Marginal Treatment Effects: Implementation

ECON 31720 Applied Microeconometrics

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Framework for Marginal Treatment Effects

- $Y \in \mathbb{R}$ is a scalar **outcome** of interest, $D \in \{0, 1\}$ is a **binary treatment**
- D and Y are linked by **potential outcomes** $Y(0), Y(1)$
- $X \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$ is a vector of predetermined, **observable** characteristics
- $U \in \mathbb{R}$ is an **unobserved** and continuously distributed **latent variable**
- $Z \in \mathcal{Z} \subseteq \mathbb{R}$ is a scalar **instrumental variable**
  - Z satisfies the conditional **exogeneity** assumption $(Y(0), Y(1), U) \perp\!\!\!\perp Z | X$

Framework for Marginal Treatment Effects

- $\nu(\cdot)$ is an **unknown function** of X and Z such that $D = \mathbb{I}[U \leq \nu(X, Z)]$
  - U, $\nu(X, Z)$ are **additively separable** (no interaction between observables and unobservables)
  - $\nu(X, Z) - U$ denotes the **net utility** from choosing treatment state $D = 1$
- Without loss, the **selection equation** can be normalized to $D = \mathbb{I}[U \leq p(X, Z)]$
  - $p(X, Z) \equiv \mathbb{P}(D = 1|X, Z)$ is the **propensity score** (also denoted as P)
  - U is a latent random variable **uniformly** distributed on $[0, 1]$
- $\mathrm{MTE}(u) \equiv \mathbb{E}[Y(1) - Y(0)|U = u]$ is the **Marginal Treatment Effect** of D on Y
- $\mathrm{MTR}(u)(d|u) \equiv \mathbb{E}[Y(d)|U = u]$ is the **Marginal Treatment Response**
  - The Marginal Treatment Effect of D on Y at $U = u$ is $\mathrm{MTE}(u) = \mathrm{MTR}(1|u) - \mathrm{MTR}(0|u)$

Identification

- Several standard parameters are **weighted averages** of marginal treatment responses
  - **Target parameters**: ATE, ATT, ATU, LATE, PRTE, Average Selection Bias
  - **Estimands**: IV, TSLS, OLS (with and without covariates)
- Multiple identification approaches have been proposed within the MTE framework
  - **Point identification**: these approaches can be broadly classified into
    - **Nonparametric**: Heckman and Vytlacil (1999)'s Local IV Estimand if Z is continuous
    - **Parametric**: linear-in-parameters and partially linear models of the MTR functions
  - **Partial identification**: Mogstad, Santos, and Torgovitsky (2018)

Point Identification: Linear-in-Parameters Models of the MTR Functions

- A general **linear-in-parameters model** of the MTR functions is

$$\mathrm{MTR}(d|u,x) \equiv \mathbb{E}\left[Y(d)|U=u,X=x\right] = \sum_{k=1}^{\overline{k}} \theta_k b_k\left(d|u,x\right) \qquad \text{for } d=0,1$$

where $\{\theta_k\}_{k=1}^{\overline{k}}$ are **unknown coefficients** and $\{b_k\}_{k=1}^{\overline{k}}$ are **known functions**

- When constructing a linear-in-parameters model, a researcher must **choose**:
  - Whether to allow for **additive separability** between U and X
  - The **order of the polynomials** of U and X and/or the **sieve** for U and X

- If observables and unobservables are assumed **not** to be **additively separable**:

$$\mathrm{MTR}(d|u,x) \equiv \mathbb{E}[Y(d)|X=x,U=u] = \alpha_d + \beta_d u + x'\gamma_d + ux'\delta_d \qquad \text{for } d=0,1$$

Point Identification: Linear-in-Parameters Models of the MTR Functions

$$
\begin{aligned}
\mathbb{E}[Y|D=1,P=u,X=x] &= \mathbb{E}[DY(1)+(1-D)Y(0)|D=1,P=u,X=x] \\
&= \mathbb{E}[Y(1)|D=1,P=u,X=x] \\
&= \mathbb{E}[Y(1)|U\leq P,P=u,X=x] \qquad (D=\mathbb{I}\left[U\leq p(X,Z)\right]) \\
&= \mathbb{E}[Y(1)|U\leq u,X=x] \qquad (Z \perp\!\!\!\perp U|X) \\
&= \frac{1}{u}\int_0^u \mathbb{E}[Y(1)|W=w,X=x]dw \qquad (U\sim\mathcal{U}\left[0,1\right]) \\
&= \frac{1}{u}\int_0^u \left[\alpha_1+\beta_1 w+x'\gamma_1+wx'\delta_1\right]dw \\
&= \frac{1}{u}\left[\alpha_1 u+\frac{\beta_1}{2}u^2+ux'\gamma_1+u^2x'\frac{\delta_1}{2}\right] \\
&= \alpha_1+\frac{\beta_1}{2}u+x'\gamma_1+ux'\frac{\delta_1}{2}
\end{aligned}
$$

Point Identification: Linear-in-Parameters Models of the MTR Functions

- Thus: $\mathbb{E}\left[Y|D=1,P=u,X=x\right]=\alpha_1+\frac{\beta_1}{2}u+x'\gamma_1+ux'\frac{\delta_1}{2}$

- Analogously: $\mathbb{E}\left[Y|D=0,P=u,X=x\right]=\left(\alpha_0+\frac{\beta_0}{2}\right)+\frac{\beta_0}{2}u+x'\left(\gamma_0+\frac{\delta_0}{2}\right)+ux'\frac{\delta_0}{2}$

- Goal: **point identify** parameters $\{\alpha_d,\beta_d,\gamma_d,\delta_d\}_{d\in\{0,1\}}$ of the linear MTR functions

- Implementation: **regress Y on 1, P, X, and PX** separately for units with $D\in\{0,1\}$

$$Y=\alpha_d^*+\beta_d^*P+X'\gamma_d^*+PX'\delta_d^*+R_d \qquad \text{for } d=0,1$$

- Back out **MTR parameters** using **regression coefficients**:

$$\begin{aligned} &\boldsymbol{\alpha_1}=\alpha_1^* && \boldsymbol{\beta_1}=2\beta_1^* && \boldsymbol{\gamma_1}=\gamma_1^* && \boldsymbol{\delta_1}=2\delta_1^* \\ &\boldsymbol{\alpha_0}=\alpha_0^*-\beta_0^* && \boldsymbol{\beta_0}=2\beta_0^* && \boldsymbol{\gamma_0}=\gamma_0^*-\delta_0^* && \boldsymbol{\delta_0}=2\delta_0^* \end{aligned}$$

Gelbach (2002)

- Example: "Public Schooling for Young Children and Maternal Labor Supply" (*AER*, 2002)
- This paper by Jonah Gelbach provides an interesting setup for the MTE framework
- **Goal**: estimate the effect of **public school enrollment** on **women's labor supply**
- Public school enrollment is **not as-good-as randomly assigned**
  - Parents may **choose** to hold their children back a year or enroll them in private school
- **Institutional framework**: parents' ability to enroll a child in public kindergarten in the academic year during which the child turns five depends on the calendar date of the **child's birth**
- **Empirical strategy**: instrument public school enrollment with child's quarter of birth

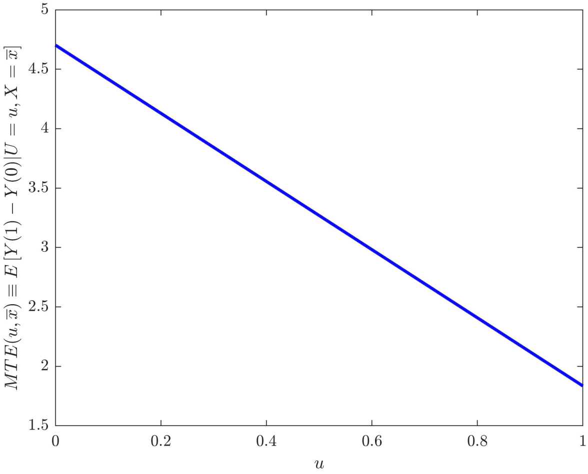
Gelbach (2002)

- The author's **TSLS estimate** is ≈ 2.71 and statistically significant at conventional levels
- However, it is hard to provide a clear **economic interpretation** to this estimate
  - The main specification conditions **linearly** on **covariates** and uses **four instruments**
  - The TSLS estimand is a weighted average (likely with **negative weights**) of treatment effects
- Let us explore treatment effect **heterogeneity** in a MTE framework
- A **linear-in-parameters** model of the MTR functions:

$$\mathrm{MTR}(d|u,x) \equiv \mathbb{E}[Y(d)|X=x, U=u] = \alpha_d + \beta_d u + x'\gamma_d + ux'\delta_d \qquad \text{for } d = 0,1$$

where $D \in \{0,1\}$ denotes public school enrollment and X is a vector of covariates

Gelbach (2002)



This figure plots the **estimated MTE function**, where the vector X is evaluated at its mean

Gelbach (2002)

- The child's **quarter-of-birth** instrument vector is defined as

$$Z \equiv \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \mathbb{I}\left[\text{QOB} = \text{Q2-1974}\right] \\ \mathbb{I}\left[\text{QOB} = \text{Q3-1974}\right] \\ \mathbb{I}\left[\text{QOB} = \text{Q4-1974}\right] \\ \mathbb{I}\left[\text{QOB} = \text{Q1-1975}\right] \end{bmatrix}$$

- The estimated MTE function can be used to compute **interpretable target parameters**:

$$\widehat{\mathrm{LATE}}_{z_4 \to z_3}(\overline{x}) = \int_0^1 \widehat{\mathrm{MTE}}(u, \overline{x}) \frac{\mathbb{I}\left[\overline{\widehat{p}}(x, z_4) < u \leq \overline{\widehat{p}}(x, z_3)\right]}{\overline{\widehat{p}}(x, z_3) - \overline{\widehat{p}}(x, z_4)} du \approx 3.45$$

$$\widehat{\mathrm{LATE}}_{z_3 \to z_2}(\overline{x}) = \int_0^1 \widehat{\mathrm{MTE}}(u, \overline{x}) \frac{\mathbb{I}\left[\overline{\widehat{p}}(x, z_3) < u \leq \overline{\widehat{p}}(x, z_2)\right]}{\overline{\widehat{p}}(x, z_2) - \overline{\widehat{p}}(x, z_3)} du \approx 2.77$$

$$\widehat{\mathrm{LATE}}_{z_2 \to z_1}(\overline{x}) = \int_0^1 \widehat{\mathrm{MTE}}(u, \overline{x}) \frac{\mathbb{I}\left[\overline{\widehat{p}}(x, z_2) < u \leq \overline{\widehat{p}}(x, z_1)\right]}{\overline{\widehat{p}}(x, z_1) - \overline{\widehat{p}}(x, z_2)} du \approx 2.38$$

Gelbach (2002)

- Enrolling a child in public school in **Q1-1975** implies the child is not even five years old
- Mothers who are willing to do so are likely to be **more sensitive to public subsidies** than mothers who are shifted into the treated arm when a child was born in **Q2-1974**
- This **unobserved heterogeneity** may explain $\widehat{\text{LATE}}_{z_4 \to z_3} > \widehat{\text{LATE}}_{z_3 \to z_2} > \widehat{\text{LATE}}_{z_2 \to z_1}$
  - A mother's opportunity cost of not working (i.e., her **return from working**) is **increasing** in her **willingness to delay** the enrollment of a five-year old child in a public kindergarten
- Modeling the MTR functions allows an empiricist to **analyze** unobserved heterogeneity
- Linear-in-parameters models of the MTR functions are not the only option...

Point Identification: Partially Linear Models of the MTR Functions

- An alternative to linear-in-parameters models is **partially linear models**

- A common partially linear model of the MTR functions is

$$\mathrm{MTR}(d|u,x) \equiv \mathbb{E}\left[Y(d)|X=x, U=u\right] = g_d\left(u\right) + x'\beta_d$$

  where g_d is an **unknown function** of the latent variable U

- This model assumes **additive separability** between observables and unobservables

- Point identification of the MTR and MTE functions follows Robinson (1988)

Point Identification: Partially Linear Models of the MTR Functions

$$
\begin{aligned}
\mathbb{E}[Y|D=1,P=u,X=x] &= \mathbb{E}[DY(1)+(1-D)Y(0)|D=1,P=u,X=x] \\
&= \mathbb{E}[Y(1)|D=1,P=u,X=x] \\
&= \mathbb{E}[Y(1)|U\leq P,P=u,X=x] \qquad (D=\mathbb{I}\left[U\leq p(X,Z)\right]) \\
&= \mathbb{E}[Y(1)|U\leq u,X=x] \qquad (Z\perp\!\!\!\perp U|X) \\
&= \frac{1}{u}\int_0^u \mathbb{E}[Y(1)|W=w,X=x]dw \qquad (U\sim\mathcal{U}\left[0,1\right]) \\
&= \frac{1}{u}\int_0^u \left(g_1\left(w\right)+x'\beta_1\right)dw \\
&= \frac{1}{u}\left(ux'\beta_1+\int_0^u g_1\left(w\right)dw\right) \\
&= x'\beta_1+\frac{1}{u}\int_0^u g_1\left(w\right)dw
\end{aligned}
$$

Point Identification: Partially Linear Models of the MTR Functions

- Thus: $\mathbb{E}[Y|D=1, P=u, X=x] = x'\beta_1 + \frac{1}{u}\int_0^u g_1(w)\, dw$

- Analogously: $\mathbb{E}[Y|D=0, P=u, X=x] = x'\beta_0 + \frac{1}{1-u}\int_u^1 g_0(w)\, dw$

- The Law of Iterated Expectations implies that

$$
\begin{aligned}
\mathbb{E}[Y|P=u, X=x] &= \mathbb{E}[Y|D=1, P=u, X=x] \times \mathbb{P}(D=1|P=u, X=x) \\
&+ \mathbb{E}[Y|D=0, P=u, X=x] \times \mathbb{P}(D=0|P=u, X=x) \\
&= \mathbb{E}[Y|D=1, P=u, X=x] \times u \\
&+ \mathbb{E}[Y|D=0, P=u, X=x] \times (1-u) \\
&= ux'\beta_1 + \int_0^u g_1(w)\, dw + (1-u)\, x'\beta_0 + \int_u^1 g_0(w)\, dw \\
&= x'\beta_0 + ux'(\beta_1 - \beta_0) + \int_0^u g_1(w)\, dw + \int_u^1 g_0(w)\, dw
\end{aligned}
$$

Point Identification: Partially Linear Models of the MTR Functions

- Under this parameterization, the **conditional mean of the observed outcome** is

$$\mathbb{E}[Y|P = u, X = x] = x'\beta_0 + ux'(\beta_1 - \beta_0) + \overline{g}(u)$$

where $\overline{g}(u) \equiv \int_0^u g_1(w)\,dw + \int_u^1 g_0(w)\,dw$ is an **unknown function** of the latent variable

- In a **linear-in-parameters** model, $\overline{g}(u)$ would be **sieved**

- In a **partially linear** model, $\overline{g}(u)$ can be estimated with a **kernel-based approach**

- The goal is to **point identify** the **Marginal Treatment Effect function**:

$$\begin{aligned}\mathrm{MTE}(u, x) &= \mathrm{MTR}(1|u, x) - \mathrm{MTR}(0|u, x)\\ &= (g_1(u) + x'\beta_1) - (g_0(u) + x'\beta_0)\\ &= x'(\beta_1 - \beta_0) + g_1(u) - g_0(u)\end{aligned}$$

Point Identification: Partially Linear Models of the MTR Functions

- Using the same derivation as Heckman and Vytlacil (1999)'s **Local IV Estimand**:

$$\mathrm{MTE}(u,x) = \frac{\partial}{\partial p} E[Y|P=p, X=x]\Big|_{p=u} = x'\left(\beta_1 - \beta_0\right) + \overline{g}'\left(u\right)$$

- Combining the two previous expressions for $\mathrm{MTE}(u,x)$:

$$\mathrm{MTE}\left(u,x\right) = x'\left(\beta_1 - \beta_0\right) + g_1\left(u\right) - g_0\left(u\right) = x'\left(\beta_1 - \beta_0\right) + \overline{g}'\left(u\right)$$

- This is not surprising if one exploits the definition of $\overline{g}\left(u\right)$:

$$\overline{g}'\left(u\right) \equiv \frac{\partial}{\partial u}\left[\int_0^u g_1\left(w\right) dw + \int_u^1 g_0\left(w\right) dw\right] = g_1\left(u\right) - g_0\left(u\right)$$

which follows from an application of **Leibniz's rule**

- **Implication**: estimating the MTE function entails estimating the **derivative** of $\overline{g}\left(U\right)$

Point Identification: Partially Linear Models of the MTR Functions

- Identification of the MTEs in this class of partially linear models follows **Robinson (1988)**

- Recall that the **conditional mean of the observed outcome** is

$$\mathbb{E}\left[Y|P,X\right] = X'\beta_0 + PX'\left(\beta_1 - \beta_0\right) + \overline{g}\left(P\right)$$

- The **Law of Iterated Expectations** implies that

$$\begin{aligned}
\mathbb{E}\left[Y|P\right] &= \mathbb{E}\left[\mathbb{E}\left[Y|P,X\right]|P\right] \\
&= \mathbb{E}\left[X'\beta_0 + PX'\left(\beta_1 - \beta_0\right) + \overline{g}\left(P\right)|P\right] \\
&= \mathbb{E}\left[X'|P\right]\beta_0 + P\mathbb{E}\left[X'|P\right]\left(\beta_1 - \beta_0\right) + \overline{g}\left(P\right)
\end{aligned}$$

- Define $\widetilde{Y} \equiv Y - \mathbb{E}\left[Y|P\right]$ and $\widetilde{X} \equiv X - \mathbb{E}\left[X|P\right]$, then add and subtract $\mathbb{E}\left[Y|X,P\right]$:

$$\widetilde{Y} = \mathbb{E}\left[Y|X,P\right] - \mathbb{E}\left[Y|P\right] + Y - \mathbb{E}\left[Y|X,P\right]$$

Point Identification: Partially Linear Models of the MTR Functions

- Replace $\mathbb{E}[Y|P,X]$ and $\mathbb{E}[Y|P]$ with their expressions above:

$$\widetilde{Y} = \widetilde{X}'\beta_0 + P\widetilde{X}'(\beta_1 - \beta_0) + R$$

where R is a **residual** defined as $R \equiv Y - \mathbb{E}[Y|X,P]$

- By the Law of Iterated Expectations, this residual has two convenient **properties**:
  - It is **mean independent** of $\boldsymbol{X}$:

$$\mathbb{E}[R|X] = \mathbb{E}[Y - \mathbb{E}[Y|X,P]|X] = \mathbb{E}[Y|X] - \mathbb{E}[\mathbb{E}[Y|X,P]|X] = \mathbb{E}[Y|X] - \mathbb{E}[Y|X] = 0$$

  - It is **mean independent** of $\boldsymbol{P}$:

$$\mathbb{E}[R|P] = \mathbb{E}[Y - \mathbb{E}[Y|X,P]|P] = \mathbb{E}[Y|P] - \mathbb{E}[\mathbb{E}[Y|X,P]|P] = \mathbb{E}[Y|P] - \mathbb{E}[Y|P] = 0$$

- If $\mathbb{E}[R|X] = \mathbb{E}[R|P] = 0$, both β_0 and $\beta_1 - \beta_0$ are **linear regression coefficients**

Point Identification: Partially Linear Models of the MTR Functions

1. **Estimate** $\widetilde{\boldsymbol{Y}} \equiv Y - \mathbb{E}[Y|P]$ and $\widetilde{\boldsymbol{X}} \equiv X - \mathbb{E}[X|P]$ **nonparametrically** (P is a scalar)
   - **Local constant** regression suffers from boundary bias, so **local linear** regression is preferable

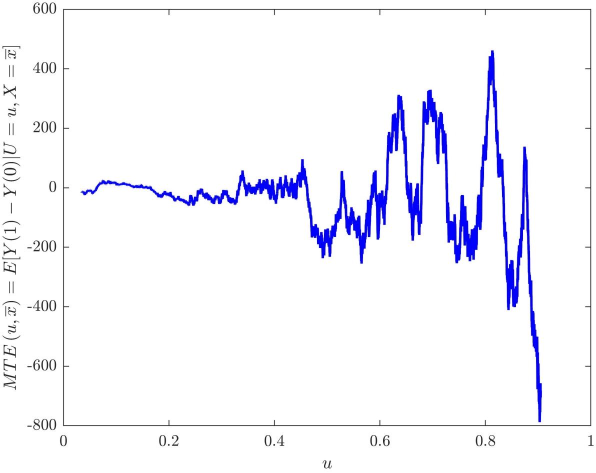
2. Perform a **linear regression** of $\widetilde{\boldsymbol{Y}}$ on $\widetilde{\boldsymbol{X}}$ and $\boldsymbol{P}\widetilde{\boldsymbol{X}}$ and store the estimated $\boldsymbol{\beta_0}$ and $\boldsymbol{\beta_1}$

3. **Estimate** $\overline{\boldsymbol{g}}(\boldsymbol{P})$
   - The mean of Y conditional on P, derived above, can be rearranged as

$$\overline{g}(P) = \mathbb{E}\left[Y - X'\beta_0 - PX'(\beta_1 - \beta_0)|P\right]$$

   - Y, X, P, β_0, and β_1 are now **known**, so $\overline{g}(P)$ can be estimated **nonparametrically**
   - Recall that $\mathrm{MTE}(u, x) = x'(\beta_1 - \beta_0) + \overline{g}'(u)$, so $\overline{g}'$ is of interest
   - **Local linear** suffers from boundary bias in the first derivative, **local quadratic** is preferable

Point Identification: Partially Linear Models of the MTR Functions



This figure plots the **estimated MTE function** ($X = \overline{x}$) using data from Gelbach (*AER*, 2002)

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

- **Target parameters** and **common estimands** are weighted averages of the **MTR** pairs
- **Target parameters** (ATE, ATT, ATU, LATE, PRTE, Average Selection Bias):

$$\beta^* = \Gamma^*(m) \equiv \mathbb{E}\left[\int_0^1 m_0\,(u, X)\,\omega_0^*\,(u, Z)\,du\right] + \mathbb{E}\left[\int_0^1 m_1\,(u, X)\,\omega_1^*\,(u, Z)\,du\right]$$

- **Common estimands** (IV, TSLS, OLS with and without covariates):

$$\beta_s = \Gamma_s(m) \equiv \mathbb{E}\left[\int_0^1 m_0\,(u, X)\,\omega_{0s}\,(u, Z)\,du\right] + \mathbb{E}\left[\int_0^1 m_1\,(u, X)\,\omega_{1s}\,(u, Z)\,du\right]$$

where $\omega_{0s}\,(u, z) \equiv s\,(0, z) \times \mathbb{I}\,[u > p(z)]$ and $\omega_{1s}\,(u, z) \equiv s\,(1, z) \times \mathbb{I}\,[u \leq p(z)]$

- Γ^* and Γ_s are **identified linear maps** of the MTR functions

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

- **Estimands** β_s are functions of the **data** and are thus **known**
- **Weights** $\omega_d^*(U, Z)$ and $\omega_{ds}(U, Z)$, for $d = 0, 1$, are functions of the **data** and **identified**
- The Marginal Treatment Response functions, $m_d(U, X)$ for $d = 0, 1$, are **unknown**
  - As a consequence, **target parameters** are **unknown**
- Intuition: **bound** target parameters such that the **implied MTR functions** are **"consistent" with the data**, i.e., they match known estimands via their (identified) weights
- Formally, these bounds solve two **convex optimization problems**:

$$\underline{\beta}^* \equiv \inf_{m \in \mathcal{M}_{\mathcal{S}}} \Gamma^*(m) \qquad \overline{\beta}^* \equiv \sup_{m \in \mathcal{M}_{\mathcal{S}}} \Gamma^*(m)$$

where $\mathcal{M}_{\mathcal{S}} \equiv \{m \in \mathcal{M} : \Gamma_s(m) = \beta_s \text{ for all } s \in \mathcal{S}\}$

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

- **Issue**: the parameter space of MTR functions, $\mathcal{M}$, is possibly **infinite-dimensional**
- **Solution**: replace $\mathcal{M}$ with a **finite-dimensional subset** $\mathcal{M}_{\text{fd}} \subseteq \mathcal{M}$
- $\mathcal{M}_{\text{fd}}$ could be specified as the **finite linear basis**

$$\mathcal{M}_{\text{fd}} \equiv \left\{ (m_0, m_1) \in \mathcal{M} : m_d(u,x) = \sum_{k=1}^{\overline{k}_d} \theta_{dk} b_{dk}(u,x) \text{ for some } \{\theta_{dk}\}_{k=1}^{\overline{k}_d}, d = 0,1 \right\}$$

where $\{\theta_{dk}\}_{k=1}^{\overline{k}_d}$ are **unknown coefficients** and $\{b_{dk}\}_{k=1}^{\overline{k}_d}$ are **known basis functions**

- This is effectively a **parameterization** of the Marginal Treatment Response functions

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

- Parameterizing MTR functions as finite linear bases **reduces** the optimization problems to

$$\overline{\beta}^{*}_{\text{fd}} \equiv \sup_{\theta_0,\theta_1\in\Theta} \sum_{k=1}^{\overline{k}_0} \theta_{0k}\Gamma_0^{*}\left(b_{0k}\right) + \sum_{k=1}^{\overline{k}_1} \theta_{1k}\Gamma_1^{*}\left(b_{1k}\right)$$

$$\text{s.t.} \quad \sum_{k=1}^{\overline{k}_0} \theta_{0k}\Gamma_{0s}\left(b_{0k}\right) + \sum_{k=1}^{\overline{k}_1} \theta_{1k}\Gamma_{1s}\left(b_{1k}\right) = \beta_s \qquad \text{for all } s \in \mathcal{S}$$

and analogously for $\underline{\beta}^{*}_{\text{fd}}$

- Recall that the (identified) **linear maps** of the MTR functions are

$$\Gamma_d^{*}\left(m_d\right) = \mathbb{E}\left[\int_0^1 m_d\left(u,X\right)\omega_d^{*}\left(u,Z\right)du\right] \qquad \Gamma_{ds}\left(m_d\right) = \mathbb{E}\left[\int_0^1 m_d\left(u,X\right)\omega_{ds}\left(u,Z\right)du\right]$$

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

Mogstad, Santos, and Torgovitsky (2018) considers two main sets of **finite linear basis**:

1. **Bernstein Polynomials**: the kth Bernstein basis polynomial of degree $\overline{k}$ is

$$b_k^{\overline{k}} : [0,1] \to \mathbb{R} \quad \text{s.t.} \quad b_k^K(u) \equiv \binom{\overline{k}}{k} u^k (1-u)^{\overline{k}-k} \quad \text{for } k = 0, 1, \ldots, \overline{k}$$

2. **Constant Splines** for **exact** computation of **nonparametric bounds**
   - Suppose Z has **discrete support** and $\omega_d^*(u,z)$, $d = 0, 1$, are **piecewise constant in u**
   - Define a **partition** $\{\mathcal{U}_j\}_{j=1}^{\bar{J}}$ of $[0,1]$ such that $\omega_d^*(u,z)$, $\mathbb{I}[u \leq p(z)]$ are constant in each $\mathcal{U}_j$
   - Construct the **basis functions**

$$b_{jl}(u,x) \equiv \mathbb{I}[u \in \mathcal{U}_j, x = x_l] \quad \text{for } 1 \leq j \leq \bar{J} \text{ and } 1 \leq l \leq \bar{l}$$

   whose **linear combinations** form **constant splines** over $[0,1]$ for each x

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)

- For illustration purposes, the **MTR functions** are assumed to be **known**:

$$m_0(u) = 0.6(1-u)^2 + 0.4u(1-u) + 0.3u^2$$
$$m_1(u) = 0.75(1-u)^2 + 0.5u(1-u) + 0.25u^2$$

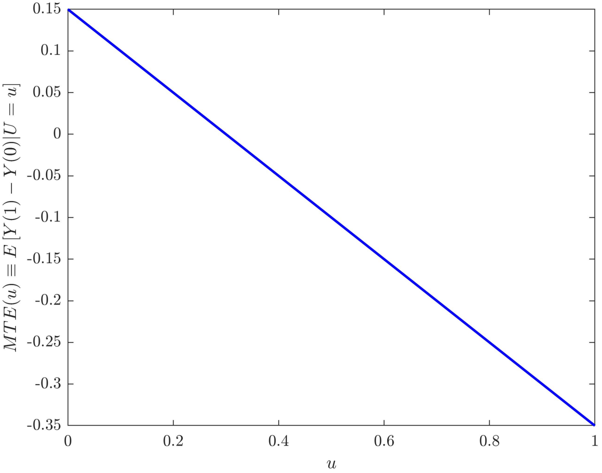
- **Outcome**: $Y \in \{0, 1\}$ is trivially **bounded**

- **Instrument**: $Z \in \{0, 1, 2\}$, with $\mathbb{P}(Z = 0) = 0.5$, $\mathbb{P}(Z = 1) = 0.4$, $\mathbb{P}(Z = 2) = 0.1$
  - Note: some of the paper's figures **incorrectly** refer to $Z \in \{1, 2, 3\}$ rather than $Z \in \{0, 1, 2\}$

- **Propensity scores**: $p(0) = 0.35$, $p(1) = 0.6$, $p(2) = 0.7$

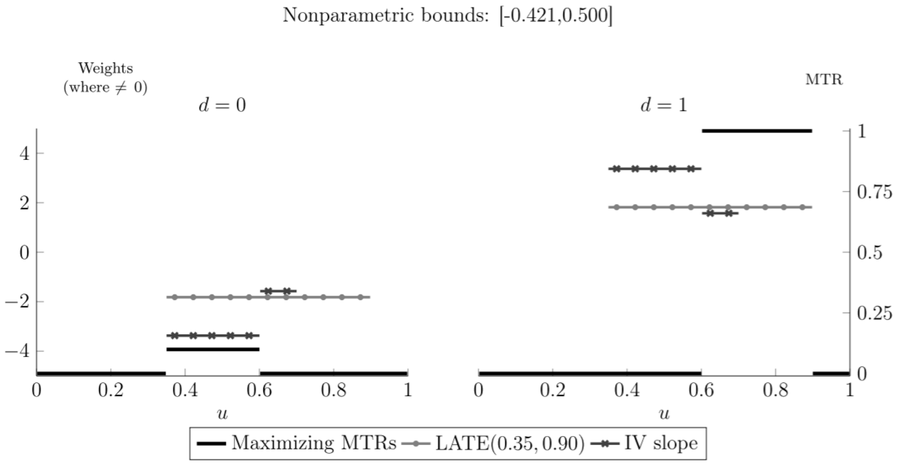
- **Target parameter**: $\mathrm{LATE}(0.35, 0.9) \equiv \mathbb{E}[Y(1) - Y(0) | U \in (0.35, 0.9]]$
  - This target parameter requires **extrapolation** since the complier subpopulation is **expanded**

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



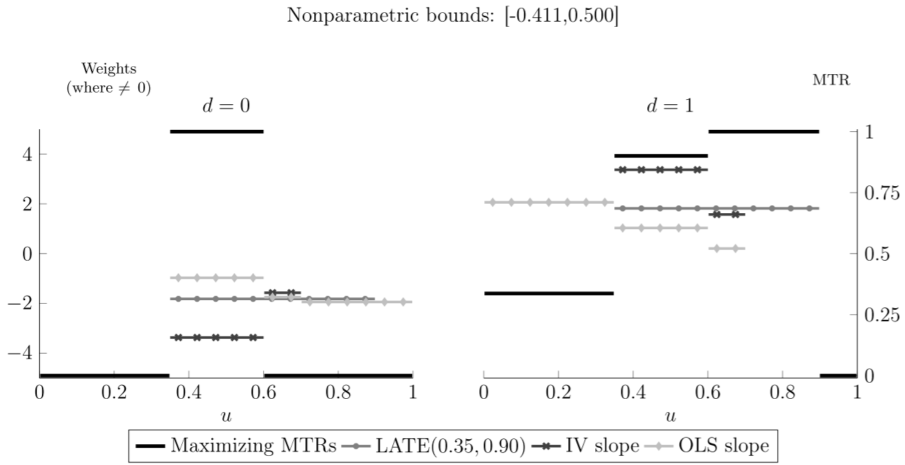
This figure plots the **DGP MTE function** in Mogstad, Santos, and Torgovitsky (2018)

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



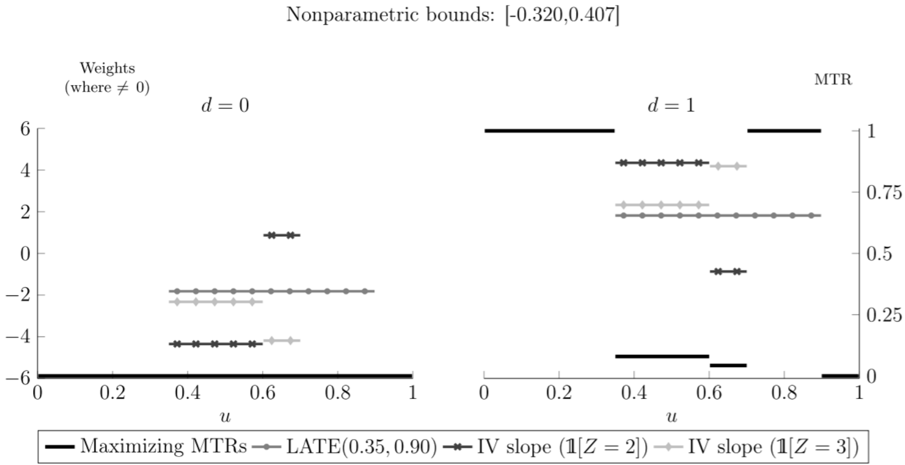
This figure plots maximizing MTRs when using **only** the **IV** slope coefficient

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



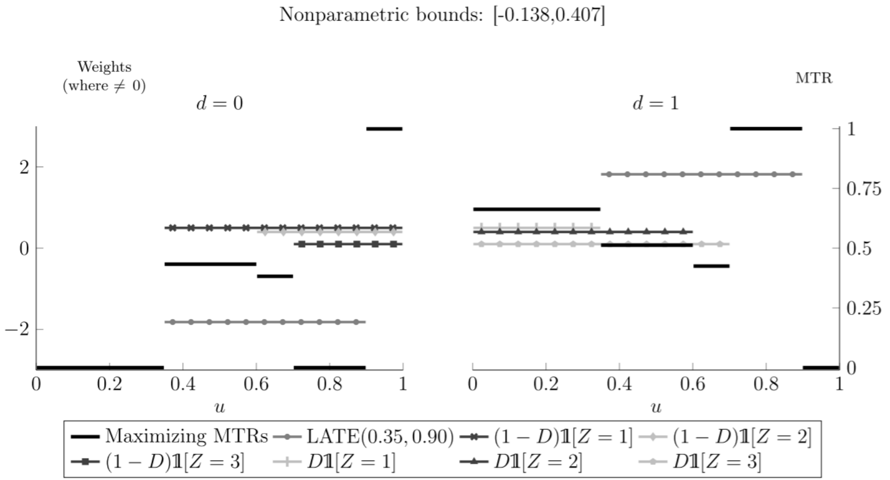
This figure plots maximizing MTRs when using both the **IV and OLS** slope coefficients

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



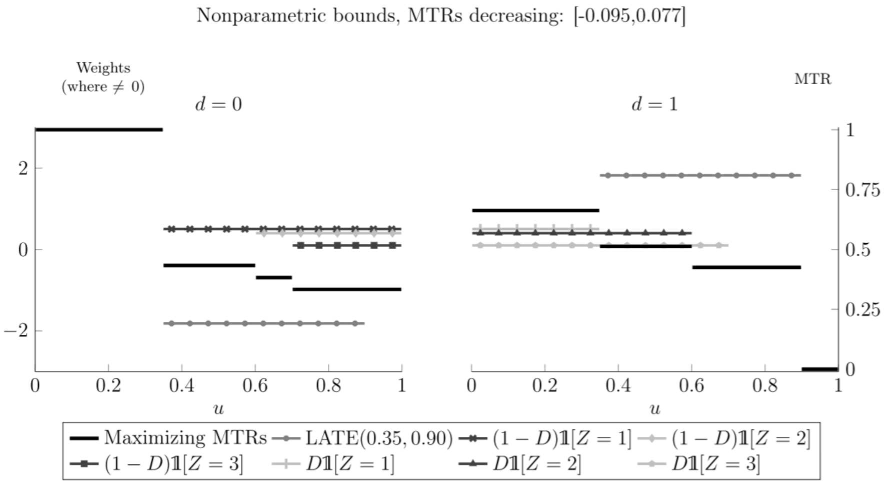
This figure plots maximizing MTRs when breaking the **IV** slope into **two components**

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



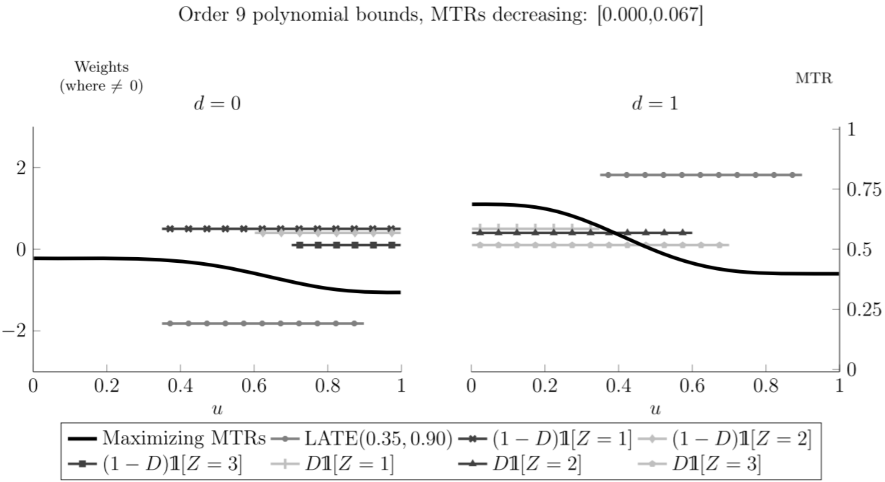
This figure plots maximizing MTRs when using **all IV-like estimands** (sharp bounds)

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



This figure plots maximizing MTRs when **restricted** to be **decreasing**

Partial Identification: Mogstad, Santos, and Torgovitsky (2018)



This figure plots maximizing MTRs when further **restricted** to be a 10th-order **polynomial**

Summary

- **Target parameters** and **common estimands** are weighted averages of MTRs
- Within a MTE framework, **point identification** of target parameters usually entails
  1. Specifying **linear-in-parameters** models of the MTR functions, or
  2. Specifying **partially linear** models of the MTR functions
- Within a MTE framework, **partial identification** of target parameters entails computing **bounds** such that the implied MTR functions are consistent with known estimands