ICFP M2 - Statistical physics 2 – Solution of the exam

Grégory Schehr, Francesco Zamponi

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1 Questions on the lectures

1. Consider an alloy made of N_x non-magnetic Copper (Cu) atoms with coordinates x_i, and N_y magnetic Manganese (Mn) atoms with coordinates y_i and magnetic moment σ_i. Assume a total Hamiltonian

$$H(x, y, \sigma) = \sum_{i<j} V_{xx}(|x_i - x_j|) + \sum_{i,j} V_{xy}(|x_i - y_j|) + \sum_{i<j} V_{yy}(|y_i - y_j|) - \sum_{i<j} J(|y_i - y_j|)\sigma_i \cdot \sigma_j \ . \tag{1}$$

Here, V_{xx}, V_{xy}, V_{yy} are standard pair-potentials for the Cu-Cu, Cu-Mn, Mn-Mn atomic interactions, while $J(|y_i - y_j|)$ is the exchange interaction between the magnetic moments of the magnetic atoms. Pure Cu ($N_y = 0$) undergoes a freezing first-order transition from the liquid phase to a face-centered cubic (FCC) crystal at a temperature scale ~ 1000 K, and it is a metal. Suppose that a few percent of Mn atoms is added, e.g. $N_y/N_x \sim 4\%$, at a preparation temperature of $T = 2000$ K, and the melt is then cooled to, say, 10 K in a few hours.

1a. Do you expect that the full system composed of the x, y, σ degrees of freedom reaches equilibrium?

Answer: no. In the liquid phase, Mn atoms diffuse in Cu and are found in random positions. When the Cu atoms crystallize, the Mn atoms remain frozen in these random positions and cannot equilibrate.

1b. How do you expect the Manganese atoms to be arranged at low temperatures, say $T \sim 10$ K?

Answer: We expect them to be frozen in the positions they had in the liquid phase, just before the Cu crystallized.

1c. What is the form of the magnetic exchange interaction $J(r)$, and why?

Answer: The magnetic interaction has the so-called Ruderman-Kittel-Kasuya-Yosida (RKKY) form, which is characterized by oscillations on the scale $1/k_F$ and a power-law decay with distance. This interaction is due to the screening of the magnetic moment of one Mn atom by the electrons of the Cu metallic matrix.

1d. What are the consequences for the magnetic interaction part of the Hamiltonian?

Answer: The magnetic interaction has a "quenched" exchange coupling $J_{ij} = J(|y_i - y_j|)$ because the Mn atoms are frozen, while their magnetic moments can still evolve. The most important aspects are that the interaction decays with distance, and can be either ferromagnetic or antiferromagnetic depending on the relative distance r between the two Mn atoms.

2. We now focus on the magnetic part of the Hamiltonian, assume for simplicity Ising spins $\sigma_i = \pm 1$ and i.i.d. Gaussian magnetic couplings J_{ij} with zero mean and variance $1/N$, i.e. the Sherrington-Kirkpatrick model with Hamiltonian

$$H(\sigma) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \ , \tag{2}$$

where h_i is an external magnetic field applied on spin i.

2a. Recall the main properties of the low-temperature phase of the model obtained for $T < 1$.

Answer: The low-temperature phase is characterized by many equilibrium states. Each of them is specified by its local magnetizations $m_i^\alpha = \langle \sigma_i \rangle_\alpha$. The global magnetization is still zero in absence of a magnetic field, because the m_i^α are positive or negative with the same probability.

2b. Explain why, in an experiment, we cannot use the external field h_i to select one of the equilibrium states of the model, like we would do in a ferromagnet.

Answer: In a ferromagnetic system, we have $m_i^{\alpha=\pm} = \pm m^*$ where m^* is the homogeneous equilibrium magnetization. Hence, adding a small field $h_i = \pm\epsilon$ can select one of the two states. But in a spin glass, we do not know the m_i that characterize a given state. A random external field h_i would typically be orthogonal to the magnetization profile, such that $\sum_i h_i m_i \sim 0$ for all states, and thus it is not effective in selecting a given state.

2c. (BONUS) Consider a uniform external field $h_i = h$. Suppose that the system is prepared in one of the equilibrium states at $h = 0$; the thermal average in that state is denoted by $\langle \bullet \rangle_\alpha$. Then, a very small field is applied and the magnetization is measured, over a small time, such that the system cannot change state. Recall why the magnetic susceptibility is

$$\chi_\alpha = \frac{dm_\alpha}{dh} = \frac{d}{dh}\frac{1}{N}\sum_i \langle \sigma_i \rangle_\alpha = \frac{\beta}{N}\sum_{ij}[\langle \sigma_i \sigma_j \rangle_\alpha - \langle \sigma_i \rangle_\alpha \langle \sigma_j \rangle_\alpha] \tag{3}$$

Answer: We have

$$\chi_\alpha = \frac{d}{dh}\frac{1}{N}\sum_i \langle \sigma_i \rangle_\alpha = \frac{1}{N}\sum_i \frac{d\langle \sigma_i \rangle_\alpha}{dh} \tag{4}$$

The derivative can be written as

$$\frac{d\langle \sigma_i \rangle_\alpha}{dh} = \frac{d}{dh}\frac{\sum_{\sigma\in\alpha} \sigma_i e^{\sum_{j<k} J_{jk}\sigma_j\sigma_k + h\sum_j \sigma_j}}{\sum_{\sigma\in\alpha} e^{\sum_{j<k} J_{jk}\sigma_j\sigma_k + h\sum_j \sigma_j}} \tag{5}$$

where $\sigma \in \alpha$ indicates that the spin configuration σ has a high probability of belonging to state α. (To be precise, this condition can be enforced by an indicator function. In a mean field system, the states are separated by regions of exponentially small probability in N, so the separation can be considered as very sharp.) Taking the derivative, we then obtain

$$\begin{aligned}\frac{d\langle \sigma_i \rangle_\alpha}{dh} &= \frac{\sum_{\sigma\in\alpha} \left(\sigma_i \sum_j \sigma_j\right) e^{\sum_{j<k} J_{jk}\sigma_j\sigma_k + h\sum_j \sigma_j}}{\sum_{\sigma\in\alpha} e^{\sum_{j<k} J_{jk}\sigma_j\sigma_k + h\sum_j \sigma_j}} - \langle \sigma_i \rangle_\alpha \langle \sum_j \sigma_j \rangle_\alpha \\ &= \sum_j \langle \sigma_i \sigma_j \rangle_\alpha - \langle \sigma_i \rangle_\alpha \langle \sigma_j \rangle_\alpha \ ,\end{aligned} \tag{6}$$

which then gives the desired result.

2d. (BONUS) Recall that in a mean field model, each state is a product $P(\sigma) = \prod_i \frac{1+m_i^\alpha \sigma_i}{2}$. Argue that as a consequence the terms with $i \neq j$ in the sum disappear. Recall the definition of the Edwards-Anderson order parameter q_{EA}, and conclude that, taking an average over the states, the linear response susceptibility is

$$\chi_{LR} = \overline{\chi_\alpha} = \beta(1 - q_{EA}) \ . \tag{7}$$

Answer: Because $P(\sigma) = \prod_i \frac{1+m_i^\alpha \sigma_i}{2}$ is a product, distinct spins with $i \neq j$ are independent and

$$\langle \sigma_i \sigma_j \rangle_\alpha = \langle \sigma_i \rangle_\alpha \langle \sigma_j \rangle_\alpha \ , \tag{8}$$

hence their contribution disappears and we get

$$\chi_\alpha = \frac{\beta}{N}\sum_i [\langle \sigma_i^2 \rangle_\alpha - \langle \sigma_i \rangle_\alpha^2] = \frac{\beta}{N}\sum_i [1 - (m_i^\alpha)^2] \ , \tag{9}$$

recalling that $\sigma_i^2 = 1$. Now, we recall the definition of the Edwards-Anderson order parameter

$$q_{EA}^\alpha = \frac{1}{N}\sum_i (m_i^\alpha)^2 \ . \tag{10}$$

Note that this quantity is self-averaging and thus we can drop the index α. This gives the desired result $\chi_{LR} = \beta(1 - q_{EA})$.

2e. (BONUS) Recall the behavior of $q_{EA}(T)$ and draw the corresponding function $\chi_{LR}(T)$.

Answer: For $T > 1$ we have $q_{EA} = 0$ and $\chi_{LR} = \beta = 1/T$ has the Curie-Weiss form. For $T \lesssim 1$, we have that $q_{EA} = C(1 - T)$ grows linearly with decreasing temperature, we thus get that $\chi_{LR} \approx \beta(1 - C + CT)$ is reduced with respect to the paramagnetic value. Thus, χ_{LR} has a cusp at the phase transition.

2 May's model and Girko's law for non-Hermitian matrices

A classical toy model in theoretical ecology is the so-called May's model. In this model, one considers a population of N distinct species with equilibrium densities ρ_i^*, with $i = 1, 2, \cdots, N$. To start with, they are non-interacting and stable in the sense that when slightly perturbed from their equilibrium densities, each density relaxes to its equilibrium value with some characteristic damping time denoted by $1/\mu$ (for simplicity, these damping times are all chosen to be the same). Hence the equations of motion for $x_i(t) = \rho_i(t) - \rho_i^*$ to linear order, are simply $\dot{x}_i(t) = -\mu\, x_i(t)$. Now, imagine switching on pairwise interactions between the species. May assumed that the interactions between pairs of species can be modeled by a random matrix J of size $N \times N$, which is generically asymmetric, $J_{ij} \neq J_{ji}$. The linearized equations of motion close to ρ_i^* in the presence of interactions read

$$\dot{x}_i(t) = -\mu\, x_i(t) + \alpha \sum_{j=1}^{N} J_{ij} x_j(t) \quad , \quad i = 1, 2, \cdots, N \; , \tag{11}$$

where α is the coupling strength. A natural question is then: what is the probability, $P_{\rm stable}(\alpha, N)$ that the system described by (46) remains stable once the interactions are switched on?

1. Such interaction matrix J_{ij} does not have any special symmetry and therefore its eigenvalues are complex. If we denote by $\{z_1, \cdots, z_N\}$ its eigenvalues, show that

$$P_{\rm stable}(\alpha, N) = \text{Prob.}\left[\max_{1 \le i \le N} \text{Re}(z_i) < \mu/\alpha\right] \; , \tag{12}$$

where $\text{Re}(z)$ denotes the real part of the complex number z.

Answer: in matrix notation the system, denoted as (S) of N equations in (46), reads

$$\text{(S)} \quad : \quad \dot{\mathbf{x}} = \mathbb{M}\,\mathbf{x} \quad , \quad \mathbb{M} = \alpha \mathbb{J} - \mu\, \mathbb{I}_{\mathbf{N}} \; . \tag{13}$$

Since the eigenvalues of $\mathbb{M}$ are simply $\{\alpha z_1 - \mu, \alpha z_2 - \mu, \cdots, \alpha z_N - \mu\}$, the standard linear stability analysis states that

$$\text{The system S is stable} \quad \Longleftrightarrow \quad \text{Re}(\alpha z_i - \mu) < 0 \quad \forall i = 1, 2, \cdots, N \quad \Longleftrightarrow \quad \max_{1 \le i \le N} \text{Re}(z_i) < \mu/\alpha \; . \tag{14}$$

This implies the relation in (12).

The computation of $P_{\rm stable}(\alpha, N)$ thus requires the study of the statistics of the eigenvalues of such $N \times N$ real non-symmetric matrices J_{ij}. It turns out that, for technical reasons, the study of such real matrices is quite difficult. However, their counterpart with complex entries is much simpler and at the same time exhibits the same qualitative behaviors. In the following we will thus restrict our analysis to complex non-Hermitian matrices.

We consider a $N \times N$ random matrix $G \equiv \{G_{jk}\}_{1 \le j,k \le N}$ with complex entries $G_{jk} = G_{jk}^R + i G_{jk}^I$, where $G_{jk}^{R,I}$ are real. Here, we study the so-called (complex) Ginibre ensemble where all these entries $G_{jk}^{R,I}$ with $1 \le j, k \le N$ are independent and identically distributed Gaussian random numbers, such that the probability weight $\mathcal{P}(G)$ of a given matrix G is given, up to a normalization factor independent of G, by

$$\mathcal{P}(G) \propto \prod_{1 \le j,k \le N} e^{-\left[(G_{jk}^R)^2 + (G_{jk}^I)^2\right]} \; . \tag{15}$$

2. Show that $\mathcal{P}(G)$ in (15) can be written as

$$\mathcal{P}(G) \propto e^{-\text{Tr}(G^\dagger G)} \; , \tag{16}$$

where we recall that $(G^\dagger)_{jk} = G_{kj}^*$ – with z^* denoting the complex conjugate of z.

Answer: Let us start from (16). We simply have, by definition of $G^\dagger$,

$$\begin{aligned} \text{Tr}\, G^\dagger G &= \sum_{1 \le j,k \le N} G_{jk}^\dagger G_{kj} = \sum_{1 \le j,k \le N} G_{kj}^* G_{kj} = \sum_{1 \le j,k \le N} |G_{kj}|^2 = \sum_{1 \le j,k \le N} |G_{jk}|^2 \\ &= \sum_{1 \le j,k \le N} (G_{jk}^R)^2 + (G_{jk}^I)^2 \; , \end{aligned} \tag{17}$$

Hence

$$e^{-\mathrm{Tr}(\mathrm{G}^\dagger \mathrm{G})} = e^{-\sum_{1\le j,k\le N}\left[(G^R_{jk})^2+(G^I_{jk})^2\right]} = \prod_{1\le j,k\le N} e^{-\left[(G^R_{jk})^2+(G^I_{jk})^2\right]} \propto \mathcal{P}(G)\;. \tag{18}$$

which shows the relation (16).

3. If $G' = UGV$ where U and V are $N\times N$ unitary matrices, i.e. $U^\dagger U = \mathbb{I}_N$ and similarly for V, show that $\mathcal{P}(G') = \mathcal{P}(G)$. This ensemble is thus called *rotationally invariant*.

   Answer: let us simply observe that $G'^\dagger = V^\dagger G^\dagger U^\dagger$ from which it follows that

$$\mathrm{Tr}(G'^\dagger G') = \mathrm{Tr}(V^\dagger G^\dagger U^\dagger U G V) = \mathrm{Tr}(V^\dagger G^\dagger G V) = \mathrm{Tr}(G^\dagger G V V^\dagger) = \mathrm{Tr}(G^\dagger G)\;, \tag{19}$$

   where in the second and fourth equality we have used that $U^\dagger U = \mathbb{I}_N$ and $VV^\dagger = \mathbb{I}_N$ respectively, while in the third equality we used that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Finally, using the relation in (16), one finds that $\mathcal{P}(G') = \mathcal{P}(G)$.

Such Ginibre matrices (15) are non-Hermitian and therefore their eigenvalues are complex. If we denote by $\{z_1,\cdots,z_N\}$ the eigenvalues of such a Ginibre matrix, one can show that their joint probability distribution function (JPDF) is given by

$$P_{\text{joint}}(z_1,\cdots,z_N) = \frac{1}{\mathcal{Z}_N}\prod_{1\le j<k\le N}|z_j-z_k|^2\prod_{l=1}^N e^{-|z_l|^2}\;, \tag{20}$$

where $\mathcal{Z}_N$ is a normalization constant (or partition function)

$$\mathcal{Z}_N = \int d^2z_1\cdots\int d^2z_N \prod_{1\le j<k\le N}|z_j-z_k|^2\prod_{l=1}^N e^{-|z_l|^2}\;. \tag{21}$$

In Eq. (21) we used the notation $d^2z = dx\,dy = r\,dr\,d\theta$ in Cartesian and polar coordinates respectively and the integrals run over the whole complex plane.

4. Comment on the differences and similarities between the JPDF in (20) and that of the eigenvalues of random Gaussian Hermitian matrices, i.e., belonging to the Gaussian Unitary Ensemble (GUE).

   Answer: the eigenvalues of a matrix belonging to the GUE are all real but their joint distribution is formally exactly similar to Eq. (20) with z_j real (and a different normalization constant $\mathcal{Z}_N$).

5. Show that the JPDF in (20) can be written as a Boltzmann weight, namely

$$P_{\text{joint}}(z_1,\cdots,z_N) = \frac{1}{\mathcal{Z}_N}e^{-E(\{z_i\})} \tag{22}$$

   and interpret $E(\{z_i\})$ as the energy of a Coulomb gas system.

   Answer: as was done during the lectures for the GOE/GUE, one uses the identity $|z| = e^{\,\ln|z|}$ to rewrite the JPDF in (20) as

$$\begin{aligned} P_{\text{joint}}(z_1,\cdots,z_N) &= \frac{1}{\mathcal{Z}_N}e^{\sum_{j<k}\ln|z_j-z_k|^2-\sum_{l=1}^N|z_l|^2} = \frac{1}{\mathcal{Z}_N}e^{\sum_{j\neq k}\ln|z_j-z_k|-\sum_{l=1}^N|z_l|^2} \\ &= \frac{1}{\mathcal{Z}_N}e^{-E(\{z_i\})} \quad,\quad E(\{z_i\}) = \sum_{l=1}^N|z_l|^2-\sum_{j\neq k}\ln|z_j-z_k|\;. \end{aligned} \tag{23}$$

   Hence the JPDF (20) can be interpreted as the Boltzmann (at weight inverse temperature $\beta=1$) of a system of N charged particles (say $+1$) on the plane, whose positions are labelled by $z_1,z_2,\cdots,z_N$. The energy of a configuration $E(\{z_i\})$ can be interpreted as follows. These particles experience an external isotropic harmonic potential $V(z) = |z|^2$ and interact (repulsively) via a logarithmic potential, which is indeed the Coulomb potential in two dimensions between charges of the same sign. Hence the name of a Coulomb gas – in the literature it is also called the one-component plasma.

6. This energy $E(\{z_i\})$ has two contributions: a confining component and an interacting one. By balancing these two contributions, show that the typical scale of z_i is $z_i = O(\sqrt{N})$.

   Answer: let us denote by $z_{\rm typ}$ the typical scale of the z_i's such that one has $z_i = z_{\rm typ}\tilde z_i$ with $\tilde z_i$ of order $O(1)$ for large N. The two terms in the energy in (23) can be written as

$$E_{\rm conf} = \sum_{l=1}^N |z_l|^2 = z_{\rm typ}^2 \sum_{i=1}^N |\tilde z_i|^2 \tag{24}$$

$$E_{\rm inter} = \sum_{1\leq j\neq k\leq N} \ln|\tilde z_j - \tilde z_k| + {\rm const.}\,, \tag{25}$$

   where const. $= N(N-1)\ln z_{\rm typ}$ denotes an unimportant constant independent of the z_i's, which can thus be simply absorbed in the normalisation constant $\mathcal{Z}_N$ in (20). Estimating the contributions that depend on $\tilde z_i$ in Eqs. (24) and (25) one finds

$$E_{\rm conf} = \sum_{l=1}^N |z_l|^2 = z_{\rm typ}^2 \sum_{i=1}^N |\tilde z_i|^2 \sim N z_{\rm typ}^2 \tag{26}$$

$$E_{\rm inter} - {\rm const.} = \sum_{1\leq j\neq k\leq N} \ln|\tilde z_j - \tilde z_k| \sim N(N-1) \sim N^2 \quad , \quad N\to\infty\,. \tag{27}$$

   Balancing these two contributions in (26) and (27), one finds $N z_{\rm typ}^2 \sim N^2 \Longrightarrow z_{\rm typ} \sim \sqrt{N}$.

We thus introduce the scaled variables $\tilde z_i = z_i/\sqrt{N}$ together with the corresponding empirical eigenvalue density $\tilde\mu_N(\tilde z)$ and $\mu_N(z)$ defined as

$$\tilde\mu_N(\tilde z) = \frac{1}{N}\sum_{i=1}^N \delta^{(2)}(\tilde z - \tilde z_i) \quad , \quad \mu_N(z) = \frac{1}{N}\sum_{i=1}^N \delta^{(2)}(z - z_i)\,, \tag{28}$$

where $\delta^{(2)}(z)$ denotes the (two-dimensional) Dirac delta function. From (28), one has $\tilde\mu_N(\tilde z) = N\,\mu_N(z)$, with $z = \sqrt{N}\,\tilde z$. We assume (and check it a posteriori) that $\tilde\mu_N(\tilde z)$ converges to a smooth a function of order $O(1)$ as $N\to\infty$. We also introduce the two-point densities

$$\tilde\mu_N^{(2)}(\tilde z, \tilde z') = \frac{1}{N(N-1)}\sum_{i\neq j}\delta^{(2)}(\tilde z - \tilde z_i)\delta^{(2)}(\tilde z' - \tilde z_j) \quad , \quad \mu_N^{(2)}(z,z') = \frac{1}{N(N-1)}\sum_{i\neq j}^N \delta^{(2)}(z-z_i)\delta^{(2)}(z'-z_i)\,. \tag{29}$$

7. Show the following identity

$$\tilde\mu_N^{(2)}(\tilde z, \tilde z') = \frac{N}{N-1}\tilde\mu_N(\tilde z)\mu_N(\tilde z') - \frac{1}{N-1}\tilde\mu_N(\tilde z)\delta^{(2)}(\tilde z - \tilde z')\,. \tag{30}$$

   Answer: one has

$$\tilde\mu_N^{(2)}(\tilde z, \tilde z') = \frac{1}{N(N-1)}\sum_{1\leq i,j\leq N}\delta^{(2)}(\tilde z - \tilde z_i)\delta^{(2)}(\tilde z' - \tilde z_j) - \frac{1}{N(N-1)}\sum_{i=1}^N \delta^{(2)}(\tilde z - \tilde z_i)\delta^{(2)}(\tilde z' - \tilde z_i) \tag{31}$$

$$= \frac{N}{(N-1)}\frac{1}{N^2}\sum_{i=1}^N \delta^{(2)}(\tilde z - \tilde z_i)\sum_{j=1}^N \delta^{(2)}(\tilde z' - \tilde z_j) - \frac{1}{N-1}\frac{1}{N}\sum_{i=1}^N \delta^{(2)}(\tilde z - \tilde z_i)\delta^{(2)}(\tilde z - \tilde z')\,, \tag{32}$$

   which, upon using the definition of $\tilde\mu_N(\tilde z)$ in (28) yields the desired relation (30).

8. Show that the JPDF can be written as in Eq. (22) with

$$E(\{z_i\}) \equiv E[\tilde\mu_N] = e_N + N^2\left[\int |\tilde z|^2\,\tilde\mu_N(\tilde z)\,d^2\tilde z - \int d^2\tilde z\int d^2\tilde z'\,\tilde\mu_N^{(2)}(\tilde z,\tilde z')\ln|\tilde z - \tilde z'|\right] + O(N)\,. \tag{33}$$

   where e_N is a constant independent of the $\tilde z_i$'s to be determined.

Answer: setting $z_i = \sqrt{N}\tilde z_i$ one finds that the energy reads, in terms of $\tilde z_i$'s:

$$E(\{z_i\}) = N\sum_{l=1}^N |\tilde z_l|^2 - \frac{N(N-1)}{2}\ln N - \sum_{1\leq j\neq k\leq N}\ln|\tilde z_j - \tilde z_k|$$

$$= N^2\int d^2\tilde z|\tilde z|^2\,\tilde\mu_N(\tilde z) - \frac{N(N-1)}{2}\ln N - N(N-1)\int d^2\tilde z\int d^2\tilde z'\,\tilde\mu_N^{(2)}(\tilde z,\tilde z')\ln|\tilde z - \tilde z'| \tag{34}$$

which, in the limit of large N, leads to Eq. (33) with $e_N = -(1/2)N(N-1)\ln N$. Using the fact that $\tilde{\mu}_N(\tilde{z})$ converges to a smooth a function of order $O(1)$ as $N \to \infty$ together with $N(N-1) = N^2 + O(N)$, one arrives at Eq. (33).

We will admit that in (33), the two-point density $\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}')$ can be replaced by its large N behavior $\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}') \approx \tilde{\mu}_N(\tilde{z})\tilde{\mu}_N(\tilde{z}')$ [see the result derived in Eq. (30)] so that $E(\{z_i\})$ can be written as

$$E(\{z_i\}) \equiv E\left[\tilde{\mu}_N\right] = \text{const} + N^2 \mathcal{E}[\tilde{\mu}_N] + O(N) \tag{35}$$

$$\mathcal{E}[\tilde{\mu}_N] = \left[\int |\tilde{z}|^2\, \tilde{\mu}_N(\tilde{z})\, d^2\tilde{z} - \int d^2\tilde{z} \int d^2\tilde{z}'\, \tilde{\mu}_N(\tilde{z})\tilde{\mu}_N(\tilde{z}') \ln|\tilde{z} - \tilde{z}'|\right] . \tag{36}$$

9. Taking for granted the fact that the Jacobian associated to the change of variable $\{z_1, \cdots, z_N\} \to \tilde{\mu}_N(\tilde{z})$ is of order $e^{O(N)}$, argue that for, large N, the typical empirical density $\tilde{\mu}_N$ is the one that minimises $\mathcal{E}\left[\tilde{\mu}_N\right]$.

   Answer: changing variables from z_i to $\tilde{\mu}_N(z_i)$ one finds

$$P_{\text{joint}}(z_1, \cdots, z_N) = P_{\text{joint}}[\tilde{\mu}_N] J[\tilde{\mu}_N] \tag{37}$$

$$\text{where} \quad J[\tilde{\mu}_N] \propto \int d^2 z_1 \cdots \int d^2 z_N \delta\left[N\tilde{\mu}_N(\tilde{z}) - \sum_{i=1}^{N} \delta^{(2)}(\tilde{z} - z_i/\sqrt{N})\right] , \tag{38}$$

   is the Jacobian associated to the change of variable $\{z_1, \cdots, z_N\} \to \tilde{\mu}_N(\tilde{z})$. Since $J[\tilde{\mu}_N] = e^{O(N)}$, one finds

$$P_{\text{joint}}[\tilde{\mu}_N] = e^{e_N + N^2 \mathcal{E}[\tilde{\mu}_N] + O(N)} . \tag{39}$$

   Hence in the large N limit, since the non-trivial part of the energy is of order $O(N^2)$, the most probable (i.e., typical) configuration is the one that minimises the energy $\mathcal{E}[\tilde{\mu}_N]$ (in the same spirit as a saddle point analysis).

This minimisation procedure leads to

$$|\tilde{z}|^2 - 2\int d^2\tilde{z}' \tilde{\mu}_N(\tilde{z}') \ln|\tilde{z} - \tilde{z}'| = C \quad , \quad \int d^2\tilde{z}\, \tilde{\mu}_N(\tilde{z}) = 1 \, , \tag{40}$$

where the first equation in (40) is valid for $\tilde{z}$ within the support of $\tilde{\mu}_N$ while C is some unknown constant.

10. Using the identity $\Delta_{\tilde{z}}\left[\ln|\tilde{z} - \tilde{z}'|\right] = 2\pi\, \delta^{(2)}(\tilde{z} - \tilde{z}')$, where $\Delta_{\tilde{z}} = \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2$ is the two-dimensional Laplacian, show that

$$\tilde{\mu}_N(\tilde{z}) = \frac{1}{\pi} \, , \quad \text{for} \quad |\tilde{z}| \leq \tilde{R}_e \, . \tag{41}$$

   Answer: Taking the Laplacian on both sides of Eq. (40), using $\Delta_{\tilde{z}}|z^2| = 2 + 2 = 4$ together with $\Delta_{\tilde{z}}\left[\ln|\tilde{z} - \tilde{z}'|\right] = 2\pi\, \delta^{(2)}(\tilde{z} - \tilde{z}')$ one finds

$$4 - 2 \times 2\pi \int\int d^2\tilde{z}' \tilde{\mu}_N(\tilde{z}') \delta^{(2)}(\tilde{z} - \tilde{z}') = 0 \quad \Longleftrightarrow \quad \tilde{\mu}_N(\tilde{z}) = \frac{1}{\pi} \, , \tag{42}$$

   for $\tilde{z}$ within the support of $\tilde{\mu}_N$.

11. From the normalisation condition in (40) show that $\tilde{R}_e = 1$.

   Answer: from the normalisation condition, i .e., the second relation in (40), one finds

$$\int d^2\tilde{z}\, \tilde{\mu}_N(\tilde{z}) = 1 \quad \Longleftrightarrow \quad \int_0^{2\pi} d\theta \int_0^{\tilde{R}_e} r\, \frac{dr}{\pi} = 1 \quad \Longleftrightarrow \quad \tilde{R}_e = 1 \, . \tag{43}$$

12. **Back to the May's model**

   12.a) From the above results for the average density of eigenvalues, argue that,

$$\max_{1 \leq i \leq N} \text{Re}(z_i)/\sqrt{N} \to 1 \quad , \quad N \to \infty \, , \tag{44}$$

   with probability one.

Answer: From Eq. (43) we deduce that the variables $\tilde{z}_i = z_i/\sqrt{N}$ are uniformly distributed over the unit disk in the limit $N \to \infty$. Therefore, with probability one, all the eigenvalues z_i are contained inside the disk of radius $\sqrt{N}$ and in particular the real part of all the eigenvalues z_i are smaller than $\sqrt{N}$, which leads to the result in (44) with probability one.

Deduce from it that

$$\text{Prob.}\left(\max_{1\leq i\leq N} \text{Re}(z_i) < M\right) \approx \begin{cases} 1 \quad , \quad M > \sqrt{N}\,, \\ \\ 0 \quad , \quad M < \sqrt{N}\,. \end{cases} \tag{45}$$

Answer: Since all the eigenvalues are contained in the disk of radius $\sqrt{N}$, the real part of all the eigenvalues is smaller than $\sqrt{N}$ – hence the first line of (45) corresponding to $M > \sqrt{N}$. On the other hand, if $M < \sqrt{N}$, there will always be at least one eigenvalue whose real part will be close to $\sqrt{N} > M$, hence the second line of (45).

12.b) Discuss the consequences of this result (45) on the probability $P_{\rm stable}(\alpha, N)$ that the dynamical system (46) remains stable: how does it depend on both α and N?

Answer: Translating this result (45) in terms of the May's model and the result obtained in (12), it can be rephrased in terms of the variable $m = \mu/(\alpha\sqrt{N})$ as

$$P_{\rm stable}(\alpha, N) = \begin{cases} 1 \quad , \quad m > 1\,, \\ \\ 0 \quad , \quad m < 1\,. \end{cases} \tag{46}$$

Hence, for fixed (large) N, the system (S) in (14) is stable is α is small enough while it becomes unstable if α exceeds a critical value $\alpha_c = \mu/\sqrt{N}$. Similarly, for fixed α, the system (S) is stable if N is small enough but becomes unstable as soon as N exceeds the critical value $N_c = (\mu/\alpha)^2$.

BONUS QUESTIONS:

In the following, we provide an alternative (more controlled) derivation of the average density $\mathbb{E}(\mu_N(z))$. To start with, we recognise in the JPDF in (20), the product of two Vandermonde determinants. Indeed one has

$$\prod_{1\leq j<k\leq N} (z_k - z_j) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{pmatrix} = \sum_{\sigma\in\mathcal{S}_N} \text{sgn}(\sigma) \prod_{j=1}^N z_j^{\sigma(j)-1}\,, \tag{47}$$

where in the second equality we have used the so-called Leibniz formula, which will be useful in the following. In that expression $\mathcal{S}_N$ denotes the group of permutations of N elements (which thus contains $N!$ distinct permutations) and $\text{sgn}(\sigma) = \pm 1$ is the signature of the permutation σ.

13. **Computation of the partition function $\mathcal{Z}_N$**

13.a) Show the following identity

$$\int d^2z\, z^{j-1} (z^*)^{k-1} e^{-|z|^2} = \delta_{j,k}\, \pi\, \Gamma(j)\,, \tag{48}$$

where $\delta_{j,k}$ is the Kronecker delta function and $\Gamma(z) = \int_0^\infty u^{z-1}\, e^{-u}\, du$.

Answer: Using the polar coordinates $d^2z = r\, dr\, d\theta$ together with $z = r\, e^{i\theta}$ and $z^* = r\, e^{-i\theta}$ one finds

$$\int d^2z\, z^{j-1} (z^*)^{k-1} e^{-|z|^2} = \int_0^{2\pi} d\theta\, e^{(j-k)\theta} \int_0^\infty dr\, r^{j+k-1} e^{-r^2} = 2\pi\delta_{j,k} \int_0^\infty dr r^{2j-1} e^{-r^2}\,, \tag{49}$$

where, in the last equality, we have used $\int_0^{2\pi} d\theta\, e^{(j-k)\theta} = 2\pi\delta_{j,k}$. Finally, performing the change of variable $u = r^2$, the interval over r can be computed explicitly, leading to the result in (48).

13.b) By using the Leibniz formula (47) for the two Vandermonde determinants in (21), and performing the integrals over z_i's using (48), show that

$$\mathcal{Z}_N = \pi^N \sum_{\sigma\in\mathcal{S}_N} \prod_{j=1}^N \Gamma\left[\sigma(j)\right]\,. \tag{50}$$

Answer: By using the Leibniz formula (47) for the two Vandermonde determinants in (21) one finds

$$\begin{aligned}
\mathcal{Z}_N &= \int d^2 z_1 \cdots \int d^2 z_N \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{l=1}^N e^{-|z_l|^2} && (51)\\
&= \int d^2 z_1 \cdots \int d^2 z_N \sum_{\sigma\in\mathcal{S}_N}\sum_{\tau\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{j=1}^N z_j^{\sigma(j)-1}(z_j^*)^{\tau(j)-1} e^{-|z_j|^2} && (52)\\
&= \sum_{\sigma\in\mathcal{S}_N}\sum_{\tau\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{j=1}^N \int d^2 z_j z_j^{\sigma(j)-1}(z_j^*)^{\tau(j)-1} e^{-|z_j|^2} && (53)\\
&= \sum_{\sigma\in\mathcal{S}_N}\sum_{\tau\in\mathcal{S}_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{j=1}^N \left[\pi \delta_{\sigma(j),\tau(j)}\, \Gamma(\sigma(j))\right] \,, && (54)
\end{aligned}$$

where in the last equality we have used the identity (48) specified to $j = \sigma(j)-1$ and $k = \tau(j)-1$. From the product of Kronecker delta functions, we see that the two permutations σ and τ coincide identically. Therefore, by using $(\mathrm{sgn}(\sigma))^2 = 1$, one obtains the relation in (50).

13.c) Obtain finally the explicit expression of $\mathcal{Z}_N$

$$\mathcal{Z}_N = \pi^N \prod_{j=1}^N j! \,, \tag{55}$$

where we recall that $j! = \Gamma(j+1)$ for an integer j.

Answer: Since σ is a permutation of the elements $1, \cdots, N$, one has

$$\prod_{j=1}^N \Gamma(\sigma(j)) = \prod_{j=1}^N \Gamma(j) = \prod_{j=0}^{N-1} j! \quad , \quad \text{independently of } \sigma \in \mathcal{S}_N \,. \tag{56}$$

Substituting this identity in Eq. (50) one finds

$$\mathcal{Z}_N = \pi^N \sum_{\sigma\in\mathcal{S}_N} \prod_{j=1}^N \Gamma(\sigma(j)) = \pi^N \sum_{\sigma\in\mathcal{S}_N} \prod_{j=0}^{N-1} j! = \pi^N N! \prod_{j=0}^{N-1} j! = \pi^N \prod_{j=1}^N j! \,, \tag{57}$$

where we have used that the group $\mathcal{S}_N$ contains $N!$ elements, together with $0! = 1$. This proves the relation (55).

14. **Direct computation of the average eigenvalue density $\mathbb{E}(\mu_N(z))$ and its large N limit**

14.a) Show that the average eigenvalue density can be written as the following $(N-1)$-dimensional integral

$$\mathbb{E}(\mu_N(z)) = \frac{1}{\pi^N \prod_{j=1}^N j!} \int d^2 z_2 \cdots \int d^2 z_N \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{l=1}^N e^{-|z_l|^2} \Bigg|_{z_1 = z} \tag{58}$$

Answer: Using the definition of $\mu_N(z)$ given in Eq. (28) one has

$$\mathbb{E}(\mu_N(z)) = \int d^2 z_1 \cdots \int d^2 z_N \frac{1}{N} \sum_{i=1}^N \delta^{(2)}(z - z_i) P_{\rm joint}(z_1, \cdots, z_N) \,. \tag{59}$$

Inserting the explicit expression of $\mathcal{Z}_N$ in (55) together with the fact that $P_{\rm joint}(z_1, \cdots, z_N)$ is invariant under any permutation of the z_i's one has

$$\mathbb{E}(\mu_N(z)) = \frac{1}{\pi^N \prod_{j=1}^N j!} \int d^2 z_1 \int d^2 z_2 \cdots \int d^2 z_N \delta^{(2)}(z - z_1) \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{l=1}^N e^{-|z_l|^2} \,, \tag{60}$$

which yields the result in Eq. (58).

14.b) Using the same kind of manipulations as above for the calculation of $\mathcal{Z}_N$, one arrives at the following expression for the average density

$$\mathbb{E}(\mu_N(z)) = \frac{1}{\pi N} \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!} e^{-|z|^2} \,. \tag{61}$$

Check that it is normalised to unity, i.e.,

$$\int d^2z\,\mathbb{E}(\mu_N(z)) = 1\;. \tag{62}$$

Answer: Using the polar coordinates $d^2z = r\,dr\,d\theta$ one has

$$\begin{aligned}\int d^2z\,\mathbb{E}(\mu_N(z)) &= \frac{1}{N\pi}\sum_{k=0}^{N-1}\frac{1}{k!}\int_0^{2\pi}d\theta\int_0^{\infty}dr\,r^{2k+1}e^{-r^2} &(63)\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{k!}\int_0^{\infty}du\,u^k\,e^{-u}\;, &(64)\end{aligned}$$

where, in the second line, we performed the integral over θ together with the change of variable $u = r^2$. Finally, using $\int_0^\infty du\,u^k\,e^{-u} = k!$ and summing over k, one arrives at the result in (62).

14.c) Show that in the limit of large N, keeping z fixed, one has

$$\lim_{N\to\infty} N\,\mathbb{E}(\mu_N(z)) = \frac{1}{\pi}\;, \tag{65}$$

which means that the average density is uniform in the limit of large N.

Answer: From Eq. (61) one has

$$\lim_{N\to\infty} N\,\mathbb{E}(\mu_N(z)) = \lim_{N\to\infty}\frac{1}{\pi}e^{-|z|^2}\sum_{k=0}^{N-1}\frac{|z|^{2k}}{k!} = \frac{1}{\pi}\;, \tag{66}$$

where, in the last equality, we have used $\sum_{k=0}^{\infty}\frac{|z|^{2k}}{k!} = e^{|z|^2}$.

14.d) Take for granted that this behavior (65) holds in fact up to a certain radius $|z| \leq R_N$ beyond which the density vanishes identically (for large N). From the normalisation condition (62) show that $R_N \sim \sqrt{N}$ for large N. Conclude that Eq. (41) holds with $\tilde{R}_e = 1$.

Answer: Using this result (65) in Eq. (62), one finds

$$\int_0^{2\pi}d\theta\int_0^{R_N} r\frac{1}{\pi} \sim N \quad\Longleftrightarrow\quad R_N \sim \sqrt{N}\;, \tag{67}$$

which implies $R_e = R_N/\sqrt{N} = 1$ in the limit $N\to\infty$.