ICFP M2 - Statistical physics 2 – TD n° 2
The Random Energy Model - Solution

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1 Preamble: concentration of random variables

1. Denoting $\mathbb{I}(E)$ the indicator function of the event E we bound the expected value of X as

$$\mathbb{E}[X] = \mathbb{E}[X\,\mathbb{I}(X \geq a)] + \mathbb{E}[X\,\mathbb{I}(X < a)] \geq \mathbb{E}[X\,\mathbb{I}(X \geq a)] \geq a\,\mathbb{E}[\mathbb{I}(X \geq a)] = a\,\mathbb{P}[X \geq a] \ ,$$

where the first inequality holds because X is a positive random variable. The Markov inequality follows by dividing by a.

2. If we apply the Markov inequality to the random variable $Y = (X - \mathbb{E}[X])^2$, which is clearly positive, we get

$$\mathbb{P}[Y \geq a] \leq \frac{1}{a}\mathbb{E}[Y] = \frac{1}{a}\mathrm{Var}[X] \ . \tag{1}$$

We can then obtain the Chebychev inequality as

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t\,\sqrt{\mathrm{Var}[X]}] = \mathbb{P}[(X - \mathbb{E}[X])^2 \geq t^2\,\mathrm{Var}[X]] \leq \frac{1}{t^2} \ , \tag{2}$$

applying (1) with $a = t^2\,\mathrm{Var}[X]$.

3. Note that for a random variable X taking values in 0,1, ..., one has $X > 0$ if and only if $X \geq 1$. Hence the Markov inequality with $a = 1$ immediately gives

$$\mathbb{P}[X > 0] \leq \mathbb{E}[X] \ . \tag{3}$$

From Chebychev inequality with $t = \frac{\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}}$ we obtain, for any random variable admitting a variance,

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \tag{4}$$

Because $|X - \mathbb{E}[X]| \geq \mathbb{E}[X] \Leftrightarrow X \leq 0 \ \text{ or } \ X \geq 2\,\mathbb{E}[X]$, this can be rewritten

$$\mathbb{P}[X \leq 0] + \mathbb{P}[X \geq 2\,\mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \tag{5}$$

For the non-negative integer valued random variable considered here $X \leq 0$ if and only if $X = 0$, and the probability of $X \geq 2\,\mathbb{E}[X]$ is a non-negative number, hence

$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ , \qquad \text{or equivalently} \qquad \mathbb{P}[X > 0] \geq 1 - \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \tag{6}$$

To obtain an improved bound we shall use the Cauchy-Schwarz inequality, which states that for two random variables A and B one has $\mathbb{E}[AB] \leq \sqrt{\mathbb{E}[A^2]}\sqrt{\mathbb{E}[B^2]}$. Applying this to $A = X$, $B = \mathbb{I}(X > 0)$ yields, for these non-negative integer valued random variables X,

$$\mathbb{E}[X] = \mathbb{E}[X\mathbb{I}(X > 0)] \leq \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[\mathbb{I}(X > 0)^2]} = \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{P}[X > 0]} \ ; \tag{7}$$

in the last step we used the fact that the square of an indicator function is equal to itself. Squaring this inequality and dividing it by $\mathbb{E}[X^2]$ gives finally

$$\frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \leq \mathbb{P}[X > 0] \ , \qquad \text{i.e.} \qquad \mathbb{P}[X > 0] \geq 1 - \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]} \ , \tag{8}$$

which is stronger than the inequality (6) obtained from Chebychev, because $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$.

2 The free-energy of the REM

1. Recall that $Z(\beta) = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}$ and $H(\underline{\sigma})$ are 2^N independent random Gaussian variables with $\mathbb{E}[H] = 0$ and $\mathbb{E}[H^2] = N/2$. Then

$$\mathbb{E}[Z] = 2^N \mathbb{E}[e^{-\beta H}] = 2^N \int \mathrm{d}H \frac{e^{-\frac{H^2}{N}}}{\sqrt{\pi N}} e^{-\beta H} = 2^N e^{N\beta^2/4} \ , \tag{9}$$

and

$$f_{\mathrm{a}}(\beta) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\beta)] = -\frac{\beta}{4} - \frac{1}{\beta} \log 2 \ . \tag{10}$$

2. Each of the H variables can independently fall in the interval $H(\underline{\sigma}) \in [Nu, N(u + \mathrm{d}u)]$ with probability

$$p = \int_{Nu}^{N(u+\mathrm{d}u)} \mathrm{d}H \frac{e^{-\frac{H^2}{N}}}{\sqrt{\pi N}} \approx \frac{e^{-Nu^2}}{\sqrt{\pi N}} N\mathrm{d}u = \sqrt{\frac{N}{\pi}} e^{-Nu^2} \mathrm{d}u \ . \tag{11}$$

Hence, $\mathcal{N}(u, \mathrm{d}u)$ is a binomial variable with parameter p,

$$\mathbb{P}[\mathcal{N}(u, \mathrm{d}u) = k] = \mathrm{Binom}(k; 2^N, p) \ , \tag{12}$$

and its average and variance are

$$\mathbb{E}[\mathcal{N}(u, \mathrm{d}u)] = 2^N p \ , \qquad \mathrm{Var}[\mathcal{N}(u, \mathrm{d}u)] = 2^N p(1-p) \ . \tag{13}$$

3. At leading exponential order we have

$$\mathbb{E}[\mathcal{N}(u, \mathrm{d}u)] \sim \mathrm{Var}[\mathcal{N}(u, \mathrm{d}u)] \sim e^{N(\log 2 - u^2)} \ . \tag{14}$$

Hence:

- If $u^2 > \log 2$, both average and variance of $\mathcal{N}(u, \mathrm{d}u)$ tend to zero exponentially fast when $N \to \infty$. From Eq. (3) it follows that $\mathbb{P}[\mathcal{N}(u, \mathrm{d}u) > 0]$ also decays exponentially with N, and in the thermodynamic limit the typical value of $\mathcal{N}(u, \mathrm{d}u)$ is zero.
- If $u^2 < \log 2$, both average and variance of $\mathcal{N}(u, \mathrm{d}u)$ grow exponentially with N, and the relative standard deviation

$$\frac{\sqrt{\mathrm{Var}[\mathcal{N}(u, \mathrm{d}u)]}}{\mathbb{E}[\mathcal{N}(u, \mathrm{d}u)]} \sim \frac{1}{\sqrt{\mathbb{E}[\mathcal{N}(u, \mathrm{d}u)]}} \sim e^{-\frac{N}{2}(\log 2 - u^2)} \to 0 \tag{15}$$

exponentially fast. Hence, $\mathcal{N}(u, \mathrm{d}u)$ concentrates around its typical value

$$\mathcal{N}(u, \mathrm{d}u) \sim \mathcal{N}_{\mathrm{typ}}(u, \mathrm{d}u) \sim \mathbb{E}[\mathcal{N}(u, \mathrm{d}u)] \sim e^{N(\log 2 - u^2)} \ , \tag{16}$$

with fluctuations that are exponentially small in N.

We conclude that, at the leading exponential order,

$$\mathcal{N}_{\mathrm{typ}}(u, \mathrm{d}u) = \begin{cases} e^{Ns_{\mathrm{m}}(u)} \mathrm{d}u & \text{if } \ u \in [-u_{\mathrm{c}}, u_{\mathrm{c}}] \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where $u_{\mathrm{c}} = \sqrt{\log 2}$ and $s_{\mathrm{m}}(u) = \log 2 - u^2$ is the microcanonical entropy.

4. The typical value of Z is then

$$Z_{\mathrm{typ}} = \int \mathrm{d}u \, \mathcal{N}_{\mathrm{typ}}(u, \mathrm{d}u) e^{-\beta N u} = \int_{-u_{\mathrm{c}}}^{u_{\mathrm{c}}} \mathrm{d}u \, e^{N(s_{\mathrm{m}}(u) - \beta u)} \approx e^{N \max_{u \in [-u_{\mathrm{c}}, u_{\mathrm{c}}]} (s_{\mathrm{m}}(u) - \beta u)} \ , \tag{18}$$

with exponentially small relative fluctuations around this value, which implies

$$Z \sim Z_{\rm typ}[1+\xi O(e^{-bN})] \quad \Rightarrow \quad \log Z \sim \log Z_{\rm typ}+\log[1+\xi O(e^{-bN})] \sim \log Z_{\rm typ}+\xi O(e^{-bN}) \ , \tag{19}$$

where ξ is a random variable of order one and b is a constant of order one, when $N \to \infty$. We deduce that

$$\begin{aligned} f_{\rm q}(\beta) &= -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \mathbb{E}[\log Z(\beta)] = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \log Z_{\rm typ} \\ &= -\frac{1}{\beta} \max_{u\in[-u_{\rm c},u_{\rm c}]} (s_{\rm m}(u) - \beta u) = \min_{u\in[-u_{\rm c},u_{\rm c}]} (u - T s_{\rm m}(u)) \ . \end{aligned} \tag{20}$$

We recognize the usual formula for the canonical free energy in terms of the microcanonical entropy, with the important remark that there are no microcanonical states of energy $|u| > u_{\rm c}$.

Recalling that $s_{\rm m}(u) = \log 2 - u^2$, we have to find the minimum over $u \in [-u_{\rm c}, u_{\rm c}]$ of $g(u) = u - T(\log 2 - u^2)$. We have that $g'(u) = 1 + 2Tu$ is linear, with $g'(-u_{\rm c}) = 1 - 2\sqrt{\log 2}T$ and $g'(u_{\rm c}) = 1 + 2\sqrt{\log 2}T > 0$. Hence, if $T > T_{\rm c} = 1/(2\sqrt{\log 2})$, we have $g'(-u_{\rm c}) < 0$ and there is a local mimimum of $g(u)$ at $u^* = -1/(2T) \in [-u_{\rm c}, 0]$. In that case, we get

$$f_{\rm q}(\beta) = u^* - T s_{\rm m}(u^*) = -\frac{\beta}{4} - \frac{\log 2}{\beta} \ , \qquad T > T_{\rm c} \ . \tag{21}$$

Note that this expression coincides with the annealed free energy. Instead, if $T < T_{\rm c}$, we have $g'(u) > 0$ for all $u \in [-u_{\rm c}, u_{\rm c}]$ and the minimum of $g(u)$ is then in $u^* = -u_{\rm c}$. Recalling that $s_{\rm m}(\pm u_{\rm c}) = 0$, we get

$$f_{\rm q}(\beta) = -u_{\rm c} = -\sqrt{\log 2} \ , \qquad T < T_{\rm c} \ . \tag{22}$$

5. The values of energy and entropy can be immediately derived from the above reasoning, recalling that if the free energy is $f_{\rm q}(\beta) = \min_{u\in[-u_{\rm c},u_{\rm c}]}(u - T s_{\rm m}(u))$, then $e = u^*$ is the energy and $s = s_{\rm m}(u^*)$ is the entropy. Alternatively, one can use the thermodynamic relations $e = {\rm d}(\beta f)/{\rm d}\beta$ and $s = -{\rm d}f/{\rm d}T$. In both cases, the results are

$$e = \begin{cases} -\frac{1}{2T} & T > T_{\rm c} \\ -\sqrt{\log 2} & T < T_{\rm c} \end{cases} \qquad\qquad s = \begin{cases} \log 2 - \frac{1}{4T^2} & T > T_{\rm c} \\ 0 & T < T_{\rm c} \end{cases} \tag{23}$$

We see that both energy and entropy are continuous at $T_{\rm c}$, but their derivatives have a jump, hence the transition is of second order thermodynamically.

6. The ground state energy density is $e_{\rm GS} = e(T = 0) = -\sqrt{\log 2} = -u_{\rm c}$. Indeed, below $-u_{\rm c}$ there are typically no states in the thermodynamic limit. Note that in the whole low temperature phase $T < T_{\rm c}$, the system is 'condensed' in its ground state and the entropy is zero.

   Remember that

$$e_{\rm GS} = \frac{1}{N} \min(H_1, \cdots, H_{2^N}) = \frac{1}{N}\sqrt{\frac{N}{2}} \min(X_1, \cdots, X_{2^N}) \ , \tag{24}$$

   where H_i are $n = 2^N$ i.i.d. Gaussian variables with zero mean and variance $N/2$, hence X_i has zero mean and unit variance. According to the results of TD1, the minimum (which is minus the maximum) $M_n = -(a_n + b_n\widehat{x})$, where at leading order $a_n \sim \sqrt{2\log n} = \sqrt{2N\log 2}$, $b_n \to 0$ and $\widehat{x}$ of order 1. Hence, $M_n \sim -\sqrt{2N\log 2}$ and

$$e_{\rm GS} = -\frac{1}{N}\sqrt{\frac{N}{2}}\sqrt{2N\log 2} = -\sqrt{\log 2} \ . \tag{25}$$

3 The structure of $p(\underline{\sigma})$

1. If $p(\underline{\sigma})$ is uniform on a subset $\mathcal{M}$ of M_{eff} configurations, we have $p(\underline{\sigma}) = \mathbb{I}(\underline{\sigma} \in \mathcal{M})/M_{\text{eff}}$. Hence,

$$Y = \sum_{\underline{\sigma}} p(\underline{\sigma})^2 = \frac{1}{M_{\text{eff}}^2} \sum_{\underline{\sigma}} \mathbb{I}(\underline{\sigma} \in \mathcal{M})^2 = \frac{1}{M_{\text{eff}}^2} M_{\text{eff}} = \frac{1}{M_{\text{eff}}} \ . \tag{26}$$

This is intuitive, because Y is the probability to find two independent copies of the system in the same configuration: the second configuration has a probability $1/M_{\text{eff}}$ to be chosen to be equal to the first.

2. Using the Gibbs form for $p(\underline{\sigma})$, we have

$$Y = \frac{1}{Z(\beta)^2} \sum_{\underline{\sigma}} e^{-2\beta H(\underline{\sigma})} = \frac{Z(2\beta)}{Z(\beta)^2} \ . \tag{27}$$

The typical value is $Z(\beta) \sim e^{-\beta N f_{\text{q}}(\beta)}$. Hence, the typical value of Y is

$$Y = e^{-2\beta N [f_{\text{q}}(2\beta) - f_{\text{q}}(\beta)]} \ . \tag{28}$$

Recall that for $\beta < \beta_{\text{c}}$ we have $f_{\text{q}}'(\beta) = -1/4 + \log 2/\beta^2 > 0$, while for $\beta > \beta_{\text{c}}$ we have that $f_{\text{q}}(\beta)$ is constant. Hence, if $\beta < \beta_c$, we have $f_{\text{q}}(2\beta) > f_{\text{q}}(\beta)$ and $Y \to 0$ exponentially fast in N. For $\beta > \beta_c$ we have $f_{\text{q}}(2\beta) = f_{\text{q}}(\beta)$ and in order to study the behavior of Y we need to compute subleading corrections.

3. The identity given in the text is proven by integration by parts. If X is a centered Gaussian of variance $\sigma^2 = \mathbb{E}[X^2]$, then

$$\begin{aligned} \mathbb{E}[XF(X)] &= \int \mathrm{d}x \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} x F(x) = \int \mathrm{d}x F(x) \left(-\sigma^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \right) \\ &= \sigma^2 \int \mathrm{d}x F(x) \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} = \mathbb{E}[X^2] \mathbb{E}[F'(X)] \ . \end{aligned} \tag{29}$$

Recalling that $H(\underline{\sigma})$ are i.i.d. Gaussians of variance $N/2$, and using the above identity with $F(H) = e^{-\beta H}/Z$ (remember that Z also depends on H), we have for the thermodynamic energy

$$e = \frac{1}{N} \sum_{\underline{\sigma}} \mathbb{E}\left[H(\sigma) \frac{e^{-\beta H(\sigma)}}{Z(\beta)} \right] = \frac{1}{N} \sum_{\underline{\sigma}} \frac{N}{2} \mathbb{E}\left[-\beta \frac{e^{-\beta H(\sigma)}}{Z(\beta)} + \beta \frac{e^{-2\beta H(\sigma)}}{Z(\beta)^2} \right] = -\frac{\beta}{2} \mathbb{E}[1 - Y] \ . \tag{30}$$

4. Recalling the result of the energy in Eq. (23), we have

$$\lim_{N\to\infty} \mathbb{E}[Y] = 1 + 2Te = \begin{cases} 0 & T > T_{\text{c}} \\ 1 - \sqrt{2\log 2}T = 1 - \frac{T}{T_{\text{c}}} & T < T_{\text{c}} \end{cases} \tag{31}$$

Hence, in the high temperature phase Y vanishes, indicating that two equilibrium configurations have zero probability of being identical, because the space of possible configurations grows exponentially in N. In the low temperature phase, Y is finite, indicating that the space of possible configurations is finite and the effective number of configurations is given by $M_{\text{eff}} = 1/Y = 1/(1 - T/T_{\text{c}})$. In particular, at zero temperature, there is a unique ground state.