Statistical Physics 2: Disordered Systems and Interdisciplinary Applications

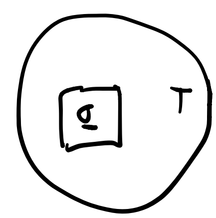
19.03.2021

Dynamics of disordered systems

I. Definitions and properties

System in contact with a heat bath

$H(\underline{\sigma})$ Hamiltonian



equilibrium $\quad p_{eq}(\underline{\sigma}) = \dfrac{e^{-\beta H(\underline{\sigma})}}{Z}$ $\qquad \beta = 1/T$

1. Dynamics

$\underline{\sigma}(t)$ conf. at time t

a) $\underline{\sigma} \in \mathbb{R}^N$ continuous

- Langevin eq. $\quad \dfrac{d\sigma_i}{dt} = -\dfrac{\partial H}{\partial \sigma_i} + \xi_i(t)$ $\quad$ Gaussian $\quad \langle \xi_i(t) \rangle = 0$

$$\langle \xi_i(t)\, \xi_j(t') \rangle = 2T\, \delta_{ij}\, \delta(t-t')$$

leads to a Fokker-Planck eq.

$$\frac{d P(\underline{\sigma},t)}{dt} = -\sum_i \frac{\partial}{\partial \sigma_i}\left(-\frac{\partial H}{\partial \sigma_i} P(\underline{\sigma},t) - T \frac{\partial P}{\partial \sigma_i} \right)$$

- Newton eq. $\quad m \dfrac{d^2\sigma_i}{dt^2} = -\dfrac{\partial H}{\partial \sigma_i} \longrightarrow$ microcanonical

b) $\underline{\sigma}$ discrete, e.g. $\underline{\sigma} \in \{-1, 1\}^N$: jump or spin flip dynamics

$$\underline{\sigma}(t) = \underline{\sigma} \qquad \underline{\sigma}(t+dt) = \begin{cases} \underline{\sigma}' \neq \underline{\sigma} & \text{with prob. } W(\underline{\sigma} \to \underline{\sigma}')\,dt \\ \underline{\sigma} & \text{with prob. } 1 - \sum_{\underline{\sigma}'(\neq \underline{\sigma})} W(\underline{\sigma} \to \underline{\sigma}')\,dt \end{cases}$$

leads to a master equation

$$\frac{dP(\underline{\sigma},t)}{dt} = \sum_{\underline{\sigma}'} W(\underline{\sigma}' \to \underline{\sigma})\, P(\underline{\sigma}',t)$$

$$W(\underline{\sigma} \to \underline{\sigma}) = -\sum_{\underline{\sigma}'(\neq \underline{\sigma})} W(\underline{\sigma} \to \underline{\sigma}')$$

Simplest choice : detailed balance

$$P_{eq}(\underline{\sigma})\, W(\underline{\sigma} \to \underline{\sigma}') = P_{eq}(\underline{\sigma}')\, W(\underline{\sigma}' \to \underline{\sigma})$$

With these choices

- conserve probability $\frac{d}{dt}\int d\underline{\sigma}\; P(\underline{\sigma},t) = 0$
- admit $P_{eq}(\underline{\sigma})$ as a stationary state

$$\frac{dP}{dt} = -\mathcal{L} P \quad \text{and} \quad \mathcal{L} P_{eq} = 0$$

Under a few assumptions , <u>for finite N</u> :

$\mathcal{L}$ has a discrete spectrum and $P_{eq}(\underline{\sigma})$ is the unique stationary state

$$P(\underline{\sigma},t) = e^{-\mathcal{L} t}\, P_{in}(\underline{\sigma}) = \sum_{\alpha} e^{-\lambda_\alpha t}\, P_\alpha(\underline{\sigma})\, \langle Q_\alpha | P_{in} \rangle$$

$$\begin{cases} \lambda_0 = 0 \\ P_0 = P_{eq} \\ Q_0 = 1 \end{cases} \qquad \lambda_\alpha > 0 \,, \forall \alpha > 0$$

$$P(\underline{\sigma},t) \xrightarrow[t\to\infty]{} P_{eq}(\underline{\sigma})$$

$$\tau_{relax} = 1/\lambda_1$$

2. Observables $\quad A(\underline{\sigma})$

average: $\langle A(t)\rangle = \langle A(\underline{\sigma}(t))\rangle = \int d\underline{\sigma}\, P(\underline{\sigma},t)\, A(\underline{\sigma})$

Correlation: $C_{AB}(t_w+t, t_w) = \langle A(t_w+t)\, B(t_w)\rangle$

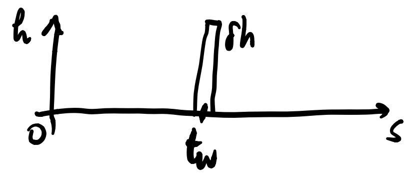
↳ average over
- initial state
- noise in the dyn

response:

$$H(\underline{\sigma}) \to H(\underline{\sigma}) - h(t)\, B(\underline{\sigma})$$

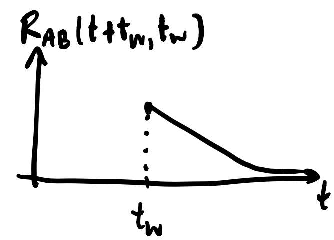
$$\langle A(t)\rangle_h = \langle A(t)\rangle_0 + \int_0^t ds\, R_{AB}(t,s)\, h(s)$$

If $\quad h(s) = \delta h \;\; \delta(s-t_w)$



$$\delta\langle A(t+t_w)\rangle = \delta h \cdot R_{AB}(t+t_w, t_w)$$

$$\frac{\delta\langle A(t+t_w)\rangle}{\delta h} = R_{AB}(t+t_w, t_w)$$



3. Special properties of equilibrium dynamics

Start at $t=0$ in eq: $\quad p_{in}(\underline{\sigma}) = p_{eq}(\underline{\sigma}) \Rightarrow p(\underline{\sigma},t) = p_{eq}(\underline{\sigma})$

- Time translation invariance (TTI)

$$\langle A(t)\rangle = \langle A\rangle_{eq} \qquad C_{AB}(t+t_w, t_w) = C_{AB}(t)$$

$$R_{AB}(t+t_w, t_w) = R_{AB}(t)$$

- Onsager reciprocity

TTI: $C_{AB}(t) = \langle A(t)B(0)\rangle = \langle A(0)B(-t)\rangle = C_{BA}(-t)$

$+$ Time reversal: $C_{AB}(t) = C_{AB}(-t) \Leftrightarrow C_{AB}(t) = C_{BA}(t)$

- Fluctuation-dissipation theorem (FDT):

$$R_{AB}(t) = -\frac{1}{T}\frac{d}{dt}C_{AB}(t)$$

- Decorrelation (at finite N)

$$\langle A(t)B(0)\rangle_{eq} - \langle A\rangle_{eq}\langle B\rangle_{eq} \sim e^{-t/\tau_{relax}}$$

$$R_{AB}(t) \propto \frac{d}{dt}\langle A(t)B(0)\rangle_{eq} \sim e^{-t/\tau_{relax}}$$

II. Equilibrium dynamics in the thermodynamic limit

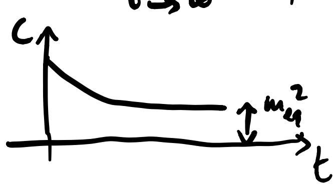
1. Ferromagnetic system

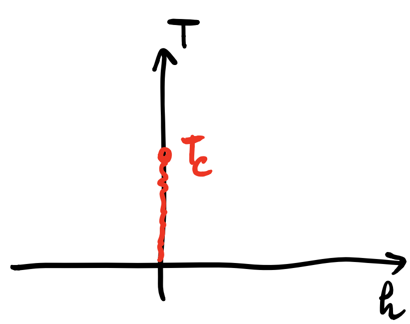
$$m(t) = \frac{1}{N}\sum_i \sigma_i(t)$$

$$C(t) = \langle m(t)m(0)\rangle$$

- $T > T_c$ or $h \neq 0$ one single equilibrium

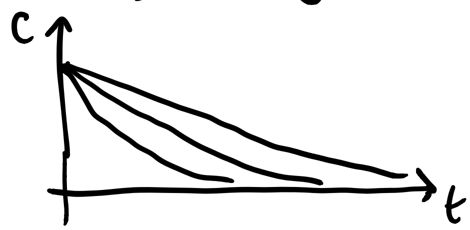
$$C(t) \xrightarrow[t\to\infty]{} m_{eq}^2$$



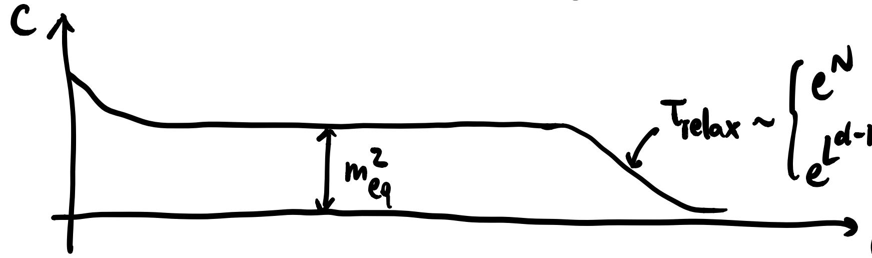


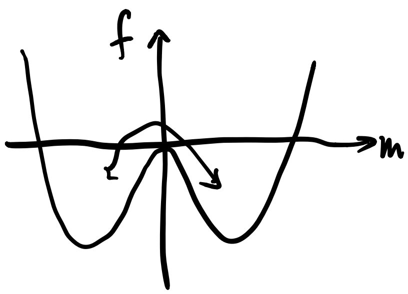
- $T \to T_c^+$ $h=0$ critical slowing down

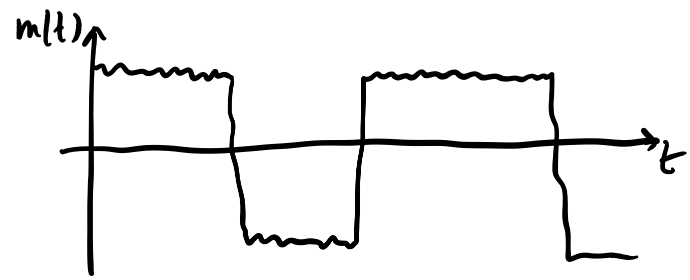
$C(t) \sim t^{-a} e^{-t/\tau}$ $\tau \sim \xi^z \sim |T-T_c|^{-\nu z}$



- $T<T_c$, $h \to 0^+$ nothing special happens to $C(t)$ at finite t

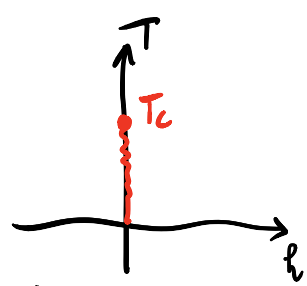






2. Disordered ferromagnet

Ex. edge diluted Ising model
Random field Ising model



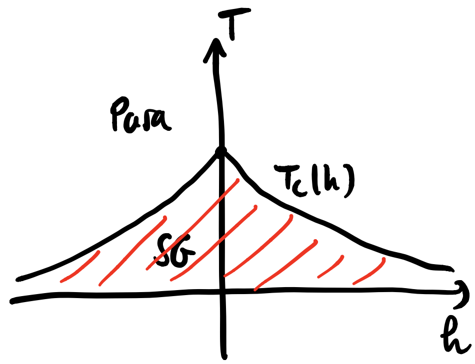
For example:

- $C(t) \sim t^{-a} e^{-t/\tau} \longrightarrow e^{-(\log t)^a}$ Griffiths phases of edge-diluted

- RFIM $\tau \sim e^{\frac{1}{|T-T_c|^{\nu\theta}}} \sim e^{\xi^\theta}$ $T \to T_c^+$, $h=0$

3. Sherrington-Kirkpatrick model (mean field)

$$C(t) = \frac{1}{N}\sum_i \langle \sigma_i(t)\,\sigma_i(0)\rangle$$



- Para: $C(t) \xrightarrow[t\to\infty]{} \frac{1}{N}\sum_i m_i^2 = q_{EA}$ exp.

- $T \to T_c^+$ critical slowing down

$$C(t) - q_{EA} \sim t^{-a}\, e^{-t/\tau}$$

$$\tau \sim |T - T_c|^{-\gamma}$$

Non-trivial exponents a, γ

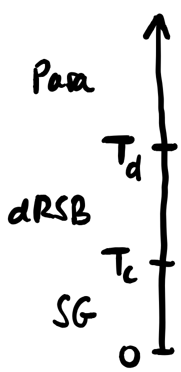
- SG phase

$$C(t) - q_{EA} \sim t^{-a}$$ marginal stability

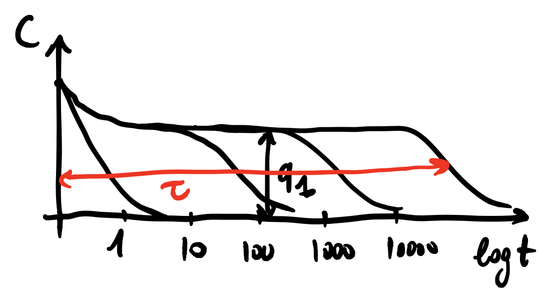
<u>But</u>: impossible to equilibrate in the SG phase

4. p-spin model $p \geq 3$ (mean field)

$$C(t) = \frac{1}{N}\sum_i \langle \sigma_i(t)\,\sigma_i(0)\rangle$$



- $T > T_d$



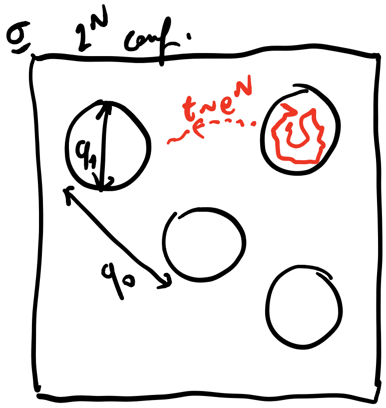
$C(t) - q_1 \propto t^{-b}$ $\quad t \ll \tau$

$C(t) - q_1 \propto -t^{a}$ $\quad t \sim$ plateau

$C(t) \sim \exp(-t/\tau)$

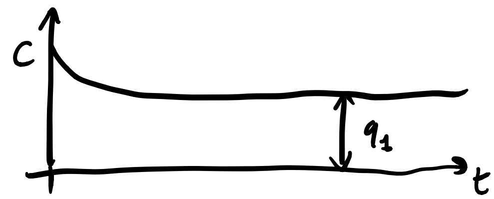
$\tau \sim |T - T_d|^{-\gamma}$

a, b, γ non trivial critical exp



e^N SG states that trap the dynamics for $t \sim e^N$

• $T < T_d$



but once again it's impossible to equilibrate!

5. Finite dimensional glass / spin glass models

Not understood, ongoing research.

III. Out of equilibrium dynamics

Simplest case: gradient descent from infinite T

1. Prepare $\underline{\sigma}$ from $p_{in}(\underline{\sigma}) = P_{eq}(\underline{\sigma} | T = \infty) \sim \text{unif}(\underline{\sigma})$
2. Run dynamics at $T = 0$

$$\begin{cases} \dfrac{\partial \sigma_i}{\partial t} = - \dfrac{\partial H}{\partial \sigma_i} & \text{(continuous)} \\ \text{accept spin flip if } \Delta H \leq 0 & \text{(discrete)} \end{cases}$$

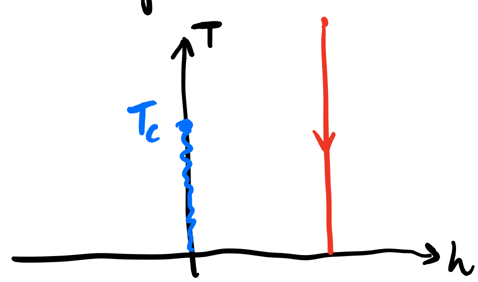
greedy minimization

Now dynamics not in eq: no TTI, no FDT

Many practical applications:

- look for stable states of particle systems
- minimize loss function (machine learning)
- look for solutions of CSP (greedy algorithm)

1. Ferromagnet



If $h \neq 0$ fast convergence to unique ground state

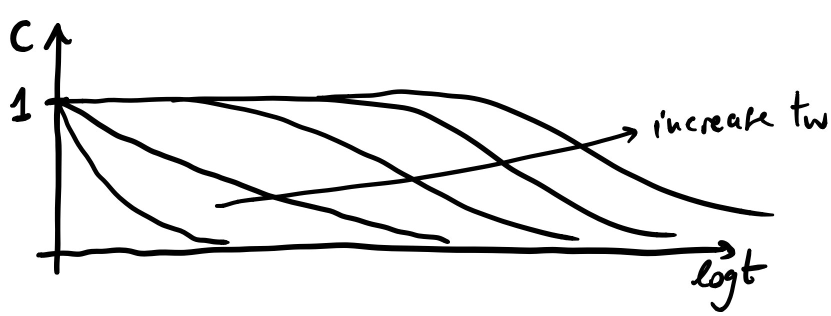
If $h = 0$ competition of the two ground states

Domains $\ell \sim t^{\alpha}$

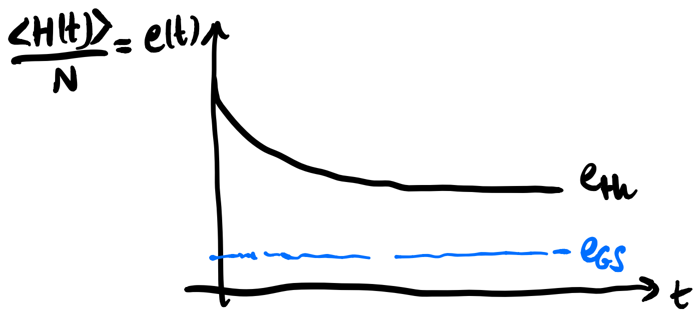
$\langle m(t) \rangle = 0$

$$C(t+t_w, t_w) = \frac{1}{N} \sum_i \langle \sigma_i(t+t_w)\, \sigma_i(t_w) \rangle$$

coarsening

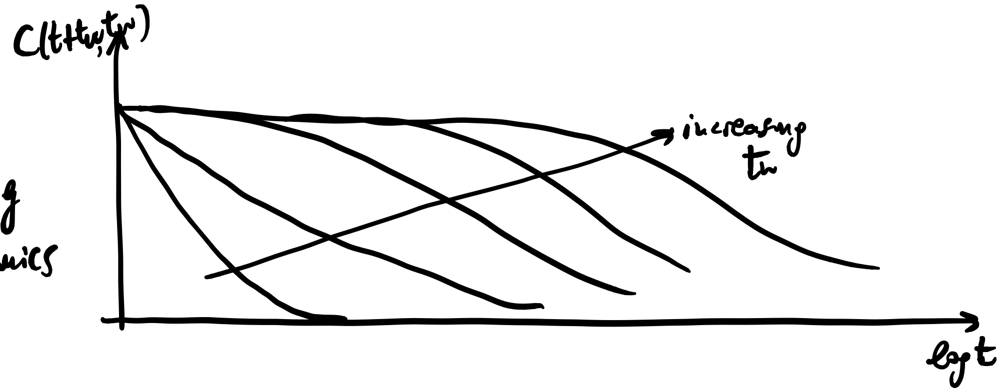


2. pspin $p \geq 3$ exact solution

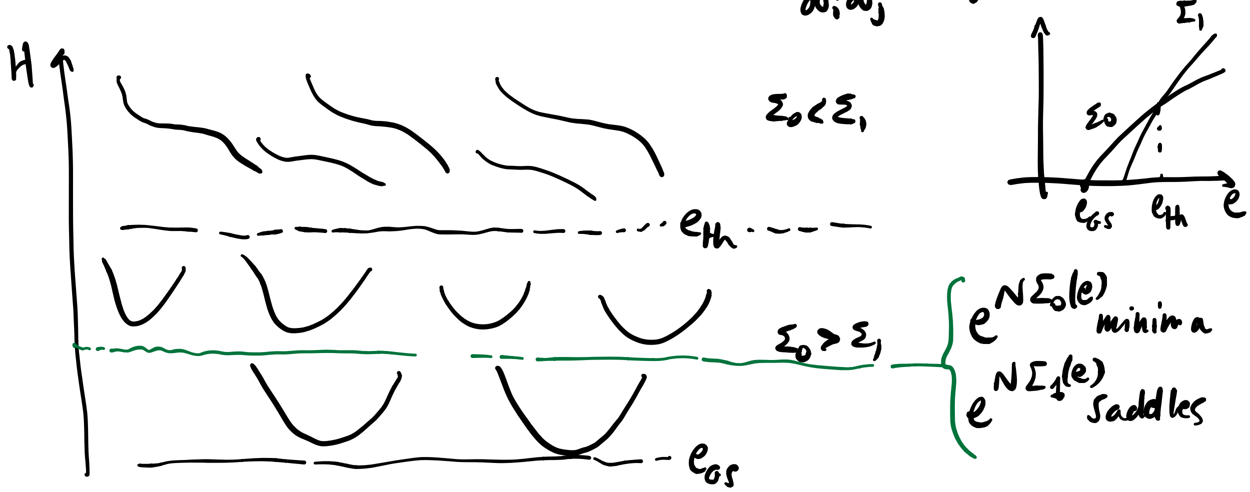


$e(t) - e_{th} \sim t^{-\alpha}$

power law relaxation to threshold energy aging dynamics



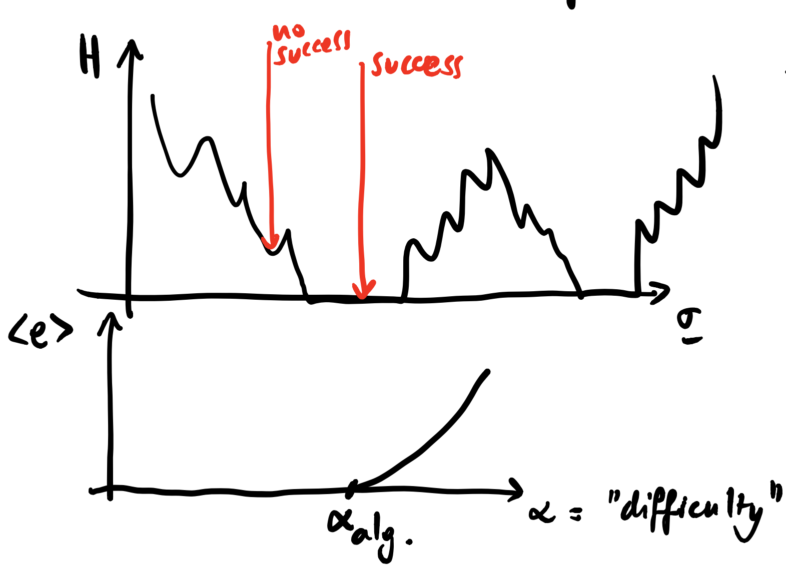
<u>Geometric picture</u> $\qquad \nabla H = 0 \qquad \dfrac{\partial^2 H}{\partial \sigma_i \partial \sigma_j} = M_{ij}$



3. CSP and machine learning

$H = 0$ solution

$H > 0$ no solution



4. Other protocols

Quench $T_i \longrightarrow T_f$

Simulated annealing : from $T = \infty$ and lower T slowly