ICFP M2 - Statistical physics 2 – Exam

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The exam is made of two parts. The first one is a series of short independent questions to check your knowledge of the contents of some of the lectures, the second one is a longer problem with partially independent subparts.

No document, calculator nor phone is allowed.

You can write your answers in English or French.

1 Questions on the lectures

1. What is an Erdös-Rényi random graph? If you sit on a site taken at random and you look at your neighborhood of finite size ℓ, how does it look like when the total number of sites of the graph diverges?
2. Give the definition of (i) a Wigner random matrix and (ii) the Gaussian Orthogonal Ensemble of random matrices. What is the shape of the density of eigenvalues in the limit of large matrices?
3. Give the Harris criterion for a second order phase transition

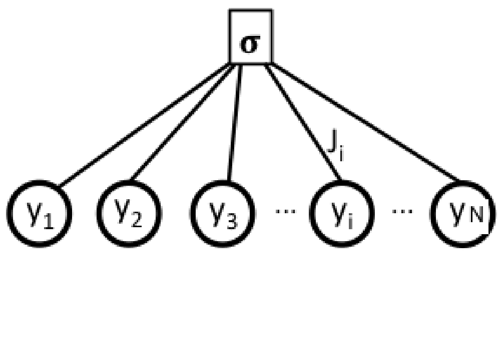
2 The capacity of a perceptron in classifying random data

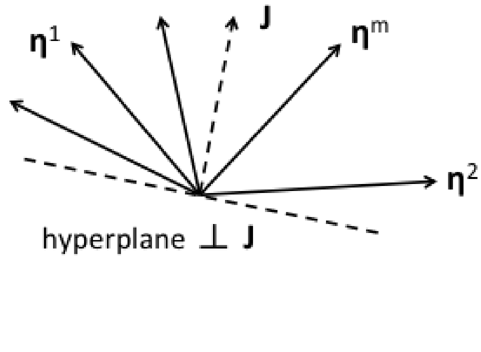
We consider in this exercise the *perceptron*, one of the simplest artificial neural networks, whose architecture is inspired by that of real neurons. A perceptron takes as input N real variables $\boldsymbol{y} = (y_1, y_2, \cdots, y_N)$, which are multiplied by weights $\boldsymbol{J} = (J_1, J_2, \cdots, J_N)$, which are also real, to construct the output as

$$\sigma(\boldsymbol{y}) = \mathrm{sign}\left(\sum_{i=1}^{N} J_i y_i\right) = \mathrm{sign}(\boldsymbol{J}\cdot\boldsymbol{y}) \ . \tag{1}$$

We now consider the following problem: we are given M training examples, $\{\boldsymbol{y}^m, \sigma^m\}$, and we want to find a set of weights $\boldsymbol{J}$ such that all examples are correctly classified, i.e., $\sigma^m = \mathrm{sign}(\boldsymbol{J}\cdot\boldsymbol{y}^m)$ for all $m = 1, \cdots, M$.

As a benchmark, we consider a totally random training set: the components y_i^m of the vectors $\boldsymbol{y}^m$ are i.i.d. Gaussian random variables with zero mean and unit variance, and the "labels" σ^m are chosen to be ± 1 with probability 1/2.





1. Show that the problem is equivalent to finding a solution to the set of inequalities:
$$\boldsymbol{\eta}^m \cdot \boldsymbol{J} \geq 0 \ , \qquad m = 1, \cdots, M \ , \tag{2}$$
for a proper choice of the "patterns" $\boldsymbol{\eta}^m$. Show that this amounts to find a hyperplane in the N dimensional space going through the origin, such that all vectors $\boldsymbol{\eta}^m$ lie on the same side of it, as shown in the figure.

2. The problem is obviously independent of the norm of the vector $\boldsymbol{J}$. Without loss of generality we then normalize $\boldsymbol{J}$ such that $|\boldsymbol{J}|^2 = N$. Consider now the following generalization of the problem in Eq. (2),
$$\boldsymbol{\eta}^m \cdot \boldsymbol{J} \geq \kappa\sqrt{N} \ , \qquad m = 1, \cdots, M \ , \qquad \kappa > 0 \ . \tag{3}$$
Suppose that a solution $\boldsymbol{J}$ of Eq. (3) has been found. Suppose then that an independent random noise $\boldsymbol{\epsilon}^m$, with $|\boldsymbol{\epsilon}^m| < \varepsilon$, is added to each pattern, $\boldsymbol{\eta}^m \to \boldsymbol{\eta}^m + \boldsymbol{\epsilon}^m$. Show that if $\varepsilon < \kappa$, then the vector $\boldsymbol{J}$ and the noisy patterns still satisfy Eq. (2). Conclude that a perceptron trained using Eq. (3) is the more robust to noise, the larger the value of κ.

3. We want to compute the *capacity* of a perceptron trained using Eq. (3), i.e. the maximum M such that a solution for $\boldsymbol{J}$ exists. We introduce a Hamiltonian and its associated partition function,
$$H[\boldsymbol{J}] = \sum_{m=1}^{M} v\left[\frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}}{\sqrt{N}} - \kappa\right] \ , \qquad Z = \int \mathrm{d}\boldsymbol{J}\delta(|\boldsymbol{J}|^2 - N)e^{-\beta H[\boldsymbol{J}]} \ , \tag{4}$$
where the function $v(x) = 0$ for $x \geq 0$ and $v(x) > 0$ for $x < 0$. The ground state energy of $H[\boldsymbol{J}]$ is then equal to zero if at least a solution of Eq. (3) exists, and is positive otherwise. Discuss the analogy with the constraint satisfaction problems introduced in the lectures. What do you expect for the ground state energy
$$e(\alpha) = \min_{\boldsymbol{J}} H[\boldsymbol{J}]/N \ , \tag{5}$$
in the thermodynamic limit $N \to \infty$ with constant $\alpha = M/N$?

4. We are now going to compute $e(\alpha)$ analytically, using the replica method. Our "dynamical variable" is the vector $\boldsymbol{J}$, while our "quenched disorder" are the pattern $\boldsymbol{\eta}^m$. Introducing auxiliary variables $r_a^m = \frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}_a}{\sqrt{N}}$, show that the replicated partition function associated to the Hamiltonian in Eq. (4) can be written as
$$\mathbb{E}[Z^n] = \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a\delta(|\boldsymbol{J}_a|^2 - N)\right] \int \left[\prod_{m=1}^{M}\prod_{a=1}^{n} \mathrm{d}r_a^m e^{-\beta v(r_a^m - \kappa)}\right] \mathbb{E}\left[\prod_{m=1}^{M}\prod_{a=1}^{n} \delta\left(r_a^m - \frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}_a}{\sqrt{N}}\right)\right] \ , \tag{6}$$
where $\mathbb{E}[\bullet]$ denotes the expectation over the random patterns, which have i.i.d. normal components.

5. Use the integral representation of a delta function,
$$\delta(x) = \int \frac{\mathrm{d}\hat{x}}{2\pi} e^{ix\hat{x}} \ , \tag{7}$$
to represent each of the $\delta\left(r_a^m - \frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}_a}{\sqrt{N}}\right)$ as an integral over another auxiliary variable $\hat{r}_a^m$, to obtain (neglecting proportionality constants):
$$\mathbb{E}[Z^n] \propto \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a\delta(|\boldsymbol{J}_a|^2 - N)\right] \int \left[\prod_{m=1}^{M}\prod_{a=1}^{n} \mathrm{d}r_a^m \mathrm{d}\hat{r}_a^m e^{-\beta v(r_a^m - \kappa) + ir_a^m \hat{r}_a^m}\right] \mathbb{E}\left[e^{-\sum_{m=1}^{M}\sum_{a=1}^{n} i\hat{r}_a^m \frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}_a}{\sqrt{N}}}\right] \ . \tag{8}$$

6. Perform the integration over the Gaussian disorder and show that the last term in the above equation is
$$\mathbb{E}\left[e^{-\sum_{m=1}^{M}\sum_{a=1}^{n} i\hat{r}_a^m \frac{\boldsymbol{\eta}^m \cdot \boldsymbol{J}_a}{\sqrt{N}}}\right] = e^{-\frac{1}{2}\sum_{m=1}^{M}\sum_{a,b}^{1,n} \hat{r}_a^m \hat{r}_b^m Q_{ab}} \ , \qquad Q_{ab} = \frac{1}{N}\boldsymbol{J}_a \cdot \boldsymbol{J}_b \ . \tag{9}$$
(To make it simple, recall that because the disorder is Gaussian, it is enough to prove the above equality at second order in $\boldsymbol{J}$.)

7. Insert the previous result into Eq. (8) to obtain
$$\mathbb{E}[Z^n] \propto \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a\delta(|\boldsymbol{J}_a|^2 - N)\right] \left\{\int \left(\prod_{a=1}^{n} \mathrm{d}r_a \mathrm{d}\hat{r}_a\right) e^{-\sum_{a=1}^{n} \beta v(r_a - \kappa) + \sum_{a=1}^{n} ir_a\hat{r}_a - \frac{1}{2}\sum_{a,b}^{1,n} \hat{r}_a\hat{r}_b Q_{ab}}\right\}^M \tag{10}$$

8. Change variable in the integration from $\boldsymbol{J}_a$ to $Q_{ab} = \frac{1}{N}\boldsymbol{J}_a \cdot \boldsymbol{J}_b$. It can be shown that the Jacobian of the change of variable is $(\det Q)^{N/2}$, i.e. for any function f,

$$\int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a\right] f[\boldsymbol{J}_a \cdot \boldsymbol{J}_b/N] = \int \mathrm{d}Q (\det Q)^{N/2} f[Q_{ab}] \ , \qquad \mathrm{d}Q = \prod_{a\leq b} \mathrm{d}Q_{ab} \ . \tag{11}$$

Note that the constraint $|\boldsymbol{J}_a|^2 = N$ is equivalent to $Q_{aa} = 1$. Conclude that

$$\mathbb{E}[Z^n] \propto \int \mathrm{d}Q e^{NS(Q)} \ , \qquad \mathrm{d}Q = \prod_{a<b} \mathrm{d}Q_{ab} \ . \tag{12}$$

Give the expression of $S(Q)$ and discuss the result, making a parallel with the lectures.

9. Recall the motivation for a replica symmetric ansatz, $Q_{ab} = \delta_{ab} + q(1-\delta_{ab})$, and use it in Eq. (12). Show that

$$\det Q = (1-q)^{n-1}[1+(n-1)q] \ . \tag{13}$$

Making use of the following Gaussian identities

$$\mathcal{D}_q z = \frac{e^{-\frac{z^2}{2q}}}{\sqrt{2\pi q}}\mathrm{d}z \ , \qquad e^{-\frac{q}{2}\left(\sum_a \hat{r}_a\right)^2} = \int \mathcal{D}_q z \, e^{-iz\sum_a \hat{r}_a} \ , \qquad \int \mathrm{d}\hat{r} e^{-\frac{1-q}{2}\hat{r}^2 + i\hat{r}r} = \sqrt{\frac{2\pi}{1-q}} e^{-\frac{r^2}{2(1-q)}} \ , \tag{14}$$

such that $\int \mathcal{D}_q z = 1$, conclude that, neglecting irrelevant constant terms (i.e., which are independent of q),

$$S(Q) = \frac{n-1}{2}\log(1-q) + \frac{1}{2}\log[1+(n-1)q] + \alpha \log\left\{\int \mathcal{D}_q z \left[\int \mathcal{D}_{1-q} r \, e^{-\beta v(r+z-\kappa)}\right]^n\right\} \ . \tag{15}$$

10. From Eq. (15) extract the replica symmetric free energy,

$$f_{\mathrm{RS}}(q) = -\frac{T}{2}\left[\log(1-q) + \frac{q}{1-q}\right] - T\alpha \int \mathcal{D}_q z \log\left[\int \mathcal{D}_{1-q} r \, e^{-\beta v(r+z-\kappa)}\right] \ . \tag{16}$$

11. We consider from now on a smooth potential $v(x)$ of the form

$$v(x) = \begin{cases} \frac{1}{2}x^2 & x<0 \ , \\ 0 & x \geq 0 \ . \end{cases} \tag{17}$$

Explain why, when the ground state of the Hamiltonian has positive energy, we expect $q \to 1$ when $T \to 0$. We call this the UNSAT phase, in which there is no solution for $\boldsymbol{J}$. Also explain why, when the ground state of the Hamiltonian has zero energy, we generically expect $q < 1$ for $T \to 0$. We call this the SAT phase.

12. Consider the UNSAT phase and assume that $q = 1 - \chi T + O(T^2)$ when $T \to 0$. By using a saddle point computation, show that

$$\lim_{T\to 0} -T\log\left[\int \mathcal{D}_{\chi T} r \, e^{-\beta v(r+z-\kappa)}\right] = \frac{(z-\kappa)^2}{2(1+\chi)}\theta(\kappa - z) \ , \tag{18}$$

where $\theta(x)$ is the Heaviside step function. Conclude that the ground state energy in the replica symmetric approximation is

$$e(\alpha) = -\frac{1}{2\chi} + \frac{\alpha}{2(1+\chi)}\int_{-\infty}^{\kappa} \mathrm{d}z \, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}(z-\kappa)^2 \ . \tag{19}$$

13. The value of χ is determined by stationarity, hence $\partial e/\partial\chi = 0$. Show that

$$\left(1+\frac{1}{\chi}\right)^2 = \frac{\alpha}{\alpha_c(\kappa)} \ , \qquad \alpha_c(\kappa) = \left[\int_{-\infty}^{\kappa} \mathrm{d}z \, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}(z-\kappa)^2\right]^{-1} \ . \tag{20}$$

Plug the result for χ in Eq. (19) and show that

$$e(\alpha) = \frac{1}{2}\left(\sqrt{\frac{\alpha}{\alpha_c(\kappa)}} - 1\right)^2 \ . \tag{21}$$

Plot the curves of $e(\alpha)$ at fixed κ, and of $\alpha_c(\kappa)$ as a function of κ. Discuss and interpret the results.