ICFP M2 - Statistical physics 2 – TD n° 5
Random XORSAT problems

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We shall consider in this problem random systems of linear equations, also known as XORSAT problems, denoted F. We recall their definition given during the lectures :

— the degrees of freedom are N Boolean variables, $\underline{x} = (x_1, \ldots, x_N) \in \{0,1\}^N$

— they have to obey M linear constraints of the form

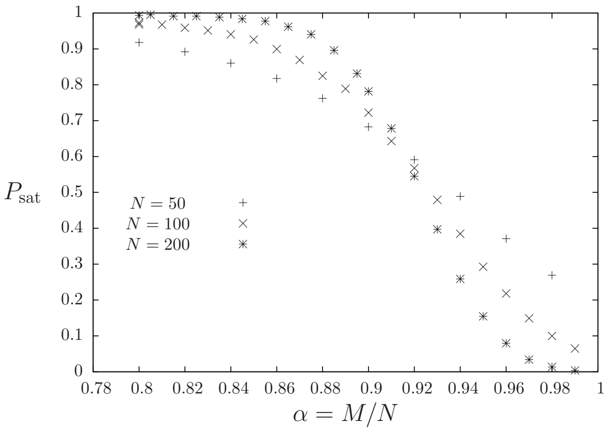
$$x_{i_a^1} + x_{i_a^2} + \cdots + x_{i_a^k} = y_a \mod 2 \ , \tag{1}$$

where $a = 1, \ldots, M$ indexes the various equations, $k \geq 3$ is an integer defining the number of variables involved in each equation, $\langle i_a^1, \ldots, i_a^k \rangle$ is a k-uplet of distinct indices in $\{1, \ldots, N\}$, and $y_a \in \{0,1\}$ fixes the right hand side of the equation.

— such a formula is said to be satisfiable if there is a configuration $\underline{x}$ that verifies all the equations simultaneously, unsatisfiable otherwise.

— a random ensemble of formulas is defined very easily by generating the M equations independently, choosing for each of them a k-uplet $\langle i_a^1, \ldots, i_a^k \rangle$ uniformly at random among the $\binom{N}{k}$ possible ones, and $y_a = 0$ or 1 with probability 1/2.

Using the Gaussian elimination algorithm one can determine whether a given formula is satisfiable or not in polynomial time. Repeating this process a large number of times one can easily obtain a numerical estimate of the probability $P_{\rm sat}(\alpha, N)$ that a random formula F with N variables and $M = \alpha N$ equations is satisfiable :



These curves, obtained for $k = 3$, suggest that a phase transition occurs in the thermodynamic limit ($N, M \to \infty$ with $\alpha = M/N$ fixed) for α around 0.92. Indeed, there exists a threshold $\alpha_{\rm sat}$ (that depends on k) such that

$$\lim_{N\to\infty} P_{\rm sat}(\alpha, N) = \begin{cases} 1 & \text{if} \quad \alpha < \alpha_{\rm sat} \\ 0 & \text{if} \quad \alpha > \alpha_{\rm sat} \end{cases} . \tag{2}$$

1 Bounds on $\alpha_{\rm sat}$

We recall a result obtained in TD2 : for a random variable Z that takes non-negative integer values,

$$\frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} \leq \mathbb{P}[Z > 0] \leq \mathbb{E}[Z] \ , \tag{3}$$

these two inequalities being called the second and first moment method, respectively.

We shall use these inequalities with Z the number of solutions of a random XORSAT formula with N variables and M equations constructed as above.

1. Compute $\mathbb{E}[Z]$, and deduce that $\alpha_{\rm sat} \leq 1$.

2. Show that

$$\mathbb{E}[Z^2] = 2^N \sum_{D=0}^{N} \binom{N}{D} \left(\frac{1}{2} \sum_{\substack{l=0 \\ l \text{ even}}}^{k} \frac{\binom{D}{l}\binom{N-D}{k-l}}{\binom{N}{k}} \right)^M . \tag{4}$$

Hint : decompose the sum over two configurations that appears in Z^2 as a sum over one configuration and over the Hamming distance D (number of spins where the two configurations differ) between the two configurations.

3. In the large N (thermodynamic) limit the sum over D is dominated, at the exponential order, by terms with $d = D/N$ of order 1. Conclude that

$$\lim_{N\to\infty} \frac{1}{N} \ln \left(\frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} \right) = \inf_{d\in[0,1]} g(\alpha, d) \ , \quad \text{with } \ g(\alpha,d) = \ln 2 + d \ln d + (1-d)\ln(1-d) - \alpha \ln(1+(1-2d)^k) \ .$$

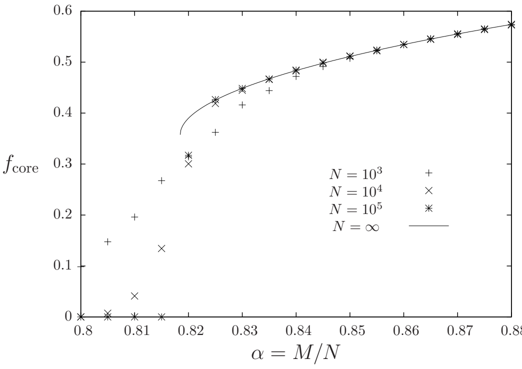
4. Draw the shape of the function g as a function of d for increasing values of α. Conclude that there exists a value $\alpha_{\rm lb} > 0$ (equal to 0.889 for $k = 3$) such that for $\alpha < \alpha_{\rm lb}$, the first term in (3) is not exponentially small. A more detailed analysis of (4) shows that in this case it actually goes to 1. Conclude that $\alpha_{\rm lb} \le \alpha_{\rm sat} \le 1$.

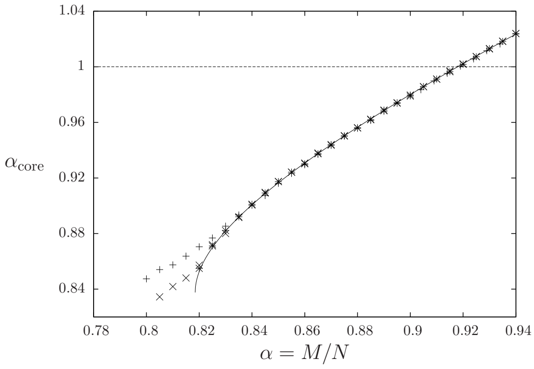
2 Leaf removal procedure

The bounds on $\alpha_{\rm sat}$ obtained above are not tight (i.e. $\alpha_{\rm lb} < 1$) because of the potentially huge fluctuations of Z, that can cause its average $\mathbb{E}[Z]$ to be much larger than its typical value. These fluctuations can be reduced by concentrating on a well-chosen subformula, as explained now.

1. Suppose that F contains a leaf, i.e. a variable that appears in a single equation, and denote F' the system of equations obtained by removing this equation. Show that F is satisfiable if and only if F' is satisfiable.

   This leaf removal procedure can be iterated as long as leaves are present. Two cases can occur : either the formula is completely emptied by this procedure, or there remains a non-trivial subset of F, called its core, in which every variable appears in at least two equations. We call $N_{\rm core}$ and $M_{\rm core}$ the number of variables and equations of the core formula, and display on the curves below the fraction $f_{\rm core} = N_{\rm core}/N$ of variables in the core and the density $\alpha_{\rm core} = M_{\rm core}/N_{\rm core}$ of equations it contains.





   These curves demonstrate a core percolation transition at $\alpha_{\rm d} = 0.818$ (for $k = 3$), and show that the density $\alpha_{\rm core}$ crosses 1 at $\alpha_* = 0.918$ (for $k = 3$).

2. A calculation (not required here) shows that

$$f_{\rm core} = 1 - e^{-\alpha k \phi^{k-1}} - \alpha k \phi^{k-1} e^{-\alpha k \phi^{k-1}} \ , \qquad \frac{1}{N} M_{\rm core} = \alpha\, \phi^k \ , \tag{5}$$

   where $\phi = \phi(\alpha, k)$ is the largest solution in $[0,1]$ of the equation

$$\phi = 1 - e^{-\alpha k \phi^{k-1}} \ . \tag{6}$$

   Study graphically this equation, show that for $k \ge 3$ the transition at $\alpha_{\rm d}$ is discontinuous, and study the behavior of ϕ for $\alpha \to \alpha_{\rm d}^+$.

3. Explain why α_* is an improved upperbound on $\alpha_{\rm sat}$. It turns out that the fluctuations of the core are much smaller than that of the full formula, hence actually $\alpha_{\rm sat} = \alpha_*$.