ICFP M2 - Statistical physics 2 – TD n° 3

The mean-field p-spin glass model - Solution

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1. For the mean:

$$\mathbb{E}[H(\underline{\sigma};\underline{J})] = -\sum_{i_1<\cdots<i_p} \mathbb{E}[J_{i_1\ldots i_p}]\sigma_{i_1}\ldots\sigma_{i_p} = 0 \quad \text{because} \quad \mathbb{E}[J_{i_1 i_2\ldots i_p}] = 0 \ . \tag{1}$$

For the variance, we recall that the couplings are independent, hence

$$\begin{aligned}
\mathbb{E}[H(\underline{\sigma};\underline{J})H(\underline{\tau};\underline{J})] &= \sum_{i_1<\cdots<i_p}\sum_{j_1<\cdots<j_p} \mathbb{E}[J_{i_1\ldots i_p}J_{j_1\ldots j_p}]\sigma_{i_1}\ldots\sigma_{i_p}\tau_{j_1}\ldots\tau_{j_p} \\
&= \sum_{i_1<\cdots<i_p} \frac{p!}{2N^{p-1}}\sigma_{i_1}\ldots\sigma_{i_p}\tau_{i_1}\ldots\tau_{i_p} \\
&= \frac{1}{2N^{p-1}}\sum_{i_1\neq\ldots\neq i_p} \sigma_{i_1}\ldots\sigma_{i_p}\tau_{i_1}\ldots\tau_{i_p} \\
&\sim \frac{1}{2N^{p-1}}\sum_{i_1,\ldots,i_p} \sigma_{i_1}\ldots\sigma_{i_p}\tau_{i_1}\ldots\tau_{i_p} \\
&= \frac{N}{2}\left(\frac{1}{N}\sum_i \sigma_i\tau_i\right)^p = \frac{N}{2}q(\underline{\sigma},\underline{\tau})^p \ .
\end{aligned} \tag{2}$$

We used that $p!\sum_{i_1<\cdots<i_p} = \sum_{i_1\neq\ldots\neq i_p}$ because the $p!$ term gives all possible permutations of the ordered indices, and $\sum_{i_1\neq\ldots\neq i_p} \sim \sum_{i_1,\ldots,i_p}$ because the terms with two equal indices are subleading in N.

2. When $p\to\infty$, we have $q(\underline{\sigma},\underline{\tau})^p \to 1$ if $\underline{\sigma}=\underline{\tau}$ and $q(\underline{\sigma},\underline{\tau})^p\to 0$ if $\underline{\sigma}\neq\underline{\tau}$. Hence, $\mathbb{E}[H(\underline{\sigma};\underline{J})H(\underline{\tau};\underline{J})] = \frac{N}{2}\delta_{\underline{\sigma},\underline{\tau}}$, which is precisely the definition of the REM.

3. The identity in Eq. (3) of the TD is a standard Gaussian integrals that can be computed in many ways, for example by completing the square in the exponent. Using it, we have

$$\mathbb{E}[Z(\beta,\underline{J})] = \sum_{\underline{\sigma}}\prod_{i_1<\cdots<i_p}\mathbb{E}\left[e^{\beta J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p}}\right] = \sum_{\underline{\sigma}}\prod_{i_1<\cdots<i_p} e^{\frac{\beta^2 p!}{4N^{p-1}}} \sim 2^N e^{\frac{\beta^2 p!}{4N^{p-1}}\frac{N^p}{p!}} = 2^N e^{N\frac{\beta^2}{4}} \ , \tag{3}$$

where we used that there are 2^N configurations and $\sim N^p/p!$ possible choices of p ordered indices $i=1,\cdots,N$, when $N\to\infty$ at fixed p. Hence,

$$f_{\mathrm{a}}(\beta) = -\frac{T}{N}\log\mathbb{E}[Z(\beta,\underline{J})] = -\frac{\beta}{4} - T\log 2 \ , \tag{4}$$

as in the REM.

4. For integer n, we have

$$\begin{aligned}
\mathbb{E}[Z(\beta,\underline{J})^n] &= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n}\prod_{i_1<\cdots<i_p}\mathbb{E}\left[e^{\beta J_{i_1\cdots i_p}(\sigma^1_{i_1}\cdots\sigma^1_{i_p}+\cdots+\sigma^n_{i_1}\cdots\sigma^n_{i_p})}\right]\\
&= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n}\prod_{i_1<\cdots<i_p} e^{\frac{\beta^2 p!}{4N^{p-1}}(\sigma^1_{i_1}\cdots\sigma^1_{i_p}+\cdots+\sigma^n_{i_1}\cdots\sigma^n_{i_p})^2}\\
&= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n} e^{\frac{\beta^2}{4N^{p-1}}\sum_{i_1,\cdots,i_p}\sum_{ab}\sigma^a_{i_1}\cdots\sigma^a_{i_p}\sigma^b_{i_1}\cdots\sigma^b_{i_p}}\\
&= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n} e^{N\frac{\beta^2}{4}\sum_{ab}q(\underline{\sigma}^a,\underline{\sigma}^b)^p}\ .
\end{aligned}\tag{5}$$

5. The verification of Eq.(5) of the TD is done by inserting the expression of $e^{NS(Q)}$:

$$\begin{aligned}
\mathbb{E}[Z(\beta,\underline{J})^n] &= \int \mathrm{d}Q e^{N\frac{\beta^2}{4}\sum_{ab}q^p_{ab}}\sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n}\prod_{a<b}\delta(q_{ab}-q(\underline{\sigma}^a,\underline{\sigma}^b))\\
&= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n}\int\left[\prod_{a<b}\mathrm{d}q_{ab}\right]e^{N\frac{\beta^2}{4}\sum_{ab}q^p_{ab}}\prod_{a<b}\delta(q_{ab}-q(\underline{\sigma}^a,\underline{\sigma}^b))\\
&= \sum_{\underline{\sigma}^1,\cdots,\underline{\sigma}^n} e^{N\frac{\beta^2}{4}\sum_{ab}q(\underline{\sigma}^a,\underline{\sigma}^b)^p}\ ,
\end{aligned}\tag{6}$$

which reproduces the previous result. Then

$$\mathbb{E}[Z(\beta,\underline{J})^n] = \int \mathrm{d}Q e^{NA(Q)} \quad\Rightarrow\quad \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}[Z(\beta,\underline{J})^n] = \sup_Q A(Q)\ ,\tag{7}$$

by the saddle point method.

6. In the RS case $q_{ab} = q(1-\delta_{ab})+\delta_{ab}$.

   (a) We start by the calculation of the non-trivial term in $A(Q)$:

$$\begin{aligned}
&\log\left(\frac{1}{2^n}\sum_{\sigma^1,\cdots,\sigma^n} e^{\frac{\beta^2}{4}p\sum_{a\neq b}q^{p-1}\sigma^a\sigma^b}\right) = \log\left(\frac{1}{2^n}\sum_{\sigma^1,\cdots,\sigma^n} e^{\frac{\beta^2}{4}pq^{p-1}[(\sum_a\sigma^a)^2-n]}\right)\\
&= \log\left(\frac{1}{2^n}e^{-n\frac{\beta^2}{4}pq^{p-1}}\sum_{\sigma^1,\cdots,\sigma^n}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2+z\beta\sqrt{\frac{pq^{p-1}}{2}}\sum_a\sigma^a}\right)\\
&= -n\frac{\beta^2}{4}pq^{p-1}+\log\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right)^n\right]\ .
\end{aligned}\tag{8}$$

   From this we get

$$\begin{aligned}
A(Q) &= n\frac{\beta^2}{4}+n\log 2-\frac{\beta^2}{4}(p-1)q^p n(n-1)-n\frac{\beta^2}{4}pq^{p-1}\\
&\quad+\log\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right)^n\right]\ ,
\end{aligned}\tag{9}$$

   and

$$\begin{aligned}
f_{\mathrm{RS}}(q;\beta) = -\lim_{n\to 0}\frac{T}{n}A(Q) &= -\frac{\beta}{4}-T\log 2-\frac{\beta}{4}(p-1)q^p+\frac{\beta}{4}pq^{p-1}\\
&\quad-\lim_{n\to 0}\frac{T}{n}\log\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right)^n\right]\ ,
\end{aligned}\tag{10}$$

The last limit is obtained by using that for small n,

$$\log \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right)^n = \log\left[1 + n\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \log\cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right)\right]$$
$$\sim n\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \log\cosh\left(z\beta\sqrt{\frac{pq^{p-1}}{2}}\right) , \tag{11}$$

which then gives the result of point 6(a) in the TD.

(b) When $q = 0$ we obtain

$$f_{\mathrm{RS}}(q = 0; \beta) = -\frac{\beta}{4} - T\log 2 \tag{12}$$

which coincides with the annealed result.

(c) We can expand $\log\cosh(x) \sim x^2/2 - x^4/12$, with $x = \beta\sqrt{\frac{pq^{p-1}}{2}}z$. Using then $\langle z^2\rangle = 1$ and $\langle z^4\rangle = 3$, where the average is over the normal distribution of z, for small q we have,

$$\begin{aligned} f_{\mathrm{RS}}(q;\beta) - f_{\mathrm{RS}}(q = 0;\beta) &\sim \frac{\beta}{4}(pq^{p-1} - (p-1)q^p) - T\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\left[\frac{x^2}{2} - \frac{x^4}{12}\right] \\ &\sim \frac{\beta}{4}(pq^{p-1} - (p-1)q^p) - \beta\frac{pq^{p-1}}{4} + \frac{T}{12}3\beta^4\frac{p^2q^{2(p-1)}}{4} \\ &= -\frac{\beta}{4}(p-1)q^p + \frac{1}{16}\beta^3p^2q^{2(p-1)} . \end{aligned} \tag{13}$$

For $p \geq 3$, the second term is subdominant, hence $q = 0$ is always a local maximum. For $p = 2$ instead, we obtain

$$f_{\mathrm{RS}}(q;\beta) - f_{\mathrm{RS}}(q = 0;\beta) \sim -\frac{\beta}{4}q^2 + \frac{1}{4}\beta^3q^2 = \frac{\beta}{4}q^2(\beta^2 - 1) . \tag{14}$$

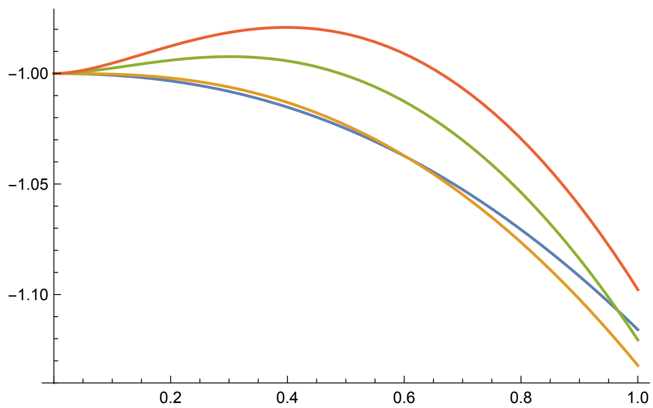
Hence, we see that $q = 0$ is a local maximum for $\beta < 1$ and a local minimum for $\beta > 1$.

(d) Assuming that $q = 0$ is the global maximum, the quenched free energy coincides with the annealed one and the entropy is

$$s = -\frac{\partial}{\partial T}\left[-\frac{\beta}{4} - T\log 2\right] = -\frac{\beta^2}{4} + \log 2 . \tag{15}$$

The entropy becomes negative when $T < 1/(2\sqrt{\log 2})$, which cannot be possible for a model of discrete spins. Hence, a phase transition must happen at a temperature higher than this value.

(e) We now restrict to the case $p = 2$. A plot of $f_{\mathrm{RS}}(q;\beta)$ as a function of q, done using Mathematica, is reported below. The curves correspond to $\beta = 0.8, 1, 1.4, 1.6$.



We then observe that the maximum is in $q = 0$ for $T > 1$ and in $q^\star > 0$ for $T < 1$. We can extend the analysis of point (c) above by noting that the next term in the expansion in x of $\log\cosh(x)$ is x^6, which then gives a term q^3 for $p = 2$, resulting in

$$f_{\mathrm{RS}}(q;\beta) - f_{\mathrm{RS}}(q=0;\beta) \sim \frac{\beta}{4} q^2(\beta^2 - 1) + \frac{C}{3} q^3 + O(q^4) \ , \tag{16}$$

which C some unknown constant. For $T \sim 1$, the maximum is then in

$$q^\star \sim \frac{\beta}{2C}(\beta^2 - 1) \propto 1 - T \ . \tag{17}$$

(f) For $T < 1$ we have a quadratic maximum of $f_{\mathrm{RS}}(q;\beta)$ in $q^\star(T) > 0$. The entropy is thus

$$s_{\mathrm{RS}}(\beta) = -\frac{\mathrm{d}}{\mathrm{d}T} f_{\mathrm{RS}}(q^\star(T);\beta) = -\partial_T f_{\mathrm{RS}}(q;\beta)|_{q=q^\star(T)} - \partial_q f_{\mathrm{RS}}(q;\beta)|_{q=q^\star(T)} \frac{\mathrm{d}q^\star(T)}{\mathrm{d}T} \ . \tag{18}$$

However, the second term vanishes: $\partial_q f_{\mathrm{RS}}(q;\beta)|_{q=q^\star(T)} = 0$ because $q^\star(T)$ is a quadratic maximum. We conclude that $s_{\mathrm{RS}}(\beta) = s_{\mathrm{RS}}(q;\beta)|_{q=q^\star(T)}$, with

$$\begin{aligned} s_{\mathrm{RS}}(q;\beta) &= -\partial_T f_{\mathrm{RS}}(q;\beta) \\ &= -\frac{\beta^2}{4} + \log 2 + \frac{\beta^2}{4}(2q - q^2) + \langle \log\cosh(\beta\sqrt{q}z)\rangle - \beta\sqrt{q}\langle z \tanh(\beta\sqrt{q}z)\rangle \ . \end{aligned} \tag{19}$$

In the plots below, done with Mathematica, it is reported $q^\star(T)$ (left) and $s_{\mathrm{RS}}(T)$ (right) versus T. We observe that $q^\star(T)$ is linear around the critical temperature $T = 1$, as predicted, and that the entropy becomes negative around $T \sim 0.27$. This result indicates that the RS solution cannot describe the spin glass phase, at least at low enough temperature.