Bayesian Learning and Montecarlo Simulation

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1 Problem Description

This report examines two time series tracking the evolution of the US GDP and CPIAUCSL from 1948 to 2021.
The US Gross Domestic Product (GDP) represents the total market value of goods and services produced within the United States. The Consumer Price Index for All Urban Consumers (CPIAUCSL) reflects the average cost of a basket of goods and services purchased by urban consumers. Both metrics are reported quarterly and have been seasonally adjusted. In particular, we analyze their percentage changes over time.
The objectives of our project are as follows:

- Fit each time series independently using AR, MA, ARMA, and GARCH models.
- Fit the two time series jointly using a VAR model.
- Use these models for in-sample and out-of-sample predictions.
- Compare different models using DIC and WAIC criteria.

All models were implemented using JAGS with the following specifications:

- 3 Chains.
- Total of 10,000 Iterations.
- 1,000 Burn-in Iterations.

The data fed into JAGS consisted of only the first 90% of each time series. The remaining 10% was used to assess the out-of-sample predictions generated by the models. Additionally, we examined trace plots to identify any potential issues and compared our results with those obtained using publicly available libraries and functions. Detailed findings are provided in the Appendix.

2 Autoregressive Model

An autoregressive model (AR model) is well-suited for time series analysis, as it operates on the premise that past values of a variable influence its current values. In essence, the AR model states that the output variable is linearly dependent on its own previous values along with a stochastic term.
Formally, an autoregressive model of order p is represented as:

$$y_t = \mu_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \epsilon_t \tag{1}$$

where p is the number of past values considered, μ_0 is a constant, α_i are the model parameters, ϵ_t is white noise, and $y_{t-1}, y_{t-2}, ..., y_{t-p}$ are the past values.
For our purposes, we decided to implement the AR model of order 1 (AR(1)).

AR(1)

The AR(1) model is defined as follows:

$$y_t = \mu_0 + \alpha y_{t-1} + \epsilon_t \tag{2}$$

In our case study, we assume that ϵ_t are independent and identically distributed variables from a normal distribution with mean 0 and variance σ^2, i.e., $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, leading to the following likelihood:

$$y_t | \mu_0, \alpha, \sigma^2, y_{t-1} \sim \mathcal{N}(\mu_0 + \alpha y_{t-1}, \sigma^2) \tag{3}$$

For the priors, we chose:

$$\begin{aligned} \mu_0 &\sim \mathcal{N}(0.0, 10000) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(2, 0.1) \\ \alpha &\sim \mathcal{U}(-1.0, 1.0) \end{aligned} \tag{4}$$

We selected these priors to ensure uninformative distributions for μ_0 and σ^2, while for α, we used a uniform distribution between -1 and 1 to satisfy the model's stationarity condition, i.e., $|\alpha| < 1$.
Running the JAGS code to implement the AR(1) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 1, with the corresponding means and 95% credible intervals reported in Table 1.

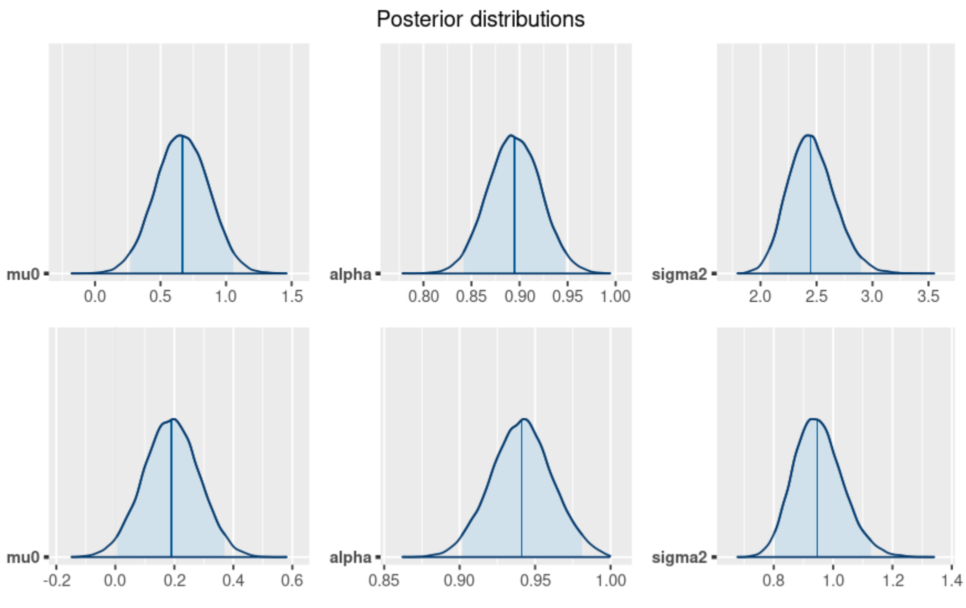


Fig. 1. Posterior distributions of the parameters for the AR(1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

Model target variable	Parameter	Posterior Mean	95% Credible Interval
GDP	μ_0	0.6646012	(0.269216192, 1.0554382)
GDP	α	0.8946932	(0.842044515, 0.9481597)
GDP	σ^2	2.4565718	(2.080629925, 2.8999602)
CPIAUCSL	μ_0	0.1891912	(0.006510675, 0.3713430)
CPIAUCSL	α	0.9411750	(0.901591825, 0.9812118)
CPIAUCSL	σ^2	0.9510602	(0.804012999, 1.1263014)

Table 1. *Posterior means and 95% credible intervals for the parameters of the AR(1) models.*

As seen from the posteriors, the α parameter is close to 1 for both GDP and CPIAUCSL. However, when using a prior for α over a larger range, i.e., $\alpha \sim \mathcal{U}(-2.0, 2.0)$, the estimated value of α did not change significantly.
We also examined the trace plots and autocorrelation plots and found no significant issues.
Lastly, we plotted the in-sample and out-of-sample predictions with 95% credible intervals and compared them with the actual data. The results are shown in Figures 2 and 3.
Analyzing the out-of-sample predictions, we observe that the AR(1) model successfully captures the initial data trend. However, the model fails to anticipate the significant impact of the COVID-19 pandemic, which led to a sharp decline in GDP and a steep rise in CPIAUCSL.

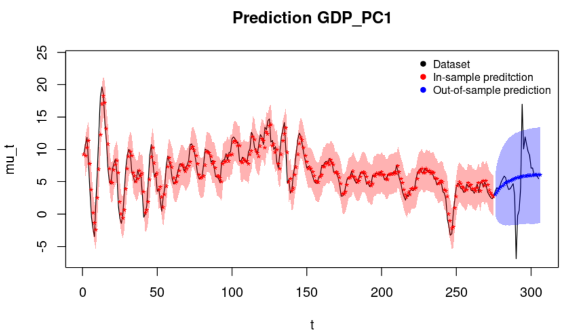


Fig. 2. In-sample and out-of-sample predictions for the GDP using the AR(1) model.

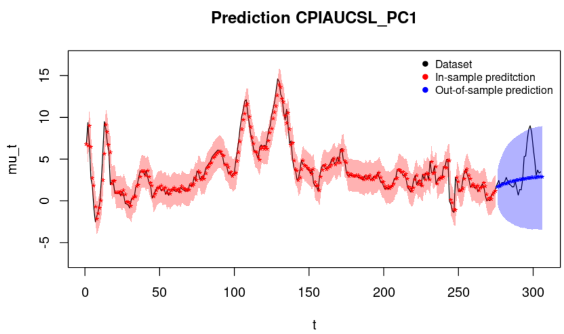


Fig. 3. In-sample and out-of-sample predictions for the CPIAUCSL using the AR(1) model.

Lastly, we compared the model's in-sample predictions and posterior distributions with the results of the AR model using the ARIMA function. This analysis revealed a strong similarity between the outcomes of both approaches.

3 Moving Average Model

A Moving Average (MA) model is another approach used in univariate time series analysis. Unlike the Autoregressive (AR) model, where the target variable depends on its past values, the MA model relies on past errors.
Formally, a Moving Average model of order q is represented as:

$$y_t = \mu_0 + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} + \epsilon_t \tag{5}$$

where q is the number of past errors considered, μ_0 is the mean of the series, $\theta_1, \theta_2, ..., \theta_q$ are the model coefficients, and $\epsilon_t, \epsilon_{t-1}, ..., \epsilon_{t-q}$ are the error terms.
For our analysis, we first implemented the MA model of order 1 (MA(1)). Then, by examining the autocorrelation plots for the two time series, we noticed a significant correlation at 2 lags. Consequently, we concluded that an MA model of order 2 (MA(2)) would perform better, and we implemented it.

MA(1)

The MA(1) model is defined as follows:

$$y_t = \mu_0 + \theta\epsilon_{t-1} + \epsilon_t \tag{6}$$

In our case study, we assume that ϵ_t are independent and identically distributed variables from a normal distribution with mean 0 and variance σ^2, i.e., $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, leading to the following likelihood:

$$y_t|\mu_0, \theta, \sigma^2, \epsilon_{t-1} \sim \mathcal{N}(\mu_0 + \theta\epsilon_{t-1}, \sigma^2) \tag{7}$$

For the priors, we chose:

$$\begin{aligned} \mu_0 &\sim \mathcal{N}(0.0, 10000) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(2, 0.1) \\ \theta &\sim \mathcal{U}(-1.0, 1.0) \end{aligned} \tag{8}$$

We selected uninformative priors for all the parameters: μ_0, σ^2, and θ.
Running the JAGS code to implement the MA(1) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 4, with the corresponding means and 95% credible intervals reported in Table 2.

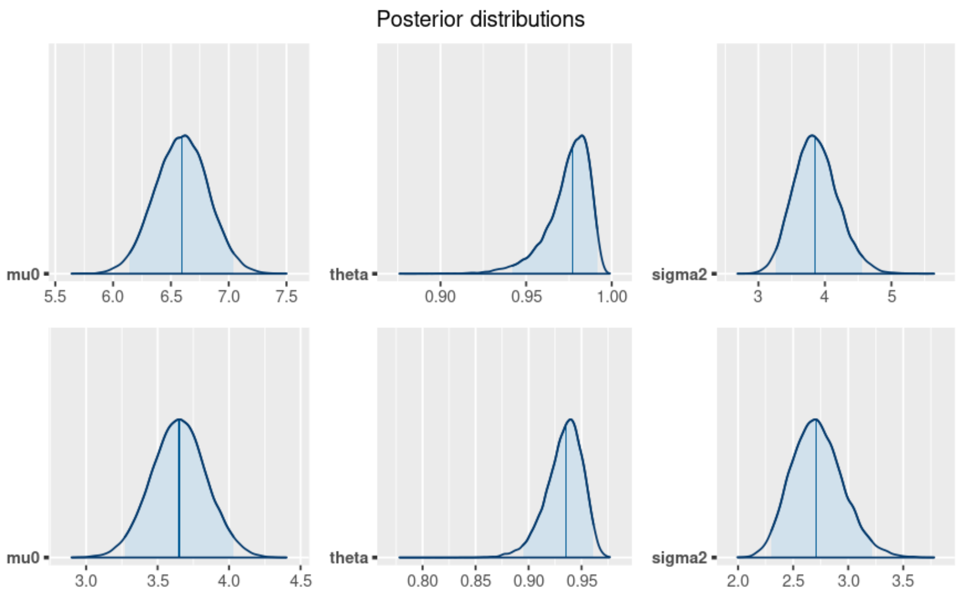


Fig. 4. Posterior distributions of the parameters for the MA(1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

Model target variable	Parameter	Posterior Mean	95% Credible Interval
GDP	μ_0	6.5934423	(6.1441382, 7.0421211)
GDP	θ	0.9745347	(0.9419309, 0.9915436)
GDP	σ^2	3.8669139	(3.2666059, 4.5663869)
CPIAUCSL	μ_0	3.6496595	(3.2718468, 4.0311333)
CPIAUCSL	θ	0.9335022	(0.8946139, 0.9613808)
CPIAUCSL	σ^2	2.7205353	(2.3038087, 3.2125894)

Table 2. *Posterior means and 95% credible intervals for the parameters of the MA(1) models.*

In both the GDP and CPIAUCSL models, the θ parameter is close to 1. However, using a prior for θ over a larger range, i.e., $\theta \sim \mathcal{U}(-2.0, 2.0)$, did not significantly change the estimated value of θ.
We also examined the trace plots and autocorrelation plots and found no significant issues.
Finally, plotting the in-sample and out-of-sample predictions with 95% credible intervals and comparing them with the actual data, we obtained the results shown in Figures 5 and 6. As observed, the MA(1) model is not good for out of sample predictions, as it fails to capture the trend of the data and returns a flat line.

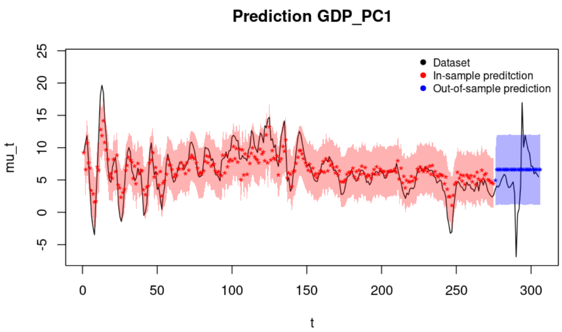


Fig. 5. In-sample and out-of-sample predictions for the GDP using the MA(1) model.

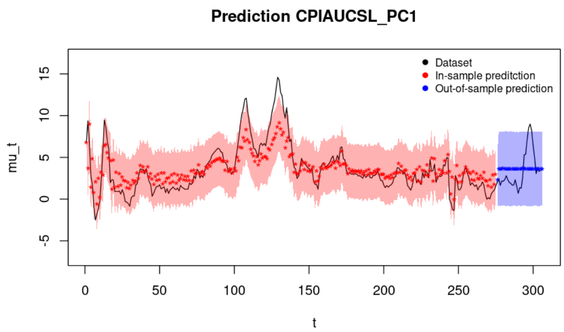


Fig. 6. In-sample and out-of-sample predictions for the CPIAUCSL using the MA(1) model.

Lastly, we compared the model's in-sample predictions and posterior distributions with the results obtained from the MA model using the ARIMA function. This analysis revealed a strong similarity between the predictions of both approaches, despite a slight difference in the parameter estimates.

MA(2)

The MA(2) model is defined as follows:

$$y_t = \mu_0 + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \epsilon_t \tag{9}$$

In our case study, we assume that ϵ_t are independent and identically distributed variables from a normal distribution with mean 0 and variance σ^2, i.e., $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, leading to the following likelihood:

$$y_t | \mu_0, \theta_1, \theta_2, \sigma^2, \epsilon_{t-1}, \epsilon_{t-2} \sim \mathcal{N}(\mu_0 + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \sigma^2) \tag{10}$$

For the priors, we chose:

$$\begin{aligned} \mu_0 &\sim \mathcal{N}(0.0, 10000) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(2, 0.1) \\ \theta_1 &\sim \mathcal{U}(-1.5, 1.5) \\ \theta_2 &\sim \mathcal{U}(-1.0, 1.0) \end{aligned} \tag{11}$$

We selected uninformative priors for μ_0, σ^2, θ_1, and θ_2.
Running the JAGS code to implement the MA(2) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 7, with the corresponding means and 95% credible intervals reported in Table 3.

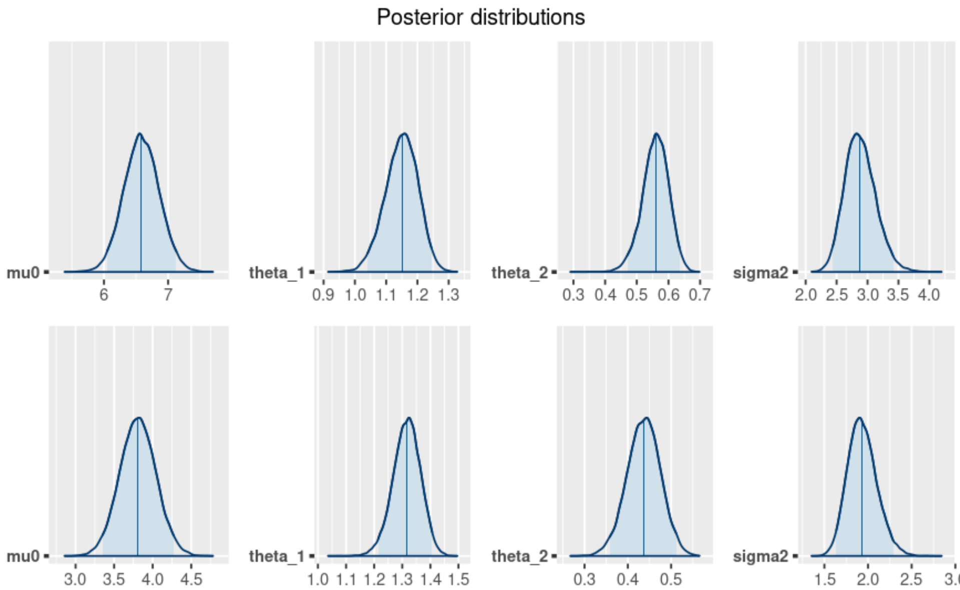


Fig. 7. Posterior distributions of the parameters for the MA(2) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

Model target variable	**Parameter**	**Posterior Mean**	**95% Credible Interval**
GDP	μ_0	6.5862752	(6.0523533, 7.1291252)
GDP	θ_1	1.1489023	(1.0408063, 1.2447200)
GDP	θ_2	0.5587509	(0.4698359, 0.6348533)
GDP	σ^2	2.8842617	(2.4383601, 3.4118375)
CPIAUCSL	μ_0	3.8081535	(3.3497252, 4.2727049)
CPIAUCSL	θ_1	1.3155230	(1.2142974, 1.4063565)
CPIAUCSL	θ_2	0.4365283	(0.3581476, 0.5111502)
CPIAUCSL	σ^2	1.9356539	(1.6371395, 2.2893457)

Table 3. *Posterior means and 95% credible intervals for the parameters of the MA(2) models.*

Plotting the in-sample and out-of-sample predictions with 95% credible intervals and comparing them with the actual data, we obtained the results shown in Figures 8 and 9.

As with the MA(1) model, the MA(2) model is not good for out-of-sample predictions, as it fails to capture the trend of the data and returns a flat line.

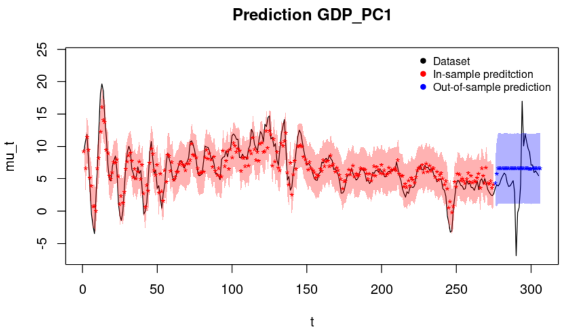


Fig. 8. In-sample and out-of-sample predictions for the GDP using the MA(2) model.

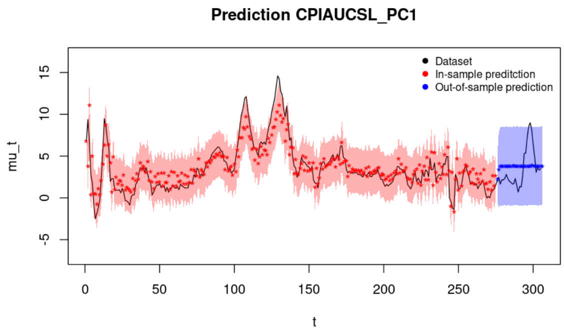


Fig. 9. In-sample and out-of-sample predictions for the CPIAUCSL using the MA(2) model.

Lastly, we compared the model's in-sample predictions and posterior distributions with the results obtained from the MA model using the ARIMA function. This analysis revealed a strong similarity between the predictions of both approaches, despite a slight difference in the parameter estimates.

4 Autoregressive Moving Average Model

The Autoregressive Moving Average Model (ARMA) is obtained by merging the AR and MA models. Formally, the ARMA model of order (p, q) is represented as follows:

$$y_t = \mu_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t \tag{12}$$

where p is the number of past values considered in the AR part, q is the number of past errors considered in the MA part, μ_0 is a constant, α_i and θ_j are the model parameters, $y_{t-1}, y_{t-2}, ..., y_{t-p}$ are the past values, and $\epsilon_t, \epsilon_{t-1}, ..., \epsilon_{t-q}$ are the error terms.
For our analysis, we decided to implement only the ARMA(1,1) model.

ARMA(1,1)

The ARMA(1,1) model is defined as:

$$y_t = \mu_0 + \alpha y_{t-1} + \theta \epsilon_{t-1} + \epsilon_t \tag{13}$$

In our case study, we assume that ϵ_t are independent and identically distributed variables from a normal distribution with mean 0 and variance σ^2, i.e., $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, leading to the following likelihood:

$$y_t | \mu_0, \alpha, \theta, \sigma^2, y_{t-1}, \epsilon_{t-1} \sim \mathcal{N}(\mu_0 + \alpha y_{t-1} + \theta \epsilon_{t-1}, \sigma^2) \tag{14}$$

For the priors, we chose:

$$\begin{aligned} \mu_0 &\sim \mathcal{N}(0.0, 10000) \\ \tau = 1/\sigma^2 &\sim \mathcal{G}(2, 0.1) \\ \alpha &\sim \mathcal{U}(-1.0, 1.0) \\ \theta &\sim \mathcal{U}(-1.0, 1.0) \end{aligned} \tag{15}$$

We selected uninformative priors for all the parameters: μ_0, σ^2, α, and θ.
Running the JAGS code to implement the ARMA(1,1) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 10, with the corresponding means and 95% credible intervals reported in Table 4.

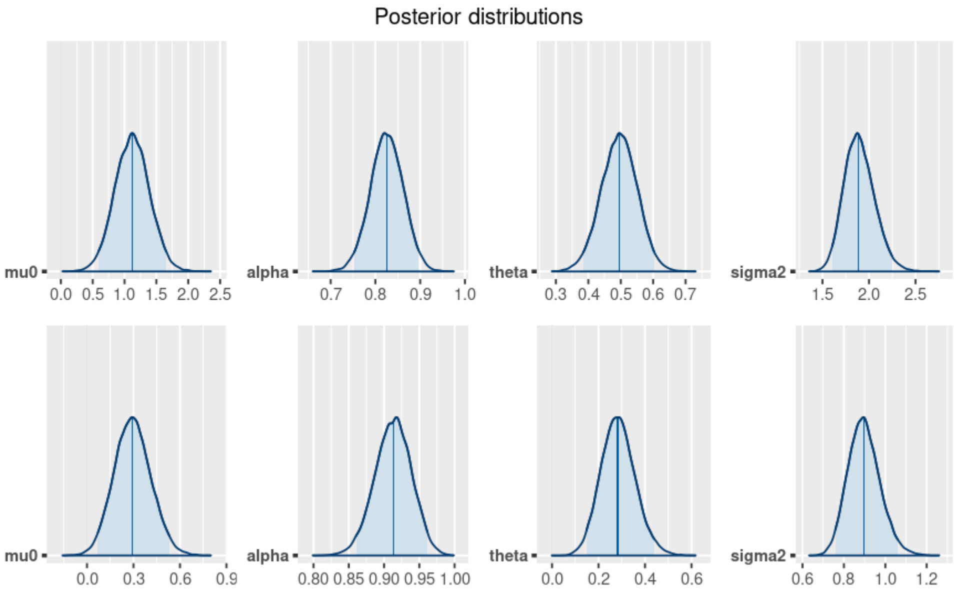
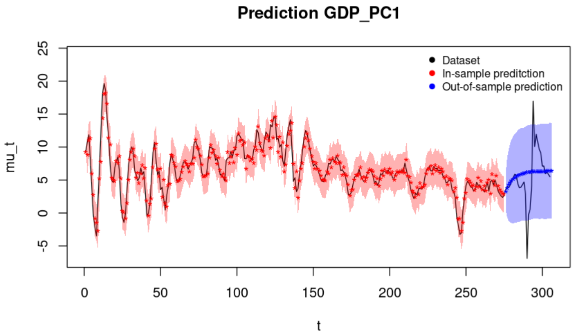


Fig. 10. Posterior distributions of the parameters for the ARMA(1,1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

Model target variable	**Parameter**	**Posterior Mean**	**95% Credible Interval**
GDP	μ_0	1.1207710	(0.59421700, 1.6538357)
GDP	α	0.8254802	(0.75268077, 0.8966752)
GDP	θ	0.4952213	(0.38504175, 0.6024967)
GDP	σ^2	1.8977382	(1.60426034, 2.2438487)
CPIAUCSL	μ_0	0.2941814	(0.07023124, 0.5314746)
CPIAUCSL	α	0.9127231	(0.86025773, 0.9612327)
CPIAUCSL	θ	0.2846962	(0.14830563, 0.4392942)
CPIAUCSL	σ^2	0.8976673	(0.75858882, 1.0607612)

Table 4. *Posterior means and 95% credible intervals for the parameters of the ARMA(1,1) models.*

Plotting the in-sample and out-of-sample predictions with 95% credible intervals and comparing them with the actual data, we obtained the results shown in Figures 11 and 12.
The out-of-sample predictions for GDP and CPIAUCSL effectively capture the initial data trends. Nevertheless, the ARMA(1,1) model fails to anticipate the significant impact of the COVID-19 pandemic, which is completely understandable.



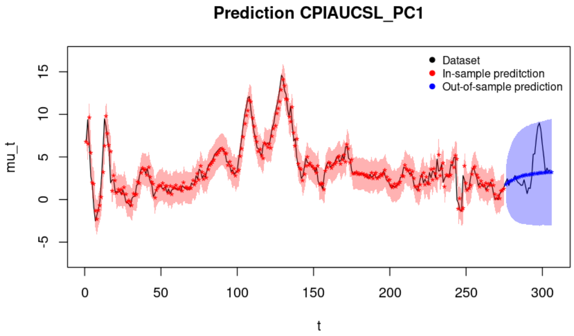


Fig. 11. In-sample and out-of-sample predictions for the GDP using the ARMA(1,1) model.

Fig. 12. In-sample and out-of-sample predictions for the CPIAUCSL using the ARMA(1,1) model.

Finally, examining the trace plots and autocorrelation plots, we found no significant issues, and comparing our model with the one obtained using the ARIMA function, we observed that the two models have similar results.

5 Generalized Autoregressive Conditional Heteroskedasticity Model

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is structured as an ARMA model applied to the variance of the error term, relying on the previous squared error terms and variances.
Formally, the GARCH model of order (p, q) is defined as:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} a_{j+p} \sigma_{t-j}^2 \tag{16}$$

Here, p is the number of past squared error terms considered, q is the number of past variances considered, $a_0, a_1, ..., a_{p+q}$ are the model parameters, $\epsilon_{t-1}^2, \epsilon_{t-2}^2, ..., \epsilon_{t-p}^2$ are the past squared error terms, and $\sigma_{t-1}^2, \sigma_{t-2}^2, ..., \sigma_{t-q}^2$ are the past variances.
For our analysis, we assumed that the error terms ϵ_t are independent and identically distributed variables from a normal distribution with mean 0 and variance σ_t^2, i.e., $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_t^2)$.

AR(1) + GARCH(1,1)

We applied the GARCH(1,1) model to the variance of the error term in an AR(1) model, leading to the following likelihood:

$$y_t | \mu_0, \alpha, y_{t-1}, a_0, a_1, a_2, \sigma_{t-1}^2 \sim \mathcal{N}(\mu_0 + \alpha y_{t-1}, a_0 + a_1 \epsilon_{t-1}^2 + a_2 \sigma_{t-1}^2) \tag{17}$$

For the priors, we chose:

$$\begin{aligned} \mu_0 &\sim \mathcal{N}(0.0, 10000) \\ \alpha &\sim \mathcal{U}(-1.0, 1.0) \\ a_0 &\sim \mathcal{G}(0.01, 0.01) \\ a_1 &\sim \mathcal{G}(0.01, 0.01) \\ a_2 &\sim \mathcal{G}(0.01, 0.01) \end{aligned} \tag{18}$$

The gamma distribution is used for a_0, a_1, a_2 to ensure positive variance values. The normal and uniform distributions for μ_0 and α, respectively, are chosen for the same reasons as in the AR(1) model discussed in Section 2.
Running the JAGS code to implement the AR(1) + GARCH(1,1) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 13, with the corresponding means and 95% credible intervals reported in Table 5. Trace plots are provided in the Appendix.

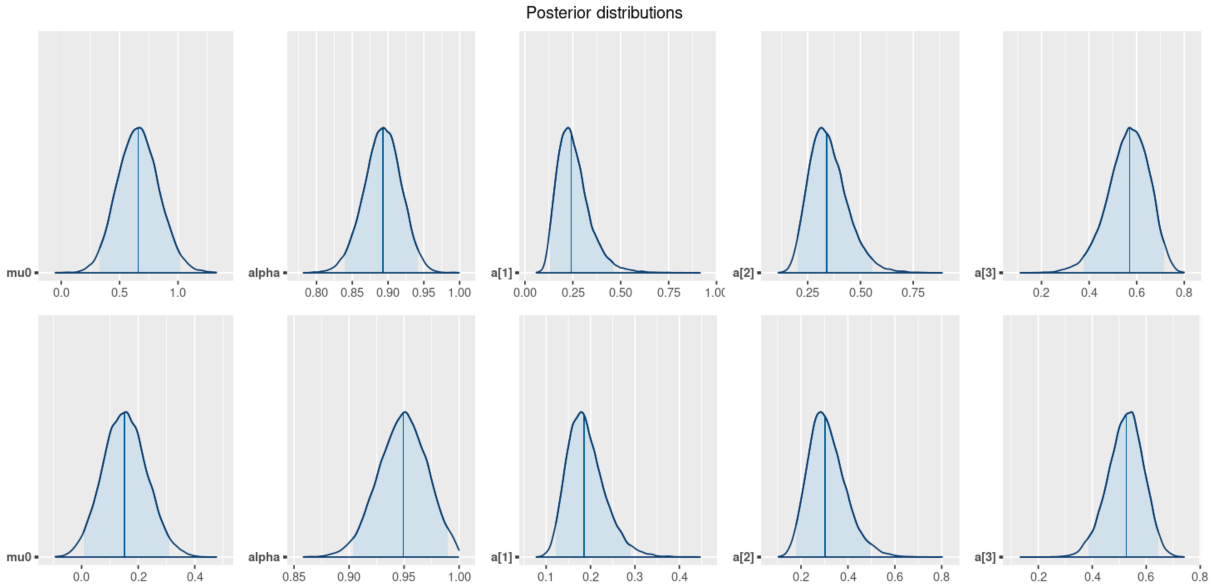


Fig. 13. Posterior distributions of the parameters for the AR(1) + GARCH(1,1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

Model target variable	**Parameter**	**Posterior Mean**	**95% Credible Interval**
GDP	μ_0	0.6607917	(0.330124347, 1.0149443)
GDP	α	0.8922917	(0.839581644, 0.9416944)
GDP	a_1	0.2543789	(0.128699457, 0.4589058)
GDP	a_2	0.3492357	(0.202666918, 0.5555133)
GDP	a_3	0.5641981	(0.377027833, 0.7144185)
CPIAUCSL	μ_0	0.1533340	(0.006694284, 0.3087467)
CPIAUCSL	α	0.9485379	(0.903837995, 0.9896255)
CPIAUCSL	a_1	0.1904433	(0.122120513, 0.2894240)
CPIAUCSL	a_2	0.3120365	(0.179010077, 0.4966238)
CPIAUCSL	a_3	0.5232054	(0.384775889, 0.6435153)

Table 5. *Posterior means and 95% credible intervals for the parameters of the AR(1) + GARCH(1,1) models.*

Plotting the in-sample and out-of-sample predictions with 95% credible intervals and comparing them with the actual data, we obtained the results shown in Figures 14 and 15. The results from the AR(1) + GARCH(1,1) model are not significantly different from those obtained using the AR(1) model, suggesting that the AR(1) model already captures most of the information present in the data.

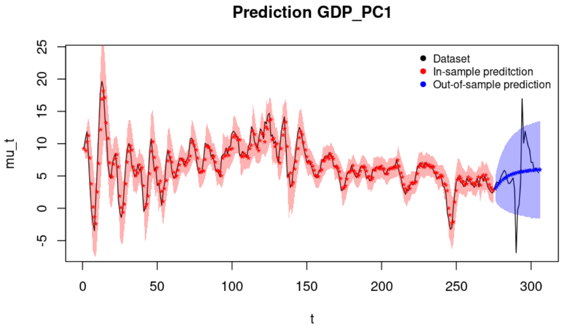


Fig. 14. In-sample and out-of-sample predictions for the GDP using the GARCH(1,1) + AR(1) model.

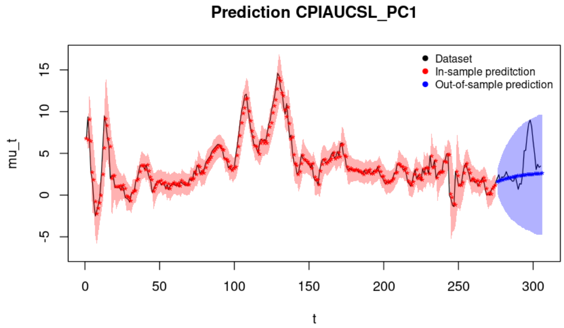


Fig. 15. In-sample and out-of-sample predictions for the CPIAUCSL using the GARCH(1,1) + AR(1) model.

Lastly, we compared our model with the results obtained from the uGarch library and found no significant differences in the in-sample predictions of both approaches. However, the analysis did reveal slight differences in the parameter estimates.

6 Vector Autoregressive Model

The Vector Autoregressive (VAR) model extends the single-variable Autoregressive (AR) model discussed in Section 2 to accommodate multiple time series. VAR models enable the analysis of dynamic relationships between interacting variables by allowing for feedback among them, making these models useful also for forecasting purposes.
Formally, a VAR model of order p (VAR(p)) for K variables is represented as:

$$\mathbf{y}_t = \boldsymbol{\mu}_0 + A_1\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-2} + \cdots + A_p\mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t \tag{19}$$

where:

- p is the number of past time periods considered.
- $\boldsymbol{\mu}_0$ is a $K \times 1$ vector of constants.
- $\mathbf{y}_t$ is a $K \times 1$ vector of variables at time t.
- A_i (for $i = 1, 2, \ldots, p$) are $K \times K$ coefficient matrices.
- $\boldsymbol{\epsilon}_t$ is a $K \times 1$ vector of error terms at time t.

For our analysis, we implement a VAR model of order 1 (VAR(1)).

VAR(1)

The VAR(1) model for $K = 2$ is defined as:

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_{0,1} \\ \mu_{0,2} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} \tag{20}$$

or more compactly:

$$\mathbf{y}_t = \boldsymbol{\mu}_0 + A\mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t \tag{21}$$

Here, $y_{1,t}$ represents the GDP_PC1 variable and $y_{2,t}$ represents the CPIAUCSL_PC1 variable at time t. Assuming that $\boldsymbol{\epsilon}_t$ follows a multivariate normal distribution with $\mathbf{0}$ mean and covariance matrix Σ, the likelihood is:

$$\mathbf{y}_t|\boldsymbol{\mu}_0, A, \Sigma, \mathbf{y}_{t-1} \sim \mathcal{N}_2(\boldsymbol{\mu}_0 + A\mathbf{y}_{t-1}, \Sigma) \tag{22}$$

For the priors, we chose:

$$\begin{aligned} \mu_{0,i} &\sim \mathcal{N}(0.0, 10000) \qquad i = 1, 2 \\ a_{ij} &\sim \mathcal{U}(-1, 1) \qquad i, j = 1, 2 \\ \Omega = \Sigma^{-1} &\sim Wishart(R, 3) \end{aligned} \tag{23}$$

where:

$$R = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \tag{24}$$

The priors are chosen to be uninformative.
Using JAGS to implement the VAR(1) model for GDP and CPIAUCSL, we obtained the posterior distributions shown in Figure 16, with the corresponding means and 95% credible intervals reported in Table 6.

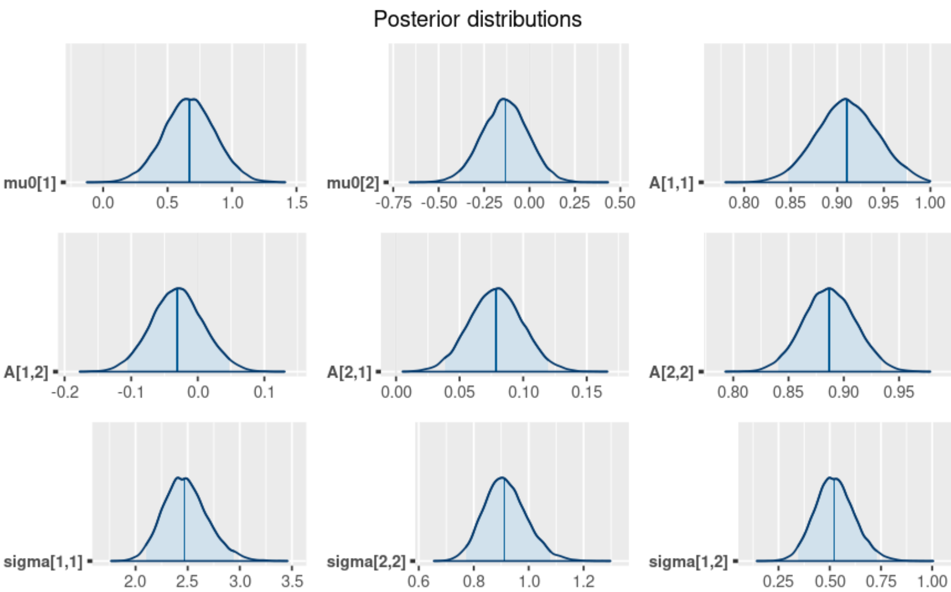


Fig. 16. Posterior distributions of the parameters for the VAR(1) model.

Parameter	Posterior Mean	95% Credible Interval
$\mu_{0,1}$	0.66929256	(0.27516565, 1.06102229)
$\mu_{0,2}$	-0.13454152	(-0.38389818, 0.11090167)
a_{11}	0.91055747	(0.84673870, 0.97466996)
a_{12}	-0.03039508	(-0.10650041, 0.04742719)
a_{21}	0.07867888	(0.03824313, 0.11987435)
a_{22}	0.88708627	(0.84108698, 0.93353337)
Σ_{11}	2.48031746	(2.09784974, 2.94229950)
Σ_{12}	0.52456983	(0.34405208, 0.72777246)
Σ_{22}	0.91495776	(0.77210667, 1.07984048)

Table 6. *Posterior means and 95% credible intervals for the parameters of the VAR(1) model.*

Lastly, we plotted the in-sample and out-of-sample predictions with 95% credible intervals and compared them with the actual data. The results are shown in Figures 17 and 18.

Analyzing the out-of-sample predictions, we observe that in the VAR(1) model, GDP values impact CPIAUCSL, resulting in slightly different predictions compared to the AR(1) model for CPIAUCSL.

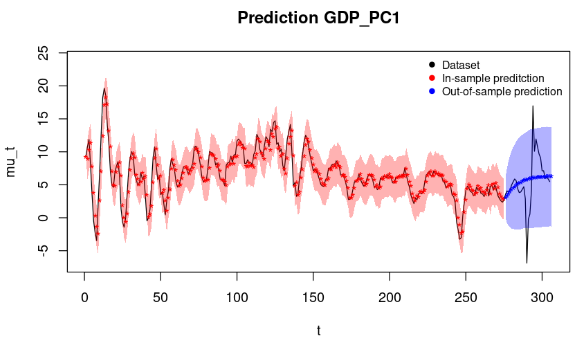


Fig. 17. In-sample and out-of-sample predictions for the GDP using the VAR(1) model.

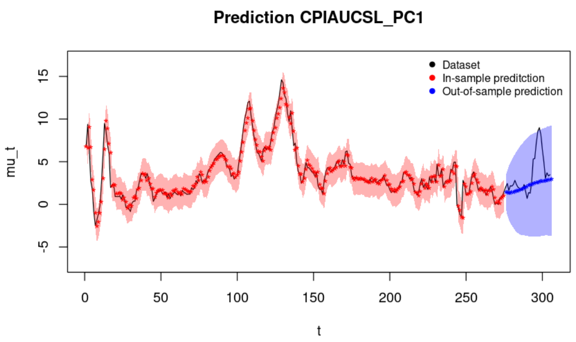


Fig. 18. In-sample and out-of-sample predictions for the CPIAUCSL using the VAR(1) model.

Lastly, we compared our results with those obtained using the VAR function in R. This comparison revealed no significant differences between the two sets of results.

Conclusions

To conclude, we calculated DIC and WAIC for each model and each target time series. The results are shown in Table 7. According to these results, it is evident that the best model to independently fit the two time series is AR(1) + GARCH(1,1), as it has the lowest DIC and WAIC values. The ARMA model also performs well.

It's worth mentioning the MA(2) model, which outperforms the MA(1) model. This result is consistent with the observations made in section 3 regarding the autocorrelation plots.

The VAR model was not considered in this comparison since it fits the two time series jointly.

	DIC		WAIC	
Model	**GDP**	**CPIAUCSL**	**GDP**	**CPIAUCSL**
AR(1)	1029.708	769.6079	1031.7	773.0
MA(1)	1154.302	1057.921	1154.5	1058.6
MA(2)	1074.749	965.4048	1068.2	949.8
ARMA(1,1)	960.1701	754.3378	962.3	759.6
AR(1) + GARCH(1,1)	932.1271	714.7022	934.5	727.5
VAR(1)	1755.134		1762.2	

Table 7. *DIC and WAIC values for each model and each target time series.*

7 Appendix

Trace Plots

In this section, we analyze the trace plots for the different models implemented. The analysis is consistent across all models due to the similarity of the trace plots. Notably, none of the trace plots display any distinct patterns. Furthermore, they appear to thoroughly explore the entire parameter space, as indicated by frequent transitions into various regions. The following plots were generated using a single chain; however, we have confirmed that the behavior remains consistent when using multiple chains.

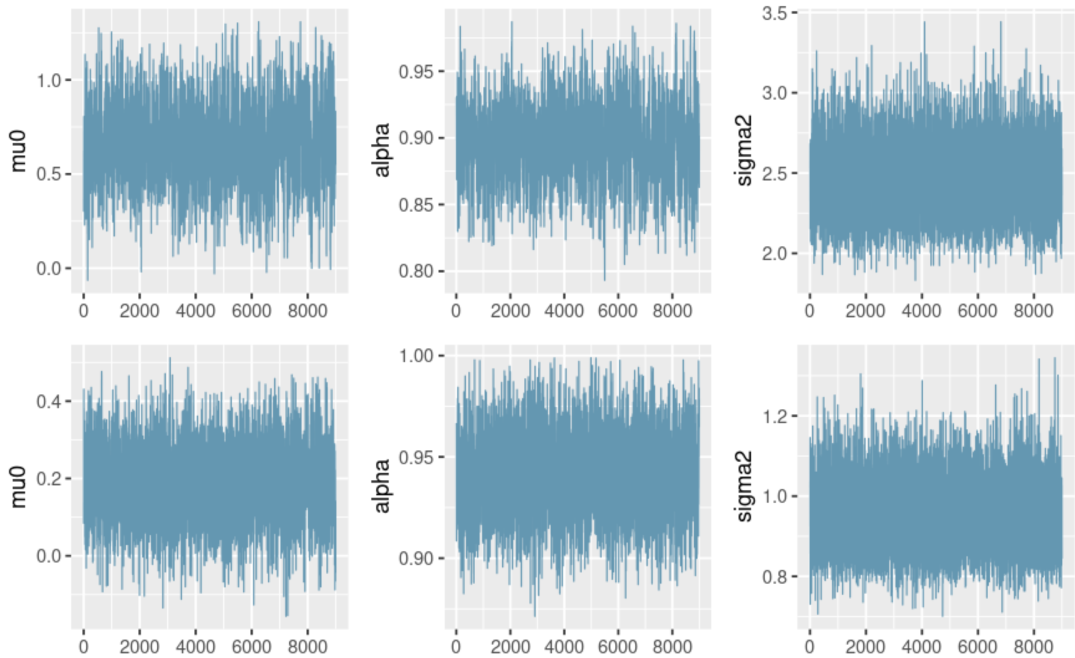


Fig. 19. Trace plots for the AR(1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

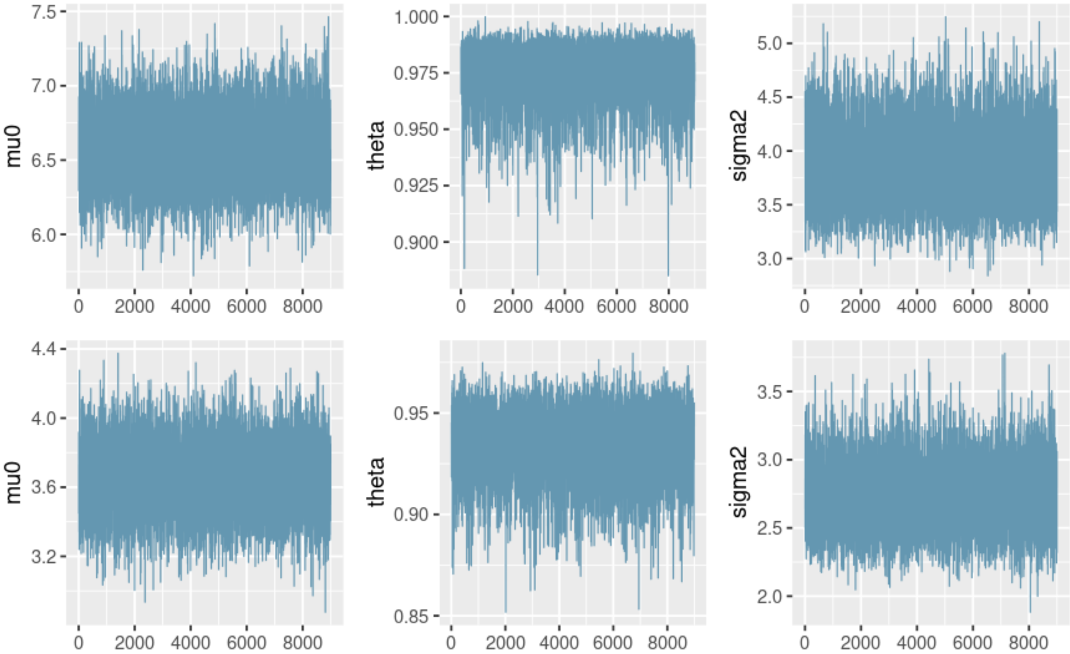


Fig. 20. Trace plots for the MA(1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

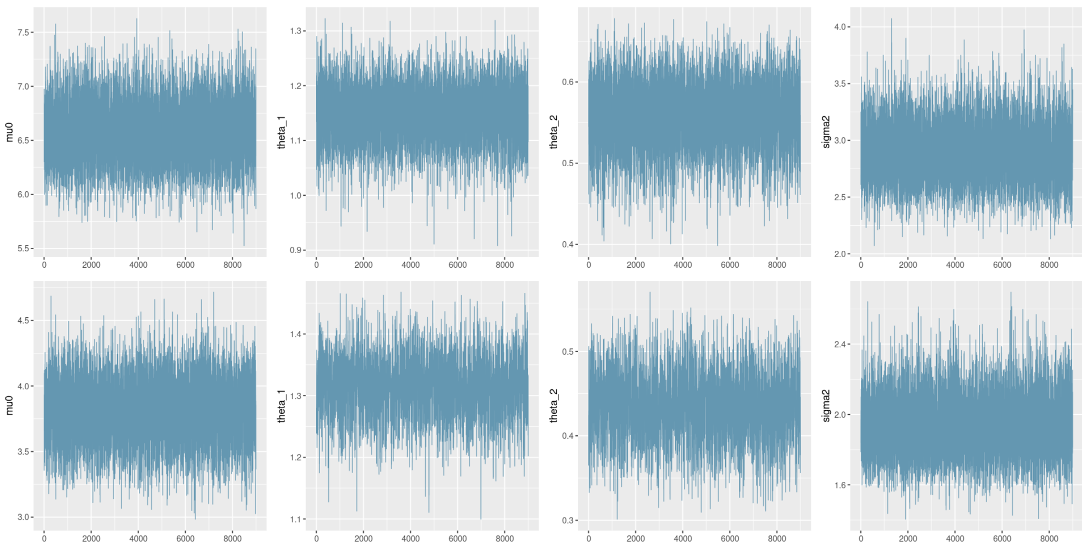


Fig. 21. Trace plots for the MA(2) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

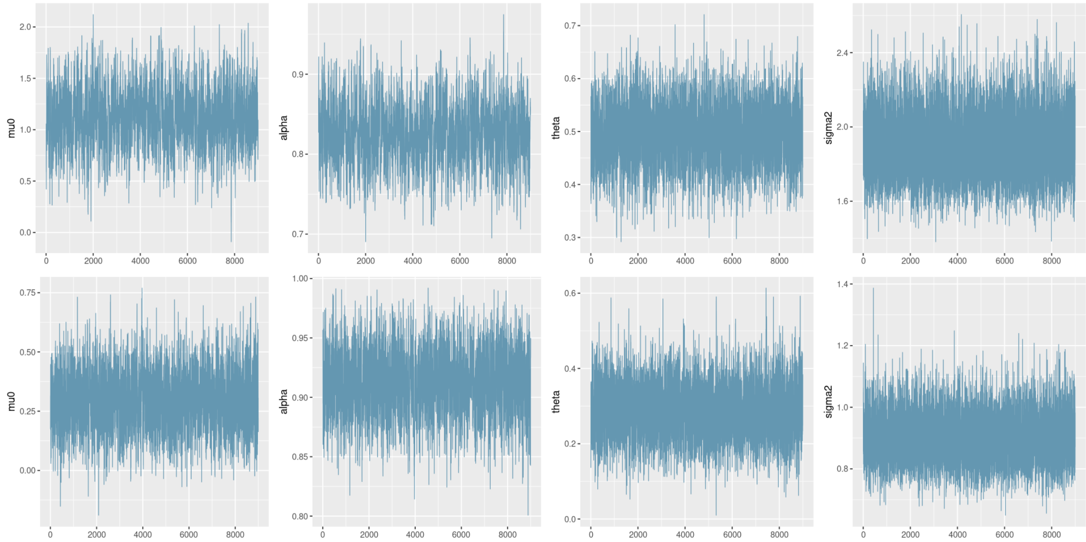


Fig. 22. Trace plots for the ARMA(1,1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

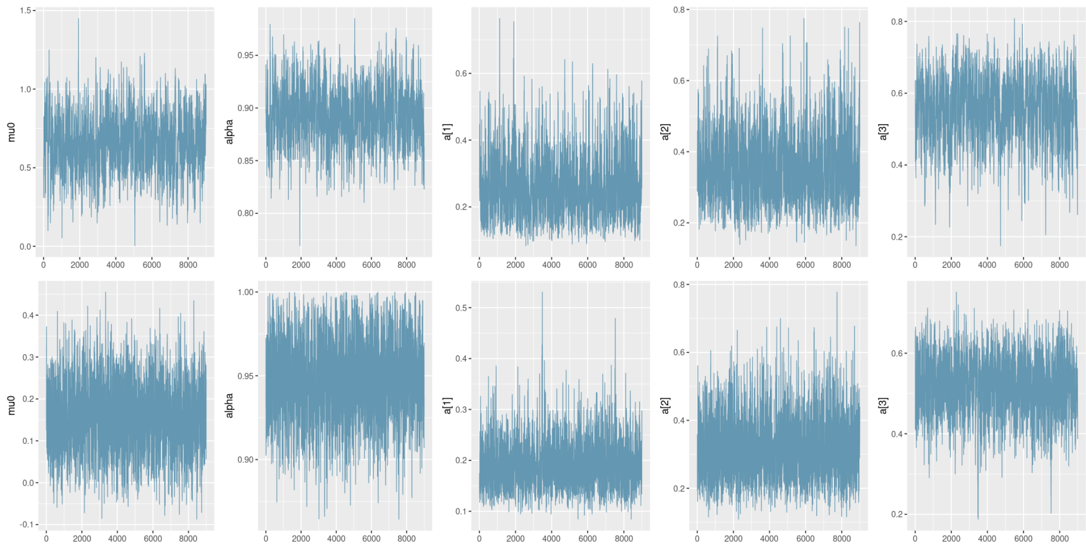


Fig. 23. Trace plots for the AR(1) + GARCH(1,1) models. The top line corresponds to the model used for GDP, while the bottom line corresponds to the model used for CPIAUCSL.

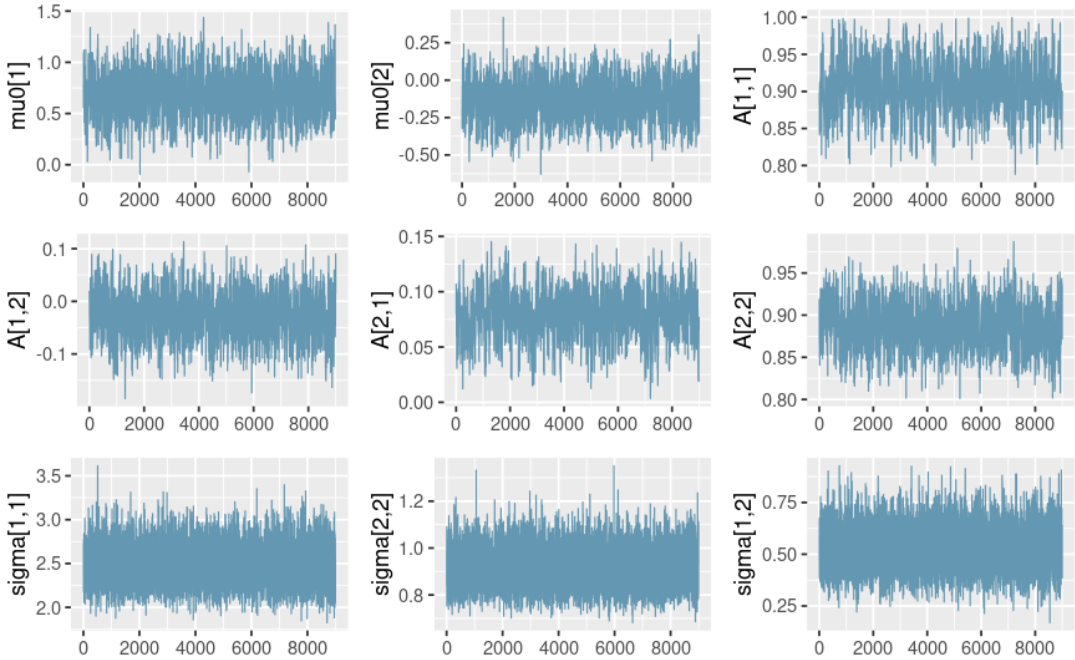


Fig. 24. Trace plots for the VAR(1) model.

Comparison with public libraries

In this section, we compared the in-sample predictions and the posterior distributions of our JAGS models with the results obtained from public libraries. The comparison do not show significant differences, with the exception of MA and GARCH models where the parameter estimates are slightly different.

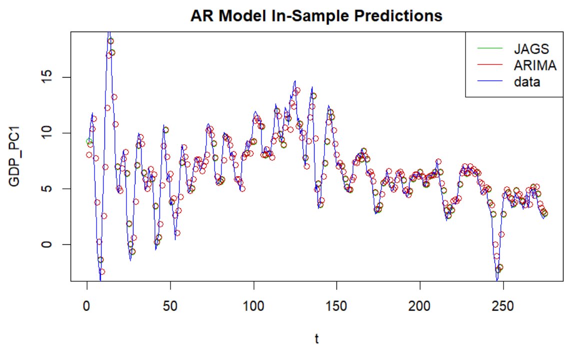


Fig. 25. In-sample predictions for GDP using AR(1) with JAGS and ARIMA.

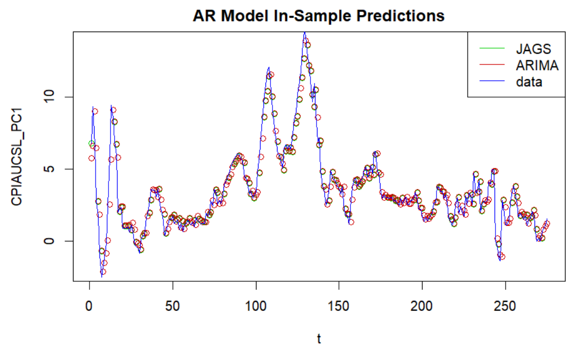


Fig. 26. In-sample predictions for CPIAUCSL using AR(1) with JAGS and ARIMA.

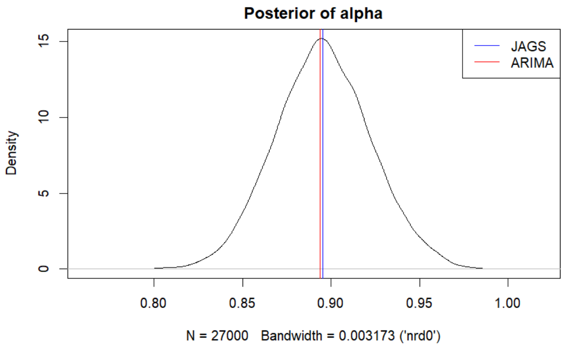


Fig. 27. Posterior distribution for the parameter of AR(1) compared with ARIMA estimated parameter for GDP.

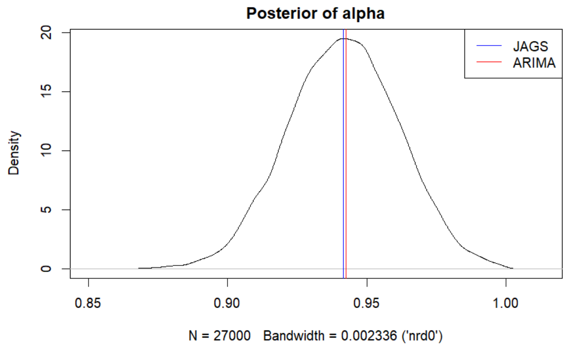


Fig. 28. Posterior distribution for the parameter of AR(1) compared with ARIMA estimated parameter for CPIAUCSL.

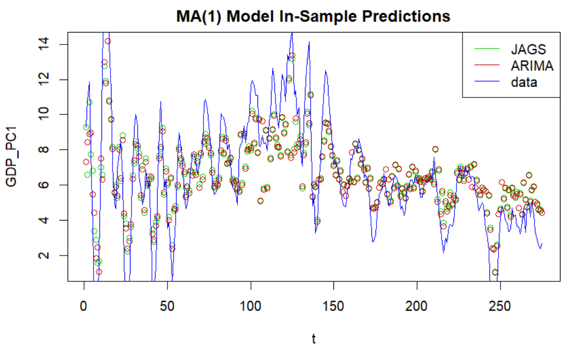


Fig. 29. In-sample predictions for GDP using MA(1) with JAGS and ARIMA.

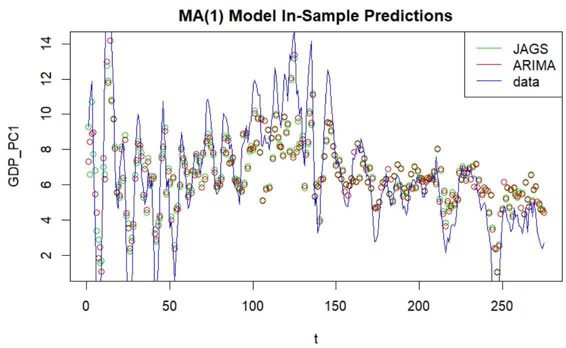


Fig. 30. In-sample predictions for CPIAUCSL using MA(1) with JAGS and ARIMA.

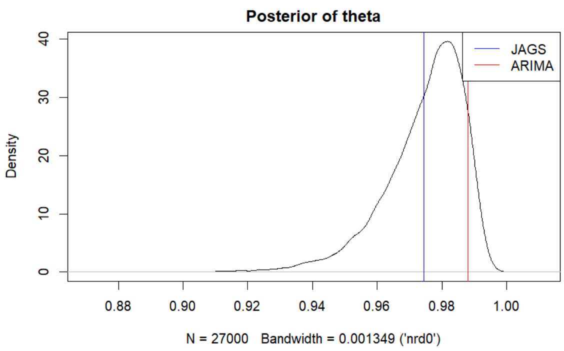


Fig. 31. Posterior distribution for the parameter of MA(1) compared with ARIMA estimated parameter for GDP.

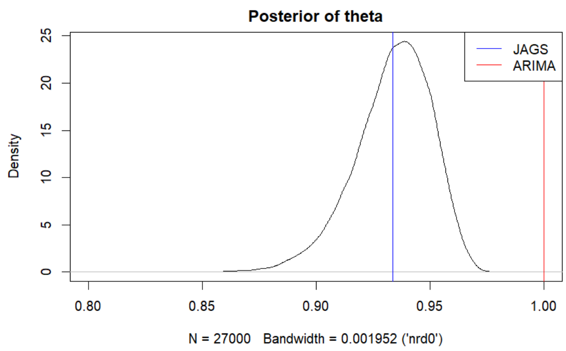


Fig. 32. Posterior distribution for the parameter of MA(1) compared with ARIMA estimated parameter for CPIAUCSL.

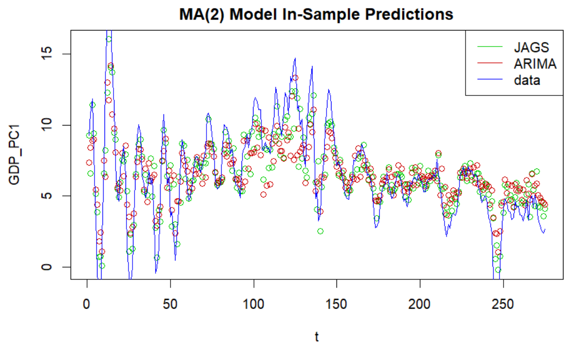


Fig. 33. In-sample predictions for GDP using MA(2) with JAGS and ARIMA.

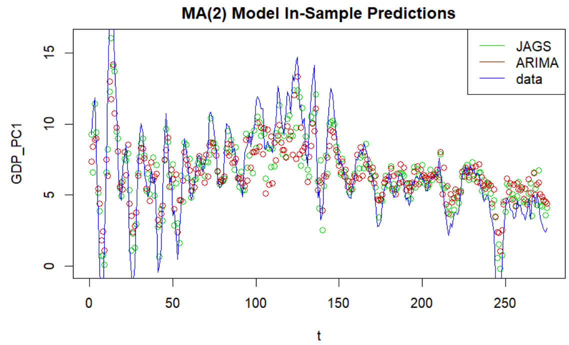


Fig. 34. In-sample predictions for CPIAUCSL using MA(2) with JAGS and ARIMA.

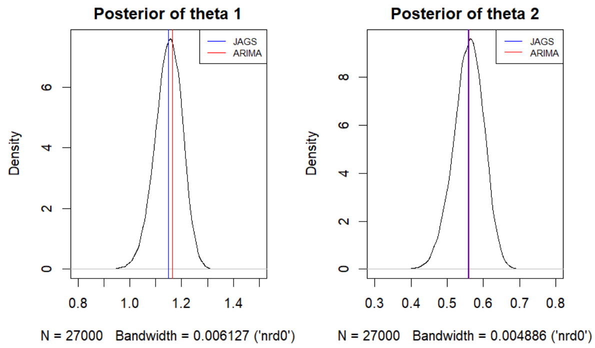


Fig. 35. Posterior distribution for the parameters of MA(2) compared with ARIMA estimated parameters for GDP.

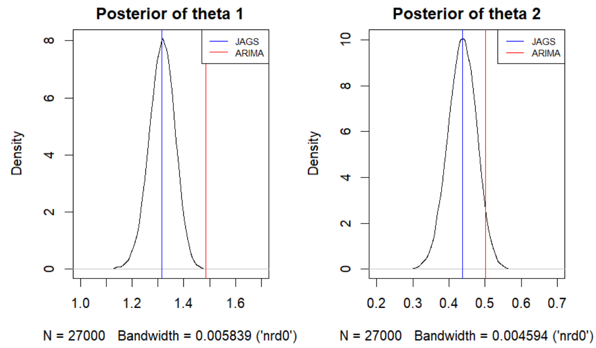


Fig. 36. Posterior distribution for the parameters of MA(2) compared with ARIMA estimated parameters for CPIAUCSL.

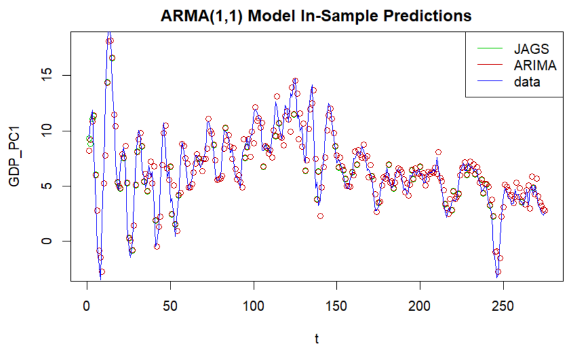


Fig. 37. In-sample predictions for GDP using ARMA(1,1) with JAGS and ARIMA.

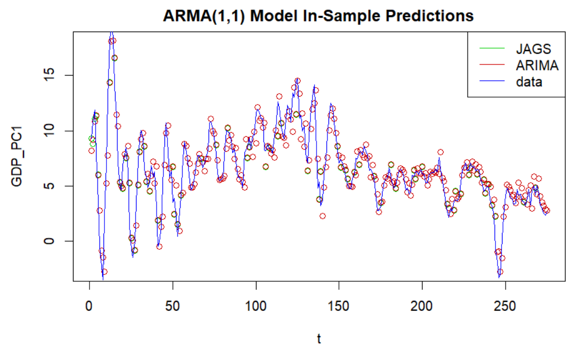


Fig. 38. In-sample predictions for CPIAUCSL using ARMA(1,1) with JAGS and ARIMA.

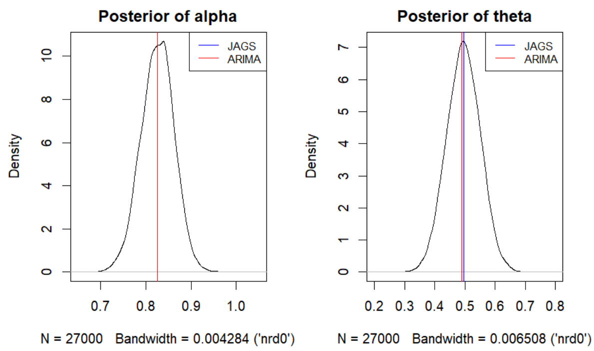


Fig. 39. Posterior distribution for the parameters of ARMA(1,1) compared with ARIMA estimated parameters for GDP.

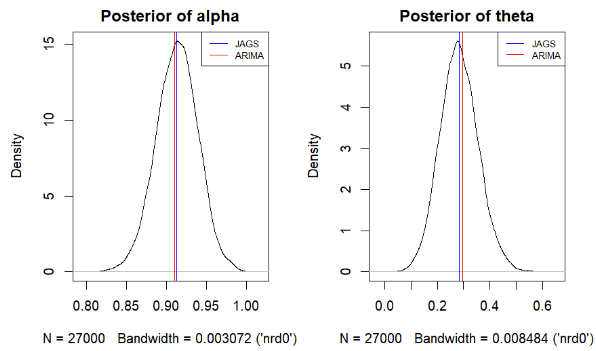


Fig. 40. Posterior distribution for the parameters of ARMA(1,1) compared with ARIMA estimated parameters for CPIAUCSL.

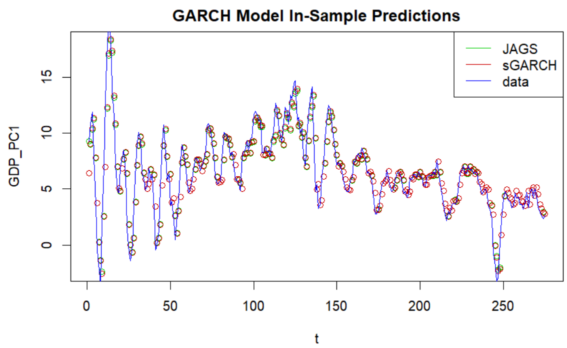


Fig. 41. In-sample predictions for GDP using GARCH(1,1) + AR(1) with JAGS and uGarch.

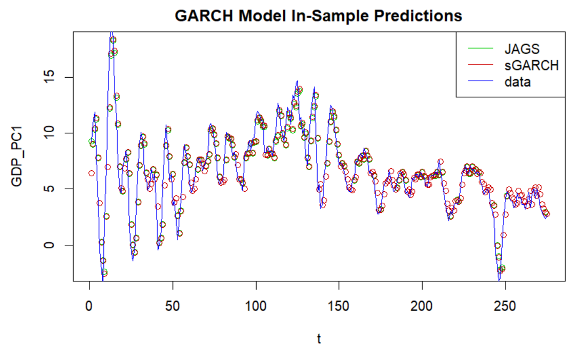


Fig. 42. In-sample predictions for CPIAUCSL using GARCH(1,1) + AR(1) with JAGS and uGarch.

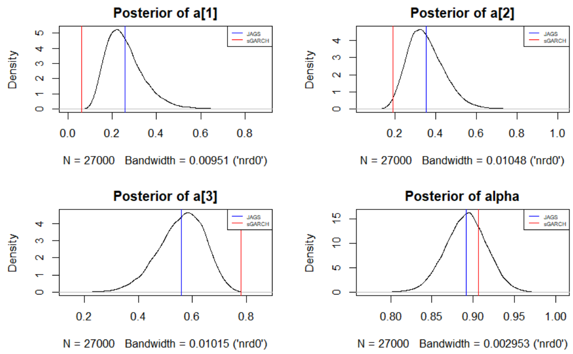


Fig. 43. Posterior distribution for the parameters of GARCH(1,1) + AR(1) compared with uGarch estimated parameters for GDP.

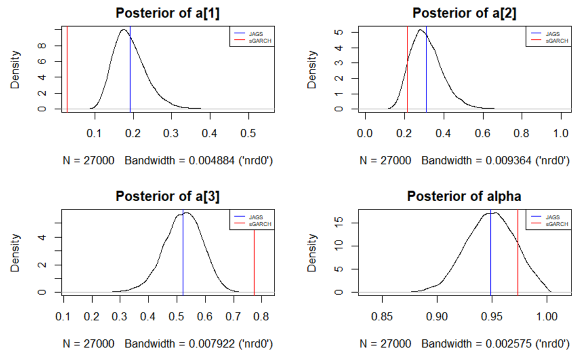


Fig. 44. Posterior distribution for the parameters of GARCH(1,1) + AR(1) compared with uGarch estimated parameters for CPIAUCSL.

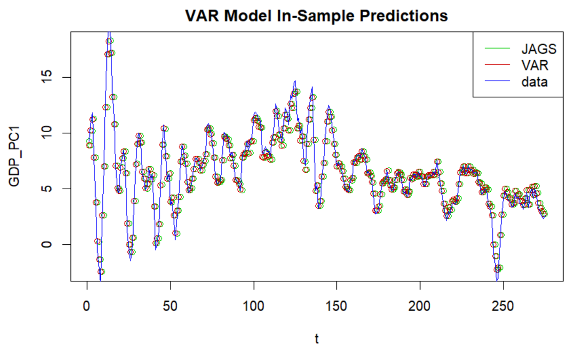


Fig. 45. In-sample predictions for GDP using VAR(1) with JAGS and VAR.

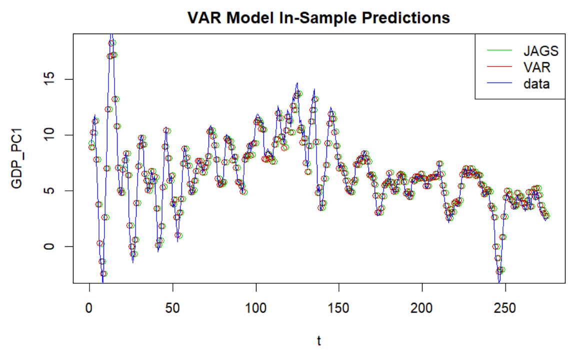


Fig. 46. In-sample predictions for CPIAUCSL using VAR(1) with JAGS and VAR.

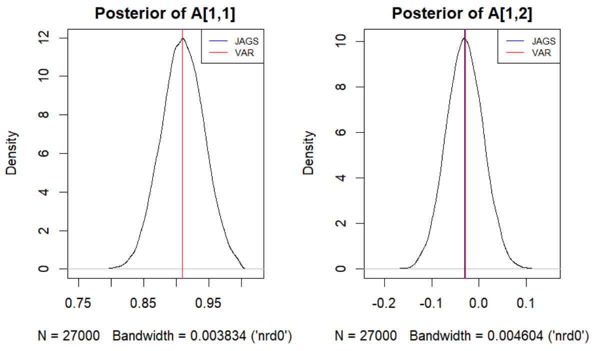


Fig. 47. Posterior distribution for the parameters of VAR(1) compared with VAR estimated parameters for GDP.

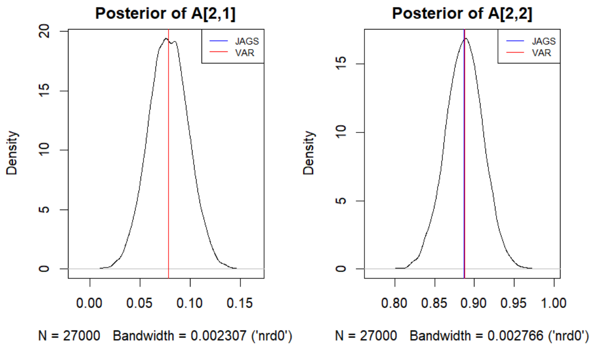


Fig. 48. Posterior distribution for the parameters of VAR(1) compared with VAR estimated parameters for CPIAUCSL.