Homework 1

* each $\hat{X}_i$ falls in $[u,v]$, independently or the others, with probability

$$\mathbb{P}\left[\hat{X}\in[u,v]\right] = \mathbb{P}\left[X\in[a_n+b_n u,\ a_n+b_n v]\right] = F_X(a_n+b_n v) - F_X(a_n+b_n u) \equiv p_n([u,v])$$

hence $N_n([u,v]) \overset{d}{=} \mathrm{Bin}\left(n,\ p_n([u,v])\right)$

when $n\to\infty$, $p_n([u,v]) = \frac{1}{n}\left(\gamma(u)-\gamma(v)\right) + o\left(\frac{1}{n}\right) \quad \Rightarrow \quad N([u,v]) \overset{d}{=} \mathcal{P}o\left(\gamma(u)-\gamma(v)\right)$

* independently or each other, each $\hat{X}_i$ falls in $[u_1,v_1]$ with proba $p_n([u_1,v_1])$

$[u_2,v_2]$ $\quad p_n([u_2,v_2])$

...

$[u_p,v_p]$ $\quad p_n([u_p,v_p])$

hence $\left(N_n([u_1,v_1]),\dots,N_n([u_p,v_p])\right) \overset{d}{=} \mathrm{Multinomial}\left(n,\ p_n([u_1,v_1]),\dots,p_n([u_p,v_p])\right)$

ie
$$\mathbb{P}\left[N_n([u_1,v_1])=k_1,\dots,N_n([u_p,v_p])=k_p\right] = \frac{n!}{k_1!\dots k_p!\,(n-k_1-\dots-k_p)!}\, p_n([u_1,v_1])^{k_1}\dots p_n([u_p,v_p])^{k_p} \times$$
$$\left(1-p_n([u_1,v_1])-\dots-p_n([u_p,v_p])\right)^{n-k_1-\dots-k_p}$$

$$\xrightarrow[n\to\infty]{} e^{-(\gamma(u_1)-\gamma(v_1))}\frac{\left(\gamma(u_1)-\gamma(v_1)\right)^{k_1}}{k_1!}\ \cdots\ e^{-(\gamma(u_p)-\gamma(v_p))}\frac{\left(\gamma(u_p)-\gamma(v_p)\right)^{k_p}}{k_p!}$$

in the limit the $N([u,v])$ are independent Poisson rando variables :

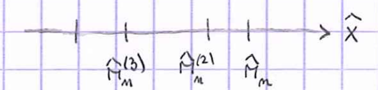
the $\{\hat{X}_i\}$ form a Poisson Point Process, with intensity measure $\mu([u,v]) = \gamma(u)-\gamma(v)$

where γ is one of the three universal extreme value distribution

* link with the question: $\mathbb{P}\left[\hat{M}_n\leq x\right] = \mathbb{P}\left[N_n([x,\infty[)=0\right] \xrightarrow[n\to\infty]{} \mathbb{P}\left[\mathcal{P}o(\mu([x,\infty[)=0\right]$

$= e^{-\gamma(x)}$ same result as in TD

call $\hat{M}_n^{(2)}$ the second largest



$$\mathbb{P}\left[\hat{M}_n^{(2)}\leq x\right] = \mathbb{P}\left[N_n([x,\infty[)=0 \text{ or } 1\right] \to \mathbb{P}\left[\mathcal{P}o\left(\mu([x,\infty[)\right)=0 \text{ or } 1\right]$$

$$= e^{-\gamma(x)}\left(1+\gamma(x)\right)$$

in general, $\mathbb{P}\left[\hat{M}_n^{(k)}\leq x\right] \to \mathbb{P}\left[\mathcal{P}o\left(\mu[x,\infty[\right)\leq k-1\right] = e^{-\gamma(x)}\left(1+\gamma(x)+\frac{1}{2!}\gamma(x)^2+\dots+\frac{1}{(k-1)!}\gamma(x)^{k-1}\right)$