Exercise M

Discuss an extension of CCS with an operator of sequential composition between processes $P; Q$. Provide an operational semantics and analyze the possibility of having an encoding in CCS of the defined operator.

Solution

CCS_{seq}

$$P, Q ::= K \mid \alpha.P \mid \sum_{i \in I} \alpha.P_i \mid (P \mid Q) \mid P[f] \mid P \setminus L \mid P; Q$$

Operational Semantics

Classical rules

$$\text{Act}\ \frac{}{\alpha.P \xrightarrow{\alpha} P} \qquad \text{Sum}\ \frac{P_j \xrightarrow{\alpha} P_j'}{\sum_{j \in J} P_j \xrightarrow{\alpha} P_j'}\ j \in J \qquad \text{Par-1}\ \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q}$$

$$\text{Par-2}\ \frac{Q \xrightarrow{\alpha} Q'}{P|Q \xrightarrow{\alpha} P|Q'} \qquad \text{Par-3}\ \frac{P \xrightarrow{\alpha} P' \qquad Q \xrightarrow{\overline{\alpha}} Q'}{P|Q \xrightarrow{\tau} P'|Q'} \qquad \text{Hide}\ \frac{P \xrightarrow{\alpha} P'}{P \setminus L \xrightarrow{\alpha} P' \setminus L}\ \alpha, \overline{\alpha} \notin L$$

$$\text{Red}\ \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(a)} P'[f]} \qquad \text{Const}\ \frac{P \xrightarrow{\alpha} P'}{K \xrightarrow{\alpha} P'}\ K \overset{\text{def}}{=} P$$

CCS_{seq} rules

$$\text{End-Zero}\ \frac{}{0 \not\rightarrow} \qquad \text{End-Par}\ \frac{P \not\rightarrow \qquad Q \not\rightarrow}{P|Q \not\rightarrow} \qquad \text{End-Hide}\ \frac{P \not\rightarrow}{P \setminus A \not\rightarrow}$$

$$\text{End-Red}\ \frac{P \not\rightarrow}{P[f] \not\rightarrow} \qquad \text{End-Const}\ \frac{P \not\rightarrow}{K \not\rightarrow}\ \text{if } K \overset{\text{def}}{=} P \qquad \text{End-Seq}\ \frac{P \not\rightarrow \qquad Q \not\rightarrow}{P; Q \not\rightarrow}$$

$$\text{Seq-L}\ \frac{P \xrightarrow{\alpha} P'}{P; Q \xrightarrow{\alpha} P'; Q} \qquad \text{Seq-R}\ \frac{P \not\rightarrow \qquad Q \xrightarrow{\alpha} Q'}{P; Q \xrightarrow{\alpha} Q'}$$

Encoding

Let $e : \text{CCS}_{\text{seq}} \to \text{CCS}$

The encoding of a CCS_{seq} process P is $e(P) \setminus \{\nu\}$

Note: ν, ν' are channels that does not appear in P

- $e(0) = \nu.0$
- $e(\alpha.P) = \alpha.e(P)$
- $e(K) = K_e,\ K_e \overset{\text{def}}{=} e(P)$ if $K \overset{\text{def}}{=} P$

- $e(P \mid Q) = \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\}$
- $e\left(\sum_{j \in J, J \neq \emptyset} \alpha_j.P_j\right) = \sum_{j \in J, J \neq \emptyset} \alpha_j.e(P_j)$
- $e(P[f]) = e(P)[f]$
- $e(P \setminus L) = e(P) \setminus L$
- $e(P;Q) = \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\}$

Lemmas

0. $\forall P \in \text{CCS}_{\text{seq}}.P \not\to \Rightarrow e(P) \xrightarrow{\tau*} P_{\text{temp}} \xrightarrow{\nu} P' \land P' \not\to$
1. if $\forall P \in \text{CCS}_{\text{seq}}.$ if $P \xrightarrow{\alpha} P'$ then $e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \land P' \mathcal{R} (P'' \setminus \{\nu\})$

   then $\forall P \in \text{CCS}_{\text{seq}}.$ if $P \xrightarrow{\alpha} P'$ and $Q \approx e(P)$ then $Q \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \land P' \mathcal{R} (P'' \setminus \{\nu\})$ and dual
3. $\forall P \in \text{CCS}_{\text{seq}}.P \not\to \Rightarrow P \approx 0$
4. $\forall P \in \text{CCS}_{\text{seq}}.e(P) \approx e(P) \setminus \{\nu'\}$

Lemma 0

$$\forall P \in \text{CCS}_{\text{seq}}.P \not\to \Rightarrow e(P) \xrightarrow{\tau*} P_{\text{temp}} \xrightarrow{\nu} P' \land P' \not\to$$

By induction on the height of the derivation tree of $P \not\to$

Base case End-Zero

If

$$\text{End-Zero}\,\frac{}{0 \not\to}$$

then $e(0) = \nu.0 \xrightarrow{\nu} 0$

Inductive case End-Par

If

$$\text{End-Par}\,\frac{P \not\to \qquad Q \not\to}{P|Q \not\to}$$

then by induction $e(P) \xrightarrow{\tau*} P_{\text{temp}} \xrightarrow{\nu} P' \land P' \not\to$ and also $e(Q) \xrightarrow{\tau*} Q_{\text{temp}} \xrightarrow{\nu} Q' \land Q' \not\to$

So

$$\begin{aligned}
e(P|Q) &= \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau*} \left(P_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid Q_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q'\left[\frac{\nu'}{\nu}\right] \mid \nu.0\right) \setminus \{\nu'\} \\
&\xrightarrow{\nu} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q'\left[\frac{\nu'}{\nu}\right] \mid 0\right) \setminus \{\nu'\} \nrightarrow
\end{aligned}$$

Inductive case End-Seq

If

$$\text{End-Seq}\,\frac{P \nrightarrow \qquad Q \nrightarrow}{P; Q \nrightarrow}$$

then by induction $e(P) \xrightarrow{\tau*} P_{\text{temp}} \xrightarrow{\nu} P' \wedge P' \nrightarrow$ and also $e(Q) \xrightarrow{\tau*} Q_{\text{temp}} \xrightarrow{\nu} Q' \wedge Q' \nrightarrow$

So

$$\begin{aligned}
e(P;Q) &= \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau*} \left(P_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau} \left(P'\left[\frac{\nu'}{\nu}\right] \mid e(Q)\right) \setminus \{\nu'\} \\
&\xrightarrow{\tau*} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q_{\text{temp}}\right) \setminus \{\nu'\} \\
&\xrightarrow{\nu} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q'\right) \setminus \{\nu'\} \nrightarrow
\end{aligned}$$

Lemma 1.1

$$\text{if } \forall P, Q \in \text{CCS}_{\text{seq}}.\ \text{if } P \xrightarrow{\alpha} P' \text{ then } e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$$
$$\text{then } \forall P, Q \in \text{CCS}_{\text{seq}}, P \mathcal{R} Q \setminus \{\nu\}.\ \text{if } P \xrightarrow{\alpha} P' \text{ then } Q \setminus \{\nu\} \overset{\alpha}{\Rightarrow} Q' \setminus \{\nu\} \wedge P' \mathcal{R} (Q' \setminus \{\nu\})$$

Proof:

$e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \overset{\text{only rule}}{\Rightarrow} e(P) \overset{\alpha}{\Rightarrow} P'' \overset{Q \approx e(P)}{\Rightarrow} Q \overset{\alpha}{\Rightarrow} Q' \wedge P'' \approx Q' \overset{\text{transitivity}}{\Rightarrow} Q' \approx e(P') \Rightarrow P' \mathcal{R} Q' \setminus \{\nu\}$

Lemma 1.2

$$\text{if } \forall P, Q \in \text{CCS}_{\text{seq}}.\ \text{if } e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\} \text{ then } P \overset{\alpha}{\Rightarrow} P' \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$$
$$\text{then } \forall P, Q \in \text{CCS}_{\text{seq}}, P \mathcal{R} Q \setminus \{\nu\}.\ \text{if } Q \setminus \{\nu\} \xrightarrow{\alpha} Q' \setminus \{\nu\} \text{ then } P \overset{\alpha}{\Rightarrow} P' \wedge P' \mathcal{R} (Q' \setminus \{\nu\})$$

Proof:

$$Q \setminus \{\nu\} \xrightarrow{\alpha} Q' \setminus \{\nu\} \overset{\text{only rule}}{\Rightarrow} Q \xrightarrow{\alpha} Q' \overset{Q \approx e(P)}{\Rightarrow} e(P) \overset{\alpha}{\Rightarrow} P'' \wedge Q' \approx P''$$

$$\overset{\text{hide}}{\Rightarrow} e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \overset{\text{hypothesis}}{\Rightarrow} P \overset{\alpha}{\Rightarrow} P' \wedge P' \mathcal{R} P'' \setminus \{\nu\} \overset{\text{relation}}{\Rightarrow} P'' \approx e(P') \overset{\text{transitivity}}{\Rightarrow} Q' \approx e(P')$$

Equivalence

$$\forall P \in \mathrm{CCS}_{\mathrm{seq}}. P \approx e(P) \setminus \{\nu\}$$

let

$$\mathcal{R} = \{(P, Q \setminus \{\nu\}) \mid P, Q \in \mathrm{CCS}_{\mathrm{seq}}, Q \approx e(P)\}$$

we need to prove that $\mathcal{R}$ is a weak bisimulation i.e.

- $\forall P \in \mathrm{CCS}_{\mathrm{seq}}$. if $P \xrightarrow{\alpha} P'$ then $e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$
- $\forall P \in \mathrm{CCS}_{\mathrm{seq}}$. if $e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}$ then $P \overset{\alpha}{\Rightarrow} P' \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$

The proof is done by induction on the height of the derivation tree, so we can rewrite it as follows:

- $\forall P \in \mathrm{CCS}_{\mathrm{seq}}. \forall h \in \mathbb{N}$. if $P \xrightarrow{\alpha} P'$ with tree of height h then $e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\} \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$
- $\forall P \in \mathrm{CCS}_{\mathrm{seq}}. \forall h \in \mathbb{N}$. if $e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}$ with tree of height h then $P \overset{\alpha}{\Rightarrow} P' \wedge P' \mathcal{R} (P'' \setminus \{\nu\})$

First point
base case h=1

The only way a process can make a transition with derivation tree of height 1 is

$$\mathrm{Act}\,\frac{}{\alpha.P \xrightarrow{\alpha} P}$$

and in this case also

$$\mathrm{Hide}\,\frac{\mathrm{Act}\,\dfrac{}{\alpha.e(P) \xrightarrow{\alpha} e(P)}}{e(\alpha.P) \setminus \{\nu\} = \alpha.e(P) \setminus \{\nu\} \xrightarrow{\alpha} e(P) \setminus \{\nu\}}\ \alpha \neq \nu \text{ by construction}$$

and $P \mathcal{R} (e(P) \setminus \{\nu\})$ because $e(P) \approx e(P)$

inductive case Const

if

$$\mathrm{Const}\,\frac{P \xrightarrow{\alpha} P'}{K \xrightarrow{\alpha} P'}\ K \overset{\mathrm{def}}{=} P$$

$\overset{\text{by induction}}{\Rightarrow} e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}$ and $P' \mathcal{R} (P'' \setminus \{\nu\})$

$\Rightarrow e(P) \xrightarrow{\alpha} P''$ so

$$\mathrm{Hide}\,\frac{\mathrm{Const}\,\dfrac{e(P) \xrightarrow{\alpha} P''}{K_e \xrightarrow{\alpha} P''}\ K_e \overset{\mathrm{def}}{=} e(P)}{e(K) \setminus \{\nu\} = K_e \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}}\ \alpha \neq \nu \text{ by construction}$$

and $P' \mathcal{R} (P'' \setminus \{\nu\})$

inductive case Hide

if

$$\text{Hide}\,\frac{P \xrightarrow{\alpha} P'}{P \setminus L \xrightarrow{\alpha} P' \setminus L}\ \alpha, \overline{\alpha} \notin L$$

$\overset{\text{by induction}}{\Rightarrow}\ e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\}$ and $P'\ \mathcal{R}\ (P'' \setminus \{\nu\})$

$\overset{\text{relation}}{\Rightarrow} P'' \approx e(P')$

$\overset{\text{only rule}}{\Rightarrow}\ e(P) \overset{\alpha}{\Rightarrow} P''$

$$\text{Hide}\,\frac{\text{Hide}\,\dfrac{e(P) \overset{\alpha}{\Rightarrow} P''}{e(P \setminus L) = e(P) \setminus L \overset{\alpha}{\Rightarrow} P'' \setminus L}\ \alpha, \overline{\alpha} \notin L}{e(P \setminus L) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus L \setminus \{\nu\}}\ \alpha, \overline{\alpha} \notin \{\nu\}$$

I have to prove that $P' \setminus L\ \mathcal{R}\ P'' \setminus L \setminus \{\nu\}$ i.e. $P'' \setminus L \approx e(P' \setminus L)$

$$P'' \setminus L \overset{P'' \approx e(P')}{\approx} e(P') \setminus L = e(P' \setminus L)$$

inductive case Red

if

$$\text{Red}\,\frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(a)} P'[f]}$$

$\overset{\text{by induction}}{\Rightarrow}\ e(P) \setminus \{\nu\} \overset{\alpha}{\Rightarrow} P'' \setminus \{\nu\}$ and $P'\ \mathcal{R}\ (P'' \setminus \{\nu\})$

$\overset{\text{only rule}}{\Rightarrow}\ e(P) \overset{\alpha}{\Rightarrow} P''$

and so

$$\text{Hide}\,\frac{\text{Red}\,\dfrac{e(P) \overset{\alpha}{\Rightarrow} P''}{e(P)[f] \overset{f(\alpha)}{\Rightarrow} P''[f]}}{e(P[f]) \setminus \{\nu\} = e(P)[f] \setminus \{\nu\} \overset{f(\alpha)}{\Rightarrow} P''[f] \setminus \{\nu\}}$$

and $P''[f] \overset{P'' \approx e(P')}{\approx} e(P')[f] = e(P'[f]) \Rightarrow P'[f]\ \mathcal{R}\ P''[f] \setminus \{\nu\}$

Inductive case Sum

Sum case is trivial because if

$$\text{Sum}\,\frac{\overset{\text{Act}}{\alpha_j.P_j \xrightarrow{\alpha_j} P_j}}{\sum_{j \in J \neq \emptyset} \alpha_j.P_j \xrightarrow{\alpha_j} P_j}\ j \in J$$

also

$$\text{Hide}\,\frac{\text{Sum}\,\dfrac{\overset{\text{Act}}{\alpha_j.e(P_j) \xrightarrow{\alpha_j} e(P_j)}}{\sum_{j \in J \neq \emptyset} \alpha_j.e(P_j) \xrightarrow{\alpha_j} e(P_j)}\ j \in J}{e\left(\sum_{j \in J \neq \emptyset} \alpha_j.P_j\right) \setminus \{\nu\} = \sum_{j \in J \neq \emptyset} \alpha_j.e(P_j) \setminus \{\nu\} \xrightarrow{\alpha_j} e(P_j) \setminus \{\nu\}}\ \alpha_j \neq \nu$$

and $P_j\ \mathcal{R}\ \left(e(P_j) \setminus \{\nu\}\right)$.

Inductive case Par-1/Par-2/Par-3

If

$$\text{Par-1}\ \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q}$$

$\overset{\text{by induction}}{\Rightarrow} \quad e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}$ and $P' \ \mathcal{R}\ (P'' \setminus \{\nu\})$

$\Rightarrow e(P) \xrightarrow{\alpha} P''$

and so

$$\cfrac{\text{Red}\ \cfrac{e(P) \xrightarrow{\alpha} P''}{e(P)\left[\frac{\nu'}{\nu}\right] \xrightarrow{\alpha} P''\left[\frac{\nu'}{\nu}\right]}\ \alpha \neq \nu \Rightarrow \left[\frac{\nu'}{\nu}\right](\alpha) = \alpha}{\cfrac{\dots}{e(P \mid Q) \setminus \{\nu\} = \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \setminus \{\nu\} \xrightarrow{\alpha} \left(P''\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \setminus \{\nu\}}}$$

we need to show that $(P'|Q)\ \mathcal{R}\ \left(P''\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \setminus \{\nu\}$

which is equivalent to show that $\left(P''\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \approx e(P'|Q)$

$$P'\ \mathcal{R}\ P'' \setminus \{\nu\} \overset{\mathcal{R}\ \text{definition}}{\Rightarrow} P'' \approx e(P')$$

$$\overset{\text{bisim properties}}{\Rightarrow} \left(P''\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} \approx \left(e(P')\left[\frac{\nu'}{\nu}\right] \mid e(Q)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.\overline{\nu'}.\nu.0\right) \setminus \{\nu'\} = e(P'|Q)$$

Par-2 and Par-3 are similar

Inductive case Seq-L

If

$$\text{Seq-L}\ \frac{P \xrightarrow{\alpha} P'}{P;Q \xrightarrow{\alpha} P';Q}$$

$\overset{\text{by induction}}{\Rightarrow} \quad e(P) \setminus \{\nu\} \xrightarrow{\alpha} P'' \setminus \{\nu\}$ and $P' \ \mathcal{R}\ (P'' \setminus \{\nu\})$

$\Rightarrow e(P) \xrightarrow{\alpha} P''$

and so

$$\cfrac{\text{Red}\ \cfrac{e(P) \xrightarrow{\alpha} P''}{e(P)\left[\frac{\nu'}{\nu}\right] \xrightarrow{\left[\frac{\nu'}{\nu}\right](\alpha)=\alpha} P''\left[\frac{\nu'}{\nu}\right]}}{\cfrac{\dots}{e(P;Q) \setminus \{\nu\} = \left(e(P)\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\} \xrightarrow{\alpha} \left(P''\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\}}}$$

we need to show that $(P';Q)\ \mathcal{R}\ \left(P''\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\}$

which is equivalent to show that $\left(P''\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} \approx e(P';Q)$

$$P'\ \mathcal{R}\ P'' \setminus \{\nu\} \overset{\mathcal{R}\ \text{definition}}{\Rightarrow} P'' \approx e(P')$$

$$\overset{\text{bisim properties}}{\Rightarrow} \left(P''\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} \approx \left(e(P')\left[\frac{\nu'}{\nu}\right] \mid \overline{v'}.e(Q)\right) \setminus \{\nu'\} = e(P';Q)$$

Inductive case Seq-R

If

$$\text{Seq-R}\ \frac{P \not\rightarrow \qquad Q \xrightarrow{\alpha} Q'}{P;Q \xrightarrow{\alpha} Q'}$$

$P \not\rightarrow \overset{\text{lemma 0}}{\Rightarrow} e(P) \xrightarrow{\tau*} P_{\text{temp}} \xrightarrow{\nu} P' \wedge P' \not\rightarrow$

$Q \xrightarrow{\alpha} Q' \overset{\text{by induction}}{\Rightarrow} e(Q) \setminus \{\nu\} \xrightarrow{\alpha} Q'' \setminus \{\nu\}$ and $Q' \ \mathcal{R}\ (Q'' \setminus \{\nu\}) \Rightarrow e(Q) \xrightarrow{\alpha} Q''$

and so

$$\cfrac{\text{Red}\ \cfrac{e(P) \xrightarrow{\tau*} P_{\text{temp}}}{e(P)\left[\frac{\nu'}{\nu}\right] \xrightarrow{\left[\frac{\nu'}{\nu}\right](\tau*)=\tau*} P_{\text{temp}}\left[\frac{\nu'}{\nu}\right]} \\ \dots}{\left(e(P)\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\} \xrightarrow{\tau*} \left(P_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\}}$$

$$\cfrac{\cfrac{\text{Red}\ \cfrac{P_{\text{temp}} \xrightarrow{\nu} P'}{P_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \xrightarrow{\left[\frac{\nu'}{\nu}\right](\nu)=\nu'} P'\left[\frac{\nu'}{\nu}\right]} \qquad \overline{\nu'}.e(Q) \xrightarrow{\overline{\nu'}} e(Q)}{\dots}}{\left(P_{\text{temp}}\left[\frac{\nu'}{\nu}\right] \mid \overline{\nu'}.e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\} \xrightarrow{\tau} \left(P'\left[\frac{\nu'}{\nu}\right] \mid e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\}}$$

$$\cfrac{\cfrac{e(Q) \xrightarrow{\alpha} Q''}{\dots}}{\left(P'\left[\frac{\nu'}{\nu}\right] \mid e(Q)\right) \setminus \{\nu'\} \setminus \{\nu\} \xrightarrow{\alpha} \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q''\right) \setminus \{\nu'\} \setminus \{\nu\}}$$

Now I have to prove that $Q' \ \mathcal{R}\ \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q''\right) \setminus \{\nu'\} \setminus \{\nu\}$

Which is equivalent to prove that $\left(P'\left[\frac{\nu'}{\nu}\right] \mid Q''\right) \setminus \{\nu'\} \approx e(Q')$

$$P' \not\rightarrow \overset{\text{End-Red}}{\Rightarrow} P'\left[\frac{\nu'}{\nu}\right] \not\rightarrow \overset{\text{lemma 3}}{\Rightarrow} P'\left[\frac{\nu'}{\nu}\right] \approx 0$$

$\Rightarrow \left(P'\left[\frac{\nu'}{\nu}\right] \mid Q''\right) \setminus \{\nu'\} \approx (0 \mid Q'') \setminus \{\nu'\} \overset{Q'' \approx e(Q')}{\approx} (0 \mid e(Q')) \setminus \{\nu'\} \overset{\text{lemma 4}}{\approx} 0 \mid e(Q') \approx e(Q')$