TD n° 7 : Wigner semi-circle law for random matrices

f abs. continuous $\Leftrightarrow$ f has a derivative f' almost everywhere, the derivative is Lebesgue integrable $f(x) = f(a) + \int_a^x f'(t)\,dt$

1.) Preamble on Gaussian variables.

- A is a ~~positive~~ symmetric invertible $N \times N$ matrix assumed to be definite positive

$\hookrightarrow$ allows to define a Gaussian proba. measure for $X_1, \dots, X_N$ such that:

$$\mathbb{E}\left[f(X_1, \dots, X_N) \right] = \frac{1}{Z} \int d\underline{x}\ f(\underline{x})\, e^{-\frac{1}{2} {}^t\underline{x} A \underline{x}}$$

positive definite $\rightarrow$ nicely converges

where $$Z = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

In particular:

$$(A^{-1})_{ii} = \mathbb{E}(X_i^2)$$

using $A_{ij} = A_{ji}$

$$= \frac{1}{Z} \int dx_i\, x_i^2\, d\underline{x}'\, e^{-\frac{1}{2} x_i^2 A_{ii} - x_i \sum_{j \neq i} A_{ij} x'_j - \frac{1}{2} {}^t\underline{x}' A^{(i)} \underline{x}'}$$

$(n-1)$ other components

where $A^{(i)}$ is obtained from A by removing i-th line & column.

Recall that for a Gaussian vector $\underline{X}'$:

$$\mathbb{E}\left(e^{\sum_i k_i X'_i}\right) = e^{\frac{1}{2} \sum_{ij} k_i k_j \mathbb{E}(X'_i X'_j)}$$

Apply this identity to perform the integral over $\underline{x}'$:

$$\int d\underline{x}'\; e^{-\frac{1}{2} {}^t\underline{x}' A^{(i)} \underline{x}' + \sum_{j \neq i} x'_j \overbrace{(-A_{ij} x_i)}^{k_j}}$$

$$= \underbrace{Z'}\; e^{\frac{1}{2} \sum_{jk \neq i} A_{ij} x_i \underset{A_{ki}}{\underset{\parallel}{A_{ik}}} x_i \left(A^{(i)-1}\right)_{jk}}$$

$$= \frac{(2\pi)^{\frac{N-1}{2}}}{\sqrt{\det A^{(i)}}}$$

$$\Rightarrow A^{-1}_{ii} = \frac{Z'}{Z} \int dx_i\, x_i^2\, e^{-\frac{1}{2} x_i^2 \left(A_{ii} - \sum_{jk \neq i} A_{ij} (A^{(i)-1})_{jk} \overset{A_{ik}}{\overset{\parallel}{A_{ki}}}\right)}$$

Note that with a similar decomposition (by isolating x_i) one obtains:

$$\mathbb{E}(1) = 1 = \frac{Z'}{Z} \int dx_i\, e^{-\frac{1}{2} x_i^2 \left(A_{ii} - \sum_{jk \neq i} A_{ij} (A^{(i)-1})_{jk} A_{ki}\right)}$$

$$\Rightarrow \boxed{A^{-1}_{ii} = \frac{1}{A_{ii} - \sum_{jk \neq i} A_{ij} \left(A^{(i)-1}\right)_{jk} A_{ki}}}$$

$\rightarrow$ holds more generally

... / ...

. Identity for a Gaussian r.v. :

$$P(x) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{1}{2a}x^2}$$

$$\mathbb{E}(X F(X)) = \int_{-\infty}^{\infty} dx \, \underbrace{\frac{x}{\sqrt{2\pi a}} e^{-\frac{1}{2a}x^2}}_{-a \frac{d}{dx} P(x)} F(x)$$

usually IPP

But one can also derive it like this :

$$\mathbb{E}(X F(X)) = \cancel{\tfrac{1}{\sqrt{2\pi a}}} \frac{d}{d\epsilon} \underbrace{\left[\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi a}} e^{-\frac{1}{2a}x^2 + \epsilon x} F(x) \right]_{\epsilon = 0}}_{I(\epsilon)}$$

write $\frac{1}{2a}\left(x^2 - 2a\epsilon\right) = \frac{1}{2a}\Big[\underbrace{(x - a\epsilon)}_{u} - a^2\epsilon^2\Big]$

$$I(\epsilon) = \underbrace{e^{\frac{a\epsilon^2}{2}}}_{1+O(\epsilon^2)} \int_{-\infty}^{\infty} \frac{du}{\sqrt{2\pi a}} e^{-\frac{1}{2a}u^2} \underbrace{F(u + a\epsilon)}_{F(u) + a\epsilon F'(u)}$$

$$= I(0) + \epsilon \underbrace{a}_{\mathbb{E}(X^2)} \mathbb{E}(F'(X)) + O(\epsilon^2)$$

$$\Rightarrow \boxed{\mathbb{E}(X F(X)) = \mathbb{E}(X^2)\, \mathbb{E}(F'(X))}$$

$\rightarrow$ Generalization to N-dimensional Gaussian vector

$$P(x_1, \dots, x_N) = \underbrace{\frac{(\det A)^{\frac{1}{2}}}{(2\pi)^{N/2}}}_{=\frac{1}{Z}} e^{-\frac{1}{2} {}^t\underline{x} A \underline{x}}$$

$$\mathbb{E}\left(X_i F(X_1, \dots, X_N)\right) = \frac{d}{d\epsilon} \underbrace{\left[\int \frac{d\underline{x}}{Z} e^{-\frac{1}{2} {}^t\underline{x} A \underline{x} + \epsilon\, {}^t e_i \underline{x}} F(\underline{x}) \right]}_{I(\epsilon)}\Bigg|_{\epsilon=0}$$

where $e_i = \begin{pmatrix} 0 \\ | \\ 0 \\ 1 \\ 0 \\ | \\ 0 \end{pmatrix} \leftarrow i^{th}$

as before: $-\frac{1}{2} {}^t\underline{x} A \underline{x} + \epsilon\, {}^t e_i \underline{x}$

$$= -\frac{1}{2} {}^t(\underbrace{\underline{x} - A^{-1} \epsilon e_i}_{\underline{y}}) A (\underline{x} - A^{-1} \epsilon e_i) + \frac{\epsilon^2}{2} {}^t e_i A^{-1} e_i$$

Performing the change of var. $\underline{x} \rightarrow \underline{y}$ (Jacobian = 1)

$$I(\epsilon) = \underbrace{e^{\frac{1}{2}\epsilon^2\, {}^t e_i A^{-1} e_i}}_{1 + O(\epsilon^2)} \int \frac{d\underline{y}}{Z} e^{-\frac{1}{2} {}^t\underline{y} A \underline{y}} \underbrace{F(\underline{y} + \epsilon \underbrace{A^{-1} e_i}_{(A^{-1}e_i)_j = A^{-1}_{ji}})}_{F(\underline{y}) + \epsilon \sum_j A^{-1}_{ji} \frac{\partial}{\partial y_j} F(\underline{y})}$$

$$\Rightarrow I(\epsilon) = I(0) + \epsilon \sum_{j=1}^{N} \mathbb{E}(X_i X_j)\, \mathbb{E}\left[\frac{\partial F}{\partial X_i}(x_1, \dots x_N)\right]$$

$$\boxed{\mathbb{E}(X_i F(X_1, \dots, X_N)) = \sum_{j=1}^{N} \mathbb{E}(X_i X_j)\, \mathbb{E}\left(\frac{\partial F}{\partial x_i}(x_1, \dots, x_N)\right)}$$

2.) The semi-circle law for GOE matrices.

M is an $N \times N$ GOE matrix

real, symmetric $\cancel{M_{ij} = M_{ji}}$

$i \le j$ M_{ij} are iid Gaussian rand. var.

$$\mathbb{E}(M_{ij}) = 0 \; ; \quad \mathbb{E}(M_{ij}^2) = \frac{1}{N} \; , \; i < j$$

$$\mathbb{E}(M_{ii}^2) = \frac{2}{N}$$

$$M = O \Lambda O^{-1}, \quad \Lambda = \mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_N}_{\text{real}})$$

empirical e.v. distribution $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$

i.e. $\mu_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$

Resolvent matrix: $G(z) = (M - z\mathbb{1})^{-1}$

$$\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}_{N\times N}$$

i.e. $z \in \mathbb{C} \mid \mathrm{Im}(z) > 0$, $g_N(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z}$

$\rightarrow$ well defined

1.)

$$g_N(\lambda + i\eta) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - \lambda - i\eta}$$

- $$\lim_{\eta \to 0^+} \frac{1}{x - i\eta} = PP\,\frac{1}{x} + i\pi\delta(x)$$

in the sense of distribution

$\rightarrow$ Sokhotski - Plemelj thm

$\rightarrow$ Kramers- Kronig relation

implies $$\mathrm{Im}\, g_N(\lambda + i\eta) \xrightarrow[\eta \to 0^+]{} \pi \times \underbrace{\frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)}_{\mu_N(\lambda)}$$

hence $$\mu_N = \frac{1}{\pi} \lim_{\eta \to 0^+} \mathrm{Im}\; g_N(\lambda + i\eta)$$

- Other proof: $$g_N(\lambda + i\eta) = \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i - \lambda + i\eta}{(\lambda_i - \lambda)^2 + \eta^2}$$

$$\frac{1}{\pi} \mathrm{Im}\; g_N(\lambda + i\eta) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\pi} \frac{\eta}{(\lambda_i - \lambda)^2 + \eta^2}$$

We want to show that : $\lim_{\eta\to 0^+} \frac{1}{\pi}\frac{\eta}{(\lambda_i-\lambda)^2+\eta^2} = \delta(\lambda-\lambda_i)$

i.e. $\forall$ functions f:

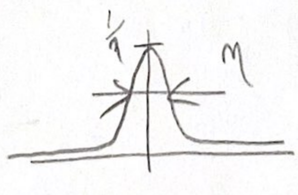
$$\underbrace{\lim_{\eta\to 0^+}\int_{-\infty}^{\infty}\frac{d\lambda}{\pi}\,\frac{\eta}{(\lambda_i-\lambda)^2+\eta^2}\,f(\lambda)}_{u=\frac{(-\lambda_i+\lambda)}{\eta}} \overset{?}{=} f(\lambda_i)$$

$$\int_{-\infty}^{\infty}\frac{du}{\pi}\,\frac{\cancel{\eta^2}}{\cancel{\eta^2}u^2+\eta^2}\,f(\lambda_i+u\eta) = \cancel{\int_{-\infty}^{\infty}\frac{du}{\pi}\,\frac{1}{u^2+1}\,f(\lambda_i)+O(\eta)}$$

$$\xrightarrow[\eta\to 0]{} \int_{-\infty}^{\infty}\frac{du}{\pi}\,\frac{1}{u^2+1}\,f(\lambda_i) = f(\lambda_i)$$

Or say it hard : convolution with Cauchy law:

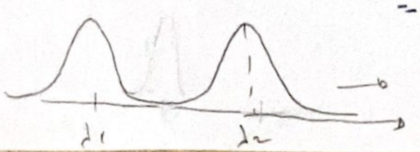
$$p_c(x) = \frac{1}{\pi\eta}\,\frac{1}{\left(\frac{x}{\eta}\right)^2+1}$$



$$\frac{1}{\pi}\,g_N(\lambda+i\eta) = \int_{-\infty}^{\infty} dz\;\mu_N(z)\,p_c(\lambda-z)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} dz\;\delta(z-\lambda_i)\,p_c(\lambda-\lambda_i)$$

$$= \frac{1}{N}\sum_{i=1}^{N} p_c(\lambda-\lambda_i)$$



$\longrightarrow$ Dirac delta

2.) A is an $N \times N$ invertible matrix

$$\sum_j (A^{-1})_{ij} A_{jk} = \delta_{ik}$$

$$\frac{\partial}{\partial A_{lm}} (\,''\,) = 0 \iff \sum_j \frac{\partial (A^{-1})_{ij}}{\partial A_{lm}} A_{jk} + \underbrace{\sum_j (A^{-1})_{ij} \delta_{jl} \delta_{km}}_{(A^{-1})_{il} \delta_{km}} = 0$$

$$\sum_k \sum_j \frac{\partial A^{-1}_{ij}}{\partial A_{lm}} A_{jk} A^{-1}_{kn} + (A^{-1})_{il} \delta_{km} A^{-1}_{kn} = 0$$

$$\Rightarrow \quad \sum_j \frac{\partial A^{-1}_{ij}}{\partial A_{lm}} \delta_{jn} + (A^{-1})_{il} (A^{-1})_{mn} = 0$$

$$\Rightarrow \quad \frac{\partial (A^{-1})_{in}}{\partial A_{lm}} + (A^{-1})_{il} (A^{-1})_{mn} = 0$$

$n \to j$
$l \to k$
$m \to l$

$$\boxed{\frac{\partial (A^{-1})_{ij}}{\partial A_{kl}} = -(A^{-1})_{ik} (A^{-1})_{lj}}$$

.../...

Rk: suppose that $A_{ij} = A_{ji}$, $i < j$ (symmetric)

↳ give directly the formula

1st line gets modified:

$$l<m: \sum_j \frac{\partial (A^{-1})_{ij}}{\partial A_{lm}} A_{jk} + \underbrace{\sum_j A^{-1}_{ij} \left[\delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl} \right]}_{(A^{-1}_{il} \delta_{km} + A^{-1}_{im} \delta_{kl}) A^{-1}_{kn}} = 0$$

$$\cdot \sum_k (\dots) A^{-1}_{kn} = 0$$

$$\Rightarrow \frac{\partial (A^{-1})_{in}}{\partial A_{lm}} + A^{-1}_{il} A^{-1}_{mn} + A^{-1}_{im} A^{-1}_{ln}$$

$n \to j$
$l \to k$
$m \to l$

$$\boxed{\frac{\partial (A^{-1})_{ij}}{\partial A_{kl}} = - (A^{-1})_{ik} (A^{-1})_{lj} - (A^{-1})_{il} (A^{-1})_{kj}}$$

for $k < l$

3.) The elements $\{M_{ij}\}_{i \leq j}$ are the components of a Gaussian vectors with independent entries

w. $\mathbb{E}(M_{ij}) = 0$; $\mathbb{E}(M_{ij}^2) = \frac{1}{N}$; $i < j$

$\mathbb{E}(M_{ii}^2) = \frac{2}{N}$.

From the Gaussian identity we get:

$$\mathbb{E}(M_{kk} \, F(M)) = \frac{2}{N} \, \mathbb{E}\left[\frac{\partial F}{\partial M_{kk}} \right]$$

$$\mathbb{E}(M_{kl} \, F(M)) = \frac{1}{N} \, \mathbb{E}\left[\frac{\partial F}{\partial M_{kl}} \right], \quad k < l.$$

Applied to $F(M) = G_{ij}(z)$, $G(z) = (M - z\mathbb{1})^{-1}$

- From question 2.2. one has:

w. $A = (M - z\mathbb{1})$

$$\frac{\partial G_{ij}(z)}{\partial M_{kk}} = - G_{ik}(z) G_{kj}(z)$$

$$\frac{\partial G_{ij}(z)}{\partial M_{kl}} = - G_{ik}(z) \, G_{lj}(z) - G_{il}(z) \, G_{kj}(z)$$

· <u>Hence</u>: $\mathbb{E}\left(M_{kk}\, G_{ij}(z)\right) = -\frac{2}{N}\,\mathbb{E}\left(G_{ik}(z)\, G_{kj}(z)\right)$

$k<l$: $\mathbb{E}\left(M_{kl}\, G_{ij}(z)\right) = -\frac{1}{N}\,\mathbb{E}\left(G_{ik}(z)\, G_{lj}(z) + G_{il}(z) G_{kj}(z)\right)$

finally:

$$\boxed{\mathbb{E}\left(M_{kl}\, G_{ij}(z)\right) = \frac{-1}{N}\,\mathbb{E}\left(G_{ik}(z)\, G_{lj}(z) + G_{il}(z)\, G_{kj}(z)\right)}$$

true for all (k,l), including $k<l$, $k=l$
$k>l$.

4.) We take this relation with $l=i$
$k=j$ and sum over i,j:

$$\mathbb{E}\,\mathrm{Tr}\left(M G(z)\right) = \frac{-1}{N}\,\mathbb{E}\left[\sum_{ij}\left(G_{ij}(z)\underbrace{G_{ij}(z)}_{=\,G_{ji}(z)} + G_{ii}(z) G_{jj}(z)\right)\right]$$

$$\boxed{\begin{array}{l}\mathbb{E}\,\mathrm{Tr}\left(M G(z)\right) = -\frac{1}{N}\,\mathbb{E}\,\mathrm{Tr}\left(G^2(z)\right) - \frac{1}{N}\,\mathbb{E}\left[\mathrm{Tr}\, G(z)\right]^2 \\ \quad \shortparallel \\ \mathbb{E}\,\mathrm{Tr}\left(G(z) M\right)\end{array}}$$

5.) By def.: $G(z)(M - z\mathbb{1}) = \mathbb{1}$

$$\Rightarrow \quad G(z)M = z\,G(z) + \mathbb{1}$$

hence Eq. (7) becomes:

$$0 = z\,\mathbb{E}\,\mathrm{Tr}\,G(z) + N + \frac{1}{N}\,\mathbb{E}\Big(\underbrace{(\mathrm{Tr}\,G(z))^2}_{N^2 g_N(z)}\Big) + \frac{1}{N}\,\mathbb{E}\,\mathrm{Tr}(G^2(z))$$

$$g_N(z) = \frac{1}{N}\,\cancel{\mathbb{E}}\,\mathrm{Tr}\,G(z)$$

$$\cancel{\Leftrightarrow} \quad \mathbb{E}\left[g_N^2(z)\right] + z\,\mathbb{E}\left[g_N(z)\right] + 1 = \frac{-1}{N^2}\,\mathbb{E}(\mathrm{Tr}\,G^2(z))$$

The rhs is of order $O(\frac{1}{N})$:

$$\frac{1}{N^2}\,\mathrm{Tr}\;G^2(z) = \frac{1}{N^2}\sum_{i=1}^{N}\frac{1}{(\lambda_i - z)^2} = \frac{1}{N}\;\underbrace{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\lambda_i - z)^2}}_{O(1)}$$

note $\quad g_N(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z - \lambda_i} = O(1)$

Assuming that $\quad g_N(z) \xrightarrow[N\to\infty]{} \cancel{\mathbb{E}}\lim_{N\to\infty}\mathbb{E}\,g_N(z)$

$$= g(z)$$

is self-averaging, i.e. concentrates around its average

- we get: $g^2(z) + z\,g(z) + 1 = 0$

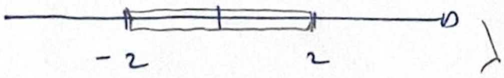
$$g(z) = -\frac{z}{2} \pm \frac{1}{2}\sqrt{z^2-4}$$

Recall that: $g_N(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{-z+\lambda_i} \underset{z\to\infty}{\sim} \frac{-1}{z} + \cdots$

hence the solution we are after is:

$$g(z) = -\frac{z}{2} + \frac{1}{2}\sqrt{z^2-4}$$

- $\mu(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \operatorname{Im}\, g(\lambda + i\eta)$



- If $|\lambda| > 2$, clearly $\mu(\lambda) = 0$ since it's real.
- If $|\lambda| \leq 2$:

$$g(\lambda+i\eta) \;\cancel{g(\lambda)}\longrightarrow -\frac{\lambda}{2} + \frac{i}{2}\sqrt{4-\lambda^2}$$

$$\Rightarrow \boxed{\mu(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad -2 \leq \lambda \leq 2}$$

ICFP M2 - Statistical physics 2 – TD n° 7
Wigner semi-circle law for random matrices – Solution to the last exercise

Grégory Schehr, Francesco Zamponi

March 4, 2021

3 The sketch of a proof by recursion for Wigner matrices

Here M is a Wigner (real symmetric) random matrix of large size N.

1. We recall that the resolvent $G(z)$ is defined as $G(z) = (M - z\mathbb{I})^{-1}$, where $\mathrm{Im}\, z > 0$. By applying the identity (2) of the TD specified to $i = 1$ [1], we immediately obtain the formula (9) of the TD, i.e.

$$G_{11}(z) = \frac{1}{M_{11} - z - \sum\limits_{j,k=2}^{N} M_{1j}\widetilde{G}_{jk}(z)M_{k1}} \;, \tag{1}$$

where $\widetilde{G}(z)$ is the resolvent for the $(N-1)\times(N-1)$ matrix $\widetilde{M}$ obtained from M by removing its first line and column.

2. By taking the average of the inverse of Eq. (1), we get

$$\mathbb{E}\left[\frac{1}{G_{11}(z)}\right] = -z - \sum_{j,k=2}^{N} \mathbb{E}[M_{1j}M_{k1}]\,\mathbb{E}[\widetilde{G}_{jk}(z)] \;, \tag{2}$$

where we have used that $\mathbb{E}[M_{11}] = 0$ together with the fact that the matrix element $\widetilde{G}_{jk}$ is independent of M_{1j} and M_{k1} for all $j,k \geq 2$. Furthermore, since M is a Wigner matrix, which implies that $\mathbb{E}[M_{1j}M_{k1}] = \delta_{j,k}/N$, the double sum over j and k in (2) reduces to a single sum which can be expressed as a trace, leading to the formula (10) of the TD, i.e.

$$\mathbb{E}\left[\frac{1}{G_{11}(z)}\right] = -z - \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\,\widetilde{G}(z)\right] \;. \tag{3}$$

3. For $N \gg 1$, $N \simeq N-1$ and therefore $\mathbb{E}\left[\mathrm{Tr}\,\widetilde{G}(z)\right] \simeq \mathbb{E}\left[\mathrm{Tr}\,G(z)\right]$. In addition, if one assumes the concentration of $G(z)$ around $g(z)\mathbb{I}$, one has $G_{11}(z) \approx 1/g(z)$ (we recall that $g(z) = \lim_{N\to\infty} N^{-1}\mathbb{E}\left[\mathrm{Tr}\,G(z)\right]$), we obtain from Eq. (3) that $g(z)$ satisfies the following equation

$$\frac{1}{g(z)} = -z - g(z) \;, \qquad \text{i.e.} \qquad g^2(z) + z\,g(z) + 1 = 0 \;, \tag{4}$$

which is the same equation found in the case of the GOE in the second exercise of the TD. Hence we also find in this case that the empirical eigenvalue distribution converges to the Wigner semi-circle law $\rho_{\mathrm{sc}}(\lambda) = \sqrt{4-\lambda^2}/(2\pi)$, for $\lambda \in [-2,2]$.

[1] Although the identity (2) was shown for symmetric definite matrix, it actually holds for a wider class of invertible matrices, such as $A(z) = M - z\mathbb{I}$, thanks to the Schur's complement lemma