Lecture Notes

Advanced Analysis

Course held by

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August 16, 2018

Disclaimer

These notes came out of the *Advanced Analysis* course, held by Professor Emanuele Paolini in the second semester of the academic year 2016/2017.

They include all the topics that were discussed in class; I added some remarks, simple proof, etc.. for my convenience.

I have used them to study for the exam; hence they have been reviewed thoroughly. Unfortunately, there may still be many mistakes and oversights; to report them, send me an email at **francescopaolo (dot) maiale (at) gmail (dot) com**.

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Chapter 1

Introduction

In this first chapter, we want to motivate the necessity of studying the so-called *distribution theory*, which is the natural generalisation of the concept of *function*. Indeed, even a simple physical problem that consists of finding the electric potential of a point-charge cannot be solved, at least formally, if we do not introduce a broader class of objects.

Electric Potential. Suppose that there is an electric charge ρ at the origin of the 3-dimensional space $\mathbb{R}^3$. The Maxwell equations describing this problem are simply given by

$$\begin{cases} \partial_x E_x + \partial_y E_y + \partial_z E_z = \rho \\ \nabla u = \vec{E}, \end{cases}$$

where $\vec{E}$ is the electric vector field generated by ρ, and u denotes the electric potential. Clearly, this is equivalent[1] to the Laplace equation

$$\Delta u = \rho, \tag{1.1}$$

and, more precisely, to the fact that the laplacian of the potential u equals 0 whenever $(x,\, y,\, z) \neq \mathbf{0}$ and ρ at the point $(x,\, y,\, z) = \mathbf{0}$.

Assume that u is radially symmetric, that is, there exists a function v defined on $[0, \infty)$ such that $u(x) = v(r)$ with $r = |x|$. Plugging the formula for the laplacian in polar coordinates into (1.1) yields to a much easier differential equation, which is

$$v_{rr} + \frac{2}{r} v_r = 0$$

for all positive $r > 0$, where v_r denotes the derivative of v with respect to r. Therefore, up to a constant, we find that

$$v(r) = \frac{1}{r} \implies u(\mathbf{x}) = \frac{1}{|\mathbf{x}|} \quad \text{for all } \mathbf{x} = (x,\, y,\, z) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}.$$

We will now compute the "distributional" laplacian of the function $|x|^{-1}$ to check whether or not u is a solution to the Laplace equation. Set

$$(\Delta u,\, \varphi) := (u,\, \Delta\varphi) = \int_{\mathbb{R}^3} u(\mathbf{x}) \Delta\varphi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

[1] Use the well-known vector identity $\operatorname{div}[\nabla(\cdot)] = \Delta(\cdot)$.

for some $\varphi \in C_c^\infty\left(\mathbb{R}^3\right)$ smooth functions with compact support, and let $R > 0$ be such that $\operatorname{spt}\varphi$ is compactly contained in the ball of radius R. The function $u(x)$ is not well-defined at the origin, and thus to compute the value of the integral we simply remove a small ball of radius $\epsilon > 0$ and take the limit for $\epsilon \to 0^+$, that is, we have

$$(\Delta u,\, \varphi) = \lim_{\epsilon\to 0^+} \int_{B_R\setminus B_\epsilon} u(\mathbf{x})\Delta\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The divergence formula implies that

$$\begin{cases} u(\mathbf{x})\Delta\varphi(\mathbf{x}) = u(\mathbf{x})\mathrm{div}(\nabla\varphi)(\mathbf{x}) = \mathrm{div}(u\nabla\varphi)(\mathbf{x}) - \nabla u\cdot\nabla\varphi(\mathbf{x}), \\ \varphi(\mathbf{x})\Delta u(\mathbf{x}) = \varphi(\mathbf{x})\mathrm{div}(\nabla u)(\mathbf{x}) = \mathrm{div}(\varphi\nabla u)(\mathbf{x}) - \nabla u\cdot\nabla\varphi(\mathbf{x}). \end{cases}$$

Subtract these two identities. Since Δu equals zero for all $\mathbf{x} \in B_R \setminus B_\epsilon$, we infer that

$$u(\mathbf{x})\Delta\varphi(\mathbf{x}) = \mathrm{div}(u\nabla\varphi)(\mathbf{x}) - \mathrm{div}(\varphi\nabla u)(\mathbf{x}). \tag{1.2}$$

Fix $\epsilon > 0$. The identity (1.2), together with the diverge theorem, immediately shows that

$$\int_{B_R\setminus B_\epsilon} u(\mathbf{x})\Delta\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial(B_R\setminus B_\epsilon)} \left[u(\mathbf{x})\varphi_r(\mathbf{x}) - \varphi(\mathbf{x})u_r(\mathbf{x})\right]\,\mathrm{d}\sigma =$$

$$= \int_{\partial B_\epsilon} \left[\varphi(\mathbf{x})u_r(\mathbf{x}) - u(\mathbf{x})\varphi_r(\mathbf{x})\right]\,\mathrm{d}\sigma,$$

since both φ and φ_r are identically equal to zero on the boundary of B_R for the support is strictly contained in the open ball of radius R. Finally, we roughly estimate the remaining terms as

$$\int_{\partial B_\epsilon} \left[\varphi(\mathbf{x})u_r(\mathbf{x}) - u(\mathbf{x})\varphi_r(\mathbf{x})\right]\,\mathrm{d}\sigma = \epsilon^{-1}\int_{\partial B_\epsilon} \varphi_r\,\mathrm{d}\sigma + \epsilon^{-2}\int_{\partial B_\epsilon} \varphi\mathrm{d}\sigma \simeq \mathrm{A}(B_\epsilon)\frac{1}{\epsilon} + \mathrm{A}(B_\epsilon)\frac{1}{\epsilon^2},$$

where $\mathrm{A}(B_\epsilon)$ denotes the surface of the ball of radius $\epsilon > 0$. Therefore, the second term only survives at the limit, and the distributional laplacian is given by

$$(\Delta u,\, \varphi) = \lim_{\epsilon\to 0^+} \int_{B_R\setminus B_\epsilon} u(\mathbf{x})\Delta\varphi(\mathbf{x})\,\mathrm{d}x = 4\pi.$$

If we denote by $\delta(\cdot)$ the delta of Dirac at the origin, then the solution is a "function" $u(x)$ satisfying the following equation:

$$\Delta u(\mathbf{x}) = 4\pi\delta(\mathbf{x}).$$

Chapter 2

Locally Convex Spaces

In this chapter, we develop the general theory of topological vector space, and we focus our attention on the main properties of locally convex topologies.

2.1 Introduction to TVS

In this section, we introduce and study the concept of *topological vector space*, which consists roughly speaking of a vector space X endowed with a compatible topology τ.

Definition 2.1 (Topological Vector Space)**.** A couple (X, τ) is a *topological vector space* over $\mathbb{C}$ (resp. $\mathbb{R}$) if the following properties are satisfied:

(a) There are operations $+$ and $\cdot$ such that X is a complex (resp. real) vector space.

(b) (X, τ) is a topological space.

(c) The vector sum $+$ and the scalar product $\cdot$ are τ-continuous.

(d) The singlet $\{0\}$ is closed in τ.

Caution!

The property **(d)** is often not included in the definition. Some authors prefer to add it because, as we shall prove later on, it implies that (X, τ) is T_2 (=Hausdorff.)

Remark 2.1. Fix $x \in X$. The translation is defined by

$$T_x : X \ni y \longmapsto x + y \in X.$$

If (X, τ) is a topological vector space, then a straight computation shows that T_x is a homeomorphism. Therefore, the neighbourhoods of a point $p \in X$ are nothing but the translations of the origin neighbourhoods, which means that τ is $\mathcal{T}$-invariant, uniquely determined by a local basis of 0, and $\mathrm{T}_1 - \mathit{regular}$.

Remark 2.2. If $\alpha \in \mathbb{C} \setminus \{0\}$, then the α-homothety $\lambda_\alpha(x) := \alpha x$ is also a homeomorphism, and its inverse is given by $\lambda_{\frac{1}{\alpha}}$.

Example 2.1. Let $(X, \|\cdot\|)$ be a normed space. The scalings $\{r \cdot B(0, 1)\}_{r>0}$ form a local basis of the origin in the (strong) topology induced by the norm.

2.1.1 Classification of TVS ($\star$)

In this brief section, we introduce several classes of function spaces and state some fundamental results connecting them with the notions introduced in this chapter.

Definition 2.2 (Locally Compact)**.** A topological space (X, τ) is *locally compact* if for every point $p \in X$ we can find a compact neighbourhood $U_p \ni p$.

Definition 2.3 (Locally Bounded)**.** A topological vector space (X, τ) is *locally bounded* if there exists a bounded[1] neighbourhood U of the origin.

Definition 2.4 (Locally Convex)**.** A topological vector space (X, τ) is *locally convex* if the origin has a local base of absolutely convex[2] absorbent[3] sets.

Definition 2.5 (F-Space)**.** A topological vector space (X, τ) is called *F-space* if the following properties are satisfied:

(a) The topology τ is induced by a translation-invariant metric d.

(b) The metric space (X, d) is complete.

Definition 2.6 (Fréchet Space)**.** A topological vector space (X, τ) is called *Fréchet space* if and only if (X, τ) is a locally convex F-space.

Definition 2.7 (Heine-Borel)**.** A topological vector space (X, τ) has the *Heine-Borel property* if any bounded closed set is also compact.

Theorem 2.8. *Let X be a topological vector space. Then X is finite-dimensional if and only if X is locally compact if and only if X is locally bounded and has the Heine-Borel property.*

Proof. See Theorem 2.46 for a detailed proof. □

Theorem 2.9. *Let X be a topological vector space. Then X is normalizable if and only if X is locally convex and locally bounded.*

Proof. It follows fairly easily from Theorem 2.44. □

Theorem 2.10. *A locally convex and locally bounded space X is linearly metrizable.*

2.2 Separation Properties of TVS

In this section, we exploit the assumption **(d)** to show that a topological vector space X is Hausdorff (=separable, T_2) provided that it is T_1. In addition, we show that closed and compact sets can be separated using only open neighbourhoods.

Definition 2.11 (Bounded Set)**.** A subset $A \subset X$ of a topological vector space is said to be *bounded* if, for all U neighbourhood of the origin, there exists $r := r(U) > 0$ such that

$$A \subset r \cdot U.$$

[1] See Definition 2.11.

[2] In other words, balanced and convex sets. See Definition 2.22.

[3] See Definition 2.23.

Lemma 2.12. *Let (X, d) be a metric space and let $\varphi : (0, +\infty) \longrightarrow (0, +\infty)$ be a strictly concave function such that $\varphi(0) = 0$. Then $d' := \varphi \circ d$ is also a distance over X.*

Proof. A function φ is strictly concave if and only if it satisfies

$$\varphi\left(\alpha a + (1-\alpha)b\right) > \alpha\varphi(a) + (1-\alpha)\varphi(b)$$

for all $a, b \in (0, +\infty)$ and all $\alpha \in (0, 1)$. The function d' is clearly symmetric as a consequence of the fact that d is symmetric:

$$d'(x, y) = \varphi \circ d(x, y) = \varphi \circ d(y, x) = d'(y, x).$$

Claim. We have $\varphi(a) = 0$ if and only if $a = 0$ as a consequence of the fact that the function φ is strictly increasing in $(0, +\infty)$.

Proof. Suppose that φ attains its unique global maximum at a finite time $\widetilde{x} < \infty$. Then the function becomes decreasing right after $\widetilde{x}$ and, eventually, becomes negative (as the concavity does not change), and this is absurd. Consequently, by definition, we have

$$d'(x, y) \geq 0,$$

and

$$d'(x, y) = 0 \iff \varphi(d(x, y)) = 0 \iff d(x, y) = 0 \iff x = y.$$

Triangular Inequality. The function φ is increasing and d is a distance; thus

$$d'(x, y) \leq \varphi \circ d(x, z) + \varphi \circ d(z, y) = d'(x, z) + d'(z, y),$$

which means that d' is a distance defined on X. $\square$

Definition 2.13 (Cauchy sequence)**.** Let (X, τ) be a topological vector space. A sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* in X if for every neighbourhood U of the origin we can find a big enough natural number $N := N(U) \in \mathbb{N}$ such that

$$x_k - x_j \in U \quad \text{for every } k,\, j \geq N.$$

Definition 2.14 (Convergence)**.** We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to some $x \in X$, and we denote it by $x_n \to_X x$, if for every neighbourhood U of the origin we can find a natural number $N := N(U) \in \mathbb{N}$ such that

$$x_n \in x + U \quad \text{for every } n \geq N.$$

Definition 2.15 (σ-Complete)**.** A topological vector space (X, τ) is said to be *σ-complete* if and only if every Cauchy sequence converges to some element of X.

2.2.1 Separation Theorem

In this brief section, we prove that closed and compact subsets of a topological vector space with empty intersection are still separated if we consider open neighbourhoods of them.

Lemma 2.16. *Let (X, τ) be a TVS, and let W be a neighbourhood of the origin. Then there exists a neighbourhood U of the origin satisfying the following properties:*

$$U = -U \quad \textit{and} \quad U + U \subseteq W.$$

Proof. The vector sum in X is τ-continuous by definition, and therefore we can always find two neighbourhoods of the origin, V_1 and V_2, such that

$$(x, y) \in V_1 \times V_2 \implies x + y \in W. \tag{2.1}$$

In particular, $V_1 + V_2 \subseteq W$ and, since the family of the neighbourhoods of a point is closed under intersection, we also have that $V := V_1 \cap V_2$ is a neighbourhood of 0. Define

$$U := V \cap (-V).$$

It is easy to prove that U is also a neighbourhood of the origin, $U \subseteq W$, and

$$u_1 + u_2 \in W \quad \text{for all } u_1,\, u_2 \in U.$$

□

Lemma 2.17. *Let (X, τ) be a TVS, let W be a neighbourhood of the origin, and let $n > 1$ be a natural number. Then there exists a neighbourhood U of the origin satisfying the following properties:*

$$U = -U \quad \textit{and} \quad \underbrace{U + \cdots + U}_{n\text{times}} \subseteq W.$$

Theorem 2.18 (Separation)**.** *Let (X, τ) be a TVS, $C \subseteq X$ a closed subset and $K \subseteq X$ a compact subset. Assume that $C \cap K = \varnothing$. Then there exists an open neighbourhood U of the origin such that*

$$(K + U) \cap (C + U) = \varnothing.$$

Proof. For any given $x \in K$, we can find an open neighbourhood W_x of 0 such that

$$(x + W_x) \cap C = \varnothing$$

since K is contained in the complement of C, which is open by assumption. It follows from Proposition 2.17 - with $n = 3$ - that, for any $x \in K$, we can find an open neighbourhood V_x of the origin satisfying $V_x = -V_x$ and $V_x + V_x + V_x \subseteq W_x$. The symmetry of V_x shows that

$$(x + V_x + V_x + V_x) \cap C = \varnothing \implies (x + V_x + V_x) \cap (C + V_x) = \varnothing.$$

Now the collection of open sets $\{x + V_x\}_{x \in K}$ is a cover of K, and thus there exists a finite subcover $\{x_i + V_i\}_{i=1,\dots,k}$ such that

$$K \subset \bigcup_{i=1}^{k} (x_i + V_i).$$

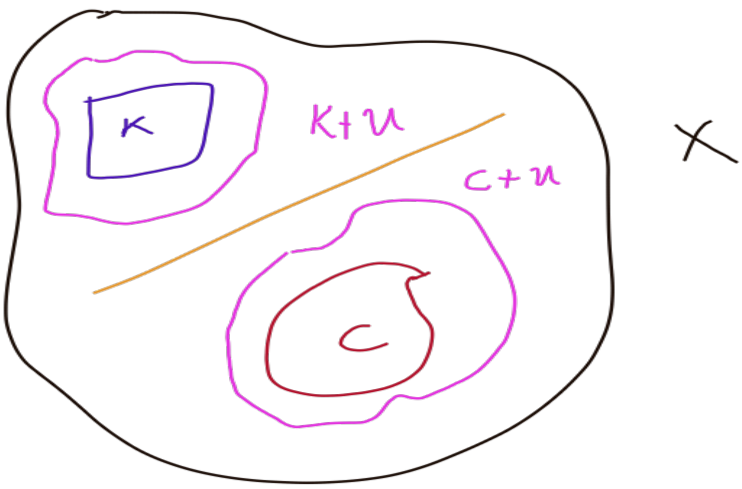

Figure 2.1: Separation Theorem Idea

Consider the intersection, which is still a neighbourhood, and denote it by

$$U := \bigcap_{i=1}^{k} V_i.$$

It is easy to see that $U = -U$ and $U + U + U \subseteq W_{x_i}$ for all $i = 1, \ldots, k$. The thesis now follows from the chain of inclusions

$$(K + U) \subset \bigcup_{i=1}^{k} (x_i + V_i + U) \subset \bigcup_{i=1}^{k} (x_i + V_i + V_i)$$

since no term in the latter union intersects $C + U$, as stated above. It finally follows that

$$(K + U) \cap (C + U) = \varnothing.$$

□

Corollary 2.19. *A topological vector space (X, τ) is a Hausdorff space.*

Proof. The singlet $\{x\}$ is (always) compact, while the singlet $\{y\}$ is closed as we required $\{0\}$ to be closed in the definition of TVS. Finally, Theorem 2.18 concludes that x and y are separated by open neighbourhoods. □

Corollary 2.20. *Let U be a neighbourhood of the origin in a topological vector space. Then there exists a closed neighbourhood V such that $U \supset V$.*

Proof. Let $W \subseteq U$ be an open neighbourhood of 0. The complement $C := X \setminus W$ is closed, while $K := \{0\}$ is compact; thus we can find a neighbourhood Z of 0 such that

$$Z \cap (C + Z) = \varnothing \implies \overline{Z} \cap (C + Z) = \varnothing.$$

We have the inclusions $Z \subseteq \overline{Z} \subseteq (C + Z)^c \subseteq C^c = W$, and this implies that $V := \overline{Z}$ is the sought closed neighbourhood of the origin. □

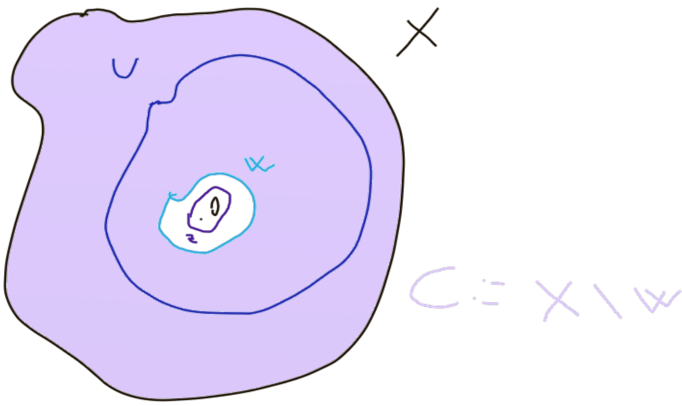

Figure 2.2: Idea of the proof

2.2.2 Balanced and Convex Basis

The main result of this short section is the following: Any topological vector space (X, τ), admits a balanced local basis (of neighbourhoods of the origin).

Definition 2.21 (Convex)**.** Let X be a vector space. A subset $C \subseteq X$ is called *convex* if

$$x,\, y \in C \implies tx + (1-t)y \in C \quad \text{for all } t \in [0,\, 1].$$

Definition 2.22 (Balanced)**.** Let X be a vector space. A subset $B \subseteq X$ is called *balanced* if

$$\alpha \cdot B \subseteq B \quad \text{for all } |\alpha| \leq 1.$$

Definition 2.23 (Absorbent Set)**.** Let (X, τ) be a topological vector space. A subset $C \subset X$ is called *absorbent* if

$$X \subseteq \bigcup_{t>0} t \cdot C.$$

Remark 2.3. Every neighbourhood of the origin in (X, τ) is absorbent.

Proof. Fix U neighbourhood of the origin and $x \in X$. By definition $0 \cdot x = 0 \in U$; therefore, the continuity of the scalar product implies that we can always find $\alpha > 0$ such that $\alpha \cdot x \in U$, that is,

$$x \in \frac{1}{\alpha} \cdot U.$$

□

Theorem 2.24 (Local Basis)**.** *Let (X, τ) be a topological vector space, and let $\mathcal{U}_0(X)$ be the set of all neighbourhood of the origin.*

(1) For all $U \in \mathcal{U}_0(X)$ there exists a balanced $V \in \mathcal{U}_0(X)$ such that $V \subseteq U$.

(2) For all convex $U \in \mathcal{U}_0(X)$ there exists a balanced convex $V \in \mathcal{U}_0(X)$ such that $V \subseteq U$.

Proof.

(1) The continuity of the scalar product shows that there exists $\delta > 0$ and $W \in \mathcal{U}_0(X)$ such that

$$\alpha \cdot W \subseteq U \quad \text{for all } \alpha \text{ such that } |\alpha| < \delta.$$

Let V be the union of all these scaling $\alpha \cdot W$, i.e.,

$$V := \bigcup_{|\alpha|<\delta} \alpha \cdot W.$$

Clearly, V is a neighbourhood of the origin such that $V \subseteq U$. To show that V is balanced, take any scalar β such that $|\beta| \leq 1$, and observe that

$$\beta \cdot V = \bigcup_{|\alpha|<\delta} (\beta\alpha) \cdot W.$$

Since $|\beta\alpha| < \delta$ for all $|\alpha| < \delta$, we infer from the definition above that V is balanced.

(2) Consider the finite intersection

$$A := \bigcap_{|\alpha|=1} \alpha \cdot U,$$

and denote by V the subset defined above, that is,

$$V := \bigcup_{|\alpha|<\delta} \alpha \cdot W.$$

We proved that V is balanced; therefore $\alpha^{-1} \cdot V = V$ for all $|\alpha| = 1$. It follows immediately that $V \subseteq \alpha \cdot U$ and $W \subset A$, which in turn implies that the interior part $\operatorname{Int} A$ belong to $\mathcal{U}_0(X)$ and satisfies the inclusion $\operatorname{Int} A \subseteq U$. Notice also that, since A is an intersection of convex sets, it is necessarily convex and so is its interior part.

To prove that $\operatorname{Int} A$ is the desired neighbourhood of the origin, we have to show that A is balanced for the same will follow for its interior. Take $0 \leq r \leq 1$ and $|\beta| = 1$. Then

$$(r\beta) \cdot A = \bigcap_{|\alpha|=1} (r\beta) \cdot \alpha \cdot U = \bigcap_{|\alpha|=1} (r\alpha) \cdot U.$$

Since $\alpha \cdot U$ is a convex set that contains the origin, we have $(r\alpha) \cdot U \subset \alpha \cdot U$. Consequently, we have proved that $(r\beta) \cdot A \subset A$, and this is enough to conclude.

□

Corollary 2.25. *Let (X, τ) be a topological vector space.*

(a) *There exists a balanced local basis, that is, a basis made up of balanced sets.*

(b) *If X is also locally convex, then there exists a balanced convex local basis.*

Proof. Immediate consequence of the result above. The reader might fill in the details as a simple exercise to get acquainted with these new notions. □

We conclude the section by showing some of the main consequences of these assertions.

Theorem 2.26 (Heine-Borel)**.** *Let (X, τ) be a topological vector space, and let $K \subset X$ be a compact subset. Then K is closed and bounded.*

Proof. Let $U \in \mathcal{U}_0(X)$ and fix $x \in X$. The sequence $\left(\frac{1}{k} \cdot x\right)_{k\in\mathbb{N}}$ obviously converges to 0 as k approaches ∞ - as the scalar multiplication is τ-continuous -.

It follows that we can find $N \in \mathbb{N}$ big enough to have $\frac{x}{k} \in U$ for all $k \geq N$. Thus $\{k \cdot U\}_{k\in\mathbb{N}}$ is a cover of K and, by compactness, it admits a finite subcover $k_i \cdot U$ for $i = 1, \ldots, J$. On the other hand, $k \cdot U$ is an increasing family of sets, and thus

$$K \subseteq \max\{r_1, \cdot, r_J\} \cdot U \implies K \text{ is bounded.}$$

Since X is Hausdorff, K compact implies K closed, and this concludes the proof. ☐

Theorem 2.27. *Let (X, τ) be a locally bounded topological vector space. Then there exists a countable local basis.*

Proof. Let $V \subset X$ denote the bounded neighbourhood of the origin. We claim that

$$\left\{\frac{1}{n} \cdot V\right\}_{n\in\mathbb{N}}$$

is the desired local basis. To prove this, let us consider an open set $A \in \tau$ and a real number $\alpha > 0$ such that $V \subset \alpha \cdot A$. If we set $N := \lfloor \alpha \rfloor + 1$, then

$$\frac{1}{N} \cdot V \subset A$$

and this is clearly enough to conclude that the claim holds true. ☐

Theorem 2.28. *Let X be a topological vector space with a countable basis of neighbourhoods of* 0*. Then X is linearly metrizable.*

Proof. See [1, Theorem 1.24]. ☐

Theorem 2.29. *Let (X, τ) be a topological vector space, and let $Y \subseteq X$ be an F-space that is also a vector subspace. Then Y is closed in X.*

Theorem 2.30. *Let (X, τ) be a topological vector space. Any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ forms a bounded subset $\{x_n\}_{n\in\mathbb{N}}$ of X.*

Proof. Let U be a balanced neighbourhood of the origin. We know that we can always find $V \in \mathcal{U}_0(X)$ satisfying $V + V \subset U$ and $V \subseteq U$. Given a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$, we can also find a natural number $N \in \mathbb{N}$ such that

$$x_n - x_m \in V \quad \text{for all } n, m \geq N.$$

In particular, setting $m = N$, we have that

$$x_n - x_N \in V \implies x_n \in x_N + V$$

for all $n \geq N$. Finally, since V is an absorbing set, there exists $r_i > 0$ such that $x_i \in r_i \cdot V$ for all $i = 1, \ldots, N$. If we set $r := \max_{0\leq i\leq N} r_i$, then $x_i \in r \cdot U$ for all $i \leq N$, and therefore

$$x_n \in x_N + V \implies x_n \in r \cdot V + V \subset r \cdot V + r \cdot V \subset r \cdot U.$$

☐

2.3 Locally Convex Spaces

The primary goal of this section is to deal with the characterisation of locally convex spaces in terms of either a balanced and convex local basis or collection of seminorms satisfying certain properties.

2.3.1 Characterization via Subbasis

Let X be a topological space with topology τ. We know already that a base of τ is nothing but a collection of sets such that $A \in \tau$ is given by the union of such elements.

Definition 2.31 (Subbasis)**.** Let X be a topological space with topology τ. A *subbase* of T is a subcollection β of τ satisfying one of the two following equivalent conditions:

(a) The subcollection β generates τ. More precisely, τ is the coarser topology containing β.

(b) The collection of open sets of the form

$$\bigcap_{i \in J} B_i,$$

where $B_i \in \beta$ and $J < \infty$, together with the set X, forms a basis for τ. More precisely, every proper open set in τ can be written as a union of finite intersections of elements of β.

Definition 2.32 (Separated)**.** Let X be a vector space, and let $\mathcal{F} \subset \mathcal{P}(X)$ be a nonempty subset of the power set. We say that $\mathcal{F}$ is *separated* if and only if for every $x \neq 0$ we can find a set $C := C(x) \in \mathcal{F}$ and a constant $r := r(x) > 0$ such that

$$x \notin r \cdot C.$$

Theorem 2.33. *Let X be a vector space, and let $\mathcal{F}$ be a nonempty separated family of convex, balanced and absorbing subsets of X. Then $\mathcal{F}$ is a subbasis of a topology τ, and*

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{N} r_i \cdot C_i \; : \; r_i > 0, \; C_i \in \mathcal{F} \right\}$$

is a neighbourhoods basis of the origin for τ. Furthermore, the generated topology τ is given by

$$\tau = \{ A \subseteq X \; : \; \textit{for all } x \in A \textit{ there is } C \in \mathcal{B} \textit{ such that } x \in x + C \subseteq A \},$$

and (X, τ) is a locally convex topological vector space.

Proof. The first, tedious, step - which is left to the reader - consists of proving that $\mathcal{B}$ is a local basis of the origin and that τ is the generated topology.

Step 1. The topology τ is invariant under translation. In fact, given any $y \in A$, we know that there exists $C \in \mathcal{B}$ such that $y \in y + C \subseteq A$ and thus, if we let $x \in X$ be an arbitrary point, we have

$$x + y \in x + y + C \subseteq x + A \iff x + A \in \tau.$$

Step 2. To prove that (X, τ) is topological vector space, we only need to show that the vector space operations are τ-continuous and $\{0\}$ is closed.

Step 2.1. First, we observe that any element of $\mathcal{B}$ is convex, balanced and absorbing. In fact, all these properties are preserved under finite intersection. To prove the continuity of the sum we want to find, given $U \in \mathcal{U}_0(X)$, a neighbourhood of the origin V such that

$$V + V \subseteq U.$$

To do this, let W be a convex and balanced neighbourhood of the origin contained in U. Then, the convexity implies the identity

$$\frac{W}{2} + \frac{W}{2} = W,$$

and thus it suffices to take $V := \frac{W}{2}$.

Step 2.2. Fix a point $(\alpha_0, x_0) \in \mathbb{C} \times X$. To prove the continuity of the scalar product at that point, given $U \in \mathcal{U}_0(X)$, we need to find $W \in \mathcal{U}_0(X)$ and $\epsilon > 0$ in such a way that

$$\alpha x - \alpha_0 x_0 \in U$$

for all $|\alpha| < \epsilon$ and all $x \in W$. Let $V \in \mathcal{U}_0(X)$ be convex, balanced and absorbing such that

$$V + V \subset U,$$

and let $r > 0$ be such that $x_0 \in r \cdot V$. Set $\epsilon := \frac{1}{r}$ and

$$W := \frac{V}{|\alpha_0| + \epsilon}.$$

It follows that, for $|\alpha - \alpha_0| < \epsilon$ and $x \in x_0 + W$, we have

$$\begin{aligned} \alpha x - \alpha_0 x_0 = \alpha(x - x_0) + (\alpha - \alpha_0)x_0 &\in \alpha \cdot W + (\alpha - \alpha_0) r \cdot V \subseteq \\ &\subseteq |\alpha| \cdot W + |\alpha - \alpha_0| r \cdot V \subseteq \\ &\subseteq (|\alpha_0| + \epsilon) \cdot W + \epsilon r \cdot V = \\ &= V + V \subseteq U. \end{aligned}$$

Step 2.3. Let $x \in X$ be an arbitrary point $x \neq 0$. Since $\mathcal{F}$ is a separated family, we can always find a subset $C \in \mathcal{F}$ and a constant $r > 0$ such that

$$x \notin \frac{1}{r} \cdot C.$$

It follows that $0 = x - x \notin x - r^{-1} \cdot C$. But $r^{-1} \cdot C \in \mathcal{U}_0(X)$, and thus x is an internal point of the complement of $\{0\}$, which means that $\{0\}$ is closed. □

2.3.2 Characterization via Seminorms

In this section, we state and prove a similar result concerning the characterisation of a locally convex space, given a family $\mathcal{P}$ of seminorms satisfying certain properties.

Definition 2.34 (Seminorms)**.** Let X be a $\mathbb{K}$-vector space. We say that $p : X \longrightarrow \mathbb{K}$ is a *seminorm* if the following properties hold true:

(a) Subadditivity. For all $x,\, y \in X$ it turns out that

$$p(x+y) \le p(x) + p(y).$$

(b) Positive Homogeneity. For every $\alpha \in \mathbb{K}$ and $x \in X$ it turns out that

$$p(\alpha x) = |\alpha| \cdot p(x).$$

Theorem 2.35. *Let X be a $\mathbb{K}$-vector space and p a seminorm. Then the following hold:*

(1) The function p is positive ($p(x) \ge 0$ for all $x \in X$) and $p(0) = 0$.

(2) The function p satisfies the following triangular-like inequality:

$$|p(x) - p(y)| \le p(x-y) \quad \text{for all } x,\, y \in X.$$

(3) The zero set $\{p(x) = 0\} \subseteq X$ is a vector subspace with the induced operations.

(4) The open unit ball $B_p := \{x \in X \,:\, p(x) < 1\}$ is convex, balanced and absorbing.

Proof. These properties follow immediately from the definition above; here we only show how to prove (**4**). The convexity of B_p is an easy consequence of the subadditivity:

$$x,\, y \in B_p \implies p(tx + (1-t)y) \le tp(x) + (1-t)p(y) < 1 \implies$$

$$\implies tx + (1-t)p(y) \in B_p$$

for all $t \in (0,\, 1)$. The open unit ball B_p is clearly balanced as a consequence of **(b)**, and hence we only need to prove that it is absorbing, i.e.,

$$X \subseteq \bigcup_{t>0} t \cdot B_p.$$

Fix $x \in X$. Given $r_x := p(x)$ and $t := 1 + r_x$, it is straightforward to prove that $x \in t \cdot B_p$. ☐

Proposition 2.36. *Let $(X,\, \tau)$ be a topological vector space, and let p be a seminorm defined on X. The following assertions are equivalent:*

(a) The function p is continuous.

(b) The function p is continuous at $x = 0$.

(c) The open unit ball B_p is open with respect to τ.

Proof. The only nontrivial implication is (**c**) $\implies$ (**a**). Let $x \in X$ be given, and let $\epsilon > 0$. We need to find a neighbourhood U of x with the property

$$p\,(U) \subseteq (p(x) - \epsilon,\, p(x) + \epsilon) \subset \mathbb{R}.$$

Set $U := x + \epsilon \cdot B_p$. Any point $y \in U$ can be written in the form $x + \epsilon \cdot u$ for some $u \in B_p$ satisfying $p(u) < 1$. Then we have the equalities

$$p(y) = p(x + \epsilon \cdot u) \leq p(x) + \epsilon,$$

$$p(y) = p(x + \epsilon \cdot u) \geq p(x) - \epsilon,$$

and this is exactly what we wanted to prove. □

Definition 2.37 (Separated)**.** A family $\mathcal{P}$ of seminorms defined over X is said to be *separated* if, for any $x \in X \setminus \{0\}$, there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Theorem 2.38. *Let X be a vector space, and let $\mathcal{P}$ be a separated family of seminorms defined over X. The collection of open unit balls induced by $\mathcal{P}$,*

$$\mathcal{F} = \{B_p \,:\, p \in \mathcal{P}\},$$

is a subbasis for a locally convex topology τ defined on X. Moreover τ is the coarser locally convex topology such that each seminorm $p \in \mathcal{P}$ is continuous.

Proof. The elements of the collection $\mathcal{F}$ are convex, balanced and absorbing. Furthermore, since $\mathcal{P}$ is a separated family, the same holds true for $\mathcal{F}$. Therefore, it follows from Theorem 2.33 that $\mathcal{F}$ generates a topology τ that makes (X, τ) locally convex. Clearly,

$$B_p \text{ open} \implies p \in \mathcal{P} \text{ is } \tau\text{-continuous}$$

as a consequence of Proposition 2.36. Furthermore, a translation-invariant topology such that $p \in \mathcal{P}$ is τ-continuous contains at least the open unit balls B_p as neighbourhoods of the origin, and thus their scalings $r \cdot B_p$ and their finite intersection. □

Remark 2.4. The topology τ defined in the previous theorem is called *initial topology* relative to the collection of seminorms $\mathcal{P} \ni p$.

Definition 2.39 (Minkowski Functional/Gauge)**.** Let X be a vector space and let $B \subseteq X$ be an absorbing subset. The *gauge functional* is defined by

$$\mu_B(x) := \inf \left\{ t > 0 \;\middle|\; \frac{x}{t} \in B \right\}. \tag{2.2}$$

Lemma 2.40. *Let X be a vector space. Then the following assertions hold:*

(a) *If $p : X \longrightarrow [0, +\infty)$ is a seminorm, then p coincides with μ_{B_p}.*

(b) *If B is a convex, balanced and absorbing set, then μ_B is a seminorm.*

(c) *If B is convex and absorbing, then*

$$\{x \in X \mid \mu_B(x) < 1\} \subseteq B \subseteq \{x \in X \mid \mu_B(x) \leq 1\}.$$

Proof.

(a) Let $x \in X$ and $T := \mu_{B_p}(x)$. Then, for every $\epsilon > 0$, it turns out that

$$\frac{x}{T+\epsilon} \in B_p \implies p(x) < T + \epsilon,$$

and thus, by taking the limit as $\epsilon \to 0^+$, we get the first inequality $p(x) \leq T$. In a similar fashion, a point $x \in X$ multiplied by a number smaller than $p(x)$ belongs to the ball, i.e.,

$$\frac{x}{p(x)+\epsilon} \in B_p \quad \text{for all } \epsilon > 0.$$

By definition $\mu_{B_p}(x) < p(x)+\epsilon$ and, by taking the limit as $\epsilon \to 0^+$, we infer that $\mu_{B_p}(x) \leq p(x)$, which concludes the proof of the first assertion.

(b) First, we notice that μ_B is well-defined and finite at all $x \in X$. Indeed, the set B is absorbing, and thus we can always find $r := r(x) > 0$ such that $x \in r \cdot B$. It follows that

$$\frac{x}{r} \in B \implies \mu_B(x) \leq r.$$

We now prove that μ_B is a subadditive function. For $\epsilon > 0$ and $\lambda \in (0,\, 1)$ we know that

$$\lambda \frac{x}{\mu_B(x)+\epsilon} + (1-\lambda)\frac{y}{\mu_B(y)+\epsilon} \in B,$$

and thus we can take

$$\lambda := \frac{\mu_B(x)+\epsilon}{\mu_B(x)+\mu_B(y)+2\epsilon} \implies 1-\lambda = \frac{\mu_B(y)+\epsilon}{\mu_B(x)+\mu_B(y)+2\epsilon}.$$

It follows that

$$\frac{x+y}{\mu_B(x)+\mu_B(y)+2\,\epsilon} \in B,$$

and thus, by definition, we obtain the desired inequality

$$\mu_B(x)+\mu_B(y)+2\epsilon \geq \mu_B(x+y)$$

since $\epsilon > 0$ was chosen as an arbitrary positive number.

We now prove that μ_B is positively homogeneous. But this is a simple consequence of the definition, since we can take the limit as $\epsilon \to 0^+$ of the following chain of inequalities:

$$\frac{\mu_B(\lambda x)+\epsilon}{|\lambda|} \geq \mu_B(x) \geq \frac{\mu_B(\lambda x)-\epsilon}{|\lambda|}$$

(c) One of the inclusions is trivial

$$\{x \in X \mid \mu_B(x) < 1\} \subseteq B.$$

Similarly, if $\mu_B(x) > 1$, it is straightforward to see that $x \notin B$. In fact, since B is convex and absorbing, the set $\{t \mid x/t \in B\}$ coincides either with $[\mu_B(x),\, +\infty)$ or with $(\mu_B(x),\, +\infty)$.

□

Corollary 2.41. *Let $(X,\, \tau)$ be a locally convex topological vector space. Then there exists a family of seminorms $\mathcal{P}$ such that τ is equal to the topology of Theorem 2.38.*

Proof. The space X is locally convex, and thus (see Corollary 2.25) we can always find a local basis $\mathcal{B} := \{B_i\}_{i\in I}$ made up of balanced, convex and absorbent sets[4]. By Lemma 2.40, it turns out that

$$\mathcal{P} := \{\mu_{B_i} \mid i \in I\}$$

is a family of seminorms, generating the same topology τ. ☐

Theorem 2.42. *Let $(X,\, \tau)$ be a locally convex topological vector space. Assume that the topology τ is induced by a family of seminorms $\mathcal{P}$. Then*

$$y_n \xrightarrow{X} y \iff p(y_n - y) \xrightarrow{n\to+\infty} 0 \quad \text{for all } p \in \mathcal{P}.$$

Proof. The implication $\implies$ is a consequence of the fact that $p \in \mathcal{P}$ is τ-continuous. Vice versa, assume that $(y_n)_{n\in\mathbb{N}} \subset X$ is a sequence such that $p(y_n) \xrightarrow{n\to+\infty} 0$ for every $p \in \mathcal{P}$. We may assume without loss of generality that

$$y_n \xrightarrow{n\to\infty} 0.$$

Let U be a neighbourhood of the origin. It follows from Theorem 2.38 that there are $p_1,\, \dots,\, p_J \in \mathcal{P}$ and $r_1,\, \dots,\, r_J > 0$ such that

$$\bigcap_{i=1}^{J} r_i \cdot B_{p_i} \subseteq U.$$

We know that $p_i(y_n)$ goes to 0 as n goes to infinity, and thus there are $N_1,\, \dots,\, N_J \in \mathbb{N}$ such that

$$p_i(y_n) < r_i \quad \text{for all } n \geq N_i.$$

Set $N := \max\{N_1,\, \dots,\, N_J\}$. We conclude that

$$y_n \in U \quad \text{for all } n \geq N,$$

which means that y_n converges to zero. ☐

Theorem 2.43. *Let $(X,\, \tau)$ be a locally convex topological vector space. If τ is generated by a separated countable family of seminorms $\mathcal{P} := \{p_k\}_{k\in\mathbb{N}}$, then X is metrizable.*

Proof. Let us set

$$d(x,\, y) := \max_{k\in\mathbb{N}} \left[2^{-k} \frac{p_k(x-y)}{1 + p_k(x-y)}\right]. \tag{2.3}$$

The reader may check by herself that (2.3) is a distance. For example, it is easy to see that

$$d(x,\, y) = 0 \implies p_k(x-y) = 0, \qquad \forall\, k \in \mathbb{N},$$

and this is enough to infer that $x = y$ using the fact that the family is separated. The function d is translation-invariant; therefore we only need to prove that open balls

$$B_r(0) := \{x \in X \mid d(x,\, 0) < r\}$$

are a local basis of neighborhoods of the origin, inducing the same topology τ.

[4] We can assume, without loss of generality, that B_i is open for every $i \in I$ since because B_i can be replaced with its interior part.

Case 1. The condition $d(x,\, 0) < r$ easily implies that

$$\left(2^{-k} - r\right) p_k(x) < r,$$

and this inequality is satisfied for any $k > \log_2(1/r)$. The finite number of remaining indices satisfy the inequality

$$p_k(x) < \frac{r}{2^{-k} - r} =: r_k,$$

and thus B_r contains the intersection of a finite number of $\mathcal{P}$-balls, that is,

$$\bigcap_{k=0}^{[\log_2(1/r)]} r_k \cdot B_{p_k} \subseteq B_r(0).$$

In particular, the metric ball $B_r(0)$ contains a τ-neighborhood of the origin.

Case 2. Vice versa, given any τ-neighborhood V of 0, it is easy to prove that there exist positive real numbers $r_j > 0$ and seminorms balls such that

$$\bigcap_{j=0}^{m} r_j \cdot B_{p_j} \subseteq U.$$

If we take r such that

$$2r < \max\left\{2^{-j} \cdot r_j \mid j = 1,\, \ldots,\, m\right\},$$

then any $x \in B_r(0)$ satisfies the inequality

$$d(x,\, 0) < r < \frac{2^{-j} \cdot r_j}{2}.$$

Hence $p_j(x) < r_j$ for any j, which means that $B_r(0) \subseteq W$. □

Theorem 2.44. *Let $(X,\, \tau)$ be a locally convex topological vector space. Then X is normable if and only if there exists a convex bounded neighbourhood U of* 0.

Proof. If X is normable, then $B := \{x \in X \mid \|x\| < 1\}$ is a bounded convex neighbourhood of 0 (since each neighbourhood U of 0 contains a rescaling of B, e.g. $r = \min_{x \in U \setminus \{0\}} \|x\|$).

Vice versa, suppose that $B \subset X$ is a convex bounded neighbourhood of 0. Then B is absorbing, since $x/k \to 0$ for any $x \in X$ and the scalar product is a continuous operation. By Lemma 2.40 we infer that μ_B is a seminorm on X.

Assume that there exists $x \neq 0$ such that $p(x) = 0$. Since $\{0\}$ is closed, there exists a neighbourhood U of 0 such that $x \notin U$. By homogeneity $p(r\,x) = 0$, thus $r\,x \in B$ but it doesn't belong to $r \cdot U$ and this means that $B \not\subset r \cdot U$ (contradiction with the boundedness).

It remains to prove that the topology induced by p, denoted by ς, is equal to τ. But B_p is a neighbourhood of 0 in τ, thus ς is coarser than τ. Similarly, given U neighbourhood of 0 in τ, there exists $r > 0$ such that $B_p \subset r \cdot U$. Dividing by r, it turns out that

$$\frac{B_p}{r} \subset U,$$

thus τ is coarser than ς. □

Theorem 2.45. *Let (X, τ) be a locally convex topological vector space. Assume that τ is induced by a family of seminorms $\mathcal{P}$. Then*

$$C \subseteq X \text{ is bounded} \iff p\big|_C \text{ is bounded for all } p \in \mathcal{P}..$$

Proof. Assume that $C \subseteq X$ is bounded. Then, for every U neighbourhood of the origin, we can find a real number $r > 0$ such that $C \subset r \cdot U$. In particular, if $U = B_p$, we have

$$C \subset r \cdot B_p \implies p(x) < r \quad \text{for all } x \in C.$$

Vice versa, assume that $p(C)$ is bounded for all $p \in \mathcal{P}$. Since the open unit balls B_p form a subbasis of τ, we have that

$$U \supseteq \bigcap_{i=1}^{k} r_i \cdot B_{p_i}$$

for all $U \in \mathcal{U}_0(X)$. By assumption, for all $i = 1, \ldots, k$ we can find a constant c_i such that $p_i(x) < c_i$ for all $x \in C$. Set

$$r := \max_{i=1, \ldots, k} \frac{c_i}{r_i}.$$

Then it is easy to show that $x \in c_i \cdot B_{p_i}$ and, consequently, that $C \subset r \cdot U$. □

Theorem 2.46. *Let (X, τ) be a topological vector space. Then X is locally compact if and only if X is a finite-dimensional space if and only if X is linearly homeomorphic to $\mathbb{K}^n$.*

Proof.

Theorem 2.47. *Let (X, τ) be a T_0 topological vector space. Then X is locally compact if and only if X is a finite-dimensional space if and only if X is linearly homeomorphic to $\mathbb{K}^n$.*

Proof. We first prove that a finite-dimensional T_0 topological vector space is linearly homeomorphic to $\mathbb{K}^n$ with the usual topology (which is locally compact), and then we show that a locally compact space is finite-dimensional.

Step 1. Suppose that X is a finite-dimensional T_0 topological vector space, and let $\{e_1, \ldots, e_n\}$ be a basis of X. There is linear isomorphism $\Phi : \mathbb{K}^n \longrightarrow X$, defined by setting

$$\mathbb{K}^n \ni (\lambda_1, \ldots, \lambda_n) \longmapsto \sum_{i=1}^{n} \lambda_i e_i \in X.$$

We only need to prove that Φ is an open map to conclude that it is an homeomorphism since Φ is clearly continuous and bijective.

Let $B := \overline{B_{\mathbb{K}^n}(0, 1)}$ be the closed unit ball, and let $S := \partial B$ be its boundary. Then S is compact in $\mathbb{K}^n$ and it does not contain the origin; hence $\Phi(S)$ is compact in X, and it does not contain the origin of X. In particular $\Phi(S)$ is closed[5] in X, and thus there exists $V \in \mathcal{U}_0(X)$ open and balanced neighborhood of the origin such that

$$V \cap \Phi(S) = \varnothing.$$

[5]Recall that a T_0 topological vector space is automatically $Hausdorff$, and a compact set in a Hausdorff space is always closed.

Let $x \in V \setminus \Phi(B)$. The map is surjective, which means that we can find $\lambda \in \mathbb{K}^n$ such that $x = \Phi(\lambda)$, with $\|\lambda\| > 1$. The rescaling $\lambda/\|\lambda\|$ belongs to S, and therefore we find a contradiction since

$$\Phi\left(\frac{\lambda}{\|\lambda\|}\right) = \frac{x}{\|\lambda\|} \implies \Phi\left(\frac{\lambda}{\|\lambda\|}\right) \in \frac{1}{\|\lambda\|} \cdot V = V,$$

that is, x belongs to both $\Phi(S)$ and V at the same time. It follows that $V \subseteq \Phi(B)$, and this implies that $\Phi(B)$ is a neighborhood of the origin in X, that is, Φ is an open mapping.

Step 2. Conversely, assume that X is locally compact, and let V be a compact neighborhood of the origin of X. Clearly $1/2 \cdot V$ is also a neighborhood of the origin and, by compactness, there are finitely many points $x_i \in V$ such that

$$V \subseteq \bigcup_{i=1}^{m} \left(x_i + \frac{1}{2} \cdot V\right).$$

Let Y be the linear span of the points $x_1, \ldots, x_m$. Then

$$V \subseteq Y + \frac{1}{2} \cdot V \implies \cdots \implies V \subseteq Y + \frac{1}{2^n} \cdot V.$$

Notice that the local compactness of X easily implies that the family $\{2^{-n} \cdot V\}_{n \in \mathbb{N}}$ is a local basis of the origin in X, and hence

$$V \subseteq \bigcap_{n \in \mathbb{N}} \left(Y + \frac{1}{2^n} \cdot V\right) = \overline{Y} = Y,$$

since Y is finite-dimensional, and thus closed. This concludes the proof since V is absorbent and each rescaling is contained in Y, i.e.

$$X = \bigcup_{t > 0} t \cdot V \subseteq Y \implies X = Y.$$

$\square$

$\square$

Chapter 3

Space of Compactly Supported Functions

The primary goal of this chapter is to introduce the space of compactly supported functions on Ω, denoted by $\mathcal{D}(\Omega)$, and endow it with a topology satisfying certain properties.

3.1 Locally Convex Topology of $C^0(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let us denote by $C^0(\Omega;\, \mathbb{C})$ the set of all continuous functions taking values in $\mathbb{C}$.

Proposition 3.1. *Let $K \subset \Omega$ be a compact subset. Then*

$$p_K(f) := \sup_{x \in K} |f(x)|$$

is a seminorm defined on $C^0(\Omega;\, \mathbb{C})$.

It follows that we can always endow $C^0(\Omega;\, \mathbb{C})$ with the topology τ induced by the family of seminorms $\mathcal{P}' := \{p_K\}_{K \subset \Omega}$.

Remark 3.1. The topology τ is locally convex but, a priori, it may not be normable since the generating family $\mathcal{P}'$ is uncountable.

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. There exists an exhaustion by compact sets, that is, an increasing sequence of compact sets $(K_n)_{n \in \mathbb{N}}$ such that $K_j \subsetneqq \operatorname{Int} K_{j+1}$ and*

$$\Omega \subseteq \bigcup_{n \in \mathbb{N}} K_n.$$

The reader should verify that the family of seminorms $\mathcal{P} = \{p_{K_n}\}_{n \in \mathbb{N}} := \{p_n\}_{n \in \mathbb{N}}$ generates the same topology τ given by $\mathcal{P}'$. Furthermore, the sequence $(K_n)_{n \in \mathbb{N}}$ is strictly increasing, and thus the finite intersections

$$\bigcap_{i=1}^{N} r_i \cdot B_{p_i}$$

contain, up to a scaling factor, the ball $r_N \cdot B_{p_N}$. In particular, the subbasis defined in Theorem 2.38 is an actual basis for the topology τ.

Note that $\mathcal{P}$ is countable, and hence it follows from Theorem 2.43 that the space $(C^0(\Omega;\ \mathbb{C}),\ \tau)$ is metrizable. The notion of convergence given by $\mathcal{P}$ coincides with the uniform convergence on compact subsets since

$$f_j \xrightarrow[\tau]{j\to+\infty} f \iff f_j \rightrightarrows_K f \quad \text{for all } K \subset \Omega \text{ compact.}$$

Theorem 3.3. *The metric space $(C^0(\Omega;\ \mathbb{C}),\ \tau)$ is complete.*

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence. We know that f_j converges uniformly to a continuous function f_K on each compact subset K of Ω, that is,

$$f_j \rightrightarrows_K f_K.$$

Let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion by compact sets of Ω. The limit f_{K_n} is defined on the whole compact set K_n and coincides with $f_{K_{n-1}}$ on K_{n-1}. Since $K_n \nearrow \Omega$ we infer that

$$f_j \rightrightarrows_{K_n} f \quad \text{for all } n \in \mathbb{N}$$

for some $f \in C^0(\Omega;\ \mathbb{C})$, and thus f_j converges with respect to τ. □

Remark 3.2. The metric space $\big(C^0(\Omega;\ \mathbb{C}),\ \tau\big)$ is a Frechét space. Indeed, the function

$$d(x,\ y) = \max_{k\in\mathbb{N}} \left[2^{-k}\, \frac{p_k(x-y)}{1+p_k(x-y)}\right],$$

is a translation-invariant distance, inducing the topology τ.

3.2 Locally Convex Topology of $C^\infty(\Omega)$

Notation. Let $\Omega \subseteq \mathbb{R}^n$ be an open subset, and let $\alpha \in \mathbb{N}^n$ be any *multi-index*. The α-th differential operator is defined by

$$\mathrm{D}^\alpha := \sum_{i=1}^n \left(\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\right),$$

where $\alpha = (\alpha_1,\ \dots,\ \alpha_n)$. The length of a multi-index is given by

$$|\alpha| := \sum_{i=1}^n |\alpha_i|.$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset, and let us denote by $C^\infty(\Omega;\ \mathbb{C})$ the space of all infinitely differentiable functions taking values in $\mathbb{C}$. The differential operator

$$\mathrm{D}^\alpha : C^\infty(\Omega;\ \mathbb{C}) \longrightarrow C^\infty(\Omega;\ \mathbb{C})$$

is well-defined for all multi-indices $\alpha \in \mathbb{N}^n$.

Proposition 3.4. *Let $K \subset \Omega$ be a compact subset. Then*

$$p_{\alpha,\,K}(f) := \max_{x\in K} |\mathrm{D}^\alpha f(x)|$$

is a seminorm defined on $C^\infty(\Omega;\, \mathbb{C})$.

It follows that we can always endow $C^\infty(\Omega;\, \mathbb{C})$ with the topology τ induced by the family of seminorms $\mathcal{P}' := \{p_{\alpha,\,K}\}_{\alpha\in\mathbb{N}^n,\,K\subset\Omega}$. Note that $\mathcal{P}'$ is a separated family since

$$p_{0,\,\{x\}}(f) = 0 \quad \text{for all } x \in \Omega \iff f \equiv 0.$$

Similarly to $C^0(\Omega;\, \mathbb{C})$, we can cover the domain Ω with an exhaustion by compact sets $(K_n)_{n\in\mathbb{N}}$ satisfying $K_n \subsetneq \operatorname{Int} K_{n+1}$ and

$$\Omega \subseteq \bigcup_{n\in\mathbb{N}} K_n.$$

The countable family of seminorms $\mathcal{P} := \{p_{\alpha,\,K_n}\}_{n\in\mathbb{N},\,\alpha\in\mathbb{N}^n} = \{p_{\alpha,\,n}\}_{(\alpha,\,n)\in\mathbb{N}^n\times\mathbb{N}}$ generates the same topology τ as the family $\mathcal{P}'$, and therefore $(C^\infty(\Omega;\, \mathbb{C}),\, \tau)$ is metrizable.

Remark 3.3. The notion of convergence given by $\mathcal{P}$ coincides with the uniform convergence of all derivatives on compact subsets of Ω, that is,

$$f_j \xrightarrow{j\to+\infty}_\tau f \iff \mathrm{D}^\alpha f_j \rightrightarrows_K \mathrm{D}^\alpha f \quad \text{for all } K \subset \Omega \text{ compact and all } \alpha \in \mathbb{N}^n.$$

Theorem 3.5. *The metric space $(C^\infty(\Omega;\, \mathbb{C}),\, \tau)$ is complete.*

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence. The idea is to repeat the argument used in Theorem 3.3 to find the limit of the sequence $g_j^\alpha := D^\alpha f_j$, $\alpha \in \mathbb{N}^n$, and conclude by noticing that

$$f_j \xrightarrow{\tau} f \implies \lim_{j\to\infty} g_j^\alpha =: g^\alpha = D^\alpha f$$

for all multi-indices $\alpha \in \mathbb{N}^n$. □

Remark 3.4. The space $(C^\infty(\Omega;\, \mathbb{C}),\, \tau)$ has the Heine-Borel property.

Proof. Let C be a bounded subset of Ω, and let $(f_j)_{j\in\mathbb{N}} \subset C$ be a sequence of functions in C.

Step 1. It follows from Theorem 2.45 that the seminorm $p_{\alpha,\,n}$ is bounded by a constant restricted to C, and thus

$$p_{\alpha,\,n}(f_j) < c < \infty \quad \text{for all } j \in \mathbb{N}.$$

In particular, the sequence f_j and all of their derivatives are uniformly bounded (the constant depends on C only!) on any compact set K_n of the exhaustion. A diagonal argument, together with the Ascoli-Arzelà theorem, allows us to extract a subsequence still denoted by f_j that converges to some $f \in C^\infty(\Omega;\, \mathbb{C})$.

Step 2. Suppose that C is closed and bounded. Then every sequence in C converges, up to subsequences, to some $f \in C$. Therefore, closed and bounded sets are compact. □

Remark 3.5. The space $C^\infty(\Omega;\, \mathbb{C})$ is not locally compact because it is not a finite-dimensional space - see Theorem 2.46 -.

3.3 Locally Convex Topology of $\mathcal{D}_K(\Omega)$

The goal of this paragraph is to define a locally convex topology on the space of infinitely differentiable functions with support contained in a **fixed** compact subset $K \subset \Omega$.

Definition 3.6 (Support)**.** Let $f : \Omega \longrightarrow \mathbb{C}$ be a function. The *support* of f is defined as the smallest closed set outside of which the function vanishes, that is,

$$\operatorname{spt} f := \overline{\{x \in \Omega \,:\, f(x) \neq 0\}}.$$

Definition 3.7. Let K be a compact subset of Ω. The space denoted by $\mathcal{D}_K(\Omega)$ is defined as the set of all C^∞ functions whose support is contained in K, that is,

$$\mathcal{D}_K(\Omega) := \{f \in C^\infty(\Omega) \,:\, \operatorname{spt} f \subseteq K\}.$$

Remark 3.6. Note that $\mathcal{D}_K(\Omega)$ is a subset of $C^\infty(\Omega)$, and therefore we can endow it with the subspace topology. A straightforward computation shows that this topology may be equivalently generated by the seminorms $p_{\alpha,\,K}$ as α ranges in $\mathbb{N}^n$. Indeed, we have

$$p_{\alpha,\,K}(f) = \max_{\Omega} |f(x)| \geq p_{\alpha,\,\widetilde{K}}(f)$$

for every $f \in \mathcal{D}_K(\Omega)$, and for every compact subset $\widetilde{K} \subset \Omega$.

The notion of convergence in $(C^\infty(\Omega;\ \mathbb{C}), \tau)$ is nothing but the uniform convergence of all derivatives on compact subsets. Therefore, the limit of a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_K(\Omega)$, with respect to the $C^\infty(\Omega)$ topology, is a function f with support contained in K, that is,

$$f_n \xrightarrow{n\to\infty} f \in \mathcal{D}_K(\Omega).$$

Corollary 3.8. *The topological space* $\left(\mathcal{D}_K(\Omega),\, \tau\,\big|_{\mathcal{D}_K(\Omega)}\right)$ *is a closed linear subspace of* $C^\infty(\Omega)$*, and thus complete, metrizable and has the Heine-Borel property.*

Proposition 3.9. *The metric space* $\mathcal{D}_K(\Omega)$ *is infinite-dimensional if and only if the interior of* K *is nonempty.*

Proof. Suppose that $\operatorname{Int} K \neq \varnothing$, and let B_ρ be a ball contained in K. It follows that

$$B_{\frac{\rho}{n}} \subset K \quad \text{for all } n \in \mathbb{N},$$

and thus K contains a countable family of balls of decreasing radii. But Theorem 3.10) asserts that we can find for all $n \in \mathbb{N}$ a function

$$f_n \in \mathcal{D}_K(\Omega)$$

with support contained in the ball of radius $\frac{\rho}{n}$. These functions cannot be linearly dependent (e.g., using the monotonicity of the radii), and thus the space is infinite-dimensional. $\square$

3.3.1 Existence of Test Functions

First we note that $\mathcal{D}_K(\Omega)$ does not depend (really) on the open set Ω (as long as it contains K), and thus, from now on, we will denote it by $\mathcal{D}_K$ to ease the notation.

Theorem 3.10. *Let $0 < r < R < \infty$. Then there exists a function $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying*

$$\varphi(x) = \begin{cases} 1 & \text{if } \|x\| \leq r, \\ 0 & \text{if } \|x\| \geq R, \end{cases}$$

and $\varphi(x) \in [0,\, 1]$ for all $x \in \mathbb{R}^n$.

Proof. We divide the argument into three different steps.

Step 1. The goal is to find an infinitely differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $0 \leq f(x) < 1$, and also that the set

$$\{x \in \mathbb{R} \,:\, f(x) = 0\}$$

is nonempty. The reader should check that the function

$$f(x) := \begin{cases} 0 & \text{if } x < 0, \\ \mathrm{e}^{-1/x} & \text{if } x > 0 \end{cases}$$

has all the required properties.

Step 2. Let f be the function above, and deifne

$$g(x) := \frac{f(x)}{f(x) + f(1-x)}.$$

The reader should check that $g : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to the class C^∞, and satisfies the following requirements:

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1, \\ \in (0,\, 1) & \text{otherwise.} \end{cases}$$

Step 3. The function φ can now be defined in terms of g as follows:

$$\varphi(x) := g\left(\frac{R - \|x\|}{R - r}\right).$$

It is an easy exercise to check that this function has the desired properties, that is, it is a function of class C^∞ with support contained in the ball B_R. □

3.4 Locally Convex Topology of $\mathcal{D}(\Omega)$

Let Ω be an open nonempty subset of $\mathbb{R}^n$. The space of the test functions on Ω is the set of all the functions whose support is compactly included in Ω, that is,

$$\mathcal{D}(\Omega) := \bigcup_{K \subset \Omega} \mathcal{D}_K.$$

Remark 3.7. The space of test functions $\mathcal{D}(\Omega)$, endowed with the subspace topology induced by the inclusion into $C^\infty(\Omega;\,\mathbb{C})$, is not closed.

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion by compact sets of Ω, and let us consider the sequence of compactly supported function $(\chi_{K_n})_{n\in\mathbb{N}}$ of the associated characteristic functions. Then

$$\chi_{K_n} \xrightarrow{n\to+\infty} \chi_\Omega,$$

and thus the support of the limit is the whole Ω, which means that $\mathcal{D}(\Omega)$ is not closed with the subspace topology. ☐

The idea is thus to see $\mathcal{D}(\Omega)$ as the inductive limit of the subsets $\mathcal{D}_K$, that is, we endow it with the finer topology that makes every inclusion map $\mathcal{D}_K \subset \mathcal{D}(\Omega)$ continuous.

Theorem 3.11. *There exists a locally convex topology σ on $\mathcal{D}(\Omega)$, which is finer than any other topology that makes the immersions*

$$\mathcal{D}_k \hookrightarrow \mathcal{D}(\Omega)$$

continuous. Furthermore, the subspace topology induced by the inclusion $\mathcal{D}_K \subset (\mathcal{D}(\Omega),\,\sigma)$ coincides with the subspace topology induced by the inclusion $\mathcal{D}_K \subset (C^\infty(\Omega;\,\mathbb{C}),\,\tau)$.

Proof. We divide the proof into two steps.

Step 1. Let τ_K be the topology induced by τ on $\mathcal{D}_K$. Recall that the immersion

$$\imath_K : \mathcal{D}_K \hookrightarrow \mathcal{D}(\Omega)$$

is continuous if and only if $A \cap \mathcal{D}_K$ belongs to τ_K for all open subset $A \subset \mathcal{D}(\Omega)$. Recall also that a locally convex topology is uniquely determined by a basis of open, convex and balanced neighbourhoods of the origin. Now define

$$\mathcal{B} := \{A \subset \mathcal{D}(\Omega)\ :\ A \text{ convex and balanced},\ A \cap \mathcal{D}_K \in \tau_K \text{ for all } K \subset \Omega \text{ compact}\}\,.$$

Notice that all $B \in \mathcal{B}$ are absorbing since, given $f \in \mathcal{D}(\Omega)$, we have $f \in \mathcal{D}_K$ for some compact set $K \subset \Omega$, and the intersection $B \cap \mathcal{D}_K$ is absorbing in $\mathcal{D}_K$. It follows from Theorem 2.33 that the topology σ generated by the basis $\mathcal{B}$ is locally convex and such that

$$\imath_K : (\mathcal{D}_K,\,\tau_K) \hookrightarrow (\mathcal{D}(\Omega),\,\sigma)$$

is continuous for all $K \subset \Omega$ compact.

Step 2.1. Let $A \in \sigma$ and $K \subset \Omega$ compact subset. By definition, the intersection $A \cap \mathcal{D}_K$ is given by the preimage of A via the continuous inclusion $\imath_K$, and thus belongs to τ_K.

Step 2.2. Vice versa, let $A \in \tau_K$. Recall that the topology τ_K is generated by the family of seminorms $\{p_{\alpha,\,K}\}_{\alpha\in\mathbb{N}^n}$, and thus we have

$$A = A' \cap \mathcal{D}_K, \quad \text{where } A' = \{f \in \mathcal{D}(\Omega) \mid p_\alpha(f) < r_\alpha\}\,.$$

☐

We are now ready to study some of the main properties of the locally convex topology σ on $\mathcal{D}(\Omega)$, with a particular focus on the notion of convergence.

Theorem 3.12. *Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. Then the following assertions hold:*

(a) *The seminorm*
$$p_\alpha(f) := \max_{x\in\Omega} |D^\alpha f|$$
is continuous on $(\mathcal{D}(\Omega), \sigma)$ for all multi-indices $\alpha \in \mathbb{N}^n$.

(b) *A subset $E \subset \mathcal{D}(\Omega)$ is bounded if and only if there exists a compact subset $K \subset \Omega$ such that $E \subset \mathcal{D}_K$. Furthermore, for all $\alpha \in \mathbb{N}^n$ we have*
$$\sup_{f\in E} \max_{x\in K} |D^\alpha f| < \infty.$$

(c) *The space $\mathcal{D}(\Omega)$ has the Heine-Borel property.*

(d) *A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ is a Cauchy sequence if and only if there exists a compact set $K \subset \Omega$ such that $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_K$. Furthermore, for all multi-indices $\alpha \in \mathbb{N}^n$ we have*
$$\lim_{m,\, n\to\infty} \max_{x\in K} |D^\alpha f_n - D^\alpha f_m| = 0.$$

(e) *A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converges to zero if and only if there exists a compact set $K \subset \Omega$ such that $\operatorname{spt} f_n \subset K$ for all $n \in \mathbb{N}$. Furthermore, for all multi-indices $\alpha \in \mathbb{N}^n$ we have*
$$D^\alpha f_n \xrightarrow{n\to\infty} 0 \quad \textit{uniformly}.$$

(f) *The space $\mathcal{D}(\Omega)$ is sequentially complete, but it is not complete.*

Proof.

(a) Fix $K \subset \Omega$ compact subset. Then
$$B_{p_\alpha} \cap \mathcal{D}_K = B_{p_{\alpha,\, K}},$$
and hence B_{p_α} is open, which means that p_α is continuous (see Proposition 2.36.)

(b) We argue by contradiction. Let $E \subset \mathcal{D}(\Omega)$ be a bounded subset, and suppose that there exists $(K_j)_{j\in\mathbb{N}}$, exhaustion by compact sets of Ω, such that
$$\text{"for all } j \in \mathbb{N} \text{ there is } n(j) \in \mathbb{N} \text{ such that } \operatorname{spt} f_{n(j)} \notin K_j\text{".}$$
In other words, for all $j \in \mathbb{N}$ we can find a point $x_j \notin K_j$ and a function $f_{n(j)} \in E$ such that $f_{n(j)}(x_j) \neq 0$. Set
$$B := \left\{ f \in \mathcal{D}(\Omega) \ : \ |f(x_j)| < \frac{|f_{n(j)}(x_j)|}{2^j} \quad \text{for all } j \in \mathbb{N} \right\}.$$
The reader should verify that B is convex and balanced. Furthermore, for all compact subsets K of Ω, only a finite number of points x_j lie in K; hence $B \cap \mathcal{D}_K$ is open as a subset of $\mathcal{D}_K$ as it contains the ball $B_{p_{0,\, K}}(m)$, where
$$m := \max_{x_j\in K} \frac{|f_{n(j)}(x_j)|}{2^j}.$$

Since E is bounded there exists $r > 0$ such that $E \subset r \cdot B$, and this implies that

$$|f_{n(j)}(x_j)| < \frac{r}{2^j} \cdot |f_{n(j)}(x_j)|.$$

In particular, we must have that $r > 2^j$ for all $j \in \mathbb{N}$, and this is only possible if $r = \infty$, which gives the desired contradiction.

Therefore, if E is bounded in $\mathcal{D}(\Omega)$, then we can find a compact subset $K \subset \Omega$ such that $E \subset \mathcal{D}_K$. The space $\mathcal{D}_K$ is endowed with the subspace topology induced by the inclusion $\mathcal{D}_K \subset \mathcal{D}(\Omega)$, and thus E is bounded also in $\mathcal{D}_K$, which, in turn, implies that the seminorms p_α are bounded on E.

The vice versa is obvious, and thus left for the reader to fill in the missing details.

(c) Let E be a closed and bounded set. By (**b**) we can find a compact subset $K \subset \Omega$ such that $E \subset \mathcal{D}_K$; thus E is closed and bounded in $\mathcal{D}_K$. Then E is compact in $\mathcal{D}_K$, and this is enough to infer that it is compact in $\mathcal{D}(\Omega)$ as a consequence of the continuity of the inclusion $\imath_K$.

(d) Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be a Cauchy sequence. By Theorem 2.30, the subset $\{f_n\}_{n\in\mathbb{N}}$ is bounded, and thus, by point (**b**), we can find a compact subset $K \subset \Omega$ such that $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_K$. In particular, $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathcal{D}_K$, and this is enough to infer the thesis.

The vice versa, as above, is obvious as a consequence of the continuity of the inclusion $\imath_K$.

(e) A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converging to zero is, in particular, a Cauchy sequence; thus the first implication follows from (**d**).

(f) Direct consequence of (**d**), (**e**) and of the completeness of $\mathcal{D}_K$.

□

Theorem 3.13. *Let Y be a locally convex topological vector space and let $L : \mathcal{D}(\Omega) \longrightarrow Y$ be a linear*[1] *operator. Then the following properties are equivalent:*

(a) *The operator L is continuous.*

(b) *The operator L is bounded.*

(c) *For all compact subsets $K \subset \Omega$, the restriction $L\big|_{\mathcal{D}_K}$ is bounded.*

(d) *For all compact subsets $K \subset \Omega$, the restriction $L\big|_{\mathcal{D}_K}$ is sequentially continuous.*

(e) *The operator L is sequentially continuous.*

(f) *For all compact subsets $K \subset \Omega$, the restriction $L\big|_{\mathcal{D}_K}$ is continuous.*

Proof.

(**a**) $\Longrightarrow$ (**b**) This implication is a consequence of a more general fact.

[1] A *linear* operator is an additive and homogeneous operator.

Theorem 3.14. *Let $L : X \longrightarrow Y$ be a continuous linear form between topological vector spaces X and Y. Then L is a bounded operator.*

Proof. Let $E \subset X$ be a bounded subset. Let U and V be neighbourhoods of the origin in Y and X respectively such that

$$L(V) \subseteq U.$$

Since E is bounded, we can find $r > 0$ such that $E \subset r \cdot V$. It follows that $L(E) \subset r \cdot U$, and hence the image $L(E)$ is also bounded. □

(b) $\Longrightarrow$ **(c)** This implication is also a consequence of a more general fact.

Theorem 3.15. *Let X be a topological vector space, and let $Y \subset X$ endowed with the subspace topology. If $E \subset Y$ is bounded in Y, then E is also bounded in X.*

Proof. Let U be any neighbourhood of the origin in X, and let $V := U \cap Y$ be a neighbourhood of the origin in Y. If E is bounded in Y, then there exists $r > 0$ such that

$$E \subset r \cdot V.$$

It follows that $E \subset r \cdot U$, and thus E is bounded as a subset of X. □

(c) $\Longrightarrow$ **(d)** This implication is a consequence of a more general fact.

Theorem 3.16. *Let X be a metrizable topological vector space, and let Y be a topological vector space. Then any bounded operator $L : X \longrightarrow Y$ maps converging sequences to converging sequences.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence converging to 0, and assume that

$$d(0,\, x_n) < \frac{1}{n^2}$$

for all $n \in \mathbb{N}$. Then, using the invariance of d, we obtain the inequality

$$d(0,\, n \cdot x_n) \leq \sum_{j=1}^{n} d\left((j-1) \cdot x_n,\, j \cdot x_n\right) =$$

$$= \sum_{j=1}^{n} d(x_n,\, 0) = n \cdot d(x_n,\, 0) \leq$$

$$\leq \frac{1}{n} \xrightarrow{n\to+\infty} 0.$$

The sequence $(n \cdot x_n)_{n\in\mathbb{N}}$ is bounded, and thus its image $n \cdot L(x_n)$ is also bounded. If V is a balanced neighbourhood of the origin in Y, we can find $r > 0$ such that $n \cdot L(x_n) \in r \cdot V$ for all $n \in \mathbb{N}$. But r is finite and $L(x_n) \in rn^{-1} \cdot V \subset V$ when $r < n$, which shows that $L(x_n) \to 0$ also in Y. □

(d) $\iff$ **(e)** Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be a converging sequence. Then we can find a compact subset $K \subset \Omega$ such that $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_K$ and

$$f_n \xrightarrow{n\to\infty} f \in \mathcal{D}_K.$$

Since $L\big|_{\mathcal{D}_K}$ is sequentially continuous, the sequence $(L(f_n))_{n\in\mathbb{N}}$ is also convergent in $\mathcal{D}_K$, and thus the operator L is sequentially continuous.

(e) $\implies$ **(f)** This is a consequence of the equivalence above and of a more general fact.

Theorem 3.17. *Let X be a metric space, and let Y be a topological space. If $L : X \longrightarrow Y$ is a sequentially continuous operator, then L is continuous.*

Proof. We argue by contradiction. Let $x_0 \in X$, let $U \subset Y$ be a neighbourhood of $f(x_0)$, and suppose that L does not send balls centered at x_0 in U. Namely, for all $k \in \mathbb{N}$ we can find a point

$$x_k \in B\left(x_0, \frac{1}{k}\right) \quad \text{and} \quad L(x_k) \notin U.$$

The operator L is sequentially continuous, and thus $x_k \to x_0$ implies $L(x_k) \to L(y) \in U$. The limit $L(y)$ must coincide with $L(x_0)$ as it follows using the usual argument

$$(x'_n) := (x_1, x_0, x_2, x_0, \dots).$$

□

(f) $\implies$ **(a)** First, notice that the composition map

$$\mathcal{D}_K \overset{\imath_K}{\hookrightarrow} \mathcal{D}(\Omega) \xrightarrow{L} Y$$

is continuous. We know that σ is the finer topology that makes the inclusion maps $\imath_K$ continuous, while

$$\widetilde{\sigma} := \left\{L^{-1}(A) \,:\, A \subset Y \text{ and } A \text{ open in } Y\right\}$$

is the coarser topology that makes L continuous. Note that $\widetilde{\sigma}$ is locally convex because Y is a locally convex TVS. Now notice that

$$\imath_K^{-1} \circ L^{-1}(A) \in \widetilde{\sigma}$$

by continuity of the composition, and thus $\widetilde{\sigma}$ is coarser than σ. Since L is continuous with respect to $\widetilde{\sigma}$, it follows that it is continuous also with respect to τ, which is exactly what we wanted to prove.

□

Definition 3.18 (Meager Set)**.** Let (X, τ) be a topological space, and let $S \subset X$. We say that S is a *meager* (or *first-category*) set if and only if there exists a countable cover made up of nowhere dense subsets of X, that is,

$$S = \bigcup_{n\in\mathbb{N}} X_n, \qquad \operatorname{Int}\overline{X}_n = \varnothing.$$

Furthermore, we say that S is a *second-category* set if it is not a first-category set.

Theorem 3.19 (Baire)**.** *Let X be either a complete metric space or a locally compact topological space. Then each open nonempty subset of X is a second-category set.*

Theorem 3.20. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $K \subset \Omega$ be a compact subset. Then the following properties hold:*

(1) The topological space $(\mathcal{D}_K, \tau_K)$ is closed in $(\mathcal{D}(\Omega), \sigma)$.

(2) The interior part of $(\mathcal{D}_K, \tau_K)$ is empty.

(3) The topological space $(\mathcal{D}(\Omega), \sigma)$ is not metrizable.

Proof.

(1) Let $f \in \mathcal{D}(\Omega) \setminus \mathcal{D}_K$. There exists a point $x \notin K$ such that $|f(x)| = \epsilon$, and thus the open set

$$U_f := \left\{ g \in \mathcal{D}(\Omega) \;\middle|\; |g(x) - f(x)| < \frac{\epsilon}{2} \right\}$$

is a neighbourhood of f such that $U_f \cap \mathcal{D}_K = \varnothing$. Indeed, notice that

$$g \in U_f \implies |g(x)| > |f(x)| - \frac{\epsilon}{2} \geq \frac{\epsilon}{2} > 0.$$

(2) This assertion is a consequence of a more general fact. If V is a subspace of a topological vector space X, then its interior is nonempty if and only if V coincides with the whole X.

(3) Since $\mathcal{D}(\Omega)$ is the union of closed subsets with empty interior part, Baire's Theorem 4.7 implies that $\operatorname{Int} \mathcal{D}(\Omega)$ is also empty, and thus it cannot be a metrizable space.

□

Part II

Distributions Theory

Chapter 4

Distribution Theory

The primary goal of this chapter is to give an overview of the distribution theory, focusing as much as possible on the properties of the convolution (e.g., regularity, associativity, etc.)

4.1 Definitions and Main Properties

Let $\Omega \subset \mathbb{R}^n$ be an open set. The space of *distributions*, or generalized functions, is the dual space of $\mathcal{D}(\Omega)$, that is, the space of all linear and continuous forms

$$f : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}.$$

Recall that the topology on $\mathcal{D}(\Omega)$ is induced by the family of seminorms $\mathcal{P} := \{p_{K,\alpha}\}$, where K ranges among all compact subsets of Ω and $\alpha \in \mathbb{N}^n$, may also be generated by the *enlarged family*

$$\tilde{\mathcal{P}} := \{\| \cdot \|_N\}_{N \in \mathbb{N}},$$

defined by setting

$$\|\varphi\|_N := \max_{|\alpha| \leq N} \max_{x \in \Omega} |D^\alpha \varphi(x)| . \tag{4.1}$$

The topological results obtained in the previous chapters will now come into use; specifically, to characterise the continuity of a linear functional.

Theorem 4.1. *Let $f : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ be a linear functional. Then f is continuous if and only if for all compact subsets $K \subset \Omega$ we can find $c(K) := c > 0$ and $N := N(K) \in \mathbb{N}$ such that*

$$|f(\varphi)| \lesssim_K \|\varphi\|_N \quad \textit{for all } \varphi \in \mathcal{D}_K.$$

Proof. We know (see Theorem 3.13) that a linear functional is continuous on $\mathcal{D}(\Omega)$ if and only if it is continuous on $\mathcal{D}_K$ for all compact subset $K \subset \Omega$. In addition, the increasing family $\{\| \cdot \|_N\}_{N \in \mathbb{N}}$ induces a local basis of the origin in $\mathcal{D}_K$ given by the balls

$$B^r_{N,K} := r \cdot B_N := \{\varphi \in \mathcal{D}_K \mid \|\varphi\|_N < r\} .$$

Thus f is continuous on $\mathcal{D}_K$ if and only if there exist $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$f(\epsilon \cdot B_N) \subset B_{\mathbb{C}}.$$

For a generic $\varphi \in \mathcal{D}_K$, we find that

$$\frac{\epsilon\varphi}{2\|\varphi\|_N} \in \epsilon \cdot B_N \implies |f(\varphi)| \le \frac{2}{\epsilon}\|\varphi\|_N$$

and this completes the proof. $\square$

Definition 4.2 (Order)**.** Let $f \in \mathcal{D}'(\Omega)$ be a distribution. The *order* of f is the minimal natural number $N \in \mathbb{N}$ such that, for all compact subsets $K \subset \Omega$, there exists a constant $c(K) := c > 0$ such that f satisfies

$$|f(\varphi)| \lesssim_K \|\varphi\|_N \quad \text{for all } \varphi \in \mathcal{D}_K.$$

Furthermore, if such a natural number does not exists, we say that f is a distribution of order ∞.

Example 4.1 (Dirac Delta)**.** Let $p \in \mathbb{R}$. The *Dirac delta* at the point p is defined by

$$\delta_p : \mathcal{D}(\mathbb{R}) \ni \varphi \longmapsto \varphi(0) \in \mathbb{C}.$$

The map obviously defined a distribution (as it is both linear and continuous) of order zero. Indeed, if we let $K \subset \Omega$ be a generic compact subset, then we have

$$|\delta_p(\varphi)| \le \|\varphi\|_0$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$, which means that δ_p is continuous as a consequence of Theorem 4.1.

Example 4.2. The linear functional defined by setting

$$f(\varphi) := \sum_{j=0}^{+\infty} \varphi'(j) \quad \text{for } \varphi \in \mathcal{D}(\mathbb{R})$$

is a distribution of order 1. To prove this assertion, we start by considering the exhaustion by compact sets of $\mathbb{R}$ given by $K_M := [-M, M]$. Indeed, if we fix $M \in \mathbb{N}$, it turns out that

$$|f(\varphi)| = \left| \sum_{j=0}^{M} \varphi'(j) \right| \le M\|\varphi\|_1 \quad \text{for all } \varphi \in \mathcal{D}_{K_M}.$$

It remains to prove that the order is not 0. Consider the function $\varphi_k \in \mathcal{D}_{K_1}$ such that $\varphi_k'(0) = 1$ and $\|\varphi_k\|_0 \le \frac{1}{k}$ for all $K \in \mathbb{N}$ - see Figure 4.1 -. The reader should check that

$$\frac{|f(\varphi_k)|}{\|\varphi_k\|_0} = k \xrightarrow{k \to +\infty} +\infty.$$

Example 4.3. The linear functional defined by setting

$$f(\varphi) := \sum_{j=0}^{+\infty} \varphi^{(j)}(j) \quad \text{for } \varphi \in \mathcal{D}(\mathbb{R})$$

is a distribution of order ∞. To prove this assertion, we start by considering the exhaustion by compact sets of $\mathbb{R}$ given by $K_M := [-M, M]$. Indeed, if we fix $M \in \mathbb{N}$, it turns out that

$$|f(\varphi)| = \left| \sum_{j=0}^{M} \varphi^{(j)}(j) \right| \le M\|\varphi\|_M \quad \text{for all } \varphi \in \mathcal{D}_{K_M}.$$

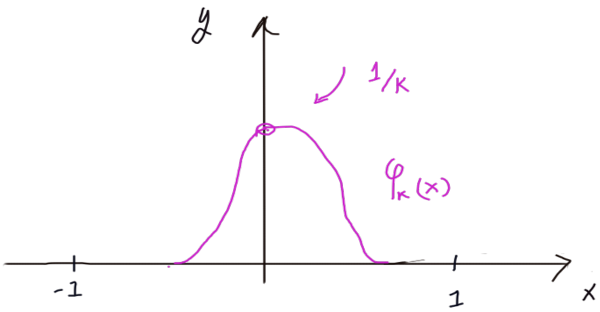

Figure 4.1: Sketch of the Function

It remains to prove that the order is not finite. Fix $M \in \mathbb{N}$, and consider the function $\varphi_{k,M} : K_M \longrightarrow \mathbb{R}$ defined in such a way that $\varphi_k^{(M)}(0) = 1$ and $\|\varphi_k\|_{M-1} \leq \frac{1}{k}$ for all $K \in \mathbb{N}$. The reader should check that

$$\frac{\left|f(\varphi_k^{(M)})\right|}{\|\varphi_k^{(M)}\|_{M-1}} \xrightarrow{k \to +\infty} +\infty \quad \text{for all } M \in \mathbb{N}.$$

Example 4.4 (Locally Summable)**.** Let $\Omega \Subset \mathbb{R}^n$ be an open set. If $f \in L^1_{\text{loc}}(\Omega)$, then we can define a distribution by setting

$$\Lambda_f(\varphi) := \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

The linearity is obvious (the integral itself is linear), while the continuity is an easy consequence of the following estimate:

$$|\Lambda_f(\varphi)| \leq \|f\|_{L^1(K)}\|\varphi\|_0.$$

Example 4.5 (Measure)**.** Let $\Omega \Subset \mathbb{R}^n$ be an open set. If μ is any Borel measure (or a locally finite positive measure) defined on Ω, then we can define a distribution by setting

$$\Lambda_\mu(\varphi) := \int_\Omega \varphi(x)\,\mathrm{d}\mu(x) \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Definition 4.3 (Derivative)**.** Let $f \in \mathcal{D}'(\Omega)$ be a distribution. The *derivative* of f is given by

$$\partial_{x_k} f(\varphi) := -f\left(\partial_{x_k}\varphi\right) \quad \text{for } k = 1, \ldots, n. \tag{4.2}$$

Remark 4.1. The notion of derivative is well-defined.

Proof. We need to show that f' is a distribution. Indeed, the linearity is obvious and the continuity is a consequence of Theorem 4.1 since

$$|\partial_{x_k} f(\varphi)| \lesssim \|\partial_{x_k}\varphi\|_N \simeq \|\varphi\|_{N+1}.$$

□

Example 4.6 (Heaviside)**.** Let $H : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

This function is not differentiable, but it admits a distributional derivative. The functional associated to H is given by Λ_H, and its derivative is

$$\partial_{x_k} \Lambda_H(\varphi) = -\int_{\mathbb{R}} H(x)\varphi'(x)\,\mathrm{d}x = \varphi(0)$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. It follows that the distributional derivative of the Heaviside function is nothing but the Dirac delta at 0.

Let us consider the family of valuations $\mathcal{F}_v := \{\lambda_\varphi\} \subset \mathcal{D}''(\Omega)$, and endow the space $\mathcal{D}'(\Omega)$ with the weak-$*$ topology induced by this choice, that is,

$$(\mathcal{D}'(\Omega),\, \sigma(\mathcal{D}'(\Omega),\, \mathcal{F}_v))\,.$$

Remark 4.2. The valuation λ_φ, defined by

$$\lambda_\varphi(f) := f(\varphi)$$

is continuous if and only if λ_φ is continuous at the point 0 if and only if the absolute value $|\lambda_\varphi|$ is continuous at the point 0.

Remark 4.3. The family of seminorms $\{|\lambda_\varphi| \,:\, \varphi \in \mathcal{D}(\Omega)\}$ is separated.

In particular, the weak-$*$ topology $\sigma(\mathcal{D}'(\Omega),\, \mathcal{F}_v)$ is locally convex. A sequence of distributions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ converges to a distribution f, and we denote it by $f_n \overset{*}{\rightharpoonup} f$, if and only if

$$|\lambda_\varphi|\,(f_n - f) \longrightarrow 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Theorem 4.4. *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ be a converging sequence, and let f be its limit. For all multi-indices $\alpha \in \mathbb{N}^n$ we have that*

$$D^\alpha f_n \overset{*}{\rightharpoonup} D^\alpha f.$$

Proof. It follows immediately from the definitions. Indeed, notice that

$$D^\alpha f_n(\varphi) = (-1)^{|\alpha|} f_n\,(D^\alpha\varphi) \xrightarrow{n\to+\infty} (-1)^{|\alpha|} f\,(D^\alpha\varphi) = D^\alpha f(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

□

Notation. Let X and Y be topological vector spaces, and let Γ be a family (eventually uncountable) of linear and continuous applications from X to Y. For every set $S \subset X$, we denote by $\Gamma(S)$ the union of the images, that is

$$\Gamma(S) := \bigcup_{T\in\Gamma} T(S),$$

and, for every $R \subset Y$, we denote by $\Gamma^{-1}(R)$ the intersection of the preimages, that is

$$\Gamma^{-1}(R) := \bigcap_{T\in\Gamma} T^{-1}(R).$$

In particular, the inclusion $\Gamma(S) \subset R$ is a compact way to express that

$$T(S) \subset R \quad \text{for every } T \in \Gamma,$$

and, similarly, it is also equivalent to the inclusion

$$S \subset T^{-1}(R) \quad \text{for every } T \in \Gamma.$$

Definition 4.5 (Equicontinuous)**.** Let X and Y be topological vector spaces. A family $\Gamma \subset \mathcal{L}(X, Y)$ of linear and continuous applications is *equicontinuous* if and only if

$$\forall\, V \in \mathcal{U}_0(Y),\ \exists U \in \mathcal{U}_0(X)\ :\ \Gamma(U) \subset V.$$

Remark 4.4. If X and Y are metric spaces, this notion is completely equivalent to the equicontinuity in the sense of ϵ-δ.

Remark 4.5. If X and Y are normed spaces, a family Γ is equicontinuous if and only if Γ is equibounded in $\mathcal{L}(X, Y)$ with respect to the operator norm.

Definition 4.6 (Meager Set)**.** Let (X, τ) be a topological space, and let $S \subset X$. We say that S is a *meager* (or *first-category*) set if and only if there exists a countable cover made up of nowhere dense subsets of X, that is,

$$S = \bigcup_{n\in\mathbb{N}} X_n, \qquad \operatorname{Int}\overline{X_n} = \varnothing.$$

Furthermore, we say that S is a *second-category* set if it is not a first-category set.

Theorem 4.7 (Baire)**.** *Let X be either a complete metric space or a locally compact topological space. Then each open nonempty subset of X is a second-category set.*

Theorem 4.8 (Banach-Steinhaus)**.** *Let X and Y be topological vector spaces and let $\Gamma \subset \mathcal{L}(X, Y)$ be a collection (eventually uncountable) of linear and continuous applications. If*

$$E := \{x \in X \mid \Gamma(\{x\}) \textit{ is bounded}\}$$

is a second-category set (i.e., Γ is pointwise bounded in a second-category set), then Γ is a equicontinuous family.

Proof. Fix $U \in \mathcal{U}_0(Y)$ neighbourhood of the origin in Y. Recall that we can always find a neighbourhood $V \in \mathcal{U}_0(Y)$ closed, balanced and satisfying the the inclusion $V + V \subset U$.

Step 1. By assumption, for all $x \in E$ there exists a positive natural number $m(x) \in \mathbb{N}$ such that $\Gamma(x) \subset m(x) \cdot V$ (or, equivalently, $x \in m(x) \cdot \Gamma^{-1}(V)$). In particular, we have that

$$E \subseteq \bigcup_{n\in\mathbb{N}} n \cdot \Gamma^{-1}(V) \quad \text{and} \quad \Gamma^{-1}(V) \text{ closed}.$$

Step 2. Since E is a second-category set, there exists $m \in \mathbb{N}$ such that $m \cdot \Gamma^{-1}(V)$ has nonempty internal part (and, hence, the same applies to $n \cdot \Gamma^{-1}(V)$ for every $n \in \mathbb{N}$). In particular,

$$\mathfrak{V} := \Gamma^{-1}(V) - \Gamma^{-1}(V) \in \mathcal{U}_0(X),$$

and by linearity of Γ we also have that

$$\Gamma(\mathfrak{V}) = \Gamma(\Gamma^{-1}(V)) - \Gamma(\Gamma^{-1}(V)) = V - V \subseteq U.$$

Therefore, the set $\mathfrak{V}$ is a neighbourhood of the origin in X whose image is contained in U, and this is exactly what we wanted to prove. □

Theorem 4.9. *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ be a sequence of distributions such that the limit*

$$f(\varphi) := \lim_{n\to+\infty} f_n(\varphi)$$

exists and is finite for all $\varphi \in \mathcal{D}(\Omega)$. Then f is also a distribution and $f_n \overset{*}{\rightharpoonup} f$.

Proof. The limit defined by f is linear. Thus, it suffices to show that f is continuous or, equivalently, that f is continuous on $\mathcal{D}_K$ for all compact subset $K \subset \Omega$. Now consider the family

$$\mathcal{F} = \{f_n : \mathcal{D}_K \longrightarrow \mathbb{C}\}_{n\in\mathbb{N}}$$

and apply the Banach-Steinhaus Theorem. It follows that $\mathcal{F}$ is equicontinuous, and thus for all $\epsilon > 0$ there exists a neighbourhood U of the origin in $\mathcal{D}_K$ such that $f_n(U) \subset B_\epsilon(0)$. Then

$$f(U) \subset \overline{B_\epsilon(0)} \implies f \text{ continuous.}$$

□

Definition 4.10 (Product Distribution)**.** Let $f \in \mathcal{D}'(\Omega)$, and let $g \in C^\infty(\Omega)$. The product of f and g is defined by setting

$$g \cdot f(\varphi) := f(g \cdot \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{4.3}$$

Remark 4.6. The notion of product is well-defined.

Proof. It is enough to prove that the product is a distribution. The linearity is obvious, and the continuity is a consequence of the Leibniz rule. Indeed, it turns out that

$$|g \cdot f(\varphi)| \lesssim_{g,\, \dots,\, g^{(N)},\, K} \|\varphi\|_{N(K)} \quad \text{for all } \varphi \in \mathcal{D}_K,$$

Notice that the Leibniz rule holds true and it is particularly simple in the case of the product between a distribution and a regular function:

$$D^\alpha(g \cdot f)(\varphi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta g D^{\alpha-\beta} f(\varphi) =$$

$$= \sum_{\beta\leq\alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} f\left(D^{\alpha-\beta}\varphi D^\beta g\right).$$

□

Theorem 4.11. *Let X be a complete metric space, and let Y and Z be topological vector spaces. If*

$$B : X \times Y \longrightarrow Z$$

is a bilinear application separately sequentially continuous, then B is jointly sequentially continuous, that is, sequentially continuous with respect to the couple.

Proof. We first prove a particular case, and then we generalise it with a simple algebraic trick.

Step 1. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence converging to 0, and let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a converging sequence. The linear mapping

$$B(\cdot, y_n) : X \longrightarrow Z$$

is clearly continuous for every $n \in \mathbb{N}$, and it is pointwise bounded. Indeed, for $u \in X$ fixed, the subset

$$\{B(u, y_n)\}_{n\in\mathbb{N}} \subset Z$$

is bounded in Z, since converging sequences are bounded. The Baire Theorem 4.7 holds in the complete metric space X, and hence the Banach-Steinhaus Theorem 4.8 implies that for every $U \in \mathcal{U}_0(Z)$ there exists $V \in \mathcal{U}_0(X)$ such that

$$B\,(V, y_n) \subset U \quad \text{for every } n \in \mathbb{N}.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ converges to 0; thus, $x_n \in V$ definitively, and this means that $B(x_n, y_n)$ belongs to U for n sufficiently large, that is, $B(x_n, y_n) \to 0$.

Step 2. If $x_n \to x \in X$, then the thesis follows from the previous step if one notices that the difference $(x_n - x)_{n\in\mathbb{N}}$ converges to 0. Then

$$B(x_n, y_n) = B(x_n - x, y_n) + B(x, y_n) \xrightarrow{n\to+\infty} 0 + B(x, y) = B(x, y).$$

$\square$

Corollary 4.12. *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ be a sequence of distributions converging to some f, and let $(g_n)_{n\in\mathbb{N}} \subset C^\infty(\Omega)$ be a sequence to a smooth function g. Then*

$$g_n \cdot f_n \overset{*}{\rightharpoonup} g \cdot f.$$

4.2 Localization and Support of a Distribution

Let $\omega \subset \Omega$ be two open subsets of $\mathbb{R}^n$. There is a natural continuous inclusion $\mathcal{D}(\omega) \subset \mathcal{D}(\Omega)$, which induces the opposite inclusion of the dual spaces

$$\mathcal{D}'(\Omega) \hookrightarrow \mathcal{D}'(\omega).$$

The inclusion is continuous, and thus a distribution $f \in \mathcal{D}'(\Omega)$ is zero in $\mathcal{D}'(\omega)$ if and only if

$$f(\varphi) = 0 \quad \text{for all } \varphi \in \mathcal{D}(\omega).$$

In this section, we are mainly interested in the "opposite" implication, that is, given a collection of compatibles distributions on a open cover of Ω, find a global distribution $f \in \mathcal{D}'(\Omega)$ such that

$$f\big|_{\Omega_\alpha} \equiv f_\alpha.$$

The fundamental tool here to define globally something that has already been defined locally is the well-known *partition of the unity*. In particular, we associate to a cover of the space a set of functions which "glue together the local properties."

Theorem 4.13 (Partition of Unity)**.** *Let $\mathcal{F}$ be a collection of open subsets of $\mathbb{R}^n$ whose union is equal to Ω, that is,*

$$\Omega = \bigcup_\alpha \Omega_\alpha.$$

Then there exist a collection of compactly supported functions $\{\psi_n\}_{n\in\mathbb{N}}$ such that the following properties hold:

(1) The function ψ_n is nonnegative for all $n \in \mathbb{N}$.

(2) For all $n \in \mathbb{N}$ there exists α such that $\Omega_\alpha \in \mathcal{F}$ satisfies

$$\operatorname{spt} \psi_n \subset \Omega_\alpha.$$

(3) For all $x \in \Omega$ we have

$$\sum_{n\in\mathbb{N}} \psi_n(x) = 1.$$

(4) For all compact subset $K \subset \Omega$ we can find an open neighbourhood $W \supset K$ such that, for all $x \in W$, we have $\psi_n(x) \neq 0$ for finitely many $n \in \mathbb{N}$.

Proof. Let us consider the family of balls

$$\mathcal{B} := \{B_{x,\,r} \;:\; B_{x,\,r} \subset \Omega_\alpha,\, x \in \mathbb{Q}^n,\, r \in \mathbb{Q}^+\}.$$

This family is obviously countable, and its union contains the whole Ω, that is,

$$\bigcup_{n\in\mathbb{N}} B_n \supset \Omega.$$

Consider now the scaled family of balls

$$\mathcal{V} := \left\{\frac{1}{2}B_{x,\,r} \;:\; B_{x,\,r} \in \mathcal{B}\right\}$$

and notice that $\mathcal{V}$ is also a cover of Ω.

Step 1. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a countable collection of functions in $\mathcal{D}(\Omega)$ defined in such a way that

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x \in V_n, \\ 0 & \text{if } x \notin B_n, \\ \in (0,\,1) & \text{if } x \in B_n \setminus V_n. \end{cases}$$

We can define the desired partition of the unity by induction as follows. Let $\psi_0 := \varphi_0$, and let

$$\psi_{n+1} := (1-\varphi_0)\dots(1-\varphi_n)\cdot\varphi_{n+1} \quad \text{for all } n \geq 1.$$

A straightforward computation shows that ψ_n is positive for all $n \in \mathbb{N}$ and, by construction, the support of ψ_n is contained in the support of φ_n, which is contained in some B_n. Furthermore,

$$\sum_{n=0}^{N} \psi_n = 1 - \prod_{n=0}^{N}(1-\varphi_n),$$

and thus, since for all $x \in \Omega$ there exists a natural number $m \in \mathbb{N}$ such that $x \in V_m$, we have

$$\sum_{n=0}^{m} \psi_n = 1 - \prod_{n=0}^{m}(1-\varphi_n) = 1.$$

In particular, for a given $x \in \Omega$, only finitely many $\psi_n(x)$ are nonzero.

Step 2. Let $K \subset \Omega$ be a compact subset. There exists a natural number $m \in \mathbb{N}$ such that $K \subset \cup_{i=1}^{m} Vn_i$. It follows that

$$\sum_{n=0}^{m} \psi_n = 1 - \prod_{n=0}^{m}(1-\varphi_n) = 1,$$

which is exactly what we wanted to prove. ☐

Theorem 4.14. *Let $\mathcal{F}$ be an open cover of Ω. Suppose that for all $\omega \in \mathcal{F}$ there is a distribution $f_\omega \in \mathcal{D}'(\omega)$, and assume also that for all ω, $\omega' \in \mathcal{F}$ the corresponding distributions satisfy*

$$f_\omega(\varphi) = f_{\omega'}(\varphi) \quad \textit{for all } \varphi \in \mathcal{D}(\omega \cap \omega').$$

Then there exists a unique $f \in \mathcal{D}'(\Omega)$ such that

$$f(\varphi) = f_\omega(\varphi) \quad \textit{for all } \varphi \in \mathcal{D}(\omega).$$

Proof. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be the partition of unity constructed in Theorem 4.13, and let ω_n be the element of $\mathcal{F}$ such that the support of ψ_n is contained in ω_n.

Step 1. Let $\varphi \in \mathcal{D}(\Omega)$ be given. Then

$$\varphi = \sum_{n\in\mathbb{N}} \psi_n \cdot \varphi,$$

and only a finite number of addendum is different from zero since we can always find a compact subset $K \subset \Omega$ such that $\operatorname{spt}\varphi \subset K$. Define the functional

$$f(\varphi) := \sum_{n\in\mathbb{N}} f_{\omega_n}(\psi_n \cdot \varphi).$$

It is easy to see that f is linear on $\mathcal{D}(\Omega)$; hence we only need to prove that it is also continuous.

Step 2. Let $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be a sequence converging to zero. We know that there exists a compact subset $K \subset \Omega$ that contains the support of φ_k for all $k \in \mathbb{N}$. It follows that

$$f(\varphi_k) := \sum_{n=0}^{m} f_{\omega_n}(\psi_n \cdot \varphi_j) \quad \text{for all } j \in \mathbb{N}$$

for some fixed $m \in \mathbb{N}$. The sequence $(\psi_n \cdot \varphi_j)_{j\in\mathbb{N}}$ converges to 0 in $\mathcal{D}(\omega_n)$ for all $n \in \mathbb{N}$ fixed; therefore

$$f_{\omega_n}(\psi_n \cdot \varphi_j) \to 0 \quad \text{for all } n \in \mathbb{N}.$$

It follows that $f(\varphi_k) \to 0$, which means that f is continuous (and thus a distribution.)

Step 3. To prove that f coincides with f_ω on each open subset $\omega \in \mathcal{F}$, notice that, given a smooth function $\varphi \in \mathcal{D}(\omega)$, the product $\psi_n \cdot \varphi$ belongs to $\mathcal{D}(\omega_n \cap \omega)$. It follows from the assumptions

$$f_{\omega_n}(\psi_n \cdot \varphi) = f_\omega(\psi_n \cdot \varphi),$$

which, in turn, implies the thesis:

$$f(\varphi) = \sum_{n\in\mathbb{N}} f_\omega(\psi_n \cdot \varphi) = f_\omega\left(\sum_{n\in\mathbb{N}} \psi_n \cdot \varphi\right) = f_\omega(\varphi).$$

□

Corollary 4.15. *Let $f, g \in \mathcal{D}'(\Omega)$, and let $\mathcal{F}$ be an open cover of Ω. Suppose that*

$$f\,\big|_{\mathcal{D}'(\omega)} = g\,\big|_{\mathcal{D}'(\omega)}$$

for all $\omega \in \mathcal{F}$. Then $f = g$ as distributions in $\mathcal{D}'(\Omega$.

Definition 4.16 (Support)**.** Let $f \in \mathcal{D}'(\Omega)$ be a distribution. The *support* of f is defined by

$$\operatorname{spt} f := \Omega \setminus \bigcup \{\omega \subset \Omega \mid \omega \text{ is an open subset such that } f \equiv 0 \text{ on } \mathcal{D}'(\omega)\}\,. \tag{4.4}$$

Remark 4.7. The notion is well-defined since f is actually zero as an element of $\mathcal{D}'(\Omega \setminus \operatorname{spt} f)$.

Theorem 4.17. *Let $f \in \mathcal{D}'(\Omega)$, and let $S := \operatorname{spt} f$. Then the following properties hold:*

(a) *If $\varphi \in \mathcal{D}(\Omega)$ has support disjoint from S, then $f(\varphi) = 0$.*

(b) *If $S = \varnothing$, then $f = 0$.*

(c) *If $g \in C^\infty(\Omega)$ is a smooth function such that $g \equiv 1$ on some open subset U containing S, then $g \cdot f = f$.*

(d) *If S is compact, then there exists $N \in \mathbb{N}$ such that*

$$|f(\varphi)| \lesssim \|\varphi\|_N.$$

In particular, the order of f is finite and it can be extended in a unique to a linear and continuous functional on $\mathcal{E} = C^\infty(\Omega)$, that is, $f \in \mathcal{E}'(\Omega)$.

Proof.

(a) If the support of φ does not intersect S, then $\varphi \in \mathcal{D}(\Omega \setminus S)$. But f is equal to 0 outside of its support, and thus $f(\varphi) = 0$.

(b) Obvious.

(c) Let $U \supset S$. Then $\mathcal{F} := \{U, \Omega \setminus S\}$ is an open cover of Ω, and it is easy to check that

$$f\big|_{\mathcal{D}'(\Omega\setminus S)} = 0 = g \cdot f\big|_{\mathcal{D}'(\Omega\setminus S)} \quad \text{and} \quad g \cdot f\big|_{\mathcal{D}'(U)} = 1 \cdot f = f\big|_{\mathcal{D}'(U)}.$$

(d) First, we prove that there exists a function $g \in \mathcal{D}(\Omega)$ such that $g\big|_U \equiv 1$ for some open set U containing the support S. Let $\{B_n\}_{n=1,\,\dots,\,N}$ be a finite collection of balls such that their closure is contained in Ω and

$$\bigcup_{n=1}^{N} \frac{1}{2} \cdot B_n \supset S.$$

Then

$$\{\Omega \setminus S,\, B_1,\, \dots,\, B_N\}$$

is an open cover of Ω, and therefore we can find a partition of unity $\{\psi_S,\, \psi_1,\, \dots,\, \psi_N\}$ such that $\psi_n\big|_{\frac{1}{2}\cdot B_n} \equiv 1$ for all $n = 1,\, \dots,\, n$. The reader should check that

$$g(x) = \sum_{n=1}^{N} \psi_n(x)$$

does the work. We now know that $g \cdot f = f$ and, for any $\varphi \in \mathcal{D}(\Omega)$, it turns out that $g \cdot \varphi \in \mathcal{D}_K$ - where $K := \operatorname{spt} g$ -. But on $\mathcal{D}_K$ we can estimate the absolute value of the functional as follows:

$$|f(g \cdot \varphi)| \lesssim \|g \cdot \varphi\|_N,$$

where $c > 0$ and $N \in \mathbb{N}$. It follows from the Leibniz rule that there exists a constant $c(g) > 0$ such that

$$|f(\varphi)| \lesssim_g \|\varphi\|_N.$$

To prove that f can be uniquely extended to a linear and continuous functional on $\mathcal{E}$, we set

$$f(\varphi) := g \cdot f(\varphi) = f(g \cdot \varphi) \quad \text{for all } \varphi \in \mathcal{E}.$$

The support of $g \cdot f$ is the intersection of $\operatorname{spt} g$ and S, which is also compact.

$\square$

Lemma 4.18. *Let X be a vector space, and let $\xi,\, \xi_1,\, \dots,\, \xi_n \in X'$. Then*

$$\ker \xi \supseteq \bigcap_{i=1}^{n} \ker \xi_i \iff \xi(x) = \sum_{i=1}^{n} \lambda_i \xi_i(x).$$

Proof. Let us consider the operator $f : X \longrightarrow \mathbb{K}^n$ defined by

$$f(x) = (\xi_1(x),\, \dots,\, \xi_n(x)).$$

Suppose that $\ker f$ is a subset (eventually proper) of the kernel of ξ. We know (algebraic geometry) that we can always find a linear mapping $h : \mathbb{K}^n \longrightarrow \mathbb{K}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\xi"'] & \mathbb{K}^n \arrow[d, "h"] \\ & \mathbb{K} \end{tikzcd}$$

commutes. Therefore, if $f(y_1, \ldots, y_n) = \sum_{i=1}^n \lambda_i y_i$, then

$$\xi(x) = \sum_{i=1}^n \lambda_i \cdot \xi_i(x).$$

$\square$

4.3 Derivative Form of Distributions

In this section, the primary goal is to prove that every distribution is, in some sense, the αth derivative of a specific continuous function.

Theorem 4.19. *Let $f \in \mathcal{D}'(\Omega)$, and let $p \in \Omega$. Assume that the support of f is contained in $\{p\}$. Then there exist a natural number $N \in \mathbb{N}$ and constants $c_\alpha > 0$ such that*

$$f = \sum_{|\alpha| \leq N} c_\alpha \cdot D^\alpha \delta_p.$$

Proof. We may assume, without loss of generality, that p is the origin.

Step 1. By Theorem 4.17 the order of f is finite, and the existence of the natural number $N \in \mathbb{N}$ is obvious. Furthermore, it follows from Lemma 4.18 that it is enough to show that $f(\varphi) = 0$ for every φ such that

$$D^\alpha \delta_0(\varphi) = 0 \quad \text{for all } \alpha \text{ such that } |\alpha| \leq N.$$

We can work in a compact set containing the origin, that is, $K = \overline{B_r(0)}$. In particular, there exists $c > 0$ such that

$$|f(\varphi)| \lesssim \|\varphi\|_N \quad \text{for all } \varphi \in \mathcal{D}_K.$$

Let $g \in \mathcal{D}(\mathbb{R}^n)$ be a function such that

$$g(x) = \begin{cases} 1 & \text{if } x \in B_1(0), \\ 0 & \text{if } x \notin B_2(0), \\ \in (0, 1) & \text{otherwise}, \end{cases}$$

and consider the scalings as $\rho > 0$

$$g_\rho(x) := g\left(\frac{x}{\rho}\right).$$

By Theorem 4.17 it follows that there exists $\rho > 0$ such that $f = g_\rho \cdot f$ and

$$\rho < \frac{r}{2} \implies \operatorname{spt} g_\rho \subset K.$$

The estimate above implies that

$$|f(\varphi)| \lesssim \|g_\rho \cdot \varphi\|_N$$

for all $\varphi \in \mathcal{D}_{\bar{B}_r}$, which means that we only need to estimate the derivative of the product between φ and g_ρ.

Step 2. For every $\epsilon > 0$ we can find $\ell > 0$ such that the N-th derivative of φ is smaller than ϵ, that is,

$$|D^\alpha \varphi(x)| \leq \epsilon \quad \text{for all } |x| \leq \ell \text{ and } |\alpha| = N.$$

A straightforward computation, together with the induction principle, shows that

$$|D^\alpha \varphi(x)| \lesssim \epsilon\, |x|^{N-|\alpha|} \quad \text{for all } |x| \leq \ell.$$

Note that the constant does not depend on φ or on the value of ϵ. Similarly, we may estimate the derivatives of the function g_ρ as follows:

$$|D^\alpha g_\rho(x)| \lesssim \frac{1}{\rho^{|\alpha|}} \quad \text{for all } \rho > 0.$$

If we now choose ρ to be exactly equal to ℓ, we obtain the chain of inequalities

$$\begin{aligned} |D^\alpha (g_\rho \cdot \varphi)(x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left|D^\beta g_\rho(x)\right| \left|D^{\alpha-\beta} \varphi(x)\right| \lesssim \\ &\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{\ell^{|\beta|}} \epsilon \cdot |x|^{N-|\alpha|-|\beta|} \lesssim \\ &\lesssim \epsilon \sum_{\beta \leq \alpha} \ell^{N-|\alpha|} \lesssim \epsilon, \end{aligned}$$

and by the arbitrariness of $\epsilon > 0$ we conclude that $f(\varphi) = 0$. □

Theorem 4.20. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in \mathcal{D}'(\Omega)$ be a distribution, and let K be a compact subset of Ω. Then there exists a continuous function $g \in C^0(\Omega;\, \mathbb{C})$ such that*

$$f(\varphi) = (-1)^{|\alpha|} \int_\Omega g(x) D^\alpha \varphi(x)\, \mathrm{d}x \quad \textit{for all } \varphi \in \mathcal{D}_K.$$

Proof. We may assume, without loss of generality, that $K \subset Q := [0,\, 1]^n$. Furthermore, given $\varphi \in \mathcal{D}_K$, we consider the null extension outside of K, still denoted by φ, in such a way that $\varphi \in \mathcal{D}_Q$.

Step 1. We use the Poincaré inequality to estimate φ by a higher-order derivative. Indeed, we known (e.g., Lagrange theorem) that for all $x \in Q$ there is a point $y \in [0, 1]$ such that

$$\varphi(x) = \varphi(x_1, \ldots, x_n) - \varphi(0, x_2, \ldots, x_n) = \partial_{x_1} \varphi(y, \ldots, x_n).$$

It follows that

$$\|\varphi\|_0 \lesssim \|\partial_{x_1}\varphi\|_0 \,,$$

and therefore (by induction) for all $\mathbf{N} := (N, \ldots, N) \in \mathbb{N}^n$ we have that

$$\|\varphi\|_N \lesssim \|D^{\mathbf{N}}\varphi\|_0.$$

We now observe that a function $\varphi \in \mathcal{D}_Q$, for $x \in Q$, satisfies

$$\varphi(x) = \int_0^{x_1} \partial_{x_1}\varphi(y, \ldots, x_n)\, \mathrm{d}y,$$

and thus, if we repeat the same process for all the directions, we obtain

$$\varphi(x) = \int_{Q(x)} D^{\mathbf{1}}\varphi(y)\, \mathrm{d}y, \tag{4.5}$$

where $Q(x) := [0, x_1] \times \cdot \times [0, x_n]$. In particular, the uniform norm of φ may be estimated with the L^1-norm of the derivatives, that is,

$$\|\varphi\|_0 \leq \|D^{\mathbf{1}}\varphi\|_{L^1(Q)}.$$

Step 2. We know that, for any fixed $K \subset \Omega$, there exist $c := c(K) > 0$ and $N := N(K) \in \mathbb{N}$ such that

$$|f(\varphi)| \lesssim_K \|\varphi\|_N \quad \text{for all } \varphi \in \mathcal{D}_K.$$

Now the previous estimates show that f can also be estimated using the L^1-norm of the $(\mathbf{N}+\mathbf{1})$-order derivative, that is,

$$|f(\varphi)| \lesssim \|\varphi^{\mathbf{N}}\|_0 \lesssim \|D^{\mathbf{N}+\mathbf{1}}\varphi\|_{L^1(Q)}.$$

The integral formula (4.5) implies the injectivity of the differential operator $D^{\mathbf{1}}$, and thus also the iterate $D^{\mathbf{N}+\mathbf{1}}$ is injective as well. If we set

$$V := D^{\mathbf{N}+\mathbf{1}}(\mathcal{D}_Q) = \left\{ D^{\mathbf{N}+\mathbf{1}}\varphi \mid \varphi \in \mathcal{D}_Q \right\},$$

then it is possible to find a linear functional $L : V \longrightarrow \mathbb{C}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{D}_Q \arrow[r, "D^{\mathbf{N}+\mathbf{1}}"] \arrow[dr, "f"'] & V \arrow[d, "L"] \\ & \mathbb{C} \end{tikzcd}$$

that is, a functional such that

$$L\left(D^{\mathbf{N}+\mathbf{1}}\varphi\right) = f(\varphi).$$

We observe that L is continuous with respect to the L^1-norm since

$$|L(\psi)| \lesssim \int_Q |\psi(x)|\, \mathrm{d}x.$$

Step 3. The Hahn-Banach theorem extends L to a functional $\tilde{L}$ linear, continuous and defined on the whole space $L^1(Q)$. Furthermore, the Riesz theorem allows us to find a representation of $\widetilde{L}$ with a function $h \in L^\infty(Q)$, which is the dual of L^1, that is,

$$\widetilde{L}(\psi) := \int_Q h(x)\psi(x)\,\mathrm{d}x.$$

Therefore, for all $\varphi \in \mathcal{D}_K$, we have that

$$f(\varphi) = \int_Q h(x) D^{\mathbf{N+1}}\varphi(x)\,\mathrm{d}x.$$

To obtain a continuous function, we need to make another integration. Set

$$g(x) := \int_{Q(x)} h(y)\,\mathrm{d}y.$$

The integral is absolutely continuous, and thus f is a Lipschitz-regular on the whole Q. It remains to prove that the distributional derivative of f is equal to g. For all $\varphi \in \mathcal{D}_Q$ we have

$$\begin{aligned}
\int_Q g(x) D^{\mathbf{1}}\varphi(x)\,\mathrm{d}x &= \int_Q \left(\int_0^{x_1} \cdots \int_0^{x_n} h(y)\,\mathrm{d}y_n \ldots \mathrm{d}y_1 \right) D^{\mathbf{1}}\varphi(x)\,\mathrm{d}x = \\
&= \int_Q h(y) \left(\int_{y_1}^1 \cdots \int_{y_n}^1 D^{\mathbf{1}}\varphi(x)\,\mathrm{d}x_n \ldots \mathrm{d}x_1 \right) \mathrm{d}y = \\
&= (-1)^n \int_Q h(y)\varphi(y)\,\mathrm{d}y.
\end{aligned}$$

Therefore, we conclude by observing that

$$f(\varphi) = \int_Q h(x) D^{\mathbf{N+1}}\varphi(x)\,\mathrm{d}x = (-1)^n \int_Q g(x) D^{\mathbf{N+2}}\varphi(x)\,\mathrm{d}x.$$

□

Theorem 4.21. *Let $f \in \mathcal{D}'(\Omega)$ be a distribution with support contained in a compact set $K \subset \Omega$, and let V be an open set such that $K \subset V \subset \Omega$. Then there are $f_1, \ldots, f_m : \Omega \longrightarrow \mathbb{C}$ continuous functions, whose support is contained in K, and multi-indices $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^n$ such that*

$$f = \sum_{k=1}^m D^{\alpha_k}\Lambda_{f_k}.$$

Proof. Let W be an open set containing K, whose closure is compact in Ω. Let $\psi \in \mathcal{D}(\Omega)$ be a smooth function such that

$$\psi(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin W, \\ \in (0,1) & \text{otherwise.} \end{cases}$$

By Theorem 4.17 it follows that $f \cdot \psi = f$, and thus

$$f(\varphi) = f(\varphi \cdot \psi) \quad \text{for all } \varphi \in \mathcal{D}'(\Omega).$$

We now apply Theorem 4.20 to the compact set $\overline{W}$; it turns out that there exist $g \in C^0(\Omega)$ and $\alpha \in \mathbb{N}^n$ such that

$$f(\varphi \cdot \psi) = (-1)^{|\alpha|} \int_\Omega g(x) D^\alpha (\psi \cdot \varphi)(x)\, \mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Set

$$g_\beta(x) := (-1)^{|\alpha|+|\beta|} g(x) D^{\alpha-\beta}\psi(x).$$

Then, using the Leibniz rule, we obtain

$$\begin{aligned} f(\varphi) &= \sum_{\beta \le \alpha} (-1)^{|\alpha|} \binom{\alpha}{\beta} \int_\Omega g(x) D^{\alpha-\beta}\psi(x) D^\beta \varphi(x)\, \mathrm{d}x = \\ &= \sum_{\beta \le \alpha} (-1)^{|\beta|} \int_\Omega g_\beta(x) D^\beta \varphi(x)\, \mathrm{d}x = \\ &= \sum_{b \le \alpha} D^\beta \Lambda_{g_\beta}(\varphi). \end{aligned}$$

□

Theorem 4.22. *Let $f \in \mathcal{D}'(\Omega)$. For all $\alpha \in \mathbb{N}^n$ we can find $g_\alpha \in C^0(\Omega;\, \mathbb{C})$ such that*

$$f = \sum_\alpha D^\alpha \Lambda_{g_\alpha}.$$

Furthermore, for all compact subset $K \subset \Omega$, only finitely many g_α are different from zero.

Proof. Let $(V_j)_{j\in\mathbb{N}}$ be a countable family of open sets with compact closure such that

$$V_j \nearrow \Omega.$$

Let $\omega_j := V_j \setminus \overline{V_{j-2}}$. It is easy to prove that $\{\omega_j\}_{j\in\mathbb{N}}$ is an open covering of Ω, and hence it admits a partition of the unity

$$\{\psi_j \in \mathcal{D}(\Omega) \mid \operatorname{spt}(\psi_j) \subset \omega_j\}_{j\in\mathbb{N}}.$$

The function $\varphi \in \mathcal{D}(\Omega)$ has compact support, and thus there are only finitely many ψ_j different from zero on the support of φ. Therefore,

$$f(\varphi) = f\left(\sum_{j\in\mathbb{N}} \psi_j \cdot \varphi\right) = \sum_{j\in\mathbb{N}} f(\psi_j \cdot \varphi) = \sum_{j\in\mathbb{N}} \psi_j \cdot f(\varphi).$$

We know that $\psi_j \cdot f$ is a distribution with compact support contained in the closure $\bar{V}_j$. Hence it follows from Theorem 4.21 that

$$\psi_j \cdot f = \sum_{k=1}^{m_j} D^{\alpha_{k,\,j}} \Lambda_{g_{k,\,j}},$$

which, in turn, implies that

$$f = \sum_{j\in\mathbb{N}} \sum_{k=1}^{m_j} D^{\alpha_{k,\,j}} \Lambda_{g_{k,\,j}}.$$

□

Theorem 4.23. *Let $f \in \mathcal{D}'(\Omega)$ be a positive distribution, whose support is contained in a compact subset $K \subset \Omega$. Then there exists a positive finite measure μ such that*

$$f(\varphi) = \int \varphi(x)\,\mathrm{d}\mu(x).$$

Moreover, if the support is not in a compact subset, then μ is σ-finite and positive.

4.4 Convolution

Let φ, $\psi \in \mathcal{D}(\mathbb{R}^n) := \mathcal{D}$. Recall that the convolution of φ and ψ is defined by

$$(\varphi * \psi)(x) := \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\,\mathrm{d}y. \tag{4.6}$$

The goal of this section is to find a well-defined notion of convolution between a distribution $f \in \mathcal{D}'$ and a function $\varphi \in \mathcal{D}$. Set

$$(f * \varphi)(x) := f\left(\tau_x \check{\varphi}\right). \tag{4.7}$$

Here τ_x denotes the translation operator $g(y) \longmapsto g(y-x)$ and $\check{\varphi}(y)$ is equal to the function at the symmetric point $\varphi(-y)$.

Definition 4.24. Let $f \in \mathcal{D}'$ be a distribution. We define the translation and the symmetric of f respectively as

$$\tau_x f(\varphi) := f\left(\tau_{-x}\varphi\right) \quad \text{and} \quad \check{f}(\varphi) := f\left(\check{\varphi}\right).$$

Theorem 4.25 (Regularity)**.** *Let $f \in \mathcal{D}'$, $\varphi \in \mathcal{D}$ and let x be a point of $\mathbb{R}^n$.*

(a) *The translation operator τ_x acting on the product can be absorbed on any of the factors, i.e.,*

$$\tau_x(f * \varphi) = (\tau_x f) * \varphi = f * (\tau_x \varphi).$$

(b) *The convolution $f * \varphi$ is a function of class $C^\infty(\mathbb{R}^n)$. Furthermore, we have*

$$D^\alpha(f * \varphi) = (D^\alpha f) * \varphi = f * (D^\alpha \varphi)$$

for all multi-indices $\alpha \in \mathbb{N}^n$.

Proof.

(a) The thesis follows immediately by noticing that for all $y \in \mathbb{R}^n$ we have

$$\tau_x(f * \varphi)(y) = \tau_x f\left(\tau_y \check{\varphi}\right) = f\left(\tau_{y-x}\check{\varphi}\right),$$

$$(\tau_x f) * \varphi(y) = \tau_x f\left(\tau_y \check{\varphi}\right) = f\left(\tau_{y-x}\check{\varphi}\right),$$

$$f * (\tau_x \varphi)(y) = f\left(\tau_y \check{\tau_x \varphi}\right) = f\left(\tau_{y-x}\check{\varphi}\right).$$

(b) First, we prove that $f * \varphi$ is a function of class C^∞. Indeed, the incremental ratio, which defines the jth directional derivative of the convolution, is given by

$$\frac{(f*\varphi)(x+he_j)-(f*\varphi)(x)}{h} = \frac{\left(f*(\tau_{-he_j}\varphi)\right)(x)-(f*\varphi)(x)}{h} =$$

$$= \tau_x\check{f}\left(\frac{\tau_{-he_j}\varphi-\varphi}{h}\right)(x).$$

The function φ belongs to $\mathcal{D}$, and hence the incremental ratio converges to the jth derivative:

$$\frac{\tau_{-he_j}\varphi-\varphi}{h}(x) = \frac{\varphi(x+he_j)-\varphi(x)}{h} \xrightarrow{h\to 0^+} \partial_{x_j}\varphi(x).$$

Furthermore, the convergence is uniform with respect to the variable x since φ is differentiable and the derivative is bounded. Similarly, we have that

$$\frac{\tau_{-he_j}\varphi-\varphi}{h} \xrightarrow{\mathcal{D}} \partial_{x_j}\varphi$$

with respect to the notion of convergence in $\mathcal{D}$. The distribution $\tau_x\check{f}$ is continuous, and therefore the limit must exist. It follows that $f*\varphi$ is differentiable in the direction e_j, $j = 1, \ldots, n$, and the desired equality holds:

$$\partial_{x_j}(f*\varphi) = \tau_x\check{f}\left(\partial_{x_j}\varphi\right) = f*(\partial_{x_j}\varphi).$$

The same holds true for any direction and derivative of φ; thus $f*\varphi \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha(f*\varphi) = \left(f*(D^\alpha\varphi)\right).$$

In conclusion, we notice that

$$\begin{aligned}\left(\partial_{x_j}f\right)*\varphi(x) &= \partial_{x_j}f\left(\tau_x\check{\varphi}\right) = \\ &= -f\left(\partial_{x_j}\tau_x\check{\varphi}\right) = \\ &= -f\left(\tau_x\partial_{x_j}\check{\varphi}\right) = \\ &= \left(f*(\partial_{x_j}\varphi)\right)(x).\end{aligned}$$

We can easily generalize this chain of equalities to a derivative of order $\alpha \in \mathbb{N}^n$, which is exactly what we needed to conclude.

□

Theorem 4.26 (Associativity)**.** *Let $f \in \mathcal{D}'$, and let φ, $\psi \in \mathcal{D}$. The convolution is associative, i.e.,*

$$(f*\varphi)*\psi = f*(\varphi*\psi).$$

Proof. The proof is rather involved. To ease the notations we will divide it into four steps.

Step 1. Let $z \in \mathbb{R}^n$ be a point. Then

$$
\begin{aligned}
\left((f*\varphi)*\psi\right)(z) &= \int_{\mathbb{R}^n} (f*\varphi)(x)\psi(z-x)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}^n} f(\tau_x\check{\varphi}\,\psi(z-x)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}^n} f\left(\tau_x\check{\varphi}\psi(z-x)\right)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}^n} f\left(y \mapsto \psi(z-x)\varphi(x-y)\right)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}^n} f\left(\Phi(x)\right)\,\mathrm{d}x,
\end{aligned}
$$

where $\Phi : \mathbb{R}^n \longrightarrow \mathcal{D}$ is defined by

$$\Phi : x \longmapsto \Phi_x \rightsquigarrow \Phi_x(y) := \psi(z-x)\varphi(x-y).$$

On the other hand, we have

$$
\begin{aligned}
\left(f*(\varphi*\psi)\right)(z) &= f\left(\tau_z(\varphi\,\check{*}\,\psi)\right) = \\
&= f\left(y \mapsto (\varphi*\psi)(z-y)\right) = \\
&= f\left(y \mapsto \int \varphi(x)\psi(z-y-x)\,\mathrm{d}x\right) = \\
&= f\left(y \mapsto \int \varphi(x-y)\psi(z-x)\,\mathrm{d}x\right) = \\
&= f\left(y \mapsto \int_{\mathbb{R}^n} \Phi(x)(y)\,\mathrm{d}x\right).
\end{aligned}
$$

Step 2. First, we notice that $\Phi(x) := \Phi_x$ is the zero function whenever $z - x \notin \operatorname{spt}\psi$, that is, when

$$x \notin z - \operatorname{spt}\psi.$$

Consequently, we have the inclusion

$$\operatorname{spt}\Phi \subset J := z - \operatorname{spt}\psi.$$

Furthermore, a point $x \in J$ satisfies $\Phi_x(y) = 0$ if and only if $x - y \notin \operatorname{spt}\varphi$, which is equivalent to requiring that

$$y \notin x - \operatorname{spt}\varphi.$$

We finally infer that

$$\operatorname{spt}\Phi_x \subset x - \operatorname{spt}(\varphi) \subset K := J - \operatorname{spt}(\varphi).$$

Step 3. The function Φ_x belongs to $\mathcal{D}_K$ and the following composition is well-defined:

$$J \xrightarrow{\Phi} \mathcal{D}_K \xrightarrow{f} \mathbb{C}.$$

We now want to compute the integral $\int_{\mathbb{R}^n} f\left(\Phi(x)\right)\, \mathrm{d}x$. The function $f \circ \Phi : J \longrightarrow \mathbb{C}$ can be uniformly approximated by a sequence of functions of the following kind:

$$\sum_{i=1}^{N} f \circ \Phi(x_i^k) \cdot \chi_{Q_i^k}.$$

Here we require Q_i^k to be the collection of cubes centered at x_i^k and with edges of length $\frac{1}{k}$ such that the disjoint union covers the set J, that is,

$$J \subseteq \bigcup_{i=1}^{N} Q_I^k.$$

It follows that

$$\int_{\mathbb{R}^n} f\left(\Phi(x)\right)\, \mathrm{d}x = \lim_{k \to +\infty} \sum_{i=1}^{N} f \circ \Phi(x_i^k) \left|Q_i^k\right| = \lim_{k \to +\infty} f(S_k),$$

where

$$S_k := \sum_{i=1}^{N} \Phi(x_i^k) \left|Q_i^k\right| = \frac{1}{k^n} \sum_{i=1}^{N} \Phi(x_i^k) \in \mathcal{D}_K.$$

Step 4. We now claim that $S_k \to \int_{\mathbb{R}^n} \Phi(x)\, \mathrm{d}x$ uniformly on K for $k \to +\infty$. Indeed, we have

$$\begin{aligned}\left| S_k(y) - \int_{\mathbb{R}^n} \Phi(x)(y)\, \mathrm{d}x \right| &\le \sum_{i=1}^{N} \int_{Q_i^k} \left|\Phi(x_i^k)(y) - \Phi(x)(y)\right|\, \mathrm{d}x \le \\ &\le \sum_{i=1}^{N} \int_{Q_i} \left|\psi(z - x_i^k)\varphi(x_i^k - y) - \psi(z-x)\varphi(x-y)\right|\, \mathrm{d}x \le \\ &\lesssim \frac{\|\varphi\|_{L^1(Q)} \|\psi\|_{L^1(Q)}}{k^n}.\end{aligned}$$

It follows immediately that $\int_{\mathbb{R}^n} \Phi(x)\, \mathrm{d}x$ is a continuous function for it is the uniform limit of continuous functions. In a similar fashion, we observe that

$$D^\alpha S_k = \frac{1}{k^n} \sum_{i=1}^{N} D^\alpha \Phi(x_i^k),$$

which, in turn, implies that

$$\|D^\alpha S_k\|_0 \le |J| \cdot \|\Phi\|_{|\alpha|}.$$

Consequently, the derivatives of the function S_k are equibounded. Thus, up to subsequences, we find from the Ascoli-Arzelà theorem that

Caution! Note that the equiboundedness of the second-order derivatives is enough to infer the equicontinuity of the first-order derivatives (and so on..)

$$D^\alpha S_k \xrightarrow{\text{uniformly on } K} D^\alpha \int_{\mathbb{R}^n} \Phi(x)\, \mathrm{d}x$$

for all multi-indices $\alpha \in \mathbb{N}^n$. This is equivalent to the notion of convergence in $\mathcal{D}_K$, and, by continuity of f, we find that

$$f(S_k) \to f\left(\int_{\mathbb{R}^n} \Phi(x)\,\mathrm{d}x\right) = f\left(y \mapsto \int_{\mathbb{R}^n} \Phi(x)(y)\,\mathrm{d}x\right),$$

which is exactly what we wanted to prove. □

The next goal is a proof of the fact that the associative property of the convolution is true under more general assumptions, (e.g., at least two out of the three elements have compact supports.)

Remark 4.8. Recall that there is a bijective correspondence between compactly supported distributions and distributions defined on C^∞ function, i.e.,

$$f \in \mathcal{D}' \text{ such that } \operatorname{spt} f \text{ is compact} \longleftrightarrow \tilde{f} \in \mathcal{E}'.$$

Lemma 4.27. *Let $f \in \mathcal{D}'$, and let $\varphi \in \mathcal{D}$. The support of the convolution is contained in the sum of the supports, that is,*

$$\operatorname{spt} f * \varphi \subseteq \operatorname{spt} f + \operatorname{spt} \varphi.$$

Proof. A point x belongs to the support of the convolution if

$$(f * \varphi)(x) \neq 0.$$

Employing the definition of convolution, this is equivalent to requiring that x satisfies

$$f\left(\tau_x \check{\varphi}\right) \neq 0,$$

which, in turn, is equivalent to the fact that the intersection of the supports is not disjoint, that is,

$$\operatorname{spt} f \cap (x + \operatorname{spt} \check{\varphi}) \neq \varnothing.$$

The symmetric $\check{\varphi}$ has support exactly equal to $-\operatorname{spt} \varphi$, and therefore

$$\operatorname{spt} f \cap (x - \operatorname{spt} \varphi) \neq \varnothing \implies x \in \operatorname{spt} f + \operatorname{spt} \varphi,$$

and this concludes the proof. □

Theorem 4.28. *Let $f \in \mathcal{E}'$ be a compactly supported distribution, and let $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{E}$. Then the convolution is associative, that is,*

$$(f * \psi) * \varphi = f * (\psi * \varphi) = (f * \varphi) * \psi. \tag{4.8}$$

Proof. As an immediate consequence of Lemma 4.27 we find that the supports of the three convolutions considered in the statement are all contained in the sum

$$\operatorname{spt} f + \operatorname{spt} \varphi + \operatorname{spt} \psi.$$

The idea is to approximate ψ with a compactly supported function $\psi_C \in \mathcal{D}$ which is sufficiently near to ψ in a specific topology. Note that, if $\psi_C \in \mathcal{D}$, then Theorem 4.26 implies the identity (4.8), i.e.,

$$(f * \psi_C) * \varphi = f * (\psi_C * \varphi) = (f * \varphi) * \psi_C.$$

Therefore, it is more than enough to prove that there is a suitable choice of ψ_C such that

$$(f * \psi_C) * \varphi = (f * \psi) * \varphi, \qquad f * (\psi_C * \varphi) = f * (\psi * \varphi)$$

$$(f * \varphi) * \psi_C = (f * \varphi) * \psi.$$

We now show that ψ_C can be chosen in such a way that the first identity holds true. The other two identities are similar, and thus they are left to the reader as an exercise. We want to prove that

$$[f * (\psi_C - \psi)] * \varphi(z) = 0$$

for all $z \in \mathbb{R}^n$. Note that this is equivalent to requiring that

$$z \notin \operatorname{spt} f + \operatorname{spt} \varphi + \operatorname{spt}(\psi_C - \psi) \iff \operatorname{spt}(\psi_C - \psi) \cap (z - \operatorname{spt} f - \operatorname{spt} \varphi) = \varnothing.$$

The supports of f and φ are compact; thus the translation of the sum,

$$z - \operatorname{spt} f - \operatorname{spt} \varphi,$$

is a compact set, which we will denote by K_z. Consequently, we can define ψ_C by setting

$$\psi_C(x) := \begin{cases} \psi(x) & \text{for all } x \in B(0,\, R), \\ \eta(x) & \text{otherwise,} \end{cases}$$

where $B(0\,R)$ is a closed ball containing K_z and $\eta(x)$ is a cut-off (smooth) function. In particular, it is easy to see that the following inclusions hold:

$$\operatorname{spt}(\psi_C - \psi) \subset B_R^c \quad \text{and} \quad B_R^c \cap (z - \operatorname{spt} f - \operatorname{spt} \varphi) = \varnothing.$$

Finally, the arbitrariness of the point z allows us to conclude that the equality at z is actually an equality between functions. □

Theorem 4.29. *Let $f,\, g \in \mathcal{D}'$ be two distributions, and let $\varphi \in \mathcal{E}$ be a smooth function. Then the convolution is associative, that is,*

$$f * (g * \varphi) = g * (f * \varphi)\,,$$

provided that one of the following conditions holds true:

- ***(a)*** *Both f and g have compact support ($f,\, g \in \mathcal{E}'$).*
- ***(b)*** *The distribution f has compact support ($f \in \mathcal{E}'$) and the function φ belongs to $\mathcal{D}$.*

The proof of this theorem relies on the following technical result, which characterises the equality of two distributions in terms of the convolution and, surprisingly, also in terms of the pointwise values.

Lemma 4.30. *Let $f,\, g \in \mathcal{D}'$ be distributions. The following properties are equivalent:*

- ***(1)*** *The distributions are equal, that is, $f = g$.*
- ***(2)*** *For all $\varphi \in \mathcal{D}$ it turns out that $f * \varphi = g * \varphi$ as functions.*
- ***(3)*** *For all $\varphi \in \mathcal{D}$ it turns out that $f * \varphi(0) = g * \varphi(0)$.*

Proof. The unique nontrivial implication is the last one. Fix $\varphi \in D$. Then

$$f(\varphi) = f\left(\tau_0 \check{\check{\varphi}}\right) = (f * \check{\varphi})(0) =$$

$$= (g * \check{\varphi})(0) = g\left(\tau_0 \check{\check{\varphi}}\right) = g(\varphi).$$

□

Proof of the Theorem 4.29. We only sketch the proof under the assumption (**b**); the other one is left to the reader as an exercise.

Proof (b). Let $\psi \in \mathcal{D}$ be a test function. By Lemma 4.30 it is enough to check that

$$[f * (g * \varphi)] * \psi = [g * (f * \varphi)] * \psi.$$

We may apply the associative results we have proved so far to both members and obtain the sought equality. Indeed, it turns out that

$$[f * (g * \varphi)] * \psi = f * [(g * \varphi) * \psi] = f * [\psi * (g * \varphi)] = (f * \psi) * (g * \varphi),$$

and also that

$$[g * (f * \varphi)] * \psi = g * [(f * \varphi) * \psi] = g * [(f * \psi) * \varphi] = (f * \psi) * (g * \varphi).$$

□

4.5 Convolution Between Distributions

The goal of this section is to generalise the notion of convolution to two distributions. The leading idea is that the resulting object should also be a distribution, and therefore we need to characterise its value on a generic test function.

Theorem 4.31. *Let $f \in \mathcal{D}'$ be a distribution. The operator defined by*

$$f_* : \mathcal{D} \to \mathcal{E}, \qquad \varphi \longmapsto f * \varphi$$

is linear, continuous and translation-invariant. Furthermore, if $L \in \mathcal{L}(\mathcal{D}, \mathcal{E})$ is a linear continuous operator which commutes with the translations, then

there exists a unique $f \in \mathcal{D}'$ such that $L = f_*$.

Proof. The operator f_* is clearly linear and translation-invariant; hence we only need to show that it is continuous with respect to the $\mathcal{D}$-topology.

Step 1. The characterisation of the continuity with respect to the $\mathcal{D}$-topology (see Theorem 3.12) asserts that f_* is continuous if and only if the restriction $f\big|_{\mathcal{D}_K}$ is continuous for all compact subsets $K \subset \mathbb{R}^n$. This is equivalent (e.g., closed graph theorem) to showing that

$$\begin{cases} \varphi_k \xrightarrow{\mathcal{D}_K} \varphi \\ f_*(\varphi_k) \xrightarrow{k \to +\infty} \psi \end{cases} \quad \overset{?}{\Longrightarrow} \quad f_*(\varphi) = \psi.$$

Now notice that the second assumption is equivalent to

$$\psi(x) = \lim_{k \to +\infty} f * \varphi_k(x) = \lim_{k \to +\infty} f\left(\tau_x \check{\varphi}_k\right).$$

The continuity of f, together with the fact that $\tau_x \varphi_k \to \tau_x \varphi$ and $\check{\varphi}_k \to \check{\varphi}$, is enough to infer that the limit above is exactly equal to $f_*(\varphi)(x)$.

Step 2. Let $L \in \mathcal{L}(\mathcal{D}, \mathcal{E})$ be a translation-invariant operator. If the distribution f exists, then it is uniquely determined by the formula

$$f(\varphi) = L(\check{\varphi})(0), \tag{4.9}$$

as a consequence of the well-known identity

$$f(\varphi) = f * \check{\varphi}(0).$$

Suppose now that f is defined by (4.9). The convolution operator is clearly linear and continuous, and therefore we only need to check that $f_* = L$. Since L is translation-invariant, we find that

$$\begin{aligned} f_*(\varphi)(x) = f * \varphi(x) = \\ = f\left(\tau_x \check{\varphi}\right) = \\ = L\left(\tau_{-x} \varphi\right) = \\ = \tau_{-x} L\varphi(0) = L\varphi(x), \end{aligned}$$

which is exactly what we wanted to prove. □

Definition 4.32 (Convolution)**.** Let $f \in \mathcal{E}'$ and let $g \in \mathcal{D}'$ be two distributions. The convolution of f and g is the unique distribution satisfying

$$(f * g)(\varphi) := f(\check{g} * \varphi) = f \circ \check{g}_*(\varphi).$$

Lemma 4.33. *Let $f \in \mathcal{E}'$ and let $g \in \mathcal{D}'$. The support of the convolution is contained in the sum of the supports, that is,*

$$\operatorname{spt} f * g \subseteq \operatorname{spt} f + \operatorname{spt} g.$$

Proof. A point φ belongs to the support of the convolution if

$$(f * g)(\varphi) \neq 0.$$

Employing the definition of convolution, this is equivalent to requiring that φ satisfies

$$f\left(\check{g}_*(\varphi)\right) \neq 0,$$

and this is clearly equivalent to requiring that the intersection of the supports is not disjoint, that is,

$$\operatorname{spt} f \cap \operatorname{spt} \check{g}_*(\varphi) \neq \varnothing.$$

The symmetric $\check{g}$ has support exactly equal to $-\operatorname{spt} g$, and therefore

$$\operatorname{spt} f \cap [\operatorname{spt} \varphi - \operatorname{spt} g] \neq \varnothing \implies (\operatorname{spt} f + \operatorname{spt} g) \cap \operatorname{spt} \varphi \neq \varnothing,$$

and this concludes the proof. □

Theorem 4.34. *Let $f, g \in \mathcal{E}'$, and let $h \in \mathcal{D}'$. Then the convolution is associative, that is,*

$$(f * g) * h = f * (g * h) .$$

Proof. By Lemma 4.30 it is enough to check that

$$(f * g) * h * \varphi = f * (g * h) * \varphi$$

for all $\varphi \in \mathcal{D}$. By definition we have that

$$(f * g) * h * \varphi = (f * g) * (h * \varphi) = f * (g * (h * \varphi)) ,$$

and, similarly, that

$$f * (g * h) * \varphi = f * ((g * h) * \varphi) = f * (g * (h * \varphi)) .$$

These expressions are identical, and thus the criterion given by the result mentioned above concludes the proof. □

Proposition 4.35. *Let $\varphi \in \mathcal{D}$, and let $f \in \mathcal{D}'$. Then*

$$\delta_0 * \varphi = \varphi \qquad \text{and} \qquad \delta_0 * f = f.$$

Proof. The first identity is obvious:

$$\delta_0 * \varphi(x) = \delta_0 (\tau_x \check{\varphi}) = \tau_x \check{\varphi}(0) = \varphi(x).$$

The second identity, on the other hand, follows immediately from the first one applying Lemma 4.30. Indeed, for any $\psi \in \mathcal{D}$ we have

$$\delta_0 * f * \psi = f * (\delta_0 * \psi) = f * \psi.$$

□

Proposition 4.36. *Let $f \in \mathcal{D}'$, and let $\alpha \in \mathbb{N}^n$ be any multi-index. Then*

$$D^\alpha f = (D^\alpha \delta_0) * f.$$

Proof. Fix $\psi \in \mathcal{D}$. A straightforward application of Proposition 4.35 shows that

$$(D^\alpha f) * \psi = (D^\alpha f) * (\delta_0 * \psi) = f * D^\alpha \delta_0 * \psi = D^\alpha \delta_0 * f * \psi,$$

and this is enough to conclude using Lemma 4.30 and the arbitrariness of ψ. □

Proposition 4.37. *Let $f \in \mathcal{D}'$, and let $g \in \mathcal{E}'$. Then*

$$D^\alpha (f * g) = (D^\alpha f) * g = f * (D^\alpha g) .$$

Proof. First, we notice that, if $g = \delta_0$, then

$$(D^\alpha f) * \varphi = f * (D^\alpha \varphi) = f * D^\alpha (\delta_0 * \varphi) = f * D^\alpha \delta_0 * \varphi,$$

from which we infer that

$$D^\alpha f = (D^\alpha \delta_0) * f.$$

It follows that, in the general case, we have

$$D^\alpha (f * g) = (D^\alpha \delta_0) * f * g = (D^\alpha f) * g,$$

which is exactly what we wanted to prove. □

4.6 Regularization by Mollification

The goal of this brief section is to use the *mollification* method to regularize a function. Let $\rho \in \mathcal{D}$ be a function with compact support

$$\operatorname{spt} \rho \subset B(0,\, 1),$$

and with unitary mass, that is,

$$\int \rho(x)\, \mathrm{d}x = 1.$$

We can define, for all $\epsilon > 0$, the *mollifier* as

$$\rho_\epsilon(x) := \frac{1}{\epsilon^n} \rho\left(\frac{x}{\epsilon}\right).$$

Clearly, the function ρ_ϵ has supported contained in the ball $B(0,\, \epsilon)$ and unitary mass.

Theorem 4.38 (Regularization)**.** *Let $\varphi \in \mathcal{D}$ and let $f \in \mathcal{D}'$. Then*

$$\varphi * \rho_\epsilon \xrightarrow{\epsilon \to 0^+} \varphi \qquad \textit{and} \qquad f * \rho_\epsilon \xrightarrow{\epsilon \to 0^+} f.$$

Proof. We first prove that the convolution product between φ and the mollifiers ρ_ϵ converges uniformly to φ as $\epsilon \to 0^+$. Fix $x \in \mathbb{R}^n$. Then

$$\begin{aligned} |\varphi * \rho_\epsilon(x) - \varphi(x)| &= \left| \int_{\mathbb{R}^n} (\varphi(x) - \varphi(y))\, \rho_\epsilon(x-y)\, \mathrm{d}y \right| \leq \\ &\leq \int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|\, \rho_\epsilon(x-y)\, \mathrm{d}y = \\ &= \int_{B_\epsilon} |\varphi(x) - \varphi(y)|\, \rho_\epsilon(x-y)\, \mathrm{d}y \leq \epsilon \cdot \|\varphi\|_1, \end{aligned}$$

and therefore $\varphi * \rho_\epsilon$ converges to φ uniformly as $\epsilon \to 0^+$. In a similar fashion, we can prove the same estimate for

$$D^\alpha\,(\varphi * \rho_\epsilon) = D^\alpha\, \varphi * \rho_\epsilon,$$

which, in turn, converges uniformly to $D^\alpha \varphi$ for all multi-indices $\alpha \in \mathbb{N}^n$. By Theorem 3.12 the uniform convergence of all derivatives is equivalent to the convergence in $\mathcal{D}$, which means that

$$\varphi * \rho_\epsilon \xrightarrow{\mathcal{D}} \varphi.$$

The second assertion now follows from the characterization Lemma 4.30. Fix $\varphi \in \mathcal{D}$. Then

$$f * \rho_\epsilon(\varphi) = f * \rho_\epsilon * \check{\varphi}(0),$$

and $\check{\rho}_\epsilon * \varphi \to \varphi$ in $\mathcal{D}$ as a consequence of the first assertion. The continuity of f is now enough to infer the thesis. $\square$

Chapter 5

The Fourier Transform

In this chapter, we introduce the Fourier transform of summable functions and prove its central properties, including the invertibility on Schwartz distributions.

5.1 Definitions and Elementary Properties

Let $\varphi \in L^1(\mathbb{R}^n)$ be a summable function. The *Fourier transform* of φ is the function defined by

$$\mathcal{F}(\varphi)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) \mathrm{e}^{-\imath \langle x,\, \xi \rangle} \, \mathrm{d}x, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product on $\mathbb{R}^n$.

Caution!

We will always denote the Fourier transform of a function φ with the symbol $\mathcal{F}(\varphi)$. The reason is that the standard notation $\hat{\varphi}$ might be confused with the overturning symbol.

Notation. We denote by e_x the exponential function

$$e_x : \mathbb{C} \ni \xi \longmapsto \mathrm{e}^{\imath \langle x,\, \xi \rangle} \in \mathbb{C},$$

and, for any $\lambda \neq 0$, we denote by h_λ the scaling

$$h_\lambda : L^1(\mathbb{R}^n) \ni \varphi \longmapsto \varphi_\lambda \in L^1(\mathbb{R}^n),$$

where

$$\varphi_\lambda(x) = \varphi\left(\frac{x}{\lambda}\right).$$

Proposition 5.1. *Let $\varphi, \psi \in L^1(\mathbb{R}^n)$ be summable functions. The Fourier transform operator satisfies the following properties:*

(a) *For all $x \in \mathbb{R}^n$ we have*

$$\mathcal{F}(\tau_x(\varphi)) = e_{-x}(\mathcal{F}(\varphi)).$$

(b) *For all $x \in \mathbb{R}^n$ we have*

$$\mathcal{F}(e_x(\varphi)) = \tau_x(\mathcal{F}(\varphi)).$$

(c) *The Fourier transform of the convolution equals a multiple of the product of the Fourier transforms, that is,*

$$\mathcal{F}(\varphi * \psi) = (2\pi)^{n/2} \left(\mathcal{F}(\varphi) \cdot \mathcal{F}(\psi)\right).$$

(d) *Let $\lambda > 0$. The Fourier transform equals the inverse scaling of the Fourier transform, that is,*

$$\mathcal{F}\left(h_\lambda(\varphi)\right) = \lambda^n h_{\frac{1}{\lambda}}(\mathcal{F}(\varphi)).$$

(e) *The Fourier transform of the overturning is the overturning of the Fourier transform, that is,*

$$\mathcal{F}(\check{\varphi}) = (-1)^n \check{\mathcal{F}}(\varphi),$$

Proof. These properties are a straightforward consequence of the definitions.

(a) The right-hand side can easily be computed using (5.1). We have

$$\begin{aligned} e_{-x}(\mathcal{F}(\varphi))(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(y) \mathrm{e}^{-\imath\langle y+x,\, \xi\rangle}\, \mathrm{d}y \overset{(*)}{=} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(z-x) \mathrm{e}^{-\imath\langle z,\, \xi\rangle}\, \mathrm{d}z = \mathcal{F}\left(\tau_x(\varphi)\right)(\xi). \end{aligned}$$

The identity $(*)$ follows from the substitution $x + y \longmapsto z$.

(b) This property is extremely similar to the previous one. Indeed, it suffices to notice that

$$\begin{aligned} \tau_x(\mathcal{F}(\varphi))(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(y) \mathrm{e}^{-\imath\langle y,\, \xi-x\rangle}\, \mathrm{d}y = \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(y) \mathrm{e}^{\imath\,\langle y,\, x\rangle} \mathrm{e}^{-\imath\langle y,\, \xi\rangle}\, \mathrm{d}y = \mathcal{F}\left(e_x(\varphi)\right)(\xi). \end{aligned}$$

(c) The left-hand side can be easily computed by using the definition of the convolution product (between functions). Indeed, it turns out that

$$\begin{aligned} \mathcal{F}(\varphi * \psi)(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\varphi * \psi)(x) \mathrm{e}^{-\imath\langle x,\, \xi\rangle}\, \mathrm{d}x = \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} (\varphi(y)\psi(x-y))\, \mathrm{d}y\right] \mathrm{e}^{-\imath\langle x,\, \xi\rangle}\, \mathrm{d}x \overset{(*)}{=} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left(\psi(x-y) \mathrm{e}^{-\imath\langle x-y,\, \xi\rangle}\right) \mathrm{d}x\right] \varphi(y) \mathrm{e}^{-\imath\langle y,\, \xi\rangle}\, \mathrm{d}y = \\ &= \mathcal{F}(\psi)(\xi) \cdot \int_{\mathbb{R}^n} \varphi(y) \mathrm{e}^{-\imath\langle y,\, \xi\rangle}\, \mathrm{d}y = (2\pi)^{\frac{n}{2}}\, \mathcal{F}(\varphi) \cdot \mathcal{F}(\psi)(\xi). \end{aligned}$$

The identity $(*)$ follows from a simple application of the Fubini theorem.

(d) This property follows straightly from the definition (5.1):

$$\begin{aligned}\mathcal{F}\left(h_\lambda(\varphi)\right)(\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi\left(\frac{x}{\lambda}\right)\mathrm{e}^{-\imath\langle x,\,\xi\rangle}\,\mathrm{d}x = \\ &= \frac{\lambda^n}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(z)\mathrm{e}^{-\imath\langle \lambda\cdot z,\,\xi\rangle}\,\mathrm{d}z = \\ &= \frac{\lambda^n}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(z)\mathrm{e}^{-\imath\langle z,\,\lambda\cdot\xi\rangle}\,\mathrm{d}z = \lambda^n h_{\frac{1}{\lambda}}\left(\mathcal{F}(\varphi)\right)(\xi).\end{aligned}$$

(e) This property also follows straightly from the definition (5.1):

$$\begin{aligned}\mathcal{F}\left(\check{\varphi}\right)(\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\check{\varphi}\,(x)\,\mathrm{e}^{-\imath\langle x,\,\xi\rangle}\,\mathrm{d}x = \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(-x)\mathrm{e}^{-\imath\langle x,\,\xi\rangle}\,\mathrm{d}x = \\ &= (-1)^n\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(x)\mathrm{e}^{-\imath\langle x,\,-\xi\rangle}\,\mathrm{d}x = (-1)^n\check{\mathcal{F}}(\varphi)(\xi).\end{aligned}$$

□

5.2 The Fourier Transform in $\mathscr{S}$ and in $\mathscr{S}'$

The goal of this section is to investigate the properties of the Fourier transform on a specific subset of $\mathcal{E}$, the Schwartz space $\mathscr{S}$, which consists of "rapidly decreasing" functions. Let $\varphi \in \mathcal{E}$ be a smooth function. Set

$$q_{\alpha,\,\beta}(\varphi) := \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta\varphi(x)\right|,$$

where $\alpha,\,\beta \in \mathbb{N}^n$ are multi-indices. The collection of seminorms $\mathcal{P} := \{q_{\alpha,\,\beta}\}_{\alpha,\,\beta\in\mathbb{N}^n}$ is well-defined on the Schwartz space

$$\mathscr{S} := \{\varphi \in \mathcal{E} \mid q_{\alpha,\,\beta}(\varphi) < +\infty,\ \forall\,\alpha,\,\beta \in \mathbb{N}^n\}\,.$$

In particular, $\mathcal{P}$ induces on $\mathscr{S}$ a structure of topological vector space $(\mathscr{S},\,\tau_{\mathscr{S}})$, where τ is a locally convex topology - see Theorem 2.38 -.

Lemma 5.2. *The inclusions*

$$(\mathcal{D},\,\tau_{\mathcal{D}}) \hookrightarrow (\mathscr{S},\,\tau_{\mathscr{S}}) \hookrightarrow \left(L^1(\mathbb{R}^n),\,\|\cdot\|_1\right)$$

are continuous.

Proof. We divide the proof into two steps.

Step 1. Let φ be a smooth compactly supported function. Then

$$q_{\alpha,\beta}(\varphi) = \sup_{x \in \operatorname{spt}\varphi} \left| x^\alpha D^\beta \varphi(x) \right| \leq C(\operatorname{spt}\varphi) \cdot \|\varphi\|_{|\beta|}$$

for all $\alpha,\, \beta \in \mathbb{N}^n$, which means that the inclusion $\mathcal{D} \hookrightarrow \mathscr{S}$ is continuous.

Step 2. Let $\varphi \in \mathscr{S}$ be any rapidly decreasing function. Recall that

$$H(x) := \left(\frac{1}{1+|x|^2} \right)^n$$

belongs to $L^1(\mathbb{R}^n)$. To obtain a bound for the L^1-norm of φ, we multiply and divide everything by $H(x)$, that is,

$$\|\varphi\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\varphi(x)| \, \mathrm{d}x = \int_{\mathbb{R}^n} \frac{(1+|x|^2)^n \, |\varphi(x)|}{(1+|x|^2)^n} \, \mathrm{d}x.$$

Now the "new" numerator is uniformly bounded and the "new" denominator is summable. More precisely, if we expand the binomial nth power (Newton formula), we find that

$$\|\varphi\|_{L^1(\mathbb{R}^n)} \lesssim \left\| \frac{1}{(1+|x|^2)^n} \right\|_{L^1(\mathbb{R}^n)} \sum_{|\alpha| \leq 2n} q_{\alpha,0}(\varphi) < +\infty,$$

that is, the inclusion is continuous. □

Theorem 5.3. *The Schwartz space $\mathscr{S}$ is a Fréchet space (=metrizable and complete).*

Proof. The space $\mathscr{S}$ is linearly metrizable because the family of seminorms $\mathcal{P} := \{q_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ is countable - see Theorem 2.28 -.

Now let $(\varphi_k)_{k \in \mathbb{N}} \subset \mathscr{S}$ be a Cauchy sequence in the Schwartz space. Then, for all $\alpha,\, \beta \in \mathbb{N}^n$, the sequence

$$\left(x^\alpha D^\beta \varphi_k \right)_{k \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$$

is a Cauchy sequence in $C^0(\mathbb{R}^n)$. By completeness, it converges uniformly to a function $\psi_{\alpha,\beta} \in C^0(\mathbb{R}^n)$, that is,

$$x^\alpha D^\beta \varphi_k \rightrightarrows \psi_{\alpha,\beta}.$$

To conclude the proof the reader should prove that for all $\alpha,\, \beta \in \mathbb{N}^n$, we have

$$\psi_{\alpha,\beta} = x^\alpha D^\beta \, \psi_{0,0} \quad \text{and} \quad \psi_{0,0} = \lim_{k \to +\infty} \varphi_k.$$

□

Proposition 5.4. *The following operators from $\mathscr{S}$ to $\mathscr{S}$ are continuous:*

$$\varphi \longmapsto (\cdot)^\gamma \varphi, \quad \psi \longmapsto \varphi \cdot \psi \quad \text{and} \quad \varphi \longmapsto D^\delta \varphi.$$

Proof. Let $\alpha,\, \beta \in \mathbb{N}^n$ be multi-indices. It follows from Leibniz formula that

$$D^\beta \left(x^\gamma \varphi \right) = \sum_{\delta \leq \beta} \binom{\beta}{\delta} D^\delta \left(x^\gamma \right) D^{\beta - \delta} \varphi.$$

therefore

$$q_{\alpha,\,\beta}\left(x^\gamma\,\varphi\right) \leq C\cdot\sum_{\delta\leq\beta}\binom{\beta}{\delta}\,q_{\gamma+\alpha-\delta,\,\beta-\delta}(\varphi),$$

i.e., the operator $\varphi\mapsto x^\gamma\,\varphi$ is continuous. On the other hand, the continuity of the latter operator (derivative) is obvious since

$$q_{\alpha,\,\beta}\left(D^\delta\,\varphi\right) = q_{\alpha,\,\beta+\delta}(\varphi),$$

therefore we only need to prove that the multiplication operator is continuous. As before, by Leibniz chain derivatives formula, it turns out that

$$D^\beta\left(\varphi\,\psi\right) = \sum_{\delta\leq\beta}\binom{\beta}{\delta}\,D^\delta\varphi\,D^{\beta-\delta}\,\psi,$$

and therefore

$$q_{\alpha,\,\beta}\left(\varphi\,\psi\right) \leq \sum_{\delta\leq\beta}\binom{\beta}{\delta}\,q_{\alpha,\,\delta}(\varphi)\cdot q_{\alpha,\,\beta-\delta}(\psi).$$

□

Theorem 5.5. *Let $\varphi\in\mathscr{S}$ be a Schwartz function. Then the following two properties hold true:*

(a) $\mathcal{F}\left(D^\alpha\,\varphi\right)(\xi) = (\imath)^{|\alpha|}\,\xi^\alpha\,\mathcal{F}(\varphi)(\xi)$, *for any multi-index $\alpha\in\mathbb{N}^n$.*

(b) $\mathcal{F}\left(x^\alpha\,\varphi\right)(\xi) = (\imath)^{|\alpha|}\,D^\alpha\,\mathcal{F}(\varphi)(\xi)$, *for any multi-index $\alpha\in\mathbb{N}^n$.*

Proof. Clearly it suffices to prove that both identities hold true when $\alpha = j\in\{1,\,\ldots,\,n\}$. Indeed, the thesis follows from a standard application of the induction principle.

(a) We compute the left-hand side using the definition of the Fourier transform (5.1):

$$\mathcal{F}\left(D^j\,\varphi(x)\right)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{x\in\mathbb{R}^n} D^j\,\left(\varphi(x)\right)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x \overset{(*)}{=}$$

$$= \int_{x\in\mathbb{R}^n}\varphi(x)\,D^j\left(\mathrm{e}^{-\imath\,(x,\,\xi)}\right)\,\mathrm{d}x = \imath\,\xi_j\,\mathcal{F}(\varphi)(\xi),$$

where the identity $(*)$ follows from a straightforward application of the integration by parts.

N.B. The latter equality is possible only because the border term vanishes (here we use that φ is a Schwartz function). More precisely, we have that

$$\int_{x\in\mathbb{R}^n} D^j\,\left(\varphi(x)\right)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x = \lim_{R\to+\infty}\int_{x\in B_R(0)} D^j\,\left(\varphi(x)\right)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x,$$

thus, integrating by parts, we obtain

$$\int_{x\in B_R(0)} D^j\,\left(\varphi(x)\right)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x = \int_{\partial\,B_R(0)} D^j\,\left(\varphi(x)\right)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\nu_j(x)\,\mathrm{d}\Sigma(x) + \ldots.$$

The border term can be estimated by $c\,R^{N-1}\,q_{\alpha,\,0}(\varphi)$, and this goes to 0 as R goes to $+\infty$ because $\varphi\in\mathscr{S}$ decreases faster than any polynomial.

(b) We compute the left-hand side using the definition of the Fourier transform (5.1):

$$\begin{aligned}\mathcal{F}\left(x_j\,\varphi(x)\right)(\xi) &= \int_{\mathbb{R}^n} x_j\,\varphi(x)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x = \\ &= \imath\int_{\mathbb{R}^n}\varphi(x)\,\frac{\partial}{\partial\,\xi_j}\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x = \\ &= \imath\,\frac{\partial}{\partial\,\xi_j}\,\mathcal{F}(\varphi)(\xi),\end{aligned}$$

where the swap between the derivative and the integral is possible as a consequence of the dominate convergence theorem (since $\varphi \in \mathscr{S}$, for any $\alpha \in \mathbb{N}^n$ the product $x^\alpha\,\varphi$ is summable).

□

Theorem 5.6. *The inclusion $\mathcal{F}(\mathscr{S}) \subseteq \mathscr{S}$ is continuous.*

Proof. Let $\varphi \in \mathscr{S}$ be a Schwartz function. For any $\alpha,\, \beta \in \mathbb{N}^n$ we have that

$$\begin{aligned}\left|\xi^\alpha\,D^\beta\,\mathcal{F}\left(\varphi(\xi)\right)\right| &= \left|\xi^\alpha\,\mathcal{F}(x^\beta\,\varphi)(\xi)\right| = \\ &= \left|\mathcal{F}\left(D^\alpha\,x^\beta\,\varphi\right)(\xi)\right| \le \\ &\le (2\pi)^{-n/2}\,\left\|D^\alpha\left(x^\beta\,\varphi(x)\right)\right\|_{L^1(\mathbb{R}^n)} < +\infty,\end{aligned}$$

and this is enough to conclude that the inclusion is continuous. □

Theorem 5.7. *The inclusion*

$$\mathcal{F}\left(L^1(\mathbb{R}^n)\right) \hookrightarrow C_0(\mathbb{R}^n)$$

is continuous.

Proof. It is a well-known fact that the inclusion

$$\mathcal{F}\left(L^1(\mathbb{R}^n)\right) \hookrightarrow L^\infty(\mathbb{R}^n),$$

is continuous. Since C_0 is the closure of $\mathscr{S}$ with respect to the L^∞-norm, it follows from Theorem 5.6 that

$$\mathcal{F}\left(L^1(\mathbb{R}^n)\right) = \mathcal{F}\left(\overline{\mathscr{S}}^{L^1}\right) \hookrightarrow \overline{\mathcal{F}(\mathscr{S})}^{L^\infty} \hookrightarrow \overline{\mathscr{S}}^{L^\infty} = C_0$$

is a continuous inclusion, being the composition of continuous inclusions. □

5.2.1 Inverse Fourier Transform from $\mathscr{S} \to \mathscr{S}$

In this subsection we want to prove that the Fourier transform $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is an invertible operator, and we can explicitly compute the inverse through the overturning operation.

Gaussian Distribution. The function

$$G(x) = \mathrm{e}^{-\frac{|x|^2}{2}}, \tag{5.2}$$

called *Gaussian distribution*, is an element of the Schwartz space $\mathscr{S}$ and it is a fixed point for the Fourier transform operator, i.e.,

$$\mathcal{F}(G)(\xi) = G(\xi) \qquad \forall\, \xi \in \mathbb{R}^n.$$

This fact is well-known, and it is usually proved in lower courses; Hence we shall give this for granted for the rest of the chapter.

Lemma 5.8. *Let φ, $\psi \in L^1(\mathbb{R}^n)$ be summable functions. The Fourier transform operator is self-adjoint, that is, the following equality holds true:*

$$\int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\, \psi(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x)\, \mathcal{F}(\psi)(x)\, \mathrm{d}x. \tag{5.3}$$

Proof. This result is a simple consequence of the Fubini theorem. Indeed, if we evaluate the left-hand side, then it turns out that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\, \psi(x)\, \mathrm{d}x &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\varphi(t)\, \mathrm{e}^{-\imath\, (x,\, t)}\right) \mathrm{d}t\, \psi(x)\, \mathrm{d}x = \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\psi(x)\, \mathrm{e}^{-\imath\, (x,\, t)}\right) \mathrm{d}x\, \varphi(t)\, \mathrm{d}t = \\
&= \int_{\mathbb{R}^n} \varphi(x)\, \mathcal{F}(\psi)(x)\, \mathrm{d}x.
\end{aligned}$$

□

Theorem 5.9 (Inversion Theorem)**.** *Let $\varphi \in \mathscr{S}$ be a Schwartz function. Then for any $x \in \mathbb{R}^n$ it turns out that*

$$\mathcal{F} \circ \mathcal{F}(\varphi)(x) = \check{\varphi}(x).$$

Proof. Let $G(x)$ be the Gaussian distribution. For any $\lambda > 0$ we define G_λ to be its λ-rescaling, that is,

$$G_\lambda(x) := G\left(\frac{x}{\lambda}\right).$$

The Fourier transform of G_λ can be explicitly computed (see Proposition 5.1) starting from the Fourier transform of G:

$$\mathcal{F}(G_\lambda)(\xi) = \lambda^n\, \mathcal{F}(G)(\lambda\, \xi).$$

If we set $\psi = G_\lambda$, then the right-hand side of (5.3) becomes

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi(x)\, \mathcal{F}(G_\lambda)(x)\, \mathrm{d}x &= \lambda^n \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|\lambda\, x|^2}{2}}\, \varphi(x)\, \mathrm{d}x = \\
&= \int_{\mathbb{R}^n} \varphi\left(\frac{s}{\lambda}\right) \mathrm{e}^{-\frac{s^2}{2}}\, \mathrm{d}s.
\end{aligned}$$

Clearly $\varphi_\lambda(s) \to \varphi(0)$ pointwise (when $\lambda \to +\infty$), hence, by the Lebesgue dominated convergence theorem, it turns out that

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathcal{F}(G_\lambda)(x)\,\mathrm{d}x \xrightarrow{\lambda\to+\infty} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{s^2}{2}}\,\varphi(0)\,\mathrm{d}s = (2\pi)^{n/2}\,\varphi(0).$$

Analogously, we compute the left-hand side of (5.3) becomes

$$\int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\,G_\lambda(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x)\,G_\lambda(x)\,\mathrm{e}^{-\imath\,(x,\,\xi)}.$$

Clearly $G_\lambda(x) \to G(0) = 1$ (when $\lambda \to +\infty$), hence, by the Lebesgue dominated convergence theorem, it turns out that

$$\int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\,G_\lambda(x)\,\mathrm{d}x \xrightarrow{\lambda\to+\infty} (2\pi)^{n/2} \int_{\mathbb{R}^n} \mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi = (2\pi)^{n/2}\,\mathcal{F}\left(\mathcal{F}(\varphi)\right)(0).$$

We infer that $\varphi(0) = \mathcal{F}^2(\varphi)(0)$, and this is enough to conclude since

$$\mathcal{F}^2(\varphi)(x) = \tau_{-x}\left(\mathcal{F}^2(\varphi)\right)(0) = \tau_x\,\varphi(0) = \varphi(-x) = \check{\varphi}(x).$$

□

Corollary 5.10. *The Fourier operator $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is invertible, and the inverse is defined by*

$$\mathcal{F}^{-1}(\varphi) = \mathcal{F}\left(\check{\varphi}\right).$$

In particular, the order of $\mathcal{F}$ is four (i.e. $\mathcal{F}^4 = \mathrm{I}_{\mathscr{S}}$).

The Fourier transform is naturally defined on the space of summable functions $L^1(\mathbb{R}^n)$, therefore it makes sense to ask if a similar result also holds in a more general setting.

The answer, surprisingly, is that there is no difference. More precisely, the operator $\mathcal{F}$ can be inverted from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, and the formula is the same (though, it holds only almost everywhere).

Theorem 5.11 (Inversion Theorem)**.** *Let $\varphi \in L^1(\mathbb{R}^n)$ be a summable function. If $\mathcal{F}(\varphi) \in L^1(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$*

$$\mathcal{F} \circ \mathcal{F}(\varphi)(x) = \check{\varphi}(x).$$

Proof. Let $\psi \in \mathscr{S}$ be any Schwartz function. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x)\,\psi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} \varphi(x)\,\mathcal{F}^2\left(\check{\psi}\right)\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n} \mathcal{F}^2(\varphi)(x)\,\check{\psi}(x)\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n} \mathcal{F}^2\left(\check{\varphi}\right)(x)\,\psi(x)\,\mathrm{d}x,\end{aligned}$$

that is, the distribution associated Λ_φ is equal to the distribution $\Lambda_{\mathcal{F}^2(\check{\varphi})}$. This final identity concludes the proof, as a consequence of the so-called fundamental lemma of calculus of variations[1].

Indeed, $\varphi \in L^1(\mathbb{R}^n)$ and $\mathcal{F}^2(\check{\varphi})$ may not be a summable function, but it surely belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$ hence the lemma can be applied. □

Theorem 5.12. *Let φ, $\psi \in \mathscr{S}$ be two Schwartz functions. Then the convolution $\varphi * \psi$ belongs to $\mathscr{S}$ and the Fourier transform of the product is, up to a constant, the convolution of the Fourier products, that is,*

$$\mathcal{F}(\varphi \cdot \psi) = \frac{1}{(2\pi)^{n/2}} \mathcal{F}(\varphi) * \mathcal{F}(\psi).$$

Proof. We have already proved (see Proposition 5.1) that

$$\mathcal{F}(\varphi * \psi) = (2\pi)^{n/2} \mathcal{F}(\varphi) \cdot \mathcal{F}(\psi),$$

therefore we can also apply this formula to $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$, since the Fourier transform operator sends $\mathscr{S}$ to $\mathscr{S}$. It follows that

$$\begin{aligned} \mathcal{F}(\mathcal{F}(\varphi) * \mathcal{F}(\psi)) &= (2\pi)^{n/2} \mathcal{F}^2(\varphi) \cdot \mathcal{F}^2(\psi) = \\ &= (2\pi)^{n/2} \check{\varphi} \cdot \check{\psi}, \end{aligned}$$

and, if we apply the Fourier transform again, it turns out to be the sought identity:

$$\mathcal{F}(\varphi \cdot \psi) = \frac{1}{(2\pi)^{n/2}} \mathcal{F}(\varphi) * \mathcal{F}(\psi).$$

□

5.2.2 Extension to $L^2(\mathbb{R}^n)$

The main goal of this brief subsection is to extend the notion of Fourier transform to $L^2(\mathbb{R}^n)$ function, in such a way that the it coincides with (5.1) whenever $\varphi \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Theorem 5.13 (Plancherel)**.** *The Fourier transform operator $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ can be uniquely extended to an isometry $\tilde{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Proof. We first show that $\mathcal{F} : \mathscr{S} \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometry. It follows easily from the

[1]**Lemma.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. If

$$\int_{\mathbb{R}^n} f(x)\,\varphi(x)\,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^n),$$

then $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

following chain of equalities:

$$\begin{aligned}\langle \varphi, \psi \rangle_{L^2} &= \int_{\mathbb{R}^n} \varphi(x)\,\overline{\psi(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{F}^2\left(\check{\varphi}\right)(x)\,\overline{\psi(x)}\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n} \mathcal{F}\left(\check{\varphi}\right)(x)\,\mathcal{F}\left(\overline{\psi}\right)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{F}\left(\check{\varphi}\right)(x)\,\overline{\mathcal{F}(\psi)(-x)}\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\,\overline{\mathcal{F}(\psi)(x)}\,\mathrm{d}x = \langle \mathcal{F}(\varphi), \mathcal{F}(\psi) \rangle_{L^2}.\end{aligned}$$

The Schwartz functions space is clearly dense in L^2, therefore the isometry $\mathcal{F} : \mathscr{S} \to L^2(\mathbb{R}^n)$ can be continuously extended in a unique way to an operator $\tilde{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is also an isometry. □

Theorem 5.14. *The extension $\tilde{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is well defined. More precisely, for any $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ it turns out that*

$$\mathcal{F}(\varphi) = \tilde{\mathcal{F}}(\varphi).$$

Proof. Let ρ_ϵ be a mollifier and let us set $\varphi_\epsilon := \varphi * \rho_\epsilon$. Clearly $\varphi_\epsilon \in \mathscr{S}$ is a Schwartz function, and it converges both in $L^2(\mathbb{R}^n)$ and in $L^1(\mathbb{R}^n)$ to φ, as $\epsilon \to 0^+$. Moreover, for any $\epsilon > 0$

$$\mathcal{F}(\varphi_\epsilon) = \tilde{\mathcal{F}}(\varphi_\epsilon),$$

therefore we conclude the proof by noticing that the first term converges in $L^\infty(\mathbb{R}^n)$ to $\mathcal{F}(\varphi)$, while the second term converges in $L^2(\mathbb{R}^n)$ to $\tilde{\mathcal{F}}(\varphi)$. □

Remark 5.1. In the previous proof, to conclude that $\mathcal{F}(\varphi) = \tilde{\mathcal{F}}(\varphi)$, we implicitly used a nontrivial fact. Indeed, for any $1 \le p \le +\infty$, from the proof of the completeness of L^p, one also shows that it is always possible to extract a subsequence converging pointwise almost everywhere.

5.3 Tempered Distributions

The primary goal of this section is to give a formal definition of Fourier transform of a distribution, which is also compatible with the one introduced already. The obvious idea would be to set

$$\mathcal{F}(f)(\psi) := f\left(\mathcal{F}(\psi)\right), \tag{5.4}$$

but, unfortunately, this relation does not make any sense if $f \in \mathcal{D}'$. To avoid this issue we restrict the Fourier transform operator to the dual of Schwartz functions $\mathscr{S}'$, which is usually referred to as *space of tempered distributions.*

Theorem 5.15. *The inclusions $\mathcal{D} \hookrightarrow \mathscr{S} \hookrightarrow \mathcal{E}$ are continuous and dense.*

Proof. Let $\varphi \in \mathscr{S}$ be a Schwartz function, and let $\psi \in \mathcal{D}$ be a cut-off function, i.e., a function such that $\psi\big|_{B(0,1)} \equiv 1$ and $\mathrm{spt}(\psi)$ is compact. Let us set

$$\psi_r(x) := \psi\left(\frac{x}{r}\right),$$

so that ψ_r is identically equal to 1 on the ball of radius r. Clearly $\varphi \cdot \psi_r$ is a compactly supported function, and it is equal to φ on the ball of radius $r > 0$. Now notice that

$$\begin{aligned}
\left|x^\alpha D^\beta \left(\varphi - \varphi \cdot \psi_r\right)(x)\right| = \left|x^\alpha D^\beta \varphi(x) \left(1 - \psi_r\right)(x)\right| \leq \\
\leq \left|x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^{\beta-\gamma} \varphi(x) D^\gamma \left(1 - \psi_r\right)(x)\right| \leq \\
\leq \sup_{|x| \geq r} \left|x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^{\beta-\gamma} \varphi(x) D^\gamma \left(1 - \psi\right)(x)\right| \leq \\
\leq C_r \, \|\psi\|_{|\beta|},
\end{aligned}$$

and this last quantity goes to 0 as $r \to +\infty$, i.e., $\mathscr{S}$ is dense in $\mathcal{D}$.

Finally recall that by Theorem 3.13 it follows that $\mathcal{D} \hookrightarrow \mathscr{S}$ is continuous if and only if $\mathcal{D}_K \hookrightarrow \mathscr{S}$ for any $K \subset \mathbb{R}^n$ compact, and this is obvious because x^α is bounded on any K compact.

The continuity can be proved in the same way, but it is much simpler, therefore we leave it to the reader. The density, on the other hand, is obvious since $\mathcal{D} \subseteq \mathcal{E}$ is dense. □

Example 5.1. We can now exhibit the first examples of tempered distribution, starting from the main examples in $\mathcal{D}'$.

(1) Let μ be a measure defined on $\mathbb{R}^n$. If there exists $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} \frac{1}{\left(1 + |x|^2\right)^N} \, \mathrm{d}\mu(x) \leq C < +\infty, \tag{5.5}$$

then $\Lambda_\mu \in \mathscr{S}'$ is a tempered distribution.

Indeed, for any $\varphi \in \mathscr{S}$ it turns out that

$$|\Lambda_\mu(\varphi)| = \left|\int_{\mathbb{R}^n} \varphi(x) \, \mathrm{d}\mu(x)\right| \leq \int_{\mathbb{R}^n} |\varphi(x)| \left(1 + |x|^2\right)^N \frac{1}{\left(1 + |x|^2\right)^N} \, \mathrm{d}\mu(x).$$

On the other hand, we can estimate the first term by the seminorms (using the binomial formula):

$$|\varphi(x)| \left(1 + |x|^2\right)^N = \sum_{i \leq N} \binom{N}{i} x^{2i} \, |\varphi(x)| \leq \sum_{i \leq N} \binom{N}{i} q_{2i,\,0}(\varphi).$$

The distribution Λ_μ is thus bounded since

$$|\Lambda_\mu(\varphi)| \leq \sum_{i \leq N} \binom{N}{i} q_{2i,\,0}(\varphi) \int_{\mathbb{R}^n} \frac{1}{\left(1 + |x|^2\right)^N} \, \mathrm{d}\mu(x) \leq C \cdot \sum_{i \leq N} \binom{N}{i} q_{2i,\,0}(\varphi) < +\infty,$$

and it is thus continuous. The linearity is obvious by the definition, so we conclude that Λ_μ is a tempered distribution provided that μ satisfies the condition (5.5).

(2) Let $f \in L^p(\mathbb{R}^n)$ for some $1 \le p \le +\infty$. Then $\Lambda_f \in \mathscr{S}'$ is a tempered distribution.

Indeed, for any $\varphi \in \mathscr{S}$ it turns out that

$$|\Lambda_f(\varphi)| = \left| \int_{\mathbb{R}^n} \varphi(x)\, f(x)\, \mathrm{d}x \right| \le \int_{\mathbb{R}^n} |\varphi(x)|\; |f(x)|\; \mathrm{d}x.$$

On the other hand, any Schwartz function belongs to $L^q(R^n)$ for $q \in [1, +\infty]$ since

$$\|\varphi\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\varphi(x)|^q \; \mathrm{d}x = \int_{\mathbb{R}^n} |\varphi(x)|^q \; \left(1 + |x|^2\right)^{nq} \frac{1}{(1 + |x|^2)^{nq}}\, \mathrm{d}x,$$

and, clearly, we can estimate the first term by the seminorms as we have already done before. If follows that Λ_f is bounded, since we can apply the Holder inequality:

$$|\Lambda_f(\varphi)| \le \|f\|_{L^p(\mathbb{R}^n)} \cdot \|\varphi\|_{L^q(\mathbb{R}^n)} < +\infty,$$

thus Λ_f is continuous with respect to the topology of $\mathscr{S}$, and thus we infer that $\Lambda_f \in \mathscr{S}'$.

(3) Let $\varphi(x) = x^\alpha$ be a monomial. Then $\Lambda_\varphi \in \mathscr{S}'$ is a distribution for any multi-index $\alpha \in \mathbb{N}^n$.

Proposition 5.16. *Let $f \in \mathscr{S}'$ be a tempered distribution.*

(1) For any multi-index $\alpha \in \mathbb{N}^n$, the product $x^\alpha f \in \mathscr{S}'$.

(2) For any multi-index $\beta \in \mathbb{N}^n$, the derivative $D^\beta f \in \mathscr{S}'$.

(3) For any Schwartz function $\psi \in \mathscr{S}'$, the product $\psi f \in \mathscr{S}'$.

Proof. We prove only one of these properties since the same argument works fine with minor adjustments. Recall that (see Proposition 5.4) the application

$$\mathscr{S} \ni \varphi \longmapsto x^\alpha \varphi \in \mathscr{S}$$

is continuous for any multi-index α. The distribution f is continuous on $\mathscr{S}$ by definition, hence the composition

$$\mathscr{S} \ni \varphi \longmapsto x^\alpha \varphi \longmapsto f\,(x^\alpha \varphi) \in \mathbb{C}$$

is continuous; we conclude by noticing that

$$f\,(x^\alpha \varphi) = x^\alpha f(\varphi), \qquad \forall\, \varphi \in \mathscr{S}.$$

□

Proposition 5.17. *Let $f \in L^1(\mathbb{R}^n)$ be a summable function. Then the Fourier transform of the distribution associated to f is exactly equal to the distribution associated to the Fourier transform of f, that is,*

$$\mathcal{F}\,(\Lambda_f) = \Lambda_{\mathcal{F}(f)}.$$

Proof. Let $\varphi \in \mathscr{S}$ be a test function. The identity is a straightforward consequence of the definitions:

$$\begin{aligned}
\Lambda_{\mathcal{F}(f)}(\varphi) &= \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi)\,\varphi(\xi)\,\mathrm{d}\xi = \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}x\,\varphi(\xi)\,\mathrm{d}\xi = \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(\xi)\,\mathrm{e}^{-\imath\,(x,\,\xi)}\,\mathrm{d}\xi\,f(x)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x)\,f(x)\,\mathrm{d}x = \Lambda_f\left(\mathcal{F}(\varphi)\right) = \mathcal{F}(\Lambda_f)(\varphi).
\end{aligned}$$

□

The same property holds true for the extended Fourier transform $\tilde{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and the proof is exactly the same.

Proposition 5.18. *Let $f \in L^2(\mathbb{R}^n)$ be a square-summable function. Then*

$$\mathcal{F}\left(\Lambda_f\right) = \Lambda_{\tilde{F}(f)}.$$

Example 5.2. Let δ_0 be the Dirac delta concentrated at the origin. The Fourier transform of δ_0 is, as expected, the distribution associated to the constant 1:

$$\mathcal{F}(\delta_0)(\varphi) = \delta_0\left(\mathcal{F}(\varphi)\right) = \mathcal{F}(\varphi)(0) = \int_{\mathbb{R}^n} 1\cdot\varphi(x)\,\mathrm{d}x = \Lambda_1(\varphi).$$

Theorem 5.19 (Inversion Theorem)**.** *The Fourier transform $\mathcal{F} : \mathscr{S}' \to \mathscr{S}'$ is a linear, continuous and bijective operator such that*

$$\mathcal{F}^2(\varphi) = \check{\varphi}, \qquad \forall\,\varphi \in \mathscr{S}.$$

Proof. The identity easily follows from the equivalent one for the Fourier transform of Schwartz functions, that is,

$$\mathcal{F}^2\,f(\varphi) = f\left(\mathcal{F}^2(\varphi)\right) = f\left(\check{\varphi}\right) = \check{f}(\varphi), \qquad \forall\,\varphi \in \mathscr{S}.$$

The operator is linear and bijective, hence it is enough to prove that it is continuous at the point 0. Recall that the topology τ' is generated by the valuations $|\lambda_\varphi|$, where

$$|\lambda_\varphi|(f) := |f(\varphi)|\,.$$

More precisely, each neighborhood W of the origin contains a finite intersection of the rescaled balls associated to the seminorms, that is,

$$W \supseteq \{f \in \mathscr{S}' \mid |f(\varphi_k)| \le \epsilon_k \text{ for } k = 1,\,\dots,\,N\} = \bigcap_{i=1}^N B_{\epsilon_i}(\varphi_i).$$

If we set

$$V := \{f \in \mathscr{S}' \mid |f\left(\mathcal{F}(\varphi_k)\right)| \le \epsilon_k \text{ for } k = 1,\,\dots,\,N\}\,,$$

then $\mathcal{F}(V) \subseteq W$, i.e., the operator $\mathcal{F}$ is continuous at the origin. □

The Fourier transform, defined on the tempered distributions space, satisfies the same properties of the operator $\mathcal{F} : \mathscr{S} \to \mathscr{S}$. We do not give the proof since they follow easily from the definitions.

Proposition 5.20. *Let $f \in \mathscr{S}'$ be a tempered distribution.*

(1) For any multi-index $\alpha \in \mathbb{N}^n$, it turns out that

$$\mathcal{F}\left(D^\alpha f\right) = (\imath\,\xi)^\alpha\,\mathcal{F}(f),$$

$$\mathcal{F}\left(x^\alpha f\right) = (\imath\,D)^\alpha\,\mathcal{F}(f).$$

(2) For any $x \in \mathbb{R}^n$, it turns out that

$$\mathcal{F}\left(\tau_x f\right) = e_{-x}\,\mathcal{F}(f),$$

$$\mathcal{F}\left(e_x f\right) = \tau_x\,\mathcal{F}(f).$$

(3) The Fourier transform of the symmetric is the symmetric of the Fourier transform, that is,

$$\mathcal{F}\left(\check{f}\right) = \check{\mathcal{F}}(f).$$

Definition 5.21 (Convolution)**.** Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi \in \mathscr{S}$ be a Schwartz function. The convolution at the point $x \in \mathbb{R}^n$ is defined by

$$f * \varphi(x) := f\left(\tau_x\,\check{\varphi}\right).$$

This definition is coherent with the one we have already given in the case $f \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$. To conclude the chapter, we need to prove that the same properties holds true.

The proofs are essentially the same, therefore we will not give any detail - except when it is necessary (or when there are stronger properties that are not totally trivial).

Theorem 5.22. *Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi \in \mathscr{S}$ be a Schwartz function. Then the convolution is a function which belongs to $\mathcal{E}$.*

Theorem 5.23. *Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi \in \mathscr{S}$ be a Schwartz function. For any multi-index $\alpha \in \mathbb{N}^n$, it turns out that*

$$D^\alpha\left(f * \varphi\right) = f * D^\alpha\,\varphi = D^\alpha\,f * \varphi.$$

Proof. The main idea is essentially the same, but we also need to use a Lemma asserting that for any $\varphi \in \mathscr{S}$ the incremental ratio converges in $\mathscr{S}$ to the directional derivative, that is,

$$\frac{\varphi(x + h\,e_k) - \varphi(x)}{h} \xrightarrow{\mathscr{S}}_{h \to 0} \frac{\partial\,\varphi}{\partial\,x_k}(x).$$

$\square$

Theorem 5.24. *Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi \in \mathscr{S}$ be a Schwartz function. The distribution associated to $f * \varphi$ is a tempered distribution, i.e., $\Lambda_{f*\varphi} \in \mathscr{S}'$.*

Proof. First, we observe that $f : \mathscr{S} \to \mathbb{C}$ is a linear and continuous functional. There exist $C_{\alpha,\,\beta}$ constants such that

$$|f(\varphi)| \leq \sum_{(\alpha,\,\beta)\in\mathbb{N}^n} C_{\alpha,\,\beta} \cdot q_{\alpha,\,\beta}(\varphi),$$

where $C_{\alpha,\,\beta} \neq 0$ for finitely many couples only. By Example 5.1, it suffices to prove that $f * \varphi$ is a function which grows less than a certain polynomial. The estimate above yields to

$$|f * \varphi(x)| \leq \sum_{(\alpha,\,\beta)\in\mathbb{N}^n} C_{\alpha,\,\beta} \cdot q_{\alpha,\,\beta}\left(\tau_x \,\check{\varphi}\right),$$

hence we conclude by noticing that the seminorm can be easily estimated:

$$\begin{aligned} q_{\alpha,\,\beta}\left(\tau_x \,\check{\varphi}\right) &= \sup_{y\in\mathbb{R}^n} \left|y^\alpha \, D_y^\beta \, \left(\tau_x \,\check{\varphi}(y)\right)\right| = \\ &= \sup_{y\in\mathbb{R}^n} \left|y^\alpha \, D_y^\beta \, \varphi(x-y)\right| = \\ &= \sup_{z\in\mathbb{R}^n} \left|(z-x)^\alpha \, D_y^\beta \, \varphi(z)\right| < +\infty. \end{aligned}$$

□

Theorem 5.25. *Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi,\, \psi \in \mathscr{S}$ be Schwartz functions. The convolution product is associative, that is,*

$$(f * \psi) * \varphi = f * (\psi * \varphi) = (f * \varphi) * \psi.$$

Moreover, the distribution associated to $f * \varphi$ computed at ψ is equal to $f(\psi * \check{\varphi})$, i.e.,

$$\Lambda_{f*\varphi}(\psi) = f\left(\psi * \check{\varphi}\right).$$

Theorem 5.26. *Let $f \in \mathscr{S}'$ be a tempered distribution, and let $\varphi \in \mathscr{S}$ be a Schwartz function.*

- ***(a)*** *The Fourier transform of the convolution is, up to a constant, the product of the Fourier transforms:*

$$\mathcal{F}\left(f * \varphi\right) = (2\pi)^{n/2} \, \mathcal{F}(\varphi) \cdot \mathcal{F}(f).$$

- ***(b)*** *The Fourier transform of the product is, up to a constant, the convolution of the Fourier transforms:*

$$\mathcal{F}\left(\varphi \cdot f\right) = (2\pi)^{-n/2} \, \mathcal{F}(f) * \mathcal{F}(\varphi).$$

Proof. We only prove the identity **(a)** since the second one follows easily by using $f \rightsquigarrow \mathcal{F}(f)$ and $\varphi \rightsquigarrow \mathcal{F}(\varphi)$. If $\psi \in \mathscr{S}$ is a Schwartz function, then

$$\begin{aligned} \Lambda_{\mathcal{F}(f*\varphi)}\left(\check{\psi}\right) &= \Lambda_{f*\varphi}\left(\mathcal{F}\left(\check{\varphi}\right)\right) = \\ &= (f * \varphi) * \mathcal{F}(\psi)(0) = \\ &= \left(f * \varphi * \mathcal{F}(\psi)\right)(0) = \\ &= \mathcal{F}(f)\left((2\pi)^{n/2} \, \mathcal{F}(\varphi) \, \check{\psi}\right) = \\ &= (2\pi)^{n/2} \, \mathcal{F}(\varphi) \cdot \mathcal{F}(f)\left(\check{\psi}\right). \end{aligned}$$

□

Chapter 6

Fourier-Laplace Transform

The primary goal of this section is to introduce the *Fourier-Laplace transform* and study some properties which are closely related to the standard Fourier transform.

6.1 Introduction to Complex Analysis

We first recall some basic definitions in several variables complex analysis. We are mainly interested in the notion of **entire** function as it will be used extensively in the next chapter.

Definition 6.1 (Holomorphic/Entire)**.** Let $\Omega \subseteq \mathbb{C}^n$ be an open subset of the complex n-plane.

(1) A function $f : \Omega \longrightarrow \mathbb{C}$ is *holomorphic* if it is continuous at every point of Ω, and the functions

$$\Omega_i \ni z_i \longmapsto f(z_1, \ldots, z_i, \ldots, z_n)$$

are holomorphic for all $i = 1, \ldots, n$.

(2) A function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ is *entire* if it is holomorphic at every point $z \in \mathbb{C}^n$.

Lemma 6.2 (Identity Principle)**.** *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be an entire function which is identically zero on the real line, that is,*

$$f(z) = 0 \quad \textit{for all } z = x + \imath y \textit{ with } y = 0.$$

Then f is the function identically zero on the whole $\mathbb{C}^n$.

Proof. Consider the k-predicate

$$P_k : f(z) = 0 \text{ for every } z = (z_1, \ldots, z_n) \in \mathbb{C}^n \text{ such that } z_1, \ldots, z_k \in \mathbb{R}.$$

By assumption P_n is true. Therefore, it is enough to prove that

$$P_k \implies P_{k-1}$$

and conclude by the finiteness of n. To achieve this implication we simply notice that

$$\mathbb{C} \ni z_k \longmapsto f(z_1, \ldots, z_k, \ldots, z_n) \in \mathbb{C}$$

is a complex function of a single variable, and hence we can apply the well-known identity theorem[1]. □

6.2 Paley-Wiener Theorem

We are now ready to state and prove the main result of this chapter (which will be followed by some kind of "inverse" result.)

Theorem 6.3 (Paley-Wiener). *Let $\varphi \in \mathcal{D}$ be a function such that* $\operatorname{spt}(\varphi) \subseteq \overline{B(0,\, r)}$. *Then*

$$f(z) := \int_{\mathbb{R}^n} \varphi(t)\, e^{-\imath\,(t,\, z)}\, dt, \tag{6.1}$$

is an entire function satisfying the estimate

$$|z^\alpha f(z)| \lesssim_\alpha e^{r|\mathfrak{Im}(z)|} \quad \textit{for all } z \in \mathbb{C}^n \textit{ and any } \alpha \in \mathbb{N}^n. \tag{6.2}$$

Vice versa, if f is an entire function satisfying the estimate (6.2), then we can find a smooth function $\varphi \in \mathcal{D}$ such that

$$\operatorname{spt}(\varphi) \subseteq \overline{B(0,\, r)}.$$

Furthermore, f is exactly given by the formula (6.1).

Proof. The argument is rather involved. Hence we divide the proof into five steps, to ease the notation for the reader.

Step 1. The function given by (6.1) is well-defined and continuous, as a consequence of the dominated convergence theorem. The single-variable complex function

$$f_k : z_k \longmapsto f(z_1,\, \dots,\, z_k,\, \dots,\, z_n)$$

is holomorphic for all $k = 1,\, \dots,\, n$ as a simple application of Moreira's Theorem.[2] Indeed, for every piecewise differentiable closed curve γ it turns out that

$$\oint_\gamma \left[z_k \mapsto \int_{\mathbb{R}^n} \varphi(t)\, e^{-\imath\,(t,\, z)}\, dt \right] dz_k = \int_{\mathbb{R}^n} \left[\oint_\gamma \left(z_k \mapsto e^{-\imath\,(t,\, z)} \right) dz_k \right] dt = 0$$

using the Fubini-Tonelli theorem together with the fact that the complex exponential is a holomorphic function.

[1]**Identity Theorem.** Let $g,\, h : D \longrightarrow \mathbb{C}$ be complex functions defined on a connected open set $D \subseteq \mathbb{C}$. If $f(x) = g(x)$ for every $x \in S$, where S is a nonempty open subset of D, then $f(x) = g(x)$ for every $x \in D$.

[2]**Theorem.** Let $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function defined on a open set of the complex plane satisfying

$$\oint_\gamma f(z)\, dz = 0$$

for every closed piecewise C^1 curve γ in D. Then f is holomorphic on D.

Step 2. We now want to show that (6.2) holds. First, notice that for any index $k = 1, \ldots, n$ we have

$$z_k f(z) = \imath \int_{\mathbb{R}^n} \varphi(t) \frac{\partial}{\partial t_k} \mathrm{e}^{-\imath\langle t,\, z\rangle}\, \mathrm{d}t =$$

$$= -\imath \int_{\mathbb{R}^n} \frac{\partial}{\partial t_k} \varphi(t) \mathrm{e}^{-\imath\langle t,\, z\rangle}\, \mathrm{d}t,$$

which can be easily generalized to a multi-index $\alpha \in \mathbb{N}^n$ as follows:

$$z^\alpha f(z) = (-\imath)^{|\alpha|} \int_{\mathrm{spt}(\varphi)} D^\alpha (\varphi)(t) \mathrm{e}^{-\imath\langle t,\, z\rangle}\, \mathrm{d}t. \tag{6.3}$$

We now observe that z can be rewritten as $\mathfrak{Re}(z) + \imath\mathfrak{Im}(z)$, and therefore

$$\left| \mathrm{e}^{-\imath\langle t,\, z\rangle} \right| = \mathrm{e}^{\langle t,\, \mathfrak{Im}(z)\rangle}.$$

Therefore, if we take the absolute value of (6.3), it turns out that

$$|z^\alpha f(z)| \leq \|D^\alpha \varphi\|_0 \int_{\mathrm{spt}(\varphi)} \left| \mathrm{e}^{\imath\langle t,\, z\rangle} \right| \mathrm{d}t \leq$$

$$\leq \|D^\alpha \varphi\|_0 \int_{\overline{B(0,\, r)}} \mathrm{e}^{\langle t,\, \mathfrak{Im}(z)\rangle}\, \mathrm{d}t \lesssim_\alpha$$

$$\lesssim_\alpha \mathrm{e}^{r|\mathfrak{Im}(z)|},$$

which is exactly what we wanted to prove.

Step 3. Vice versa, suppose that f is an entire function satisfying (6.2). Denote by F the restriction of f to the real line, that is,

$$F : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad F(\mathfrak{Re}(z)) := f\left(\mathfrak{Re}(z)\right).$$

The estimate (6.2), applied to the function F, shows immediately that F is rapidly decreasing, which means that $F \in \mathscr{S}$. The Fourier transform is an invertible operator on $\mathscr{S}$, and hence we have

$$\mathcal{F}^2(F) = \check{F}.$$

More precisely, if φ exists, then it is necessarily unique, and it is given by the formula

$$F(x) = \mathcal{F} \circ \mathcal{F}\left(\check{F}\right) \implies \varphi = \frac{1}{(2\pi)^{n/2}} \mathcal{F}\left(\check{F}\right) \tag{6.4}$$

since the Fourier transform is, up to a constant, the restriction to the real line of the Paley-Weiner transform (6.1).

Step 4. We now want to show that the support of φ, defined in (6.4), is contained in the closed ball of radius r. We claim that

$$\varphi(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x + \imath y) \mathrm{e}^{\imath\langle t,\, x+\imath y\rangle}\, \mathrm{d}x \tag{6.5}$$

for all $y \in \mathbb{R}^n$. In particular, the function φ can be extended to the whole complex n-plane $\mathbb{C}^n$ since it only depends on the values attained on the real line.

Now fix $k \in \{1, \ldots, n\}$ and consider the curve γ_N defined in Figure 6.1. The function is holomorphic, and therefore Cauchy theorem[3] shows that

$$0 = \int_{\gamma_N} f(z) e^{\imath \langle t,\, z \rangle} \, dz_k = \int_{-N}^{N} f(x,\, 0) e^{\imath \langle t,\, x \rangle} \, dx_k - \int_{-N}^{N} f(x,\, h) e^{\imath \langle t,\, x + \imath h \rangle} \, dx_k + \ldots$$

$$\cdots + \int_0^h f(N,\, y) e^{\imath \langle t,\, N + \imath y \rangle} \, dy_k - \int_0^h f(-N,\, y) e^{\imath \langle t,\, -N + \imath y \rangle} \, dy_k.$$

The estimate (6.2) immediately implies that the last two terms goes to zero as N goes to $+\infty$, and this is enough to conclude that the claim holds.

Step 5. We are finally ready to prove that the support of φ is contained in $\overline{B(0,\, r)}$. Let t be a real number such that $|t| > r$, and notice that

$$\begin{aligned}
|\varphi(t)| &\le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |f(x + \imath y)| \left| e^{\imath \langle t,\, x + \imath y \rangle} \right| dx \le \\
&\le \frac{1}{(2\pi)^n} e^{-\langle t,\, y \rangle} \int_{\mathbb{R}^n} |f(x + \imath y)| \, dx \lesssim_\alpha \\
&\lesssim_\alpha \frac{1}{(2\pi)^n} e^{-\langle t,\, y \rangle} \int_{\mathbb{R}^n} \frac{e^{r|y|}}{(1 + |z|^2)^n} \, dx \lesssim_\alpha \\
&\lesssim_\alpha e^{r|y| - \langle t,\, y \rangle}.
\end{aligned}$$

If we take $y_k := k \frac{t}{|t|}$, then

$$r|y_k| - \langle t,\, y_k \rangle < 0 \implies |\varphi(t)| \xrightarrow{k \to +\infty} 0,$$

which means that $\varphi(t) = 0$ for all $|t| > r$. In conclusion, it remains to prove that (6.1) holds true for any $z \in \mathbb{C}^n$, but this follows easily from Lemma 6.2 since (6.1) holds for all $x \in \mathbb{R}^n$ by construction. $\square$

Lemma 6.4. *Let $g \in \mathcal{E}'$ be a compactly supported distribution. Then the Fourier transform $\mathcal{F}(g)$ is a function of class $\mathcal{E}$ and*

$$\mathcal{F}(g)(x) = \frac{1}{(2\pi)^{n/2}} g\left(\exp_{-\imath x}\right), \tag{6.6}$$

where

$$\exp_z : t \longmapsto e^{\langle t,\, z \rangle} \in \mathcal{E}.$$

[3] **Cauchy Theorem.** Let U be an open subset of $\mathbb{C}$ which is simply connected, let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function and let γ be a rectifiable closed path in U. Then

$$\oint_\gamma f(z) \, dz = 0.$$

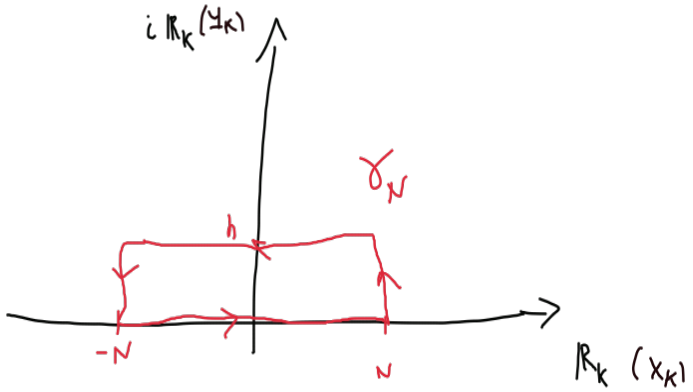

Figure 6.1: Path γ used in the previous proof.

Proof. Let $r > 0$ be a real number such that

$$\mathrm{spt}(g) \subseteq \overline{B(0,\, r)}.$$

Let $\Psi \in \mathcal{D}$ be a cutoff function satisfying

$$\Psi\,\big|_{\overline{B(0,\, r+1)}} \equiv 1,$$

so that (see Theorem 4.17) we have the distributional identity

$$g = \Psi \cdot g.$$

We now apply the Fourier transform to both sides

$$\mathcal{F}(g) = \underbrace{(2\pi)^{n/2}\mathcal{F}(\Psi)}_{:=\varphi\in\mathscr{S}} * \mathcal{F}(g),$$

and we find that

$$\mathcal{F}(g)(x) = \mathcal{F}(g)\left(\tau_x\varphi\right) = g\left(\mathcal{F}(\tau_x\varphi)\right) = \frac{1}{(2pi)^{n/2}} g\left(\exp_{-\imath x}\right),$$

which is exactly what we wanted to prove. $\square$

Theorem 6.5 (Paley-Wiener, II). *Let $g \in \mathcal{E}'$ be a compactly supported distribution such that*

$$\mathrm{spt}(g) \subseteq \overline{B(0,\, r)}.$$

Then the function

$$f(z) := g\left(\mathrm{e}_{-\imath z}\right), \tag{6.7}$$

is entire and it satisfies the estimate

$$|f(z)| \lesssim (1+|z|)^N \, \mathrm{e}^{r|\mathfrak{Im}(z)|} \quad \text{for all } z \in \mathbb{C}^n \text{ and } \alpha \in \mathbb{N}^n, \tag{6.8}$$

where N is the order of the distribution g. Vice versa, if f is an entire function that satisfies the estimate (6.8), then we can always find a distribution $g \in \mathcal{E}'$ such that

$$\mathrm{spt}(g) \subseteq \overline{B(0,\, r)},$$

and f is exactly given by the formula (6.7).

Proof. The argument is rather involved. Hence we divide the proof into five steps, to ease the notation for the reader.

Step 1. The function given by (6.7) is well-defined and continuous since it is given by the composition of continuous maps:

$$z \longmapsto \exp_{\imath z} \longmapsto g\left(\exp_{\imath z}\right).$$

Moreover, it is holomorphic with respect to each variable z_i (again, Moreira Theorem). Indeed, for every piecewise differentiable closed curve $z : [a, b] \longrightarrow \mathbb{C}$ we have

$$\int_a^b f\left(z(s)\right)\, \mathrm{d}s = \int_a^b g\left(t \mapsto \exp_{-\imath z(s)}(t)\right)\, \mathrm{d}s.$$

If we mimic the proof of Theorem 4.26, we find that

$$\int_a^b f\left(z(s)\right)\, \mathrm{d}s = g\left(t \longmapsto \int_a^b \exp^{\imath\langle t,\, z(s)\rangle}\, \mathrm{d}s\right) = 0$$

since the complex exponential is a holomorphic function.

Step 2. We now want to show that (6.7) holds. Let h be the cutoff function defined by

$$h(x) = \begin{cases} 1 & x \leq 1, \\ \textit{linear interpolation} & x \in (1,\, 2), \\ 0 & x \geq 2, \end{cases}$$

and let us consider the auxiliary function

$$\varphi_z(t) := \mathrm{e}^{\imath\langle t,\, z\rangle} h\left(\left(|t| - r\right)|z|\right).$$

By definition φ_z is identically equal to the exponential function $\mathrm{e}^{\imath\langle t,\, z\rangle}$ on the closed ball of radius $r + \frac{1}{|z|}$ and its support is contained in the bigger closed ball of radius $r + \frac{2}{|z|}$. Now Theorem 4.17 shows that

$$\varphi_z \cdot g = g \implies f(z) = g\left(t \longmapsto \varphi_z(t)\right). \tag{6.9}$$

Recall that g is a compactly supported distribution, and thus the order of g is necessarily finite and equal to some $N \in \mathbb{N}$. It follows from (6.9) that

$$|f(z)| \lesssim \|\varphi_z\|_N.$$

The Leibniz rule implies that

$$D^\alpha \varphi_z = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \left(\mathrm{e}^{-\imath\langle t,\, z\rangle}\right) D^{\alpha-\beta}\left(h\left(\left(|t| - r\right)|z|\right)\right) =$$

$$= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(-\imath z\right)^\beta \mathrm{e}^{-\imath\langle t,\, z\rangle} D^{|\alpha|-|\beta|}\left(h\left(\left(|t| - r\right)|z|\right)\right) \left(\frac{t}{|t|} z\right)^{|\alpha|-|\beta|},$$

from which it follows that

$$|D^\alpha \varphi_z| \lesssim \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{e}^{\langle \mathfrak{Im}(z),\, t\rangle} z^{|\alpha|} \lesssim$$

$$\lesssim |z|^\alpha \mathrm{e}^{\left(r + \frac{2}{|z|}\right)|\mathfrak{Im}(z)|} \le$$

$$\le \underbrace{\mathrm{e}^2}_{:=C} |z|^\alpha \mathrm{e}^{r|\mathrm{Im}(z)|}.$$

Step 3. Vice versa, suppose that f is an entire function satisfying (6.8). Denote by F the restriction of f to the real line, that is,

$$F : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad F(\mathfrak{Re}(z)) := f\left(\mathfrak{Re}(z)\right).$$

Now Lemma 6.4 shows that

$$f(x) = g\left(\exp_{-\imath x}\right) = (2\pi)^{n/2} \mathcal{F}(g)(x).$$

The estimate (6.8), applied to the function F, shows immediately that F is rapidly decreasing, which means that $F \in \mathscr{S}$. The Fourier transform is an invertible operator on $\mathscr{S}$, and hence we have

$$\mathcal{F}^2(F) = \check{F}.$$

Consequently, if g exists, it must be unique and, more precisely, given by the formula

$$F(x) = \mathcal{F} \circ \mathcal{F}\left(\check{F}\right) \implies g = \frac{1}{(2\pi)^{n/2}} \mathcal{F}\left(\check{F}\right), \tag{6.10}$$

since the Fourier transform is, up to a constant, the restriction to the real line of the Paley-Weiner transform (6.7).

Step 4. We will now show that the support of g, defined in (6.10), is contained in the closed ball of radius r. Let ρ_ϵ be a mollifier supported in B_ϵ, and recall that

$$\mathcal{F}(g * \rho_\epsilon) = (2\pi)^{n/2} \mathcal{F}(g)\mathcal{F}(\rho_\epsilon) = f\mathcal{F}(\rho_\epsilon).$$

Using the first version of the Paley-Wiener Theorem 6.3 we find that $\mathcal{F}(\rho_\epsilon)$ is an entire function, which satisfies the estimate (6.1). In particular, we have

$$|z|^\alpha \left| f\mathcal{F}(\rho_\epsilon)(z)\right| \lesssim \mathrm{e}^{(r+\epsilon)|\mathfrak{Im}(z)|}.$$

We can thus apply the second statement contained in the Paley-Wiener Theorem 6.3. It follows that

$$f\mathcal{F}(\rho_\epsilon) = \mathcal{F}(g_\epsilon) \quad \text{and} \quad \mathrm{spt} g_\epsilon \subseteq \overline{B_{r+\epsilon}}.$$

To conclude observe that

$$g(\psi) = \lim_{\epsilon \to 0^+} g_\epsilon(\psi) = 0$$

for all $\psi \in \mathcal{D}$ such that the support of ψ does not intersect the closed ball of radius r. Therefore, $\mathrm{spt} g$ is contained in $\overline{B_r}$.

Step 5. Finally, it remains to prove that (6.7) holds true for any $z \in \mathbb{C}^n$, but this follows easily from Lemma 6.2 since (6.7) holds for any $x \in \mathbb{R}^n$ by definition. $\square$

Part III

Applications to Partial Differential Equations

Chapter 7

Regularity Theory of Elliptic Operators

The goal of this chapter is to investigate elliptic operators L and show that solutions of the equation $Lu = 0$ are necessarily smooth.

Definition 7.1 (Linear PDE)**.** A *linear partial differential equation* with constant coefficients is a functional equation of the form

$$\sum_{|\alpha|\leq m} c_\alpha \left(-\imath D\right)^\alpha u = \varphi, \tag{7.1}$$

where $c_\alpha \in \mathbb{C}$ are complex constants, while u and φ are (a priori) distributions. The function

$$P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \, \xi^\alpha$$

is known as the *characteristic polynomial* associated to the equation (7.1). It turns out that

$$P\left(-\imath D\right) u = \varphi. \tag{7.2}$$

There is a special class of solutions of PDEs, which is extremely useful whenever we are dealing with a problem of the form $Lu = f(u)$.

Definition 7.2 (Fundamental Solution)**.** Let $P(-\imath D)$ be a partial differential operator with constant coefficients. A distribution u_0 is a *fundamental solution* for P if

$$P\left(-\imath D\right) u_0 = \delta_0,$$

where δ_0 denotes the Dirac distribution centred at the origin.

Remark 7.1. Suppose that u_0 is a fundamental solution for P. Then

$$P\left(-\imath D\right)\left(u_0 * \varphi\right) = \left(P(-\imath D)u_0\right) * \varphi = \delta_0 * \varphi = \varphi,$$

which means that, at least formally, a fundamental solution produces a nontrivial solution to the equation with right-hand side given by φ.

We now apply the Fourier transform operator $\mathcal{F}$ to (7.1). We obtain the identity

$$\sum_{|\alpha|\leq m} c_\alpha \left(-\imath D\right)^\alpha u = \varphi \implies P(t)\mathcal{F}(u)(t) = \mathcal{F}(\varphi)(t). \tag{7.3}$$

In particular, if u_0 is a fundamental solution of (7.1), then it must satisfy

$$P(t)\mathcal{F}(u_0)(t) = 1 \quad \text{for all } t \in \mathbb{R}.$$

This approach suggests that a fundamental solution u_0 can never be a compactly supported distribution. In fact, if we had $u_0 \in \mathcal{E}'$, then a straightforward application of the Paley-Wiener Theorem 6.3 would imply that both $\mathcal{F}(u_0)$ and $P(t)\mathcal{F}(u_0)(t)$ are entire functions. Consequently,

$$P(t) \text{ is not constant} \implies \exists\, t\, :\, P(t)\mathcal{F}(u_0)(t) = 0,$$

which is in contradiction with the identity above.

7.1 Malgrange Theorem

In this section, we set the ground to state and prove the well-known Malgrange theorem, which asserts that the problem (7.3) always admits a fundamental solution.

Lemma 7.3. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonzero polynomial of degree d. Then there exists a positive constant $C > 0$ such that the estimate*

$$|f(z)| \leq \frac{C}{r^d} \int_{\mathbb{T}^n} |P \cdot f| \left(z + r\mathrm{e}^{\imath\theta}\right) \,\mathrm{d}\theta \tag{7.4}$$

holds for every entire function f, $z \in \mathbb{C}^n$ and $r > 0$. Here $\mathbb{T}^n$ denotes the n-torus $[0,\, 2\pi]^n$.

Proof. We first prove that the estimate (7.4) holds at the origin for a polynomial $Q(\xi)$ of a single complex variable. Next, we reduce the general case to it using a simple trick.

Step 1. Let $Q \in \mathbb{C}[\xi]$ be a polynomial of degree d, and let $F(\xi)$ be an entire function. We can factorise Q over $\mathbb{C}$ and write it as

$$Q(\xi) = a \prod_{j=1}^{d} \left(\xi - \xi_j\right),$$

where $\xi_1, \ldots, \xi_d \in \mathbb{C}$ are the (eventually repeated) roots of Q. We also consider the auxiliary polynomial defined by

$$Q_0(\xi) := a \prod_{j=1}^{d} \left(1 - \bar{\xi}_j \xi\right).$$

We now notice that, if $\xi \in \mathbb{C}$ is a complex number of norm equal to one, then

$$\begin{aligned} |\xi - \xi_j| &= \left|\overline{\xi - \xi_j}\right| = \\ &= |\bar{\xi}|^{-1} \left|1 - \frac{\bar{\xi}_j}{\bar{\xi}}\right| = \\ &= \left|1 - \frac{\bar{\xi}_j \xi}{|\xi|}\right| = \left|1 - \bar{\xi}_j \xi\right|, \end{aligned}$$

which means that

$$|\xi| = 1 \implies |Q_0(\xi)| = |Q(\xi)|\,.$$

A straightforward application of the Cauchy formula[1] shows that

$$F \cdot Q_0(z) = \frac{1}{2\pi} \int_0^{2\pi} F \cdot Q_0\left(z + r e^{\imath\theta}\right)\, d\theta. \tag{7.5}$$

It follows that

$$|F \cdot Q_0(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |F \cdot Q_0| \left(e^{\imath\theta}\right)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |F \cdot Q| \left(e^{\imath\theta}\right)\, d\theta$$

using the fact that $|Q_0(e^{\imath\theta})| = |Q(e^{\imath\theta})|$ for all $\theta \in [0,\, 2\pi]$. We finally infer that

$$|a \cdot F(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |F \cdot Q| \left(e^{\imath\theta}\right)\, d\theta, \tag{7.6}$$

where a is a constant whose exact value is given by the formula

$$|a| = \lim_{|\xi| \to +\infty} \frac{|Q(\xi)|}{|\xi|^d}.$$

Step 2. Fix $\theta \in \mathbb{T}^n$ and $z \in \mathbb{C}^n$. We consider the entire function

$$F : \mathbb{C} \ni \xi \longmapsto f(z + r e^{\imath\theta}\xi) \in \mathbb{C},$$

and the d-polynomial

$$Q : \mathbb{C} \ni \xi \longmapsto P(z + r e^{\imath\theta}\xi) \in \mathbb{C}.$$

The corresponding constant $|a|$ is given by the following limit evaluation:

$$\begin{aligned}
|a| &= \lim_{|\xi| \to +\infty} \frac{|Q(\xi)|}{|\xi|^d} = \lim_{|\xi| \to +\infty} \frac{\left|P(z + r e^{\imath\theta}\xi)\right|}{|\xi|^d} = \\
&= \lim_{|\xi| \to +\infty} \left[\sum_{|\alpha| \leq d} |c_\alpha| \frac{\left|z + r e^{\imath\theta}\xi\right|^{|\alpha|}}{|\xi|^d} \right] = \\
&= \lim_{|\xi| \to +\infty} \left[\sum_{|\alpha| = d} |c_\alpha| \left| \frac{z}{\xi} + r e^{\imath\theta} \right| \right] = \\
&= P_d\left(r e^{\imath\theta}\right),
\end{aligned}$$

where P_d is the maximal-degree part of the polynomial P, that is,

$$P_d(z) = \sum_{|\alpha| = d} c_\alpha z^\alpha.$$

[1]Let $f : A \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function defined on an open subset of $\mathbb{C}$. Let γ be a simple closed curve contained in A. Let S be the region enclosed in γ (counterclockwise) and let z be any point in the internal of S which does not belong to the curve. Then

$$f(z) = \frac{1}{2\pi\imath} \oint_\gamma \frac{f(\xi)}{\xi - z}\, d\xi.$$

The estimate (7.6) immediately implies that

$$\left|P_d(r\mathrm{e}^{\imath\theta})\right| |f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |P \cdot f| \left(z + r\mathrm{e}^{\imath\theta}\mathrm{e}^{\imath\tau}\right) \, \mathrm{d}\tau, \tag{7.7}$$

and, by taking the average of the polynomial over the n-dimensional torus, it turns out that

$$|f(z)| \int_{\mathbb{T}^n} \left|P_d(r\mathrm{e}^{\imath\theta})\right| \, \mathrm{d}\theta \leq \frac{1}{2\pi}\int_{\mathbb{T}^n}\int_0^{2\pi} |P \cdot f| \left(z + r\mathrm{e}^{\imath\theta}\mathrm{e}^{\imath\tau}\right) \, \mathrm{d}\tau \, \mathrm{d}\theta.$$

The left-hand side, by homogeneity, satisfies the following equality

$$|f(z)| \cdot \int_{\mathbb{T}^n} \left|P_d(r\mathrm{e}^{\imath\theta})\right| \, \mathrm{d}\theta = r^d \cdot |f(z)| \int_{\mathbb{T}^n} \left|P_d(\mathrm{e}^{\imath\theta})\right| \, \mathrm{d}\theta,$$

and hence it is enough to prove that

$$\int_{\mathbb{T}^n} \left|P_d(\mathrm{e}^{\imath\theta})\right| \, \mathrm{d}\theta = c \neq 0$$

to infer that the estimate (7.4) holds. But the integral cannot be equal to zero since the integrand is positive and the P_d is assumed to be nonzero.

Note. The substitution $\tau + \theta \mapsto \sigma$ is enough to deal with the right-hand side since we obtain the following identity:

$$\frac{1}{2\pi}\int_{\mathbb{T}^n}\int_\theta^{2\pi+\theta} |P \cdot f| \left(z + r\mathrm{e}^{\imath\sigma}\right) \, \mathrm{d}\sigma \, \mathrm{d}\theta = \frac{1}{2\pi}\int_{\mathbb{T}^n} |P \cdot f| \left(z + r\mathrm{e}^{\imath\theta}\right) \, \mathrm{d}\theta.$$

□

Theorem 7.4 (Solution Property). *Let $\varphi \in \mathcal{E}'$ be a compactly supported distribution.*

(a) *If there exists $u \in \mathcal{E}'$ such that*

$$P(-\imath D)u = \varphi,$$

then there exists an entire function f satisfying

$$P(x)f(x) = \mathcal{F}(\varphi)(x).$$

(b) *If there exists an entire function f such that*

$$P(x)f(x) = \mathcal{F}(\varphi)(x),$$

then the equation (7.3) admits a solution $u \in \mathcal{E}'$ whose support is contained in the convex hull generated by $\mathrm{spt}(\varphi)$*, that is,*

$$\mathrm{spt}(u) \subset \mathrm{Conv}\left(\mathrm{spt}(\varphi)\right).$$

Proof. The assertion **(a)** has already been proved. Suppose that there exists an entire function f satisfying

$$P(x)f(x) = \mathcal{F}(\varphi)(x)$$

for all x. Since φ belongs to $\mathcal{E}'$ it follows from the Paley-Wiener Theorem 6.5 that $\mathcal{F}(\varphi)$ is an entire function satisfying the estimate

$$|\mathcal{F}(\varphi)(z)| \leq c\,(1+|z|)^d \cdot \mathrm{e}^{r|\mathfrak{Im}(z)|}.$$

We now plug this into the inequality (7.4) given by Lemma 7.3. It turns out that

$$\begin{aligned}|f(z)| &\leq \int_{\mathbb{T}^n} |\mathcal{F}(\varphi)|\left(z+\mathrm{e}^{\imath\theta}\right)\,\mathrm{d}\theta \leq \\ &\leq c\int_{\mathbb{T}^n}\left(1+\left|z+\mathrm{e}^{\imath\theta}\right|\right)^d \mathrm{e}^{r\left|\mathfrak{Im}(z+\mathrm{e}^{\imath\theta})\right|}\,\mathrm{d}\theta \leq \\ &\leq \underbrace{c'\,\mathrm{vol}\,(\mathbb{T}^n)}_{=c''}\,(1+|z|)^d\cdot \mathrm{e}^{r|\mathfrak{Im}(z)|}.\end{aligned}$$

The Paley-Wiener Theorem 6.5 shows that f is the Fourier transform of a function $u \in \mathcal{E}'$ with support satisfying

$$\operatorname{spt}(u) \subset \overline{B_r}.$$

Putting together all these facts we conclude that u is the sought solution and

$$\operatorname{spt}(u) \subset \overline{B_r} \quad \text{for all } B_r \text{ s.t. } \overline{B_r} \supset \operatorname{spt}(\varphi) \rightsquigarrow \operatorname{spt}(u) \subset \operatorname{Conv}\left(\operatorname{spt}(\varphi)\right).$$

□

Theorem 7.5 (Malgrange)**.** *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonzero d-polynomial. Then there exists a distribution $u \in \mathcal{D}'$ such that*

$$P(-\imath D)\,u = \delta_0.$$

Proof. We may equivalently show that

$$P(-\imath D)u(\varphi) = \delta_0(\varphi) = \varphi(0) \quad \text{for all } \varphi \in \mathcal{D}. \tag{7.8}$$

Step 1. We first observe that for every multi-index $\alpha \in \mathbb{N}^n$ we have

$$D^\alpha u(\varphi) = (-1)^{|\alpha|} u\left(D^\alpha \varphi\right) = \check{u}\left(D^\alpha \check{\varphi}\right),$$

and hence (7.8) holds if and only if

$$\check{u}\left(P(-\imath D)\check{\varphi}\right) = \varphi(0) \quad \text{for all } \varphi \in \mathcal{D}$$

holds. Since $\varphi(0) = \check{\varphi}(0)$ the thesis (7.8) is also equivalent to requiring that

$$\check{u}\left(P(-\imath D)\varphi\right) = \varphi(0) \quad \text{for all } \varphi \in \mathcal{D}. \tag{7.9}$$

Step 2. To show that (7.9) is true, we will first prove the existence of a unique map $U : \mathfrak{Y} \longrightarrow \mathbb{C}$ with the property that the following diagram is commutative

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathcal{P}"] \arrow[dr, "\delta_0"'] & \mathfrak{Y} := \operatorname{Ran}(\mathcal{P}) \arrow[d, dotted, "U"] \\ & \mathbb{C} \end{tikzcd}$$

where $\mathcal{P}$ denotes the map sending $\psi \in \mathcal{D}$ to $P(-\imath\, D)\,\psi \in Y$.

Step 3. The existence of the map U follows from the fact that $\mathcal{P}$ is an injective operator. Let $\psi \in \mathcal{D}$ be an element in the kernel of $\mathcal{P}$, that is,

$$P(-\imath D)\psi = 0.$$

We apply the Fourier transform to both the left-hand side and the right-hand side and we obtain the identity

$$P(\xi) \cdot \mathcal{F}(\psi)(\xi) = 0 \quad \text{for all } \xi \in \mathbb{C}^n.$$

The Paley-Wiener Theorem 6.5 asserts that $\mathcal{F}(\psi)$ is an entire function, and thus the identity above implies that

$$\mathcal{F}(\psi)(\xi) = 0 \quad \text{for all } \xi \in \{P(\xi) \neq 0\}.$$

The set of points where $P(\xi)$ is equal to zero has empty interior part; hence $\mathcal{F}(\psi)$ vanishes on a dense set and, by continuity, $\mathcal{F}(\psi)$ must be identically equal to zero, which means that $\mathcal{P}$ is injective.

Step 4. The existence of the map U follows from the previous point so, if we can prove that U is continuous, then the Hahn-Banach theorem would allow us to extend it to a linear continuous map satisfying

$$\mathcal{D} \ni \varphi \longmapsto P(-\imath D)u(\varphi) \longmapsto \varphi(0).$$

Step 5. Let $(\psi_j)_{j\in\mathbb{N}} \subset Y$ be a sequence converging to zero with respect to the $\mathcal{D}$-topology. We can always find a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{D}$ such that

$$\psi_j = P(-\imath D)\varphi_j \quad \text{and} \quad U(\psi_j) = \varphi_j(0).$$

If we apply the Fourier transform, we obtain

$$P(\xi) \cdot \mathcal{F}(\varphi_j)(\xi) = \mathcal{F}(\psi_j)(\xi)$$

for all $j \in \mathbb{N}$ and, as a consequence of the Paley-Wiener Theorem 6.5, we infer that the function $\mathcal{F}(\varphi_j)$ is entire. It follows from Lemma 7.3 that

$$|\varphi_j(0)| = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\mathcal{F}(\varphi_j(\xi)| \, \mathrm{d}\xi \lesssim \int_{\mathbb{R}^n} \left[\int_{\mathbb{T}^n} |P \cdot \mathcal{F}(\varphi_j)| \, (\xi + \mathrm{e}^{\imath\theta}) \, \mathrm{d}\theta \right] \mathrm{d}\xi, \tag{7.10}$$

where

$$|P \cdot \mathcal{F}(\varphi_j)| \, (\xi + \mathrm{e}^{\imath\theta}) = \mathcal{F}(\psi_j)(\xi + \mathrm{e}^{\imath\theta}).$$

Recall that, by definition of the exponential notation, we have

$$\mathcal{F}(\psi_j)(\xi + \mathrm{e}^{\imath\theta}) = \mathcal{F}\left(\exp_{\imath\theta}\psi_j\right).$$

Set $\psi_{\theta,\,j} := \exp_{\imath\theta}\psi_j$. If we plug this into (7.10), we obtain the estimate

$$|\varphi_j(0)| \lesssim \int_{\mathbb{R}^n} \left[\int_{\mathbb{T}^n} |\mathcal{F}(\psi_{\theta,\,j})\,(\xi) \, \mathrm{d}\theta \right] \mathrm{d}\xi. \tag{7.11}$$

We now apply the Schwartz-Hölder inequality and find that

$$\begin{aligned}
\|\mathcal{F}(\psi_{\theta,\,j})\|_{L^1(\mathbb{R}^n)} &= \left\| \frac{1}{(1+|x|^2)^n}(1+|x|^2)^n \cdot \mathcal{F}(\psi_{j,\,\theta}) \right\|_{L^1(\mathbb{R}^n)} \leq \\
&\leq \left\| \frac{1}{(1+|x|^2)^n} \right\|_{L^2(\mathbb{R}^n)} \left\| (1+|x|^2)^n \mathcal{F}(\psi_{j,\,\theta}) \right\|_{L^2(\mathbb{R}^n)} \lesssim \\
&\lesssim C' \left\| \mathcal{F}\left(Q(-\imath D)\psi_{j,\,\theta}\right)\right\|_{L^2(\mathbb{R}^n)} = \\
&= \|Q(-\imath D)\psi_{j,\,\theta}\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where $Q(x) := (1+|x|^2)^n$, while the red identity follows from Theorem 5.13. It remains to prove that

$$\|Q(-\imath D)\psi_{j,\,\theta}\|_{L^2(\mathbb{R}^n)} \xrightarrow{j\to+\infty} 0$$

uniformly with respect to θ. However, for all θ, we have

$$\psi_{\theta,\,j} \xrightarrow{\mathcal{D}} 0,$$

which means that there exists a compact set K such that the support of every such function is contained there and

$$D^\alpha \psi_{\theta,\,j} \xrightarrow{\text{uniformly in } K} 0$$

for every multi-index $\alpha \in \mathbb{N}^n$. Since θ ranges in a compact set, it turns out that

$$\|Q(-\imath D)\,\psi_{j,\,\theta}\|_{L^2(\mathbb{R}^n)} \xrightarrow{j\to+\infty} 0$$

uniformly with respect to θ, which is exactly what we needed to prove to conclude. □

7.2 Introduction to Sobolev Spaces

We will now discuss some basic results concerning $(m,\,2)$-Sobolev spaces, e.g. embedding-type theorems, which will be essential to prove the main result of the elliptic operator regularity theory.

Definition 7.6 (Sobolev Space)**.** The $(m,\,p)$-Sobolev space on $\mathbb{R}^n$ is defined by setting

$$W^{m,\,p}(\mathbb{R}^n) := \{ f \in \mathcal{D}'(\mathbb{R}^n) \mid D^\alpha f \in L^p(\mathbb{R}^n) \text{ for every } |\alpha| \leq m \}.$$

Furthermore, if $\Omega \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, the *local* Sobolev space is defined by

$$W^{m,\,p}_{\text{loc}}(\mathbb{R}^n) := \{ f \in \mathcal{D}'(\mathbb{R}^n) \mid D^\alpha f \in L^p_{\text{loc}}(\Omega) \text{ for every } |\alpha| \leq m \}$$

Note

The main idea is to use the fact that the Fourier transform is an isometry from L^2 to L^2.

The primary goal of this paragraph is to prove an embedding-type theorem for Sobolev spaces with p equal to 2.

Theorem 7.7 (Sobolev)**.** *The inclusion*

$$W^{m,\,2}(\mathbb{R}^n) \subseteq C^r(\mathbb{R}^n),$$

holds, provided that

$$r < m - \frac{n}{2}.$$

Theorem 7.8 (Sobolev). *The inclusion*

$$W^{m,\,2}_{\text{loc}}(\Omega) \subseteq C^r(\Omega),$$

holds, provided that

$$r < m - \frac{n}{2}.$$

Lemma 7.9. *Let $f \in L^1(\mathbb{R}^n)$ be a summable function, and assume that $x_j f \in L^1(\mathbb{R}^n)$ for all $j = 1, \ldots, n$. Then*

$$\mathcal{F}(f) \in C^1(\mathbb{R}^n) \quad \textit{and} \quad \partial_j \mathcal{F}(f) = D^j \mathcal{F}(f),$$

which means that the weak partial derivative coincides with the classical partial derivative.

Proof. Recall that the Fourier transform of a summable function $g \in L^1(\mathbb{R}^n)$ is a continuous function vanishing at infinity, that is,

$$g \in L^1(\mathbb{R}^n) \implies \mathcal{F}(g) \in C^0_0(\mathbb{R}^n).$$

In our case it turns out that

$$\mathcal{F}(f) \in C^0_0(\mathbb{R}^n) \quad \text{and} \quad \mathcal{F}\left(x_j f\right) = D^j \mathcal{F}(f) \in C^0_0(\mathbb{R}^n).$$

The thesis will now follow if we are able to show that the incremental ratio

$$\lim_{h \to 0} \frac{\mathcal{F}(f)(\xi + h e_j) - \mathcal{F}(f)(\xi)}{h}$$

coincides with the weak derivative $D^j \mathcal{F}(f)$ for all j. A straightforward computation shows that

$$\begin{aligned}
\lim_{h \to 0} \frac{\mathcal{F}(f)(\xi + h e_j) - \mathcal{F}(f)(\xi)}{h} &= \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\imath \langle x,\, \xi \rangle} \left(\frac{\mathrm{e}^{-\imath x_j h} - 1}{h} \right) \mathrm{d}x \overset{*}{=} \\
&\overset{*}{=} -\imath \int_{\mathbb{R}^n} x_j f(x) \mathrm{e}^{-\imath \langle x,\, \xi \rangle} \, \mathrm{d}x = \\
&= -\imath \mathcal{F}\left(x_j f\right)(\xi) = D^j \mathcal{F}(f),
\end{aligned}$$

where the equality $*$ follows from the Lebesgue dominated convergence theorem. □

Proposition 7.10. *Let $r \in \mathbb{N}$ be a positive integer. Then*

$$(1 + |x|)^r f(x) \in L^1(\mathbb{R}^n) \implies \mathcal{F}(f) \in C^r(\mathbb{R}^n).$$

Proof. We argue by induction. The base step has already been discussed above, so we only deal with the inductive step.

Inductive Step. Suppose that the thesis holds true for some integer k such that $k - 1 < r$ with

$$(1 + |x|)^k f(x) \in L^1(\mathbb{R}^n).$$

The inductive assumption tells us that

$$D^\alpha \mathcal{F}(f) \in C^0(\mathbb{R}^n) \quad \text{for all } \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq r - 1,$$

and therefore it is enough to check the regularity of the derivatives

$$D^j\left[D^\alpha \mathcal{F}(f)\right] \quad \text{for } j = 1, \ldots, n.$$

Now $D^j D^\alpha \mathcal{F}(f)$ is equal to $\mathcal{F}(x_j x^\alpha f)$ and

$$|x^\alpha f| \leq (1+|x|)^{|\alpha|}|f| \in L^1(\mathbb{R}^n) \quad \text{for all } |\alpha| \leq r.$$

We finally apply Lemma 7.9 and infer that

$$D^j\left[D^\alpha \mathcal{F}(f)\right] \in C^0(\mathbb{R}^n) \quad \text{for } j = 1, \ldots, n.$$

□

Lemma 7.11. *Let $m \in \mathbb{R}_{\geq 0}$. The estimate*

$$(1+|x|)^{2m} \leq 2^m(1+n)^m\left(1+|x_1|^{2m}+\cdots+|x_n|^{2m}\right) \tag{7.12}$$

holds for all $x \in \mathbb{R}^n$.

Proof. Let $k \in \{1, \ldots, n\}$ be an index realising the maximum absolute value, that is,

$$|x_k|^2 = \max_{i=1,\ldots,n} |x_i|^2,$$

and notice that

$$|x|^2 \leq n|x_k|^2.$$

It follows that

$$(1+|x|)^2 \leq \left(1+\sqrt{n}|x_k|\right)^2 \leq 2\left(1+n|x_k|^2\right),$$

where the last inequality comes from the well-known estimate

$$\frac{1}{2}(a+b)^2 \leq a^2+b^2.$$

We now take the mth power of the inequality above and obtain the following chain that leads to the desired estimate:

$$\begin{aligned}(1+|x|)^{2m} &\leq 2^m\left(1+n|x_k|^2\right)^m \leq \\ &\leq 2^m \sum_{j=0}^m \binom{m}{j} n^j |x_i|^{2j} \leq \\ &\leq 2^m \sum_{j=0}^m \binom{m}{j} n^j \left(1+|x_k|^{2m}\right) = \\ &= 2^m(1+n)^m\left(1+|x_k|^{2m}\right) \leq \\ &\leq 2^m(1+n)^m\left(1+|x_1|^{2m}+\cdots+|x_n|^{2m}\right).\end{aligned}$$

□

Proof of Theorem 7.7. By assumption $D^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leq m$. Therefore

$$\mathcal{F}\left(D^\alpha f\right) = (\imath x)^\alpha\, \mathcal{F}(f) \in L^2(\mathbb{R}^n)$$

since the Fourier transform maps L^2 into L^2 isometrically. The assumption on r allows us to estimate the $L^1(\mathbb{R}^n)$-norm of $(1+|x|^r)\mathcal{F}(f)$ as follows:

$$\begin{aligned}
\|(1+|x|)^r \mathcal{F}(f)\|_{L^1(\mathbb{R}^n)} &= \left\|(1+|x|)^{r-m}\,(1+|x|)^m\, \mathcal{F}(f)\right\|_{L^1(\mathbb{R}^n)} \leq \\
&\leq \left\|(1+|x|)^{r-m}\right\|_{L^2(\mathbb{R}^n)} \|(1+|x|)^m\, \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} \leq \\
&\lesssim \|(1+|x|)^m\, \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} .
\end{aligned}$$

To conclude we need to estimate this L^2-norm, but it is worth noticing that

$$\left\|(1+|x|)^{r-m}\right\|_{L^2(\mathbb{R}^n)} \leq C$$

provided that $2(r-m) < -n$, which is exactly our assumption. Using the algebraic result (7.12), it turns out that

$$\begin{aligned}
\|(1+|x|)^m\, \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|x|)^{2m}\, |\mathcal{F}(f)|^2\,(x)\,\mathrm{d}x \lesssim \\
&\lesssim \int_{\mathbb{R}^n} \left(1+|x_1|^{2m} + \cdots + |x_n|^{2m}\right) |\mathcal{F}(f)\|^2\,(x)\,\mathrm{d}x \lesssim \\
&\lesssim \left[\|\mathcal{F}(f)\|^2_{L^2(\mathbb{R}^n)} + \sum_{j=1}^n \left\|x_j^m \mathcal{F}(f)\right\|^2_{L^2(\mathbb{R}^n)}\right] \simeq \\
&\simeq \left[\|f\|^2_{L^2(\mathbb{R}^n)} + \sum_{j=1}^n \|D^{\alpha_j} f\|^2_{L^2(\mathbb{R}^n)}\right],
\end{aligned}$$

where $\alpha_j = m e_j$. In particular, the function $(1+|x|)^r$ belongs to $L^1(\mathbb{R}^n)$, and hence the function f is of class $C^r(\mathbb{R}^n)$ as a consequence of Proposition 7.10. ☐

Proof of Theorem 7.8. Let $f \in W^{m,2}_{\mathrm{loc}}(\Omega)$. Let $x \in \Omega$ be an arbitrary point, and let $U_x \ni x$ be an open neighborhood of x such that

$$\overline{U_x} \Subset \Omega.$$

Let ψ be a cutoff function satisfying $\psi\big|_{U_x} \equiv 1$ and $\mathrm{spt}(\psi) \subset \Omega$. The function

$$\tilde{f} := \psi \cdot f$$

belongs to $W^{m,2}(\mathbb{R}^n)$ by construction. Indeed, the usual Leibniz rule implies that

$$D^\alpha\left(\psi \cdot f\right) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^\gamma \psi D^{\alpha-\gamma} f \in L^2(\mathbb{R}^n)$$

for all $|\alpha| \leq m$. The global Sobolev Embedding Theorem 7.7 implies that $\tilde{f} \in C^r(\mathbb{R}^n)$, that is,

$$f \in C^r\left(U_x\right),$$

and we conclude using the arbitrariness of $x \in \Omega$. ☐

7.3 Fractional Sobolev Spaces

Let $s \in \mathbb{R}$. We denote by $L^2_{\mu_s}(\mathbb{R}^n)$ the weighted L^2-space with respect to the measure

$$\mathrm{d}\mu_s(x) := \left(1 + |x|^2\right)^s \, \mathrm{d}x$$

that is, the space of square-integrable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |x|^2\right)^s \, \mathrm{d}x < +\infty.$$

Definition 7.12. The s-Sobolev space, denoted by $H^s(\mathbb{R}^n)$, is the space of all the tempered distributions $f \in \mathscr{S}'$ such that

$$\mathcal{F}(f) \in L^2_{\mu_s}(\mathbb{R}^n).$$

In other words, we define

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathscr{S}' \;\middle|\; \int_{\mathbb{R}^n} |\mathcal{F}(f)(x)|^2 \left(1 + |x|^2\right)^s \, \mathrm{d}x < +\infty \right\}.$$

Exercise 7.1. Prove that

$$\|f\|_{H^s} := \|\mathcal{F}(f)\|_{L^2_{\mu_s}(\mathbb{R}^n)} \tag{7.13}$$

is a well-defined norm on the space $H^s(\mathbb{R}^n)$.

Definition 7.13. The local space $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by all tempered distributions f satisfying the following property: for any $x \in \Omega$ we can find an open neighborhood $U_x \ni x$, with

$$\overline{U_x} \Subset \Omega,$$

such that $f\big|_{U_x} = g\big|_{U_x}$ for some $g \in H^s(\mathbb{R}^n)$.

Theorem 7.14. *For all $m \in \mathbb{N}$ the m-Sobolev space corresponds to the W-space, that is,*

$$W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n).$$

Proof. Let $f \in W^{m,2}(\mathbb{R}^n)$ be any Sobolev function. Then

$$\|f\|^2_{H^m} = \int_{\mathbb{R}^n} |\mathcal{F}(f)|^2(x) \left(1 + |x|^2\right)^s \, \mathrm{d}x =$$

$$= \sum_{|\alpha| \le m} c_\alpha \|D^\alpha f\|^2_{L^2(\mathbb{R}^n)} < +\infty,$$

and this implies that $f \in H^m(\mathbb{R}^n)$. In a similar fashion, if $f \in H^m(\mathbb{R}^n)$, then we can use the properties of the Fourier transform to infer that

$$\|D^\alpha f\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(D^\alpha f)\|^2_{L^2(\mathbb{R}^n)} =$$

$$= \|x^\alpha \mathcal{F}(f)\|^2_{L^2(\mathbb{R}^n)} < +\infty$$

for any $|\alpha| \le m$. Note that we used the obvious inequality

$$|x^\alpha|^2 \le |x|^{2|\alpha|} \le \left(1 + |x|^2\right)^{|\alpha|}$$

$\square$

Remark 7.2. We now list a few properties of s-Sobolev spaces that are immediate from the definition and some well-known facts.

(1) We have
$$H^0(\mathbb{R}^n) = W^{0,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n).$$

(2) The Fourier transform
$$\mathcal{F} : H^s(\mathbb{R}^n) \to L^2_{\mu_s}(\mathbb{R}^n)$$
is an isometry. Consequently $H^s(\mathbb{R}^n)$ is a Hilbert space for every $s \in \mathbb{R}$.

(3) If $s \geq t$, then
$$H^s(\mathbb{R}^n) \subseteq H^t(\mathbb{R}^n).$$

Definition 7.15 (Order)**.** Let L be an operator defined on $H^*(\mathbb{R}^n) := \cup_{s\in\mathbb{R}} H^s(\mathbb{R}^n)$. The *order* of L is the minimal integer k, if there exists one, such that
$$L\left(H^s(\mathbb{R}^n)\right) \subseteq H^{s-k}(\mathbb{R}^n) \quad \text{for all } s \in \mathbb{R}.$$

Lemma 7.16. *Let $\varphi \in \mathcal{E}'$ be a distribution of order N. Then $\varphi \in H^s(\mathbb{R}^n)$ for all s satisfying*
$$s < -N - \frac{n}{2}.$$

Proof. The Fourier transform of φ satisfies the estimate
$$|\mathcal{F}(\varphi)|(x) \lesssim (1 + |x|)^N$$
as a consequence of the Paley-Wiener Theorem 6.5. Therefore
$$\|\varphi\|^2_{H^s(\mathbb{R}^n)} = \|\mathcal{F}(\varphi)\|_{L^2_{\mu_s}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} (1 + |x|)^{2N} \left(1 + |x|^2\right)^s \, \mathrm{d}x,$$
and this integral is finite if and only if
$$2N + 2s + n - 1 < -1,$$
which is exactly what we wanted to show. ☐

Lemma 7.17 (Operators Order)**.**

(a) *Let $t \in \mathbb{R}$. The operator $L_t : H^*(\mathbb{R}^n) \longrightarrow H^*(\mathbb{R}^n)$ defined by*
$$L_t(u) := \mathcal{F}\left(\left(1 + |x|^2\right)^{\frac{t}{2}} \mathcal{F}(u)\right)$$
has order t.

(b) *Let $f \in L^\infty(\mathbb{R}^n)$. The operator $L_f : H^*(\mathbb{R}^n) \longrightarrow H^*(\mathbb{R}^n)$ defined by*
$$L_f(u) := \mathcal{F}\left(f \cdot \mathcal{F}(u)\right)$$
has order zero.

(c) *The derivative operator $D^\alpha : H^*(\mathbb{R}^n) \longrightarrow H^*(\mathbb{R}^n)$ has order $|\alpha|$.*

Proof.

(a) Let $s \in \mathbb{R}$. Then

$$\begin{aligned}
\|L_t(u)\|^2_{H^s(\mathbb{R}^n)} &= \left\| \left(1+|x|^2\right)^{\frac{t}{2}} \mathcal{F}(u) \right\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} = \\
&= \int_{\mathbb{R}^n} |\mathcal{F}(u)|^2 (x) \left(1+|x|^2\right)^t \left(1+|x|^2\right)^s \, \mathrm{d}x = \\
&= \|\mathcal{F}(u)\|^2_{L^2_{\mu_{s+t}}(\mathbb{R}^n)} = \\
&= \|u\|^2_{H^{s+t}(\mathbb{R}^n)}.
\end{aligned}$$

(b) A straightforward computation proves that

$$\begin{aligned}
\|L_f(u)\|^2_{H^s(\mathbb{R}^n)} &= \|f \cdot \mathcal{F}(u)\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} = \\
&= \int_{\mathbb{R}^n} |f \cdot \mathcal{F}(u)|^2 (x) \left(1+|x|^2\right)^s \, \mathrm{d}x \leq \\
&\leq \|f\|^2_{L^\infty(\mathbb{R}^n)} \|\mathcal{F}(u)\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} = \\
&= \|f\|^2_{L^\infty(\mathbb{R}^n)} \|u\|^2_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

(c) A straightforward computation proves that

$$\begin{aligned}
\|D^\alpha u\|^2_{H^s(\mathbb{R}^n)} &= \|x^\alpha \mathcal{F}(u)\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} = \\
&= \int_{\mathbb{R}^n} |\mathcal{F}(u)|^2 (x) |x|^{2\alpha} \left(1+|x|^2\right)^s \, \mathrm{d}x \leq \\
&\leq \int_{\mathbb{R}^n} |\mathcal{F}(u)|^2 (x) \left(1+|x|^2\right)^{s+|\alpha|} \, \mathrm{d}x = \\
&= \|\mathcal{F}(u)\|^2_{L^2_{\mu_{s+|\alpha|}}(\mathbb{R}^n)} = \\
&= \|u\|^2_{H^{s+|\alpha|}(\mathbb{R}^n)}.
\end{aligned}$$

□

Proposition 7.18. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a function satisfying*

$$\left(1+|\cdot|^2\right)^{-\frac{N}{2}} f \in L^\infty(\mathbb{R}^n).$$

Then the operator defined by

$$F_f(u) := \mathcal{F}\left(f \cdot \mathcal{F}(u)\right)$$

has order N.

Proof. The main idea is to prove that F is, up to overturning, the composition of the operators L_g and L_N introduced in Lemma 7.17. Let

$$g := \left(1 + |\cdot|^2\right)^{-\frac{N}{2}} f \in L^\infty(\mathbb{R}^n).$$

We claim that the identity

$$F(u) = L_g\left(\check{L}_N(u)\right)$$

holds for all u. Indeed, we have that

$$\begin{aligned}
L_g\left(\check{L}_N(u)\right) &= \mathcal{F}\left(\left(1 + |x|^2\right)^{-\frac{N}{2}} f(x)\mathcal{F}\left(L_N(u)\right)\right) = \\
&= \mathcal{F}\left(\left(1 + |x|^2\right)^{-\frac{N}{2}} f(x)\mathcal{F}\left(\check{\mathcal{F}}\left(\left(1 + |x|^2\right)^{\frac{N}{2}} \mathcal{F}(u)\right)\right)\right) = \\
&= \mathcal{F}\left(\left(1 + |x|^2\right)^{-\frac{N}{2}} f(x) \underbrace{\mathcal{F} \circ \check{\mathcal{F}}}_{=\mathrm{id}}\left(\left(1 + |x|^2\right)^{\frac{N}{2}} \mathcal{F}(u)\right)\right) = \\
&= \mathcal{F}\left(f \cdot \mathcal{F}(u)\right) = F(u),
\end{aligned}$$

and this concludes the proof since the order of the composition is the sum of the orders. □

Proposition 7.19. *Let $u \in H^s(\mathbb{R}^n)$ be a Sobolev function and $\varphi \in \mathscr{S}$ a Schwartz function. Then the product is a Sobolev function, that is,*

$$\varphi u \in H^s(\mathbb{R}^n).$$

Proof. We may equivalently show that

$$\|\varphi u\|_{H^s(\mathbb{R}^n)} < +\infty.$$

We know (Theorem 5.26) that the Fourier of the product is, up to a constant, the convolution of the Fourier transforms. Hence, we have the following chain of equalities:

$$\begin{aligned}
\|\varphi u\|_{H^s(\mathbb{R}^n)} &= \|\mathcal{F}(\varphi u)\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} \simeq \\
&\simeq \|\mathcal{F}(\varphi) * \mathcal{F}(u)\|^2_{L^2_{\mu_s}(\mathbb{R}^n)} = \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathcal{F}(u)(x-y)\mathcal{F}(\varphi)(y)\,\mathrm{d}y \right| \mathrm{d}\mu_s(x).
\end{aligned}$$

We now notice that a Schwartz function $\psi \in \mathscr{S}$ belongs to $H^t(\mathbb{R}^n)$ for all $t \in \mathbb{R}$. Indeed, we can always estimate the H^t-norm as follows:

$$\|\psi\|_{H^t(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathcal{F}(\psi)|^2 (x)(1 + |x|^2)^t \,\mathrm{d}x < +\infty$$

since the Fourier transform $\mathcal{F}$ maps $\mathscr{S}$ into $\mathscr{S}$. In particular, it turns out that

$$\|\varphi u\|_{H^s(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathcal{F}(u)(x-y)\mathcal{F}(\varphi)(y)\,\mathrm{d}y \right| \mathrm{d}\mu_s(x) =$$

$$= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{\mathcal{F}(u)(x-y)}{(1+|y|^2)^{t/2}} \mathcal{F}(\varphi)(y)(1+|y|^2)^{\frac{t}{2}}\,\mathrm{d}y \right| \mathrm{d}\mu_s(x) \leq$$

$$\leq \int_{\mathbb{R}^n} \left[\|\mathcal{F}(\varphi)\|_{L^2_{\mu_t}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\mathcal{F}(u)|^2(x-y)}{(1+|y|^2)^t}\,\mathrm{d}y \right]$$

$$mathrmd\mu_s(x) \leq$$

$$\leq \|\varphi\|^2_{H^t(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |\mathcal{F}(u)|^2(x)(1+|x-y|^2)^s\,\mathrm{d}x \right] (1+|y|^2)^{-t}\,\mathrm{d}y,$$

where the last inequality follows from the change of variables $x - y \longmapsto x$ and the Fubini-Tonelli theorem. At this point, we need to state and use an useful algebraic inequality:

Lemma 7.20. *For every x, $y \in \mathbb{R}^n$ and every $s \in \mathbb{R}$ it turns out that*

$$\left(1+|x+y|^2\right)^s \leq 2^{|s|}\left(1+|x|^2\right)^s \left(1+|y|^2\right)^{|s|}. \tag{7.14}$$

Proof. It suffices to check the inequality (7.14) for $s = \pm 1$. This is a trivial exercise, and it is left to the reader to fill in the details. □

If we apply (7.14) to the inequality above, we find that

$$\|\varphi u\|_{H^s(\mathbb{R}^n)} \lesssim \|\varphi\|^2_{H^t(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |\mathcal{F}(u)|^2(x)(1+|x-y|^2)^s\,\mathrm{d}x \right] (1+|y|^2)^{-t}\,\mathrm{d}y \lesssim$$

$$\lesssim \|\varphi\|^2_{H^t(\mathbb{R}^n)} \|u\|^2_{H^s(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{(1+|y|^2)^s}{(1+|y|^2)^t}\,\mathrm{d}y,$$

and the last term is finite since $t \in \mathbb{R}$ is arbitrarily big. □

7.4 Elliptic Regularity Theorem

In this final section, we want to state and prove the regularity theorem for an elliptic operator. From now on, we denote by Ω an open subset of $\mathbb{R}^n$.

Theorem 7.21 (Regularity)**.** *Let L be an elliptic operator of order N and suppose that*

$$L(u) = \sum_{|\alpha|<N} g_\alpha(-\imath D)^\alpha u + \sum_{|\alpha|=N} c_\alpha(-\imath D)^\alpha u,$$

for some smooth functions $g_\alpha \in C^\infty(\Omega)$ and complex constants $c_\alpha \in \mathbb{C}$. Then

$$L(u) \in H^s_{\mathrm{loc}}(\Omega) \implies u \in H^{s+N}_{\mathrm{loc}}(\Omega).$$

In particular, a function u such that $L(u) = 0$ is necessarily smooth, that is, $u \in C^\infty(\Omega)$.

Proof. Let $p(x)$ denote the polynomial associated to the maximum-order terms of the elliptic operator L, that is,

$$p(x) := \sum_{|\alpha|=N} c_\alpha x^\alpha.$$

We also introduce the auxiliary function

$$r(x) := \frac{1+|x|^N}{|x|^N} p(x).$$

Step 1. We now observe that the function

$$r(x)\cdot(1+|x|^2)^{-\frac{N}{2}} = \frac{(1+|x|^N)}{(1+|x|^2)^{N/2}}\frac{p(x)}{|x|^N}$$

belongs to $L^\infty(\Omega)$ since it is given by the product of two L^∞ functions. In fact, the polynomial $p(x)$ is homogeneous of degree N and L is elliptic, which means that

$$\min_{x\in S^n} |p(x)| > 0 \implies c_1|x|^N < |p(x)| < c_2|x|^N$$

for every $x \in \mathbb{R}^n \setminus \{0\}$. In a similar fashion, one can prove that the function

$$\frac{1}{r(x)}\cdot(1+|x|^2)^{\frac{N}{2}}$$

also belongs to $L^\infty(\Omega)$. It follows from Proposition 7.18 that the operator

$$R(w) := \check{\mathcal{F}}\left(r\cdot\mathcal{F}(w)\right)$$

has order equal to N, while the operator

$$R^{-1}(w) := \check{\mathcal{F}}\left(\frac{1}{r}\cdot\mathcal{F}(w)\right)$$

has order equal to $-N$. Furthermore, we can check that R^{-1} is the inverse of R:

$$R^{-1}\left(R(w)\right) = \check{\mathcal{F}}\left(\frac{1}{r}\cdot\mathcal{F}\circ\check{F}\left(r\cdot\mathcal{F}(w)\right)\right) = w,$$

$$R\left(R^{-1}(w)\right) = \check{\mathcal{F}}\left(r\cdot\mathcal{F}\circ\check{F}\left(\frac{1}{r}\cdot\mathcal{F}(w)\right)\right) = w.$$

We now claim that the operator $P-R$ has order zero. Indeed, it suffices to notice that

$$\begin{aligned}(P-R)(w) &= \check{\mathcal{F}}\left(p\cdot\mathcal{F}(w)\right) - \check{\mathcal{F}}\left(r\cdot\mathcal{F}(w)\right) = \\ &= \check{\mathcal{F}}\left((p-r)\cdot\mathcal{F}(w)\right) = \\ &= \check{\mathcal{F}}\left(-\frac{1}{|x|^N}\,p\cdot\mathcal{F}(w)\right),\end{aligned}$$

and the latter is an operator of order zero as a consequence of Lemma 7.17 since

$$-\frac{1}{|x|^N}\cdot p(x) \in L^\infty(\Omega).$$

Step 2. Fix $x \in \Omega$ and let $U_x \ni x$ be a neighbourhood of x. Let $\varphi_0 \in \mathcal{D}(\Omega)$ be a cutoff function such that φ_0 is identically one over U_x. Clearly, the function

$$u_0 := \varphi_0 \cdot u : \mathbb{R}^n \to \mathbb{C}$$

belongs to $\mathcal{E}'$, and therefore it belongs to $H^t(\mathbb{R}^n)$ for some t.

Step 3. The idea is to use a recursive process (long $k = s + N - t$) and, at each step, gain one order of regularity. Let $\varphi_1, \dots, \varphi_k$ be cutoff functions satisfying

$$\varphi_j(x') = 1 \quad \text{for all } x' \in U_x^{(j)} \quad \text{and} \quad \operatorname{spt} \varphi_j \subset U_x^{(j-1)}.$$

Step 4. Suppose that $u_j := u \cdot \varphi_j$ belongs to $H^{t+j}(\Omega)$. By construction we have $u_{j+1} = \varphi_{j+1} \cdot u_j$ since $\varphi_{j+1} \cdot \varphi_j = \varphi_{j+1}$. Let us consider the operator

$$T[w] := L[\varphi_{j+1} \cdot w] - \varphi_{j+1} \cdot L[w].$$

A straightforward computation proves that

$$D^\alpha \left(\varphi_{j+1} \cdot w\right) - \varphi_{j+1} \cdot D^\alpha(w) = \sum_{\beta < \alpha} c_\beta \, D^\beta(w) \, D^{\alpha-\beta}(\varphi_{j+1}),$$

that is,

$$T[w] = \sum_{|\alpha| < N} \widetilde{g}_\alpha \, D^\alpha w,$$

which means that the order of T is at most $N - 1$. On the other hand,

$$T[u_j] = L[\varphi_{j+1} \cdot u_j] - \varphi_{j+1} \cdot L[u_j] =$$

$$= L\left[u_{j+1}\right] - \varphi_{j+1} \cdot L[u],$$

from which it follows that

$$L\left[u_{j+1}\right] = T[u_j] + \varphi_{j+1} \cdot L[u].$$

Since $T[u_j] \in H^{t+k-(N-1)}(\mathbb{R}^n)$ and $L[u] \in H^s(\mathbb{R}^n)$, it turns out that

$$L\left[u_{j+1}\right] \in H^{t+j-N+1}(\mathbb{R}^n).$$

In a similar fashion, one can prove that

$$S\left[u_{j+1}\right] \in H^{t+j-N+1}(\mathbb{R}^n) \qquad \text{and} \qquad (P - R)[u_{j+1}] \in H^{t+j}(\mathbb{R}^n) \subset H^{t+j-N+1}(\mathbb{R}^n),$$

and hence

$$R[u_{j+1}] = \left(L - S - (P - R)\right)\left[u_{j+1}\right] \in H^{t+j-N+1}(\mathbb{R}^n).$$

In conclusion, we apply the inverse of R (which is an operator of order $-N$, and we obtain

$$u_{j+1} = R^{-1} \circ R[u_{j+1}] \in H^{t+j+1}(\mathbb{R}^n),$$

which is exactly what we wanted to prove. $\square$

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Bibliography

[1] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Science/Engineering/Math, 1991.