Statistical Physics 2: Disordered Systems and Interdisciplinary Applications 29.01.2021

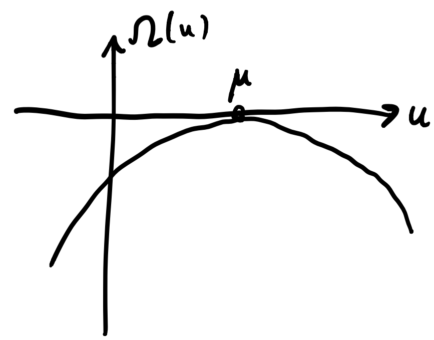
I Large deviations and the replica method

X st. $\mathbb{E}(X) = \mu \qquad Var(X) = \sigma^2$

$$u_N = \frac{1}{N}(X_1 + \dots + X_N) = \mu + \frac{1}{\sqrt{N}}\sigma z^{\sim \mathcal{N}(0,1)}$$ Typical fluctuations

Large deviations $u_N - \mu \sim O(1)$

$$p(u) \sim e^{N\Omega(u)}$$



$$\Omega(u) = \min_\lambda [m(\lambda) - \lambda u]$$

$$m(\lambda) = \log M(\lambda)$$

$$M(\lambda) = \mathbb{E}[e^{\lambda X}]$$

For a disordered system:

$$f_N = -\frac{T}{N}\log Z_N$$

1) is a random variable

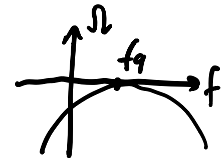
2) $\mathbb{E}(f_N) \sim O(1)$

3) At least for finite $Var(f_N) \sim \frac{1}{N}$

$$f_q = \lim_{N\to\infty} \mathbb{E}[f_N]$$

$$f_a = \lim_{N\to\infty} -\frac{T}{N}\log \mathbb{E}[Z_N]$$

Assume $\quad p(f) \sim e^{N\Omega(f)} \quad$ for large N



$$Z = e^{-N\beta f} \quad \Rightarrow \quad \mathbb{E}(Z^n) = \mathbb{E}(e^{-\beta n N f})$$
$$= \int df \; e^{N[\Omega(f) - \beta n f]}$$

Define

$$f_n = \lim_{N\to\infty} -\frac{T}{nN} \log \mathbb{E}(Z^n) = -\frac{T}{n} \max_f \left[\Omega(f) - \beta n f\right]$$
$$= \min_f \left[f - \frac{T}{n}\Omega(f)\right]$$

$$f_a = f_1$$

<u>Quenched f:</u>

a) $$\mathbb{E}(Z^n) = \mathbb{E}(e^{n\log Z}) \sim 1 + n\,\mathbb{E}(\log Z)$$

$$-\frac{T}{nN}\log \mathbb{E}[Z^n] \underset{n\sim 0}{\sim} -\frac{T}{\cancel{n}N}\cancel{n}\,\mathbb{E}(\log Z) = f_q$$

b) $$\Omega(f) \sim -\frac{A}{2}(f - f_q)^2$$

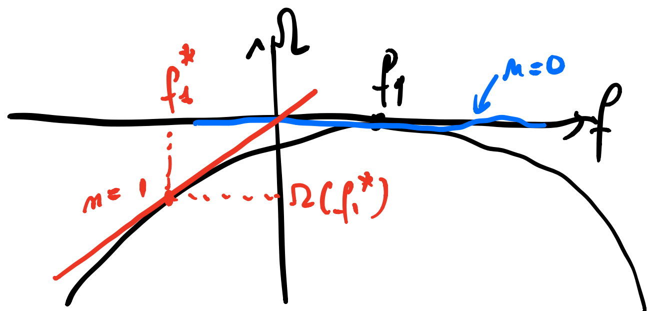
$$f_n = \min_f \left[f + \frac{T}{n}\frac{A}{2}(f - f_q)^2\right]$$

$$1 + \frac{T}{n}A\,(f - f_q) = 0 \qquad f = f_q - \frac{n}{TA}$$

$$f_n = f_q - \frac{n}{TA} + \frac{T}{\cancel{n}}\frac{A}{2}\frac{n^{\cancel{2}}}{T^2A^2} \xrightarrow[n\to 0]{} f_q$$

More generally:

$$\Omega'(f) = \frac{n}{T}$$



$$f_a = f_1^* - T\,\Omega(f_1^*) \qquad \text{with} \quad \Omega'(f_1^*) = 1/T$$

$$f_a - f_q = f_1^* - \frac{\Omega(f_1^*)}{\Omega'(f_1^*)} - f_q = 0 \iff \Omega(f_1^*) = \Omega'(f_1^*)(f_q - f_1^*)$$

only if $\Omega(f)$ is linear in $[f_1^*, f_q]$

If instead:

$$p(f) \sim e^{N^\alpha \omega(f)} \qquad \alpha > 1 \qquad \Rightarrow f_a = f_q$$

narrower than a large deviation principle

Recap:

1) $$f_q = \lim_{N\to\infty}\left[-\frac{T}{N}\lim_{n\to 0}\frac{1}{n}\log \mathbb{E}(z^n)\right] \quad \text{(trivial)}$$

2) If $p(f) \sim e^{N\Omega(f)}$ we can exchange $\lim\limits_{n\to 0}\lim\limits_{N\to\infty}$

$$f_q = \lim_{n\to 0} \lim_{N\to\infty} -\frac{T}{Nn} \log \mathbb{E}(Z^n) \neq f_a$$

3) If $p(f) \sim e^{N^\alpha w(f)}$ $\alpha > 1$

$$f_q = f_a$$

II The REM with replicas

1. Calculation of f_n

$\underline{\sigma} = (\pm 1 \cdots \pm 1)$ 2^N configurations

$H(\underline{\sigma})$ i.i.d. Gaussians $\mathbb{E}[H(\underline{\sigma})] = 0$

$$\mathbb{E}[H(\underline{\sigma})H(\underline{\tau})] = \frac{N}{2}\delta_{\underline{\sigma}\underline{\tau}}$$

$n = 1, 2, 3, \ldots$

$$\mathbb{E}(Z^n) = \mathbb{E}\left[\left(\sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}\right)^n\right] = \sum_{\underline{\sigma}_1 \cdots \underline{\sigma}_n} \mathbb{E}\left(e^{-\beta H(\underline{\sigma}_1) \cdots - \beta H(\underline{\sigma}_n)}\right)$$

We need to count how many replicas are in the same config.

Encoded in a matrix $\Delta_{ab} = \begin{cases} 1 & \text{if } \underline{\sigma}_a = \underline{\sigma}_b \\ 0 & \text{if } \underline{\sigma}_a \neq \underline{\sigma}_b \end{cases} = \delta_{\underline{\sigma}_a \underline{\sigma}_b}$

$$\Delta_{ab} = \begin{array}{c} \begin{matrix} \underbrace{\hphantom{1\ 1}}_{m_1} & \underbrace{\hphantom{0\ 0\ 0}}_{m_2} & \underbrace{\hphantom{0\ 0}}_{m_3} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

a $n \times n$ matrix

$$\mathbb{E}\left(e^{-\beta \sum_a H(\underline{\sigma}_a)}\right) = \mathbb{E}\left(e^{-\beta(n_1 E_1 + n_2 E_2 + \cdots + n_k E_k)}\right)$$

$$= \mathbb{E}\left(e^{-\beta n_1 E_1}\right) \cdots \mathbb{E}\left(e^{-\beta n_k E_k}\right)$$

$$= e^{\frac{\beta^2 N}{4}(n_1^2 + \cdots + n_k^2)}$$

$$= e^{\frac{\beta^2 N}{4} \sum_{ab} \Delta_{ab}}$$

$$\mathbb{E}\left(e^{\lambda X}\right) = e^{\frac{\lambda^2}{2}\mathbb{E}(X^2)}$$

centered Gaussian

$$\mathbb{E}(Z^n) = \sum_{\Delta} \underbrace{e^{\frac{\beta^2 N}{4}\sum_{ab}\Delta_{ab}}}_{\text{energy}} \underbrace{\sum_{\underline{\sigma}_1 \cdots \underline{\sigma}_n} \mathbb{1}\left[\Delta_{ab} = \delta_{\underline{\sigma}_a \underline{\sigma}_b}\right]}_{\text{entropy}}$$

$e^{NS(\Delta)}$

how many ways to arrange the n configs. in the way defined by Δ

$$e^{NS(\Delta)} = 2^N (2^N - 1)(2^N - 2)\cdots(2^N - k) \sim 2^{NK}$$

$$S(\Delta) = K \log 2$$

$$f_n = \lim_{N\to\infty} -\frac{T}{Nn} \log \mathbb{E}(Z^n) = \lim_{N\to\infty} -\frac{T}{Nn} \log \sum_{\Delta} e^{N\left[\frac{\beta^2}{4}\sum_{ab}\Delta_{ab} + S(\Delta)\right]}$$

$$= -\frac{T}{n} \max_{\Delta} \left[\frac{\beta^2}{4} \sum_{ab} \Delta_{ab} + S(\Delta) \right]$$

(minus) energy: energy is minimized by all replicas in the same block $(K=1)$

entropy is maximized by each replica in a different block $(K=n)$

- We reduced a 2^N sum to a max over $n \times n$ matrices
- but: only for integer $n = 1, 2, 3, \dots$

How to take $n \to 0$?

2. The replica symmetric ansatz

All replicas equivalent $\quad \Delta_{ab} = \delta_{ab} \quad \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$K = n$ blocks

$$f_n = -\frac{T}{n}\left(\frac{\beta^2}{4} n + n \log 2\right) = -\frac{\beta}{4} - T \log 2 = f_a$$

RS $\Rightarrow f_n = f_a$ independent of $n \Rightarrow f_q = \lim_{n \to 0} f_n = f_a$

$$P(f \neq f_q) \sim \exp N^{\alpha} \quad \alpha > 1$$

Correct for $T > T_c = \frac{1}{2\sqrt{\log 2}}$

3. Replica symmetry breaking

The RS solution maximizes the entropy
(all replicas in different blocks)

Form a few blocks to lower the energy

Simplest choice: $\frac{n}{m}$ blocks, each of m replicas

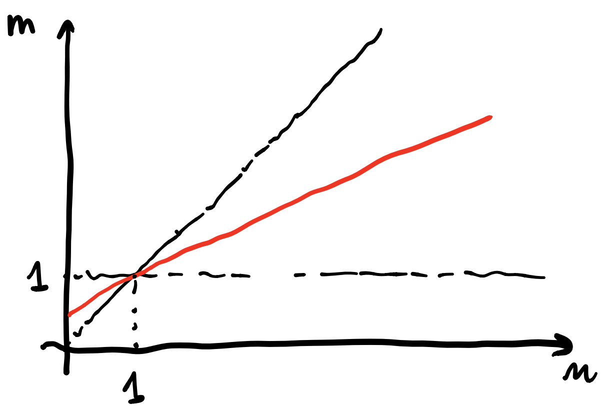
$$\Delta_{ab} = \begin{array}{c} \overbrace{\quad\quad}^{m} \quad\quad\quad\quad\quad\quad \\ \begin{pmatrix} \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \end{pmatrix} \end{array}$$

m variational parameter

$$S = \frac{n}{m} \log 2$$

$$\sum_{ab} \Delta_{ab} = \frac{n}{m} m^2 = nm$$

$$f_n = -\frac{T}{n} \max_m \left[\frac{\beta^2}{4} mn + \frac{n}{m} \log 2 \right] = \min_m \left[-\frac{\beta}{4} m - \frac{T}{m} \log 2 \right]$$



$1 \leq m \leq n$

when $n \leq 1$

$n \leq m \leq 1$

Actually it is a max

$$f_q = \underset{m \in [0,1]}{\overset{\max}{\cancel{\min}}} \left[-\frac{\beta}{4} m - \frac{T}{m} \log 2 \right]$$

Saddle point over $\frac{n(n-1)}{2}$ elements

$T \le T_c$

$$-\frac{\beta}{4} + \frac{T}{m^2}\log 2 = 0$$

$$m^* = \frac{T}{T_c} \qquad T_c = \frac{1}{2\sqrt{\log 2}}$$

1) $T = T_c \quad m^* = 1$

2) $f_q = -\sqrt{\log 2}$

Remember from tb2:

$Y = 1 - \frac{T}{T_c}$ prob. of extracting twice the same σ from Gibbs

n replicas $\qquad \frac{n}{m}$ blocks of m replicas

$\frac{n(n-1)}{2}$ pairs : $\frac{n(m-1)}{2}$ in same state , $\frac{n(n-m)}{2}$ in diff. states

$$Y = \frac{m-1}{n-1} \xrightarrow[n\to 0]{} 1-m = 1 - \frac{T}{T_c} \Rightarrow \boxed{0 \le m \le 1}$$

$$P(\Delta) = P(\Delta_{ab} = \Delta) = \frac{2}{n(n-1)} \sum_{a<b} \delta_{\Delta,\Delta_{ab}}$$

$$= \frac{m-1}{n-1}\delta_{\Delta,1} + \frac{n-m}{n-1}\delta_{\Delta,0} \xrightarrow[n\to 0]{} (1-m)\delta_{\Delta,1} + m\,\delta_{\Delta,0}$$