ICFP M2 - Statistical physics 2 – TD n° 7
Directed Polymers and Interfaces in Random Media

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In this TD we shall study and discuss the physics of directed polymers and one dimensional interfaces in random media. This problem emerges in several different situations such as interfaces in random magnets, crack interfaces in fracture, DNA sequences alignement. As we shall find, quenched disorder changes drastically the behavior of these systems compared to their non-disordered counterpart.

1 From one dimensional interfaces in random media to quantum interacting particles via the replica method

The energy of an interface is proportional to its length. The proportionality coefficient is called the surface tension. In the following we consider a one dimensional interface, i.e. a line $x(t)$ in the (x, t) plane, that starts at $(0, 0)$ and ends at (x, L). Its energy reads :

$$E = \sigma \int dl = \sigma \int_0^L dt \sqrt{1 + \left(\frac{dx}{dt}\right)^2} \simeq \sigma L + \frac{\sigma}{2} \int_0^L dt \left(\frac{dx}{dt}\right)^2 \qquad ,$$

where $x(t)$ is the height of the interface. We consider the temperature low enough so that the interface does not fluctuate too much, i.e. it has no overhangs (so $x(t)$ is a single valued function) and the expansion of the square root above to first order is justified.

The partition function of the interface is obtained by performing a functional integral :

$$\begin{aligned} Z &= \int dx Z(x, L; 0, 0) \qquad (1) \\ Z(x, L; 0, 0) &= \int_{x(0)=0}^{x(L)=x} \mathcal{D}[x(t)] \exp\left(-\beta\sigma L - \beta\frac{\sigma}{2} \int_0^L dt \left(\frac{dx}{dt}\right)^2\right) \end{aligned}$$

As you found in homework 4, in the absence of disorder the thermal fluctuations of the interface, in particular of its endpoint x, are of the order $\sqrt{L}$ but we expect the disorder to drastically change the physical behavior, as we now show.

The partition function in presence of disorder and with a free boundary condition in L reads :

$$Z = \int dx Z(x, L; 0, 0) = \int dx \int_{x(0)=0}^{x(L)=x} \mathcal{D}[x(t)] \exp\left(-\beta\sigma L - \beta \int_0^L dt \left[\frac{\sigma}{2}\left(\frac{dx}{dt}\right)^2 + V(x(t), t)\right]\right) \quad (2)$$

where the disordered potential is represented by Gaussian i.i.d. random variables at all points x, t : its continuum version is a Gaussian random field with zero mean and covariance :

$$\overline{V(x, t)V(x', t')} = \Delta^2 \delta(x - x')\delta(t - t')$$

In order to compute the free energy and its fluctuations we use the replica method :

$$\log \overline{Z^n} = n\overline{\log Z} + \frac{n^2}{2}\overline{(\log Z)^2_c} + \frac{n^3}{3!}\overline{(\log Z)^3_c} + \dots$$

where $\overline{(\log Z)^k_c}$ denotes the k-th cumulant of $\log Z$.

1. Expressing Z^n for integer n by replicating the configurations into $(x^1(t), \ldots, x^n(t))$, show that the term containing the disorder can be rewritten as

$$\exp\left(-\beta \int dx dt \rho(x,t) V(x,t)\right) \qquad , \qquad \rho(x,t) = \sum_{a=1}^{n} \delta(x - x^a(t))$$

2. This expression makes clear that the average over the disorder of this term can be performed using the classical identity on the exponential of Gaussian i.i.d random variables, which for just one Gaussian random variable g reads $\langle e^{gy} \rangle = e^{\sigma_g^2 y^2/2}$ where $\langle g \rangle = 0$ and $\langle g^2 \rangle = \sigma_g^2$.
   Using this trick show that $\overline{Z^n(x^1, \cdots x^n, L; 0, \cdots 0, 0)}$ equals

$$\int_{x^a(0)=0}^{x^a(L)=x^a} \prod_{a=1}^{n} \mathcal{D}[x^a(t)] \exp\left(-n\beta\sigma L - \frac{\beta\sigma}{2} \sum_{a=1}^{n} \int_0^L dt \left(\frac{dx^a}{dt}\right)^2 + \frac{\beta^2 \Delta^2}{2} \sum_{a,b=1}^{n} \int_0^L dt \delta(x^a(t) - x^b(t))\right)$$

   Clearly the continuum approximation breaks down for the delta function term $a = b$, which leads to a contribution $\frac{\beta^2\Delta^2}{2\Delta x} nL$ where Δx is the underlying discretized unit of length. Henceforth we shall replace the first term, $n\beta\sigma L$, with nL/ξ where $1/\xi = \beta\sigma - \frac{\beta^2\Delta^2}{2\Delta x}$ and only sum over $a \neq b$.
3. The functional integral above can be interpreted as the imaginary time propagator of an n-particle quantum mechanics problem. It is indeed the functional integral expression of the following matrix element :

$$\overline{Z^n(x^1, \cdots, x^n, L; 0, \cdots, 0, 0)} = \langle x^1, \cdots x^n | e^{-\widehat{H}L} | 0, \cdots, 0 \rangle$$

$$\widehat{H} = \frac{n}{\xi} - \frac{1}{2\beta\sigma} \sum_a \left(\frac{\partial}{\partial x^a}\right)^2 - \frac{\beta^2\Delta^2}{2} \sum_{a \neq b} \delta(x^a - x^b)$$

   Using this result and decomposing $e^{-\widehat{H}L}$ on the eigenvectors of $\widehat{H}$ show that in the thermodynamic limit, i.e. for large L, one obtains at leading order in L :

$$\log \overline{Z^n} = -E_{GS}(n) L$$

   where E_{GS} is the ground state energy of $\widehat{H}$, which corresponds to the Hamiltonian of n quantum particles interacting with an attractive delta potential.

2 Solution of the quantum problem and average free energy

1. This Hamiltonian is integrable, i.e. it is one of the few many-body Hamiltonian which is exactly solvable for any n. The ground state wavefunction is known and reads :

$$\psi_{GS}(x^1, \ldots, x^n) = \mathcal{N} e^{-\frac{k}{2} \sum_{a \neq b} |x^a - x^b|} \qquad k = \frac{\Delta^2 \beta^3 \sigma}{2}$$

   For each ordering of the particles on the line the wave-function can be written as a product of exponentials $\psi_{GS} \sim \exp\left[\sum_\alpha -k_\alpha x_\alpha\right]$, with the "momenta" k_α getting permuted for different orderings. Show that for example for $x_1 < x_2 < \cdots < x_n$ the momenta are $k_\alpha = k[2\alpha - (n+1)]$.
2. We can evaluate the ground-state energy considering configurations in which $x_1 < x_2 < \cdots < x_n$. Show that

$$E_{GS} = \left[n\frac{1}{\xi} - \frac{k^2}{2\beta\sigma} \sum_{\alpha=1}^{n} [2\alpha - (n+1)]^2\right]$$

3. By performing the sum above one finds

$$E_{GS} = n\frac{1}{\xi} - \frac{k^2}{6\beta\sigma} n(n^2 - 1)$$

   Use this result to compute $-\beta f = \lim_{L \to +\infty} \frac{\overline{\log Z}}{L}$.

3 A paradox, its solution and the wandering of the interface

1. Since we have the full n dependence of $\log \overline{Z^n}$ we have also access to all the cumulants of $\log Z$. Using this argue why the fluctuations of the free energy are of the order $L^{1/3}$.
2. *Paradox :* Argue why it is in general impossible to have a distribution that has only the first and the third cumulant different from zero.
3. The way out of this paradox is that the limit $n \to 0$ and $L \to \infty$ do not commute. It has been verified by exactly solving the problem at finite L that $-\log \overline{Z^n}/L$ takes the scaling form, in the limit $n \to 0$, $L \to \infty$ but keeping $nL^{1/3}$ fixed

$$-\frac{\log \overline{Z^n}}{L} = \left(\frac{1}{\xi} + \frac{k^2}{6\beta\sigma}\right) n + \frac{1}{L} f(nL^{1/3}) \ .$$

   Extensivity of the generalized free energy $-\frac{1}{\beta}\log \overline{Z^n}$ implies that $f(nL^{1/3}) \sim L$ when $L \to \infty$ at fixed n. On the other hand the replica method giving the cumulants of the free energy is based on $n \to 0$ at fixed L. Use these considerations to solve the paradox.
4. The free-energy fluctuations of the order $L^{1/3}$ are due to wanderings of the interface that sample favourable disorder configurations in space. We therefore expect that the transverse fluctuations, in particular the fluctuations of its endpoint also scale with L as L^ζ. In order to obtain ζ we use again a scaling argument. We consider

$$-\beta\delta F = \overline{\log Z(x, L; 0, 0)} - \overline{\log Z(0, L; 0, 0)}$$

   If x is in the range of the typical fluctuations we assume the scaling behavior $-\beta\delta F = L^{1/3} g_\beta(x/L^\zeta)$. When instead x is much larger, but x/L remains small, we expect a quadratic "elastic" behavior : $-\beta\delta F = C_\beta L \left(\frac{x}{L}\right)^2$. This has been shown by explicit computation. By matching these two scaling regimes find that $\zeta = 2/3$.

The results we found show that the effect of disorder on interfaces and directed polymer is crucial. The disorder induces a wandering which is much wider than the one induced by simple thermal fluctuations because the system try to catch particularly favorable energetic configurations. This in turn leads to free energy fluctuations which are much larger than the temperature $L^{1/3} \gg T$ and which comes from the competition between the elasticity, that hampers stretching, and disorder that instead favours it. This is an example of a "zero-temperature" fixed point that governs the behavior at large length-scale : contrary to usual cases where it is the competition between entropy and energy that rules the system, here it is the competition between disorder and elasticity that matters.