ICFP M2 - Statistical physics 2 – Exam

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The exam is made of two parts. The first one is a series of short questions to check your knowledge of the contents of some of the lectures, the second one is a longer problem with partially independent subparts.

No document, calculator nor phone is allowed.

You can write your answers in English or French. Please make an effort to write clearly in order to facilitate the correction, thanks!

1 Questions on the lectures

1. Consider an alloy made of N_x non-magnetic Copper (Cu) atoms with coordinates x_i, and N_y magnetic Manganese (Mn) atoms with coordinates y_i and magnetic moment σ_i. Assume a total Hamiltonian

$$H(x,y,\sigma) = \sum_{i<j} V_{xx}(|x_i - x_j|) + \sum_{i,j} V_{xy}(|x_i - y_j|) + \sum_{i<j} V_{yy}(|y_i - y_j|) - \sum_{i<j} J(|y_i - y_j|)\sigma_i \cdot \sigma_j \ . \tag{1}$$

Here, V_{xx}, V_{xy}, V_{yy} are standard pair-potentials for the Cu-Cu, Cu-Mn, Mn-Mn atomic interactions, while $J(|y_i - y_j|)$ is the exchange interaction between the magnetic moments of the magnetic atoms. Pure Cu ($N_y = 0$) undergoes a freezing first-order transition from the liquid phase to a face-centered cubic (FCC) crystal at a temperature scale ~ 1000 K, and it is a metal. Suppose that a few percent of Mn atoms is added, e.g. $N_y/N_x \sim 4\%$, at a preparation temperature of $T = 2000$ K, and the melt is then cooled to, say, 10 K in a few hours.

1a. Do you expect that the full system composed of the x, y, σ degrees of freedom reaches equilibrium?

1b. How do you expect the Manganese atoms to be arranged at low temperatures, say $T \sim 10$ K?

1c. What is the form of the magnetic exchange interaction $J(r)$, and why?

1d. What are the consequences for the magnetic interaction part of the Hamiltonian?

2. We now focus on the magnetic part of the Hamiltonian, assume for simplicity Ising spins $\sigma_i = \pm 1$ and i.i.d. Gaussian magnetic couplings J_{ij} with zero mean and variance $1/N$, i.e. the Sherrington-Kirkpatrick model with Hamiltonian

$$H(\sigma) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_i h_i \sigma_i \ , \tag{2}$$

where h_i is an external magnetic field applied on spin i.

2a. Recall the main properties of the low-temperature phase of the model obtained for $T < 1$.

2b. Explain why, in an experiment, we cannot use the external field h_i to select one of the equilibrium states of the model, like we would do in a ferromagnet.

2c. (**BONUS**) Consider a uniform external field $h_i = h$. Suppose that the system is prepared in one of the equilibrium states at $h = 0$; the thermal average in that state is denoted by $\langle \bullet \rangle_\alpha$. Then, a very small field is applied and the magnetization is measured, over a small time, such that the system cannot change state. Recall why the magnetic susceptibility is

$$\chi_\alpha = \frac{dm_\alpha}{dh} = \frac{d}{dh}\frac{1}{N}\sum_i \langle \sigma_i \rangle_\alpha = \frac{\beta}{N}\sum_{ij}[\langle \sigma_i\sigma_j\rangle_\alpha - \langle\sigma_i\rangle_\alpha\langle\sigma_j\rangle_\alpha] \tag{3}$$

2d. (**BONUS**) Recall that in a mean field model, each state is a product $P(\sigma) = \prod_i \frac{1+m_i^\alpha \sigma_i}{2}$. Argue that as a consequence the terms with $i \neq j$ in the sum disappear. Recall the definition of the Edwards-Anderson order parameter q_{EA}, and conclude that, taking an average over the states, the linear response susceptibility is

$$\chi_{LR} = \overline{\chi_\alpha} = \beta(1 - q_{EA}) \ . \tag{4}$$

2e. (**BONUS**) Recall the behavior of $q_{EA}(T)$ and draw the corresponding function $\chi_{LR}(T)$.

2 May's model and Girko's law for non-Hermitian matrices

A classical toy model in theoretical ecology is the so-called May's model. In this model, one considers a population of N distinct species with equilibrium densities ρ_i^*, with $i = 1, 2, \cdots, N$. To start with, they are non-interacting and stable in the sense that when slightly perturbed from their equilibrium densities, each density relaxes to its equilibrium value with some characteristic damping time denoted by $1/\mu$ (for simplicity, these damping times are all chosen to be the same). Hence the equations of motion for $x_i(t) = \rho_i(t) - \rho_i^*$ to linear order, are simply $\dot{x}_i(t) = -\mu\, x_i(t)$. Now, imagine switching on pairwise interactions between the species. May assumed that the interactions between pairs of species can be modeled by a random matrix J of size $N \times N$, which is generically asymmetric, $J_{ij} \neq J_{ji}$. The linearized equations of motion close to ρ_i^* in the presence of interactions read

$$\dot{x}_i(t) = -\mu\, x_i(t) + \alpha \sum_{j=1}^{N} J_{ij} x_j(t) \quad , \quad i = 1, 2, \cdots, N \; , \tag{5}$$

where α is the coupling strength. A natural question is then: what is the probability $P_{\rm stable}(\alpha, N)$ that the system described by (5) remains stable once the interactions are switched on?

1. Such interaction matrix J_{ij} does not have any special symmetry and therefore its eigenvalues are complex. If we denote by $\{z_1, \cdots, z_N\}$ its eigenvalues, show that

$$P_{\rm stable}(\alpha, N) = \text{Prob.}\left[\max_{1 \leq i \leq N} \text{Re}(z_i) < \mu/\alpha\right] \; , \tag{6}$$

where $\text{Re}(z)$ denotes the real part of the complex number z.

The computation of $P_{\rm stable}(\alpha, N)$ thus requires the study of the statistics of the eigenvalues of such $N \times N$ real non-symmetric matrices J_{ij}. It turns out that, for technical reasons, the study of such real matrices is quite difficult. However, their counterpart with complex entries is much simpler and at the same time exhibits the same qualitative behaviors. In the following we will thus restrict our analysis to complex non-Hermitian matrices.

We consider a $N \times N$ random matrix $G \equiv \{G_{jk}\}_{1 \leq j,k \leq N}$ with complex entries $G_{jk} = G_{jk}^R + i G_{jk}^I$, where $G_{jk}^{R,I}$ are real. Here, we study the so-called (complex) Ginibre ensemble where all these entries $G_{jk}^{R,I}$ with $1 \leq j, k \leq N$ are independent and identically distributed Gaussian random numbers, such that the probability weight $\mathcal{P}(G)$ of a given matrix G is given, up to a normalization factor independent of G, by

$$\mathcal{P}(G) \propto \prod_{1 \leq j,k \leq N} e^{-\left[(G_{jk}^R)^2 + (G_{jk}^I)^2\right]} \; . \tag{7}$$

2. Show that $\mathcal{P}(G)$ in (7) can be written as

$$\mathcal{P}(G) \propto e^{-\text{Tr}(G^\dagger G)} \; , \tag{8}$$

where we recall that $(G^\dagger)_{jk} = G_{kj}^*$ – with z^* denoting the complex conjugate of z.

3. If $G' = UGV$ where U and V are $N \times N$ unitary matrices, i.e. $U^\dagger U = \mathbb{I}_N$ and similarly for V, show that $\mathcal{P}(G') = \mathcal{P}(G)$. This ensemble is thus called *rotationally invariant.*

Such Ginibre matrices (7) are non-Hermitian and therefore their eigenvalues are complex. If we denote by $\{z_1, \cdots, z_N\}$ the eigenvalues of such a Ginibre matrix, one can show that their joint probability distribution function (JPDF) is given by

$$P_{\rm joint}(z_1, \cdots, z_N) = \frac{1}{\mathcal{Z}_N} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \prod_{l=1}^{N} e^{-|z_l|^2} \; , \tag{9}$$

where $\mathcal{Z}_N$ is a normalization constant (or partition function)

$$\mathcal{Z}_N = \int d^2 z_1 \cdots \int d^2 z_N \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \prod_{l=1}^{N} e^{-|z_l|^2} \; . \tag{10}$$

In Eq. (10) we used the notation $d^2 z = dx\, dy = r\, dr\, d\theta$ in Cartesian and polar coordinates respectively and the integrals run over the whole complex plane.

4. Comment on the differences and similarities between the JPDF in (9) and that of the eigenvalues of random Gaussian Hermitian matrices, i.e., belonging to the Gaussian Unitary Ensemble (GUE).

5. Show that the JPDF in (9) can be written as a Boltzmann weight, namely

$$P_{\rm joint}(z_1, \cdots, z_N) = \frac{1}{\mathcal{Z}_N} e^{-E(\{z_i\})} \tag{11}$$

   and interpret $E(\{z_i\})$ as the energy of a Coulomb gas system.

6. This energy $E(\{z_i\})$ has two contributions: a confining component and an interacting one. By balancing these two contributions, show that the typical scale of z_i is $z_i = O(\sqrt{N})$.

We thus introduce the scaled variables $\tilde{z}_i = z_i/\sqrt{N}$ together with the corresponding empirical eigenvalue density $\tilde{\mu}_N(\tilde{z})$ and $\mu_N(z)$ defined as

$$\tilde{\mu}_N(\tilde{z}) = \frac{1}{N} \sum_{i=1}^N \delta^{(2)}(\tilde{z} - \tilde{z}_i) \quad , \quad \mu_N(z) = \frac{1}{N} \sum_{i=1}^N \delta^{(2)}(z - z_i) \;, \tag{12}$$

where $\delta^{(2)}(z)$ denotes the (two-dimensional) Dirac delta function. From (12), one has $\tilde{\mu}_N(\tilde{z}) = N\,\mu_N(z)$, with $z = \sqrt{N}\,\tilde{z}$. We assume (and check it a posteriori) that $\tilde{\mu}_N(\tilde{z})$ converges to a smooth a function of order $O(1)$ as $N \to \infty$. We also introduce the two-point densities

$$\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}') = \frac{1}{N(N-1)} \sum_{i \neq j} \delta^{(2)}(\tilde{z} - \tilde{z}_i)\delta^{(2)}(\tilde{z}' - \tilde{z}_j) \quad , \quad \mu_N^{(2)}(z, z') = \frac{1}{N(N-1)} \sum_{i \neq j}^N \delta^{(2)}(z - z_i)\delta^{(2)}(z' - z_i) \;. \tag{13}$$

7. Derive the following identity

$$\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}') = \frac{N}{N-1} \tilde{\mu}_N(\tilde{z})\tilde{\mu}_N(\tilde{z}') - \frac{1}{N-1} \tilde{\mu}_N(\tilde{z})\delta^{(2)}(\tilde{z} - \tilde{z}') \;. \tag{14}$$

8. Show that the JPDF can be written as in Eq. (11) with

$$E(\{z_i\}) \equiv E\left[\tilde{\mu}_N\right] = e_N + N^2 \left[\iint |\tilde{z}|^2 \, \tilde{\mu}_N(\tilde{z}) \, d^2\tilde{z} - \int d^2\tilde{z} \int d^2\tilde{z}' \, \tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}') \ln|\tilde{z} - \tilde{z}'|\right] + O(N) \;. \tag{15}$$

   where e_N is a constant independent of the $\tilde{z}_i$'s to be determined.

We will admit that in (15), the two-point density $\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}')$ can be replaced by its large N behavior $\tilde{\mu}_N^{(2)}(\tilde{z}, \tilde{z}') \approx \tilde{\mu}_N(\tilde{z})\tilde{\mu}_N(\tilde{z}')$ [see the result derived in Eq. (14)] so that $E(\{z_i\})$ can be written as

$$E(\{z_i\}) \equiv E\left[\tilde{\mu}_N\right] = e_N + N^2 \mathcal{E}[\tilde{\mu}_N] + O(N) \tag{16}$$

$$\mathcal{E}[\tilde{\mu}_N] = \left[\iint |\tilde{z}|^2 \, \tilde{\mu}_N(\tilde{z}) \, d^2\tilde{z} - \int d^2\tilde{z} \int d^2\tilde{z}' \, \tilde{\mu}_N(\tilde{z})\tilde{\mu}_N(\tilde{z}') \ln|\tilde{z} - \tilde{z}'|\right] \;. \tag{17}$$

9. Taking for granted the fact that the Jacobian associated to the change of variable $\{z_1, \cdots, z_N\} \to \tilde{\mu}_N(\tilde{z})$ is of order $e^{O(N)}$, argue that for, large N, the typical empirical density $\tilde{\mu}_N$ is the one that minimises $\mathcal{E}\left[\tilde{\mu}_N\right]$.

   This minimisation procedure leads to

$$|\tilde{z}|^2 - 2\int d^2\tilde{z}' \tilde{\mu}_N(\tilde{z}') \, \ln|\tilde{z} - \tilde{z}'| = C \quad , \quad \int d^2\tilde{z} \, \tilde{\mu}_N(\tilde{z}) = 1 \;, \tag{18}$$

   where the first equation in (18) is valid for $\tilde{z}$ within the support of $\tilde{\mu}_N$ while C is some unknown constant.

10. Using the identity $\Delta_{\tilde{z}}\left[\ln|\tilde{z} - \tilde{z}'|\right] = 2\pi\,\delta^{(2)}(\tilde{z} - \tilde{z}')$, where $\Delta_{\tilde{z}} = \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2$ is the two-dimensional Laplacian, show that

$$\tilde{\mu}_N(\tilde{z}) = \frac{1}{\pi} \;, \quad \text{for} \quad |\tilde{z}| \leq \tilde{R}_e \;. \tag{19}$$

11. From the normalisation condition in (18) show that $\tilde{R}_e = 1$.

12. **Back to the May's model**

12.a) From the above results for the average density of eigenvalues, argue that,

$$\max_{1\leq i\leq N} \mathrm{Re}(z_i)/\sqrt{N} \to 1 \quad , \quad N \to \infty \;, \tag{20}$$

with probability one. Deduce from it that

$$\mathrm{Prob.}\left(\max_{1\leq i\leq N} \mathrm{Re}(z_i) < M\right) \approx \begin{cases} 1 \quad , \quad M > \sqrt{N} \;, \\ \\ 0 \quad , \quad M < \sqrt{N} \;. \end{cases} \tag{21}$$

12.b) Discuss the consequences of this result (21) on the probability $P_{\rm stable}(\alpha, N)$ that the dynamical system (5) remains stable: how does it depend on both α and N?

BONUS QUESTIONS:

In the following, we provide an alternative (more controlled) derivation of the average density $\mathbb{E}(\mu_N(z))$. To start with, we recognise in the JPDF in (9), the product of two Vandermonde determinants. Indeed one has

$$\prod_{1\leq j<k\leq N} (z_k - z_j) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{pmatrix} = \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \prod_{j=1}^N z_j^{\sigma(j)-1} \;, \tag{22}$$

where in the second equality we have used the so-called Leibniz formula, which will be useful in the following. In that expression $\mathcal{S}_N$ denotes the group of permutations of N elements (which thus contains $N!$ distinct permutations) and $\mathrm{sgn}(\sigma) = \pm 1$ is the signature of the permutation σ.

13. **Computation of the partition function $\mathcal{Z}_N$**

13.a) Show the following identity

$$\int d^2z\, z^{j-1} (z^*)^{k-1} e^{-|z|^2} = \delta_{j,k}\, \pi\, \Gamma(j) \;, \tag{23}$$

where $\delta_{j,k}$ is the Kronecker delta function and $\Gamma(z) = \int_0^\infty u^{z-1}\, e^{-u}\, du$.

13.b) By using the Leibniz formula (22) for the two Vandermonde determinants in (10), and performing the integrals over z_i's using (23), show that

$$\mathcal{Z}_N = \pi^N \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N \Gamma\left[\sigma(j)\right] \;. \tag{24}$$

13.c) Obtain finally the explicit expression of $\mathcal{Z}_N$

$$\mathcal{Z}_N = \pi^N \prod_{j=1}^N j! \;, \tag{25}$$

where we recall that $j! = \Gamma(j+1)$ for an integer j.

14. **Direct computation of the average eigenvalue density $\mathbb{E}(\mu_N(z))$ and its large N limit**

14.a) Show that the average eigenvalue density can be written as the following $(N-1)$-dimensional integral

$$\mathbb{E}(\mu_N(z)) = \frac{1}{\pi^N \prod_{j=1}^N j!} \int d^2z_2 \cdots \int d^2z_N \prod_{1\leq j<k\leq N} |z_j - z_k|^2 \prod_{l=1}^N e^{-|z_l|^2} \Bigg|_{z_1 = z} \tag{26}$$

14.b) Using the same kind of manipulations as above for the calculation of $\mathcal{Z}_N$, one arrives at the following expression for the average density

$$\mathbb{E}(\mu_N(z)) = \frac{1}{\pi N} \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!}\, e^{-|z|^2} \;. \tag{27}$$

Check that it is normalised to unity, i.e.,

$$\int d^2z \, \mathbb{E}(\mu_N(z)) = 1 \; . \tag{28}$$

14.c) Show that in the limit of large N, keeping z fixed, one has

$$\lim_{N\to\infty} N \, \mathbb{E}(\mu_N(z)) = \frac{1}{\pi} \; , \tag{29}$$

which means that the average density is uniform in the limit of large N.

14.d) Take for granted that this behavior (29) holds in fact up to a certain radius $|z| \leq R_N$ beyond which the density vanishes identically (for large N). From the normalisation condition (28) show that $R_N \sim \sqrt{N}$ for large N. Conclude that Eq. (19) holds with $\tilde{R}_e = 1$.