<u>1650 HW #5</u> Francesco Serraino

1. $E(X) = \int_0^\infty x f(x)\,dx$ where $f(y) = f(x)$

$$= \int_0^\infty y f(y)\,dy$$

$$= \int_0^\infty \int_0^y dx\, f(y)\,dy$$

$$= \int_0^\infty \int_0^y f(y)\,dx\,dy$$

$$= \int_0^\infty \underbrace{\left[\int_x^\infty f(y)\,dy\right]}_{= P(X \geq x)} dx$$

$$= \int_0^\infty P(X \geq x)\,dx \quad //$$

2. a) $E(Y) = \int_0^4 \frac{3}{64} y^3 (4-y)\,dy$

$$= \frac{3}{64}\left(y^4 - \frac{y^5}{5}\right)\Big|_0^4$$

$$= \frac{3}{64}\left[256 - 204.8\right]$$

$$= \frac{3}{64}(51.2)$$

$$= 2.4 \;//$$

$$E(Y^2) = \frac{3}{64}\int_0^4 y^4(4-y)\,dy$$

$$= \frac{3}{64}\left[\frac{4y^5}{5} - \frac{y^6}{6}\right]\Bigg|_0^4$$

$$= \frac{32}{5}$$

$$\therefore V(Y) = \frac{32}{5} - (2.4)^2$$

$$= 0.64 \;//$$

b) $E(cY) = c \cdot E(Y) = 200 \cdot 2.4 = 480$ //

$V(cY) = c^2 \cdot V(Y) = (200)^2 \cdot 0.64 = 25600$ //

c) $P(Y > 3) = P(Y \geq 3) = P(3 \geq Y \geq 4)$

where $P(3 \geq Y \geq 4) = \int_3^4 f(y)\,dy$

$$= \frac{3}{64} \int_3^4 y^2 (4-y)\,dy$$

$$= \frac{3}{64} \left[\frac{4y^3}{3} - \frac{y^4}{4} \right) \Bigg|_3^4$$

$$= \frac{3}{64} \left[\frac{4^4}{3} - \frac{4^4}{4} - 4 \cdot 3^2 + \frac{3^4}{4} \right]$$

$$= \frac{3}{64} \cdot \frac{67}{12}$$

$= \frac{67}{256} \simeq 26.2\%$ $\therefore$ about $\frac{1}{4}$ of the time //

3\. a) $P(Z \leq z)$ assuming $Z = \frac{X-\mu}{\sigma}$

$$= p\left(\frac{X-\mu}{\sigma} \leq z\right)$$

$$= p(X \leq \mu + z\sigma)$$

$\because X \sim N(0,1)$

$$= \int_{-\infty}^{\mu+z\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

change of variable $y = \frac{x-\mu}{\sigma}$ $\therefore$ $dy = \frac{dx}{\sigma}$

$$\therefore = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy \rightarrow \text{cdf of } Z \text{, QED}$$

b) $Y = e^X$ $\therefore$ $y = e^x$, $\therefore$ $x = \ln y$

$$F_Y(y) = P(Y \leq y) = p(e^x \leq y) = p(x \leq \ln(y)) = F_X(\ln(y))$$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln(y)) = \frac{1}{y} \frac{d}{dx}\left[F_X(\ln(y))\right]$$

$$= \frac{1}{y} f_X(\ln(y)) = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}$$

4. a) Area of center $= 0$ $\because$ its continuous

$\therefore \quad p = 0$

b) Let r = radius of board

$$\therefore \quad p(E) = \frac{\pi \left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

c) $$P(b \geq Y \geq a) = \frac{\pi b^2}{\pi r^2} - \frac{\pi a^2}{\pi r^2}$$

where $r = 1$

$\therefore p = b^2 - a^2$

5.a) consider X - bernouli random variable

where X = candidate will vote for A

$P(X) = 0.5$

$P(\bar{X}) = 0.5$

$\therefore$ follows binomial distribution

$\therefore \mu = np = 400 \times 0.5 = 200$

$V = np(1-p) = 400 \times (0.5)^2 = 100$

$\therefore \sigma = 10$

by central limit theorem we can assume a normal distribution

$\therefore$ propability at least 52.5% of candidates vote for A

= probability that at least 210 candidates vote for A

$\therefore Z = 1$

using table 4, Appendix 3

$p = 0.1587 = 15.87\%$

b). using binomial again

$p(CDE) = 0.3$

$p(\overline{CDE}) = 0.7$

$\therefore$ for 400 candidates

$\mu = 0.3 \times 400 = 120$

$\sigma = \sqrt{np(1-p)} = \sqrt{(400)(0.3)(0.7)} = \sqrt{84}$

25% of 400 = 100

$\therefore \quad z = \frac{-20}{\sqrt{84}} \simeq -2.18$

$P(Z < -2.182) \equiv P(Z > 2.182)$

using table 4, Appendix 3 for $z = 2.182$

$p \simeq 14.6\%$