Exercise G

Show that the encoding of the operator Even in the mu-calculus captures the property of interest:

$$[\![\mathrm{Even}(\varphi)]\!]_\eta = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$$

Solution

As discussed during the lectures:

$$\mathrm{Even}(\varphi) = \mu X.(\varphi \vee ([\mathrm{Act}]X \wedge \langle \mathrm{Act}\rangle T))$$
$$\Rightarrow [\![\mathrm{Even}(\varphi)]\!]_\eta = [\![\mu X.(\varphi \vee ([\mathrm{Act}]X \wedge \langle \mathrm{Act}\rangle T))]\!]_\eta = \mathrm{fix}(f_\varphi)$$

where $f_\varphi(S) = [\![\varphi \vee ([\mathrm{Act}]X \wedge \langle \mathrm{Act}\rangle T)]\!]_{\eta_{[X \to S]}} = [\![\varphi \vee ([\mathrm{Act}]S \wedge \langle \mathrm{Act}\rangle T)]\!]_\eta$

The proof can be divided in two parts:

1. $[\![\mathrm{Even}(\varphi)]\!]_\eta \subseteq \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$
2. $[\![\mathrm{Even}(\varphi)]\!]_\eta \supseteq \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$

In fact

1. $\wedge$ 2. $\Rightarrow [\![\mathrm{Even}(\varphi)]\!]_\eta = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$

Part 1

$$S = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$$
$$= \{P \mid P \vDash \varphi \vee P \text{ can make a transition and } \forall P_1 . P \to P_1 \Rightarrow \forall \text{ complete computation } P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$$
$$= \{P \mid P \vDash \varphi\} \cup \left(\left\{P \mid \forall P \xrightarrow{\mathrm{Act}} P' . P \in S\right\} \cap \left\{P \mid \exists P \xrightarrow{\mathrm{Act}} P' . P' \in \mathrm{Proc}\right\}\right)$$
$$= [\![\varphi]\!]_\eta \cup ([\![[\mathrm{Act}]S]\!]_\eta \cap [\![\langle \mathrm{Act}\rangle T]\!]_\eta)$$
$$= [\![\varphi \vee ([\mathrm{Act}]S \wedge \langle \mathrm{Act}\rangle T)]\!]_\eta = f_\varphi(S) = S$$

$\Rightarrow S$ is a fixed point of f_φ , but $[\![\mathrm{Even}(\varphi)]\!]_\eta$ is the lfp of f_φ
$\Rightarrow [\![\mathrm{Even}(\varphi)]\!]_\eta \subseteq S = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i \text{ s.t. } P_i \vDash \varphi\}$

Part 2

Let

$$S_n = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i < n \text{ s.t. } P_i \vDash \varphi\}$$

I want to prove that $\forall n \in \mathbb{N}. S_n \subseteq f_\varphi^n(\emptyset)$

By induction on n:

Base case $n = 1$

$$S_1 = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i < 1 \text{ s.t. } P_i \vDash \varphi\} = \{P \mid P \vDash \varphi\}$$
$$\subseteq \{P \mid P \vDash \varphi\} \cup [\![[\mathrm{Act}]\emptyset \wedge \langle \mathrm{Act}\rangle T]\!]_\eta = [\![\varphi \vee ([\mathrm{Act}]\emptyset \wedge \langle \mathrm{Act}\rangle T)]\!]_\eta = f_\varphi(\emptyset)$$

Inductive case $n \Rightarrow n+1$

$$S_{n+1} = \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \ldots \exists i < n+1 \text{ s.t. } P_i \vDash \varphi\}$$

It is easy to see that

$$
\begin{aligned}
S_{n+1} &= \{P \mid P \vDash \varphi \vee P \text{ can make a transition and } \forall P_1 . P \to P_1 \Rightarrow \forall \text{ complete computation } P_1 \to \dots \exists i < n \text{ s.t. } P_i \vDash \varphi\} \\
&= \{P \mid P \vDash \varphi\} \cup \left(\left\{P \mid \forall P \overset{\mathrm{Act}}{\to} P' . P \in S_n\right\} \cap \left\{P \mid \exists P \overset{\mathrm{Act}}{\to} P' . P' \in \mathrm{Proc}\right\}\right) \\
&= [\![\varphi]\!]_\eta \cup \left([\![[\mathrm{Act}] S_n]\!]_\eta \cap [\![\langle \mathrm{Act} \rangle T]\!]_\eta\right)
\end{aligned}
$$

By induction

$$
\begin{aligned}
& S_n \subseteq f_\varphi^n(\emptyset) \\
\Rightarrow\ & [\![[\mathrm{Act}] S_n]\!]_\eta \subseteq [\![[\mathrm{Act}] f_\varphi^n(\emptyset)]\!]_\eta \\
\Rightarrow\ & S_{n+1} = [\![\varphi]\!]_\eta \cup \left([\![[\mathrm{Act}] S_n]\!]_\eta \cap [\![\langle \mathrm{Act} \rangle T]\!]_\eta\right) \subseteq [\![\varphi]\!]_\eta \cup \left([\![[\mathrm{Act}] f_\varphi^n(\emptyset)]\!]_\eta \cap [\![\langle \mathrm{Act} \rangle T]\!]_\eta\right) = f_\varphi^{n+1}(\emptyset)
\end{aligned}
$$

Since 2^{Proc} is a dcpo, By Kleene fixed-point theorem

$$
[\![\mathrm{Even}(\varphi)]\!]_\eta = \mathrm{fix}(f_\varphi) = \sup\left(\left\{f_\varphi^n(\emptyset) \mid n \in \mathbb{N}\right\}\right)
$$

and so

$$
\begin{aligned}
& \forall n \in \mathbb{N} . S_n \subseteq f_\varphi^n(\emptyset) \subseteq \mathrm{fix}(f_\varphi) = [\![\mathrm{Even}(\varphi)]\!]_\eta \\
\Rightarrow\ & \{P \mid \forall \text{ complete computation } P = P_0 \to P_1 \to P_2 \to \dots \exists i \text{ s.t. } P_i \vDash \varphi\} \subseteq [\![\mathrm{Even}(\varphi)]\!]_\eta
\end{aligned}
$$