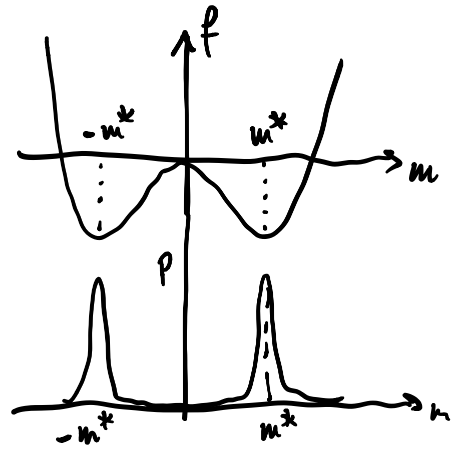
Statistical Physics 2: Disordered Systems and Interdisciplinary Applications — 5.2.2021

I Interpretation of RSB in mean field models

1) Decomposition of the Gibbs measure in pure states

Ferromagnet



$$P[\underline{\sigma}] = \frac{1}{2} P_+[\underline{\sigma}] + \frac{1}{2} P_-[\underline{\sigma}]$$

$$P(m) \sim e^{-\beta N f(m)}$$

General (at fixed disorder)

$$P_J[\underline{\sigma}] = \sum_\alpha w_\alpha P_\alpha[\underline{\sigma}] \qquad \sum_\alpha w_\alpha = 1$$

In mean field a state is fully specified by local magnetization

$$m_i^\alpha = \langle \sigma_i \rangle_\alpha \qquad P_\alpha(\underline{\sigma}) = \prod_i \frac{1 + m_i^\alpha \sigma_i}{2}$$

2) The overlap distribution

Overlap between two states

$$q_{\alpha\beta} = \frac{1}{N} \sum_i m_i^\alpha m_i^\beta = \frac{1}{N} \sum_i \langle \sigma_i \rangle_\alpha \langle \tau_i \rangle_\beta$$

Average overlap in the Gibbs ensemble

$$q_{EA} = \overline{\langle q \rangle_J} = \overline{\frac{1}{N}\sum_i \langle \sigma_i \rangle_J \langle \tau_i \rangle_J} = \overline{\frac{1}{N}\sum_i \sum_\alpha w_\alpha \langle \sigma_i \rangle_\alpha \sum_\beta w_\beta \langle \tau_i \rangle_\beta}$$

$$= \frac{1}{N} \overline{\sum_{\alpha\beta} w_\alpha w_\beta q_{\alpha\beta}}$$

Edwards Anderson order parameter

With replicas

$$q_{EA} = \overline{\langle \frac{1}{N}\sum_i \sigma_i \tau_i \rangle_J} = \overline{\frac{1}{Z^2}\sum_{\underline{\sigma}\underline{\tau}} e^{-\beta H(\underline{\sigma})} e^{-\beta H(\underline{\tau})} \frac{1}{N}\sum_i \sigma_i \tau_i}$$

$$= \lim_{n\to 0} \sum_{\underline{\sigma}^1 \dots \underline{\sigma}^n} \overline{e^{-\beta[H(\underline{\sigma}^1) + \dots + H(\underline{\sigma}^n)]}} \frac{1}{N}\sum_i \sigma_i^1 \sigma_i^2$$

or any other choice of $a \neq b$

In mean field models:

$$\overline{e^{-\beta \sum_a H(\underline{\sigma}^a)}} = e^{N f(Q)} \qquad q_{ab} = \frac{1}{N}\sum_i \sigma_i^a \sigma_i^b$$

$$q_{EA} = \lim_{n\to 0} \sum_{\{\underline{\sigma}^a\}} e^{N f(Q)} q_{12} = \lim_{n\to 0} \int dQ\, e^{N A(Q)} q_{12} \sim q_{12}^*$$

$$\boxed{q_{EA} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a<b} q_{ab}^* = \overline{\sum_{\alpha\beta} w_\alpha w_\beta q_{\alpha\beta}}} \qquad q_{EA} = \int dq\, q\, P(q)$$

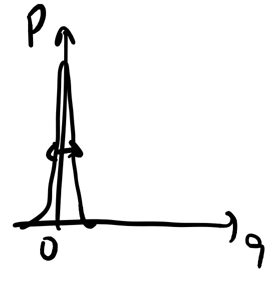
$$\boxed{P(q) = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a<b} \delta(q - q_{ab}^*) = \overline{\sum_{\alpha\beta} w_\alpha w_\beta \delta(q - q_{\alpha\beta})}}$$

Examples:

a) RS case $\quad q^* = 0$

$$Q^* = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad \begin{cases} \text{REM } T > T_c \\ \text{SK } T > T_c \\ \text{p spin high } T \end{cases}$$

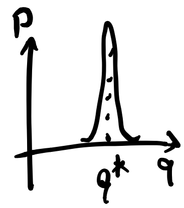
$$P(q) = \delta(q)$$



Typical configurations from Gibbs are very different

b) RS case $\quad q^* > 0$ $\qquad$ SK model for $T < T_c = 1$ (but wrong solution)

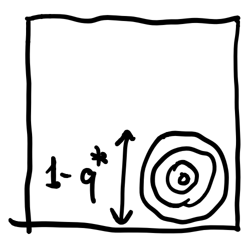
$$Q^* = \begin{pmatrix} 1 & & q^* \\ & \ddots & \\ q^* & & 1 \end{pmatrix} \qquad P(q) = \delta(q - q^*)$$



(spin flip symmetry has been implictly broken)

In a ferromagnet $\quad P(q) = \delta(q - m^2)$

Here a "disordered ferromagnet" $\quad q^* = \overline{\frac{1}{N} \sum_i m_i^2}$



space of 2^N spin configurations

c) 1RSB case

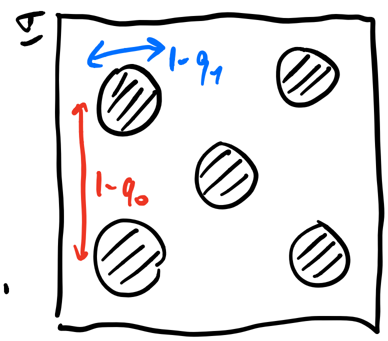
$$Q^* = \begin{pmatrix} \begin{pmatrix} 1 & q_1 \\ q_1 & 1 \end{pmatrix} & q_0 & q_0 \\ q_0 & \begin{pmatrix} 1 & q_1 \\ q_1 & 1 \end{pmatrix} & q_0 \\ q_0 & q_0 & \begin{pmatrix} 1 & q_1 \\ q_1 & 1 \end{pmatrix} \end{pmatrix}$$

$$P(q) = m\,\delta(q-q_0) + (1-m)\,\delta(q-q_1) = \overline{\sum_{\alpha\beta} w_\alpha w_\beta\, \delta(q-q_{\alpha\beta})}$$

Interpretation:

with prob m fall into distinct states $\alpha \neq \beta \Rightarrow q_{\alpha\beta} = q_0$

with prob $1-m$ fall into the same state $\alpha = \beta \Rightarrow q_{\alpha\alpha} = q_1$



REM: $m = \frac{T}{T_c}$, $q_1 = 1$, $q_0 = 0$

3) Summary and dRSB phase

a) RS phase : one state or a few states related by a symmetry

b) RSB phase: many disordered states (a finite number) not related by a symmetry

c) dRSB phase : p spin $p \geq 3$

$T > T_d$ RS $q = 0$

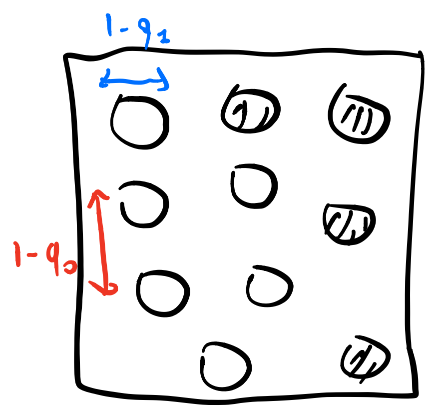
$T < T_c$ 1RSB $1 > m > 0,\ q_0 = 0,\ 1 > q_1 > 0$

$T_c < T < T_d$ strange phase

formally 1RSB but $m = 1$

$q_0 = 0$

$0 < q_1 < 1$



exp(N) states

$1 - m \sim e^{-N} \to 0 \Rightarrow m = 1$

Dynamics not ergodic

REM is in this phase for $T > T_c$

References:

FZ arXiv: 1008.4844

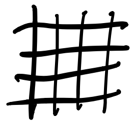
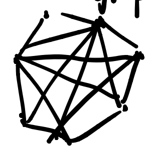
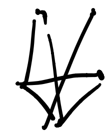
Theory of simple glasses Ch. 5 (Parisi Urbani FZ)

Spin glass theory and beyond (Mézard Parisi Virasoro)

Statistical physics of spin glasses (Nishimori)

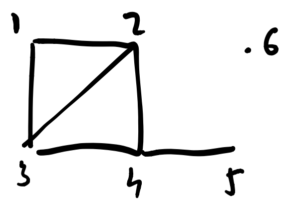
II Introduction to random graphs

Motivation:

2d  mean field  complete graph random graph 

1 Definitions

- graph $G = (V, E)$ $\qquad V = \{1, 2, 3, 4, 5, 6\}$



$E = \{ \langle 12 \rangle, \langle 23 \rangle, \langle 13 \rangle, \langle 34 \rangle, \langle 45 \rangle \}$

- complete graph K_N, all possible $\binom{N}{2}$ edges
- neighbors of i, $\partial i = \{ j \in V, \langle ij \rangle \in E \}$
- degree of i, $d_i = |\partial i|$

  empirical degree distribution $\hat{q}_K = \frac{1}{N} \sum_i \delta_{d_i, K}$

- a walk of length L a sequence of vertices

  $i_1 i_2 \cdots i_L$ such that $\langle i_1 i_2 \rangle \in E, \langle i_2 i_3 \rangle \in E, \cdots$

- a path is a walk with all vertices distinct
- a cycle is a path with $\langle i_L i_1 \rangle \in E$

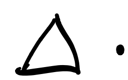
- i and j are said to be connected if there is a path that includes them

divide V into connected component

- G is connected if there is only one connected component
- G is a tree if it is connected and has no cycles

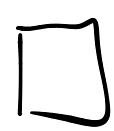
tree $\Rightarrow |E| = |V| - 1$

$|E| = |V| - 1 \not\Rightarrow$ tree

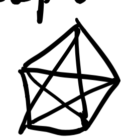


$|E| = |V| - 1$ and is connected $\Rightarrow$ tree

- G is called a forest if it has no cycles (collection is trees)
- G is planar if you can draw it on paper without edges crossing



planar



not planar

- $d(i,j)$ minimal length of a path between i,j
- diameter of a graph $\max_{ij \in E} d(ij)$

- matrices

adjancency matrix $A_{ij} = \begin{cases} 1 & \text{if } \langle ij \rangle \in E \\ 0 & \text{otherwise} \end{cases}$

$A_{ii} = 0$

$D_{ij} = d_i \, \delta_{ij}$

- random walk

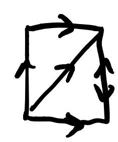
$$p_i(t+1) = \sum_{j \in \partial i} \frac{1}{d_j} p_j(t) \qquad \vec{p}(t+1) = A D^{-1} \vec{p}(t)$$

stationary state $\pi_i \propto d_i$

Spectrum of A is related to the rapidity of convergence to stationary state.

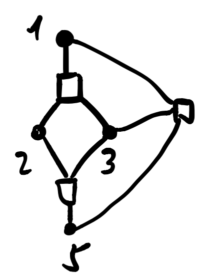
- extensions of graphs:

* weighted graphs w_{ij}
* directed graphs $i \to j$



* hypergraphs

$$E = \{ \langle 123 \rangle, \langle 235 \rangle, \langle 135 \rangle \}$$



2. Real-world networks

- www V = pages E = links
- social networks V = user E = friendship

- economy $V =$ banks $E =$ i has a loan from j
  $V =$ firms $E =$ i needs j to produce
- transports $V =$ airport $E =$ flight from i to j
- biology $V =$ gene $E =$ regulatory function

<u>Features</u>:

- $\hat{q}_k$ often "scale free" $\hat{q}_k \sim k^{-a}$
- short cycles
- small world effects
- corelations between degrees