Image Reconstruction via EP with auxiliary informative variables

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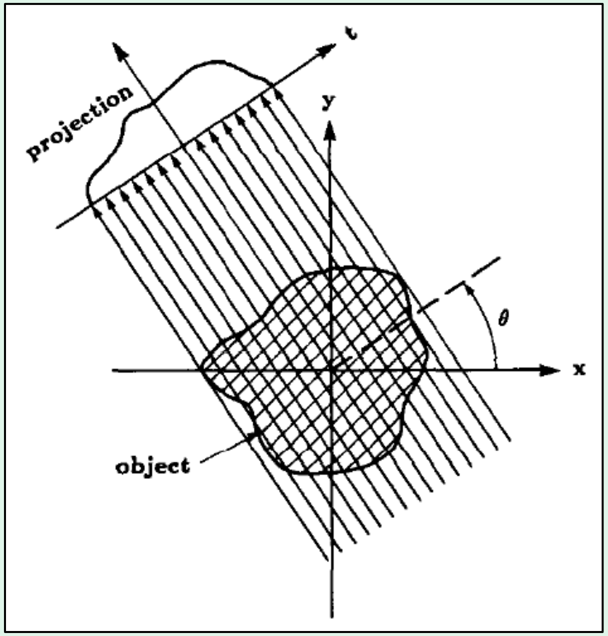
Tomography

Tomography is a medical imaging procedure used to reconstruct the cross section of an object from a set of measurements at different angles.

Source: A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging*. IEEE Press, 1988.

Measurement process



The interaction between the radiation and the material of which the object is composed produces projection data.

Mathematical tools

Linear Estimation Problem

$$\boldsymbol{Ax} = \boldsymbol{p}$$

Limited data regime

$$M < N$$

Total Variation

$$\boldsymbol{x}^* = argmin_{\boldsymbol{x}} \left|\left|\boldsymbol{Ax} - \boldsymbol{p}\right|\right|_2 + \lambda \left|\left|\nabla_{img}\boldsymbol{x}\right|\right|_1$$

Bayesian Inference

Bayes' theorem

$$\mathcal{P}(\boldsymbol{x}|\boldsymbol{p}) = \frac{\mathcal{P}(\boldsymbol{p}|\boldsymbol{x})\mathcal{P}_0(\boldsymbol{x})}{\mathcal{P}(\boldsymbol{p})}$$

Likelihood function

$$\mathcal{P}(\boldsymbol{p}|\boldsymbol{x}) \propto e^{-\frac{\beta}{2}(\boldsymbol{Ax}-\boldsymbol{p})^2}$$

Prior term

$$\mathcal{P}_0(\boldsymbol{x}) = \prod_j \psi_j(x_j)$$

Standard Optimization Problems?

Expectation Propagation

The drawback of Bayesian Inference with non-convex prior is the computational cost, but there are algorithms that make these non-convex optimization problems computationally feasible through approximations.

«True» posterior distribution

$$\mathcal{P}(\boldsymbol{x}|\boldsymbol{p}) = \frac{1}{Z}\underbrace{e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}}_{\text{Multivariate Gaussian}}\prod_j \psi_j(x_j)$$

Expectation Propagation

***EP** is an iterative algorithm that approximates intractable posterior probability distributions.*

$$Q(\boldsymbol{x}|\boldsymbol{p}) \propto \underbrace{e^{-\frac{1}{2}(\boldsymbol{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}}_{\text{Multivariate Gaussian}} \prod_j \phi_j(x_j)$$

$$Q^{(i)}(\boldsymbol{x}|\boldsymbol{p}) \propto \underbrace{e^{-\frac{1}{2}(\boldsymbol{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}}_{\text{Multivariate Gaussian}} \psi_i(x_i) \prod_{j \neq i} \phi_j(x_j)$$

Gaussian approximated posterior	Tilted distribution

$$\phi_j(x_j) = \frac{1}{\sqrt{2\pi b_j}} e^{-\frac{(x_j - a_j)^2}{2b_j}}$$

Expectation Propagation

$$a_i, b_i = argmin_{(a_i,b_i)} D_{KL}\left[Q^{(i)}(\boldsymbol{x}|\boldsymbol{p}) || \mathcal{Q}(\boldsymbol{x}|\boldsymbol{p})\right]$$

Minimizing the KL distance

$$< x_i >_{Q^{(i)}(\mathbf{x})} = < x_i >_{\mathcal{Q}(\mathbf{x})}$$
$$< x_i^2 >_{Q^{(i)}(\mathbf{x})} = < x_i^2 >_{\mathcal{Q}(\mathbf{x})}$$

Moment matching condition

Expectation Propagation

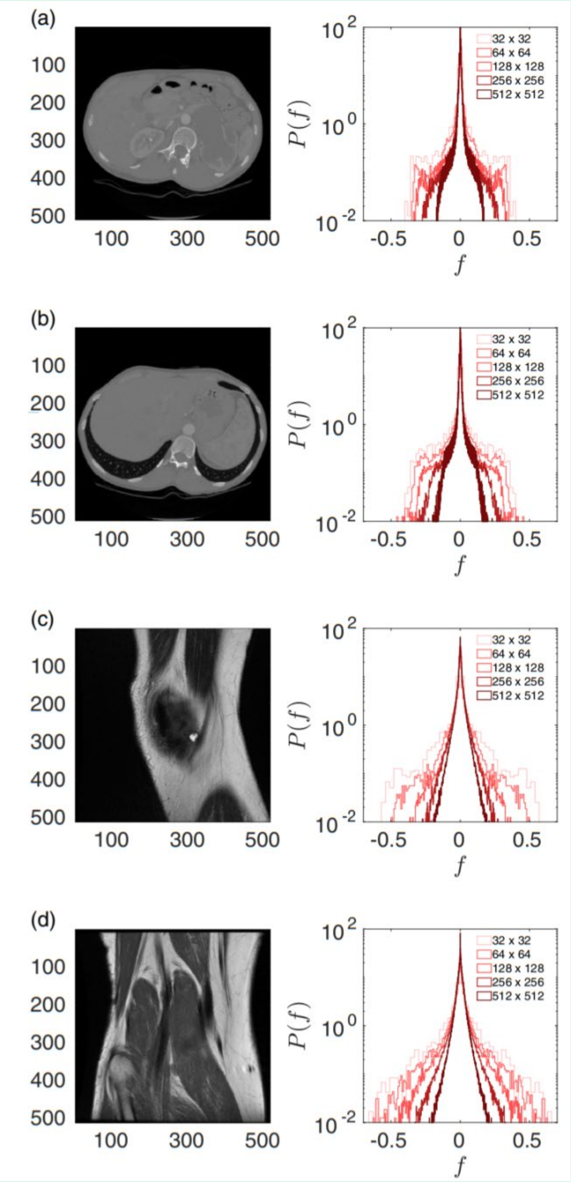
In EP we can also include some auxiliary variables y_i, obtained through some linear transformation $\boldsymbol{y} = \boldsymbol{F}\boldsymbol{x}$ of the pixel intensity variables.

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{p}) \propto e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}\ \delta(\boldsymbol{y} - \boldsymbol{F}\boldsymbol{x}) \prod_j \psi_j(x_j) \prod_i \psi_i(y_i)$$

EP with difference variables

Consider the information obtained through the empirical distributions of the pixel ***difference variables****.*

- *Accuracy*
- *Few measurements*
- *Validity empirical priors in EP*



Source: A. P. Muntoni, R. D. H. Rojas, A. Braunstein, A. Pagnani, and I. P. Castillo, *"Non-convex image reconstruction via expectation propagation",* Physical Review E, vol. 100, no. 3, Sep 2019.

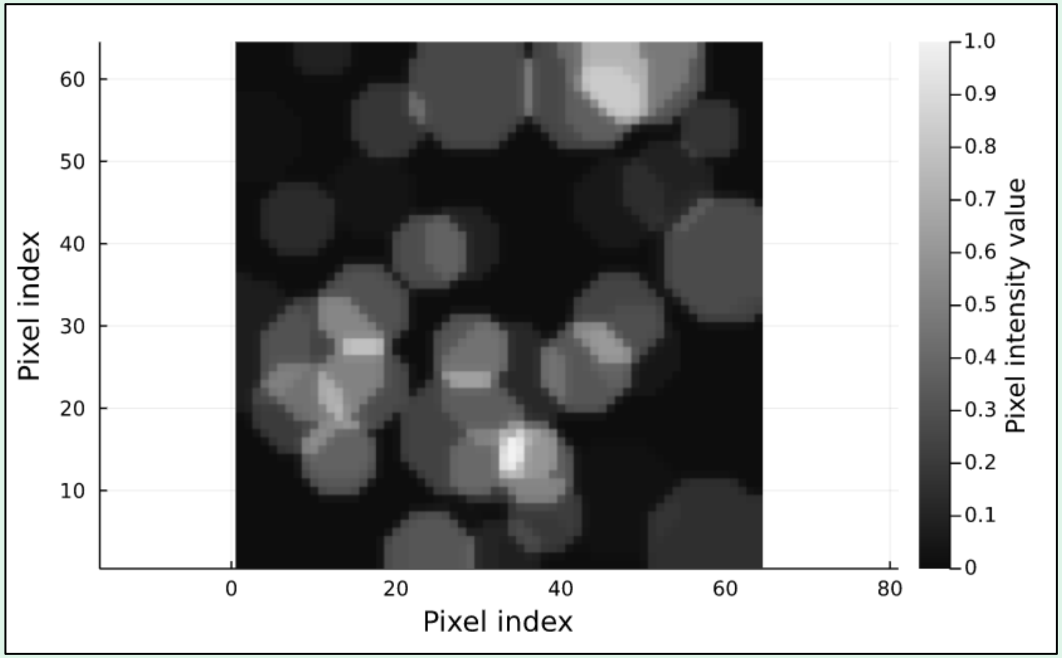
Training dataset composed by 20.000 images, reproducing the typical features of tomographic images.

Training dataset

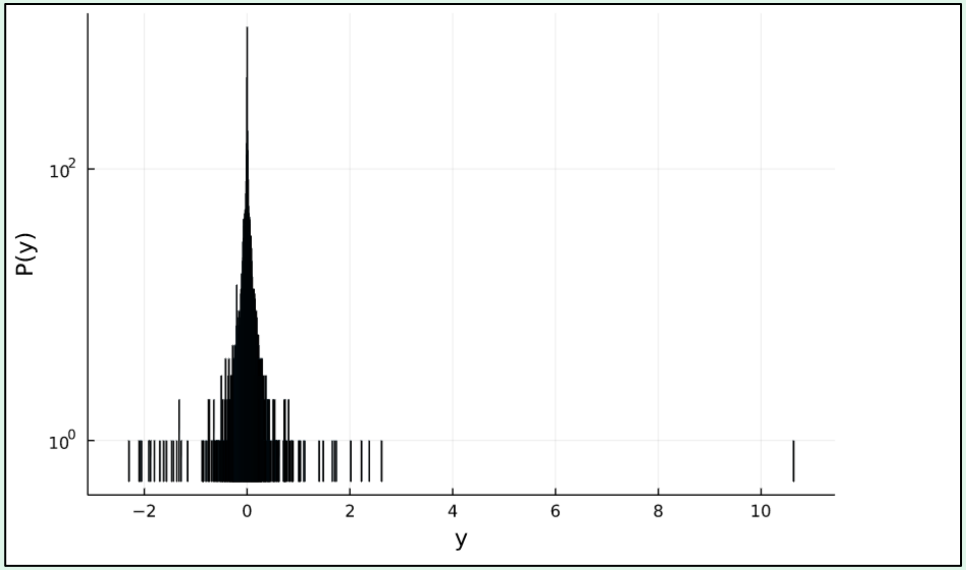
The Haar transform

Curve fitting

Priors assignment



*Applying the linear **Haar transform** to each image of the dataset, we analyze the statistics of the auxiliary variables.*



Training dataset

The Haar transform

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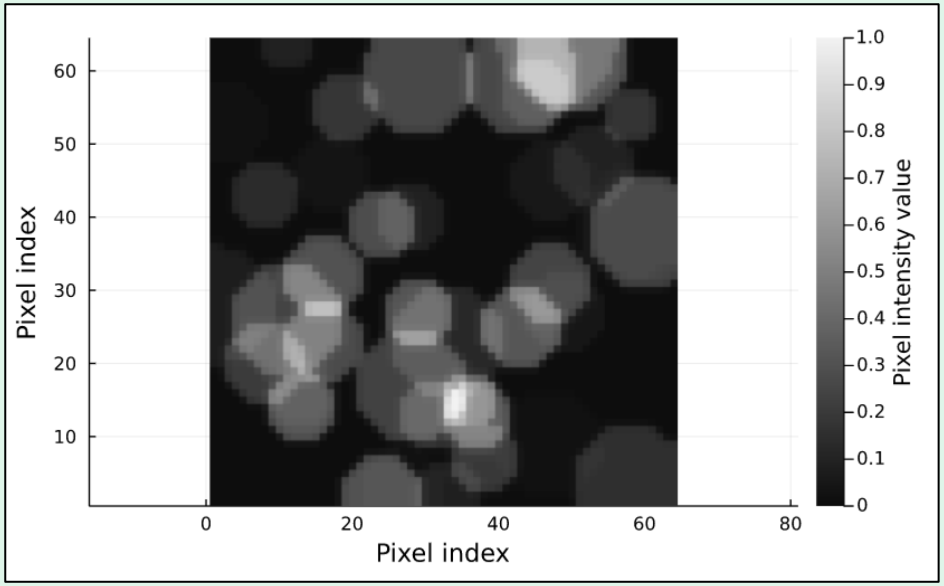
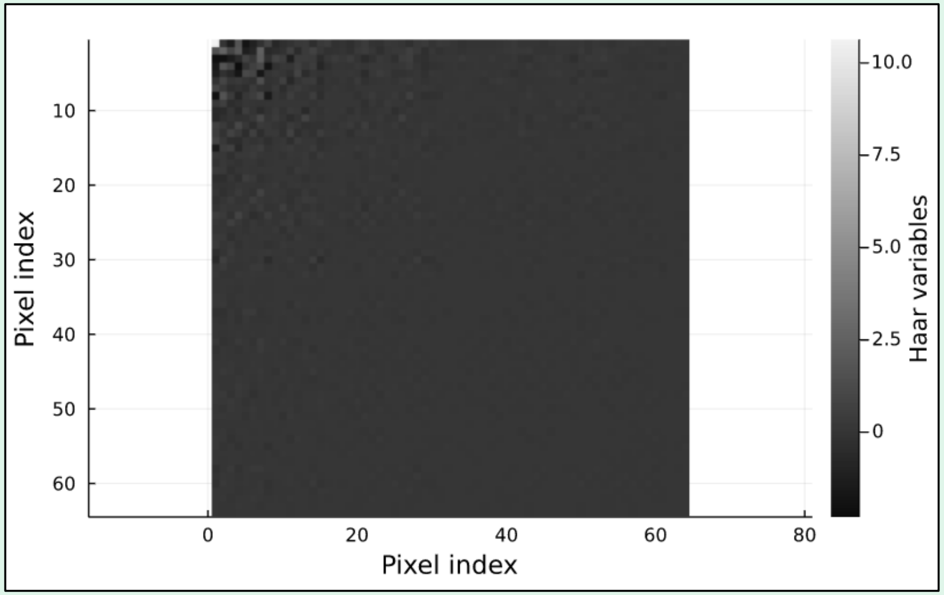
Priors assignment





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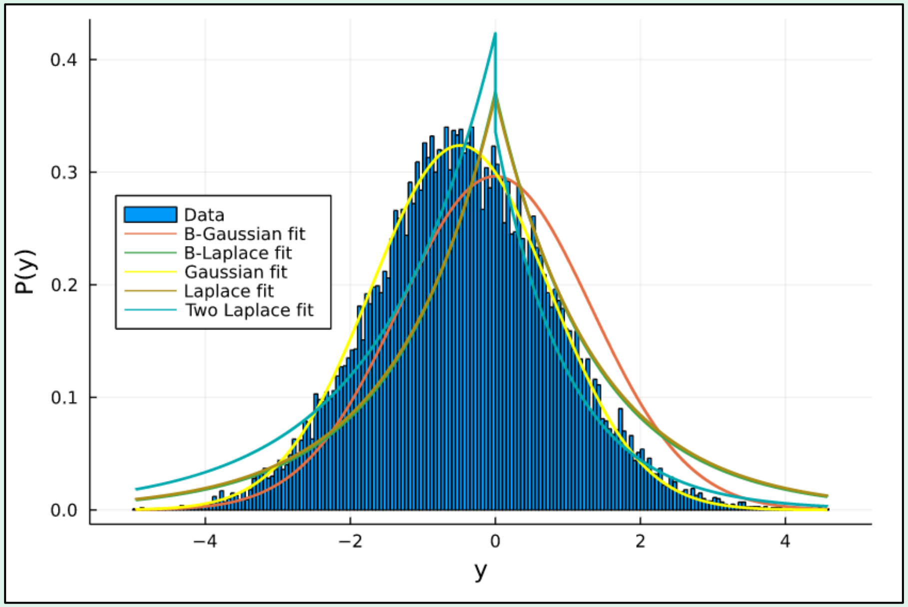
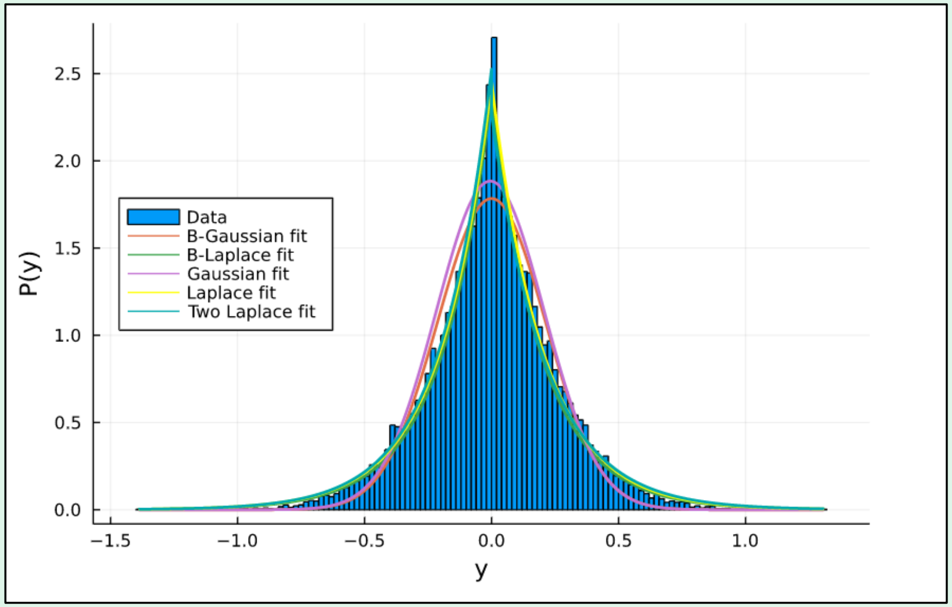




Training dataset

Unimodal distributions

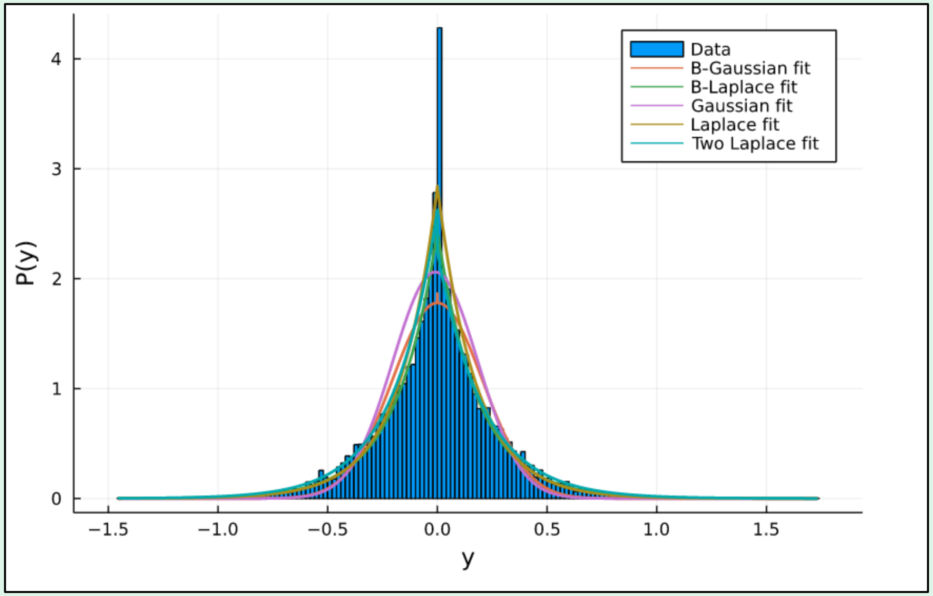
$$\mathcal{P}^{GA}(y) = \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

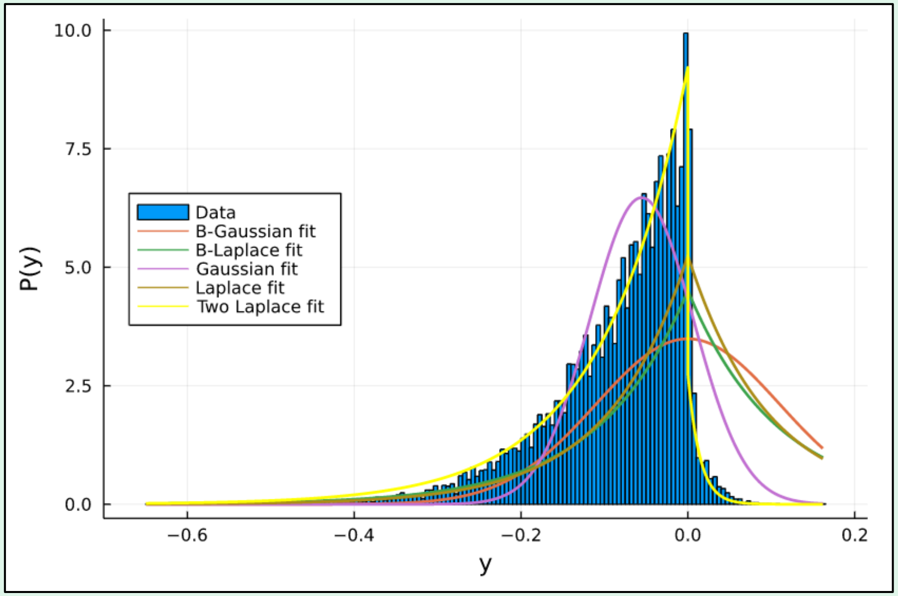
$$\mathcal{P}^{\mathcal{L}A}(y) = \frac{1}{2}\lambda e^{-\lambda|y|}$$

The Haar transform

Curve fitting

Priors assignment





$$\mathcal{P}^{BG}(y) \propto \rho\delta(y) + (1-\rho)\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

$$\mathcal{P}^{\mathcal{BL}}(y) \propto \rho\delta(y) + (1-\rho)\frac{1}{2}\lambda e^{-\lambda|y|}$$

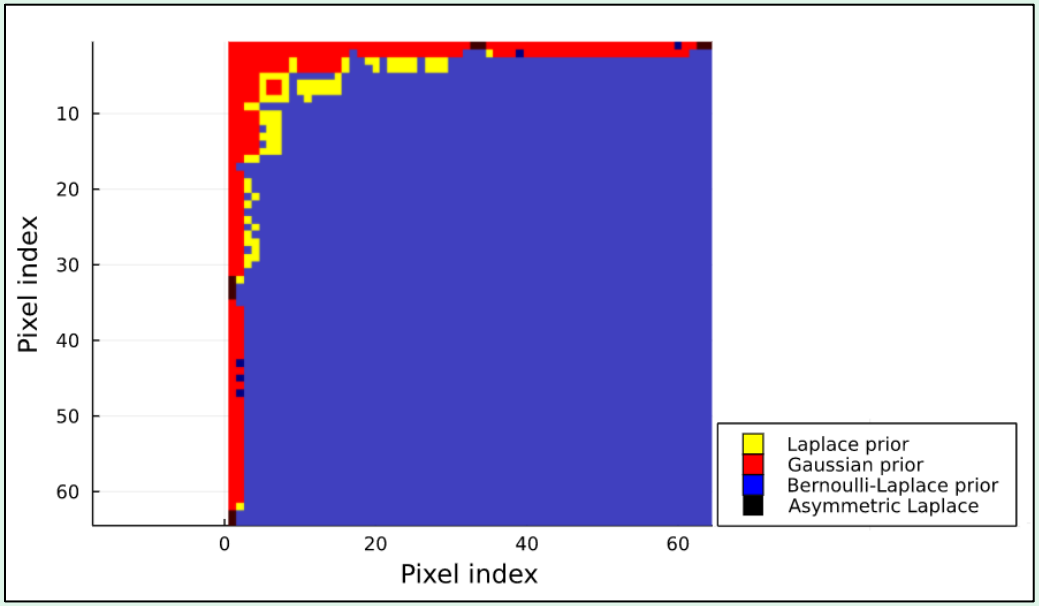
$$\mathcal{P}^{AS}(y) = \mathbb{I}[y>0]\varrho e^{-\lambda_1|y|} + \mathbb{I}[y<0](1-\varrho)e^{-\lambda_2|y|}$$

Training dataset

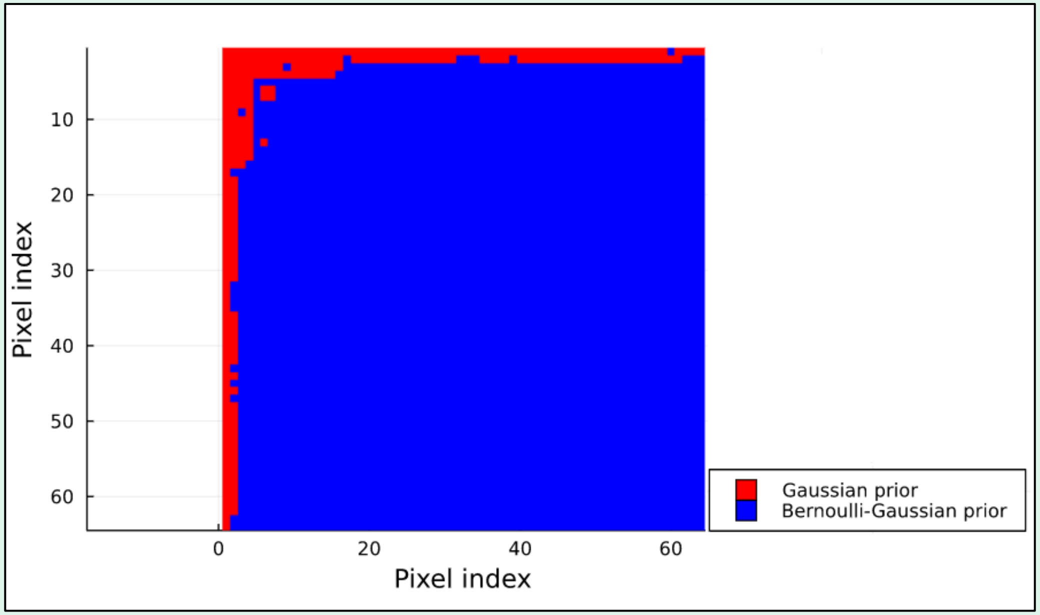
The Haar transform

Curve fitting

Priors assignment



Replacement of the Laplace priors due to the numerical instability of EP algorithm.



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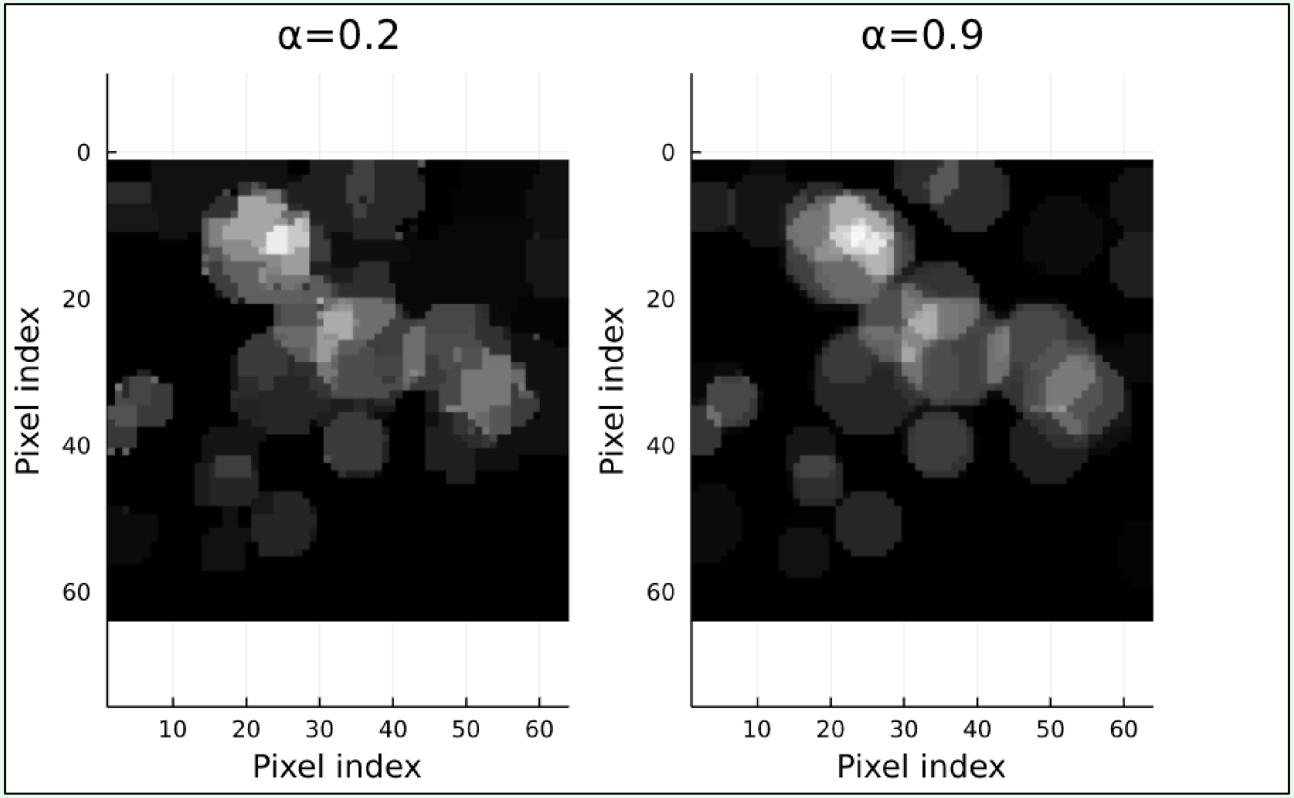
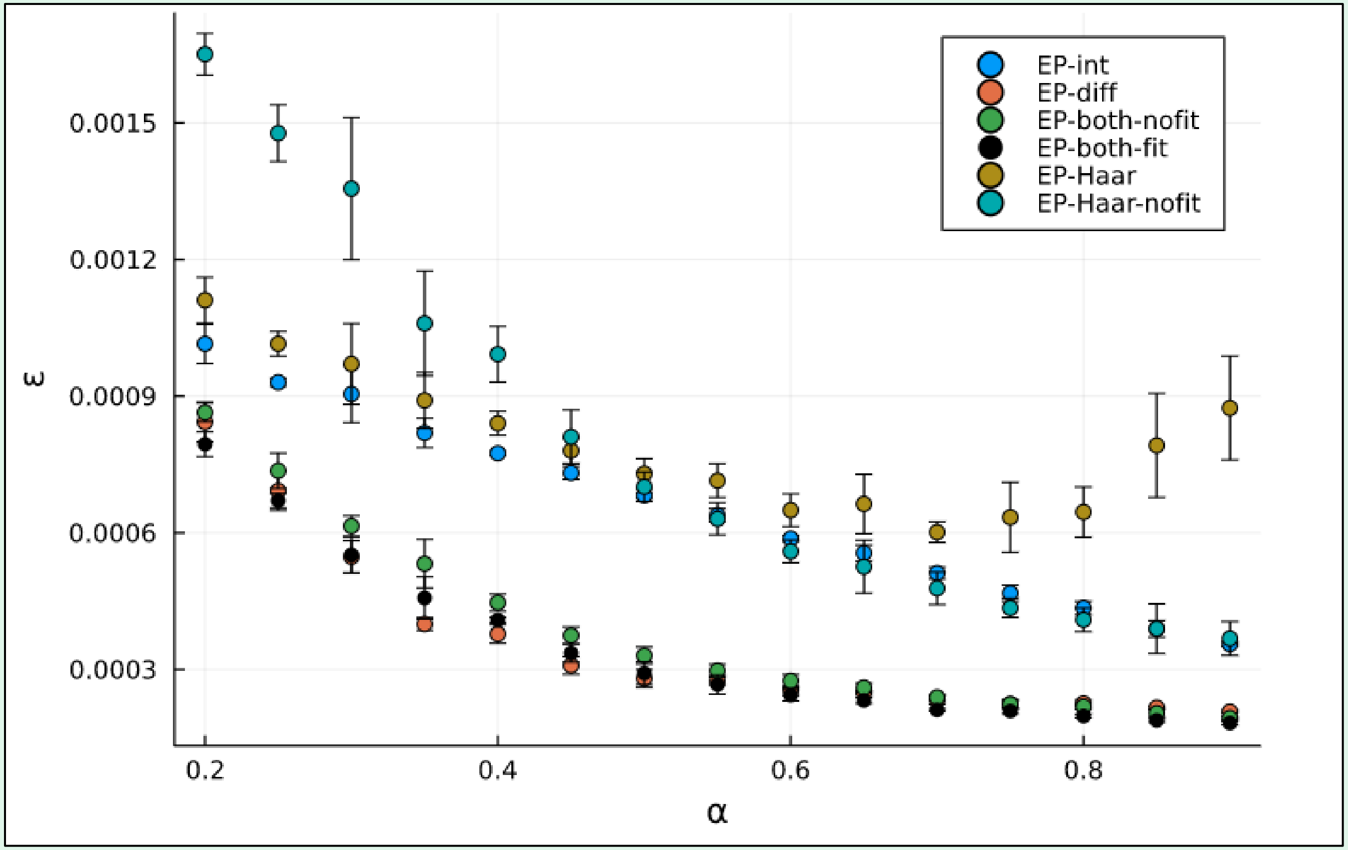


Sampling rate

$$\alpha = \frac{M}{N}$$

Different projection matrix, trying to recontruct the same image.

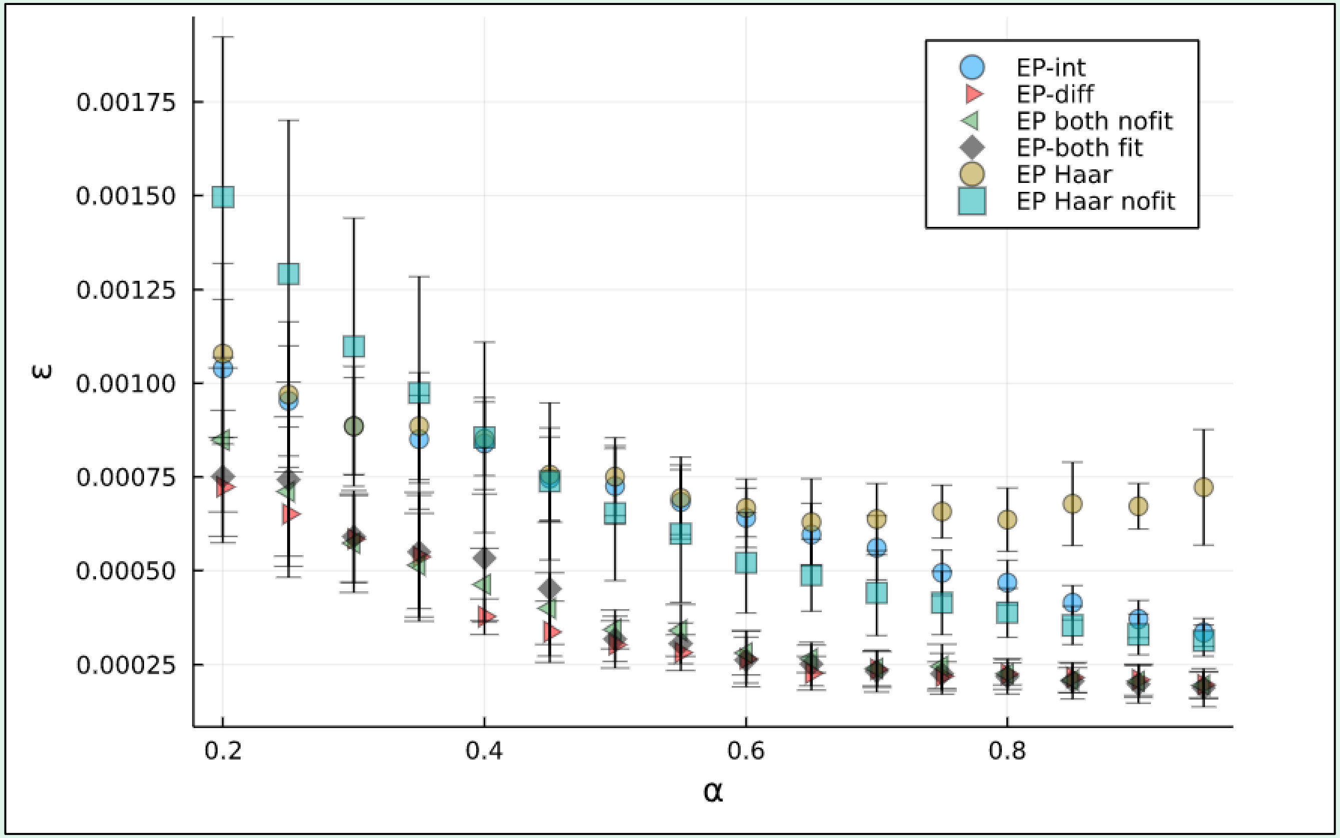
$$\varepsilon = \frac{||\boldsymbol{x}^{*} - \boldsymbol{x}||_{2}}{N}$$



$$\alpha = \frac{M}{N}$$

Same projection matrix, aiming to reconstruct different images.

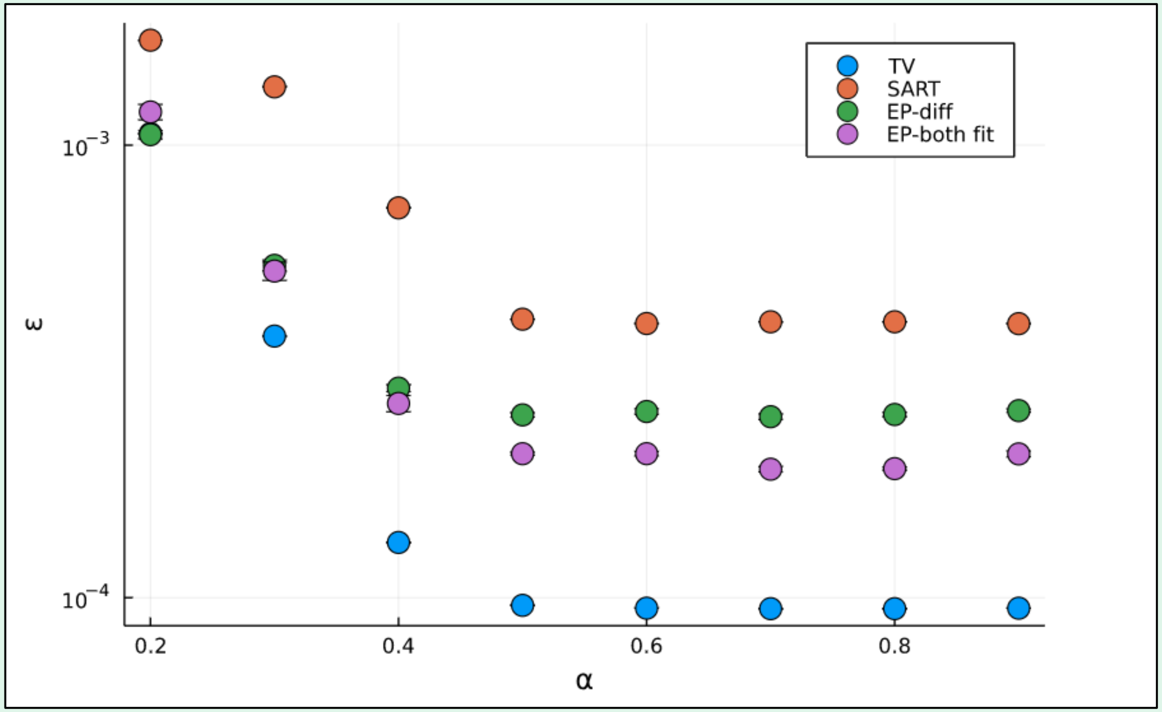
$$\varepsilon = \frac{\left|\left|\boldsymbol{x}^* - \boldsymbol{x}\right|\right|_2}{N}$$



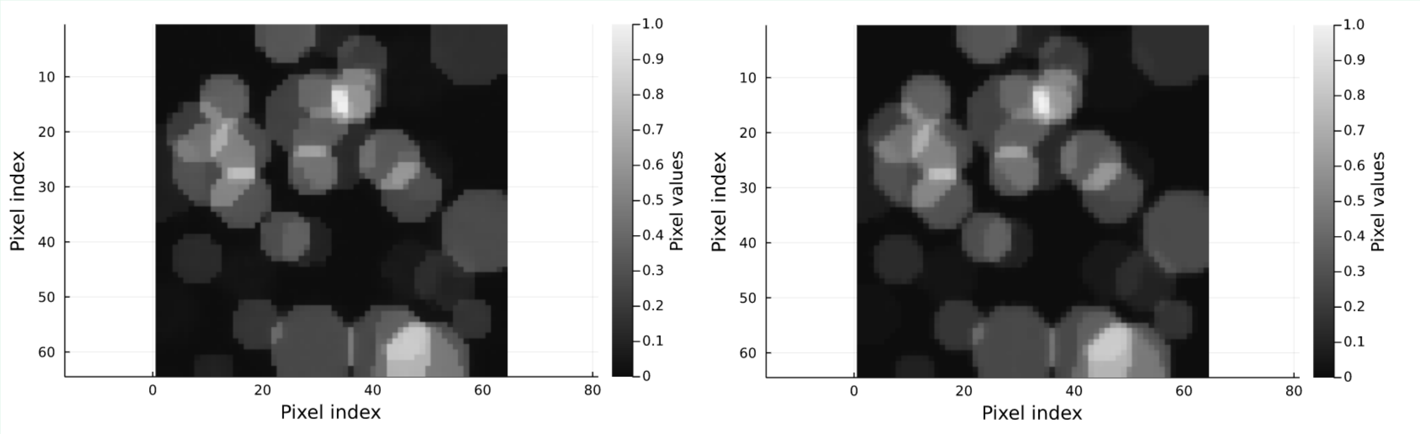
$$\alpha = \frac{M}{N}$$

01.Methods 02.Empirical priors **03.Results**

$$\varepsilon = \frac{||\boldsymbol{x}^* - \boldsymbol{x}||_2}{N}$$



$$\alpha = \frac{M}{N}$$



Conclusions

- *The confirm of introducing the auxiliary difference variables in the EP algorithm*
- *A slightly improvement with respect to EP-diff in some measurement regimes*
- *The possibility to implement the Laplace priors*
- *Same approach with other auxiliary variables connected through a linear transform to the pixel intensity variables*

Thank you for your attention