap_{\alpha,\, \beta \in \mathbb{C}} \; \bigcap_{x,\, y \in \mathscr{X}} \left\{ f \in \mathbb{C}^{\mathscr{X}} \mid \langle f,\, \alpha x + \beta y\rangle = \alpha\langle f,\, x\rangle + \beta\langle f,\, y\rangle \right\}.$$

Definition 5.9 (Polar)**.** Let X be a topological vector space, and let $S \subset X$ be any subset. The *polar* set associated to S is defined by

$$S^\circ := \{f \in X^* \mid |f(x)| \le 1 \text{ for all } x \in S\}$$

Remark 5.5. Let X be a topological vector space and let $S \subset X$ be any subset. The polar S° is clearly convex and weakly-$*$ closed since it is the intersection of weakly-$*$ closed convex sets:

$$S^\circ := \bigcap_{x \in S} \{x\}^\circ.$$

Furthermore, if V is a neighborhood of the origin in X, then the polar V° is convex, weakly-$*$ closed and also closed in the product space $\mathbb{C}^X$ (since it is closed in the algebraic dual X', which is closed in the product).

Theorem 5.10 (Banach-Alaoglu)**.** *Let X be a topological vector space, and let V be a neighborhood of the origin. Then the polar V° is convex, weakly-$*$ closed and weakly-$*$ compact.*

Proof. First, notice that as a consequence of the definition of polar set we have

$$|\langle f, x\rangle| \leq 1 \quad \text{for all } x \in V \text{ and all } f \in V^\circ.$$

In a topological vector space, a neighborhood of the origin is always absorbing. Therefore, there exists a function $\nu : X \longrightarrow \mathbb{N}$ such that, for any $f \in V^\circ$, we have

$$|\langle f, x\rangle| \leq \nu(x) \quad \text{for all } x \in X.$$

Let $x \in X$ be a point. The image via $f \in V^\circ$ belongs to a ball, whose radius depends only on x itself only, and this proves that

$$V^\circ \subset X' \cap \prod_{x \in X} \overline{B_{\mathbb{C}}(0,\, \nu(x))}.$$

By Tychonov Theorem 5.29, it follows that

$$X' \cap \prod_{x \in X} \overline{B_{\mathbb{C}}(0,\, \nu(x))}$$

is compact with respect to the product topology. On the other hand, the product topology coincides with the subspace topology induced by the inclusion

$$X' \subset \mathbb{C}^X.$$

In conclusion, note that V° is a weakly-$*$ closed subset of a Hausdorff weakly-$*$ compact set, and it is thus compact. □

Corollary 5.11. *Let X be a Banach space. The closed ball $\overline{B_{X^*}(0,\, 1)}$ in X^* is weakly-$*$ compact, that is, it is compact in the $\sigma(X^*,\, X)$ topology.*

Proof. It follows from the Banach-Alaoglu Theorem 5.10 since

$$V = B_X(0,\, 1) \implies V^\circ = \overline{B_{X^*}(0,\, 1)}.$$

□

Theorem 5.12. *Let X be a infinite-dimensional Banach space. The topological spaces $(X,\, \sigma(X,\, X^*))$ and $(X^*,\, \sigma(X^*,\, X))$ are not metrizable, that is, they are not* A_1.

Optional Proof. We may always assume without loss of generality that X is an infinite-dimensional space, that is, $\dim X \geq \aleph_0$. It is a well-known fact that the dual has (at least) the dimension of the continuum $\mathfrak{c}$, i.e., the dimension of X^* is uncountable.

Step 1. A subbasis of the neighborhoods of the origin in the weak topology $\sigma(X, X^*)$ is given by the sets of the form

$$W_{f,\epsilon} = \{x \in X \mid |\langle f, x\rangle| < \epsilon\} \quad \text{for some } f \in X^* \text{ and } \epsilon > 0.$$

Clearly the family of the finite intersection of these elements is a local basis for the origin, since for any $U \in \mathcal{U}_0(X)$ there are $f_1, \ldots, f_N \in X^*$ and $\epsilon_1, \ldots, \epsilon_N > 0$ such that

$$U \supseteq \bigcap_{i=1}^{N} W_{f_i, \epsilon_i}.$$

As a consequence, we may always assume that the finite intersections are made up of **linearly independent** elements.

Step 2. Suppose that X is metrizable, i.e. that X is second-countable, and let

$$\mathcal{B} := \left\{ W_{f_i, \frac{1}{k}} \mid i, k \in \mathbb{N} \right\}$$

be a local countable basis made up of linearly independent functionals.

The dimension of X^* is uncountable, and therefore we can always find a nontrivial functional $g \in X^* \setminus \operatorname{Span}\langle f_i \mid i \in \mathbb{N}\rangle$. Furthermore, by the Hahn-Banach Theorem 3.1, we can find a sequence of points $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$g(x_n) = 1 \quad \text{and} \quad f_i(x_n) = \frac{1}{n} \text{ for } i = 1, \ldots, n.$$

Any finite intersection of the form

$$W_{f_{i_1}, \frac{1}{k_1}} \cap \cdots \cap W_{f_{i_N}, \frac{1}{k_N}}$$

contains all but finitely many x_n's, while the neighborhood $W_{g,1}$ does not contain any point of the sequence. In particular

$$W_{f_{i_1}, \frac{1}{k_1}} \cap \cdots \cap W_{f_{i_N}, \frac{1}{k_N}} \not\subset W_{g,1}$$

for any choice of the indices, and thus there exists a neighborhood of the origin that does not belong to $\mathcal{B}$: a contradiction.

Step 3. Assume that the dimension of X is uncountable, that is, $\dim X > \aleph_0$. A subbasis of the neighborhoods of the origin in the weak-$*$ topology $\sigma(X^*, X)$ is given by the sets of the form

$$W_{x,\epsilon} = \{f \in X^* \mid |\langle f, x\rangle| < \epsilon\} \quad \text{for some } x \in X \text{ and } \epsilon > 0.$$

The family of the finite intersections of these elements is a local basis for the origin, since for any $U \in \mathcal{U}_0(X)$ there are $x_1, \ldots, x_N \in X$ and $\epsilon_1, \ldots, \epsilon_N > 0$ such that

$$U \supseteq \bigcap_{i=1}^{N} W_{x_i, \epsilon_i}.$$

As a consequence, we may always assume that the finite intersections are made up of **linearly independent** elements.

Step 4. Suppose that X is metrizable, i.e. that X is second-countable, and let

$$\mathcal{B} := \left\{ W_{x_i,\, \frac{1}{k}} \mid i,\, k \in \mathbb{N} \right\}$$

be a local countable basis made up of linearly independent functionals. The dimension of X is uncountable by assumption, and hence there exists a point $y \in X \setminus \operatorname{Span}\langle x_i \mid i \in \mathbb{N}\rangle$. Furthermore, by the Hahn-Banach Theorem 3.1, we can always find a sequence of functionals $(f_n)_{n\in\mathbb{N}} \subset X$ such that

$$f_n(y) = 1 \quad \text{and} \quad f_n(x_i) = \frac{1}{n} \text{ for } i = 1,\, \dots,\, n.$$

Any finite intersection of the form

$$W_{x_{i_1},\frac{1}{k_1}} \cap \cdots \cap W_{x_{i_N},\frac{1}{k_N}}$$

contains all but finitely many f_n's, while the neighborhood $W_{y,\,1}$ does not contain any point of the sequence. In particular

$$W_{x_{i_1},\frac{1}{k_1}} \cap \cdots \cap W_{x_{i_N},\frac{1}{k_N}} \not\subset W_{y,\,1}$$

for any choice of the indices, and thus there is a neighborhood of the origin that does not belong to $\mathcal{B}$: a contradiction.

Step 5. Assume now that the dimension of X is countable, that is, $\dim X = \aleph_0$. Then the completion $\widetilde{X}$ has dimension at least $\mathfrak{c}$, and the weak-$*$ topologies of X^* and $\widetilde{X}^*$ coincide. Therefore, the previous steps conclude the proof. □

Theorem 5.13 (Banach-Alaoglu)**.** *Let X be a separable Banach space. Then the closed ball*

$$\overline{B_{X^*}(0,\, 1)} \subset X^*$$

is weakly-$*$ metrizable and sequentially weakly-$*$ compact.

Proof. Let $\{x_k\}_{k\in\mathbb{N}} \subset \overline{B_{X^*}(0,\, 1)}$ be a dense countable subset, and let us consider

$$D := \operatorname{Span}\langle x_k \ :\ k \in \mathbb{N}\rangle\,.$$

Metric. For any $f,\, g \in \overline{B_{X^*}(0,\, 1)}$, let us set

$$d(f,\, g) := \sum_{k\in\mathbb{N}} 2^{-k}\, |\langle f - g,\, x_k\rangle|\,. \tag{5.1}$$

We can easily check that (5.1) defines an actual metric on the closed ball of radius one. Indeed, one can readily prove that:

(1) The function $d(-,\, \cdot)$ is positive and symmetric with respect to both variables.

(2) The function $d(-, \cdot)$ is subadditive since

$$\sum_{k=0}^{M} 2^{-k} \, |\langle f - h, \, x_k \rangle| = \sum_{k=0}^{M} 2^{-k} \, |\langle f - g + g - h, \, x_k \rangle| \leq$$

$$\leq \sum_{k=0}^{M} 2^{-k} \, |\langle f - g, \, x_k \rangle| + \sum_{k=0}^{M} 2^{-k} \, |\langle g - h, \, x_k \rangle| \, .$$

(3) If $f, \, g \in \overline{B_{X^*}(0, \, 1)}$ and $d(f, \, g) = 0$, then

$$\langle f - g, \, x_k \rangle = 0 \quad \forall \, k \in \mathbb{N} \implies \langle f - g, \, d \rangle = 0 \qquad \forall \, d \in D,$$

and we conclude that $f = g$ using the fact that D is dense and $f - g$ is uniformly continuous.

Equivalence. To conclude the proof, we need to show that the weak-$*$ convergence is equivalent to the notion of convergence induced by the metric d. Let us consider

$$(f_n)_{n \in \mathbb{N}} \subset \overline{B_{X^*}(0, \, 1)}.$$

Step 1. Suppose that f_n converges to some f w.r.t. the weak-$*$ topology. It follows that

$$d(f_n, \, f) := \sum_{k \in \mathbb{N}} 2^{-k} \underbrace{|\langle f - f_n, \, x_k \rangle|}_{\to 0},$$

as a consequence of the Lebesgue domination theorem.

Step 2. Vice versa, if $f_n \xrightarrow{d} f$, then $d(f_n, \, f) \xrightarrow{n \to +\infty} 0$. In particular, it turns out that

$$\sum_{k \in \mathbb{N}} 2^{-k} \, |\langle f - f_n, \, x_k \rangle| \xrightarrow{n \to +\infty} 0 \iff \langle f - f_n, \, x_k \rangle \xrightarrow{n \to +\infty} \quad \text{for all } k \in \mathbb{N},$$

which means that $f_n - f$ converges weakly at every x_k to 0. On the other hand, the f_n are uniformly continuous, and therefore the sequence converges on the closure of D as well, that is, it converges at every $x \in X$. More precisely, it turns out that

$$f_n \overset{*}{\rightharpoonup} f \in \overline{B_{X^*}(0, \, 1)}.$$

Compactness. We consider the identity mapping between two copies of the closed ball, endowed with different topologies, i.e.

$$\mathrm{id} : \left(\overline{B_{X^*}(0, \, 1)}, \, d \right) \longrightarrow \left(\overline{B_{X^*}(0, \, 1)}, \, \sigma(X^*, \, X) \right).$$

This map is sequentially continuous, the domain is compact, and so is the codomain. We can finally infer that the closed ball is sequentially compact w.r.t. the weak-$*$ topology. $\square$

Theorem 5.14 (Separability Criterion)**.** *Let X be a Banach space. If $\overline{B_{X^*}(0, \, 1)} \subset X^*$ is weakly-$*$ metrizable, then X is separable.*

Proof. Let d be the metric on $\overline{B_{X^*}(0,\,1)}$. For all $n \in \mathbb{N}$, we consider the metric ball of radius $1/n$, that is,

$$U_n := \left\{ f \in \overline{B_{X^*}(0,\,1)} \;\middle|\; d(f,\,0) < \frac{1}{n} \right\}.$$

For all $n \in \mathbb{N}$, let V_n be a neighborhood of the origin w.r.t. $\sigma\,(X^*,\,X)$ satisfying the inclusion $V_n \subset U_n$. We may assume that V_n has the form

$$V_n := \left\{ f \in \overline{B_{X^*}(0,\,1)} \mid |\langle f,\,x\rangle| < \epsilon_n \text{ for all } x \in E_n \right\},$$

where $\epsilon_n > 0$ and $E_n \subset X$ is some finite subset. Then

$$D := \bigcup_{n \in \mathbb{N}} E_n \subset X,$$

is countable, and we now prove that D is also dense in X. Indeed, if

$$\langle f,\,x\rangle = 0 \quad \text{for all } x \in D,$$

then it turns out that $f \in V_n \subset U_n$ for any $n \in \mathbb{N}$, and thus $f \equiv 0$ since

$$\bigcap_{n \in \mathbb{N}} U_n = \{0\}.$$

□

Theorem 5.15. *Let $(X,\,\tau)$ be a T_0 topological vector space. Then X is locally compact if and only if X is a finite-dimensional space if and only if X is linearly homeomorphic to $\mathbb{K}^n$.*

Proof. We first prove that a finite-dimensional T_0 topological vector space is linearly homeomorphic to $\mathbb{K}^n$ with the usual topology (which is locally compact), and then we show that a locally compact space is finite-dimensional.

Step 1. Suppose that X is a finite-dimensional T_0 topological vector space, and let $\{e_1,\,\dots,\,e_n\}$ be a basis of X. There is linear isomorphism $\Phi : \mathbb{K}^n \longrightarrow X$, defined by setting

$$\mathbb{K}^n \ni (\lambda_1,\,\dots,\,\lambda_n) \longmapsto \sum_{i=1}^{n} \lambda_i e_i \in X.$$

We only need to prove that Φ is an open map to conclude that it is an homeomorphism since Φ is clearly continuous and bijective.

Let $B := \overline{B_{\mathbb{K}^n}(0,\,1)}$ be the closed unit ball, and let $S := \partial B$ be its boundary. Then S is compact in $\mathbb{K}^n$ and it does not contain the origin; hence $\Phi(S)$ is compact in X, and it does not contain the origin of X. In particular $\Phi(S)$ is closed[2] in X, and thus there exists $V \in \mathcal{U}_0(X)$ open and balanced neighborhood of the origin such that

$$V \cap \Phi(S) = \varnothing.$$

[2]Recall that a T_0 topological vector space is automatically *Hausdorff*, and a compact set in a Hausdorff space is always closed.

Let $x \in V \setminus \Phi(B)$. The map is surjective, which means that we can find $\lambda \in \mathbb{K}^n$ such that $x = \Phi(\lambda)$, with $\|\lambda\| > 1$. The rescaling $\lambda/\|\lambda\|$ belongs to S, and therefore we find a contradiction since

$$\Phi\left(\frac{\lambda}{\|\lambda\|}\right) = \frac{x}{\|\lambda\|} \implies \Phi\left(\frac{\lambda}{\|\lambda\|}\right) \in \frac{1}{\|\lambda\|} \cdot V = V,$$

that is, x belongs to both $\Phi(S)$ and V at the same time. It follows that $V \subseteq \Phi(B)$, and this implies that $\Phi(B)$ is a neighborhood of the origin in X, that is, Φ is an open mapping.

Step 2. Conversely, assume that X is locally compact, and let V be a compact neighborhood of the origin of X. Clearly $1/2 \cdot V$ is also a neighborhood of the origin and, by compactness, there are finitely many points $x_i \in V$ such that

$$V \subseteq \bigcup_{i=1}^{m} \left(x_i + \frac{1}{2} \cdot V\right).$$

Let Y be the linear span of the points $x_1, \ldots, x_m$. Then

$$V \subseteq Y + \frac{1}{2} \cdot V \implies \cdots \implies V \subseteq Y + \frac{1}{2^n} \cdot V.$$

Notice that the local compactness of X easily implies that the family $\{2^{-n} \cdot V\}_{n \in \mathbb{N}}$ is a local basis of the origin in X, and hence

$$V \subseteq \bigcap_{n \in \mathbb{N}} \left(Y + \frac{1}{2^n} \cdot V\right) = \overline{Y} = Y,$$

since Y is finite-dimensional, and thus closed. This concludes the proof since V is absorbent and each rescaling is contained in Y, i.e.

$$X = \bigcup_{t>0} t \cdot V \subseteq Y \implies X = Y.$$

□

Corollary 5.16. *Let (X, τ) be a T_0 topological vector space, and let $Y \subset X$ be a finite-dimensional subspace. Then Y is closed.*

5.4 Iteration Lemma

In this brief section, we investigate the *iteration lemma*, which we implicitly exploited to prove the open mapping theorem in the previous chapter.

Lemma 5.17 (Iteration Lemma)**.** *Let $T \in \mathcal{L}(X, Y)$ be a continuous linear mapping between two Banach spaces. Suppose that there exists a neighborhood of the origin $U \in \mathcal{U}_0(Y)$ and a positive real number $t \in (0, 1)$ satisfying the inclusion*

$$U \subseteq TB_X + t \cdot U. \tag{5.2}$$

Then

$$(1-t) \cdot U \subseteq TB_X \quad \text{and} \quad T \text{ is open and surjective.}$$

Proof. The assumption (5.2) implies that

$$(1-t)\cdot U \subseteq (1-t)\cdot T(B_X) + (1-t)t\cdot U \subseteq T(B_X) + t\cdot T(B_X) + t^2\cdot U.$$

If we iterate this process n times, we find the following inclusion:

$$(1-t)\cdot U \subseteq \left[(1-t)\sum_{k=0}^{n-1} t^k\right]\cdot T(B_X) + (1-t)t^n\cdot U.$$

Since t belongs to $(0, 1)$, we can easily compute the sum of the series multiplying the first term on the right-hand side, that is,

$$\lim_{n\to+\infty}\left[\sum_{k=0}^{n-1} t^k\right] = \frac{1}{1-t}.$$

Therefore, if we take the limit as $n\to+\infty$ of the inclusion above, we find that

$$(1-t)\cdot U \subseteq \lim_{n\to+\infty}\left[(1-t)\sum_{k=0}^{n-1} t^k\right]\cdot T(B_X) \subseteq T(B_X),$$

where the last inclusion follows from the fact that X is a Banach space (thus the sequence of points converges). Indeed, one can easily notice that

$$(1-t)\cdot U \subseteq T(B_X) \implies T \text{ open},$$

and

$$Y = \bigcup_{n\in\mathbb{N}} n\cdot U \implies Y \subseteq \bigcup_{n\in\mathbb{N}} \frac{n}{1-t}\cdot T(B_X) \implies T \text{ surjective}.$$

□

Applications of the Iteration Lemma. In this paragraph, we state and prove two extension results that follow from an easy application of the iteration lemma.

Theorem 5.18. *Let A be a closed subset of a metric space M, and let $f : A \longrightarrow E$ be a continuous map with values in a Banach space E. Then there exists a continuous map*

$$\widetilde{f} : M \longrightarrow E$$

which extends f, that is $\widetilde{f}\big|_A \equiv f$.

Proof. Notice that we may always assume, without loss of generality, that f is a bounded map.

Step 0. The thesis is clearly equivalent to the surjectivity of the operator

$$\begin{aligned} T : C_b^0(M, E) &\longrightarrow C_b^0(A, E) \\ g &\longmapsto g\big|_A. \end{aligned}$$

The idea here is to apply the Iteration Lemma 5.17 to the operator T. In particular, it is enough to prove that for every $f \in C_b^0(A, E)$ there is $g \in C_b^0(M, E)$ such that $\|g\| \leq 1$ and

$$\|g\,|_A - f\| \leq \frac{1}{2}.$$

Indeed, any function $f \in C_b^0(A, E)$ can be rewritten as

$$f = g\,|_A - \left(g\,|_A - f\right),$$

and this is enough to infer that that the map is open and surjective.

Step 1. The function f is continuous at every point $p \in A$, which means that we can always find a neighborhood U_p of p in M such that

$$f\,|_{U_p \cap A} \equiv f_p\,|_{U_p \cap A},$$

where $f_p : U_p \longrightarrow E$ is a continuous local approximation of f at p.

Step 2. Consider the open covering

$$\Omega := \{U_p\}_{p \in A} \cup \{M \setminus A\}$$

of the metric space M, and let let $\{\psi_p\}_{p \in A} \cup \{\psi_0\}$ be the associated partition of unity[3].

Step 3. For every $p \in A$ there are only finitely many functions ψ_p that does not vanishing at p, and therefore we can define the approximation in the obvious way:

$$\widetilde{f}(x) := \sum_{p \in A} \psi_p(x) f_p(x).$$

The reader may check that, not only $\widetilde{f}(x)$ is the sought approximation, but also that it can be chosen to be zero outside of an arbitrarily small neighborhood of A. □

Theorem 5.19. *Let E and F be Banach spaces, and let $L \in \mathcal{L}(E, F)$ linear, continuous and surjective operator. If (M, d) is a metric space and $f : M \longrightarrow F$ a mapping, then it can be lifted to a map $\widetilde{f} : M \longrightarrow E$ in such a way that the following diagram commutes:*

$$\begin{tikzcd} & E \arrow[d, "L"] \\ M \arrow[ur, dotted, "\widetilde{f}"] \arrow[r, "f"'] & F \end{tikzcd}$$

Proof. The argument is very much similar to the previous one. First, we assume, without loss of generality, that f is a bounded mapping.

[3]Every metric space is paracompact, and therefore every metric space admits a partition of unity associated to an open covering.

Step 0. The thesis is clearly equivalent to the surjectivity of the operator

$$\begin{aligned} T : C_b^0(M, E) &\longrightarrow C_b^0(M, F) \\ g &\longmapsto L \circ g. \end{aligned}$$

In particular, the Iteration Lemma 5.17 implies again that it is enough to show that for every $f \in C_b^0(M, F)$ there is a $g \in C_b^0(M, E)$ such that $\|g\| \le 1$ and

$$\|L \circ g - f\| \le \frac{1}{2}.$$

We shall not prove this result here because it is enough to produce minor changes to the argument used in the previous theorem. □

5.5 Closed Rank Theorem

In this final section, we study the set of all operators between two Banach spaces in more depth; more precisely, we investigate necessary and sufficient condition for an operator to be injective (or surjective) and the relation with the dual operator.

Proposition 5.20. *Let X and Y be Banach spaces. The subspace of all surjective continuous linear mappings from X to Y is open.*

Remark 5.6. More precisely, if $T \in \mathcal{L}(X, Y)$ is surjective, the operator

$$\widetilde{T} : X/_{\operatorname{Ker} T} \longrightarrow Y$$

is invertible, and the positive constant $k := \left\|\widetilde{T}^{-1}\right\| > 0$ is well-defined. Then, the proposition above can be restated in a compact way as follows:

$$k \cdot B_Y \subseteq T(B_X) \implies T + H \text{ surjective for all } H \text{ such that } \|H\| \le k.$$

Proof. The proof is a fairly simple consequence of the Iteration Lemma 5.17. Indeed, the inclusion

$$k \cdot B_Y \subseteq T(B_X)$$

immediately implies that

$$\begin{aligned} k \cdot B_Y &\subseteq (T + H)(B_X) + H(B_X) \subseteq \\ &\subseteq (T + H)(B_X) + \|H\| \cdot B_Y \subseteq \\ &\subseteq (T + H)(B_X) + \frac{\|H\|}{k} \cdot (k \cdot B_Y). \end{aligned}$$

Since $k > \|H\|$ by assumption, the Iteration Lemma 5.17 implies that

$$k \cdot \left(1 - \frac{\|H\|}{k}\right) \cdot B_Y \subseteq (T + H)(B_X),$$

and hence

$$(k - \|H\|) \cdot B_Y \subseteq (T + H)(B_X),$$

which means that $T + H$ is open and, hence, surjective. □

Remark 5.7. If X and Y are infinite-dimensional Banach spaces, then the result does not hold for the set of all injective linear continuous operators.

Lemma 5.21. *Let $T \in \mathcal{L}(X, Y)$ be a linear continuous injective operator. Then*

$$\mathrm{Ran}(T) = \overline{\mathrm{Ran}(T)} \iff \exists\, c > 0 \ : \ \|Tx\| \geq c\|x\|.$$

Proof. First, assume that $\mathrm{Ran}(T)$ is closed. Then $(\mathrm{Ran}(T), \|\cdot\|_Y)$ is a Banach space, and the Open Mapping Theorem implies that

$$T^{-1} : \mathrm{Ran}(T) \longrightarrow X$$

is a continuous well-defined operator. Therefore

$$\|T^{-1}x\|_Y \leq c_1\|x\|_X \implies \|Tx\|_Y \geq \frac{1}{c_1}\|x\|_X.$$

Vice versa, notice that

$$\exists\, c > 0 \ : \ \|Tx\| \geq c\|x\| \implies T \text{ injective}.$$

Let $(y_n)_{n\in\mathbb{N}} \subset \mathrm{Ran}(T)$ be a sequence converging to some $y \in Y$. Then

$$\|y_n - y_m\|_Y = \|TT^{-1}y_n - TT^{-1}y_m\|_Y \gtrsim \|T^{-1}y_n - T^{-1}y_m\|_X$$

for all $n,\, m \in \mathbb{N}$. The sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy in Y, and thus the sequence $(T^{-1}y_n)_{n\in\mathbb{N}} \subset X$ is Cauchy in X, which means that

$$T^{-1}y_n \xrightarrow{n\to+\infty} x \in X.$$

In conclusion, notice that

$$T(x) = \lim_{n\to+\infty} T(T^{-1}(y_n)) = \lim_{n\to+\infty} y_n = y,$$

and this proves that $y \in \mathrm{Ran}(T)$. □

On the other hand, under the simple assumption that the images are closed, we can find an equivalent proposition for injective operators.

Proposition 5.22. *Let X and Y be Banach spaces. The subspace of injective continuous linear mappings from X to Y, with closed rank, is open.*

Proof. Let $L \in \mathcal{L}(X, Y)$ be an injective operator and assume that $\mathrm{Ran}(T)$ is closed (in norm). The previous lemma implies that we can always find a positive constant $c > 0$ such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in X,$$

and this is an open condition. Indeed, the triangular inequality implies that

$$\|(T + H)\,x\| \geq \|Tx\| - \|H\|\|x\| \geq (c - \|H\|)\,\|x\|,$$

and thus $T + H \in \mathcal{L}(X, Y)$ is injective and its rank is closed for every H whose norm is strictly less than c. □

Proposition 5.23. *Let X and Y be Banach spaces, and let $T \in \mathcal{L}(X, Y)$ be an operator such that the adjoint $T^* : Y^* \longrightarrow X^*$ satisfies the estimate*

$$\|T^* y^*\| \geq \|y^*\| \quad \textit{for all } y^* \in Y^*.$$

Then $T : X \longrightarrow Y$ is an open and surjective mapping, that is,

$$k \cdot B_Y \subseteq T\,(B_X)\,.$$

Proof. Let $t \in (0,\,1)$. By the Iteration Lemma 5.17, it suffices to prove that

$$B_Y \subseteq T\,(B_X) + t \cdot B_Y.$$

Suppose that there is $y_0 \in B_Y$ such that $y \notin T\,(B_X) + t \cdot B_Y$. Since $C := T\,(B_X) + t \cdot B_Y$ is an open convex set, the Hahn-Banach Theorem 3.6 implies that there exists $y^* \in Y^*$ such that

$$\langle y^*,\, Tx + t \cdot y\rangle < \langle y^*,\, y_0\rangle \quad \text{for all } x \in B_X \text{ and all } y \in B_Y.$$

If we take $y = y_0$, then it turns out that

$$(1-t) \cdot \langle y^*,\, y_0\rangle > \langle y^*,\, Tx\rangle = \langle T^* y^*,\, x\rangle \quad \text{for all } x \in B_X.$$

We take the supremum w.r.t. $x \in B_X$ and $y \in B_Y$ respectively, and we obtain a contradiction with the assumption on T^* since

$$(1-t) \cdot \|y^*\| \geq (1-t) \cdot \langle y^*,\, y_0\rangle \geq \|T^*\, y^*\| \geq \|y^*\|$$

for some $t \in (0,\,1)$, which is clearly impossible. □

Remark 5.8. The condition stated in the previous Proposition 5.23 is not only sufficient for the operator T to be surjective, but it is also necessary.

Corollary 5.24. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be a linear continuous operator. If T^* is injective and the rank* $\operatorname{Ran} T^*$ *is closed, then the map T is open and surjective.*

Annihilator. Let X be a Banach space and let M be a subspace of X. The *annihilator* of M is defined by

$$M^\perp = \{ f \in X^* \mid \langle f,\, x\rangle = 0 \text{ for all } x \in M \}\,.$$

Notice that $M^\perp$ is a weakly-$*$ closed subspace of the dual X^*, since it is equal to an intersection (possibly infinite) of closed sets:

$$M^\perp = \bigcap_{x \in M} \{ f \in X^* \mid \langle f,\, x\rangle = 0 \}\,.$$

In a similar fashion, given subspace N of X^*, we define the *pre-annihilator* of N as the set of points of x which are zero against each element of N, that is,

$$N_\perp = \{ x \in X \mid \langle f,\, x\rangle = 0 \text{ for all } f \in N \}\,.$$

Clearly, the pre-annihilator $N_\perp \subset X$ is closed w.r.t. the norm topology or, equivalently, w.r.t. the weak topology $\sigma(X,\, X^*)$ since it is convex (see Lemma 5.8). Indeed, we can always write it as the intersection of closed sets:

$$N_\perp = \bigcap_{f \in N} \operatorname{Ker} f.$$

Notice that $^{\perp}$ and $_{\perp}$ are both inclusion-inversing operation, that is,

$$A \subset B \subset X \implies A^{\perp} \supset B^{\perp} \quad \text{and} \quad A \subset B \subset X^* \implies A_{\perp} \supset B_{\perp}.$$

Remark 5.9. If N is a subspace of the dual X^*, the annihilator $N^{\perp}$ is a subset of the bidual X^{**}, while the pre-annihilator $N_{\perp}$ is a subset of X, which can be identified with its image via the canonical immersion $\imath : X \hookrightarrow X^{**}$. As a consequence, it turns out that

$$N_{\perp} = N^{\perp} \cap X = \imath^{-1}\left(N^{\perp}\right).$$

Lemma 5.25. *Let X be a Banach space, and let $M \subset X$ and $N \subset X^*$ be two subspaces.*

- ***(a)*** $M \subseteq \left(M^{\perp}\right)_{\perp}$.
- ***(b)*** $\left(M^{\perp}\right)_{\perp} = \overline{M}^{\|\cdot\|} = \overline{M}^{\tau_w}$.
- ***(c)*** $N \subseteq (N_{\perp})^{\perp}$.
- ***(d)*** $(N_{\perp})^{\perp} = \overline{N}^{\tau_w^*}$.

Proof.

- **(a)** This inclusion follows from the definitions.
- **(b)** By Lemma 5.8, it follows that $\left(M^{\perp}\right)_{\perp}$ is closed with respect to both the strong topology and the weak topology, and thus

$$\left(M^{\perp}\right)_{\perp} \supseteq \overline{M}^{\|\cdot\|} = \overline{M}^{\tau_w}.$$

  Let $x \notin \overline{M}^{\|\cdot\|}$. It follows from Hahn-Banach Theorem there exists a functional f such that

$$\langle f, x\rangle = 1 \quad \text{and} \quad f \in M^{\perp},$$

  and thus $x \notin \left(M^{\perp}\right)_{\perp}$, which is enough to infer that the opposite inclusion holds.
- **(c)** This inclusion follows from the definitions.
- **(d)** We noted above that $(N_{\perp})^{\perp}$ is closed with respect to both the weak-$*$ topology, and thus

$$(N_{\perp})^{\perp} \supseteq \overline{N}^{\tau_w^*}.$$

  Let $f \notin \overline{N}^{\tau_w^*}$. It follows from Hahn-Banach Theorem there exists a valuation j_x such that

$$\langle j_x, f\rangle = 1 \quad \text{and} \quad x \in N_{\perp},$$

  and thus $f \notin (N_{\perp})^{\perp}$, which is enough to infer that the opposite inclusion holds.

□

Lemma 5.26. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then the following inclusions hold true:*

(a) $\operatorname{Ker} T = (\operatorname{Ran} T^*)_\perp$.

(b) $\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp$.

(c) $\operatorname{Ran} T \subseteq (\operatorname{Ker} T^*)_\perp$.

(d) $\operatorname{Ran} T^* \subseteq (\operatorname{Ker} T)^\perp$.

(e) $\overline{\operatorname{Ran} T}^{\|\cdot\|} = \overline{\operatorname{Ran} T}^{\tau_w} = (\operatorname{Ker} T^*)_\perp$.

(f) $\overline{\operatorname{Ran} T^*}^{\tau_w^*} = (\operatorname{Ker} T)^\perp$.

Proof.

(a) Let $x \in \operatorname{Ker} T$. For every $f \in Y^*$, it turns out that

$$\langle f, Tx\rangle = 0 \implies \langle T^* f, x\rangle = 0,$$

that is, $x \in (\operatorname{Ran} T^*)_\perp$. Vice versa, let $x \in (\operatorname{Ran} T^*)_\perp$ and notice that

$$\langle f, Tx\rangle = 0 \impliedby \langle T^* f, x\rangle = 0$$

for all $f \in Y^*$, which means that $x \in \operatorname{Ker} T$.

(b) Let $f \in \operatorname{Ker} T^*$. For every $x \in X$, it turns out that

$$\langle T^* f, x\rangle = 0 \implies \langle f, Tx\rangle = 0,$$

that is, $x \in (\operatorname{Ran} T)^\perp$. Vice versa, let $f \in (\operatorname{Ran} T)^\perp$ and notice that

$$\langle T^* f, x\rangle = 0 \impliedby \langle f, Tx\rangle = 0,$$

for all $x \in X$, which means that $f \in \operatorname{Ker} T^*$.

(c) Let $y \in \operatorname{Ran} T$, and let $x \in X$ be a point such that $Tx = y$. Thus, for every $f \in Y^*$, it turns out that

$$\langle f, Tx\rangle = \langle f, y\rangle \implies \langle f, y\rangle = \langle T^* f, x\rangle$$

and this concludes the proof.

(d) This is similar to the previous point.

(e) This equality follows from the point **(c)** and Lemma 5.25.

(f) This equality follows from the point **(d)** and Lemma 5.25.

□

Lemma 5.27. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be an operator.*

(a) *The operator T is injective if and only if $\operatorname{Ran} T^*$ is weakly-$*$ dense.*

(b) *The operator T^* is injective if and only if $\operatorname{Ran} T$ is weakly/strongly dense.*

Proof.

(a) The operator T is injective if and only if $\operatorname{Ker} T = 0$ if and only if (Lemma 5.26) $(\operatorname{Ran} T^*)_\perp = 0$ if and only if the weak-$*$ closure of $\operatorname{Ran} T^*$ is the whole space Y^*, that is, the rank is τ_w^*-dense.

(b) The operator T^* is injective if and only if $\operatorname{Ker} T^* = 0$ if and only if (Lemma 5.26) $(\operatorname{Ran} T)^\perp = 0$ if and only if the weak closure (or norm closure) of $\operatorname{Ran} T$ is the whole space Y, that is, the rank is τ_w-dense (or dense in norm).

□

Theorem 5.28 (Closed Rank)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be a continuous linear operator. The following properties are equivalent:*

(1) *The rank* $\operatorname{Ran} T$ *is closed in norm.*

(2) *The rank* $\operatorname{Ran} T$ *is weakly closed.*

(3) *The rank* $\operatorname{Ran} T$ *is equal to* $(\operatorname{Ker} T^*)_\perp$*.*

(4) *The rank* $\operatorname{Ran} T^*$ *is closed in norm.*

(5) *The rank* $\operatorname{Ran} T^*$ *is weakly-$*$ closed.*

(6) *The rank* $\operatorname{Ran} T^*$ *is equal to* $(\operatorname{Ker} T)^\perp$*.*

Proof. Here we only prove the two nontrivial implications. The remaining ones follows easily from the theory we have developed so far.

"**(4)** $\implies$ **(1)**" Suppose that $\operatorname{Ran} T^*$ is closed w.r.t. the topology induced by the norm as a subset of X^*. Let $Z = \overline{\operatorname{Ran} T}^{\|\cdot\|}$ and consider the decomposition of the operator T as follows:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "S"'] & & Y \\ & Z \arrow[ur, hook, "j"'] & \end{tikzcd}$$

We may equivalently prove that S is a surjective operator. If we consider the dual diagram

$$\begin{tikzcd} X^* & & Y^* \arrow[ll, "T^*"'] \arrow[dl, "j^*"] \\ & Z^* \arrow[ul, "S^*"] & \end{tikzcd}$$

then it is easy to see that j^* is the restriction operator, which is surjective by the Hahn-Banach Theorem 3.1. Consequently, we have the equality

$$\operatorname{Ran} S^* = \operatorname{Ran} T^*,$$

which means that $\operatorname{Ran} S^*$ is closed w.r.t. the norm topology. In conclusion, since S^* is an injective operator with closed rank, it follows from Corollary 5.24 that S is surjective, and thus $Z = \operatorname{Ran} T$.

"(1) $\Longrightarrow$ (6)" Suppose that $\operatorname{Ran} T$ is closed w.r.t. the topology induced by the norm. We already know that the inclusion

$$\operatorname{Ran}(T^*) \subseteq (\operatorname{Ker}(T))^{\perp}$$

always holds. Therefore, it is enough to prove the opposite one. Let $f \in (\operatorname{Ker} T)^{\perp}$, and notice that

$$\operatorname{Ker} f \supset \operatorname{Ker} T.$$

The usual algebraic lemma proves that there exists $\widetilde{f} : \operatorname{Ran} T \longrightarrow \mathbb{K}$ such that the small triangle in the diagram below commutes:

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[rr, bend left, "T"] \arrow[d, "f"] & \mathrm{Ran}(T) \arrow[r, hook] \arrow[dl, "\tilde{f}"] & Y \arrow[dll, dotted, bend left, "F"] \\
\mathbb{K} & &
\end{tikzcd}$$

By Hahn-Banach Theorem 3.1 there is an extension F of $\widetilde{f}$ to the whole Y satisfying the identity $F \circ T = f$. Since $F \circ T = T^*(F)$, this implies that f belongs to the image of T^*, that is,

$$\operatorname{Ran} T^* \supseteq (\operatorname{Ker} T)^{\perp}.$$

□

5.6 Appendix

In this appendix, we introduce the notion of *filter* and, specifically, *ultrafilter* to give a relatively simple proof of the Tychonov Theorem, which is a result we used in the Banach-Alaoglu section.

5.6.1 Tychonov Theorem. Ultrafilter Approach

The main purpose of this section is to prove the Tychonov theorem using the basic definitions and the basic properties of **ultrafilters**.

Theorem 5.29 (Tychonov)**.** *The topological product $\prod_i X_i$ of an arbitrary collection of compact topological spaces X_i is compact.*

Recall that a set E is compact (in the Heine-Borel sense) if and only if any open covering $\mathcal{U}$ admits a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that

$$E \subseteq \bigcup_{V \in \mathcal{V}} V.$$

This notion of compactness can be easily reformulated in terms of closed coverings. Namely, a set $Y \subset X$ is compact if and only if every collection of closed set $\mathcal{F} = \{F_i := U_i^c\}_{i \in I}$ has the finite intersection property, that is,

$$\bigcap_{i \in J} F_i \neq \varnothing \quad \text{for all } J \subset I \text{ finite} \implies \bigcap_{i \in I} F_i \neq \varnothing.$$

Definition 5.30 (Filter)**.** Let $\mathfrak{X}$ be a set. A filter $\mathcal{F}$ on $\mathfrak{X}$ is a subset of $\mathcal{P}(\mathfrak{X})$ satisfying the following properties:

(a) If $F \in \mathcal{F}$, then F is nonempty.

(b) The collection is increasing, that is, if $F \in \mathcal{F}$ and $F' \supset F$, then $F' \in \mathcal{F}$ as well.

(c) The collection is closed under intersection, that is, for all $F, F' \in \mathcal{F}$, the intersection $F \cap F'$ still belongs to $\mathcal{F}$.

Example 5.2. We now present a short list of filters.

(a) If (X, τ) is a topological space and $x_0 \in X$ a point, then $\mathcal{F} = \mathcal{U}_{x_0}(X)$ is a filter.

(b) If $X = \mathbb{N}$, the family $\mathcal{F} = \{A \subset \mathbb{N} \mid A \text{ is finite}\}$ is a filter.

(c) If $A \subset X$ is nonempty, then the *principal filter* associated to A is defined by

$$\langle A \rangle := \{B \in \mathcal{P}(X) \mid B \supseteq A\}.$$

Definition 5.31 (Filter Basis)**.** Let $\mathfrak{X}$ be a set. A basis for a filter is a collection $\mathcal{B} \subset \mathcal{P}(\mathfrak{X})$ satisfying the following properties:

(a) If $B \in \mathcal{B}$, then B is nonempty.

(b) The collection is closed under intersection, that is, for all $B, B' \in \mathcal{B}$ there exists $B'' \in \mathcal{B}$ such that $B'' \subseteq B \cap B'$.

Furthermore, the filter generated by the basis $\mathcal{B}$ is defined by

$$\mathcal{F}_{\mathcal{B}} := \{A \in \mathcal{P}(\mathfrak{X}) \mid \text{there is } B \in \mathcal{B} \text{ such that } B \subseteq A\}.$$

Definition 5.32 (Coarser Filter)**.** Let $\mathfrak{X}$ be a set, and let $\mathcal{F}$ and $\mathcal{F}'$ be filters on $\mathfrak{X}$. We say that $\mathcal{F}$ is coarser than $\mathcal{F}'$ if $\mathcal{F} \subset \mathcal{F}'$.

Definition 5.33 (Coarser Basis)**.** Let $\mathfrak{X}$ be a set, and let $\mathcal{B}$ and $\mathcal{B}'$ be filter bases on $\mathfrak{X}$. We say that $\mathcal{B}$ is coarser than $\mathcal{B}'$ if $\mathcal{F}_{\mathcal{B}} \subset \mathcal{F}'_{\mathcal{B}'}$.

Remark 5.10. Let $\mathfrak{X}$ be a set, and let $\mathcal{F}$ and $\mathcal{F}'$ be filters on $\mathfrak{X}$. The *infimum* is defined as the maximum filter coarser than both, it always exists and it is equal to

$$\mathcal{F} \wedge \mathcal{F}' := \mathcal{F} \cap \mathcal{F}'.$$

In a similar fashion, the *supremum* is defined as the minimum filter finer than both, but it does not always exists. When it does, it is denoted by

$$\mathcal{F} \vee \mathcal{F}'.$$

Lemma 5.34. *Let $\mathfrak{X}$ be a set, and let $\mathcal{F}$ and $\mathcal{F}'$ be filters on $\mathfrak{X}$. The following properties are equivalent:*

(1) There exists a filter $\mathcal{G}$ finer than both.

(2) The supremum $\mathcal{F} \vee \mathcal{F}'$ exists.

(3) For every $A \in \mathcal{F}$ and every $A' \in \mathcal{F}'$, the intersection $A \cap A'$ is nonempty.

Proof. The chain of implications (**2**) $\implies$ (**1**) $\implies$ (**3**) is trivial. If (**3**) holds true, then the supremum may be directly defined as

$$\mathcal{F} \vee \mathcal{F}' = \{A \cap A' \mid A \in \mathcal{F},\ A' \in \mathcal{F}'\}.$$

□

Notice that the set of all the filters on $\mathfrak{X}$ is partially ordered by the inclusion $\subset$. Furthermore, it has the ascendant chain property, and thus (Zorn's lemma) each filter is contained in a maximal filter, called *ultrafilter.*

Lemma 5.35. *Let $\mathfrak{X}$ be a set. The filter $\mathcal{M}$ on $\mathfrak{X}$ is maximal if and only if for every $A \in \mathcal{P}(\mathfrak{X})$ either $A \in \mathcal{M}$ or $A^c \in \mathcal{M}$.*

Proof. We divide the proof into two steps.

" $\implies$ " Suppose that $\mathcal{M}$ is an ultrafilter. If A is any subset of $\mathfrak{X}$, then it turns out that either $A \cap F \neq \varnothing$ for any $F \in \mathcal{M}$ or there exists $F \in \mathcal{M}$ such that the intersection is trivial.

In the first case, the previous lemma shows that the supremum between $\mathcal{M}$ and $\langle A \rangle$ is well-defined, and it is finer than $\mathcal{M}$, which is absurd. In the second case, $A \cap F = \varnothing$ implies that $F \subset A^c$, and thus $A^c \in \mathcal{M}$.

" $\impliedby$ " Vice versa, suppose that for any $A \in \mathcal{P}(\mathfrak{X})$ either $A \in \mathcal{M}$ or $A^c \in \mathcal{M}$. If $\mathcal{F} \supset \mathcal{M}$, then each $F \in \mathcal{F}$ either belongs also to $\mathcal{M}$ (which is fine) or the complement belongs to $\mathcal{M}$ (which is absurd, since $F^c \in \mathcal{M} \subset \mathcal{F}$ implies that $\varnothing = F \cap F^c \in \mathcal{F}$). □

Image of a filter. Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a function and let $\mathcal{F}$ be a filter on $\mathfrak{X}$. The image of $\mathcal{F}$ through f is the filter $f(\mathcal{F})$ generated by the basis

$$\mathcal{B} = \{f(F) \mid F \in \mathcal{F}\}.$$

Lemma 5.36. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a function and let $\mathcal{F}$ be an ultrafilter on $\mathfrak{X}$. Then the image filter $f(\mathcal{F})$ is always an ultrafilter on $\mathfrak{Y}$.*

Topological Filters. We are finally ready to restate the main topological definitions in terms of filters and ultrafilters.

Definition 5.37 (Convergence)**.** Let $(\mathfrak{X}, \tau)$ be a topological space. A filter $\mathcal{F}$ is said to be *convergent* at $x \in \mathfrak{X}$ if and only if

$$\mathcal{F} \supset \mathcal{U}_x(\mathfrak{X}).$$

Definition 5.38 (Continuity)**.** Let $(\mathfrak{X}, \tau)$ and $(\mathfrak{Y}, \sigma)$ be topological spaces. A function $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is *continuous* at $x \in \mathfrak{X}$ if and only if

$$f(\mathcal{U}_x(\mathfrak{X})) \supset \mathcal{U}_{f(x)}(\mathfrak{Y}).$$

In particular, a sequence $f : \mathbb{N} \longrightarrow X$ is convergent, and its limit is $x_0 \in \mathfrak{X}$, if and only if the image filter of the Frechèt filter $\mathcal{F}$ on $\mathbb{N}$ is convergent at x_0.

Remark 5.11. Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a function, and let $\mathcal{F}$ be a filter on $\mathfrak{Y}$. If f is *surjective*, then

$$\left\{f^{-1}(F) \mid F \in \mathcal{F}\right\}$$

is a filter basis on $\mathfrak{X}$, and we denote by $f^{-1}(\mathcal{F})$ the generated filter.

Remark 5.12. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be sets, and let $\mathcal{F}$ and $\mathcal{G}$ be filters on $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a function, then

$$f^{-1}(\mathcal{F}) \subseteq \mathcal{G} \iff \mathcal{F} \subseteq f(\mathcal{G}).$$

Indeed, if $f^{-1}(\mathcal{F}) \subseteq \mathcal{G}$, then for every $F \in \mathcal{F}$ there is $G \in \mathcal{G}$ such that $G \subseteq f^{-1}(F)$. But this implies that $f(G) \subseteq F$, and thus $\mathcal{F} \subseteq f(\mathcal{G})$.

Lemma 5.39. *Let $(\mathfrak{X}_i, \tau_i)_{i \in I}$ be topological spaces. A filter $\mathcal{F}$ on $\mathfrak{X} := \prod_{i \in I} X_i$ converges at $x \in X$ if and only if for each $i \in I$ the projection $\pi_i(\mathcal{F})$ converges at $\pi_i(x)$.*

Proof. The filter of the neighborhoods of a point $x \in X$ can be expressed as the supremum of the filters $\pi_i^{-1}(\mathcal{U}_{x_i}(\mathfrak{X}_i))$, where $x_i = \pi_i(x)$, and

$$\pi_i^{-1}(\mathcal{U}_{x_i}(\mathfrak{X}_i)) = \text{Filter generated by } \left\{\prod_{j \neq i} \mathfrak{X}_j \times U_i \mid U_i \in \mathcal{U}_{x_i}(\mathfrak{X}_i)\right\}.$$

Therefore, a filter $\mathcal{F}$ on $\mathfrak{X} := \prod_{i \in I} \mathfrak{X}_i$ converges at $x \in \mathfrak{X}$ if and only if

$$\mathcal{F} \supseteq \mathcal{U}_x = \sup_{i \in I} \pi_i^{-1}(\mathcal{U}_{x_i}(\mathfrak{X}_i)),$$

which is equivalent to for any $i \in I$

$$\pi_i(\mathcal{F}) \supseteq \mathcal{U}_{x_i}(\mathfrak{X}_i) \quad \text{for all } i \in I,$$

also equivalent to $\pi_i(\mathcal{F})$ converges at $\pi_i(x) = x_i$ for all $i \in I$. □

Definition 5.40 (Adherent)**.** Let $(\mathfrak{X}, \tau)$ be a topological space and let $\mathcal{F}$ be a filter on $\mathfrak{X}$. A point $p \in \mathfrak{X}$ is *adherent* to the filter $\mathcal{F}$ if and only if

$$\forall\, U \in \mathcal{U}_p(X),\ \forall\, F \in \mathcal{F} \rightsquigarrow U \cap F \neq \varnothing.$$

Remark 5.13. Equivalently, the set of the adherent point to a filter $\mathcal{F}$ can be expressed as the intersection of all the closures of the elements of $\mathcal{F}$. More precisely, it turns out that

$$\{p \in \mathfrak{X} \mid p \text{ adherent to } \mathcal{F}\} = \bigcap_{F \in \mathcal{F}} \overline{F}.$$

Moreover, $p \in \mathfrak{X}$ is adherent to $\mathcal{F}$ if and only if there is a finer filter than both $\mathcal{U}_x(\mathfrak{X})$ and $\mathcal{F}$ if and only if there exists the supremum $\mathcal{F} \vee \mathcal{U}_x(\mathfrak{X})$ if and only if $\mathcal{F}$ admits a refinement convergent at p.

Remark 5.14. Let $(\mathfrak{X}, \tau)$ be a compact topological space. As we have already pointed out, it is equivalent to require that every family of closed sets with nonempty finite intersection, has nonempty intersection.

On the other hand, we can prove that $\mathfrak{X}$ is compact if and only if each filter on $\mathfrak{X}$ has adherent points (at least one) if and only if each ultrafilter is convergent if and only if each filter has adherent points and admits a convergent refinement.

Proof of Tychonov Theorem. Let $\mathcal{F}$ be an ultrafilter on the product $X := \prod_{i \in I} X_i$. Then for all $i \in I$ the image $\pi_i(\mathcal{F})$ is an ultrafilter on X_i and thus, by compactness, it is also convergent. $\square$

Remark 5.15. The Thychonov theorem requires the choice axiom to be proved, but it turns out that it is actually equivalent. More precisely, if we assume the Thychonov theorem to be true, then it turns out that for any family $(X_i)_{i \in I}$ of nonempty set, the product $\prod_{i \in I X_i}$ is nonempty.

5.7 Exercises

Definition 5.41 (σ-convexity)**.** The set $C \subseteq X$ is σ-convex if for any bounded sequence $(x_k)_{k \in \mathbb{N}} \subset C$ and any sequence $(\lambda_k)_{k \in \mathbb{N}}$ of positive numbers such that $\sum_k \lambda_k = 1$, it turns out that

$$\sum_{k \in \mathbb{N}} \lambda_k \cdot x_k \in C.$$

Exercise 5.1. Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$.

(a) If C is convex and closed, then C is σ-convex.

(b) If C is convex and open, then C is σ-convex.

(c) If C is σ-convex and bounded, then $T(C)$ is also σ-convex.

(d) If C is σ-convex, then also the translation and the preimage are σ-convexes.

(e) If C is σ-convex and bounded, then for any $t \in (0, 1)$ the inclusion $E \subseteq C + t \cdot E$ implies that $(1 - t) \cdot E \subseteq C$.

Moreover, this property characterize σ-convex sets in a Banach space.

Part II

Operators and Functional Calculus

Chapter 6

Compact Operators on Banach Spaces

In this chapter, we are mainly concerned with *compact operators* and *symmetric (compact) operators* between Banach spaces.

In the first half, we study the ideal of compact operators and the spectral theory (e.g., the Fredholm alternative theorem), which gives us an excellent estimate of the spectral radius.

In the second half, we investigate the fundamental properties of compact symmetric operators, for which the spectral theory gives us more precise pieces of information.

6.1 Main Definitions and Elementary Properties

Definition 6.1 (Compact Operator)**.** Let X and Y be two Banach spaces. An operator $T : X \longrightarrow Y$ is *compact* if and only if the following properties hold:

(a) The operator T is continuous with respect to the strong topologies.

(b) The image $T(E)$ of every bounded subset $E \subseteq X$ is relatively compact[1] in Y.

Remark 6.1. If the operator $T : X \longrightarrow Y$ is linear and continuous, we do not need to check it maps every bounded set in a relatively compact one. Indeed, it is enough to verify whether or not the image of the unit ball is relatively compact, that is, if $\overline{T(B_X)}$ is compact in Y.

Lemma 6.2. *Let $T \in \mathcal{L}(X, Y)$ be a linear continuous operator between Banach spaces. Then T is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset B_X(0, c)$ there exists an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$\exists\, y \in Y \,:\, \lim_{k\to+\infty} \|Tx_{n_k} - y\|_Y = 0.$$

[1] Let $\mathfrak{X}$ be a topological space. A subset $\mathfrak{Y} \subset \mathfrak{X}$ is relatively compact if the topological closure $\overline{\mathfrak{Y}}$ is compact in $\mathfrak{X}$.

The next goal is to give two compactness criteria, easy to check, that hold in reflexive ($=X \cong X^{**}$) Banach spaces. Before we can do that, we need a characterization of the reflexivity in terms of the compactness of the ball with respect to a suitable topology.

Lemma 6.3. *A Banach space $\mathfrak{X}$ is reflexive if and only if the closed unit ball $\overline{B_{\mathfrak{X}}(0,\,1)}$ is weakly compact.*

Proof. First, we notice that the closed unit ball $\overline{B_{\mathfrak{X}}(0,\,1)}$ is weakly compact if $\mathfrak{X}$ is a reflexive Banach space, as a consequence of the Banach-Alaoglu Theorem.

Vice versa, if B_X is weakly compact, then it is weakly-$*$ closed in X^{**}. Since the weakly-$*$ closure of B_X in X^{**} is $B_{X^{**}}$, and we get

$$B_X = B_{X^{**}} \implies X \text{ is reflexive.}$$

□

Lemma 6.4. *Let X be a reflexive (separable) Banach space, and let M be a closed linear subspace of X. Then $(M,\, \|\cdot\|_M)$ is also a reflexive (separable) Banach space, where*

$$\|\cdot\|_M = \|\cdot\|\big|_M.$$

First Proof. Recall that X is reflexive if and only if the closed unit ball is compact in the weak topology, as a consequence of Lemma 6.3.

Therefore, it is enough to notice that the closed unit ball of M is the intersection of the unit ball of X with the closed subspace M, that is,

$$\overline{B_M} = \overline{B_X} \cap M \implies \overline{B_M} \text{ is closed.}$$

□

Second Proof. Let $j_M : M \hookrightarrow X$ be the inclusion map. The dual operator

$$j_M^* : X^* \twoheadrightarrow M^*$$

is surjective as a consequence of the Hahn-Banach Theorem 3.1, since it is nothing more than the restriction map:

$$j_M^* : X^* \ni f \longmapsto f\big|_M \in M^*.$$

The dual of this operator, which is given by the bidual of the inclusion

$$j_M^{**} : M^{**} \hookrightarrow X^{**},$$

is injective, and its image is weakly-$*$ closed[2], that is, the rank is $\sigma(X^{**},\, X^*)$-closed. On

[2]The rank $\mathrm{Ran} j_M^*$ is weakly closed because j_M^* is surjective and M is closed w.r.t. the subspace topology. It follows from the Closed Rank Theorem 5.28 that the rank of dual operator j_M^{**} is weakly-$*$ closed.

the other hand, it can be easily identified with the closure of M by noticing that

$$\begin{aligned} M^* \cong {X^*}/{M^\perp} \implies M^{**} &\cong \left({X^*}/{M^{\perp,\,\sigma(X,\,X^*)}}\right)^* \cong \\ &\cong \left(M^{\perp,\,\sigma(X,\,X^*)}\right)^{\perp,\,\sigma(X^*,\,X^{**})} \cong \\ &\cong \left(M_{\perp,\,\sigma(X^{**},\,X^*)}\right)^{\perp,\,\sigma(X^*,\,X^{**})} \cong \\ &\cong \overline{M}^{\sigma(X^{**},\,X^*)}. \end{aligned}$$

If X is a reflexive Banach space, it turns out that $\sigma(X^{**},\,X^*) = \sigma(X,\,X^*)$ is the weak topology, and therefore

$$M^{**} \cong \overline{M}^{\sigma(X^{**},\,X^*)} \cong \overline{M}^{\tau_W} = M$$

since we assumed M to be a closed subspace of X. □

Lemma 6.5. *Let X be a Banach space. Every weakly converging sequence in X is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a weakly convergent sequence in X, and let $T_n \in X^{**}$ be defined by

$$T_n(\ell) := \ell(x_n) \quad \text{for all } \ell \in X^* \text{ and } n \in \mathbb{N}.$$

If we consider $\Gamma := \{T_n \ : \ n \in \mathbb{N}\}$, then the weak convergence of x_n implies that Γ is a pointwise bounded family in a second-category set. It follows from the Banach-Steinhaus Theorem that

$$\sup_{n\in\mathbb{N}} \|x_n\|_X = \sup_{n\in\mathbb{N}} \|T_n\|_{X^{**}} < +\infty,$$

which means that $(x_n)_{n\in\mathbb{N}} \subset X$ is bounded. □

Lemma 6.6. *Let X be a Banach space, and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. If every subsequence $(n_k)_{k\in\mathbb{N}}$ admits a sub-subsequence $(n_{k_\ell})_{\ell\in\mathbb{N}}$ such that*

$$\lim_{\ell\to+\infty} \|x_{n_{k_\ell}}\|_X = 0,$$

then the whole sequence converges strongly to 0, that is,

$$\lim_{n\to+\infty} \|x_n\|_X = 0.$$

Proof. We argue by contradiction. Fix $\epsilon > 0$. We claim that for every $k \in \mathbb{N}$, there exists $n_k > k$ such that

$$\|x_{n_k}\| \geq \epsilon. \tag{6.1}$$

If the claim holds, then it suffices to notice that the subsequence x_{n_k} does not admit any converging sub-subsequence, which yields to a contradiction.

To prove that the claim (6.1) holds true, notice that, if for some $k \in \mathbb{N}$ there is no bigger index $n_k > k$ such that $\|x_{n_k}\| \geq \epsilon$, then

$$\|x_m\| \leq \epsilon \quad \text{for all } m \geq k,$$

and therefore the sequence x_n converges strongly to 0, in contradiction with what we have assumed. □

Lemma 6.7. *Let X be a reflexive Banach space. A linear continuous operator $T \in \mathcal{L}(X, Y)$ is compact if and only if every sequence $(x_n)_{n\in\mathbb{N}} \subset X$, weakly converging to 0, has the property that the image $(Tx_n)_{n\in\mathbb{N}} \subset Y$ converges strongly (=w.r.t the norm $\|\cdot\|_Y$) to 0 in Y.*

Proof. We divide the argument into two steps to ease the notation.

" $\implies$ " Suppose that $T : X \longrightarrow Y$ is compact and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence such that

$$x_n \rightharpoonup 0.$$

The set $\{x_n \mid n \in \mathbb{N}\}$ is bounded in X, as a consequence of Lemma 6.5, and thus (since T is compact) there exists an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$Tx_{n_k} \xrightarrow{\|\cdot\|_Y} y \in Y.$$

On the other hand, the operator T is continuous; hence $Tx_n \rightharpoonup 0$ since

$$\langle \varphi, Tx_n\rangle = \langle T^*\varphi, x_n\rangle \xrightarrow{n\to+\infty} 0.$$

The uniqueness of the limit immediately implies that $y = 0$, and thus the usual argument based on the behavior of the sub-subsequences (see Lemma 6.6) proves the whole sequence $(T\,x_n)_{n\in\mathbb{N}}$ converges strongly to 0.

" $\impliedby$ " Vice versa, let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence weakly converging to 0, and let

$$X_0 := \overline{\text{Span}\,\langle x_n \ : \ n \in \mathbb{N}\rangle}$$

be the closure of the linear span generated by the sequence.

By Lemma 6.4, it follows that X_0 is a separable reflexive Banach space, and thus, by Banach-Alaoglu Theorem 5.13, the unit ball of X_0^* is sequentially weakly-$*$ compact. On the other hand, since X_0 is a reflexive space, the weak-$*$ topology on X_0^* is equal to the weak topology on X_0, and hence the unit ball of X_0 is weakly compact, that is, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$x_{n_k} \rightharpoonup x_\infty \in X.$$

The sequence $(x_{n_k} - x_\infty)_{k\in\mathbb{N}}$ converges weakly to 0. Therefore, by assumption, it follows immediately that

$$\|Tx_{n_k} - Tx_\infty\| \xrightarrow{n\to+\infty} 0$$

in Y, which is enough to infer that the set $\{Tx_n \mid n \in \mathbb{N}\}$ is relatively compact with respect to the strong topology in Y. □

Lemma 6.8. *Let X and Y be reflexive Banach spaces. A linear and continuous operator $T : X \longrightarrow Y$ is compact if and only if for any sequence $(x_n)_{n\in N} \subset X$ weakly converging to 0 and any sequence $(y_n^*)_{n\in\mathbb{N}} \subset Y^*$ weakly-$*$ converging to 0, it turns out that*

$$\langle y_n^*, Tx_n\rangle \xrightarrow{n\to+\infty} 0.$$

6.1.1 Ideals of Finite-Rank Operators and Compact Operators

In this section, we prove that compact operators give rise to an ideal and we investigate some of its main properties (e.g., if it is closed), included the relation with finite-rank operators.

Let X, Y be Banach spaces. We denote by $\mathcal{L}_c(X, Y)$ the subset of the linear and continuous operators which are also compact.

Definition 6.9 (Totally Bounded)**.** A metric space (M, d) is *totally bounded* if and only if for every $\epsilon > 0$ there is a finite collection of open balls $B(x_1, \epsilon), \dots, B(x_N, \epsilon)$ of radius ϵ covering the space, that is,

$$M \subseteq \bigcup_{i=1}^{N} B\left(x_i, \epsilon\right).$$

Lemma 6.10. *Let (M, d) be a complete metric space. A subset $Y \subseteq M$ is totally bounded if and only if Y is relatively compact.*

Lemma 6.11 (Ideal of Compact Operators)**.**

(1) *The set of linear continuous compact operators $\mathcal{L}_c(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.*

(2) *Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ be two operators. If either of them is compact, then the composition $S \circ T$ is also compact.*

(3) *The subspace $\mathcal{L}_c(X)$ is a closed bilateral ideal in the operator algebra with respect to the composition.*

(4) *The subspace $\mathcal{L}_f(X)$ of finite-dimensional rank operators is an ideal in the operator algebra, but it is, in general, not closed. Moreover, it is the minimal nonzero ideal.*

Proof.

(1) Let $T \in \overline{\mathcal{L}_c(X, Y)}$ and let $S \in \mathcal{L}_c(X, Y)$ be a compact operator. The inclusion

$$T\left(B_X\right) \subseteq S\left(B_X\right) + \|T - S\| \cdot B_Y \tag{6.2}$$

holds as a consequence of the triangular inequality. For any $\epsilon > 0$ there exist $S_\epsilon \in \mathcal{L}_c(X, Y)$ such that $\|T - S_\epsilon\| < \epsilon/2$, and a finite[3] subset $F \subseteq S\left(B_X\right)$ such that

$$S_\epsilon\left(B_X\right) \subseteq F + \frac{\epsilon}{2} \cdot B_Y.$$

Therefore, the inclusion (6.2) may be rewritten as follows

$$T\left(B_X\right) \subseteq F + \epsilon \cdot B_Y,$$

and this is enough to conclude that $T\left(B_X\right)$ is totally bounded. Since Y is a complete space, we infer from Lemma 6.10 that $T\left(B_X\right)$ is also relatively compact, i.e., T is a compact operator.

[3]The operator S_ϵ is compact and Y is a Banach (complete) space. It follows that the image of the unit ball B_X (a bounded set) is relatively compact in Y, and hence also totally bounded.

(2) This assertion is a straightforward consequence of the fact that any linear continuous operator sends bounded (resp. relatively compact) sets in bounded (resp. relatively compact) sets.

(3) The subspace $\mathcal{L}_c(X)$ is closed by **(1)**, and it is a bilateral ideal by **(2)**.

(4) Let $\mathcal{I}$ be a nonzero ideal of $\mathcal{L}(X)$, and let $T \neq 0$ be an element of $\mathcal{I}$. Let $x_0 \in X$ be any point such that $Tx_0 \neq 0$, and let $\alpha,\, \beta \in X^*$ be linear continuous forms.

Recall that, for every fixed $y \in X$, the operators of the form

$$x \longmapsto y\,\langle \alpha,\, x\rangle$$

have rank equal to 1. Therefore, if we consider the composition with T (which is still an element of $\mathcal{I}$ by assumption), then for $y_0 \in X$ fixed it turns out that

$$[y_0\langle \beta,\, \cdot\rangle] \circ T \circ [x_0\langle \alpha,\, \cdot\rangle] \in \mathcal{I}.$$

If we let $\langle \beta,\, Tx_0\rangle = 1$, then the composition above is given by

$$x \longmapsto y_0\langle \alpha,\, x\rangle,$$

and thus $\mathcal{I}$ contains all the operators with rank of dimension 1.

□

In particular, the minimal closed ideal in the operator algebra is $\overline{\mathcal{L}_f(X)}$. It has been for long an open problem to determine whether it is equal to $\mathcal{L}_c(X)$ or if the inclusion can be strict.

In the $'73$ the mathematician Enflo showed that the equality could be strict in a particular Banach space [2].

Lemma 6.12. *Let X and Y be normed spaces, and let $T \in \mathcal{L}_c(X,\, Y)$ be a compact operator. Then*

$$T(X) \subset Y \text{ is a separable Banach space.}$$

Proof. Note that

$$X = \bigcup_{n\in\mathbb{N}} B(0,\, n),$$

and every ball $B(0,\, n)$ is a bounded subset of X, which means that its image via T is relatively compact in Y. On the other hand, a relatively compact set is always separable and

$$T(X) = \bigcup_{n\in\mathbb{N}} T\left(B(0,\, n)\right)$$

is a countable union of separable set, meaning that it is also separable. □

Lemma 6.13. *Let X be a Hilbert space. Then the closure of the finite-rank operators is the ideal of the compact operators, i.e.,*

$$\mathcal{L}_c(X) = \overline{\mathcal{L}_f(X)}.$$

Proof. First, we notice that every finite-rank operator is compact. Since the ideal $\mathcal{L}_c(X)$ is closed, the first inclusion follows:

$$\mathcal{L}_c(X) \supseteq \overline{\mathcal{L}_f(X)}.$$

Let $T : X \longrightarrow X$ be a compact operator. The closure of the image $\overline{T(X)}$ is a separable Hilbert space (see Lemma 6.12). Thus, we can always consider a sequence of projections $(P_n)_{n\in\mathbb{N}}$ over the n-dimensional subspaces

$$P_n : X \longrightarrow \operatorname{Span}\langle e_1, \dots, e_n\rangle,$$

in such a way that

$$\overline{\bigcup_{n\in\mathbb{N}} P_n(X)} = \overline{T(X)} \cong \ell_2.$$

The convergence criterion (see Lemma 6.14 below) implies that $P_n T \to T$ strongly (in norm), which means that

$$\lim_{n\to+\infty} \|P_n T - T\|_{X^*} = 0.$$

On the other hand, by construction $P_n T$ is a finite-rank operator for all $n \in \mathbb{N}$. Therefore, we just proved that T is the limit of a sequence of finite-rank operators with respect to the operator norm, which means that

$$\mathcal{L}_c(X) \subseteq \overline{\mathcal{L}_f(X)}.$$

□

Lemma 6.14. *Let H be a Hilbert space and let $(x_n)_{n\in N} \subset H$ be a sequence. Then $x_n \to x \in H$ strongly if and only if $x_n \rightharpoonup x$ weakly and $\|x_n\| \to \|x\|$.*

Proof. Suppose that $x_n \to x$ strongly (in norm). The weak topology is coarser than the topology induced by the norm; therefore strong convergence always implies weak convergence.

Moreover, the norm $\|\cdot\|_H$ is a continuous function with respect to the norm topology as a consequence of the reversed triangular inequality:

$$|\|x_n\| - \|x\|| \le \|x_n - x\| \xrightarrow{n\to+\infty} 0.$$

Vice versa, suppose that $x_n \rightharpoonup x$ weakly and that $\|x_n\| \to \|x\|$. By definition we have that

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \left(\|x_n\|^2 + \|x\|^2\right) - 2\langle x_n, x\rangle,$$

and it is easy to see that the first term converges to $2\|x\|^2$, while the second converges to $-2\|x\|^2$. □

Lemma 6.15. *Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be a continuous linear operator. If T is not compact, then there exists an infinite-dimensional closed subspace $M \subset H$ such that*

$$T\big|_M : M \xrightarrow{\sim} T(M) \text{ is an isomorphism}$$

and $T(M) \subset H$ is a closed infinite-dimensional subset of H.

Proposition 6.16. *Let H be a separable Hilbert space. The compact operators form the unique bilateral closed ideal in the operators algebra.*

Proof. Let $\mathcal{I}$ be a nonempty bilateral closed ideal in the operator algebra $\mathcal{L}(H)$, and suppose that there exists $T \in \mathcal{I}$ that is not compact.

By Lemma 6.15 there exists an infinite-dimensional closed subspace $M \subset H$ such that $T\big|_M$ is invertible onto its image. On the other hand, since M and $T(M)$ are both infinite-dimensional spaces, it turns out that

$$H \text{ separable } \implies M \cong T(M) \cong H.$$

Consequently, the operator given by the composition

$$S : H \xrightarrow{\sim} M \xrightarrow{T} T(M) \xrightarrow{\sim} H$$

belongs to $\mathcal{I}$, and this concludes the proof since S is invertible and $S \in \mathcal{I}$ means that $\mathcal{I}$ is equal to the whole operator algebra. □

6.1.2 Schauder Theorem

Theorem 6.17 (Schauder)**.** *Let X, Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is compact if and only if $T^* \in \mathcal{L}(Y^*, X^*)$ is compact.*

Proof. We divide the argument into three steps to ease the notation.

Step 1. Suppose that T is a compact operator and let $(y_n^*)_{n\in\mathbb{N}} \subset Y^*$ be a bounded sequence, that is, there exists $C > 0$ such that

$$\|y_n^*\| \leq C.$$

Consider the sequence defined by the restrictions on the closure of the image of the unit ball

$$(z_n^*)_{n\in\mathbb{N}} := \left(y_n^* \,|_{\overline{T(B_X)}}\right)_{n\in\mathbb{N}},$$

and notice that z_n^* is a continuous function defined on the compact metric space $\overline{T(B_X)}$ (since B_X is a bounded set in X and T is compact).

Step 2. The sequence $(z_n^*)_{n\in\mathbb{N}}$ is clearly equibounded and equi-Lipschitz continuous with constant $0 < L \leq C$. It follows from the Ascoli-Arzelà theorem that one can find an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$z_{n_k}^* \rightrightarrows z \in \left(\overline{T(B_X)}\right)^*.$$

In particular, it is a Cauchy sequence with respect to the uniform norm (since the convergence is uniform) on the closure of the image of the unit ball, and therefore

$$\begin{aligned}
\left\|y_{n_k}^* - y_{n_j}^*\right\|_{\infty,\, \overline{T(B_X)}} &= \left\|y_{n_k}^* - y_{n_j}^*\right\|_{\infty,\, T(B_X)} = \\
&= \left\|y_{n_k}^* \circ T - y_{n_j}^* \circ T\right\|_{\infty,\, B_X} = \\
&= \left\|T^* y_{n_k}^* - T^* y_{n_j}^*\right\|_{\infty,\, B_X} = \left\|T^* y_{n_k}^* - T^* y_{n_j}^*\right\|_{X^*}.
\end{aligned}$$

In conclusion, we infer that $\left(T^* y^*_{n_k}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in X^*, which is complete, and thus it converges.

Step 3. Vice versa, assume that the dual operator T^* is compact. The previous steps prove that the bidual operator T^{**} is compact, that is,

$$T^{**}(B_X) \text{ has compact closure in } Y^{**}.$$

Since $T(B_X) = T^{**}(B_X)$ and Y is closed in Y^{**}, we conclude that $T(B_X)$ has compact closure in Y, which is exactly what we wanted to prove. □

6.2 Riesz-Fredholm Spectral Theory

In this section, we state and prove the so-called *Fredholm alternative theorem* for compact operators, which asserts that compact perturbation of the identity behaves in a very controlled manner.

Theorem 6.18 (Fredholm alternative)**.** *Let $T \in \mathcal{L}_c(X)$ be a compact operator and let X be a Banach space.*

- ***(a)*** *The kernel* $\mathrm{Ker}(\mathrm{id_X} - T)$ *is a closed, T-invariant and finite-dimensional subspace of X.*
- ***(b)*** *The kernels $K_n := \left(\mathrm{Ker}\,(\mathrm{id_X} - T)^n\right)_{n\in\mathbb{N}}$ form an increasing and stable sequence of closed, T-invariant and finite-dimensional subspaces of X.*
- ***(c)*** *The rank* $\mathrm{Ran}(\mathrm{id_X} - T)$ *is a closed, T-invariant and finite-codimensional subspace of X.*
- ***(d)*** *The ranks $R_n := \left(\mathrm{Ran}\,(\mathrm{id_X} - T)^n\right)_{n\in\mathbb{N}}$ form a decreasing and stable sequence of closed, T-invariant and finite-codimensional subspaces of X.*

Proof.

- **(a)** The kernel $\mathrm{Ker}(\mathrm{id_X} - T)$ is clearly closed, while the T-invariance follows from a simple computation:

$$x \in \mathrm{Ker}(\mathrm{id_X} - T) \implies x - Tx = 0 \implies Tx \in \mathrm{Ker}(\mathrm{id_X} - T).$$

Therefore, the restriction operator

$$T\Big|_{K_1} : \mathrm{Ker}(\mathrm{id_X} - T) \longrightarrow \mathrm{Ker}(\mathrm{id_X} - T)$$

is compact and equal to the identity ($x = Tx$). On the other hand, if $\mathcal{B}$ is a Banach space, then the identity $\mathrm{id}_{\mathcal{B}}$ is compact if and only if $\mathcal{B}$ is locally compact if and only if $\mathcal{B}$ is a finite-dimensional space.

(b) If we expand the binomial $(\mathrm{id}_X - T)^n$ using the Newton formula, we obtain an operator of the form $\mathrm{id}_X - T'$, for some $T' \in \mathcal{L}_c(X)$. In particular, the sequence $(K_n)_{n\in\mathbb{N}}$ is increasing and made up of closed, T-invariant and finite-dimensional subspaces.

We only need to prove that the sequence is stable, that is, there exists $n_0 \in \mathbb{N}$ such that

$$K_{n_0} = K_m \quad \text{for all } m \geq n_0.$$

We argue by contradiction. Namely, we suppose that there exists a strictly increasing subsequence $(n_j)_{j\in\mathbb{N}}$ such that $K_{n_j} \subset K_{n_{j+1}}$ is a strict inclusion. First, we claim that for all $j \in \mathbb{N}$ there exists a point $x_{n_j} \in K_{n_j}$ such that

$$\|x_{n_j}\| \leq 2 \quad \text{and} \quad d\left(x_{n_j},\, x_{n_{j-1}}\right) = 1. \tag{6.3}$$

Proof Claim. Let $K \subset L$ be closed subspaces of a Banach space $\mathcal{B}$. Then there exists a point $x \in L \setminus K$ such that

$$d\,(x,\, K) = 1.$$

Since the distance between a point and a set is given by the infimum of the distances, there must be a point $k \in K$ such that $d(x,\, k) \leq 2$. Hence the point $u := x - k$ belongs to L and it turns out that

$$\|u\| = d(x,\, k) \leq 2 \quad \text{and} \quad d\,(u,\, K) = d(x,\, K) = 1.$$

Back to the proof of **(b)**, for all $i < j$ we have that

$$\left\|Tx_{n_j} - Tx_{n_i}\right\| = \| \underbrace{x_{n_j}}_{\in K_{n_j}} - (\underbrace{x_{n_j} - Tx_{n_j}}_{\in K_{n_{j-1}}} + \underbrace{Tx_{n_i}}_{\in K_{n_i} \subset K_{n_{j-1}}})\| \geq 1,$$

since $\mathrm{id}_X - T : K_n \longrightarrow K_{n-1}$ and $d(x_{n_j},\, K_{n_{j-1}}) = 1$.

This is in contradiction with the assumption that T is compact since the sequence $\left(Tx_{n_j}\right)_{j\in\mathbb{N}}$ does not admit any converging subsequence.

(c) The kernel $\mathrm{Ker}(\mathrm{id}_X - T)$ is a finite-dimensional closed subspace of a Banach space. Therefore, there exists $X_0 \subseteq X$ such that

$$X_0 \text{ is closed and } X = X_0 \oplus \mathrm{Ker}(\mathrm{id}_X - T) \text{ topological direct sum},$$

that is, the application $+ : X_0 \times K_1 \longrightarrow X$ is linearly invertible.

N.B. If X is a Banach space and $K \subset X$ is a closed infinite-dimensional subspace of X, then the existence of such a decomposition is, in general, not certain.

On the other hand, if K and the algebraic supplement K' are closed, then it holds true (e.g, as a consequence of the Open Mapping Theorem 4.20).

Back to the proof of **(c)**, notice that the restriction $(\mathrm{id}_X - T)\big|_{X_0}$ is injective and it has the same rank. Therefore, the operator

$$(\mathrm{id}_X - T)\big|_{X_0} : X_0 \longrightarrow \mathrm{Ran}(\mathrm{id}_X - T)$$

is bijective, linear and continuous. If we can prove that $\mathrm{Ran}(\mathrm{id_X} - T)$ is complete, then we will be able to conclude that it is also closed. Clearly, it suffices to prove that

$$(\mathrm{id_X} - T)^{-1} : \mathrm{Ran}(\mathrm{id_X} - T) \longrightarrow X_0$$

is continuous. We argue by contradiction. If the restriction is not a homeomorphism, then there is a sequence $(x_k)_{k\in\mathbb{N}} \subset X_0$ of elements, whose norm is uniformly equal to 1, such that

$$\|x_k - Tx_k\| \xrightarrow{k\to+\infty} 0.$$

Since T is a compact operator, up to subsequences, it turns out that $Tx_k \to \xi$. It follows from the relation above that also $x_k \to \xi \in X_0$, and the norm of ξ is also equal to 1. This yields to a contradiction since

$$\xi \in \partial B_X(0,\,1) \cap X_0 \cap \mathrm{Ker}(\mathrm{id_X} - T) = \partial B_X(0,\,1) \cap \{0\} = \varnothing.$$

(d) If we expand the binomial $(\mathrm{id_X} - T)^n$, we obtain an operator of the form $\mathrm{id_X} - T'$ for a certain $T' \in \mathcal{L}_c(X)$. In particular, the sequence $(R_n)_{n\in\mathbb{N}}$ is decreasing and made up of closed, T-invariant and finite-codimensional (see Theorem 6.20) subspaces.

We only need to prove that the sequence is stable, that is, there exists $n_0 \in \mathbb{N}$ such that

$$R_{n_0} = R_m \quad \text{for all } m \geq n_0.$$

We argue by contradiction. Namely, we suppose that there exists a strictly increasing subsequence $(n_j)_{j\in\mathbb{N}}$ such that $R_{n_j} \supset R_{n_{j+1}}$ is a strict inclusion. First, we claim that for every $j \in \mathbb{N}$ there is a point $x_{n_j} \in R_{n_j}$ such that

$$\|x_{n_j}\| \leq 2 \qquad \text{and} \qquad d\left(x_{n_j},\, x_{n_{j+1}}\right) = 1. \tag{6.4}$$

For every $i < j$, it turns out that

$$\left\|Tx_{n_j} - Tx_{n_i}\right\| = \| \underbrace{x_{n_i}}_{\in R_{n_i}} - (\underbrace{x_{n_i} - Tx_{n_i}}_{\in R_{n_{i+1}}} + \underbrace{Tx_{n_j}}_{\in R_{n_j} \subset R_{n_{i+1}}})\| \geq 1,$$

since $\mathrm{id_X} - T : R_n \longrightarrow R_{n+1}$ and $d(x_{n_i},\, R_{n_{i+1}}) = 1$.

This is in contradiction with the assumption that T is compact since the sequence $(Tx_{n_i})_{i\in\mathbb{N}}$ does not admit any converging subsequence.

□

Lemma 6.19. *Let $T \in \mathcal{L}_c(X)$ be a compact operator. Then*

$$\mathrm{id_X} - T \textit{ is injective if and only if } \mathrm{id_X} - T \textit{ is surjective.}$$

Proof.

" $\implies$ " Suppose that $\mathrm{id_X} - T$ is injective, but not surjective, and let $y \in X$ be a point that does not belongs to the image, i.e., $y \notin \mathrm{Ran}(\mathrm{id_X} - T)$. Then it is easy to prove that

$$(\mathrm{id_X} - T)^n y \in R_n \setminus R_{n+1} \implies \text{the sequence } (R_n)_{n\in\mathbb{N}} \text{ is not stable,}$$

and this is absurd, as a consequence of the Fredholm Alternative Theorem 6.18.

" $\Longleftarrow$ " Suppose now that $\mathrm{id}_X - T$ is surjective, but not injective, and let $y \in X$ be a nonzero point of the kernel, i.e., $y \neq 0$ and $(\mathrm{id}_X - T)\, y = 0$. For every $n \in \mathbb{N}$ there exists $y_n \in X$ such that

$$(\mathrm{id}_X - T)^n\, y_n = y \neq 0 \quad \text{and} \quad (\mathrm{id}_X - T)^{n+1}\, y_n = 0.$$

It follows that the sequence $(K_n)_{n \in \mathbb{N}}$ is not stable, and this is absurd for the same reason mentioned above. ☐

Theorem 6.20. *Let $T \in \mathcal{L}_c(X)$ be a compact operator. Then*

$$\dim K_n = \operatorname{codim} R_n = \operatorname{codim} R_n^* = \dim K_n^*,$$

where

$$K_n^* = \operatorname{Ker}(\mathrm{id}_X - T^*)^n \quad \textit{and} \quad R_n^* = \operatorname{Ran}(\mathrm{id}_X - T^*)^n.$$

Proof. Let us consider the projection

$$P : X \longrightarrow K_1$$

onto the finite-dimensional subspace $\operatorname{Ker}(\mathrm{id}_X - T)$, corresponding to the direct sum $X = X_0 \oplus K_1$ mentioned above. Let $s : K_1 \longrightarrow Y \subset X$ be a surjective (or injective[4]) mapping between K_1 and the algebraic supplement Y of $\operatorname{Ran}(\mathrm{id}_X - T)$, that is, the subspace Y such satisfying the following properties:

$$Y \cap \operatorname{Ran}(\mathrm{id}_X - T) = \varnothing \qquad \text{and} \qquad X = Y + \operatorname{Ran}(\mathrm{id}_X - T).$$

If we prove that s is a bijection, then the following identity will arise automatically:

$$\dim \operatorname{Ker}(\mathrm{id}_X - T) = \dim Y = \operatorname{codim} \operatorname{Ran}(\mathrm{id}_X - T).$$

Step 1. Let us consider the operator defined by

$$\mathrm{id}_X - T' := \mathrm{id}_X - (T - s \circ P).$$

Since the composition $s \circ P \in \mathcal{L}_f(X)$ is a finite-rank operator, we can apply Lemma 6.19 and infer that

$$s \text{ is surjective} \iff \mathrm{id}_X - T' \text{ is surjective} \iff \mathrm{id}_X - T' \text{ is injective} \iff s \text{ is injective},$$

and this concludes the first part of the proof.

Step 2. Recall that, if T is a compact operator, then T^* is also a compact operator. Therefore, it is enough to prove that

$$\dim \operatorname{Ker}(\mathrm{id}_X - T) = \dim \operatorname{Ker}(\mathrm{id}_X - T^*).$$

[4]Both the kernel K_1 and the algebraic supplement Y of the rank are vector spaces (one of which finite-dimensional), and therefore we can always find an injective or surjective map, depending on the respective dimensions.

Indeed, a straightforward computation shows that

$$\begin{aligned}\dim \operatorname{Ker}(\mathrm{id_X} - T) &= \operatorname{codim} \operatorname{Ran}(\mathrm{id_X} - T) = \\ &= \dim \left({}^{X}\!\diagup_{\operatorname{Ran}(\mathrm{id_X} - T)} \right) = \\ &\overset{(*)}{=} \dim \left({}^{X}\!\diagup_{\operatorname{Ker}(\mathrm{id_X} - T^*)_\perp} \right) = \\ &\overset{(*)}{=} \dim \left({}^{X}\!\diagup_{\operatorname{Ker}(\mathrm{id_X} - T^*)_\perp} \right)^* = \\ &= \dim \left(\operatorname{Ker}(\mathrm{id_X} - T^*)_\perp \right)^\perp = \dim \left(\operatorname{Ker}(\mathrm{id_X} - T^*) \right),\end{aligned}$$

where $(*)$ follows from the Closed Rank Theorem 5.28, and $(*)$ follows from the fact that the quotients are finite-dimensional (and $V^* \cong V$ if $\dim(V) < +\infty$).

Step 3. In conclusion, the generalization to $n > 1$ is an easy consequence of the fact that

$$(\mathrm{id_X} - T)^n = \mathrm{id_X} + \sum_{i=1}^{n} (-1)^i \binom{n}{i} T^i = \mathrm{id_X} - T \circ \sum_{i=1}^{n} (-1)^i \binom{n}{i} T^{i-1},$$

since the composition between an operator and a compact operator is still compact. $\square$

Corollary 6.21. *Let $T \in \mathcal{L}_c(X)$ be a compact operator defined on a Banach space, and let $n_0 \in \mathbb{N}$ be the natural number such that, for all $n \geq n_0$, it turns out that*

$$\operatorname{Ker}(\mathrm{id_X} - T)^n = \operatorname{Ker}(\mathrm{id_X} - T)^{n_0} \quad \text{and} \quad \operatorname{Ran}(\mathrm{id_X} - T)^n = \operatorname{Ran}(\mathrm{id_X} - T)^{n_0}.$$

Then, for any $n \geq n_0$, the space X admits the T-invariant decomposition given by

$$X = \operatorname{Ker}(\mathrm{id_X} - T)^n \oplus \operatorname{Ran}(\mathrm{id_X} - T)^n.$$

Proof. Let $x \in \operatorname{Ker}(\mathrm{id_X} - T)^n \cap \operatorname{Ran}(\mathrm{id_X} - T)^n$, and let $y \in X$ be such that $(\mathrm{id_X} - T)^n\, y = x$. The sequences of subspaces are both stable after n_0; thus

$$(\mathrm{id_X} - T)^{2n}\, y = (\mathrm{id_X} - T)^n\, x = 0 \implies y \in \operatorname{Ker}(\mathrm{id_X} - T)^{2n} = \operatorname{Ker}(\mathrm{id_X} - T)^n,$$

and this implies that $x = 0$. Moreover, it is easy to check that

$$X = \operatorname{Ker}(\mathrm{id_X} - T)^n + \operatorname{Ran}(\mathrm{id_X} - T)^n,$$

since, by Theorem 6.20, the dimension of the kernel is exactly equal to the codimension of the rank. $\square$

6.3 Spectrum of a Compact Operator

In this section, we investigate the basic properties of the spectrum of a compact operator, and we give an estimate from above of the spectral radius.

Introduction. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be any complex number. The results obtained in the previous section can be easily generalized to operators of the form $\lambda \cdot \mathrm{id}_X - T$ since one could always consider the compact perturbation

$$\lambda\left(\mathrm{id}_X - \frac{1}{\lambda} \cdot T\right).$$

Definition 6.22 (Spectrum)**.** Let $T \in \mathcal{L}(X)$ be a linear and continuous operator. The *spectrum* of T is defined as the space of all λ such that $(\lambda \cdot \mathrm{id}_X - T)$ is not invertible, that is,

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda \cdot \mathrm{id}_X - T \notin \mathrm{G_L}(X)\}.$$

The *eigenvalue spectrum* of T is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} \mid \mathrm{Ker}\,(\lambda \cdot \mathrm{id}_X - T) \neq 0\}.$$

Furthermore, the elements of the eigenvalue spectrum are called *eigenvalues*, while the nonzero elements of $\mathrm{Ker}\,(\lambda \cdot \mathrm{id}_X - T)$ are called *eigenvectors* (or eigenstates).

Definition 6.23 (Multiplicity)**.** Let $T \in \mathcal{L}_c(X)$ be a compact operator defined on a Banach space X, and let $\lambda \in \mathbb{C} \setminus \{0\}$. The *algebraic multiplicity* of λ is defined by

$$m_a(\lambda) := m_a(\lambda, T) = \dim \mathrm{Ker}(\lambda \cdot \mathrm{id}_X - T)^n, \tag{6.5}$$

where $n \geq n_0$. The *geometric multiplicity* of λ is defined as

$$m_g(\lambda) := m_g(\lambda, T) = \dim \mathrm{Ker}(\lambda \cdot \mathrm{id}_X - T). \tag{6.6}$$

Notation. Let $T \in \mathcal{L}_c(X)$ be a compact operator, and let $n_0 \in \mathbb{N}$ be a natural number big enough that the sequences $(K_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ are stable at n_0. We denote by $V(\lambda)$ the generalized autospace associated to λ, that is,

$$V(\lambda) := \mathrm{Ker}(\lambda \cdot \mathrm{id}_X - T)^n,$$

and we denote by $R(\lambda)$ the generalized rank associated to λ, that is,

$$R(\lambda) := \mathrm{Ran}(\lambda \cdot \mathrm{id}_X - T)^n.$$

Remark 6.2. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be a nonzero complex number.

(a) The algebraic multiplicity $m_a(\lambda)$ is always bigger than or equal to the geometric multiplicity $m_g(\lambda)$, as it follows from the Fredholm Theorem 6.18.

(b) The complex number λ is an eigenvalue for T if and only if $m_a(\lambda) > 0$ if and only if $m_g(\lambda) > 0$.

(c) If T is a compact operator, then $m_a(\lambda, T) = m_a(\lambda, T^*)$ and, similarly, $m_g(\lambda, T) = m_g(\lambda, T^*)$.

In particular, as a consequence of Corollary 6.21, the Banach space X admits a *spectral decomposition* with respect to λ, that is X is the direct sum of the generalized spaces:

$$X = V(\lambda) \oplus R(\lambda).$$

The decomposition is T-invariant, and the restriction $(\lambda \cdot \mathrm{id}_X - T)\big|_{V(\lambda)} : V(\lambda) \longrightarrow V(\lambda)$ is a nilpotent operator, while the restriction $(\lambda \cdot \mathrm{id}_X - T)\big|_{R(\lambda)} : R(\lambda) \longrightarrow R(\lambda)$ is an invertible operator.

Remark 6.3. If T is defined on an infinite-dimensional Banach space, then, in general, the spectrum $\sigma(T)$ contains the eigenvalue spectrum $\sigma_e(T)$ strictly (see, e.g., Exercise 6.3).

Remark 6.4. If $T \in \mathcal{L}_c(X)$ is a compact operator, then

$$\sigma(T) \setminus \{0\} = \sigma_e(T) \setminus \{0\}.$$

Lemma 6.24. *Let $T \in \mathcal{L}(X)$ be a linear continuous operator. The spectrum $\sigma(T)$ is a closed subset of $\mathbb{C}$, and it is contained in the closed ball of center 0 and radius $\|T\|$, i.e.,*

$$\sigma(T) \subseteq \overline{B_{\mathbb{C}}(0,\, \|T\|)}.$$

Proof. The first assertion is an easy consequence of the fact that

$$A \in \mathrm{G_L}(X) \implies B\left(A,\, \|A^{-1}\|^{-1}\right) \subseteq \mathrm{G_L}(X),$$

that is, the complement of $\sigma(T)$ is an open subset of $\mathbb{C}$. On the other hand, if λ is a complex number such that $|\lambda| > \|T\|$ strictly, then

$$\lambda \cdot \mathrm{id_X} - T = \lambda \cdot \left(\mathrm{id_X} - \frac{T}{\lambda}\right),$$

and it is easy to prove that the operator $\mathrm{id_X} - \frac{T}{\lambda}$ is invertible (e.g., notice that the modulus of any eigenvalue is bigger than or equal to $1 - \|T\|/\lambda > 0$.) $\square$

Remark 6.5 (Spectrum of Compact Operators)**.**

(a) If X is an infinite-dimensional Banach space, then $0 \in \sigma(T)$ for any $T \in \mathcal{L}_c(X)$.

Indeed, a compact operator is invertible if and only if X is locally compact if and only if X is finite-dimensional.

(b) There are compact operators such that $\sigma(T) = \{0\}$ and $T \not\equiv 0$ (see, e.g., Exercise 6.4).

Lemma 6.25. *Let T be a compact operator. The eigenvalue spectrum $\sigma_e(T)$ is made up of isolated points, i.e., for any $\lambda \in \sigma_e(T)$ there exists $\epsilon > 0$ such that*

$$B_{\mathbb{C}}(\lambda,\, \epsilon) \cap \sigma_e(T) = \{\lambda\}.$$

Proof. Let $\lambda,\, \mu \in \sigma_e(T)$ be two complex numbers such that $\lambda \neq \mu$. The spectral decomposition associated to λ, i.e.

$$X = V(\lambda) \oplus R(\lambda),$$

is invariant under the action of the operator $\mu \cdot \mathrm{id_X} - T$. Moreover, the identity

$$\mu \cdot \mathrm{id_X} - T = (\mu - \lambda) \cdot \mathrm{id_X} + (\lambda \cdot \mathrm{id_X} - T)$$

implies that $\mu \cdot \mathrm{id_X} - T$ is equal to a nonzero multiple of the identity, minus a nilpotent operator on $V(\lambda)$. Hence, a simple algebraic lemma proves that $\mu \cdot \mathrm{id_X} - T$ is invertible on $V(\lambda)$.

In conclusion, since the operator $\mu \cdot \mathrm{id_X} - T$ is invertible for λ sufficiently near to μ (the condition is open, as mentioned before), the unique possibility is that λ is an isolated point. $\square$

If $T \in \mathcal{L}_c(X)$ is a compact operator defined on a infinite-dimensional Banach space, then $\sigma(T)$ is given by the singlet $\{0\}$ and it contains all the nonzero eigenvalues whose algebraic multiplicity is strictly greater than 0, that is,

$$T \in \mathcal{L}_c(X) \implies \sigma(T) = \{0\} \cup \sigma_e(T).$$

Moreover, as a consequence of Lemma 6.25, it turns out that 0 is an cluster point for the eigenvalue spectrum of T, that is, the nonzero eigenvalues of T form a sequence which converges to 0.

Remark 6.6. In the proof of Lemma 6.25, we have proved that, if μ and λ are "far enough", then

$$V(\lambda) \subset R(\mu) \qquad \text{and} \qquad V(\mu) \subset R(\lambda).$$

Iterated Decomposition. Let $\lambda,\, \mu \in \mathbb{C} \setminus \{0\}$ be two complex numbers such that $\lambda \neq \mu$, and let $P_\lambda : X \longrightarrow V(\lambda)$ and $P_\mu : X \longrightarrow V(\mu)$ be the projections. It is easy to see that there is a spectral decomposition associated to both λ and μ, that is,

$$X = V(\lambda) \oplus V(\mu) \oplus (R(\lambda) \cap R(\mu)).$$

Since $P_\lambda \circ P_\mu = P_\mu \circ P_\lambda = 0$, it tuns out that $(\mathrm{id}_X - P_\mu - P_\lambda)$ is an idempotent operator (a projection), and it is clearly the one associated with the decomposition above.

Notation. Let $\lambda \in \sigma_e(T)$. If T is a compact operator, then the projection associated with the spectral decomposition is denoted by $E(\lambda) : X \longrightarrow V(\lambda)$, and it is called *spectral projection*.

In a similar fashion, the projection onto the rank is denoted by $D(\lambda) : X \longrightarrow R(\lambda)$, and it is defined as the orthogonal projection to the spectral one, i.e. $D(\lambda) := \mathrm{id}_X - E(\lambda)$.

In particular, for any finite subset $F \subset \sigma(T)$ (e.g., $F = \sigma(T) \setminus B(0,\, \epsilon)$) we have a spectral decomposition

$$X = \left(\bigoplus_{\lambda \in F} V(\lambda) \right) \bigoplus \left(\bigcap_{\lambda \in F} R(\lambda) \right).$$

Complexification. Let X be a real Banach space. The complexification of X is the space

$$X_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} X,$$

with a complex structure induced by the action of the operator

$$J = \begin{pmatrix} 0 & -\mathrm{id}_X \\ \mathrm{id}_X & 0 \end{pmatrix},$$

where $(a + \imath\, b) \cdot z := az + b\, Jz$, as z ranges in $X \oplus X$.

Let $T \in \mathcal{L}^{\mathbb{R}}(X,\, Y)$ be an operator between real topological vector spaces. The complexification of T is the operator defined by

$$T_{\mathbb{C}} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \in \mathcal{L}^{\mathbb{C}}(X_{\mathbb{C}},\, Y_{\mathbb{C}}).$$

6.4 Symmetric Operators

In this section, we introduce a new class of operators, called *symmetric operators*, defined on a Hilbert space H. We investigate some of their main properties, and we prove that the kernel and the rank behave nicely with respect to the orthogonal.

Definition 6.26 (Hilbert Adjoint)**.** Let $A : H \longrightarrow K$ be a linear continuous operator between two Hilbert spaces H and K. The hilbertian adjoint of A is the operator $A^* : K \longrightarrow H$ defined by the formula

$$(Ax,\, y)_K = (x,\, A^* y)_H\,, \tag{6.7}$$

for all $x \in H$ and all $y \in K$.

Remark 6.7. Let $A : H \longrightarrow K$ be a linear continuous operator between Hilbert spaces H and K.

(1) For every complex number $\lambda \in \mathbb{C} \setminus \{0\}$ it turns out that

$$(\lambda \cdot A)^* = \overline{\lambda} \cdot A^*.$$

Indeed, if we replace A with $\lambda \cdot A$ in the left-hand side of (6.7), then

$$(\lambda \cdot Ax,\, y)_K = \lambda\,(A\,x,\, y)_K =$$

$$= \lambda\,(x,\, A^* y)_H = \left(x,\, \overline{\lambda} \cdot A^* y\right)_H.$$

(2) The map $\mathcal{L}(H,\, K) \ni A \longmapsto A^* \in \mathcal{L}(K,\, H)$ is semi-linear.

Definition 6.27 (Symmetric Operator)**.** Let $A : H \longrightarrow H$ be a linear continuous operator defined on a Hilbert space H. We say that A is *symmetric* if $A = A^*$, and we denote by $\mathcal{L}^{sym}(H)$ the space of all symmetric operator defined on a Hilbert space H.

Proposition 6.28. *Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator.*

(a) $(\mathrm{Ran}(A))^{\perp} = \mathrm{Ker}(A)$.

(b) $(\mathrm{Ker}(A))^{\perp} = \overline{\mathrm{Ran}(A)}$.

(c) *If $H_0 \subseteq H$ is an A-invariant subset, then $\overline{H_0}$ and $H_0^{\perp}$ are also A-invariant.*

(d) *The spectrum is contained in the open ball of radius $\|A\|$ of $\mathbb{R}$, that is,*

$$\sigma(A) \subset [-\|A\|,\, \|A\|] \subset \mathbb{R}.$$

(e) *If $\lambda,\, \mu \in \sigma_e(A)$ are eigenvalues of A such that $\lambda \neq \mu$, then the corresponding eigenvectors are orthogonal.*

(f) *The eigenvalues are semisimple, that is,*

$$m_a(\lambda) = m_g(\lambda), \qquad \forall\, \lambda \in \sigma_e(A).$$

Proof.

(a) This assertion follows from Lemma 5.26 since the operator is symmetric (i.e., it is equal to its dual.)

(b) This assertion follows from Lemma 5.26. Indeed, the operator A is symmetric and one can show that in a Hilbert space H, given two closed subsets M and N, it turns out that[5]

$$\left(M^{\perp}\right)_{\perp} = M \quad \text{and} \quad (N_{\perp})^{\perp} = N.$$

(c) Both the assertion follows easily from the definition; we prove the second one, and leave the first one as a simple exercise for the reader.

Let $x \in H_0^{\perp}$ be any point. For any $y \in H_0$, it turns out that

$$(Ax,\, y)_H = (x,\, Ay)_H = 0,$$

and hence $Ax \in H_0^{\perp}$, which is what we wanted to prove.

(d) First, we prove that the any element of the spectrum belongs to $\mathbb{R}$. Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ be given; the idea is to show that operator $(a + \imath b) \cdot \mathrm{id}_{\mathrm{X}} + A$ is always invertible.

We claim that the (composition) operator

$$[(a - \imath b) \cdot \mathrm{id}_{\mathrm{X}} + A] \circ [(a + \imath b) \cdot \mathrm{id}_{\mathrm{X}} + A]$$

is invertible. Indeed, if we evaluate the product, it turns out that

$$[(a - \imath b) \cdot \mathrm{id}_{\mathrm{X}} + A] \circ [(a + \imath b) \cdot \mathrm{id}_{\mathrm{X}} + A] = b^2 \left(\mathrm{id}_{\mathrm{X}} + \left(\frac{a + A}{b} \right)^2 \right),$$

and thus it is enough to prove that $\mathrm{id}_{\mathrm{X}} + A^2$ is invertible for every symmetric operator $A \in \mathcal{L}^{sym}(H)$. The estimate

$$\left\| (\mathrm{id}_{\mathrm{X}} + A^2)x \right\|_H^2 \geq \|x\|^2 > 0 \qquad \forall\, x \neq 0,$$

proves that $\mathrm{id}_{\mathrm{X}} + A^2$ is injective with closed image (see Corollary 5.24). In conclusion, the assertion **(b)** implies $\mathrm{id}_{\mathrm{X}} + A^2$ surjective, and thus bijective.

(e) This assertion follows from a straightforward computation:

$$\begin{aligned} (\lambda x,\, y) &= (Ax,\, y) = (x,\, Ay) = \\ &= (x,\, \mu y) = \\ &= (\mu x,\, y) \iff (\lambda - \mu)(x,\, y) = 0 \iff (x,\, y) = 0 \iff x \perp y. \end{aligned}$$

(f) It is enough to prove that $\mathrm{Ker}(\lambda \cdot \mathrm{id}_{\mathrm{X}} - T)^2 = \mathrm{Ker}(\lambda \cdot \mathrm{id}_{\mathrm{X}} - T)$; the assertion will follow by means of a simple induction.

Let $x \in H$ be an element such that $(\lambda \cdot \mathrm{id}_{\mathrm{X}} - T)^2 x = 0$. Then

$$\begin{cases} (\lambda \cdot \mathrm{id}_{\mathrm{X}} - A)x \in \mathrm{Ker}(\lambda \cdot \mathrm{id}_{\mathrm{X}} - A) \\ (\lambda \cdot \mathrm{id}_{\mathrm{X}} - A)x \in \mathrm{Ran}(\lambda \cdot \mathrm{id}_{\mathrm{X}} - A) \end{cases} \implies (\lambda \cdot \mathrm{id}_{\mathrm{X}} - A)x = 0 \implies x \in \mathrm{Ker}(\lambda \cdot \mathrm{id}_{\mathrm{X}} - A).$$

□

[5] It suffices to notice that, for example, $\left(M^{\perp}\right)_{\perp} = \left(M^{\perp}\right)^{\perp} = M$, as we have proved in the first chapter.

Quadratic Form. Let $q_A : H \longrightarrow \mathbb{R}$ be the quadratic form associated to an operator A, that is,

$$q_A(x) := (Ax,\, x)_H\,.$$

It is easy to prove that the following inequality holds (even in a more general setting than Hilbert spaces, e.g., Banach spaces):

$$r_A := \sup_{\|x\|\le 1} |q_A(x)| \le \sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle Ax,\, y\rangle| = \sup_{\|x\|=1} \|Ax\| = \|A\|.$$

On the other hand, in our particular case of Hilbert spaces, the *polarization equality* characterizes the scalar product in terms of the quadratic form:

$$(Ax,\, y)_H = \frac{1}{4}\, [q_A(x+y) - q_A(x-y)]\,. \tag{6.8}$$

It follows that

$$|(Ax,\, y)_H| \le \frac{1}{4}\left[\|x+y\|^2 + \|x-y\|^2\right] \cdot r_A = \frac{1}{2} r_A \left(\|x\|^2 + \|y\|^2\right),$$

and thus we infer that r_A is equal to the operator norm of A.

Variational Characterization. Let $A \in \mathcal{L}^{sym}(H)$. The *Rayleigh quotient* is the function

$$f_A(x) = \frac{(Ax,\, x)_H}{(x,\, x)_H} : H \setminus \{0\} \longrightarrow \mathbb{R}, \tag{6.9}$$

which is clearly a C^∞ function (in the sense of Fréchet derivatives). Then

$$\nabla f_A(x) = \frac{2}{\|x\|^2}\, (Ax - f_A(x),\, x)_H\,,$$

and hence a couple $(\lambda,\, x) \in \mathbb{R} \times (H \setminus \{0\})$ is a (eigenvalue, eigenvector) couple if and only if x is a critical point for f_A and $\lambda = f_A(x)$.

6.5 Compact Symmetric Operators

In this section, we investigate the compact symmetric operators, and we prove that the standard *spectral theorem* holds true even in infinite-dimensional spaces.

Lemma 6.29. *Let $A \in \mathcal{L}_c(H)$ be a compact operator defined on a Hilbert space H. If*

$$\alpha := \sup_{\|x\|=1} \langle Ax,\, x\rangle \neq 0,$$

then there exists $x_0 \in \partial B_H(0,\, 1)$ such that $\alpha = (Ax_0,\, x_0)$.

Proof. We can assume that H is a separable Hilbert space. Indeed, we can always consider the decomposition

$$H = \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)},$$

and restrict, without loss of generality, the quadratic form q_A to $\overline{\mathrm{Ran}(A)}$, which is always a separable Hilbert space (see Lemma 6.12). Let $(x_n)_{n\in\mathbb{N}} \subset H$ be a maximizing sequence, that is,

$$\|x_n\| = 1 \qquad \text{and} \qquad \langle Ax_n,\, x_n\rangle \xrightarrow{n\to+\infty} \alpha.$$

By Banach-Alaoglu Theorem 5.13 it turns out that there exists an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$ weakly. Notice also that the limit point x has norm $\|x\| \le 1$ since

$$\|x\|^2 = \langle x,\, x\rangle = \lim_{n\to+\infty} \langle x_n,\, x\rangle \le \|x\|\|x_{n_0}\| = \|x\|.$$

To prove that the norm of x is exactly one, let us set $y = x/\|x\|$ and observe that

$$\alpha \ge \langle Ay,\, y\rangle = \alpha\frac{1}{\|x\|^2} \iff \|x\| = 1,$$

The operator A is compact. Therefore (see Lemma 6.2) there exists an increasing subsequence such that $Ax_{n_k} \to Ax$ strongly (in norm), and this implies that

$$\langle Ax,\, x\rangle_H = \alpha,$$

that is, the limit point x is the sought one. □

Lemma 6.30. *Let H be a Hilbert space. If $(x_n)_{n\in\mathbb{N}} \subset H$ is a sequence weakly converging to x, and $(y_n)_{n\in\mathbb{N}} \subset H$ is a sequence strongly converging to y, then*

$$(x_n,\, y_n) \xrightarrow{n\to+\infty} (x,\, y)\,.$$

Theorem 6.31. *Let $A \in \mathcal{L}_c^{\mathrm{sym}}(H)$ be a symmetric compact operator defined on a Hilbert space H, and let q_A be the associated quadratic form. Then either $\|A\|$ or $-\|A\|$ is an element of $\sigma_e(A)$.*

Proof. In the previous section, we have proved that

$$A \in \mathcal{L}^{sym}(H) \implies \|A\| = \sup_{\|x\|=1} |q_A(x)|\,.$$

It follows that

$$\|A\| = \sup_{\|x\|=1} |(Ax,\, x)| = \sup_{\|x\|=1} \max\left\{(Ax,\, x)\,,\, -(Ax,\, x)\right\} =$$

$$= \max\left\{\sup_{\|x\|=1} (Ax,\, x)\,,\, -\inf_{\|x\|=1} (Ax,\, x)\right\},$$

and hence either $\|A\| = \sup_{\|x\|=1}(Ax,\, x)$ or $-\|A\| = \inf_{\|x\|=1}(Ax,\, x)$. We now assume, without loss of generality, that

$$\|A\| = \sup_{\|x\|=1} (Ax,\, x)\,.$$

The operator A is compact; hence, by Lemma 6.29, there exists $x \in H$ such that $\|x\| \le 1$, and there is a maximizing sequence $x_n \rightharpoonup x$ such that

$$\langle Ax_n,\, x_n\rangle \xrightarrow{n\to+\infty} \langle Ax,\, x\rangle = \|A\| \ne 0 \implies x \ne 0.$$

To prove that the norm of x is exactly one, let us set $y = x/\|x\|$ and observe that

$$\alpha \geq \langle Ay, y \rangle = \alpha \frac{1}{\|x\|^2} \iff \|x\| = 1,$$

so that, by Lemma 6.14, the convergence $x_n \to x$ is strong (it is not needed for the proof, but it is an interesting fact worth remarking on).

The variational characterization (in terms of the Rayleigh quotient) concludes the proof, since we have proved that x is a critical point of f_A and $\|A\|$ is exactly equal to $f_A(x)$. □

Corollary 6.32. *Let $A \in \mathcal{L}_c^{\text{sym}}(H)$ be a symmetric compact operator defined on a Hilbert space H. Each nonempty, closed and invariant subspace $H_0 \subseteq H$ contains the eigenvector of A associated either to the eigenvalue $\|A\|_{H_0}$ or to $-\|A\|_{H_0}$.*

Theorem 6.33 (Spectral Theorem)**.** *Let $A \in \mathcal{L}_c^{\text{sym}}(H)$ be a symmetric compact operator defined on a Hilbert space H. There exists an orthogonal basis of eigenvectors, i.e., the operator A is diagonalizable.*

Proof. We can assume without loss of generality that H is a separable Hilbert space, since the same argument of Lemma 6.29 holds.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal system of eigenvectors for A, and suppose that it is maximal with respect to the inclusion (simple application of Zorn lemma). We claim that

$$\overline{\text{Span}\,\langle e_k \,:\, k \in \mathbb{N}\rangle} = H.$$

If not, by Corollary 6.32, there would be an eigenvector in $H_0 := (\text{Span}\,\langle e_k \,:\, k \in \mathbb{N}\rangle)^{\perp} \neq \varnothing$, and this is absurd since the system $\{e_k\}_{k\in\mathbb{N}}$ is maximal. □

In particular, any $A \in \mathcal{L}_c^{\text{sym}}(H)$ is unitary equivalent to a multiplication operator defined on $\ell_2(\mathbb{N})$ by an element of c_0, i.e., there is $\lambda \in c_0$ such that

$$\begin{array}{ccc} H & \xrightarrow{A} & H \\ \updownarrow & & \updownarrow \\ \ell_2 & \xrightarrow{\cdot\lambda} & \ell_2 \end{array}$$

6.6 Applications of the Spectral Theorem

Minimax Principle. Let $A \in \mathcal{L}_c^{\text{sym}}(H)$ be a symmetric compact operator defined on a Hilbert space H. The Spectral Theorem 6.33 allows us to index the positive eigenvalues with positive integers, that is,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \ldots,$$

where an eigenvalue λ is repeated as many times as its multiplicity. In a similar fashion, we can index the negative eigenvalues

$$\lambda_{-1} \leq \lambda_{-2} \leq \cdots \leq \lambda_{-m} \leq \ldots.$$

The set of indices $I \subset \mathbb{Z}$ allows us to identify the Hilbert space H with $\ell_2(I)$, and the operator A with the multiplication operator

$$x \longmapsto \sum_{i \in I} x_i \lambda_i \, e_i.$$

Theorem 6.34 (Courant-Fisher)**.** *Let $A \in \mathcal{L}_c^{\mathrm{sym}}(H)$. For every $n \in I$ it turns out that*

$$\lambda_n = \inf_{\substack{E \subseteq H \\ \operatorname{codim}(E) < n}} \sup_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) = \sup_{\substack{E \subseteq H \\ \dim(E) \geq n}} \inf_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) \, .$$

Similarly, for any $-n \in I$ it turns out that

$$\lambda_{-n} = \sup_{\substack{E \subseteq H \\ \operatorname{codim}(E) < n}} \inf_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) = \inf_{\substack{E \subseteq H \\ \dim(E) \geq n}} \sup_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) \, .$$

Proof. The second assertion follows immediately from the first one since

$$\lambda_{-n}(A) = -\lambda_n(-A).$$

Let us identify H with $\ell_2(I)$ and A with the multiplication operator $\cdot\lambda$. Let $\{e_i\}_{i \in I}$ be the set of eigenvectors of A, and let us consider

$$E_n := \operatorname{Span} \langle e_1, \, \dots, \, e_n \rangle \, , \qquad E_{n-1}^{\perp} = \overline{\operatorname{Span} \langle e_i \mid i \in I \setminus \{1, \, \dots, \, n-1\} \rangle}.$$

We consider the restriction operators

$$A \big|_{E_n} : x \longmapsto \sum_{i=1}^{n} \lambda_i x_i \, e_i \quad \text{has eigenvalues } \lambda_1, \, \dots, \, \lambda_n$$

$$A \big|_{E_{n-1}^{\perp}} : x \longmapsto \sum_{i \in I \setminus \{1, \, \dots, \, n-1\}} \lambda_i x_i \, e_i \quad \text{has eigenvalues } \lambda_n, \, \dots$$

from which it follows that

$$\sup_{\substack{x \in E_{n-1}^{\perp} \\ \|x\|=1}} (Ax, \, x) = \lambda_n = \inf_{\substack{x \in E_n \\ \|x\|=1}} (Ax, \, x) \, .$$

If we take the infimum of the left-hand side and the supremum of the right-hand side, we obtain the first inequality:

$$\inf_{\substack{E \subseteq H \\ \operatorname{codim}(E) < n}} \sup_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) \leq \lambda_n \leq \sup_{\substack{E \subseteq H \\ \dim(E) \geq n}} \inf_{\substack{x \in E \\ \|x\|=1}} (Ax, \, x) \, .$$

Let $E \subseteq H$ be a subspace of codimension strictly less than n, and let $F \subseteq H$ be a subspace of dimension at least n. It follows easily that

$$E \cap E_n \cap \partial B(0, \, 1) \neq \varnothing$$

$$F \cap E_{n-1}^{\perp} \cap \partial B(0, \, 1) \neq \varnothing,$$

and thus there exist points x_0 in the first intersection and y_0 in the second intersection. We conclude the proof by noticing that

$$
\begin{aligned}
\sup_{\substack{x\in E\\ \|x\|=1}} (Ax,\,x) \geq (Ax_0,\,x_0) \geq \\
&\geq \min_{\substack{x\in E_n\\ \|x\|=1}} (Ax,\,x) = \lambda_n = \\
&= \max_{\substack{x\in E_{n-1}^{\perp}\\ \|x\|=1}} (Ax,\,x) \geq \\
&\geq (Ay_0,\,y_0) \geq \inf_{\substack{x\in F\\ \|x\|=1}} (Ax,\,x)\,.
\end{aligned}
$$

□

Theorem 6.35 (Alternating Eigenvalues Principle)**.** *Let $A \in \mathcal{L}_c^{\mathrm{sym}}(H)$, and let $H_0 \subset H$ be a closed hyperplane of H. Let $P_0 : H \longrightarrow H_0$ be the projection, and set $A_0 := P_0 \circ A\big|_{H_0} : H_0 \longrightarrow H_0$. Then $A_0 \in \mathcal{L}_c^{\mathrm{sym}}(H_0)$, and for any $n \in I$*

$$\lambda_{n+1}(A) \leq \lambda_n(A_0) \leq \lambda_n(A).$$

Proof. It is a simple consequence of the Courant-Fisher Theorem 6.34. More precisely, it turns out that

$$
\begin{aligned}
\lambda_{n+1}(A) &= \sup_{\substack{E\subseteq H\\ \dim(E)\geq n+1}} \inf_{\substack{x\in E\\ \|x\|=1}} (Ax,\,x) \leq \\
&\leq \sup_{\substack{E\subseteq H\\ \dim(E\cap H_0)\geq n}} \inf_{\substack{x\in E\\ \|x\|=1}} (Ax,\,x) \leq \\
&\leq \sup_{\substack{E\subseteq H\\ \dim(E\cap H_0)\geq n}} \inf_{\substack{x\in E\cap H_0\\ \|x\|=1}} (Ax,\,x) = \\
&= \sup_{\substack{E\subseteq H_0\\ \dim(E)\geq n}} \inf_{\substack{x\in E\\ \|x\|=1}} (Ax,\,x) = \\
&= \lambda_n(A_0) \leq \sup_{\substack{E\subseteq H\\ \dim(E)\geq n}} \inf_{\substack{x\in E\\ \|x\|=1}} (Ax,\,x) = \lambda_n(A).
\end{aligned}
$$

□

6.7 Spectral Theory of Banach Spaces

Let X be a Banach space. The space of linear continuous operators $(\mathcal{L}(X), +, \circ)$ is an algebra satisfying the inequality

$$\|A \circ B\| \leq \|A\| \cdot \|B\|, \tag{6.10}$$

for any $A, B \in \mathcal{L}(X)$.

Definition 6.36 (Banach Algebra)**.** The normed space $(X, \|\cdot\|_X)$ is a *Banach algebra* if the following properties are satisfied:

(a) X is a Banach space.

(b) There are $+ : X \times X \longrightarrow X$ and $\cdot : \mathbb{K} \times X \longrightarrow X$ such that $(X, +, \cdot)$ is an associative algebra over $\mathbb{K}$.

(c) For any $x, y \in X$, it turns out that

$$\|x \cdot y\| \leq \|x\| \, \|y\|.$$

Notation. The *resolvent set* associated to a linear operator $A \in \mathcal{L}(X)$ is defined as the set of complex numbers such that $A - \lambda \cdot \mathrm{id}_X$ is invertible, that is,

$$\rho(A) := \{\lambda \in \mathbb{C} \mid A - \lambda \cdot \mathrm{id}_X \text{ is invertible}\}.$$

The *spectrum* is the complement of the resolvent, and it is denoted by

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

The *resolvent operator*, defined on $\rho(A)$, it denoted by $R(\lambda, A)$ and it is defined by setting

$$R(\lambda, A) := (\lambda \cdot \mathrm{id}_X - A)^{-1}.$$

Finally, the *spectral radius* is defined as the supremum of $\sigma(A)$, i.e.,

$$r_A := \sup_{\lambda \in \sigma(A)} |\lambda|.$$

Lemma 6.37. *Let X be a Banach space, and let $A \in \mathcal{L}(X)$ be a linear continuous operator on X.*

(1) *The resolvent $\rho(A)$ is an open subset of $\mathbb{C}$.*

(2) *The spectral radius r_A is less or equal than $\|A\|$.*

(3) *The map defined by the resolvent operator*

$$\rho(A) \ni \lambda \mapsto R(\lambda, A) =: R(\lambda) \in \mathcal{L}(X)$$

is analytic. If $\lambda > \|A\|$, then its power series is given by

$$R(\lambda) = \sum_{i=0}^{+\infty} \lambda^{-i-1} A^i.$$

(4) The map defined by the resolvent operator admits a local representation as

$$R(\lambda) = \sum_{i=0}^{+\infty} (\lambda - \lambda_0)^i \, R(\lambda_0)^{i+1}, \qquad \forall \lambda \in B\left(\lambda_0, \, \|R(\lambda_0)\|^{-1}\right).$$

(5) Resolvent Identity. *For any μ, $\lambda \in \rho(A)$, it turns out that*

$$R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\mu)\, R(\lambda). \tag{6.11}$$

The central goal of this section is to prove that the spectral radius is equal to the limit of the sequence $\|A^n\|^{1/n}$. We will give two proofs of this fact: the first one relies on holomorphic function theory, while the second one relies on a variational argument (Palais-Smale sequences).

Lemma 6.38. *Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a subadditive sequence ($a_{m+n} \leq a_m + a_n$). Then the limit of the sequence $(1/n \cdot a_n)_{n\in\mathbb{N}}$ exists and it is equal to the infimum, that is,*

$$\lim_{n\to+\infty} \left(\frac{1}{n}\, a_n\right) = \inf_{n\in\mathbb{N}} \left(\frac{1}{n}\, a_n\right).$$

Proof. Let d be a positive natural number. For any $n \in \mathbb{N}$ there are $p \in \mathbb{N}$ and $k \in \{0, \dots, d-1\}$ such that

$$n = p \cdot d + k, \qquad \text{where } p = \left\lfloor \frac{n}{d} \right\rfloor.$$

For every $n \in \mathbb{N}$ it turns out that

$$\begin{aligned} \inf_{m\in\mathbb{N}} \left(\frac{1}{m}\, a_m\right) \leq \frac{a_n}{n} \leq \frac{a_{p\cdot d} + a_k}{n} &\leq \\ &\leq \frac{1}{n}\,(p \cdot a_d + a_k) = \left(\frac{1}{d} + \mathcal{O}(1)\right) \cdot a_d + \mathcal{O}(1) = \\ &= \frac{a_d}{d} + \mathcal{O}(1). \end{aligned}$$

Thus we can take the superior limit of the left-hand side of the inequality above

$$\inf_{m\in\mathbb{N}} \left(\frac{1}{m}\, a_m\right) \leq \limsup_{n\to+\infty} \left(\frac{1}{n}\, a_n\right) \leq \frac{a_d}{d} \implies \inf_{n\in\mathbb{N}} \left(\frac{1}{n}\, a_n\right) = \lim_{n\to+\infty} \left(\frac{1}{n}\, a_n\right),$$

and this concludes the proof. □

Theorem 6.39 (Spectral Radius)**.** *Let $A \in \mathcal{L}(X)$ be any linear continuous operator defined on a Banach space X. The spectral radius of A is given by*

$$r_A = \lim_{n\to+\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

Holomorphic Proof. Let us consider the sequence $a_n = \log\left(\|A^n\|\right)$.

Step 1. The logarithm is subadditive; thus we can apply Lemma 6.38 and infer that

$$\lim_{n\to+\infty} \frac{1}{n}\cdot \log\left(\|A^n\|\right) = \inf_{n\to+\infty} \frac{1}{n}\cdot \log\left(\|A^n\|\right),$$

that is, the second identity is true:

$$\lim_{n\to+\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

It remains to prove that this quantities coincide with the spectral radius r_A of A.

Step 2. One inequality is trivial since $\lambda \in \sigma(A)$ implies $\lambda \cdot \mathrm{id}_X - A$ not invertible; hence, for every $n \in \mathbb{N}$, the operator $\lambda^n \cdot \mathrm{id}_X - A^n$ is not invertible, and this easily implies that

$$\lambda^n \in \sigma\left(A^n\right) \text{ and } r_{A^n} \le \|A^n\| \implies |\lambda|^n \le \|A^n\| \implies |\lambda| \le \|A^n\|^{\frac{1}{n}}.$$

Vice versa, let us fix $x \in X$, $x^* \in X^*$ and let us consider the **holomorphic** function

$$\mathbb{C} \supseteq \left\{ t\in\mathbb{C} \;\middle|\; |t| < \frac{1}{r_A} \right\} \ni t \longmapsto \left\langle x^*,\, \left(\mathrm{id}_X - t\cdot A\right)^{-1} x \right\rangle \in \mathbb{C},$$

where $< \cdot, \cdot >$ is the duality $(X,\, X^*)$. Consequently, the series

$$\Phi(t) = \sum_{k=0}^{+\infty} \left\langle x^*,\, A^k x\right\rangle t^k$$

converges at any point $x \in B$, where B is the maximal ball contained in the domain. Therefore, when $r > r_A$, it turns out that

$$\sup_{k\in\mathbb{N}} \left| \left\langle x^*,\, \left(\frac{A}{r}\right)^k x \right\rangle \right| \le C < +\infty$$

for every $x \in X$ and $x^* \in X^*$. By Banach-Steinhaus Theorem 4.17 it turns out that $\|(A/r)^k\|$ is bounded, and thus

$$\left\|A^k\right\|^{\frac{1}{k}} \le c\cdot r \implies \inf \left\|A^k\right\|^{\frac{1}{k}} \le r$$

for every $r > r_A$, which is enough to conclude that $\inf \left\|A^k\right\|^{\frac{1}{k}} \le r_A$. □

Remark 6.8. The same result holds true in the more general setting of Banach algebras.

Theorem 6.40 (Spectral Map)**.** *Let $p(z) \in \mathbb{C}[z]$ be a complex polynomial of degree n, and let $A \in \mathcal{L}(X)$ be a linear continuous operator defined on a Banach space X. Then*

$$p\left(\sigma(A)\right) = \sigma\left(p(A)\right),$$

where $p\left(\sigma(A)\right) = \{p(\lambda) \mid \lambda \in \sigma(A)\}$.

Proof. Suppose that $p(z)$ is a monic polynomial. For any $\lambda \in \mathbb{C}$, we can factorize the polynomial $p(z) - \lambda$ as follows:

$$p(z) - \lambda = \prod_{i=1}^{n} \left(z - \mu_j^{(\lambda)}\right),$$

where $\mu_j^{(\lambda)}$ are the roots of the polynomial, eventually repeated. Clearly, by construction, we have that

$$p(\mu) = \lambda \iff \exists\, j \in \{1, \ldots, n\} \;:\; \mu = \mu_j^{(\lambda)}.$$

Therefore $\lambda \in \sigma(p(A))$ if and only if $p(A) - \lambda \cdot \mathrm{id}_X$ is not invertible if and only if there is a factor which is not invertible, that is, there exists $j \in \{1, \ldots, n\}$ such that $A - \mu_j^{(\lambda)} \cdot \mathrm{id}_X$ is not invertible. □

Symmetric Operators. Let H be a Hilbert space, and let $A \in \mathcal{L}^{sym}(H)$ be a symmetric continuous linear operator. We can introduce a partial ordering on $\mathcal{L}^{sym}(H)$ by setting

$$A \geq 0 \iff \langle Ax, x\rangle \geq 0 \qquad \forall\, x \in H,$$

from which it follows that $A \geq B$ if and only if $A - B \geq 0$.

Proposition 6.41. *Let H be a Hilbert space, and let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator. Then*

$$m_A := \inf_{\|x\|=1} \langle Ax, x\rangle = \min \sigma(A),$$

$$M_A := \sup_{\|x\|=1} \langle Ax, x\rangle = \max \sigma(A).$$

Proof. First, we observe that any positive operator $A \in \mathcal{L}^{sym}(H)$ (i.e., such that $A \geq 0$) has the following property: the operator $\mathrm{id}_X + A$ is invertible or, equivalently, minus one is not an element of the spectrum $\sigma(A)$. Indeed, for every $x \in H$ it turns out that

$$\|(\mathrm{id}_X + A)(x)\|^2 = \|x\|^2 + \underbrace{\|Ax\|^2}_{\geq 0} + 2\underbrace{\langle Ax, x\rangle}_{\geq 0} \geq \|x\|^2,$$

and hence by Remark 5.22 it is injective and with closed rank. By Proposition 6.28 it implies that the operator is injective and surjective, that is, bijective. In particular, if $t < m_A$, then

$$-1 \notin \sigma\left(\frac{A - m_A}{m_A - t}\right)$$

since it is a positive symmetric operator, and thus $A - m_A \cdot \mathrm{id}_X + (m_A - t) \cdot \mathrm{id}_X = A - t \cdot \mathrm{id}_X$ is invertible. If we replace m_A with M_A, a similar argument proves that

$$t < m_A \text{ or } t > M_A \implies t \notin \sigma(A) \implies \sigma(A) \subseteq [m_A, M_A].$$

It remains to prove that the extremal points m_A and M_A are actually attained, i.e., they belong to $\sigma(A)$. Clearly, it is enough to prove it for M_A (we obtain m_A by replacing A with $-A$). Let

$$a(u, v) := (M_A u - Au, v),$$

and notice that $a(\cdot, -)$ is symmetric and such that

$$a(u, u) \geq 0 \quad \text{for all } u \in H.$$

It follows from the Cauchy-Schwartz inequality that

$$|a(u, v)| \leq a(u, u)^{\frac{1}{2}} a(v, v)^{\frac{1}{2}} \quad \text{for all } u, v \in H,$$

and this implies that

$$|M_A u - Au| \leq C \cdot a(u, u)^{\frac{1}{2}} \quad \text{for all } u \in H.$$

Let $(u_n)_{n\in\mathbb{N}} \subset H$ be a sequence such that

$$\|u_n\| = 1 \quad \text{and} \quad (Au_n, u_n) \xrightarrow{n\to+\infty} M_A,$$

and notice that the estimate above yields to

$$|M_A u_n - Au_n| \xrightarrow{n\to+\infty} 0.$$

We infer that $M_A \in \sigma(A)$ since, if $M_A \in \rho(A)$, then

$$u_n = (M_A - A)^{-1} (M_A u_n - Au_n) \xrightarrow{n\to+\infty} 0,$$

and this is impossible because $\|u_n\| = 1$ for all $n \in \mathbb{N}$. In conclusion, we want to prove that the maximum between $|M_A|$ and $|m_A|$ is equal to $\|A\|$. A simple computation shows that

$$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} \langle Ax, Ax \rangle = \sup_{\|x\|=1} \left\langle A^2 x, x \right\rangle = \|A^2\|.$$

In particular, we can consider the subsequence $(2^n)_{n\in\mathbb{N}} \subset (n)_{n\in\mathbb{N}}$ and observe that

$$\|A\| = \left\|A^{2^n}\right\|^{\frac{1}{2^n}} \implies r_A = \lim_{n\to+\infty} \|A^n\|^{\frac{1}{n}} = \lim_{n\to+\infty} \left\|A^{2^n}\right\|^{\frac{1}{2^n}} \implies \max\{|M_A|, |m_A|\} = r_A = \|A\|.$$

□

Variational Approach to Spectral Radius. Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator defined on a Hilbert space H. The couple $(x, \lambda) \in H \times \mathbb{C}$ is an eigenvector-eigenvalue couple if and only if x is a critical point of the Rayleigh quotient f_A and $\lambda = f_A(x)$.

Lemma 6.42. *Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator defined on a Hilbert space H. A complex number λ is an eigenvalue of A if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \partial B_X(0, 1)$ such that $(x_n)_{n\in\mathbb{N}} \in (\mathrm{PS})_\lambda$, i.e., it is a Palais-Smale sequence at the level λ.*

Proof. The details are left to the reader. The main idea is to prove that Rayleigh quotient f_A is lower semi-continuous and that it satisfies the Palais-Smale condition. □

Behind this Lemma, there is the following general variational principle (named after **Ekeland**).

Theorem 6.43 (Ekeland)**.** *Let $f : X \to \mathbb{R}$ be a lower semi-continuous function, and let X be a complete metric space. If $\inf_{x\in X} f(x) > -\infty$, then for any $x_0 \in X$ and $\delta, \epsilon > 0$ such that*

$$f(x_0) < \inf_{x\in X} f(x) + \epsilon,$$

there exists $y \in X$ satisfying the following properties:

(a) *$d(y, x_0) < \delta$;*

(b) $\displaystyle\sup_{\substack{u\neq y\\ u\in X}} \frac{f(y) - f(u)}{d(y, u)} < \frac{\epsilon}{\delta}.$

Therefore, if $(x_n)_{n\in\mathbb{N}} \subset H$ is a minimizing sequence, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$, equivalent in the sense of Cauchy, such that $(y_n)_{n\in\mathbb{N}}\, (\mathrm{PS})_{\inf f}$.

6.8 Exercises

Exercise 6.1. Let $k \in L^2([0,\,1] \times [0,\,1])$. The integral operator

$$T_k : L^2([0,\,1] \times [0,\,1]) \longrightarrow L^2([0,\,1]),$$

defined by setting

$$T_k(u)(x) := \int_0^1 k(x,\,y)u(x,\,y)\,\mathrm{d}y$$

is well-defined, linear, continuous and compact.

Solution. First, we notice that the operator T_k is well-defined as a consequence of both the Fubini theorem and the Hölder inequality. Namely, a simple computation shows that

$$\begin{aligned}
\|T_k(u)\|^2_{L^2([0,\,1])} &= \int_0^1 \left| \int_0^1 k(x,\,y)u(x,\,y)\,\mathrm{d}y \right|^2 \mathrm{d}x \le \\
&\le \int_0^1 \|k(x,\,\cdot)\|^2_{L^2([0,\,1])} \|u(x,\,\cdot)\|^2_{L^2([0,\,1])}\,\mathrm{d}x \le \\
&\le \left[\big\| \|k(x,\,\cdot)\|_{L^2([0,\,1])} \big\|_{L^2([0,\,1])} \big\| \|u(x,\,\cdot)\|_{L^2([0,\,1])} \big\|_{L^2([0,\,1])} \right]^2 = \\
&= \|k\|^2_{L^2([0,\,1]\times[0,\,1])} \|u\|^2_{L^2([0,\,1]\times[0,\,1])} < +\infty,
\end{aligned}$$

which means that T_k sends $L^2([0,\,1] \times [0,\,1])$ in $L^2([0,\,1])$, and the operator norm of T_k is less than or equal to the L^2-norm of k.

Compactness The operator T_k is clearly linear, and therefore the computation above shows that it is also continuous (since it is bounded). To prove that T_k is compact, we shall show that there is a sequence $(T_k^n)_{n\in\mathbb{N}}$ of finite-rank operator such that

$$\lim_{n\to+\infty} \|T_k - T_k^n\| = 0.$$

Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a orthonormal basis of $L^2([0,\,1])$. Then one can easily show that

$$\{\phi_i\phi_j\}_{i,\,j\in\mathbb{N}}$$

is a orthonormal basis of the product space $L^2([0,\,1] \times [0,\,1])$. In particular, if we set

$$k_{i,\,j} := \iint_{[0,\,1]^2} k(x,\,y)\phi_i(x)\phi_j(y)\,\mathrm{d}x\mathrm{d}y \quad \text{for all } i,\,j \in \mathbb{N},$$

then the kernel k is given by the following sum:

$$k(x,\,y) = \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} k_{i,\,j}\phi_i(x)\phi_j(y).$$

We now define the partial kernel

$$k_n(x,\, y) := \sum_{i=1}^{n}\sum_{j=1}^{+\infty} k_{i,\,j}\phi_i(x)\phi_j(y)$$

and the associated operator

$$T_k^n(u)(x) := \int_0^1 k_n(x,\, y)u(x,\, y)\,\mathrm{d}y.$$

The operator T_k^n maps $L^2([0,\,1]\times[0,\,1])$ into a finite-dimensional linear subspace of $L^2([0,\,1])$, and therefore T_k^n is a finite-rank operator. In conclusion, note that

$$\|T_k(u) - T_k^n(u)\|^2_{L^2([0,\,1])} = \left\| \int_0^1 k_n(x,\, y)u(x,\, y)\,\mathrm{d}y - \int_0^1 k_n(x,\, y)u(x,\, y)\,\mathrm{d}y \right\|^2_{L^2([0,\,1])} \le$$

$$\le \|k - k_n\|^2_{L^2([0,\,1]\times[0,\,1])}\|u\|_{L^2([0,\,1]\times[0,\,1])} =$$

$$= \left[\sum_{i=n+1}^{+\infty}\sum_{j=1}^{+\infty} |k_{i,\,j}|^2\right] \|u\|_{L^2([0,\,1]\times[0,\,1])},$$

and the last term goes to zero as $n \to +\infty$ since the sum is finite

$$\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} |k_{i,\,j}|^2 = \|k\|^2_{L^2([0,\,1]\times[0,\,1])} < +\infty.$$

$\square$

Exercise 6.2. Let $\lambda \in c_b$ be a bounded sequence and let us consider the operator

$$T_\lambda : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{N})$$

defined by setting

$$\mathbf{x} := (x_1,\, x_2,\, \dots) \longmapsto (\lambda_1 x_1,\, \lambda_2 x_2, \dots) =: \lambda \cdot \mathbf{x}.$$

Prove that

$$\|T_\lambda\| = \|\lambda\|_\infty,$$

and T_λ is compact if and only if $\lambda \in c_0$.

Solution. The operator norm $\|T_\lambda\|$ can be computed by standard means. Here we only show that T_λ is compact if and only if λ is an infinitesimal sequence.

" $\Longrightarrow$ " This implication is left as an easy exercise for the reader. The main idea is to use the definition of compact operator, choose an adequate sequence in $(\mathbf{z}^n)_{n\in\mathbb{N}} \subset \ell_2(\mathbb{N})$, and show that λ must be infinitesimal for $T(\mathbf{z}^n)$ to have a converging subsequence.

" $\Longleftarrow$ " To prove that T_λ is compact, we simply need to show that it is the limit (in the operator norm) of a sequence of finite-rank operators T_λ^n. Let us define

$$\lambda^n := (\lambda_1,\, \lambda_2,\, \dots,\, \lambda_n,\, 0,\, \dots) = \sum_{i=1}^{n} \lambda_i e_i \in c_0,$$

and consider the operator

$$T_\lambda^n(\mathbf{x}) := \lambda^n \cdot \mathbf{x}.$$

The operator T_λ^n maps $\ell_2(\mathbb{N})$ into the finite-dimensional linear subspace of $\ell_2(\mathbb{N})$ spanned by $e_1,\, \dots,\, e_n$, and therefore T_λ^n is a finite-rank operator. In conclusion, note that

$$\begin{aligned}\|T_\lambda(\mathbf{x}) - T_\lambda^n(\mathbf{x})\|_{\ell_2(\mathbb{N})}^2 &= \|\lambda \cdot \mathbf{x} - \lambda^n \cdot \mathbf{x}\|_{\ell_2(\mathbb{N})}^2 \leq \\ &\leq \|\lambda - \lambda^n\|_\infty^2 \|\mathbf{x}\|_{\ell_2(\mathbb{N})}^2 = \\ &= \sup_{i>n} |\lambda_i|^2 \|\mathbf{x}\|_{\ell_2(\mathbb{N})}^2,\end{aligned}$$

and the last term goes to zero as $n \to +\infty$ since the λ is an infinitesimal (c_0) sequence by assumption. $\square$

Exercise 6.3. Let $S : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{N})$ be the *shift* operator defined by

$$(x_1,\, x_2,\, \dots) \longmapsto (x_2,\, x_3,\, \dots).$$

Prove that $\sigma(S)$ is equal to $\overline{B_{\mathbb{C}}(0,\, 1)}$, but none of them is an eigenvalue.

Solution. The spectrum $\sigma(S)$ is contained in the closure of the ball of center zero and radius 1 because the spectral radius of S is equal to 1. The opposite implication is left to the reader.

Eigenvalues. We argue by contradiction. Suppose that S admits an eigenvalue $\lambda \in \mathbb{C}\backslash\{0\}$ with eigenstate $\mathbf{x} \in \ell_2(\mathbb{N})$. Then

$$(\lambda x_1,\, \lambda x_2,\, \dots) = (x_2,\, x_3,\, \dots),$$

which means that

$$x_i = \frac{x_{i+1}}{\lambda} \quad \text{for all } i = 1,\, \dots$$

In particular, the sequence $\mathbf{x}$ is uniquely determined by the value of the first coefficient x_1, i.e.

$$\mathbf{x} = (x_1,\, \frac{x_1}{\lambda},\, \frac{x_1}{\lambda^2},\, \dots),$$

and therefore $\mathbf{x}$ cannot be an element of $\ell_2(\mathbb{N})$ since

$$\|\mathbf{x}\|_{\ell_2(\mathbb{N})} = \sum_{i=1}^{+\infty} \left| \frac{x_1}{\lambda^{i-1}} \right|^2 = |x_1|^2 \sum_{i=0}^{+\infty} |\lambda^{-i}|^2 = +\infty$$

for all values of $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda| \leq 1$, and this is exactly what we wanted to prove. $\square$

Exercise 6.4. Let $X := C^0([0, 1])$, and let us consider the Volterra operator $T : C^0([0, 1]) \to C^0([0, 1])$, defined by

$$u \longmapsto \int_0^x u(t)\,\mathrm{d}t.$$

Prove that T is a compact operator with no eigenvalues, i.e.,

$$\sigma_e(T) = \{0\}.$$

Furthermore, find an explicit formula for the inverse operator $(\lambda \cdot \mathrm{id}_X - T)^{-1}$, as λ ranges in the resolvent set $\rho(T)$.

Exercise 6.5. Let $H = L^2(X, \Sigma, \mu)$, and let $f \in L^\infty(X, \Sigma, \mu)$. The operator

$$M_f : H \longrightarrow H, \qquad g \mapsto fg$$

is linear and continuous. Prove that:

(a) The norm operator of M_f is equal to $\|f\|_\infty$.

(b) The spectrum of M_f is given by

$$\sigma(M_f) = \bigcap_{N \text{ null-set}} \overline{g(X \setminus N)},$$

where $g \in [f]$ is any function in the equivalence class. More precisely, $\lambda \in \sigma(M_f)$ if and only if for any $\epsilon > 0$ the support of $g_*(\mu)$ intersects $B(\lambda, \epsilon)$ in a set of positive measure.

(c) The eigenvalues spectrum of M_f is given by

$$\sigma_e(M_f) = \left\{\lambda \in \mathbb{C} \setminus \{0\} \;\middle|\; \mu(f^{-1}(\lambda)) \neq 0\right\}.$$

Chapter 7

Functional Calculus

From Wikipedia: *"In mathematics, a functional calculus is a theory allowing one to apply mathematical functions to mathematical operators. It is now a branch (more accurately, several related areas) of the field of functional analysis, connected with spectral theory."*

7.1 Continuous Functional Calculus

In this section, we only develop the functional calculus of continuous functions defined on the spectrum of linear bounded symmetric operators on a Hilbert space H. Therefore, unless stated otherwise, we assume that $A \in \mathcal{L}^{sym}(H)$ throughout this paragraph.

First, we present a few motivational examples of functional calculus in a Hilbert space with finite dimension, and we also discuss what may happen in an infinite-dimensional setting using a computation that was already considered in the previous chapter.

Example 7.1.

(i) The Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases}$$

admits one and only one solution, that is,

$$x(t) = \exp(tA)\, x_0.$$

It follows that

$$\mathrm{e}^A := \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

is the result of the exponential function $\exp(\cdot)$ applied to an operator, and it is still an operator.

(ii) If $\|A\| < 1$, then we have already proved that

$$(\mathrm{id}_X - A)^{-1} = \sum_{n=0}^{+\infty} A^n.$$

More precisely, we can obtain the inverse of the operator $(\mathrm{id}_X - A)$ simply by applying a function, defined by its Taylor series, to the operator A.

(iii) Assume that H is a finite-dimensional Hilbert space. Then A is diagonalizable, that is, there exists a unitary operator U such that

$$A = UDU^{-1}, \qquad \text{where } D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

If $f : \sigma(A) \subset \mathbb{C} \longrightarrow \mathbb{C}$ is a function defined on the spectrum of A, then we can easily infer that

$$f(A) = Uf(D)U^{-1}, \qquad \text{where } f(D) = \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n)).$$

Real Polynomials. Let $p(x) \in \mathbb{R}[x]$ be a real polynomial defined by

$$p(x) = a_n x^n + \ldots a_1 x + a_0, \quad \text{with } a_i \in \mathbb{R} \text{ for all } i = 0, \ldots, n.$$

Set $B := p(A)$. The operator B is linear continuous and symmetric (i.e. $B \in \mathcal{L}^{sym}(H)$) because

$$(Bx, y) = \sum_{i=1}^{n} a_i (A^i x, y) + a_0(x, y) = \sum_{i=1}^{n} a_i (x, A^i y) + a_0(x, y) = (x, By),$$

where the last equality holds because a_i is real, and hence

$$a_i(x, A^i y) = (x, \bar{a}_i A^i y) = (x, a_i A^i y).$$

Therefore, the operator norm of B is given by the maximum absolute value of the eigenvalues of B, that is,

$$\|B\| = \sup_{\|x\| \leq 1} |\langle Bx, x\rangle| = \max_{\lambda \in \sigma(B)} |\lambda| .$$

On the other hand, the Spectral Map Theorem 6.40 shows that the spectrum of B is related to the spectrum of A via the polynomial p, that is,

$$\sigma(B) = p(\sigma(A)).$$

It follows that the operator norm of B is also equal to

$$\|B\| = \max_{\lambda \in p(\sigma(A))} |\lambda| = \max_{\lambda \in \sigma(A)} |p(\lambda)| = \|p\|_{\infty, \sigma(A)}, \tag{7.1}$$

that is, it is equal to the uniform norm of the restriction to the spectrum $\sigma(A)$ of the polynomial p.

Complex Polynomials. Let $p(z) \in \mathbb{C}[z]$ be a complex polynomial defined by

$$p(z) = a_n z^n + \ldots a_1 z + a_0, \quad \text{with } a_i \in \mathbb{C} \text{ for all } i = 0, \ldots, n.$$

It induces:

(a) A function $\sigma(A) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, also denoted by p, defined on the spectrum of the operator A.

(b) A bounded symmetric operator $p(A)$ if p is a real-valued polynomial, and a linear bounded (but, generally, not symmetric) operator if $p(z) \in \mathbb{C}[z] \setminus \mathbb{R}[x]$.

In particular, there exists an isometric correspondence between **(a)** and **(b)**. More precisely, it turns out that by sending a polynomial p to $p(A)$ we have a one-to-one map

$$\{\text{polynomial functions } \sigma(A) \to \mathbb{C}\} \longleftrightarrow \{\text{operators of the form } p(A)\} = \mathbb{C}[A],$$

where $\mathbb{C}[A]$ is the **commutative** algebra spanned by A. Furthermore, the correspondence is also an isometry as a consequence of (7.1), since they are endowed with the subspace norms induced by $\left(C^0\left(\sigma(A), \mathbb{C}\right), \|\cdot\|_{\infty, \sigma(A)}\right)$ and $(\mathcal{L}(H), \|\cdot\|)$ respectively.

Proposition 7.1. *Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric bounded operator. There exists a unique continuous homomorphism of algebras*

$$\Phi : C^0\left(\sigma(A), \mathbb{C}\right) \longrightarrow \overline{\mathbb{C}[A]}$$

with the additional requirement that $\Phi\left(\mathrm{id}_{\sigma(A)}\right) = A$. Furthermore, the homomorphism Φ satisfies the following properties:

(a) *The map Φ is an isometry.*

(b) *If $f \in C^0\left(\sigma(A), \mathbb{C}\right)$ is a positive function, then the operator $\Phi(f) := f(A)$ is symmetric and positive, that is,*

$$\lambda \in \sigma\left(f(A)\right) \implies \lambda \geq 0.$$

(c) *For any $f \in C^0\left(\sigma(A), \mathbb{C}\right)$ it turns out that*

$$\Phi(f)^* = \Phi\left(\overline{f}\right).$$

(d) *For all $B \in \mathcal{L}(H)$ such that $[A, B] = 0$ and for all continuous function $f \in C^0\left(\sigma(A), \mathbb{C}\right)$, it turns out that*

$$[\Phi(f), B] = 0.$$

Proof. To ease the notation for the reader, we divide the proof into many little steps.

Uniqueness. Suppose that there exists a homomorphism of algebras Φ, and suppose that it satisfies the properties (**a**) – (**d**). Φ. The requirement that $\Phi\left(\mathrm{id}_{\sigma(A)}\right) = A$ allows us to extend Φ to all the polynomials as follows:

$$p \longmapsto p(A).$$

Since Φ is an isometry, it follows from Stone-Weierstrass[1] (note that $\sigma(A)$ is compact) that Φ can be uniquely extended by density to the whole space $C^0\left(\sigma(A), \mathbb{C}\right)$.

[1]Suppose that X is a compact Hausdorff space and A is a subalgebra of $C^0\left(X; \mathbb{C}\right)$ which contains a non-zero constant function. Then A is dense if and only if it separates points.

Existence. The map Φ is already well-defined from the complex polynomials to $\mathbb{C}[A]$. To prove that it can be extended up to the closures of these spaces, we simply need to show the property (**a**), that is, Φ is an isometry.

(a) Let $p(z) \in \mathbb{C}[z]$ be a complex polynomial, and let $B := p(A)$. A straightforward computation shows that

$$\begin{aligned} \|p(A)\|^2_{\mathcal{L}(H)} = \|p(A)^* \, p(A)\|_{\mathcal{L}(H)} &= \\ = \|\overline{p}(A) \, p(A)\|_{\mathcal{L}(H)} &= \\ = \|(\overline{p} \, p) \, (A)\|_{\mathcal{L}(H)} &= \\ = \max_{\lambda \in \sigma(A)} |\overline{p} \, p(\lambda)| = \|p\|^2_{\infty, \, \sigma(A)}&, \end{aligned}$$

that is, Φ is an isometric homomorphism that sends $\mathrm{id}_{\sigma(A)}$ to A. Therefore, there exists an unique extension of Φ up to the closures, that is,

$$\Phi : C^0 \left(\sigma(A), \, \mathbb{C}\right) \to \overline{\mathbb{C}[A]}.$$

Notice that the closure of the complex polynomials defined on $\sigma(A)$ is $C^0 \left(\sigma(A), \, \mathbb{C}\right)$ as a consequence of the Stone-Weierstrass and the compactness of $\sigma(A)$.

To conclude the proof of this first property (and, thus, of the existence of Φ), it remains to justify the identities we used to infer that Φ is an isometry.

The red identity is true under more general assumptions, that is, it is satisfied by any linear bounded operator B. Indeed, the operator $B^* B$ is always symmetric and thus

$$\|B\|^2_{\mathcal{L}(H)} = \left(\sup_{\|x\|=1} \|Bx\| \right)^2 = \sup_{\|x\| \leq 1} \langle B^* Bx, \, x \rangle = \|B^* B\|_{\mathcal{L}(H)}$$

The blue identity follows easily from $(\imath A)^* = -\imath A^*$, while the green identity follows from the fact that Φ is a homomorphism of algebras.

In a similar fashion, the properties (**b**), (**c**) and (**d**) are certainly true when $f \in C^0 \left(\sigma(A), \, \mathbb{C}\right)$ is a polynomial, and they are preserved under the isometric extension.

The reader may fill the details as an exercise. Notice that (**b**) follows immediately from (**c**) and the Spectral Map Theorem 6.40. □

In this paragraph we introduce a more general class of operators, and we show that a similar (but slightly more technical) result holds true.

Definition 7.2 (Normal Operator)**.** A linear bounded operator A defined on a Hilbert space H is *normal* if and only if

$$[A, \, A^*] := AA^* - A^* A = 0.$$

Example 7.2. Here we give some examples of normal operators.

(1) If $A \in \mathcal{L}^{sym}(H)$, then A is normal.

(2) If A is a linear bounded unitary operator, then A is also normal.

(3) If $A \in \mathcal{L}^{sym}(H)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then λA is a normal operator which is, generally, not symmetric.

(4) If $A \in \mathcal{L}^{sym}(H)$ and $f \in C^0(\sigma(A), \mathbb{C})$, then $f(A)$ is a normal operator, but it may fail to be symmetric as we proved above.

Lemma 7.3 (Spectral Radius)**.** *Let $A \in \mathcal{L}(H)$ be a normal operator. Then the spectral radius of A, denoted by r_A, is equal to $\|A\|$.*

Proof. Since A is a linear bounded operator, Theorem 6.39 proves that the spectral radius r_A is given by

$$r_A = \lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

Therefore, it is enough to prove that $\lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}}$ is equal to $\|A\|$. Recall that this property has already been proved for symmetric operators (see Proposition 6.41). It follows that

$$\begin{aligned}
\|A\| &= \|A^* A\|^{\frac{1}{2}} = \\
&= \lim_{n \to +\infty} \|(A^* A)^n\|^{\frac{1}{2n}} = \\
&\overset{(\star)}{=} \left[\lim_{n \to +\infty} \|(A^*)^n A^n\|^{\frac{1}{n}} \right]^{\frac{1}{2}} \le \\
&\le \left[\lim_{n \to +\infty} \|(A^*)^n\|^{\frac{1}{n}} \lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}} \right]^{\frac{1}{2}} = \\
&= (r_{A^*} \cdot r_A)^{\frac{1}{2}} = r_A,
\end{aligned}$$

where the equality $(\star)$ follows from the normality of A. The proof is now concluded because the opposite one ($\|A\| \ge r_A$) is trivially satisfied by any linear bounded operator A. $\square$

7.2 Borel Functional Calculus

The functional calculus of symmetric operators does not work anymore since $p(A)^* p(A)$ is not a polynomial in the sole variable A, but it depends on **both** A and A^*.

Moreover, $\sigma(A)$ may have a nonempty internal part, which, in turn, would imply that the polynomials are not dense anymore.

The fundamental idea here is to first develop a functional calculus in one variable, assuming that f is a Borel function (that is, there is no need for f to be continuous.)

Remark 7.1. Let H be a Hilbert space. Then there is a canonical isomorphism

$$\mathcal{L}(H) = \mathcal{L}_c(H)^{**}.$$

We will not prove this fact, but it is important to know because it proves that $\mathcal{L}(H)$ is the dual of a Banach space, and thus we can endow it with a weak-$*$ topology.

Theorem 7.4 (Riesz-Markov)**.** *Let Σ be a compact metric space. The map*

$$\mathcal{M}(\Sigma;\ \mathbb{C}) \ni \mu \longrightarrow \Lambda_\mu \in C^0(\Sigma;\ \mathbb{C})^* \ :\ \Lambda_\mu(f) := \int_\Sigma f(x)\,\mathrm{d}\mu(x)$$

is an isometry, where $\mathcal{M}(\Sigma;\ \mathbb{C})$ is equipped with the total variation norm.

Proposition 7.5. *Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator. There exists a sesquilinear application*

$$H \times H \ni (x,\, y) \longmapsto \mu_{x,\,y} \in \mathcal{M}(\sigma(A),\, \mathbb{C}),$$

where $\mu_{x,\,y}$ is a complex-valued Borel measure, whose support is contained in $\sigma(A)$, such that the following properties are satisfied:

(a) *The mapping*

$$\Psi_{x,\,y} : C^0(\sigma(A);\ \mathbb{C}) \ni f \longmapsto (f(A)\,x,\, y)_H \in \mathbb{C}$$

is linear and continuous, that is, $\Psi_{x,\,y} \in C^0(\sigma(A);\ \mathbb{C})^*$. Furthermore, for all $x,\, y \in H$ we have

$$(f(A)x,\, y) = \int_{\sigma(A)} f(t)\,\mathrm{d}\mu_{x,\,y}(t).$$

(b) *For all $x,\, y \in H$ it turns out that*

$$\mu_{y,\,x} = \overline{\mu_{x,\,y}}.$$

(c) *For all $x,\, y \in H$ it turns out that*

$$\|\mu_{x,\,y}\| := |\mu_{x,\,y}|\,(\sigma(A)) \le \|x\|\|y\|,$$

where the absolute value of a measure ν is defined by

$$|\nu|\,(E) := \sup_\pi \sum_{A\in\pi} |\nu(A)| \quad \text{for all } E \subset \sigma(A).$$

(d) *The measure $\mu_x := \mu_{x,\,x}$ is positive and the total variation of μ_x is exactly equal to the norm of x squared, that is,*

$$|\mu_x|\,(\sigma(A)) = \|x\|^2.$$

(e) *For any $f \in C^0(\sigma(A);\ \mathbb{C})$, it turns out that the Radon-Nikodym derivative is given by*

$$\frac{\mathrm{d}\mu_{f(A)x,\,y}}{\mathrm{d}\mu_{x,\,y}} = f.$$

Proof. The existence of the sesquilinear application follows immediately from the first property and the Riesz-Markov representation theorem, as mentioned below.

(a) For any couple $(x,\, y) \in H \times H$, the mapping

$$\Psi_{x,\, y} : C^0\left(\sigma(A);\; \mathbb{C}\right) \ni f \longmapsto \left(f(A)\, x,\, y\right) \in \mathbb{C}$$

is linear since it is equal to the composition of the following linear operators:

$$f \longmapsto f(A) \longmapsto f(A)x \longmapsto (f(A)x,\, y)\,.$$

Furthermore, $\Psi_{x,\, y}$ is bounded as a consequence of the definition of operator norm and the Cauchy-Schwartz inequality:

$$\begin{aligned} |\Psi_{x,\, y}(f)| = |(f(A)x,\, y)| &\leq \\ &\leq \|f(A)x\|\|y\| \leq \\ &\leq \|f(A)\|\|x\|\|y\| = \|f\|_{\infty,\, \sigma(A)}\|x\|\|y\|, \end{aligned}$$

and hence $\Psi_{x,\, y}$ belongs to the dual space $C^0\left(\sigma(A);\; \mathbb{C}\right)^*$. It follows from the Riesz-Markov Theorem 7.4 that, for any x and y in H, there exists a unique complex-valued measure $\mu_{x,\, y} \in \mathcal{M}\left(\sigma(A),\, \mathbb{C}\right)$ such that

$$(f(A)x,\, y) = \int_{\sigma(A)} f(t)\, \mathrm{d}\mu_{x,\, y}(t).$$

(b) The identification with the dual space $C^0\left(\sigma(A);\; \mathbb{C}\right)^*$ immediately implies that the two measures are equal if and only if they are equal against every continuous function f. A simple computation shows that

$$\begin{aligned} \langle \mu_{y,\, x},\, f\rangle = (f(A)y,\, x) &= \\ &= (y,\, f(A)^*x) = \\ &= \left(y,\, \overline{f}(A)\, x\right) = \\ &= \overline{\left(\overline{f}(A)x,\, y\right)} = \left\langle \overline{\mu}_{x,\, y},\, f\right\rangle, \end{aligned}$$

where the red identity follows from the following, general, property:

$$\overline{\int_\Sigma f(t)\, \mathrm{d}\mu(t)} = \int_\Sigma \overline{f(t)}\, \mathrm{d}\overline{\mu}(t).$$

(c) This estimate follows immediately from the argument used above to show (**a**).

(d) The measure μ_x is positive if and only if for any positive function $f \in C^0\left(\sigma(A);\; \mathbb{C}\right)$ it turns out that

$$\int_{\sigma(A)} f(t)\, \mathrm{d}\mu(t) \geq 0.$$

This is an easy consequence of the fact that $f \geq 0 \implies f(A)$ is a positive operator (see Proposition 7.1).

(e) For all functions $g \in C^0(\sigma(A);\, \mathbb{C})$ it turns out that

$$\begin{aligned}\langle \mu_{f(A)x,\,y},\, g\rangle &= \int_{\sigma(A)} g(t)\,\mathrm{d}\mu_{f(A)x,\,y}(t) = \\ &= \left(g(A)\left(f(A)x\right),\, y\right) = \\ &= \left(g \cdot f(A)x,\, y\right) = \langle \mu_{x,\,y},\, g \cdot f\rangle\,.\end{aligned}$$

Therefore, the Radon-Nikodym density of $\mu_{f(A)x,\,y}$ with respect to $\mu_{x,\,y}$ is the function f.

□

Lax-Milgram Theorem. We now introduce a fundamental result in functional analysis and partial differential equations, which is also the main ingredient needed to extend the functional calculus from continuous to Borel functions.

Theorem 7.6 (Lax-Milgram)**.** *Let H be a complex Hilbert space, and let $b : H \times H \longrightarrow \mathbb{C}$ be a sesquilinear form. Suppose that b is bounded, that is, there exists a constant $C > 0$ such that*

$$|b(x,\, y)| \leq C \cdot \|x\|\|y\| \quad \textit{for all } (x,\, y) \in H \times H.$$

Then there exists a continuous linear operator $B \in \mathcal{L}(H,\, H)$ such that

$$b(x,\, y) = (Bx,\, y) \quad \textit{for all } (x,\, y) \in H \times H. \tag{7.2}$$

Proof. For $x \in H$ fixed, the application

$$H \ni y \longmapsto \overline{b(x,\, y)} \in \mathbb{C}$$

is linear and continuous. Therefore, a straightforward application of the Riesz Representation Theorem 1.22 proves that there exists a unique element $v \in H$ such that

$$\overline{b(x,\, y)} = (v,\, y) \quad \text{for every } y \in H.$$

If we denote by $B(x)$ the element $v \in H$, then we find an operator $B : H \longrightarrow H$ satisfying the identity

$$b(x,\, y) = (B(x),\, y)\,,$$

and thus it is enough to prove that B is linear and continuous. The uniqueness of the vector $v \in H$ (given by the representation theorem) is enough to infer that B is linear since the two elements

$$\alpha B(x) + \beta B(y) \quad \text{and} \quad B\left(\alpha x + \beta y\right)$$

both satisfies the relation (7.2). More precisely, we have that

$$\begin{aligned}(B(\alpha x + \beta y),\, z) &= b(\alpha x + \beta y,\, z) = \\ &= \alpha b(x,\, z) + \beta b(y,\, z) = \\ &= \alpha(B(x),\, z) + \beta(B(y),\, z) = (\alpha B(x) + \beta B(y),\, z)\end{aligned}$$

for all $x,\, y \in H \times H$ and $z \in H$. It follows that (since B is linear) to prove the continuity, it suffices to prove the boundedness of B. It follows from the assumption that for any $x \in H$ we have

$$|b(x,\, y)| \leq C \cdot \|x\| \|y\| \implies \|Bx\| \leq C \cdot \|x\|,$$

and this concludes the proof. □

Notions of Convergence. In this brief paragraph, we introduce a different notion of convergence on the space of bounded Borel functions, and we also recall the strong convergence for continuous linear operators.

Notation. Let Σ be a compact subset of $\mathbb{C}$. From now on, we shall denote by $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ the set of all bounded Borel functions defined on Σ.

Definition 7.7. Let $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ be a sequence of functions. We say that f_n converges *dominantly pointwise* to f if and only if the following properties are satisfied:

(1) For every $x \in \Sigma$ it turns out that

$$f_n(x) \xrightarrow{n\to+\infty} f(x).$$

(2) There exists a positive constant $C > 0$ such that

$$\|f_n\|_{\infty,\, \Sigma} \leq C < +\infty \tag{7.3}$$

Remark 7.2. Let μ be a finite measure, and let $\Sigma \subset \mathbb{C}$ be a compact set. Then every sequence of function $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ converging to some f dominantly pointwise, also converges to f with respect to the $L^2(\Sigma,\, \mu)$ norm.

Proof. It suffices to notice that the difference $|f_n(x) - f(x)|^2$ is uniformly bounded, and it converges μ-almost everywhere (in Σ) to zero. □

Lemma 7.8. *Let $\mathscr{P}(\Sigma;\, \mathbb{C})$ be the set of all polynomials defined on Σ. The sequential closure of $\mathscr{P}(\Sigma;\, \mathbb{C})$ with respect to the dominantly pointwise convergence is the space $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$.*

Road Map. We do not prove this statement here, but the reader may try to fill in the details as an exercise.

(a) If $V \subseteq \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ is a subspace, then $\overline{V}^{seq} \subseteq \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ is also a subspace.

(b) If $V \subseteq \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ is a lattice, then $\overline{V}^{seq} \subseteq \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ is also a lattice.

(c) If $V = C^0(\Sigma;\, \mathbb{C})$, then

$$\left\{ E \subseteq \Sigma \;\middle|\; \chi_E \in \overline{V}^{seq} \right\}$$

is a σ-algebra containing the Borel σ-algebra.

□

Definition 7.9. Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ be a sequence of linear bounded operators. We say that T_n converges *strongly* to T if and only if

$$T_n(x) \xrightarrow{n\to+\infty} T(x) \quad \text{for all } x \in H, \tag{7.4}$$

that is, if and only if T_n is pointwise convergent to T.

Remark 7.3. The Banach-Steinhaus theorem implies that, if $T_n x$ converges for all $x \in H$, then there exists a linear bounded operator T such that $T_n \to T$ strongly.

Remark 7.4. A sequence of linear bounded operators $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ converges weakly to some $T \in \mathcal{L}(H)$ if and only if

$$\langle T_n x,\, y\rangle_H \xrightarrow{n\to+\infty} \langle Tx,\, y\rangle \quad \text{for every } x,\, y \in H. \tag{7.5}$$

Lemma 7.10. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ be a weakly converging sequence, and let $T \in \mathcal{L}(H)$ be its limit. Then the following assertions hold:*

(a) *If $S \in \mathcal{L}(H)$ is a linear bounded operator, then $T_n S$ converges weakly to the product TS.*

(b) *If $S \in \mathcal{L}(H)$ is a linear bounded operator, then BT_n converges weakly to the product ST.*

Proof.

(a) The sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ is weakly convergent; therefore, for every $x,\, y \in H$, it turns out that
$$\langle T_n Sx,\, y\rangle = \langle T_n\left(Sx\right),\, y\rangle \xrightarrow{n\to+\infty} \langle T\left(Sx\right),\, y\rangle\,,$$
which is exactly what we wanted to prove.

(b) The sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ is weakly convergent; therefore, for all $x,\, y \in H$, it turns out that

$$\langle ST_n x,\, y\rangle = \langle T_n x,\, S^* y\rangle \xrightarrow{n\to+\infty} \langle Tx,\, S^* y\rangle = \langle STx,\, y\rangle\,.$$

□

Definition 7.11 ($\star$-Homomorphism)**.** Let A and B be $\star$-algebras. A $\star$-homomorphism $f : A \longrightarrow B$ is an algebra homomorphism that is compatible with the involutions of A and B, that is,
$$f(a^*) = f(a)^* \quad \text{for all } a \in A.$$

We are finally ready to state and prove the main result concerning the functional calculus for bounded Borel functions.

Proposition 7.12. *Let $A \in \mathcal{L}^{sym}(H)$ be a symmetric operator, and let $\sigma(A) := \Sigma$. There exists a unique $\star$-homomorphism of $\star$-algebras*

$$\Phi : \mathscr{L}^\infty\left(\Sigma;\, \mathbb{C}\right) \longrightarrow \mathcal{L}(H)$$

such that $\Phi\left(\mathrm{id}_\Sigma\right) = A$. Moreover, it satisfies the following properties:

(a) *The mapping Φ is sequentially continuous if we endow $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ with the dominate pointwise convergence and $\mathcal{L}(H)$ with the weak convergence of operators.*

(b) *The mapping*

$$\Psi_{x,\,y} : \mathscr{L}^\infty(\Sigma;\, \mathbb{C}) \ni f \longmapsto (\Phi(f)x,\, y) \in \mathbb{C}$$

is linear and continuous, that is, $\Psi_{x,\,y} \in \mathscr{L}^\infty(\Sigma;\, \mathbb{C})^*$. Furthermore, there exists a unique complex-valued measure $\mu_{x,\,y}$ satisfying

$$(\Phi(f)x,\, y) = \int_\Sigma f(t)\, \mathrm{d}\mu_{x,\,y}(t).$$

(c) *For any $f \in \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ it turns out that*

$$\|\Phi(f)\| \le \|f\|_{\infty,\,\Sigma}.$$

(d) *For any $f \in \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ it turns out that*

$$B \in \mathcal{L}(H) \;:\; [A,\, B] = 0 \implies [\Phi(f),\, B] = 0.$$

(e) *For any $f \in \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ it turns out that*

$$f \ge 0 \implies \Phi(f) \ge 0.$$

(f) *The mapping Φ is sequentially continuous if we endow $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ with the dominate pointwise convergence and $\mathcal{L}(H)$ with the strong convergence of operators.*

Proof. We first prove that if such a map exists, then it must be unique. Then, we use the formula obtained in the uniqueness step and show that it actually defines a map with the desired properties.

Uniqueness. The map Φ is unique because it is the unique isometric extension of the continuous functional calculus (see Proposition 7.1). Moreover, from the formula

$$(\Phi(f)x,\, y) = \int_\Sigma f(t)\, \mathrm{d}\mu_{x,\,y}(t) \tag{7.6}$$

valid for all $f \in C^0(\Sigma;\, \mathbb{C})$, it follows that the functional $\Psi_{x,\,y}$, defined in the statement, can be extended to the whole $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ by (sequential) continuity.

Existence. The mapping

$$H \times H \ni (x,\, y) \longmapsto \Psi_{x,\,y}(f) \in \mathbb{C},$$

defined by formula (7.6), is a sesquilinear application satisfying the assumptions of the Lax-Milgram Theorem 7.6. Therefore, there exists a continuous linear operator, denoted by $\Phi(f)$, such that

$$\Psi_{x,\,y}(f) = (\Phi(f)x,\, y)_H\,,$$

which is exactly what we wanted to prove.

Sequential Continuity (a). If we endow $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ with the dominate pointwise convergence and $\mathcal{L}(H)$ with the weak convergence of operators, then the sequential continuity is an immediate consequence of the properties of the spectral measures (see Proposition 7.5).

Properties (b) – (e) . Recall that the properties (b) – (e) have already been proved in Proposition 7.1 for the continuous functional calculus. Since Φ is a $\star$-homomorphism, we infer that these properties are already satisfied when $f \in C^0(\Sigma;\, \mathbb{C})$.

The reader may complete the proof of this step by noticing that these properties can all be extended by continuity to $\mathscr{L}^\infty(\Sigma;\, \mathbb{C})$.

Sequential Continuity (f). Let $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^\infty(\Sigma;\, \mathbb{C})$ be a sequence of functions such that

$$f_n(x) \xrightarrow{n\to+\infty} f(x) \quad \text{for all } x \in \Sigma \quad \text{and} \quad \|f_n\|_{\infty,\,\Sigma} \le C.$$

The goal is to prove that the sequence of continuous linear operators $(\Phi(f_n))_{n\in\mathbb{N}} \subset \mathcal{L}(H)$ converges to the continuous linear operator $\Phi(f)$ pointwise. Indeed, if $\|\cdot\|_H$ denotes the norm on the Hilbert space H, then it turns out that

$$\begin{aligned} \|\Phi(f_n)(x) - \Phi(f)(x)\|_H^2 &= \left((\Phi(f_n) - \Phi(f))\, x,\, (\Phi(f_n) - \Phi(f))\, x \right)_H = \\ &= \left(\Phi\left(f_n - f\right) \Phi\left(\overline{f_n - f}\right) x,\, x \right)_H = \\ &= \left(\Phi\left((f_n - f)\left(\overline{f_n - f}\right)\right) x,\, x \right)_H = \\ &= \left(\Phi\left(|f_n - f|^2 \right) x,\, x \right)_H = \\ &= \int_\Sigma |f_n(t) - f(t)|^2 \,\mathrm{d}\mu_x(t) = \\ &= \|f_n - f\|^2_{L^2(\Sigma,\,\mu_x)} \xrightarrow{n\to+\infty} 0 \end{aligned}$$

since by Remark 7.2 the dominated pointwise convergence is stronger than the L^2 convergence.

The red identity follows from the fact that $\Phi(f)^* = \Phi\left(\overline{f}\right)$, while the blue identity follows from the fact that Φ is a $\star$-homomorphism. □

7.3 Multivariable Functional Calculus

The primary goal of this section is to generalize the functional calculus to a multivariable setting, that is, when we have to deal with more than one operator.

Remark 7.5. Let $A \in \mathcal{L}(H)$ be a normal operator. Then A is the sum of symmetric operators $B,\, C \in \mathcal{L}^{sym}(H)$, defined in the following way:

$$A = B + \imath C, \qquad \text{where} \quad B = \frac{A + A^*}{2} \quad \text{and} \quad C = \frac{A - A^*}{2\imath}.$$

Since A commutes with its adjoint A^*, a simple computation shows that B commutes with C, i.e.,

$$[A,\, A^*] = 0 \implies [B,\, C] = 0.$$

In particular, the functional calculus for normal operators will follows immediately from the multivariable functional calculus, which we shall develop throughout this section.

Proposition 7.13. *Let $A_1, \dots, A_n \in \mathcal{L}^{sym}(H)$ be symmetric operators such that*

$$[A_i,\, A_j] = 0 \quad \textit{for all } i,\, j \in \{1, \dots, n\}.$$

For each index i let $I_i := [-\|A_i\|,\, \|A_i\|]$, and let

$$I := \prod_{i=1}^{n} I_i.$$

There exists a unique $\star$-homomorphism of $\star$-algebras

$$\Phi : \mathscr{L}^\infty(I;\, \mathbb{C}) \longrightarrow \mathcal{L}(H)$$

sequentially continuous with respect to the dominated pointwise convergence and the strong convergence of operators respectively, sending the projection π_i to A_i for each index i, that is,

$$\pi_i : I \longrightarrow I_i \implies \Phi(\pi_i) = A_i.$$

Furthermore, it satisfies the following properties:

(a) *The mapping*

$$\Psi_{x,\,y} : \mathscr{L}^\infty(I;\, \mathbb{C}) \ni f \longmapsto (\Phi(f)x,\, y) \in \mathbb{C}$$

is linear and continuous, that is, $\Psi_{x,\,y} \in \mathscr{L}^\infty(I;\, \mathbb{C})^*$. Furthermore, there exists a unique complex-valued measure $\mu_{x,\,y}$ such that

$$(\Phi(f)x,\, y) = \int_I f(t_1, \dots, t_n)\, \mathrm{d}\mu_{x,\,y}(t_1, \dots, t_n).$$

(b) *For any $f \in \mathscr{L}^\infty(I;\, \mathbb{C})$ it turns out that*

$$\|\Phi(f)\| \le \|f\|_{\infty,\, I}.$$

(c) *For any $f \in \mathscr{L}^\infty(I;\, \mathbb{C})$ it turns out that*

$$B \in \mathcal{L}(H) \,:\, [A_i,\, B] = 0 \quad \textit{for every } i = 1, \dots, n \implies [\Phi(f),\, B] = 0.$$

(d) *For any $f \in \mathscr{L}^\infty(I;\, \mathbb{C})$ it turns out that*

$$f \ge 0 \implies \Phi(f) \ge 0.$$

Proof. We first suppose that such an operators satisfying these properties exists, and we prove that it must be uniquely determined. Then, we prove the existence of Φ and give the idea on how it can be extended to the space of Borel bounded functions.

Uniqueness. First, we notice that if f is a function that depends on a single variable x_i, then $\Phi(f)$ coincides with the result of the usual Borel functional calculus

$$f \longmapsto f(A_i).$$

On the other hand, since Φ is a homomorphism, for every collection $E_1 \subset I_1, \dots, E_n \subset I_n$ of Borel subsets, it turns out that

$$\Phi\left(\chi_{E_1 \times \cdots \times E_n}\right) = \prod_{i=1}^{n} \chi_{E_i}(A_i). \tag{7.7}$$

Therefore, Φ is uniquely determined by its value on simple functions (finite sums of characteristics of Borel subsets $E \subset I$) and, by pointwise convergence density, it turns out that Φ can be extended in a unique way to the whole space $\mathscr{L}^{\infty}(I;\, \mathbb{C})$.

Existence, Step 1. Let $\mathcal{P}_i$ be a finite and measurable partition of I_i for all $i \in \{1, \dots, n\}$, and let us denote by $\mathcal{P}$ the product partition of I, that is,

$$\mathcal{P} := \{E_1 \times \cdots \times E_n \mid E_i \in \mathcal{P}_i \text{ for every } i \in \{1, \dots, n\}\}.$$

Let $\mathscr{F}_i$ be the finite algebra generated by the partition $\mathcal{P}_i$ (i.e., its power set), and let $\mathscr{F}$ be the finite algebra generated by $\mathcal{P}$. Let Φ be a Borel function defined on $\mathscr{F}$. By construction, it turns out that

$$\Phi \in \mathcal{B}_{\mathscr{F}} := \operatorname{Span}\{\chi_E \mid E \in \mathcal{P}\},$$

as the reader may readily check using (7.7). The map Φ is certainly a $\star$-homomorphism since

$$\left[\chi_{E_i}(A_i),\, \chi_{E_j}(A_j)\right] = 0$$

for all $i, j \in \{1, \dots, n\}$. Indeed, it follows easily from Proposition 7.12, which asserts that in the Borel functional calculus each operator $B \in \mathcal{L}(H)$ such that $[B, A] = 0$ also commutes with $\Phi(f)$, that is, $[\Phi(f), B] = 0$.

The map Φ is also *ordinate*, that is, it sends positive functions $f \geq 0$ to positive operators $\Phi(f)$. Indeed, since Φ is a $\star$-homomorphism it turns out that

$$f = \left(\sqrt{f}\right)^2 \implies \Phi(f) = \Phi\left(\sqrt{f}\right)^2 = \Phi(f)^* \,\Phi(f),$$

and the latter operator is always positive.

Existence, Step 2. A refinement of the partition $\mathcal{P}_i$ for all $i \in \{1, \dots, n\}$ yields to a different function Φ, which extends the one constructed previously. Hence Φ is well-defined on the set

$$\bigcup_{\mathcal{P}} \mathcal{B}_{\mathscr{F}} \;:\; \mathscr{F} \text{ algebra generated by the finite partition } \mathcal{P}.$$

There are now two different, but equivalent, ways to extend Φ to the set of Borel bounded functions:

(1) We first extend Φ to $C^0(I;\, \mathbb{C})$ by density. Then we extend it to the space $\mathscr{L}^{\infty}(I;\, \mathbb{C})$, as we have already done in the case of a single variable (that is, we define suitable spectral measures and prove an equivalent assertion to Proposition 7.12).

(2) We extend Φ directly to $\mathscr{L}^{\infty}(I;\, \mathbb{C})$ by density.

Properties. The map $\Phi : \mathscr{L}^\infty(I;\, \mathbb{C}) \to \mathcal{L}(H)$ is clearly a $\star$-homomorphism, and a similar argument to the one used in the previous section, is enough to prove that the properties (**a**) – (**d**) hold true. For example, if $f \in \mathscr{L}^\infty(I)$ is a real-valued function, then

$$-\|f\|_{\infty,\, I} \le f \le \|f\|_{\infty,\, I} \overset{\Phi}{\Longrightarrow} -\|f\|_{\infty,\, I}\mathrm{id}_H \le \Phi(f) \le \|f\|_{\infty,\, I}\mathrm{id}_H \implies$$

$$\implies \|\Phi(f)\| \le \|f\|_{\infty,\, I}.$$

If $f \in \mathscr{L}^\infty(I;\, \mathbb{C})$ is a bounded Borel function, then the previous inequality implies that

$$\|\Phi\left(|f|\right)\| \le \|f\|_{\infty,\, I},$$

and this is enough to conclude that **(b)** holds true since

$$\Phi\left(|f|\right) = \sqrt{\Phi^*(f)\,\Phi(f)} \implies \|\Phi(f)\| = \|\Phi\left(|f|\right)\|.$$

□

Multiplicative Operator. In this paragraph, we briefly introduce one of the most important applications of the functional calculus and, in the next section, we will see why it is so important.

Proposition 7.14. *Let $(X,\, \Theta,\, m)$ be a measure space, and assume that $m(X) < +\infty$. If $a : X \to \mathbb{R}$ is a measurable function such that*

$$a \cdot L^2\left(X,\, \Theta,\, m\right) \subseteq L^2\left(X,\, \Theta,\, m\right),$$

then a is essentially bounded, that is, $a \in L^\infty\left(X,\, \Theta,\, m\right)$.

Proposition 7.15. *Let $(X,\, \Theta,\, m)$ be a measure space, and assume that $m(X) < +\infty$. Let $H = L^2\left(X,\, \Theta,\, m\right)$ and let $\alpha \in L^\infty\left(X,\, \Theta,\, m\right)$ be a class of equivalence of measurable essentially bounded real-valued functions $X \to \mathbb{R}$. The multiplicative operator*

$$M_\alpha : H \to H, \qquad u \longmapsto \alpha \cdot u$$

satisfies the following properties:

(a) *The kernel of $M_\alpha - \lambda\,\mathrm{Id}_H$ is equal to $L^2\left(\mathrm{spt}(\lambda),\, \Theta,\, m\right)$, where*

$$\mathrm{spt}(\lambda) := \alpha^{-1}(\lambda) \subseteq X.$$

(b) *The rank of $M_\alpha - \lambda\,\mathrm{Id}_H$ is equal to*

$$\mathrm{Ran}\left(M_\alpha - \lambda\,\mathrm{Id}_H\right) = \left\{u \in L^2\left(X,\, \Theta,\, m\right) \,\middle|\, \exists\, v \in L^2\left(X,\, \Theta,\, m\right),\ u = (\alpha - \lambda)\, v\right\}.$$

(c) *The operator $M_\alpha - \lambda\,\mathrm{Id}_H$ is surjective if and only if*

$$\frac{1}{\alpha - \lambda} \in L^\infty\left(X,\, \Theta,\, m\right)$$

if and only if there exists $\epsilon > 0$ such that

$$m\left(\{|\alpha - \lambda| < \epsilon\}\right) = 0$$

if and only if there exists $\epsilon > 0$ such that

$$\alpha_\#\, m\left(B(\lambda,\, \epsilon)\right) = 0.$$

(d) *For any* $f \in \mathscr{L}^\infty(\sigma(M_\alpha);\, \mathbb{C})$ *it turns out that*

$$f\,(M_\alpha) = M_{f\circ\alpha}.$$

(e) *For any* $u,\, v \in H$ *and for any* $f \in \mathscr{L}^\infty(\sigma(M_\alpha);\, \mathbb{C})$ *the spectral measure is given by*

$$\mu_{u,\,v} = \alpha_{\#}\,(uv \cdot m)\,.$$

7.4 Unitary Conjugation of Symmetric Operators

Introduction. In this final section, we show that any symmetric operator A defined on a complex Hilbert space is conjugated, via a unitary operator, to the multiplication operator in $L^2(\sigma(A),\, \mu_\xi)$ if its spectrum is simple. We also show how this property can be generalized if the spectrum is not simple, using a particular orthogonal decomposition.

Let $A \in \mathcal{L}^{sym}(H)$ be a linear bounded symmetric operator defined on a Hilbert space H. For any $\xi \in H$ we denote by H_ξ the minimal closed A-invariant subspace of H containing ξ, that is,

$$H_\xi = \overline{\operatorname{Span}\langle A^n\,\xi \;:\; n \in \mathbb{N}\rangle}.$$

Let $p \in \mathbb{C}[z]$ be a complex-valued polynomial. The reader may easily check that the subspace H_ξ is defined in such a way that it is also invariant under the action operator $p(A)$, that is,

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 \implies p(A)\eta = a_n A^n \eta + \cdots + a_1 A\eta + a_0\eta \in H_\xi.$$

Furthermore, the subspace

$$\{T \in \mathcal{L}(H) \mid H_\xi \text{ is } T\text{-invariant}\} \subset \mathcal{L}(H)$$

is closed with respect to the weak topology of the operators, since

$$T_n \rightharpoonup T \iff \langle T_n,\, \eta\rangle \xrightarrow{n\to+\infty} \langle T,\, \eta\rangle,$$

and this implies that, for any $\eta \in H_\xi$, we have

$$\langle T_n,\, \eta\rangle \in H_\xi \quad \text{for all } n \in \mathbb{N} \implies \langle T,\, \eta\rangle \in H_\xi.$$

Therefore, it follows from Proposition 7.12 that the subspace H_ξ is invariant under the operator $f(A)$ for every $f \in \mathscr{L}^\infty(\sigma(A);\, \mathbb{C})$.

Lemma 7.16. *Let* $\xi \in H$ *be any element. For any* $\xi' \in H$ *it turns out that*

$$\xi' \perp H_\xi \iff H_{\xi'} \perp H_\xi.$$

Proof. Recall that the operator A is symmetric, which means that $(A^n)^* = A^n$ for any $n \in \mathbb{N}$. A straightforward computation shows that

$$\begin{aligned}
\xi' \perp H_\xi &\iff \left\langle \xi',\, A^{n+m}\,\xi \right\rangle = 0 \quad \text{for all } n,\, m \in \mathbb{N} \\
&\iff \langle A^m\,\xi',\, A^n\,\xi\rangle = 0 \quad \text{for all } n,\, m \in \mathbb{N} \\
&\iff H_{\xi'} \perp H_\xi,
\end{aligned}$$

and this concludes the proof. $\square$

Orthogonal Decomposition. The previous Lemma 7.16 suggest the possibility that the complex Hilbert space H may be decomposed as the direct orthogonal sum of closed A-invariant subspaces, that is, there exists $\Xi \subset H$ such that

$$H = \bigoplus_{\xi \in \Xi} H_\xi.$$

We shall give this decomposition for granted, but the interested reader may try to fill in the details using the statement of Theorem 1.28. Indeed, if we consider H as a Hilbert module with scalar ring $\mathbb{C}[z]$ and action defined by

$$\mathbb{C}[z] \times H \ni p \cdot v \longmapsto p(A)v \in H,$$

then the mentioned theorem proves the existence of a maximal family Ξ.

Remark 7.6. We also observe that, if H is a separable Hilbert space, then Ξ is at most countable.

Relation between H_ξ and μ_ξ. Set $\mu_\xi := \mu_{\xi,\xi}$. For all $f \in C^0(\sigma(A);\, \mathbb{C})$, it turns out that $f(A)\,\xi \in H_\xi$, and a straightforward computation yields to

$$\begin{aligned} \|f(A)\xi\|_H^2 &= (f(A)\xi,\, f(A)\,\xi)_H = \\ &= (f(A)^* f(A)\xi,\, \xi)_H = \\ &= \left(|f|^2(A)\,\xi,\, \xi\right)_H = \\ &= \int_{\sigma(A)} |f|^2 \mathrm{d}\mu_\xi = \|f\|_{L^2(\mu_\xi)}^2. \end{aligned}$$

In particular, the map defined by setting

$$C^0(\sigma(A);\, \mathbb{C}) \to H_\xi, \qquad f \longmapsto f(A)\xi$$

is an isometry, and thus it can be extended to the closures of the sets, that is,

$$U : L^2(\sigma(A),\, \mu_\xi) \to H_\xi, \qquad f \longmapsto f(A)\xi.$$

Let M_x be the multiplicative operator that acts on $L^2(\sigma(A),\, \mu_\xi)$. Then there is a commutative diagram

$$\begin{array}{ccc} L^2(\sigma(A),\, \mu_\xi) & \xrightarrow{U} & H_\xi \\ \downarrow{\scriptstyle M_x} & & \downarrow{\scriptstyle A} \\ L^2(\sigma(A),\, \mu_\xi) & \xrightarrow{U} & H_\xi \end{array}$$

as the reader may easily verify that

$$f \longmapsto f(A)\xi \longmapsto A\left[f(A)\xi\right],$$

$$f \longmapsto x f(x) \longmapsto A\left[f(A)\xi\right]$$

are, respectively, the composition $A \circ U$ and $U \circ M_x$.

The brief discussion above has many important consequences. We first study a possible outcome when the spectrum of A has a particularly simple form.

Definition 7.17 (Simple Spectrum)**.** The spectrum of a linear bounded symmetric operator $A \in \mathcal{L}^{sym}(H)$, defined on a complex Hilbert space H, is *simple* if there exists $\xi \in H$ such that

$$H_\xi = H.$$

Corollary 7.18. *If $A \in \mathcal{L}^{sym}(H)$ has a simple spectrum, then A is conjugate to the multiplication operator M_x, defined on $L^2\left(\sigma(A),\, \mu_\xi\right)$, via a unitary operator U.*

In the general case, as we mentioned above, the space H may be written as the orthogonal sum of A-invariant subspaces as follows:

$$H = \bigoplus_{\xi \in \Xi} H_\xi.$$

Let us consider the topological product $\sigma(A) \times \Xi$, endowed with the σ-algebra product

$$\mathcal{B}((\sigma(A)) \otimes \mathcal{P}(\Xi)$$

where $\mathcal{B}((\sigma(A))$ is the Borel σ-algebra generated by the spectrum $\sigma(A)$, and $\mathcal{P}(\Xi)$ is the power set of Ξ. The measure μ that makes it a measure space is defined in the usual way, that is,

$$\mu(E) = \sum_{\xi \in \Xi} \mu_\xi\left(E_\xi\right), \qquad \text{where} \quad E = \bigcup_{\xi \in \Xi} E_\xi \times \{\xi\}.$$

In this case we have a slightly more complex commutative diagram

$$\begin{array}{ccc}
L^2\left(\sigma(A) \times \Xi,\, \mathcal{B} \times P(\Xi),\, \mu\right) & \xrightarrow{U = \oplus_{\xi \in \Xi} U_\xi} & H \\
\Big\downarrow {\scriptstyle M_x} & & \Big\downarrow {\scriptstyle A} \\
L^2\left(\sigma(A) \times \Xi,\, \mathcal{B} \times P(\Xi),\, \mu\right) & \xrightarrow{U = \oplus_{\xi \in \Xi} U_\xi} & H
\end{array}$$

where M_x is the multiplicative operator defined by setting

$$M_x\left(f(x,\, \xi)\right) := x\, f(x,\, \xi).$$

As a consequence of the decomposition and of what we have already proved above, a simple generalization of the previous corollary holds.

Corollary 7.19. *If $A \in \mathcal{L}^{sym}(H)$, then A is conjugate to the multiplication operator M_x, defined on $L^2\left(\sigma(A) \times \Xi,\, \mathcal{B} \times P(\Xi),\, \mu\right)$, via a unitary operator U.*

Chapter 8

Fredholm Operators

In this brief chapter, we introduce the class of *Fredholm operators* and the notion of *Fredholm index*, and we investigate their relation with compact operators.

8.1 Definitions and Main Properties

Definition 8.1 (Fredholm Operator)**.** A linear continuous operator $T \in \mathcal{L}(X, Y)$ between two Banach spaces is a *Fredholm operator* if and only if the following properties hold:

1) The dimension of the kernel $\operatorname{Ker} T$ is finite.

2) The codimension of the rank $\operatorname{Ran} T$ is also finite.

Remark 8.1. If $T \in \mathcal{L}(X, Y)$ is a Fredholm operator, then $\operatorname{Ran} T$ is closed as a subspace of Y.

Proof. Let $F \subset Y$ be a finite-dimensional direct algebraic addendum of the range of T in Y, that is,

$$Y = F \oplus \operatorname{Ran} T, \qquad \oplus\ :\ \text{algebraic direct sum.}$$

Let $\widetilde{T} : X \times F \longrightarrow Y$ be the operator defined by sending

$$X \times F \ni (x, f) \longmapsto Tx + f \in Y.$$

By construction $\widetilde{T}$ is a surjective operator; hence the Open Mapping Theorem 4.20 implies that $\widetilde{T}$ is an open map and also that it can be identified to a quotient map

$$\begin{tikzcd} X \times F \arrow[r, two heads, "\widetilde{T}"] \arrow[d, "\pi"] & Y \\ X \times F \big/ \operatorname{Ker} \widetilde{T} \arrow[ur, bend right, "\alpha"'] & \end{tikzcd}$$

where α is a linear isomorphism. Moreover, the kernel of the operator is given by

$$\operatorname{Ker} \widetilde{T} = \operatorname{Ker} T \times \{0\} \subset X \times \{0\},$$

and it is thus closed in $X \times F$. On the other hand, we have that

$$\operatorname{Ran} T = \widetilde{T}(X \times \{0\}) = \alpha \circ \pi(X \times \{0\})$$

as a consequence of the decomposition above, and this is enough to infer that $\operatorname{Ran} T$ is a closed subspace of Y since α is a linear isomorphism and Lemma 8.2 applies here. □

Lemma 8.2. *Let X be a Banach space, and let N be a closed subspace of X. For every $V \supset N$ closed subspace, the image of V via the projection*

$$p_N : X \twoheadrightarrow X/N,$$

denoted by $p_N(V)$, is a closed subspace of the quotient X/N.

Definition 8.3 (Semi-Fredholm Operator)**.** A linear continuous operator $T \in \mathcal{L}(X, Y)$ between Banach spaces is a *semi-Fredholm operator* if and only if the following properties hold:

1) The rank $\operatorname{Ran} T$ is closed.

2) Either the dimension of $\operatorname{Ker} T$ or the codimension of $\operatorname{Ran} T$ is finite.

Definition 8.4 (Fredholm Index)**.** Let $T \in \mathcal{L}(X, Y)$ be a (semi-)Fredholm operator. The Fredholm index of T is the integer number (eventually infinite) defined by the formula

$$i(T) := \dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Ran} T \in \mathbb{Z} \cup \{\pm\infty\}. \tag{8.1}$$

Example 8.1.

(1) If $X = Y$ is a Banach space and $K \in \mathcal{L}_c(X)$ is a compact operator, then $T := \mathrm{id}_X - K$ is a Fredholm operator with Fredholm index equal to 0.

(2) Let $X = Y = \ell_2$. We denote by S the injective shift operator

$$S : \ell_2 \to \ell_2, \qquad (x_0, x_1, \dots) \longmapsto (0, x_0, x_1, \dots),$$

and we denote by S^* the surjective shift operator

$$S^* : \ell_2 \to \ell_2, \qquad (x_0, x_1, \dots) \longmapsto (x_1, x_2, x_3, \dots).$$

The reader may check by herself that S^* is the adjoint of S (i.e., the notation is coherent), and also that the Fredholm indexes are respectively equal to

$$i(S) = -1 \qquad \text{and} \qquad i(S^*) = +1.$$

Moreover, the operator S is the essential inverse (see Theorem 8.5) of S^* (and vice versa) since

$$S^* \circ S = \mathrm{id}_{\ell_2} \qquad \text{and} \qquad S \circ S^* = \mathrm{id}_{\ell_2} - \pi_{\langle e_1 \rangle},$$

where π is the projection onto the 1-dimensional subspace $\langle e_1 \rangle$ of ℓ_2. Notice also that the Fredholm index seems to satisfy an additive property because

$$i(\mathrm{id}_{\ell_2}) = 0 = i(S) + i(S^*).$$

Remark 8.2. Let $T : X \longrightarrow Y$ be a linear continuous operator between Banach spaces. Then T is a Fredholm operator if and only if there are finite dimensional spaces E and F such that

$$0 \to E \to X \xrightarrow{T} Y \to F \to 0$$

is a short exact sequence.

Theorem 8.5 (Atkinson)**.** *Let $T \in \mathcal{L}(X, Y)$ be a linear continuous operator between Banach spaces. Then T is a Fredholm operator if and only if T is essentially invertible, that is, there exists $S \in \mathcal{L}(Y, X)$ such that*

$$T \circ S = \mathrm{id}_Y + K_1 \qquad \textit{and} \qquad S \circ T = \mathrm{id}_X + K_2,$$

where $K_1 \in \mathcal{L}_c(Y)$ and $K_2 \in \mathcal{L}_c(Y)$ are compact operators.

Proof. To ease the notation, we divide the argument into two paragraphs.

" $\Longrightarrow$ ": Suppose that T is a Fredholm operator. By definition, the subspace $E_0 := \mathrm{Ker}\, T$ is closed and finite-dimensional; hence it follows from Exercise 8.1 that E_0 is a direct algebraic addendum of X, that is,

$$X = E_0 \oplus E_1.$$

Similarly, the subspace $F_1 := \mathrm{Ran}(T)$ is closed and finite-codimensional; hence it is an algebraic direct addendum of Y, that is,

$$Y = F_0 \oplus F_1.$$

Let $P_i : X \twoheadrightarrow E_i$ and $Q_i : Y \twoheadrightarrow F_i$ be the projections associated to the decompositions above, and let us consider the invertible operator

$$T_1 : E_1 \longrightarrow F_1, \qquad T_1 := Q_1 \circ T \circ P_1.$$

If we set $S := T_1^{-1} \circ Q_1 \in \mathcal{L}(Y, X)$, then it turns out that

$$S \circ T = T_1^{-1} \circ Q_1 \circ T = P_1 = \mathrm{id}_X - P_0,$$

$$T \circ S = T \circ T_1^{-1} \circ Q_1 = Q_1 = \mathrm{id}_Y - Q_0,$$

and this proves the thesis (since the projections are compact operators).

In other words, the argument of this arrow can be equivalently stated by seeing the operator T as a 2×2 matrix defined on the decompositions of X and Y as follows:

$$T : E_0 \oplus E_1 \longrightarrow F_0 \oplus F_1, \qquad T = \begin{pmatrix} Q_0 \circ T \circ P_0 & Q_1 \circ T \circ P_0 \\ Q_0 \circ T \circ P_1 & Q_1 \circ T \circ P_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}.$$

" $\Longleftarrow$ " Suppose that T is essentially invertible, and let S be its essential inverse. It is immediate to verify that the following inclusions hold:

$$\mathrm{Ker}\, T \subset \mathrm{Ker}\, S \circ T = \mathrm{Ker}\,(\mathrm{id}_X + K_2)\,,$$

$$\mathrm{Ran}\, T \supset \mathrm{Ran}\, T \circ S = \mathrm{Ran}\,(\mathrm{id}_Y + K_1)\,.$$

It follows from Theorem 6.18 that the right-hand side of the first (resp. second) inclusion has finite dimension (resp. codimension), and therefore the same is true for the left-hand side. □

Remark 8.3. If S_s and S_d are respectively the left essential inverse and the right essential inverse of a Fredholm operator T, then $S_d - S_s \in \mathcal{L}_c(X, Y)$. Indeed, it turns out that

$$\begin{cases} TS_d = \mathrm{id}_Y + K_1 \\ S_s T = \mathrm{id}_X + K_2 \end{cases} \rightsquigarrow \begin{cases} S_s T S_d = S_s + S_s \circ K_1 \\ S_s T S_d = S_d + K_2 \circ S_d, \end{cases}$$

and this proves the claim. In particular, both S_s and S_d are essential inverses of T.

Let X and Y be Banach spaces. From now on, we denote by $\mathscr{F}(X, Y)$ the set of all the Fredholm operators between X and Y.

Proposition 8.6. *The set of all Fredholm operators $\mathscr{F}(X, Y)$ is an open subset of $\mathcal{L}(X, Y)$.*

Proof. Let $T \in \mathscr{F}(X, Y)$ be a Fredholm operator, let $S \in \mathscr{F}(Y, X)$ be an essential inverse and set

$$r := \|S\|^{-1}.$$

We claim that

$$T + H \in \mathscr{F}(X, Y) \qquad \forall\, H \in \mathcal{L}(X, Y) \,:\, \|H\| < r.$$

Indeed, consider the operator $S(\mathrm{id}_Y + HS)^{-1}$, and notice that it is well-defined. Furthermore, a simple computation shows that

$$\begin{aligned} (T + H)\left[S(\mathrm{id}_Y + HS)^{-1}\right] &= (TS + HS)\,(\mathrm{id}_Y + HS)^{-1} = \\ &= (\mathrm{id}_Y + HS)\,(\mathrm{id}_Y + HS)^{-1} + K' = \\ &= \mathrm{id}_Y + K', \end{aligned}$$

where $K' := K_1\,(\mathrm{id}_Y + HS)^{-1} \in \mathcal{L}_c(Y)$. In a similar fashion one can prove that

$$\left[(\mathrm{id}_X + SH)^{-1}\,S\right](T + H) = \mathrm{id}_X + K'',$$

and this is enough to infer that $T + H \in \mathscr{F}(X, Y)$, as a consequence of Remark 8.3. □

Lemma 8.7. *Let $T \in \mathscr{F}(X, Y)$ be a Fredholm operator of index $i(T) = 0$. There exist $U : X \longrightarrow Y$ invertible operator and $L : X \longrightarrow Y$ finite-rank operator such that*

$$T = U + L.$$

Proof. Let E_i, F_i, P_i, Q_i and T_i be the objects defined in the proof of Theorem 8.5. Since E_0 and F_0 are finite-dimensional vector spaces (with the same dimension), there exists a linear isomorphism $L_0 : E_0 \longrightarrow F_0$; hence we can define

$$L := L_0 \circ P_0 \qquad \text{and} \qquad U := T - L.$$

The operator U is invertible because both of the diagonal blocks are invertible, that is,

$$T - L : E_0 \oplus E_1 \longrightarrow F_0 \oplus F_1, \qquad T - L = \begin{pmatrix} -L_0 & 0 \\ 0 & T_1 \end{pmatrix}.$$

□

Proposition 8.8. *The Fredholm index*

$$i : \mathscr{F}(X, Y) \longrightarrow \mathbb{Z}$$

is a continuous map (and thus locally constant.)

Proof. We divide the argument into two steps.

Step 1. Suppose that the Fredholm index of $T \in \mathscr{F}(X, Y)$ is 0. By Lemma 8.7 there exist $U \in \mathcal{L}(X, Y)$ invertible and $L \in \mathcal{L}_f(X, Y)$ such that

$$T = U + L.$$

Let $S \in \mathscr{F}(Y, X)$ be an essential inverse of T, and let $H \in \mathcal{L}(X, Y)$ be an operator such that $\|H\| < r$, where $r := \|S\|^{-1}$. Clearly

$$U + H \text{ is invertible because } U\left(\mathrm{id}_x + U^{-1}H\right) \text{ is invertible,}$$

and hence the reader may easily check that

$$T + H = (\mathrm{id}_Y + L') \circ U',$$

where $U' := U + H$ is invertible and $L' := L \circ (U + H)^{-1}$ is a finite-rank operator. The operator U' is invertible, and therefore

$$i\,(\mathrm{id}_Y + L') = 0 \implies i(T + H) = 0$$

which is exactly what we wanted to prove.

Step 2. In the general case ($i(T) = k$), one may always consider the operator

$$\widetilde{T} : X \oplus E \longrightarrow Y \oplus F, \qquad (x, e) \longmapsto (Tx, 0),$$

where E and F are finite-dimensional spaces such that

$$-k = \dim E - \dim F.$$

By definition, the operator $\widetilde{T}$ has Fredholm index equal to zero. Thus what we have proved in the first step immediately implies that

$$i\left(\widetilde{T} + (H \oplus 0_{E \to F})\right) = 0 \qquad \forall\, H \in \mathcal{L}(X, Y) \;:\; \|H\| < r,$$

from which we easily infer that $i(T + H) = k$. □

Proposition 8.9 (Invariance Properties)**.** *Let X and Y be Banach spaces.*

(a) The sum of a Fredholm operator and a compact operator is still a Fredholm operator, that is,

$$\mathscr{F}(X, Y) + \mathcal{L}_c(X, Y) \subseteq \mathscr{F}(X, Y).$$

(b) The index is invariant under compact perturbations, that is,

$$i(T + K) = i(T), \qquad \forall T \in \mathscr{F}(X, Y)\ \forall K \in \mathcal{L}_c(X, Y).$$

Proof.

(a) Let K be a compact operator, let $T \in \mathscr{F}(X, Y)$ be a Fredholm operator and let S an the essential inverse, given by Theorem 8.5. Then S is also an essential inverse of the operator $T + K$ since one can easily check that

$$S(T + K) = ST + SK = \mathrm{id}_X + K_2 + SK \in \mathrm{id}_X + \mathcal{L}_c(X)$$

$$(T + K)S = TS + KS = \mathrm{id}_Y + K_1 + KS \in \mathrm{id}_Y + \mathcal{L}_c(Y)$$

since the operators SK and KS are both compact as a result of Lemma 6.11.

(b) Let $t \in [0, 1]$ be a real number, and let us consider the operator

$$L_t := T + t \cdot K.$$

The operator $t \cdot K$ is compact; hence **(a)** implies that $L_t \in \mathscr{F}(X, Y)$ for every $t \in [0, 1]$. In conclusion, it suffices to notice that by Proposition 8.8 the map

$$(0, 1) \ni t \longmapsto i(T + t \cdot K) \in \mathbb{Z}$$

is continuous and with values in a discrete set, that is, it is constant.

□

Corollary 8.10. *Let* $\mathrm{GL}(X, Y) \subset \mathcal{L}(X, Y)$ *be the subset of all the invertible operators from* X *o* Y*. Then the Fredholm operator of index* 0 *are given by*

$$\mathscr{F}_0(X, Y) = \mathrm{GL}(X, Y) + \mathcal{L}_c(X, Y) = \mathrm{GL}(X, Y) + L_f(X, Y).$$

Remark 8.4. The set of all Fredholm operators $\mathscr{F}(X, Y)$ is open (Proposition 8.6). Moreover, it is equal to the countable union of open subsets

$$\mathscr{F}(X, Y) = \bigcup_{k \in \mathbb{N}} \mathscr{F}_k(X, Y)$$

where $\mathscr{F}_k(X, Y)$ is the subset of all the operators whose index is equal to k.

Notice that, a priori, the $\mathscr{F}_k(X, Y)$'s are not the connected components of $\mathscr{F}(X, Y)$ (they may be disconnected), but, if $X = Y$ is an infinite-dimensional separable Hilbert space, then they are.

Proposition 8.11. *Let X, Y, Z be Banach spaces, and let $T \in \mathscr{F}(X, Y)$ and $S \in \mathscr{F}(Y, Z)$. Then the composition ST belongs to $\mathscr{F}(X, Z)$ and the Fredholm index is multiplicative, that is,*

$$i(ST) = i(S) + i(T).$$

Proof 1. We split the argument into two steps.

Step 1. The reader may prove as an exercise that there is an exact sequence

$$0 \to \operatorname{Ker} T \to \operatorname{Ker} S \circ T \xrightarrow{T} \operatorname{Ker} S \to 0,$$

hence the dimension of $\operatorname{Ker} S \circ T$ is finite since the dimensions of $\operatorname{Ker} T$ and $\operatorname{Ker} S$ are finite.

In a similar fashion, one could prove that the codimension of $\operatorname{Ran} S \circ T$ is finite, and thus that $ST \in \mathscr{F}(X, Z)$ is a Fredholm operator.

Step 2. The argument presented here works only if $X = Y = Z$. We consider the path of Fredholm operators $(L_\theta)_{\theta \in [0, \pi/2]}$ defined by

$$L_\theta := \begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & \mathrm{id}_X \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

A simple computation proves that

$$L_0 = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \qquad \text{and} \qquad L_{\frac{\pi}{2}} = \begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & ST \end{pmatrix},$$

are Fredholm operators linked via a continuous path. In particular, it turns out that

$$i\left(L_0\right) = i\left(L_{\frac{\pi}{2}}\right) \implies i(T) + i(S) = \underbrace{i(\mathrm{id}_X)}_{=0} + i(ST),$$

which is exactly what we wanted to prove. $\square$

Proof 2. The key idea is to decompose X, Y and Z in 3, 4 and 3 closed spaces respectively in such a way that T and S induce, on corresponding spaces, either the null map or an invertible map.

Decomposition of X. Let $X_0 := \operatorname{Ker} T$ be a finite-dimensional closed subspace of X. Since the kernel of T is included in the kernel of ST, it turns out that there exists a finite-dimensional subspace $X_1 \subset X$ such that

$$X_0 \oplus X_1 = \operatorname{Ker} S \circ T.$$

Finally, let $X_2 \subset X$ be the complement of $\operatorname{Ker} S \circ T \subset X$ so that

$$X_0 \oplus X_1 \oplus X_2 = X.$$

Moreover, observe that the operator T, restricted to X_0, is the null map. In a similar fashion, one can prove that $T\big|_{X_i}$ is injective for $i = 1, 2$.

Decomposition of Y. Let $Y_1 := T(X_1)$ and notice that

$$T(X_1) = T(X_0 + X_1) = T\left(\operatorname{Ker} S \circ T\right) =$$

$$= \operatorname{Ran} T \cap \operatorname{Ker} S$$

Let $Y_2 := T(X_2)$. Since $T(X_1 + X_2) = T(X_0 + X_1 + X_2) = \operatorname{Ran} T$ it follows that Y_1 and Y_2 are closed, disjoint and that their direct sum is equal to the image of T. Moreover, there exists a finite-dimensional closed subset $Y_0 \subset Y$ such that

$$Y_0 \oplus Y_1 = \operatorname{Ker}(S),$$

and there exists a finite-dimensional closed subset $Y_3 \subset Y$ such that

$$Y_0 \oplus Y_1 \oplus Y_2 \oplus Y_3 = Y.$$

In particular, observe that T restricted to $Y_0 \oplus Y_1$ is the null map (by definition), and also that the restriction $S\big|_{Y_2 \oplus Y_3}$ is an injective operator.

Decomposition of Z. In a similar fashion, let us set $Z_2 := S(Y_2)$ and $Z_3 := S(Y_3)$. These subspaces are closed, and they do not intersect, as follows from an argument similar to the one used to decompose Y. Thus it turns out that

$$Z_2 \oplus Z_3 = \operatorname{Ran} S.$$

Finally, let $Z_1 \subset Z$ be the closed finite-dimensional subspace of Z such that

$$Z = Z_1 \oplus Z_2 \oplus Z_3.$$

Computation of the indices. The operators T and S assume a particularly simple form as matrices 4×3 and 3×4 respectively, as a consequence of the decompositions. More precisely, it turns out that

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T\big|_{X_1} & 0 \\ 0 & 0 & T\big|_{X_2} \\ 0 & 0 & 0 \end{pmatrix} \implies i(T) = \dim(X_0) - \left(\dim(Y_0) + \dim(Y_3)\right),$$

and also that

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S\big|_{Y_2} & 0 \\ 0 & 0 & 0 & S\big|_{Y_3} \end{pmatrix} \implies i(S) = \dim(Y_0) + \dim(Y_1) - \dim(Z_1).$$

The composition is given by

$$ST = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S\big|_{Y_2} T\big|_{X_2} \\ 0 & 0 & 0 \end{pmatrix} \implies i(ST) = \dim(X_0) + \dim(X_1) - \dim(Z_1) - \dim(Z_3),$$

and this is exactly what we wanted to prove since

$$\begin{aligned} i(T) + i(S) &= \dim(X_0) + \dim(Y_1) - \dim(Y_3) - \dim(Z_1) = \\ &= \dim(X_0) + \dim(X_1) - \dim(Z_3) - \dim(Z_1) = \\ &= i(ST). \end{aligned}$$

□

8.2 Calkin Algebra

Let X be a Banach space. In the previous chapters we proved that $\mathcal{L}(X)$ is an algebra, and $\mathcal{L}_c(X)$ is a closed bilateral ideal (see Lemma 6.11).

The quotient space is a Banach algebra (since it is a Banach space and also an algebra satisfying the inequality (6.10)). It is called *Calkin algebra*, and it is denoted by $C(X)$, i.e.,

$$C(X) := {\mathcal{L}(X)}\big/{\mathcal{L}_c(X)}.$$

The set of all the invertible Calkin operator, denoted by $\mathcal{G}\left(C(X)\right)$, is an open subset and also a subgroup of $C(X)$. Therefore, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{L}(X) \arrow[r, "\pi"] & C(X) \\
\pi^{-1}\left(\mathcal{G}\left(C(X)\right)\right) \arrow[u, hook] \arrow[r, "\pi"] & \mathcal{G}\left(C(X)\right) \arrow[u, hook]
\end{tikzcd}$$

and it is clear that

$$\pi^{-1}\left(\mathcal{G}\left(C(X)\right)\right) = \mathscr{F}(X),$$

that is, it is equal to the set of all the Fredholm operators. The reader may prove the following result as an exercise:

Lemma 8.12. *Let $\mathscr{F}_0(X)$ be the set of all the Fredholm operators of index 0. Then $\pi\left(\mathscr{F}_0(X)\right)$ is a normal subgroup and an open subset of $\mathcal{G}\left(C(X)\right)$. Moreover, the quotient*

$${\mathcal{G}\left(C(X)\right)}\big/{\pi\left(\mathscr{F}_0(X)\right)}$$

is a discrete group.

Remark 8.5. If X is a separable Hilbert space, then the composition

$$\mathscr{F}(X) \xrightarrow{\pi} \mathcal{G}\left(C(X)\right) \to {\mathcal{G}\left(C(X)\right)}\big/{< 0 >} \cong \mathbb{Z}$$

is equal to the Fredholm index, where $< 0 >$ denotes the connected component containing the neutral element of the group.

Essential Spectrum. There is a notion of spectrum in every Banach algebra, and it is always a compact subset of $\mathbb{C}$ (as a result of a more general theory).

In particular, there is a notion of spectrum associated with the Banach algebra $C(X)$, called *essential spectrum*, defined by

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} \mid \lambda\, \mathrm{id}_X - T \notin \mathscr{F}(X)\} = \sigma\left(\pi(T)\right),$$

where $\sigma(\cdot)$ is the spectrum associated to the Banach algebra $\mathcal{L}(X)$.

8.3 Exercises

Finite Dimension. Let X be a Banach space, and let N be a finite-dimensional (or finite-codimensional) closed subset of X.

Exercise 8.1. Prove that there exists a linear projection or, equivalently, that N is an algebraic direct addendum of X.

Solution. Let $\{e_1, \dots, e_n\}$ be a basis of N, that is,

$$N = \mathrm{Span}\langle e_1, \dots, e_n\rangle.$$

Every $x \in N$ can be written, uniquely, as a sum

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n,$$

and therefore we can always consider the linear continuous functionals $\alpha_1, \dots, \alpha_n \in N^*$ such that

$$x = \alpha_1(x)e_1 + \cdots + \alpha_n(x)e_n \quad \text{for all } x \in N.$$

By the Hahn-Banach Theorem we can find linear continuous extensions of $\alpha_1, \dots, \alpha_n \in X^*$, and therefore we can define a projection

$$P : X \longrightarrow N, \qquad x \longmapsto \alpha_1(x)e_1 + \cdots + \alpha_n(x)e_n.$$

Let $M := \mathrm{Ker}P$. Then one can easily prove that

$$X = M \oplus N$$

since every $x \in X$ can be written as

$$x = (x - P(x)) + P(x) \in M + N.$$

$\square$

Exercise 8.2. Prove that there is no linear projection associated to the inclusion

$$c_0 \subset \ell_\infty.$$

Explain why, in this case, the statement of Exercise 8.1 does not hold.

Proof. The reader may consult this thread for a collection of references and proofs of this assertion. $\square$

Exercise 8.3. Let X be a Banach space. The inclusion

$$X^* \hookrightarrow X^{***}$$

always admits a direct addendum.

Hint. The reader may try to consider the following maps:

$$\imath_X : X \hookrightarrow X^{**} \rightsquigarrow (\imath_X)^* : X^{***} \to X^*,$$

$$\imath_{X^*} : X^* \hookrightarrow X^{***}.$$

$\square$

As a corollary of Exercise 8.2 and Exercise 8.3 we find again that c_0 is not the dual of a Banach space (see Subsection 3.3.3).

Inverse Operators. Let X and Y be Banach spaces. An operator $L \in \mathcal{L}(X, Y)$ admits a right (left) inverse if there exists $S \in \mathcal{L}(Y, X)$ such that $ST = \mathrm{id}_X$ $(TS = \mathrm{id}_Y)$.

Exercise 8.4. Let $T \in \mathcal{L}(X, Y)$ be a right-invertible operator, and let $S \in \mathcal{L}(Y, X)$ be the right inverse. Prove that:

(1) The operator TS is a projection.

(2) The kernel of TS coincides with the kernel of S, that is, $\mathrm{Ker}(TS) = \mathrm{Ker}(S)$.

(3) The range of TS coincides with the range of T, that is, $\mathrm{Ran}(TS) = \mathrm{Ran}(T)$.

Solution.

(1) To prove that TS is a projection, we simply employ the associative property of the composition:
$$(TS)^2 = TSTS = T(ST)S = TS.$$

(2) The kernel of the operator S is alway contained in the kernel of the operator TS, and hence we only need to prove the opposite inclusion.

Let $y \in \mathrm{Ker}(TS)$. It follows that $TS(y) = 0$, and therefore
$$0 = TS(y) \implies 0 = S(0) = \underbrace{ST}_{=\mathrm{id}_X} S(y) \implies S(y) = 0,$$
which means that $y \in \mathrm{Ker}(S)$.

(3) In a similar way, the range of the operator TS is always contained in the range of the operator T, and hence we only need to prove the opposite inclusion.

Let $y \in \mathrm{Ran}(T)$. It follows that $y = T(x)$, and therefore
$$S(y) = ST(x) = x \implies y = TS(x),$$
which means that $y \in \mathrm{Ran}(TS)$.

□

Essential Spectrum. Let X be a Banach space, and let $C(X)$ be the Calkin algebra.

Exercise 8.5. Let S be the injective shift operator
$$S : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \qquad (x_0, x_1, \dots) \longmapsto (0, x_0, x_1, \dots).$$

(1) Prove that the spectrum of S is given by the closed ball of radius 1 in $\mathbb{C}$, i.e. $\sigma(S) = \overline{B_C(0, 1)}$.

(2) Prove that the eigenvalue spectrum of S is given by the empty set. Namely, the shift operator S does not admit any eigenvalue.

(3) Find an explicit description of the essential spectrum of S.

Part III

Sobolev Spaces

Chapter 9

Sobolev Spaces

In this final chapter, we use all the theory we have developed so far to introduce and study the main properties of the Sobolev spaces $W^{m,p}(\Omega)$, whose importance is well-known in the partial differential equations field.

9.1 Introduction and Elementary Properties

Introduction. In this paragraph, we denote by I an open interval (a, b) of the real line $\mathbb{R}$ (eventually unbounded), unless stated otherwise.

Definition 9.1 (Weak Derivative)**.** Let $f \in L^1_{\text{loc}}(I)$ be a locally summable function. A function $g \in L^1_{\text{loc}}(I)$ is the *weak derivative* of f if and only if

$$\int_I f(x)\varphi'(x)\,\mathrm{d}x = -\int_I g(x)\varphi(x)\,\mathrm{d}x \quad \text{for every } \varphi \in C^\infty_{\text{c}}(I). \tag{9.1}$$

Proposition 9.2. *Let $f \in L^1_{\text{loc}}(I)$. The following properties hold:*

(a) *If $g \in L^1_{\text{loc}}(I)$ is the weak derivative of f, then it is unique up to the a.e. equivalence relation. More precisely, if $g_1 \in L^1_{\text{loc}}(I)$ and $g_2 \in L^1_{\text{loc}}(I)$ are both weak derivatives of f, then*

$$g_1(x) = g_2(x) \quad \text{for almost every } x \in I.$$

(b) *If $f \in C^1(I)$, then a representative of the weak derivative equivalence class coincides with the usual derivative f'.*

(c) *Let $f \in C^0(I)$ be a continuous function such that the usual derivative $f'(x)$ exists at all $x \in I \setminus D$, where D is at most countable. Then the weak derivative of f exists, and a representative of the equivalence class is given by f'.*

Proof.

(a) Suppose that g_1 and g_2 are both weak derivatives of f. It follows from (9.1) that

$$\int_I [g_1(x) - g_2(x)]\, \varphi(x)\, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(I),$$

and hence $g_1(x) = g_2(x)$ for almost every $x \in I$ as a consequence of the fundamental lemma in calculus of variations[1].

(b) It suffices to integrate by parts

$$\int_I f(x)\varphi'(x)\, \mathrm{d}x = 0,$$

and apply the same argument of the previous point.

□

Remark 9.1. Let $f \in C^0(\Omega)$ be a continuous function, and suppose that $f'(x)$ exists at almost every $x \in I$. Then

$$g(x) := f'(x) \quad \text{a.e. } x \in I \;\not\Longrightarrow\; g \text{ is the weak derivative of } f.$$

Proof. Recall that the Cantor function $f_C : [0, 1] \to \mathbb{R}$ is the unique continuous increasing function satisfying the following relations:

$$f_C(x) + f_C(1 - x) = 1 \quad \text{and} \quad f_C\left(\frac{x}{3}\right) = \frac{1}{2} f_C(x),$$

for every $x \in [0, 1]$. The Cantor function is locally constant on the complement of the Cantor set C since

$$f \text{ constant on the interval } J \implies f \text{ constant on } \frac{1}{3}J \text{ and } 1 - \frac{1}{3}J.$$

Therefore f_C is locally constant on a set of full measure (note that the Cantor set C is uncountable and has measure zero).

Conclusion. The usual derivative f_C' exists at almost all $x \in [0, 1]$, and it must be equal to 0 a.e. because f_C is locally constant almost everywhere. If the function identically equal to zero were the weak derivative of f_C, then a variation of the fundamental lemma in the calculus of variation[2] would imply f_C constant, which gives the sought contradiction. □

[1] **Lemma.** Let $f \in L^1_{\mathrm{loc}}(I)$. If

$$\int_I f(x)\varphi(x)\, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(I) \implies f(x) = 0 \text{ for almost every } x \in I.$$

[2] **Lemma.** (Paul du Bois-Reymond.) Let $f \in L^1_{\mathrm{loc}}(I)$. If

$$\int_I f(x)\, \varphi(x)\, \mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^\infty(I) \,:\, \int_I \varphi(x)\, \mathrm{d}x = 0,$$

then $f(x) = c$ for almost every $x \in I$.

Alternative Proof. We argue by contradiction. If $f_C \in L^1_{\text{loc}}([0, 1])$ is a weakly differentiable function, then Proposition 9.7 proves that the Cantor function f_C admits an absolutely continuous representative, and thus it maps null-sets to null-sets.

But f_C maps the Cantor set (a set of measure zero) to its complement (a set of strictly positive measure), and this is yields to a contradiction. $\square$

Definition 9.3 (Absolutely Continuous). A function $f : I \longrightarrow \mathbb{R}$ is *absolutely continuous* if and only if for all $\epsilon > 0$ there exists $\delta(\epsilon) := \delta > 0$ such that, for any finite **disjoint** family $J_1, \dots, J_n \subset I$ of open intervals (a_i, b_i) satisfying the property

$$\sum_{i=1}^{n} |J_i| = \sum_{i=1}^{n} |b_i - a_i| \le \delta,$$

it turns out that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \le \epsilon.$$

Definition 9.4 (Oscillation). Let $f : I \longrightarrow \mathbb{R}$ be a continuous function, and let $J \subset I$ be a subset. The *oscillation* of f in J is defined by setting

$$\operatorname{osc}(f, J) := \sup \{ |f(x) - f(y)| \ : \ x, y \in J \}.$$

Remark 9.2. There are many equivalent definitions of absolute continuity of a function, whose proof is left to the reader.

(1) A function $f : I \longrightarrow \mathbb{R}$ is *absolutely continuous* if and only if for all $\epsilon > 0$ there exists $\delta(\epsilon) := \delta > 0$ such that, for any finite **disjoint** family $J_1, \dots, J_n \subset I$ of open intervals (a_i, b_i) satisfying the property

$$\sum_{i=1}^{n} |J_i| = \sum_{i=1}^{n} |b_i - a_i| \le \delta,$$

it turns out that

$$\sum_{i=1}^{n} \operatorname{osc}(f, J_i) \le \epsilon.$$

(2) A function $f : I \longrightarrow \mathbb{R}$ is *absolutely continuous* if and only if for all $\epsilon > 0$ there exists $\delta(\epsilon) := \delta > 0$ such that, for any countable **disjoint** family $J_1, \dots, J_n, \dots \subset I$ of open intervals (a_i, b_i) satisfying the property

$$\sum_{i=1}^{+\infty} |J_i| = \sum_{i=1}^{+\infty} |b_i - a_i| \le \delta,$$

it turns out that

$$\sum_{i=1}^{+\infty} \operatorname{osc}(f, J_i) \le \epsilon.$$

Proposition 9.5. *A finite measure μ defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is absolutely continuous with respect to the Lebesgue measure if and only if the cumulative distribution function*

$$F_\mu(x) = \mu((-\infty, x])$$

is absolutely continuous.

Theorem 9.6. *A function $f \in L^1_{\mathrm{loc}}(I)$ is (locally) absolutely continuous if and only if there exists $g \in L^1_{\mathrm{loc}}(I)$ such that*

$$f(x) - f(y) = \int_x^y g(t)\,\mathrm{d}t. \tag{9.2}$$

More precisely, any absolutely continuous function is a.e. differentiable, and the usual derivative is a representative of the weak derivative equivalence class.

Proof. Let $J := [a,\, b] \subset I$ be a closed interval. The function f is absolutely continuous in J, and hence we can define a measure μ by setting

$$\mu\left([x,\, y)\right) := f(y) - f(x).$$

We can easily prove that the cumulative distribution function is absolutely continuous, and therefore μ is absolutely continuous with respect to the Lebesgue measure, as a consequence of Proposition 9.5. It follows from the Radon-Nikodym Theorem that there exists $g \in L^1(J)$ such that

$$f(y) - f(x) = \int_x^y g(t)\,\mathrm{d}t \quad \text{for all } x,\, y \in J.$$

Therefore, f is differentiable at almost every $x \in J$, and we can easily prove that g is a representative of the weak derivative equivalence class (using the integration by parts formula). □

Proposition 9.7. *Let $f \in L^1_{\mathrm{loc}}(I)$ be a weakly differentiable function. There exists an absolutely continuous representative in the equivalence class of f.*

Proof. Suppose that $f \in L^1_{\mathrm{loc}}(I)$ is a weakly differentiable function, and let $g \in L^1_{\mathrm{loc}}(I)$ be the weak derivative. It is easy to prove that

$$(9.1) \iff \int_I f(x)\varphi'(x)\,\mathrm{d}x = -\int_I g(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{PWC}^1_c(I),$$

where $\mathrm{PWC}^1_c(I)$ denotes the space of all piecewise continuously differentiable functions with compact support in I.

Conclusion. Let $x < y \in I$, and let us consider the function

$$\Phi^\epsilon_\eta(t) := \begin{cases} 0 & t \le x - \epsilon, \\ 1 & x + \epsilon < t < y - \eta, \\ 0 & t \ge y + \eta. \end{cases}$$

Denote by Φ^ϵ_η the obvious piecewise differentiable extension to the whole interval I. It is easy to compute the left-hand side of (9.1) since

$$\int_I f(t)\left(\Phi^\epsilon_\eta\right)'(t)\,\mathrm{d}t = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)\,\mathrm{d}t - \frac{1}{2\eta}\int_{y-\eta}^{y+\eta} f(t)\,\mathrm{d}t,$$

and, similarly, the right-hand side is given by

$$\int_I g(t)\Phi^\epsilon_\eta(t)\,\mathrm{d}t = \int_x^y g(t)\,\mathrm{d}t + o(1) \quad \text{for } \epsilon,\, \eta \to 0^+.$$

If we let

$$\chi_\epsilon(t) := \frac{1}{2\epsilon}\chi_{[-\epsilon,\,\epsilon]} \quad \text{and} \quad \chi_\eta(t) := \frac{1}{2\eta}\chi_{[-\eta,\,\eta]}$$

then the left-hand side may be rewritten as follows:

$$\int_I f(t)\left(\Phi_\eta^\epsilon\right)'(t)\,\mathrm{d}t = f * \chi_\epsilon(x) - f * \chi_\eta(y).$$

If we take the limit as $\epsilon \to 0^+$, and then we take the limit as $\eta \to 0^+$, it turns out that

$$f(x) - f(y) = \int_x^y g(t)\,\mathrm{d}t \implies f(y) - f(x) = \int_x^y (-g)(t)\,\mathrm{d}t,$$

and Theorem 9.6 concludes the proof. □

9.2 $W^{1,\,p}$ Spaces

Introduction. In this section, we denote by I an open interval $(a,\,b)$ of the real line $\mathbb{R}$ (eventually unbounded), unless stated otherwise.

Definition 9.8 (Sobolev Space)**.** Let $p \in [1,\,+\infty]$. The $(1,\,p)$-Sobolev space, denoted by $W^{1,\,p}(I)$, is the space of all the $L^p(I)$ functions with weak derivative in $L^p(I)$, that is,

$$W^{1,\,p}(I) := \{f \in L^p(I) \mid Df \in L^p(I)\}\,.$$

Normed Space. In this paragraph, we briefly discuss the main idea that allows one to define a norm on $W^{1,\,p}(I)$ that makes it a Banach space.

Remark 9.3. The Sobolev space $W^{1,\,p}(I)$ is a vector space, and the derivative map

$$D : W^{1,\,p}(I) \longrightarrow L^p(I), \qquad f \longmapsto Df$$

is well-defined and linear.

Lemma 9.9. *The graph of the operator $D : W^{1,\,p}(I) \longrightarrow L^p(I)$ is closed, that is,*

$$\Gamma(D) := \left\{(f,\,Df) \;\middle|\; f \in W^{1,\,p}(I)\right\} \subset L^p(I) \times L^p(I)$$

is closed with respect to the subspace topology.

Proof. Let $(f_n,\,Df_n)_{n\in\mathbb{N}} \subset \Gamma(D)$ be a converging sequence, and let $(f,\,g) \in L^p \times L^p$ be its limit in the product norm, that is,

$$f_n \xrightarrow{L^p} f \quad \text{and} \quad Df_n \xrightarrow{L^p} g.$$

The identity

$$\int_I f_n(t)\varphi'(t)\,\mathrm{d}t = -\int_I Df_n(t)\varphi(t)\,\mathrm{d}t$$

holds for every $n \in \mathbb{N}$ and for every $\varphi \in C_c^\infty(I)$. Recall that the L^p convergence implies the pointwise convergence of a subsequence, and therefore we can take the limit as n goes to $+\infty$ of the identity above to obtain

$$\int_I f(t)\varphi'(t)\,\mathrm{d}t = -\int_I g(t)\varphi(t)\,\mathrm{d}t,$$

which means that $g = Df$, i.e., the graph is closed. □

Corollary 9.10. *The mapping*

$$W^{1,p}(I) \longrightarrow L^p(I) \times L^p(I), \qquad f \longmapsto (f,\, Df)$$

is linear, injective and onto the rank. More precisely, it turns out that

$$W^{1,p}(I) \longrightarrow \Gamma(D)$$

is linear and bijective.

In conclusion, the product $L^p \times L^p$ induces on $\Gamma(D)$ the subspace topology by taking the restriction of the product norm $\|\cdot\|_p + \| - \|_p$. More precisely, if we endow $W^{1,p}$ with the topology generated by the norm

$$\|f\|_{W^{1,p}(I)} := \|f\|_{L^p(I)} + \|Df\|_{L^p(I)}, \tag{9.3}$$

then one can easily prove that $W^{1,p}$ is complete (and thus a Banach space).

Remark 9.4. Clearly (9.3) may be replaced by any equivalent norm in the product $L^p \times L^p$. In particular, when $p = 2$, it is particularly useful to consider the equivalent norm

$$\|f\|_{H^1(I)} := \left(\|f\|^2_{L^2(I)} + \|Df\|^2_{L^p(I)}\right)^{\frac{1}{2}} \tag{9.4}$$

since it makes $H^1(I) := W^{1,2}(I)$ a Hilbert space, with scalar product given by the formula

$$(u,\, v)_{H^1(I)} = (u,\, v)_{L^2(I)} + (Du,\, Dv)_{L^2(I)}. \tag{9.5}$$

Remark 9.5. The method employed in this paragraph to define a complete norm can be easily extended to any linear subspace of a Banach space. Indeed, let

$$T : \mathfrak{Y} \longrightarrow \mathcal{B}$$

be a linear operator between a linear space $\mathfrak{Y} \subseteq \mathcal{B}$ and a Banach space $(\mathcal{B},\, \|\cdot\|_{\mathcal{B}})$. If T is a closed operator, then the operator

$$\mathfrak{Y} \longrightarrow \mathrm{Grap}(D) \subset \mathcal{B} \times \mathcal{B}, \qquad f \longmapsto (f,\, Tf)$$

is linear and bijective. In particular, it turns out that $\mathfrak{Y}$ is a Banach space endowed with the norm

$$\|f\|_{\mathfrak{Y}} := \|f\|_{\mathcal{B}} + \|T\,f\|_{\mathcal{B}}.$$

Proposition 9.11. *Let $p \in [1,\, +\infty]$. Then for every $f \in W^{1,p}(I)$ there is an absolutely continuous representative of the equivalence class. In other words, the inclusion*

$$W^{1,p}(I) \subseteq C^0(I)$$

is a linear and continuous, i.e., there exists a positive constant C such that

$$\|f\|_{C^0(I)} \leq C\|f\|_{W^{1,p}(I)}.$$

Proof. This follows directly from Proposition 9.7. □

Proposition 9.12.

(a) *A function $f : I \longrightarrow \mathbb{R}$ is Lipschitz if and only if it belongs to $W^{1,\infty}(I)$.*

(b) *If $p \in (1, +\infty)$, then*

$$f \in W^{1,p}(I) \implies f \in C^{0,\frac{1}{p'}}(I),$$

where p' is the conjugate of p. Vice versa,

$$f \in C^{0,\alpha}(I) \not\Longrightarrow f \in W^{1,p}(I).$$

(c) *The Sobolev space $W^{1,p}(I)$ is isomorphic to a closed subset of $L^p(I) \times L^p(I)$. In particular, it is separable for $p \in [1, +\infty)$ and it is reflexive for $p \in (1, +\infty)$.*

Proof.

(a) This assertion is left as a simple exercise. For the solution, the reader may refer to this post.

(b) Let $f \in W^{1,p}(I)$ be a p-summable function, for some $p \in (1, +\infty)$. By Proposition 9.7 there is an absolutely continuous representative, which we still denote by f, such that

$$f(x) - f(y) = \int_y^x Df(t)\,\mathrm{d}t.$$

If we take the absolute values and apply the Hölder inequality, we find that

$$|f(x) - f(y)| \leq \|Df\|_{L^p(I)} \cdot |x - y|^{\frac{1}{p'}},$$

which means that f is $1/p'$-Hölder continuous.

Vice versa, the reader may check for herself that the Weierstrass function is Hölder continuous, but is not absolutely continuous, and is not of bounded variation either.

□

Dual Space. Let $p \in [1, +\infty)$. The dual space of $W^{1,p}(I)$ can be easily represented as

$$W^{1,p}(I) \ni u \longmapsto \int_I [u(x)f(x) + (Du(x))\, g(x)]\,\mathrm{d}x,$$

as f and g range in $L^{p'}$ (i.e., the dual space L^p for $p \neq +\infty$). This is an easy consequence of the fact that the dual of a product is the product of the duals

$$(L^p \times L^p)^* = L^{p'} \times L^{p'},$$

endowed with a suitable norm (see Exercise 3.4).

Remark 9.6. The representation is **not** unique. Indeed, the following Proposition 9.13 explains partially why (since ϕ can be chosen almost arbitrarily).

Proposition 9.13. *Let $x \in I$, and let $p \in [1, +\infty)$. Then there exists a function $\Psi_x \in L^{p'}$ such that the valuation functional $j_x \in \left(W^{1,p}(I)\right)^*$ can be represented as*

$$j_x(u) = \int_I u(t)\phi(t)\,\mathrm{d}t + \int_I Du(t)\Psi_x(t)\,\mathrm{d}t,$$

where $\phi : I \longrightarrow \mathbb{R}$ is a continuous function with compact support.

Proof. Let $\phi : I \longrightarrow \mathbb{R}$ be a continuous function with compact support and unitary mass

$$\int_I \phi(t)\,\mathrm{d}t = 1.$$

Let u be the absolutely continuous representative of the equivalence class $u \in W^{1,p}(I)$. In particular, we have that

$$u(x) = u(y) + \int_y^x Du(t)\,\mathrm{d}t \quad \text{for every } y < x \in I,$$

and thus, if one multiplies by $\phi(y)$ and integrate in dy both members, then it turns out that

$$u(x) = \int_I u(y)\phi(y)\,\mathrm{d}y + \int_I \left(\int_y^x Du(t)\,\mathrm{d}t \right) \phi(y)\,\mathrm{d}y.$$

By Fubini-Tonelli theorem

$$u(x) = \int_I u(y)\phi(y)\,\mathrm{d}y + \int_I \left[\mathbb{1}_{(-\infty,\,x]}(t) \int_I \mathbb{1}_{(-\infty,\,t]}(y)\phi(y)\,\mathrm{d}y \right] Du(t)\,\mathrm{d}t,$$

and this implies that

$$u(x) = \int_I u(y)\phi(y)\,\mathrm{d}y + \int_I Du(y)\Psi_x(y)\mathrm{d}y,$$

where

$$\Psi_x(y) := \mathbb{1}_{(-\infty,\,x)}(y) \int_{-\infty}^y \phi(t)\,\mathrm{d}t.$$

$\square$

Inclusion in Bounded Functions. It follows from Proposition 9.13 that, given $p \in [1, +\infty)$ and $u \in W^{1,p}(I)$, we have the estimate

$$|u(x)| \leq \|u\|_{L^p(I)} \|\phi\|_{L^{p'}(I)} + \|D\,u\|_{L^p(I)} \|\Psi_x\|_{L^{p'}(I)}.$$

Moreover, the $L^{p'}$-norm of Ψ_x does not depend on x, and hence there exists a positive constant $C > 0$ such that

$$|u(x)| \leq C\|u\|_{W^{1,p}(I)},$$

which means that the inclusion

$$W^{1,p}(I) \hookrightarrow \left(C_b^0(I), \, \|\cdot\|_{\infty,\,I} \right)$$

is continuous with respect to the uniform norm.

Proposition 9.14. *Let I be a* ***bounded*** *interval of $\mathbb{R}$, and let $p \in (1, +\infty]$. Then the inclusion*

$$W^{1,p}(I) \hookrightarrow C^0\left(\overline{I}\right)$$

is continuous and compact.

Proof. We have already proved in Proposition 9.12 that the inclusion is continuous, and

$$|u(x) - u(y)| \leq \|Du\|_{L^p(I)} \cdot |x - y|^{\frac{1}{p'}}.$$

Namely, the elements of the unitary ball in $W^{1,p}(I)$ are equicontinuous (as they are also equi-Hölder of parameter $1/p'$). Moreover, the L^p-norm is bounded by 1, and thus it follows from the main-value theorem that for all $u \in B_{W^{1,p}(I)}(0, 1)$ there exists a point $x_0 := x_0(u) \in I$ such that

$$|u(x_0)| \leq C < +\infty,$$

which, in turn, implies that

$$|u(x)| \leq |u(x_0)| + \|Du\|_{L^p(I)} \cdot |x - x_0|^{\frac{1}{p'}}.$$

In particular, every sequence $(u_n)_{n\in\mathbb{N}} \in B_{W^{1,p}(I)}(0, 1)$ is equibounded and equicontinuous. The interval I is bounded; hence the closure is compact and we conclude that the inclusion

$$W^{1,p}(I) \hookrightarrow C^0\left(\overline{I}\right)$$

is compact as a straightforward corollary of the Ascoli-Arzelà theorem. □

Remark 9.7. The inclusion

$$W^{1,p}(\mathbb{R}) \hookrightarrow C^0_b(\mathbb{R})$$

is not compact. Indeed, the action of the group of translations $\mathcal{T}_\mathbb{R}$ preserves the norm, that is,

$$u \in W^{1,p}(\mathbb{R}) \implies \|T_h u\|_{W^{1,p}(\mathbb{R})} = \|u\|_{W^{1,p}(\mathbb{R})}.$$

Let $u \in W^{1,p}(\mathbb{R})$ be any function of norm 1, and let $(\tau_n u)_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R})$ be a sequence. The weak limit is clearly the function identically zero, but

$$\|\tau_n u\|_{W^{1,p}(\mathbb{R}^n)} = 1 \quad \text{for all } n \in \mathbb{N} \implies u_n \not\to 0 \text{ strongly}.$$

9.2.1 Extension Operator

Let $r : W^{1,p}(\mathbb{R}) \longrightarrow W^{1,p}(I)$ be the restriction operator defined by setting

$$r(f) := f\big|_I.$$

Definition 9.15 (Extension Operator)**.** A linear continuous operator

$$E : W^{1,p}(I) \longrightarrow W^{1,p}(\mathbb{R})$$

is called *extension operator* if it is a right inverse of r, i.e., if

$$r(E(f)) = f.$$

Example 9.1. Let $u \in W^{1,p}(I)$, and assume that I is an open interval of the form (a, b). We have proved earlier that there always is an absolutely continuous function in the class of equivalence u, which we still denote by u. It turns out that the value at the extremal points of the interval can be computed as follows:

$$u(a) = \lim_{n\to+\infty} u\left(a + \frac{1}{n}\right) = \lim_{x\to a^+} u(x) \quad \text{and} \quad u(b) = \lim_{n\to+\infty} u\left(b - \frac{1}{n}\right) = \lim_{x\to b^-} u(x).$$

Let $\delta > 0$ be a real number, and consider the following extension of u, given by

$$\tilde{u}(x) := \begin{cases} 0 & x \in (-\infty,\, a-\delta], \\ \text{linear interpolation} & x \in (a-\delta,\, a], \\ u(x) & x \in (a,\, b] \\ \text{linear interpolation,} & x \in (b,\, b+\delta], \\ 0 & x \in (b+\delta,\, +\infty). \end{cases}$$

The reader may check by herself, as an exercise, that $\tilde{u}$ belongs to $W^{1,p}(\mathbb{R})$. More precisely, it is enough to check that

(1) $\tilde{u}$ belongs to $L^p(\mathbb{R})$, and

(2) $\tilde{u}$ is the primitive (in the sense of formula (9.2)) of some $\tilde{v} \in L^p(\mathbb{R})$.

Example 9.2. Let $u \in W^{1,p}(I)$ be any function, and assume that $I = [0,\, a)$. We can always consider the extension by reflection, that is,

$$\tilde{u}(x) := \begin{cases} u(x) & x \in (-a,\, 0] \\ u(x) & x \in [0,\, a). \end{cases}$$

Again, the reader may verify, as an exercise, that

$$\tilde{u} \in W^{1,p}\left((-a,\, a)\right).$$

The following theorem asserts that for all $p \in [1,\, +\infty]$, one can always find an extension operator P satisfying the additional property that the inclusions

$$P : W^{1,p}(I) \hookrightarrow L^p(\mathbb{R}) \quad \text{and} \quad P : W^{1,p}(I) \hookrightarrow W^{1,p}(\mathbb{R})$$

are both continuous.

Theorem 9.16 ([1])**.** *Let $1 \leq p \leq \infty$. There exists a bounded linear operator $P : W^{1,p}(I) \longrightarrow W^{1,p}(\mathbb{R})$, called an extension operator, satisfying the following properties:*

(a) *The map P is an actual extension, that is,*

$$Pu\big|_I = u \quad \textit{for every } u \in W^{1,p}(I).$$

(b) *The inclusion $P : W^{1,p}(I) \hookrightarrow L^p(\mathbb{R})$ is continuous with respect to the L^p-norm, that is,*

$$\|Pu\|_{L^p(\mathbb{R})} \leq C\|u\|_{L^p(I)} \quad \textit{for every } u \in W^{1,p}(I).$$

(c) *The inclusion $P : W^{1,p}(I) \hookrightarrow W^{1,p}(\mathbb{R})$ is continuous, that is,*

$$\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{L^p(I)} \quad \textit{for every } u \in W^{1,p}(I).$$

Furthermore, the constant C depends only on the length of the interval I.

9.2.2 Characterizations of $W^{1,p}(I)$

In this section, we investigate some possible characterization of the space $W^{1,p}(I)$, for a suitable range of p's, in terms of the L^p-continuity property.

Proposition 9.17. *Let $p \in (1, +\infty]$. If $u \in L^p(I)$, then the following assertions are equivalent:*

- ***(a)*** *The function u belongs to $W^{1,p}(I)$.*
- ***(b)*** *For every $\varphi \in C_c^\infty(I)$ it turns out that*

$$\left| \int_I u(t)\varphi'(t)\,\mathrm{d}t \right| \le C(u)\|\varphi\|_{L^{p'}(I)}.$$

Proof. We divide the proof into two steps.

Step 1. Assume that $u \in W^{1,p}(I)$ is a Sobolev function, and let $\varphi \in C_c^\infty(I)$ be a test function. By definition, we have the identity

$$\left| \int_I u(t)\varphi'(t)\,\mathrm{d}t \right| = \left| - \int_I Du(t)\varphi(t)\,\mathrm{d}t \right|,$$

and thus by the Hölder inequality it turns out that

$$\left| \int_I u(t)\varphi'(t)\,\mathrm{d}t \right| \le \|D\,u\|_{L^p(I)} \cdot \|\varphi\|_{L^{p'}(I)},$$

which is exactly what we wanted to prove.

Step 2. The functional

$$C_c^\infty(I) \ni \varphi \longmapsto \int_I u(t)\varphi'(t)\,\mathrm{d}t \in \mathbb{R}$$

is linear and, by assumption, continuous with respect to the $L^{p'}$ norm. The inclusion

$$C_c^\infty(I) \subset L^{p'}(I)$$

is dense, which means that the functional can be extended to the whole $L^{p'}$ isometrically. By the Riesz Representation Theorem 1.22 there exists[3] an element $f \in L^p(I)$ such that

$$\int_I f(t)\varphi(t)\,\mathrm{d}t = \langle f, \varphi \rangle_2 = \int_I u(t)\varphi'(t)\,\mathrm{d}t \quad \text{for all } \varphi \in C_c^\infty(I),$$

and therefore $-f \in L^p(I)$ is the weak derivative of u. □

Proposition 9.18 (L^p-continuity)**.** *Let $p \in [1, +\infty)$, and let $u \in L^p(\mathbb{R})$. Then there exists a modulus of L^p-continuity $\omega : [0, +\infty] \longrightarrow [0, +\infty]$ such taht*

$$\|\tau_h f - f\|_{L^p(\mathbb{R})} = \omega(|h|) \searrow 0. \tag{9.6}$$

Proof. We divide the proof into two steps.

[3] Here we need $p \neq 1$ since L^p is the dual space of $L^{p'}$ whenever $p' \neq +\infty$.

Step 1. The property (9.6) clearly holds true for all compactly supported continuous functions.

Step 2. Let us consider the subspace

$$\mathcal{G} := \{g \in L^p(\mathbb{R}) \mid (9.6) \text{ is satisfied by } g\} \subset L^p(\mathbb{R}).$$

The first step proves that $C_c^0(\mathbb{R}) \subseteq \mathcal{G}$, which means that if we can prove (9.6) for any function on the closure with respect to the L^p norm of $\mathcal{G}$, then (9.6) will be true for the closure of $C_c^0(\mathbb{R})$ in $L^p(\mathbb{R})$, which coincides with $L^p(\mathbb{R})$ itself.

Step 3. Let $f \in \overline{\mathcal{G}}$ be any element of the closure so that for any positive $\epsilon > 0$ there exists a function $g_\epsilon \in \mathcal{G}$ such that

$$\|f - g_\epsilon\|_{L^p(\mathbb{R})} \leq \epsilon.$$

It follows that

$$\begin{aligned}\|\tau_h f - f\|_{L^p(\mathbb{R})} &\leq \|\tau_h f - \tau_h g_\epsilon\|_{L^p(\mathbb{R})} + \|\tau_h g_\epsilon - g_\epsilon\|_{L^p(\mathbb{R})} + \|f - g_\epsilon\|_{L^p(\mathbb{R})} = \\ &= 2\|f - g_\epsilon\|_{L^p(\mathbb{R})} + \|\tau_h g_\epsilon - g_\epsilon\|_{L^p(\mathbb{R})} \leq \\ &\leq 2\epsilon + \|\tau_h g_\epsilon - g_\epsilon\|_{L^p(\mathbb{R})}.\end{aligned}$$

By taking the limit as $|h| \to 0$, it turns out that

$$\limsup_{|h|\to 0} \|\tau_h f - f\|_{L^p(\mathbb{R})} \leq 2\,\epsilon,$$

and this is enough to conclude the proof since ϵ was chosen to be arbitrarily small. □

We are finally ready to state the main result of this section, namely the characterization of $W^{1,p}(I)$ in terms of the L^p-continuity property stated above.

Proposition 9.19. *Let $p \in (1, +\infty)$. If $u \in L^p(I)$, then the following assertions are equivalent:*

- ***(a)*** *The function u belongs to $W^{1,p}(\mathbb{R})$.*
- ***(b)*** *For every translation $\tau_h \in \mathcal{T}_{\mathbb{R}}$, it turns out that*

$$\|\tau_h u - u\|_{L^p(\mathbb{R})} \lesssim |h|.$$

Remark 9.8. Let $f \in L^p(\mathbb{R}^n)$ be any p-summable function. As we mentioned above f satisfies, for all $p \neq +\infty$, the L^p-continuity property, that is,

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} = o(1) \quad \text{as } |h| \to 0.$$

Equivalently, f admits a *modulus of L^p-continuity*, i.e., there exists an increasing continuous function $\omega : [0, +\infty] \longrightarrow [0, +\infty]$ satisfying the properties

$$\omega(0) = 0 \quad \text{and} \quad \omega(t) = o(1) \text{ for } t \to 0,$$

such that

$$\|\tau_h\, f - f\|_{L^p(\mathbb{R}^n)} \leq \omega\left(|h|\right). \tag{9.7}$$

Proof of Proposition 9.19. We divide the proof into two steps.

Step 1. Assume that $u \in W^{1,p}(\mathbb{R})$, and let u be its absolutely continuous representative. For any positive number $h > 0$, we have that

$$u(x+h) - u(x) = \int_x^{x+h} Du(t)\, \mathrm{d}t.$$

Hence, we can easily estimate the absolute value of the left-hand side using the Hölder inequality

$$|u(x+h) - u(x)| \le \int_x^{x+h} |Du(t)| \, \mathrm{d}t \le \|Du\|_{L^p(\mathbb{R})} \, |h|^{1/p'},$$

where p' is the conjugate of p. Consequently, we have

$$\|u - \tau_h u\|^p_{L^p(\mathbb{R})} \le \overbrace{|h|^{p' \cdot p}}^{=h^{p-1}} \iint_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{[x,\, x+h]}(t) \, |Du(t)| \, \mathrm{d}t\mathrm{d}x =$$

$$= h \cdot |h|^{p-1} \int_{\mathbb{R}} |Du(t)|^p \, \mathrm{d}t =$$

$$= \|Du\|^p_{L^p(\mathbb{R})} \cdot |h|^p,$$

where the red identity follows from the Fubini-Tonelli theorem. Taking the pth root of both the left-hand side and the right-hand side, it turns out that

$$\|u - \tau_h \, u\|_{L^p(\mathbb{R})} \lesssim |h|,$$

where the constant depends on u only $c := \|Du\|_{L^p(\mathbb{R})}$, and is finite by assumption.

Step 2. Vice versa, we use the characterization given by Proposition 9.17 and prove instead that

$$\|u - \tau_h u\|_{L^p(\mathbb{R})} \lesssim |h| \implies \left| \int_{\mathbb{R}} u(t)\varphi'(t)\, \mathrm{d}t \right| \le C(u)\|\varphi\|_{L^{p'}(\mathbb{R})}$$

for every $\varphi \in C_c^\infty(\mathbb{R})$. Recall that the translation is a self-adjoint operator, that is,

$$\int_{\mathbb{R}} \tau_h u(t)\varphi(t)\, \mathrm{d}t = \int_{\mathbb{R}} u(t)\tau_{-h}\varphi(t)\, \mathrm{d}x,$$

and hence

$$\int_{\mathbb{R}} u(t) \frac{\varphi(t) - \varphi(t-h)}{h} \, \mathrm{d}t = -\frac{1}{h} \int_{\mathbb{R}} \left[\tau_h u(t) - u(t)\right] \varphi(t)\, \mathrm{d}t.$$

If we take the absolute value, then it turns out that

$$\left| \int_{\mathbb{R}} u(t) \frac{\varphi(t) - \varphi(t-h)}{h} \, \mathrm{d}t \right| \le \frac{1}{|h|} \left| \int_{\mathbb{R}} \left[\tau_h u(t) - u(t)\right] \varphi(t)\, \mathrm{d}t \right| \le$$

$$\le \frac{1}{|h|} \|\tau_h u - u\|_{L^p(\mathbb{R})} \cdot \|\varphi\|_{L^{p'}(\mathbb{R})} \lesssim$$

$$\lesssim \frac{1}{|h|} |h| \|\varphi\|_{L^{p'}(\mathbb{R})} \simeq \|\varphi\|_{L^{p'}(\mathbb{R})},$$

and we conclude by taking the limit as $h \to 0$ since the right-hand side of the inequality does not depend on h anymore. $\square$

9.3 Compactness in $L^p(\mathbb{R}^n)$

First, we recall a definition that will be extremely useful in what follows in this section.

Definition 9.20 (Modulus of Continuity)**.** A modulus of continuity is any real-extended valued function

$$\omega : [0, \infty] \longrightarrow [0, \infty],$$

vanishing at 0 and continuous at 0, that is,

$$\lim_{|h|\to 0} \omega(|h|) = 0.$$

Let $f \in L^p(\mathbb{R}^n)$ be any p-summable function defined on the whole real line. The following properties are trivial:

(1) The L^p-norm of f is finite, i.e., $\|f\|_{L^p(\mathbb{R}^n)} < +\infty$.

(2) There is a modulus of L^p-continuity $\omega : [0, +\infty] \to [0, +\infty]$ such that

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \leq \omega(|h|). \tag{9.8}$$

(3) There is a modulus of L^p-continuity $\alpha : [0, +\infty] \to [0, +\infty]$ such that

$$\|f\|_{L^p(B_R^{\complement})} \leq \alpha\left(\frac{1}{R}\right). \tag{9.9}$$

Remark 9.9. If $\mathcal{F} = \{f_1, \dots, f_n\} \subset L^p(\mathbb{R}^n)$ is a finite family of functions, then the properties **(1)**, **(2)** and **(3)** hold uniformly. More precisely, we have that:

(1) The L^p-norm of the f_i's is bounded by a uniform constant. Namely, it turns out that

$$\|f_i\|_{L^p(\mathbb{R}^n)} \leq \max_{j=1,\dots,n} \|f_j\|_{L^p(\mathbb{R}^n)} =: C < +\infty$$

for all $i = 1, \dots, n$.

(2) There is a modulus of L^p-continuity $\omega : [0, +\infty] \to [0, +\infty]$ such that

$$\|\tau_h f_i - f_i\|_{L^p(\mathbb{R}^n)} \leq \omega(|h|) \quad \text{for all } i \in \{1, \dots, n\},$$

and it is given by the maximum of the moduli of continuity of the f_i's, that is,

$$\omega(t) := \max_{j=1,\dots,n} \omega_j(t).$$

(3) There is a modulus of L^p-continuity $\omega : [0, +\infty] \to [0, +\infty]$ such that

$$\|f_i\|_{L^p(B_R^{\complement})} \leq \alpha\left(\frac{1}{R}\right) \quad \text{for all } i \in \{1, \dots, n\},$$

and it is given by the maximum of the moduli of continuity of the f_i's, that is,

$$\alpha(r) := \max_{j=1,\dots,n} \alpha_j(r).$$

Remark 9.10. In a similar fashion, one could prove that any compact family $\mathcal{F} \subset L^p(\mathbb{R}^n)$ satisfies the properties **(1)**, **(2)** and **(3)** uniformly.

We are now ready to state a complete characterization of relatively compact sets (and thus compact sets) in the L^p space for all $p \in [1, +\infty)$.

Theorem 9.21 (Fréchet-Kolmogorov)**.** *A set $E \subset L^p(\mathbb{R}^n)$, for $p \in [1, +\infty)$, is relatively compact if and only if E satisfies the following properties:*

(1) *The set E is equibounded in $L^p(\mathbb{R}^n)$, that is, there exists a constant $c > 0$ such that*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c \quad \text{for all } f \in E.$$

(2) *The set E is equicontinuous in $L^p(\mathbb{R}^n)$, that is, there exists a modulus of L^p-continuity $\omega : [0, +\infty] \longrightarrow [0, +\infty]$ such that*

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \leq \omega(|h|) \quad \text{for all } f \in E.$$

(3) *The set E is equiconcentrated in $L^p(\mathbb{R}^n)$, that is, there exists a modulus of L^p-continuity $\alpha : [0, +\infty] \longrightarrow [0, +\infty]$ such that*

$$\|f\|_{L^p\left(B_R^{\mathsf{c}}\right)} \leq \alpha\left(\frac{1}{R}\right) \quad \text{for all } f \in E.$$

Before we give the complete proof of this result, we discuss briefly the case $p = +\infty$ and why the same statement does not hold.

Proposition 9.22. *A set $E \subseteq C_0^0(\mathbb{R}^n)$ is relatively compact with respect to the uniform topology if and only if E satisfies the following properties:*

(a) *The set E is equibounded in $L^\infty(\mathbb{R}^n)$, that is, there exists a constant $c > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq c \quad \text{for all } f \in E.$$

(b) *The set E is equicontinuous in $L^\infty(\mathbb{R}^n)$, that is, there exists a modulus of L^∞-continuity $\omega : [0, +\infty] \longrightarrow [0, +\infty]$ such that*

$$\|\tau_h f - f\|_{L^\infty(\mathbb{R}^n)} \leq \omega(|h|) \quad \text{for all } f \in E.$$

(c) *The set E is equiconcentrated in $L^\infty(\mathbb{R}^n)$, that is, there exists a modulus of L^∞-continuity $\alpha : [0, +\infty] \longrightarrow [0, +\infty]$ such that*

$$\|f\|_{L^\infty\left(B_R^{\mathsf{c}}\right)} \leq \alpha\left(\frac{1}{R}\right) \quad \text{for all } f \in E.$$

Proof. The idea is to first prove uniform convergence on compact sets (closed balls), and then generalize it to the whole space.

Step 1. Let $(f_n)_{n\in\mathbb{N}} \subset E$ be an arbitrary sequence, and fix $R > 0$. The properties (**a**) and (**b**) allows us to apply the Ascoli-Arzelà theorem and find a subsequence $(f_{n_k,R})_{k\in\mathbb{N}}$ uniformly converging to some f_R in the closed ball of center 0 and radius R.

The argument does not depend on the particular value of R, therefore we can find such a subsequence for all $R > 0$. The diagonal method allows us to extract a sub-subsequence, still denoted by $(f_{n_k})_{k\in\mathbb{N}} \subset E$, such that

$$f_{n_k} \rightrightarrows f\,\big|_{\overline{B_R}} \quad \text{for all } R > 0, \text{ uniformly in } \overline{B_R}.$$

Step 2. In conclusion, the property (**c**) implies that E is relatively compact with respect to the uniform norm since one can always write

$$\|f_n - f\|_{\infty,\mathbb{R}^n} \le \|f_n - f\|_{\infty,\overline{B_R}} + 2\,\alpha(R) = \|f_n - f\|_{\infty,\overline{B_R}} + o(1) \quad \text{as } R \to +\infty.$$

□

Proof of Theorem 9.21. The proof presented here is rather involved, and requires a lot of work. Thus we divide it into five steps to ease the notation a little.

Step 1. First, observe that, for any $R > 0$, the property (**3**) implies that

$$\|f - \chi_{B_R} f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(B_R^{\complement})} \le \alpha\left(\frac{1}{R}\right), \tag{9.10}$$

where χ_{B_R} is the indicator function of the ball of radius R.

Step 2. In a similar fashion, for every positive mollifier $\varphi \in C_c^0(\mathbb{R}^n)$ with mass equal to 1 and diameter of the support $d > 0$, it turns out that

$$\|f * \varphi - f\|_{L^p(\mathbb{R}^n)} \le \sup_{|h|\le d} \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \le \omega(d). \tag{9.11}$$

The estimate (9.11) requires a little bit of work to be justified completely. Indeed, for every such mollifier $\varphi \in C_c^0(\mathbb{R}^n)$, we have

$$\begin{aligned}
|f * \varphi(x) - f(x)| &= \left|\int_{\mathbb{R}^n} \varphi(y)\,[f(x) - f(x-y)]\,\mathrm{d}y\right| = \\
&= \left|\int_{\mathbb{R}^n} \varphi^{1/p}(y)\,[f(x) - f(x-y)]\,\varphi^{1/p'}(y)\,\mathrm{d}y\right| \le \\
&\le \left(\int_{\mathrm{spt}(\varphi)} \varphi(y)\,|f(x) - f(x-y)|^p\,\mathrm{d}y\right)^{1/p} \left(\underbrace{\int_{\mathbb{R}^n} |\varphi(y)|\,\mathrm{d}y}_{=1}\right)^{1/p'}.
\end{aligned}$$

It follows that

$$\|f * \varphi - f\|_{L^p(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n} \left[\int_{\mathrm{spt}(\varphi)} \varphi(y) \, |f(x) - f(x-y)|^p \, \mathrm{d}y \right] \mathrm{d}x =$$

$$= \int_{\mathrm{spt}(\varphi)} \left[\int_{\mathbb{R}^n} |\tau_y f(x) - f(x)|^p \, \mathrm{d}x \right] \varphi(y) \, \mathrm{d}y \leq$$

$$\leq \sup_{|y| \leq d} \|\tau_y f - f\|_{L^p(\mathbb{R}^n)}^p,$$

where the last inequality follows easily from the fact that the support of φ has diameter d.

Step 3. The space $L^p(\mathbb{R}^n)$ is a complete normed space, which means that, as a consequence of Lemma 6.10, it is enough to prove that

$$(\mathbf{1}) + (\mathbf{2}) + (\mathbf{3}) \implies E \text{ is totally bounded.}$$

Let B denotes the unitary ball $B_{L^p(\mathbb{R}^n)}(0,\, 1)$. Note that the estimates (9.10) and (9.11) respectively prove the following inclusions:

$$\begin{cases} E \subseteq \chi_{B_R} \cdot E + \alpha\left(\frac{1}{R}\right) \cdot B, \\ E \subseteq \varphi * E + \omega(d) \cdot B. \end{cases}$$

Moreover, if E is equicontinuous and $R > 0$, then the family of functions $\chi_{B_R} \cdot E$ is also equicontinuous (although, with a different modulus of L^p-continuity, which we denote by ω^R). More precisely,

$$\|\varphi * (\chi_{B_R} f) - f\|_{L^p(\mathbb{R}^n)} \leq \|\varphi * (\chi_{B_R} f) - \chi_{B_R} f\|_{L^p(\mathbb{R}^n)} + \|\chi_{B_R} f - f\|_{L^p(\mathbb{R}^n)} \leq$$

$$\leq \omega^R(d) + \alpha\left(\frac{1}{R}\right),$$

which means that

$$E \subseteq \chi_{B_R} \cdot E + \alpha\left(\frac{1}{R}\right) \cdot B \subseteq \varphi * (\chi_{B_R} \cdot E) + \left(\omega^R(d) + \alpha\left(\frac{1}{R}\right)\right) \cdot B.$$

Step 4. Fix $\epsilon > 0$. One can always find real numbers $R > 0$ and $d > 0$ such that the constant on the right-hand side can be estimated by 2ϵ, that is,

$$E \subseteq \varphi * (\chi_{B_R} \cdot E) + 2\,\epsilon \cdot B.$$

Therefore, we can equivalently prove that the set $\varphi * (\chi_{B_R} \cdot E)$ is totally bounded in $C_c^0(B_{R+d})$ with respect to the uniform norm. In fact, this is enough to come to the same conclusion in the L^p topology (since it is weaker, and thus there are less open sets to check).

Step 5. For any $f \in L^p(\mathbb{R}^n)$ and any $\varphi \in C_c^0(\mathbb{R}^n)$, it turns out that

$$\|\tau_h(\varphi * f) - \varphi * f\|_\infty = \|\varphi * (\tau_h f - f)\|_\infty \leq$$

$$\leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \|\tau_h f - f\|_{L^p(\mathbb{R}^n)},$$

where the red inequality follows from a straightforward application of Young inequality[4]. In particular, for every $f \in E$, it turns out that

$$\|\tau_h(\varphi * f) - \varphi * f\|_\infty \leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \omega(|h|),$$

from which it also follows that

$$\|\varphi * f\|_\infty \leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \|f\|_{L^p(\mathbb{R}^n)},$$

i.e., the set $\varphi * (\chi_{B_R} \cdot E)$ is equibounded and equicontinuous with respect to the uniform topology (=uniform norm). A straightforward application of the Ascoli-Arzelà theorem proves that the set $\varphi * (\chi_{B_R} \cdot E)$ is relatively compact in $C_c^0(B_{R+d})$ with respect to the uniform norm, and hence it is relatively compact in $L^p(\mathbb{R}^n)$. □

There is a different way to prove this theorem using the regularization by convolution only, but we will not give the details here. The interested reader can try to fill in the missing information in the following road-map.

Alternative Proof. Consider the inclusion

$$E \subseteq \varphi * E + \omega(d) \cdot B$$

proved in the third step of the previous proof. The set $\varphi * E$ is compact in $C_0^0(\mathbb{R}^n)$ since one can easily prove that the following properties hold true:

(a) For every $f \in E$ it turns out that

$$\|\varphi * f\|_\infty \leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \|f\|_{L^p(\mathbb{R}^n)}.$$

(b) For every $f \in E$ and for every $h \in \mathbb{R}$ it turns out that

$$\|\tau_h(\varphi * f) - \varphi * f\|_\infty \leq \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \omega(|h|).$$

(c) For every $f \in E$ and for every R sufficiently large it turns out that

$$\|\varphi * f\|_{B_R^{\complement}, \infty} \leq \alpha\left(\frac{1}{R-d}\right).$$

In particular, the set $\varphi * E$ is compact in $L^p_{\text{loc}}(\mathbb{R}^n)$, and we can conclude that it is also continuously included in $L^p(\mathbb{R}^n)$ by using the property (**3**). □

[4]**Young Inequality.** Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, and assume that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

The convolution $f * g$ belongs to $L^r(\mathbb{R}^n)$, and the following inequality holds:

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}. \tag{9.12}$$

9.4 Sobolev Space $W^{m,p}(\Omega)$

The primary goal of this section is to generalize the notion of Sobolev space from the interval $I \subseteq \mathbb{R}$ to an arbitrary (and eventually unbounded) open set $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$.

Remark 9.11.

(1) Let $\alpha \in \mathbb{N}^n$ be a multi-index. The αth derivative operator, denoted by D^α, is defined on the space $C_c^\infty(\Omega)$ as follows:

$$D^\alpha u(x_1, \ldots, x_n) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x_1, \ldots, x_n).$$

(2) For every $f \in C^1(\Omega)$ and for every $g \in C_c^\infty(\Omega)$, it turns out that

$$\int_\Omega \frac{\partial}{\partial x_i} f(x) g(x)\,\mathrm{d}x = -\int_\Omega f(x) \frac{\partial}{\partial x_i} g(x)\,\mathrm{d}x.$$

In particular, for any multi-index of length $|\alpha| \leq k$ and any $f \in C^k(\Omega)$, the formula above can be generalized as follows:

$$\int_\Omega D^\alpha f(x) g(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha g(x)\,\mathrm{d}x. \tag{9.13}$$

Definition 9.23 (Weak Derivative)**.** Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a locally summable function. A function $g \in L^1_{\mathrm{loc}}(\Omega)$ is the αth *weak derivative* of f, and we denote g by $D^\alpha f$, if and only if

$$\int_\Omega f(x) \partial^\alpha \varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g(x)\varphi(x)\,\mathrm{d}x, \qquad \forall\, \varphi \in C_c^\infty(\Omega). \tag{9.14}$$

Proposition 9.24. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then the following properties hold:*

(a) *If $g \in L^1_{\mathrm{loc}}(\Omega)$ is the αth weak derivative of f, then it is unique up to the almost everywhere equivalence relation.*

(b) *The definition (9.14) is local. Namely, if there exists a neighborhood $U_x \subset \Omega$ where f is α-weakly differentiable for every $x \in \Omega$, then f admits a global α-weak derivative g in Ω.*

Proof.

(a) Suppose that g_1 and g_2 are both α-th weak derivatives of f. It follows from the definition formula (9.14) that

$$\int_\Omega [g_1(x) - g_2(x)]\,\varphi(x)\,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^\infty(\Omega),$$

and therefore $g_1(x) = g_2(x)$ for almost every $x \in \Omega$ as a consequence of the fundamental lemma in calculus of variations[5].

[5]**Lemma.** Let $f \in L^1_{\mathrm{loc}}(\Omega)$. If

$$\int_\Omega f(x)\,\varphi(x)\,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^\infty(\Omega),$$

then $f(x) = 0$ for almost every $x \in \Omega$.

(b) Let us consider an open covering

$$\mathcal{U} := \{U_i\}_{i\in\mathbb{N}}$$

of Ω, and the collection $g_i \in L^1_{\mathrm{loc}}(U_i)$ of weak derivatives of $f\,\big|_{U_i}$. Let us consider also the partition of unity

$$\{\rho_i : \Omega \longrightarrow \mathbb{R}_{\geq 0}\}_{i\in\mathbb{N}}$$

subordinated to the covering $\mathcal{U}$. We claim that the function

$$g(x) := \sum_{i=0}^{+\infty} \rho_i(x) \cdot g_i(x)$$

is the (global) weak derivative of f, that is, the following properties are satisfied:

(1) The function is locally summable, that is, $g \in L^1_{\mathrm{loc}}(\Omega)$.

(2) The function g is the weak derivative of f, that is, it satisfies (9.14).

Let $K \subset \Omega$ be a compact subset. The partition is locally finite; hence the first assertion follows easily from the estimate

$$\|g\|_{L^1(K)} = \left\| \sum_{i=1}^{\ell} \rho_i \cdot g_i \right\|_{L^1(K)} \leq \sum_{i=1}^{\ell} \|g_i\|_{L^1(K)} < +\infty,$$

since the g_i's are locally summable by assumption.

Let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$, and let $K \subset \Omega$ be a compact set containing the support of φ. Then

$$\begin{aligned}
\int_\Omega f(x) D^\alpha \varphi(x)\,\mathrm{d}x = \int_K f(x) D^\alpha \varphi(x)\,\mathrm{d}x &= \\
&= \int_K \left(\sum_{i=1}^{\ell} \rho_i(x)\right) f(x) D^\alpha \varphi(x)\,\mathrm{d}x = \\
&= \sum_{i=1}^{\ell} \int_K \left(\rho_i(x) f(x)\right) D^\alpha \varphi(x)\,\mathrm{d}x = \\
&= \sum_{i=1}^{\ell} \int_{U_i} \left(\rho_i(x) f(x)\right) D^\alpha \varphi(x)\,\mathrm{d}x = \\
&= (-1)^{|\alpha|} \sum_{i=1}^{\ell} \int_{U_i} \left(\rho_i(x) g_i(x)\right) \varphi(x)\,\mathrm{d}x = \\
&= (-1)^{|\alpha|} \int_K g(x)\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

which is exactly what we wanted to prove.

□

Definition 9.25 (Sobolev Space)**.** Let $p \in [1, +\infty]$, and let $m \in \mathbb{N}$. The (m, p)-Sobolev space, denoted by $W^{m,p}(\Omega)$, is the space of all $L^p(\Omega)$ functions with α weak derivatives in $L^p(\Omega)$ for every $|\alpha| \leq m$, that is,

$$W^{k,p}(I) := \{ f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq m \}.$$

Normed Space. In this paragraph, we discuss briefly the main idea that allows us to define a norm on $W^{m,p}(\Omega)$ that makes it a Banach space.

Remark 9.12. The Sobolev space $W^{m,p}(\Omega)$ is a vector space, and the application

$$D^\alpha : W^{m,p}(\Omega) \to L^p(\Omega), \qquad f \longmapsto D^\alpha f$$

is well-defined and linear for all multi-indices $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$.

Lemma 9.26. *The graph of the operator* $D^\alpha : W^{k,p}(\Omega) \longrightarrow L^p(\Omega)$ *is closed for every* $\alpha \in \mathbb{N}^n$ *of length* $|\alpha| \leq m$*, that is,*

$$\Gamma(D^\alpha) := \left\{ (f, D^\alpha f) \;\middle|\; f \in W^{k,p}(\Omega) \right\} \subset L^p(\Omega) \times L^p(\Omega)$$

is closed with respect to the subspace topology.

Lemma 9.27. *Let* N *denote the cardinality of the set* $\mathfrak{N} := \{\alpha \in \mathbb{N}^n \,:\, |\alpha| \leq m\}$*. The graph of the operator* T*, defined by setting*

$$T : W^{k,p}(\Omega) \longrightarrow (L^p(\Omega))^N, \qquad f \longmapsto (D^\alpha f)_{\alpha \in \mathfrak{N}},$$

is closed with respect to the subspace topology.

Proof. Let $(f_j)_{j \in \mathbb{N}} \subset W^{m,p}(\Omega)$ be a converging sequence, i.e.,

$$\begin{cases} f_j \xrightarrow{L^p} f \in L^p(\Omega), \\ D^\alpha f_j \xrightarrow{L^p} f_\alpha \in L^p(\Omega) \quad \text{for all } \alpha \in \mathfrak{N}. \end{cases}$$

If we are able to prove that f_α is nothing else than the αth weak derivative of f for all $\alpha \in \mathfrak{N}$, then we can easily infer that f belongs to $W^{m,p}(\Omega)$. By definition of weak derivative, it turns out that

$$\int_\Omega f_j(x) D^\alpha \varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega D^\alpha f_j(x) \varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in C_c^\infty(\Omega)$$

holds for every $j \in \mathbb{N}$ and for every admissible α.

Therefore, we can pass the identity to the limit using the Lebesgue dominated convergence theorem. This is possible because the L^p convergence implies (up to a subsequence) the a.e. pointwise convergence of the sequence, i.e.

$$\begin{cases} f_{j_k}(x) \xrightarrow{k \to +\infty} f(x) & \text{for almost every } x \in \Omega, \\ D^\alpha f_{j_k}(x) \xrightarrow{k \to +\infty} f_\alpha(x) & \text{for almost every } x \in \Omega \text{ and for all } \alpha \in \mathfrak{N}. \end{cases}$$

□

Corollary 9.28. *The operator*

$$T : W^{m,\,p}(\Omega) \longrightarrow \left(L^p(\Omega)\right)^N$$

is linear, injective and onto its rank. More precisely, it turns out that

$$T : W^{m,\,p}(\Omega) \longrightarrow \Gamma(T)$$

is linear and bijective.

In particular, the product $\left(L^p(\Omega)\right)^N = L^p(\Omega) \times \cdots \times L^p(\Omega)$ induces on the graph $\Gamma(T)$ the subspace topology, that is, the topology generated by the restriction of the product norm

$$\| -_1 \|_p + \cdots + \| -_N \|_p.$$

More precisely, for $p \neq +\infty$, if we endow $W^{m,\,p}(\Omega)$ with the topology generated by the norm

$$\|f\|_{m,\,p,\,\Omega} := \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \tag{9.15}$$

then $W^{m,\,p}(\Omega)$ is a Banach space. Similarly, if $p = +\infty$, then the norm

$$\|f\|_{m,\,\infty,\,\Omega} := \sup_{\substack{|\alpha| \leq m \\ x \in \Omega}} |D^\alpha f(x)|\,, \tag{9.16}$$

gives to $W^{m,\,\infty}(\Omega)$ the structure of a Banach space. In any case, there is an isometry

$$\left(W^{m,\,p}(\Omega),\, \|\cdot\|_{m,\,p,\,\Omega}\right) \xrightarrow{\sim} \left(\Gamma(T),\, \|\cdot\|_{(L^p(\Omega))^N}\big|_{\Gamma(T)}\right)$$

which makes the following properties trivially true:

(a) The Sobolev space $W^{m,\,p}(\Omega)$ is isomorphic to a closed subset of $\left(L^p(\Omega)\right)^N$.

(b) If $1 \leq p < +\infty$, then $W^{m,\,p}(\Omega)$ is a separable Banach space. If $p = +\infty$, then $W^{m,\,\infty}(\Omega)$ is not separable.

(c) If $1 < p < +\infty$, then $W^{m,\,p}(\Omega)$ is a reflexive Banach space.

Dual Space. The elements g of the dual space $\left(W^{m,\,p}(\Omega)\right)^*$ can be easily represented by

$$g \;:\; W^{m,\,p}(\Omega) \ni u \longmapsto \sum_{|\alpha| \leq m} \int_\Omega g_\alpha(x) D^\alpha f(x)\,\mathrm{d}x,$$

where $g_\alpha \in L^{p'}(\Omega)$, and p' is the conjugate of p.

Remark 9.13. Contrarily to the one-variable case, in general, it is not true that

$$W^{m,\,p}(\Omega) \subset C^0(\Omega).$$

For example, the space $W^{0,\,p}(\Omega) = L^p(\Omega)$ is not contained in the space of all continuous functions on Ω since a L^p function need not be continuous (nor there is a continuous representative in each class) when the dimension is at least $n \geq 2$. In a similar fashion (see Exercise 9.10), one can show that

$$W^{m,\,p}(\Omega) \not\subset L^\infty(\Omega).$$

Theorem 9.29 (Sobolev Embedding Theorem, [3]). *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Let $f \in W^{m,\,p}(\Omega)$.*

(a) *If*

$$m < \frac{n}{p},$$

then f belongs to $L^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|f\|_{L^q(\Omega)} \leq C(m,\, p,\, n,\, \Omega)\, \|f\|_{W^{m,\,p}(\Omega)}.$$

(b) *If*

$$m > \frac{n}{p},$$

then $f \in C^{k-\lfloor n/p \rfloor -1,\,\gamma}(\bar{\Omega})$, where

$$\gamma = \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|f\|_{C^{k-\lfloor n/p \rfloor -1,\,\gamma}(\bar{\Omega})} \leq C(m,\, \gamma,\, p,\, n,\, \Omega)\, \|f\|_{W^{m,\,p}(\Omega)}.$$

Remark 9.14. The space $W^{m,\,2}(\Omega)$ is a Hilbert space for any $m \in \mathbb{N}$.

9.4.1 Operations on Sobolev Space $W^{m,p}(\Omega)$

Product. In this brief paragraph, we introduce the main properties of the multiplication operator in Sobolev spaces.

Lemma 9.30. *Let $f \in W^{m,\,p}(\Omega)$, and let $\varphi \in C_c^\infty(\Omega)$.*

(a) *The support of the product is smaller than the support of the test function, that is,*

$$\operatorname{spt}(f\varphi) \subseteq \operatorname{spt}(\varphi).$$

(b) *The product is a closed operator, that is,*

$$f\varphi \in W^{m,\,p}(\Omega).$$

Moreover, the Leibniz rule holds true for any (integer) derivative, that is,

$$D^\alpha\,(f \cdot \varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta\, f\, D^{\alpha-\beta}\, \varphi. \tag{9.17}$$

Proof.

(a) Obvious.

(b) First, we check the two assertions for the derivatives of order one, and then we generalize it by induction. Indeed, for every test function ψ it turns out that

$$\int_\Omega \left[f(x)\frac{\partial \varphi(x)}{\partial x_j} + \varphi(x) D^j\left(f(x)\right)\right]\psi(x)\,\mathrm{d}x =$$

$$= \int_\Omega f(x)\frac{\partial \varphi(x)}{\partial x_j}\psi(x)\,\mathrm{d}x + \int_\Omega \varphi(x) D^j\left(f(x)\right)\psi(x)\,\mathrm{d}x =$$

$$= \int_\Omega f(x)\left(\frac{\partial \varphi(x)}{\partial x_j}\psi(x)\right)\,\mathrm{d}x + \int_\Omega \varphi(x) D^j\left(f(x)\right)\psi(x)\,\mathrm{d}x =$$

$$= -\int_\Omega \left(f(x)\varphi(x)\right)\frac{\partial \psi(x)}{\partial x_j}\,\mathrm{d}x,$$

where the red equality follows from the formula

$$\frac{\partial \varphi(x)}{\partial x_j}\psi(x) = \frac{\partial\left(\varphi\psi\right)(x)}{\partial x_j} - \varphi(x)\frac{\partial \psi(x)}{\partial x_j}.$$

The reader may check by herself that Sobolev spaces can be equivalently defined as

$$f \in W^{m,p}(\Omega) \iff \begin{cases} D^j f \in W^{m-1,p}(\Omega) & \forall\, j = 1, \dots, n \\ f \in L^p(\Omega), \end{cases}$$

and hence the thesis follows by induction on the length of the multi-indices $\alpha \in \mathbb{N}^n$.

$\square$

Regularization by Convolution $(1 \le p < +\infty)$**.** Let $f \in W_c^{m,p}(\Omega)$ be a Sobolev function with compact support, and let $\varphi \in C_c^\infty(\Omega)$ be a function such that

$$\operatorname{diam}\left(\operatorname{spt}(\varphi)\right) < d\left(\operatorname{spt}(f),\, \Omega^c\right).$$

It is a well-known fact that the support of the convolution is contained in the sum of the supports. As a consequence of the condition above, we have that

$$\operatorname{spt}\left(f * \varphi\right) \subset\subset \Omega,$$

and the inclusion is compact. Furthermore, one can easily prove that

$$f * \varphi \in C_c^\infty(\Omega),$$

that is, the usual informal assertion *"the regularity of the convolution is the maximum between the regularity of the elements"* holds true.

Suppose that $\Phi \in C_c^\infty(B(0,1))$ is a mollifier (that is, a positive function with total mass equal to 1), and let Φ_ϵ be the rescaled associated function defined by setting

$$\Phi_\epsilon(x) := \frac{1}{\epsilon^n}\,\Phi\left(\frac{x}{\epsilon}\right).$$

A natural question now arises: *The convolution product* $\Phi_\epsilon * f$ *converges to* f *in the* $W^{m,\,p}(\Omega)$ *topology (=with respect to the* $\|\cdot\|_{m,\,p,\,\Omega}$ *norm) when* $\epsilon \to 0^+$*?*

Lemma 9.31. *Let* $\Phi \in L^1(\mathbb{R}^n)$*, and let* $f \in W^{m,\,p}(\mathbb{R}^n)$*. The convolution product belongs to* $W^{m,\,p}(\mathbb{R}^n)$ *and, for every* $|\alpha| \leq m$*, it turns out that*

$$D^\alpha (f * \Phi) = D^\alpha f * \Phi = f * D^\alpha \Phi. \tag{9.18}$$

Proof. Suppose that (9.18) holds true. By Young's inequality (9.12) it turns out that

$$\|D^\alpha (f * \Phi)\|_{L^p(\mathbb{R}^n)} = \|D^\alpha f * \Phi\|_{L^p(\mathbb{R}^n)} \leq \|\Phi\|_{L^1(\mathbb{R}^n)} \cdot \|D^\alpha f\|_{L^p(\mathbb{R}^n)}$$

for every $|\alpha| \leq m$. Therefore the convolution product belongs to $W^{m,\,p}(\mathbb{R}^n)$. ☐

The answer to the question raised above is thus affirmative. Indeed, it follows from (9.18) that

$$D^\alpha (\Phi_\epsilon * f) = \Phi_\epsilon * D^\alpha f,$$

and this converges to $D^\alpha f$ in the $L^p(\Omega)$ topology (= with respect to the L^p norm) for every $|\alpha| \leq m$, which means that

$$\|\Phi_\epsilon * f - f\|_{m,\,p,\,\Omega} \xrightarrow{\epsilon \to 0^+} 0.$$

9.5 Sobolev Spaces: $W_0^{m,\,p}(\Omega)$ and $H^{m,\,p}(\Omega)$

In this final section, we introduce the spaces $W_0^{m,\,p}(\Omega)$ and $H^{m,\,p}(\Omega)$, and we also give a sketch of the proof of the well-known result "$H = W$".

Definition 9.32. The $(m,\,p,\,0)$-Sobolev space is the closure of the space of test functions with respect to the $(m,\,p)$-norm, that is,

$$W_0^{m,\,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{m,\,p,\,\Omega}}.$$

Remark 9.15. The primary idea behind the notion of $W_0^{m,\,p}(\Omega)$ is to give a meaning to the Dirichlet boundary condition $u(x) = 0$ in the weak formulation of partial differential equations.

Remark 9.16. The inclusion

$$W_0^{m,\,p}(\Omega) \subseteq W^{m,\,p}(\Omega)$$

is always true since $C_c^\infty(\Omega)$ is a subspace of $W^{m,\,p}(\Omega)$, and so is its closure. On the other hand, if $\Omega = \mathbb{R}^n$, then it turns out that

$$W_0^{m,\,p}(\mathbb{R}^n) = W^{m,\,p}(\mathbb{R}^n),$$

coherently with the intuitive meaning of "being zero at the boundary".

Proof. It suffices to prove the inclusion

$$W_0^{m,\,p}(\mathbb{R}^n) \supseteq W^{m,\,p}(\mathbb{R}^n),$$

that is, every function $f \in W^{m,\,p}(\mathbb{R}^n)$ is the limit of a sequence of functions $C_c^\infty(\mathbb{R}^n)$ with respect to the Sobolev norm $\|\cdot\|_{m,\,p,\,\Omega}$.

Step 1. Let us consider a cut-off function such that

$$\eta \in C_c^\infty\left(B(0,\,2)\right), \quad \eta\big|_{B(0,\,1)} \equiv 1 \quad \text{and} \quad \eta(x) \in [0,\,1],$$

and let us consider the rescaling function

$$\eta_R(x) := \eta\left(\frac{x}{R}\right).$$

By Lemma 9.30 it turns out that the product function $\eta_R \cdot f$ belongs to $W^{k,\,p}(\Omega)$, and from the Leibniz formula we infer that

$$D^\alpha\left(\eta_R \cdot f\right) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta(\eta_R)\, D^{\alpha-\beta}(f).$$

The γth derivative of η_R goes to 0 as $R^{-|\gamma|}$ for $R \to +\infty$, and therefore one can easily prove that

$$\sum_{\substack{\beta \le \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} D^\beta(\eta_R)\, D^{\alpha-\beta}(f) \xrightarrow{R \to +\infty} 0,$$

e.g., as a consequence of the dominated convergence theorem.

Step 2. In particular, for any fixed $\epsilon > 0$ there exists $R > 0$ such that

$$\|\eta_R \cdot f - f\|_{m,\,p} \le \epsilon.$$

Let $\Phi \in C_c^\infty(\mathbb{R}^n)$ be a smooth mollifier, and let Φ_r be its rescaling. The convolution $\Phi_r * (\eta_R \cdot f)$ is smooth, and we find that

$$\Phi_r * (\eta_R \cdot f) \xrightarrow{r \to 0^+} \eta_R \cdot f \quad \text{with respect to the } W^{m,\,p} \text{ norm.}$$

Set $r := 1/R$. We infer that

$$\Phi_{\frac{1}{R}} * (\eta_R \cdot f) \xrightarrow{R \to +\infty} f \quad \text{with respect to the } W^{m,\,p} \text{ norm,}$$

and this is exactly what we wanted to prove. □

Definition 9.33 (Nikol'skii space)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The space $H^{m,\,p}(\Omega)$ is the completion of the space

$$\{f \in C^m(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \le m\}$$

with respect to the Sobolev norm $\|\cdot\|_{m,\,p,\,\Omega}$.

Theorem 9.34 (Meyers-Serrin)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open subset, and let $p \in [1,\,+\infty)$. Then*

$$H^{m,\,p}(\Omega) = W^{m,\,p}(\Omega).$$

Remark 9.17. The identity, in general, does not hold for the value $p = +\infty$. Indeed, in this particular case, the statement depends also on the regularity of the boundary of Ω.

Proof. The inclusion

$$H^{m,\,p}(\Omega) \subseteq W^{m,\,p}(\Omega)$$

is trivial because

$$\{f \in C^m(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq m\} \subseteq W^{m,\,p}(\Omega),$$

and therefore the completion with respect to its norm is also contained in $W^{m,\,p}(\Omega)$.

The opposite inclusion, on the other hand, requires a little bit more work. We divide the proof into four steps to ease the notation, and we leave to the reader to fill in the missing details.

Step 1. Let us consider the family (as k ranges in $\mathbb{N}$) of open sets

$$\mathcal{U}_k := \left\{ x \in \Omega \;\middle|\; d(x,\,\Omega^c) > \frac{1}{k} \text{ and } |x| < k \right\},$$

and notice that it is increasing

$$\mathcal{U}_k \subset\subset \mathcal{U}_{k+1} \subset\subset \cdots \subset\subset \Omega \implies \overline{\mathcal{U}_k} \subset \mathcal{U}_{k+1},$$

and such that every $\mathcal{U}_k$ has compact closure.

Step 2. Let us consider a collection of open sets $\{V_k\}_{k\in\mathbb{N}}$ satisfying the following property:

$$\mathcal{U}_{k+1} \supset V_k \supset \overline{U_k}.$$

For every $k \geq 1$ there exists a cut-off function $\eta_k \in C_c^\infty(\mathcal{U}_k)$ such that

$$\eta_k \big|_{V_{k-1}} \equiv 1 \quad \text{and} \quad \eta_k \big|_{\mathcal{U}_k^c} \equiv 0.$$

Let $\Omega_k := \mathcal{U}_k \setminus \overline{\mathcal{U}_{k-2}}$, and let

$$\varphi_k := \eta_k - \eta_{k-2}$$

be a partition of unity associated with the covering $\{\Omega_k\}_{k\geq 2}$. Indeed, it is easy to check that

$$\sum_{k\geq 2} \varphi_k(x) = 1 \quad \text{for all } x \in \Omega,$$

and also that the sum is locally finite since

$$\Omega_k \cap \Omega_j \neq \emptyset \iff |k-j| = 1.$$

Step 3. Let $f \in W^{m,\,p}(\Omega)$. If we set $f_k(x) := f(x)\cdot\varphi_k(x) \in W_c^{m,\,p}(\Omega_k)$, then we can write

$$f(x) = \sum_{k\geq 2} f_k(x).$$

Let $\epsilon > 0$ be a fixed real number. By definition, one can always find a collection of functions $g_k \in C_c^\infty(\Omega_k)$ such that

$$\|g_k - f_k\|_{m,\,p,\,\Omega_k} = \|g_k - f_k\|_{m,\,p,\,\mathbb{R}^n} \leq \frac{\epsilon}{2^k}.$$

Step 4. Let

$$g(x) := \sum_{k \geq 2} g_k(x).$$

The reader may prove by herself, as an exercise, that

$$g \in W^{m,\,p}(\Omega) \quad \text{and} \quad \|f - g\|_{m,\,p,\,\Omega} = \mathcal{O}(\epsilon),$$

that is, the test functions are dense in $W^{m,\,p}(\Omega)$ with respect to the norm $\|\cdot\|_{m,\,p,\,\Omega}$. □

9.6 Exercises

In this section, we denote by I an open interval $(a,\, b)$ of the real line $\mathbb{R}$ (eventually unbounded), unless otherwise stated.

Exercise 9.1. Let $I = (0,\, 1)$, and let $p \in (1,\, +\infty]$.

(1) The inclusion $W^{1,\,1}(I) \subseteq C^0\left(\overline{I}\right)$ is not compact.

(2) The inclusion

$$W^{1,\,1}(I) \hookrightarrow L^q(I)$$

is compact for every $1 \leq q < \infty$.

Proof.

(1) Let us consider the sequence $(u_n)_{n \geq 2} \subset W^{1,\,1}\left((0,\, 1)\right)$ defined by

$$u_n(x) = \begin{cases} 0 & \text{if } x \in [0,\, \frac{1}{2}], \\ n \cdot \left(x - \frac{1}{2}\right) & \text{if } x \in (\frac{1}{2},\, \frac{1}{2} + \frac{1}{n}), \\ 1 & \text{if } x \in [\frac{1}{2} + \frac{1}{n},\, 1]. \end{cases}$$

The weak derivatives sequence is clearly given by

$$D\, u_n(x) = \begin{cases} 0 & \text{if } x \in [0,\, \frac{1}{2}], \\ n & \text{if } x \in (\frac{1}{2},\, \frac{1}{2} + \frac{1}{n}), \\ 0 & \text{if } x \in [\frac{1}{2} + \frac{1}{n},\, 1], \end{cases}$$

hence

$$\begin{aligned} \|u_n\|_{1,\,1,\,I} &= n \cdot \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left(x - \frac{1}{2}\right) \mathrm{d}x + \frac{1}{2} - \frac{1}{n} + n \cdot \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \mathrm{d}x \\ &= \frac{1}{n} + \frac{1}{2} - \frac{1}{n} + 1 = \frac{3}{2}, \end{aligned}$$

that is, the sequence is bounded in $W^{1,\,1}$.

We now observe that sequence $u_n(x)$ converges pointwise to the bump function

$$u(x) = \begin{cases} 0 & \text{if } x \in [0,\, \frac{1}{2}], \\ 1 & \text{if } x \in [\frac{1}{2},\, 1]. \end{cases}$$

On the other hand, for any $n \geq 2$ it turns out that

$$\|u_n - u\|_\infty = 1,$$

hence no subsequence of $(u_n)_{n\geq 2}$ can converge to u in the uniform norm.

(2) Let B be the unit ball in $W^{1,1}(I)$. Let P be the extension operator of Theorem 9.16 and set $\mathcal{F} := P(B)$, so that $B = \mathcal{F}\big|_I$.

The set $\mathcal{F}$ is bounded in $W^{1,1}(\mathbb{R})$ by construction; hence it is bounded also in $L^q(\mathbb{R})$ for all $1 \leq q < +\infty$. If we are able to prove that for every $f \in \mathcal{F}$ there is a modulus of L^q-continuity such that

$$\|\tau_h f - f\|_{L^q(\mathbb{R})} \leq \omega(|h|),$$

then we can apply the compactness result (Theorem 9.21) to infer that B has compact closure in $L^q(\mathbb{R})$. From Proposition 9.17 it turns out that

$$\|\tau_h f - f\|_{L^1(\mathbb{R})} = \mathcal{O}(|h|),$$

and hence

$$\|\tau_h f - f\|^q_{L^q(\mathbb{R})} \leq \left(2\,\|f\|_{L^\infty(\mathbb{R})}\right)^{q-1} \cdot \|\tau_h f - f\|_{L^1(\mathbb{R})} \leq C \cdot |h|.$$

If we take the q-th root, then we obtain the estimate

$$\|\tau_h f - f\|_{L^q(\mathbb{R})} \leq \; C \cdot |h|^{\frac{1}{q}},$$

and this concludes the proof since we can set $\omega(|h|) := C \cdot |h|^{\frac{1}{q}}$ for every $1 \leq q < +\infty$.

□

Exercise 9.2. The Sobolev space $W^{1,p}(I)$ is a function algebra, that is,

$$u,\, v \in W^{1,p}(I) \implies u \cdot v \in W^{1,p}(I),$$

and the Leibniz rule also holds true for the weak derivative operator.

Exercise 9.3 (Convolution)**.** Let $u \in W^{1,p}(\mathbb{R})$, and let $v \in L^1(\mathbb{R})$ be a convolution kernel. Prove that

$$u * v \in W^{1,p}(\mathbb{R}) \quad \text{and} \quad D\,(u * v) = u * v'.$$

Exercise 9.4 (Mollification)**.** Let $\varphi \in C_c^\infty(\mathbb{R})$ be a convolution kernel, and let us set

$$\varphi_\epsilon(x) := \frac{1}{\epsilon}\,\varphi\left(\frac{x}{\epsilon}\right).$$

For any $p \in [1,\, +\infty)$ and for every $u \in W^{1,p}(\mathbb{R})$, it turns out that

$$u_\epsilon := u * \varphi_\epsilon \xrightarrow{\epsilon \to 0^+} u \qquad \text{in } W^{1,p}(\mathbb{R}),$$

that is,

$$\begin{cases} u_\epsilon \to u & \text{in } L^p(\mathbb{R}) \\ Du_\epsilon \to Du & \text{in } L^p(\mathbb{R}). \end{cases}$$

Exercise 9.5 (Density)**.** For any $p \in [1, +\infty)$, the inclusion

$$W^{1,p}(I) \cap C_c^{\infty}(I) \subset W^{1,p}(I)$$

is dense.

Exercise 9.6. Let $g \in C^1(I)$ be a differentiable function such that $\|g'\|_{\infty, I} < +\infty$. Prove that, for any $u \in W^{1,p}(I)$, it turns out that

$$g \circ u \in W^{1,p}(I) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[g\left(u(x)\right)\right] = g'(u(x))Du(x).$$

Exercise 9.7. Prove that, as a particular case of the previous exercise, it turns out that

$$u \in W^{1,p}(I) \implies |u| \in W^{1,p}(I).$$

Exercise 9.8 (*)**.** Let I be a bounded interval. If $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function, then

$$u \in W^{1,p}(I) \implies \frac{\mathrm{d}}{\mathrm{d}x}\left[g\left(u(x)\right)\right] = g'(u(x))Du(x) \quad \text{for almost every } x \in I.$$

If I is an unbounded interval, then the same formula holds true if we require g to be essentially bounded, that is, if $\|g\|_{L^{\infty}(I)} < +\infty$.

Solution. The reader may refer to [4, Theorem 2.1.11] for a proof of this statement. □

Exercise 9.9. Prove that the Sobolev space $W^{1,p}(I)$ is a lattice.

Exercise 9.10. Prove that there exists a function $f \in W^{1,2}(\mathbb{R}^2)$ such that f is unbounded.

Solution. Here we only present a road map of the solution. The reader may try to fill in the missing details as an easy computational exercise.

Step 1. The function

$$g(x, y) = \log\left[\log\left(1 + \frac{1}{\sqrt{x^2 + y^2}}\right)\right]$$

is unbounded, but one can prove as an exercise that it belongs to $W^{1,2}\left(\mathrm{Int}(B(0, 1))\right)$.

Hint. Take the derivative of g, find an estimate from above of its absolute value and then compute the integral using the polar coordinates.

Step 2. Let Φ be a cut-off function such that $\phi\big|_{B(0,1)} \equiv 1$ and $\operatorname{spt}\Phi \subset B(0, 2)$. The function

$$f(x, y) := (g \cdot \Phi)(x, y)$$

is an unbounded function on $\mathbb{R}^2$ with compact support. On the other hand, based on what we have proved in the first step, it follows immediately that $f \in W^{1,2}(\mathbb{R}^2)$. □

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