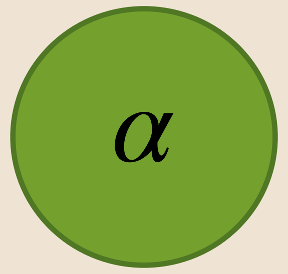
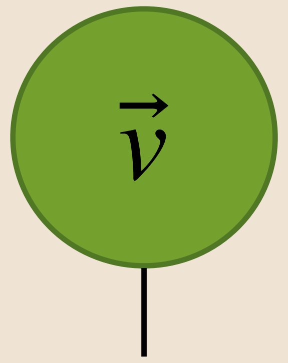
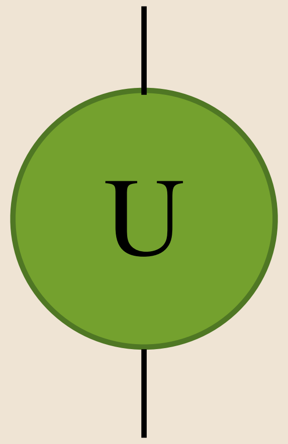
Tensors



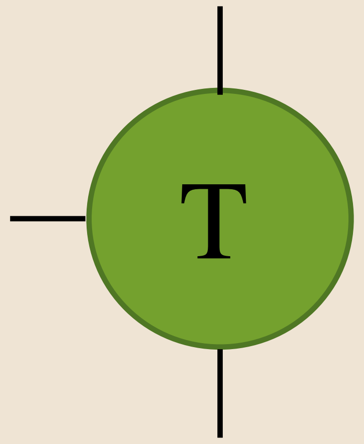
$=$ order-0 tensor $=$ **scalar**



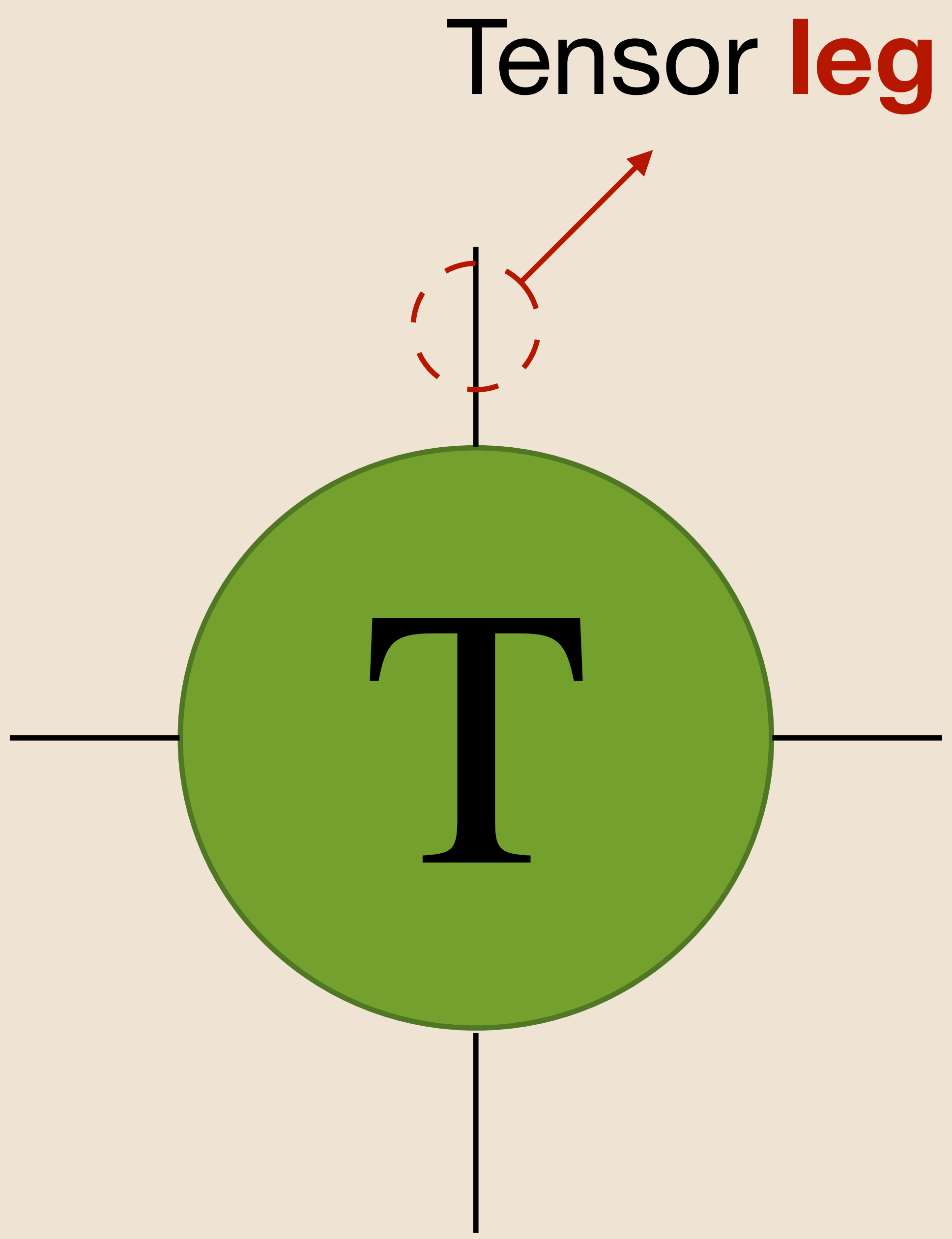
$=$ order-1 tensor $=$ **vector**



$=$ order-2 tensor $=$ **matrix**

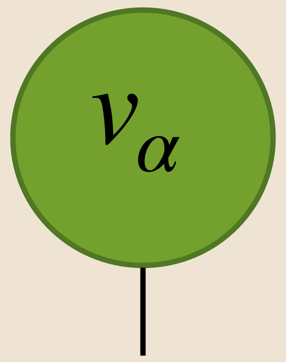


$=$ order-3 tensor $=$ **tensor**

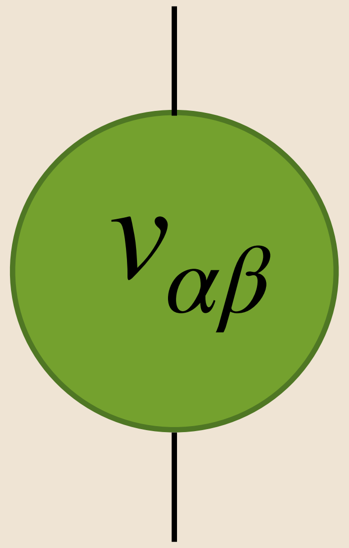


Reshape a tensor

We can manipulate **Tensors** and **reshape** their indexes (legs) as we prefer:



$$= \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \xrightarrow{\textbf{Reshape}}$$



$$= \begin{pmatrix} v_0 & v_1 \\ v_2 & v_3 \end{pmatrix}$$

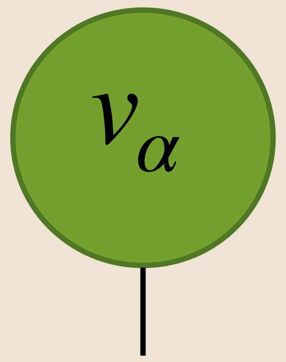
$$[v_1, v_2, v_3, \ldots v_N] \qquad N = mn$$

$$\alpha \in [0,N-1] \rightarrow \beta \in [0,m-1], \gamma \in [0,n-1]$$

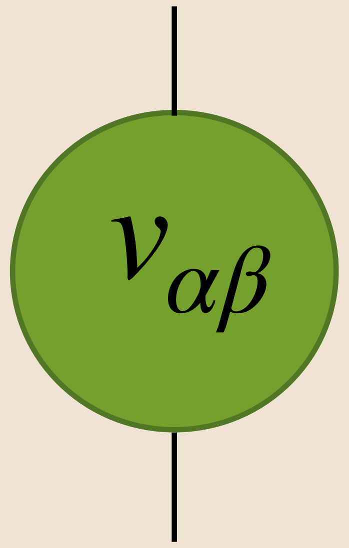
$$\alpha = \gamma m + \beta$$

Reshape a tensor

We can manipulate **Tensors** and **reshape** their indexes (legs) as we prefer:

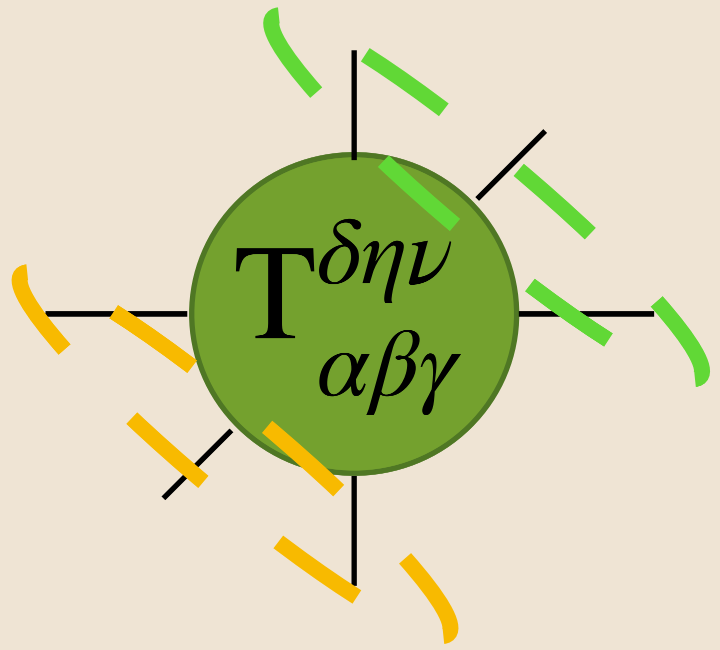


$$= \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \xrightarrow{\textbf{Reshape}}$$

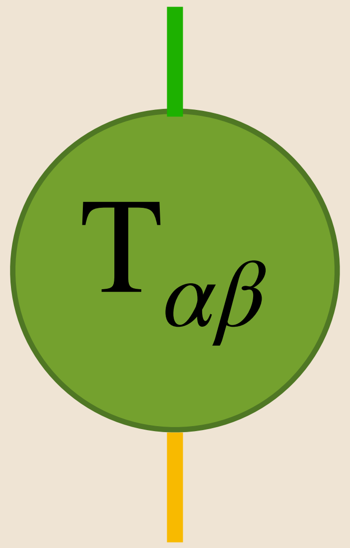


$$= \begin{pmatrix} v_0 & v_1 \\ v_2 & v_3 \end{pmatrix}$$

This means that a tensor of **any order** can be mapped to a **matrix**. So, we can use linear algebra to work with tensors.



$$\xrightarrow{\textbf{Reshape}}$$



Permute a tensor

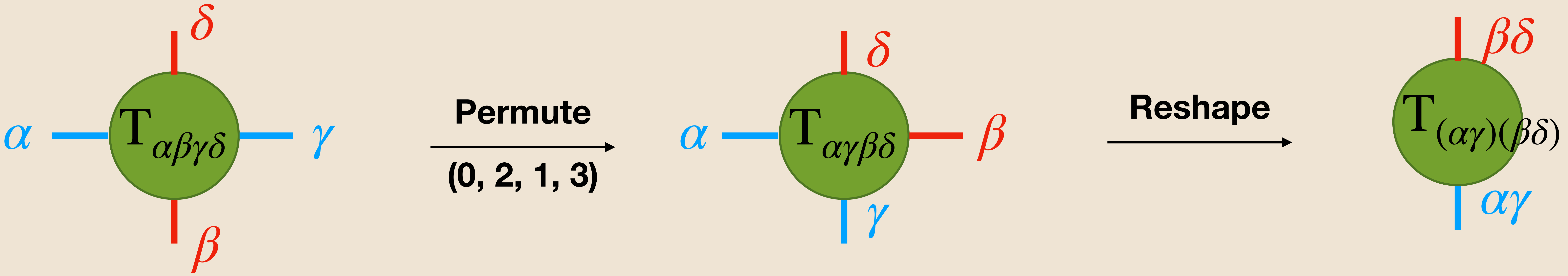
To reshape a tensor in the correct way we have to group together the legs involved in the following operation.
To do so, we have to know how to permute the legs of a tensor. In the case of a matrix it is a simple **transposition.**

$$v_{\alpha\beta} = \begin{pmatrix} v_0 & v_1 \\ v_2 & v_3 \end{pmatrix} \xrightarrow{\textbf{Transpose}} v_{\beta\alpha} = \begin{pmatrix} v_0 & v_2 \\ v_1 & v_3 \end{pmatrix}$$

We can generalise this to n-legged tensors. Suppose we want to reshape $T_{\alpha\beta\gamma\delta}$ in a matrix

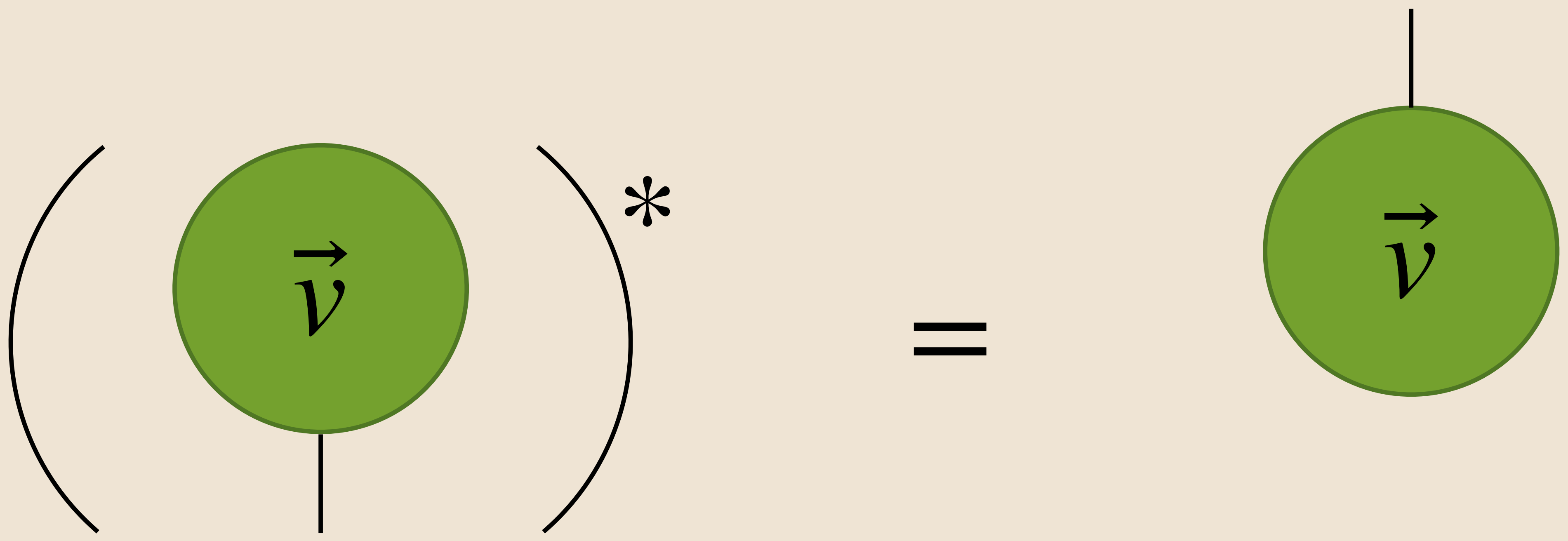
$T_{(\alpha\gamma),(\beta\delta)}$

$O(n)$



Complex conjugate of a tensor

We start by introducing the **complex conjugate** of a order-1 tensor:

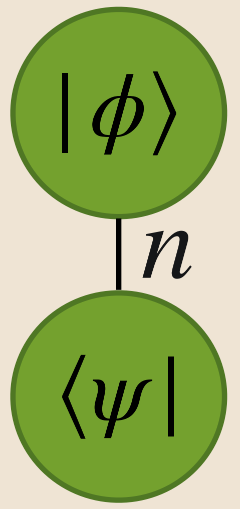


Tensor contraction

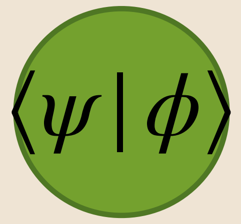
Then we introduce the **contraction** between two tensor along their **legs.**

- We start from two order-1 tensor, and it is equivalent to the scalar product between two vectors:

$$\langle\psi|\phi\rangle = \sum_{i}^{n} \psi_i^* \phi_i =$$



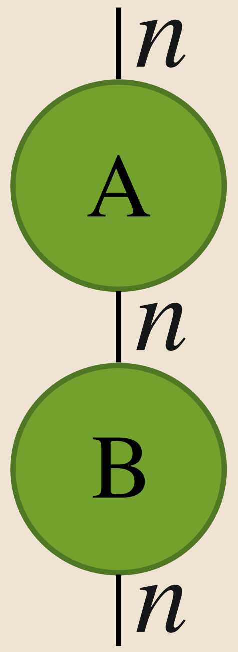
$=$



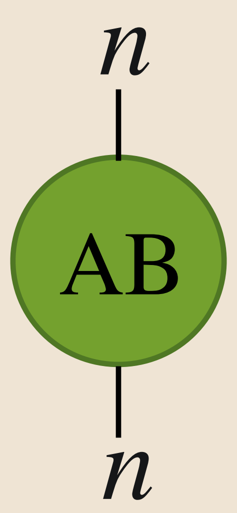
$$O(n)$$

- Then, the contraction between two order-2 tensor is simply the matrix-matrix multiplication:

$$(\mathrm{AB})_{ik} = \sum_{j} a_{ij} b_{jk} =$$



$=$

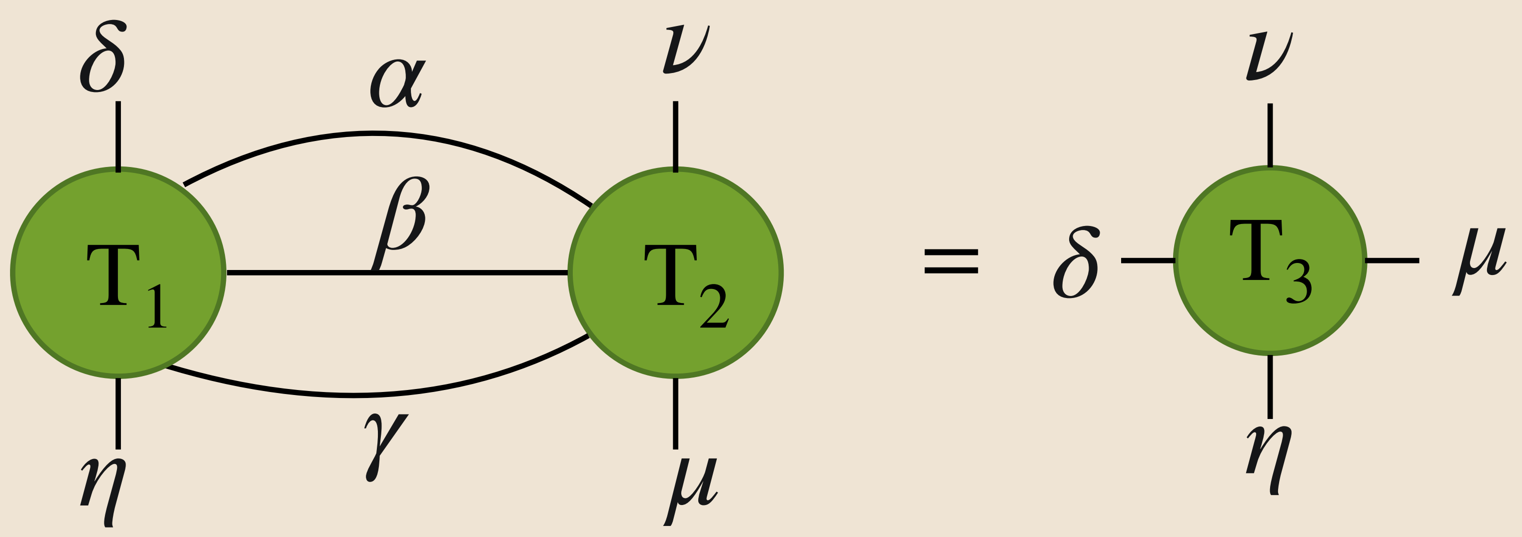


$$O(n^3)$$

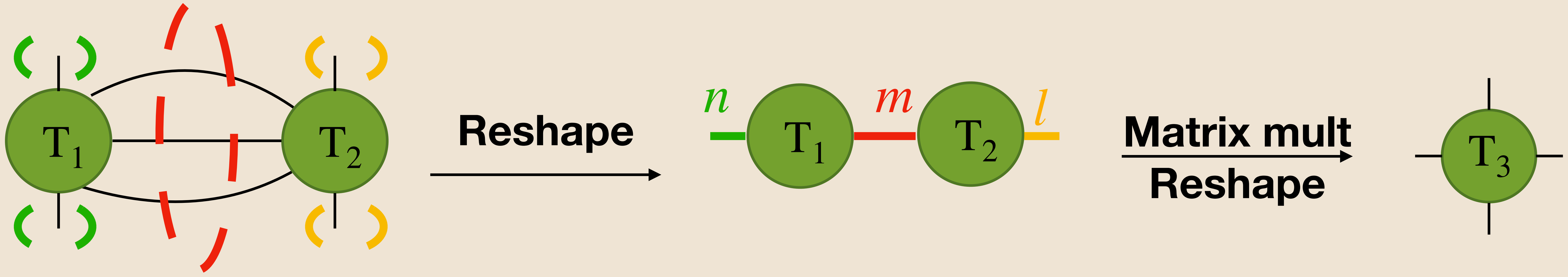
Tensor contraction

- In general, we can contract any leg of an order-n tensor:

$$T_3 = \sum_{\alpha\beta\gamma} \mathrm{T}_{1,\alpha\beta\gamma\delta\eta}\mathrm{T}_{2,\alpha\beta\gamma\mu\nu} =$$



- What will be done in practice by the simulator, however, will be a little different:



$O(nml)$