Discretization of Wave Equation

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The focus here is devoted to the discretization of the following wave equation:

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2} \tag{1}$$

Starting by looking at the spatial domain $[0, L]$, this one is divided into N intervals, obtaining a spatial interval length $\Delta x = \frac{L}{N}$.
The function Ψ is now Taylor expanded around the point $x + \Delta x$, obtaining:

$$\Psi(x + \Delta x, t) = \Psi(x, t) + \Delta x \frac{\partial \Psi}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Psi}{\partial x^2} + O(\Delta x^3)$$

By defining Ψ_i=$\Psi(x_i, t)$ where $x_i = i\Delta x$, we can then rewrite Equation 2 for clarity throughout the process of discretization:

$$\Psi_{i+1} = \Psi_i + \Delta x \frac{\partial \Psi}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Psi}{\partial x^2} + O(\Delta x^3) \tag{2}$$

By doing the same in the opposite direction ($-\Delta x$, also referred to as the backward scheme, where the previous one was the forward one), the following can be obtained:

$$\Psi_{i-1} = \Psi_i - \Delta x \frac{\partial \Psi}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Psi}{\partial x^2} + O(\Delta x^3) \tag{3}$$

where Ψ_{i-1}corresponds to $\Psi(x - \Delta x, t)$ according to the notation explained above.
Summing Equations 2 and 3 leads to:

$$\Psi_{i+1} + \Psi_{i-1} = \Psi_i + \Psi_i + \Delta x \frac{\partial \Psi}{\partial x} - \Delta x \frac{\partial \Psi}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Psi}{\partial x^2} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Psi}{\partial x^2} + O(\Delta x^4)$$

$$\Psi_{i+1} + \Psi_{i-1} = 2\Psi_i + \Delta x^2 \frac{\partial^2 \Psi}{\partial x^2} + O(\Delta x^4) \tag{4}$$

Here, the last term is $O(\Delta x^4)$ and not $O(\Delta x^3)$ because the Δx^3 terms (and any other term with x raised to the power of any odd-exponent) cancel out when the forward and backward approximations are summed together (as in Equation 4. Rearranging the terms in Equation 4 brings to:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

or

$$\frac{\partial^2 \Psi}{\partial x^2} \approx \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta x^2} \tag{5}$$

The above result can be referred to as the central difference approximation for $\frac{\partial^2 \Psi}{\partial x^2}$ (second order accurate).

The same discretization process is also applied at the time level. Considering that the time domain is divided into intervals of size Δt, $\Psi(x_i, t_n)$ is referred to as Ψ_i^n where $t_n = n\Delta t$. Similarly to the steps above, it is first taken the forward approximation of Ψ_i^{n+1} (as it was done for the spatial component in Equation 2), then the backward one Ψ_i^{n-1} (as it was done for the spatial component in Equation 3), subsequently added together and simplified (as it was done for the spatial component in Equation 4) to obtain:

$$\frac{\partial^2 \Psi}{\partial t^2} \approx \frac{\Psi_i^{n+1} - 2\Psi_i^n + \Psi_i^{n-1}}{\Delta t^2} \tag{6}$$

comparable to Equation 5, just applied to the time domain instead of the space one, in other words the the central difference approximation for $\frac{\partial^2 \Psi}{\partial t^2}$ (again, second order accurate).

As a final step, Equations 5 and 6 are substituted into Equation 1:

$$\frac{\Psi_i^{n+1} - 2\Psi_i^n + \Psi_i^{n-1}}{\Delta t^2} = c^2 \frac{\Psi_{i+1}^n - 2\Psi_i^n + \Psi_{i-1}^n}{\Delta x^2} \tag{7}$$

Rearranging Equation 7, the following is obtained:

$$\Psi_i^{n+1} = 2\Psi_i^n - \Psi_i^{n-1} + c^2 \frac{\Delta t^2}{\Delta x^2}(\Psi_{i+1}^n - 2\Psi_i^n + \Psi_{i-1}^n) \tag{8}$$

With Equation 8, it is now possible to compute the wave's amplitude at the next time step (t_{n+1}) for any space interval.