

# ICFP M2 - STATISTICAL PHYSICS 2 – TD n° 9

## Dyson Brownian Motion – Solution

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1. The matrix  $M$  satisfies

$$\frac{dM}{dt} = -M(t) + \eta(t) \quad (1)$$

starting from  $M(0) = M_0$ . This equation (1) implies a set of ordinary differential equations for the matrix element  $M_{ij}(t)$  with  $i \leq j$ , i.e.,  $dM_{ij}(t)/dt = -M_{ij}(t) + \eta_{ij}(t)$  whose solutions can be obtained by “varying the constant”. The solution of (1) can be written in a matrix form as

$$M(t) = M_0 e^{-t} + X(t), \quad X(t) = \int_0^t e^{-(t-t')} \eta(t') dt'. \quad (2)$$

Since  $\eta_{ij}(t')$  is a Gaussian random variable, each matrix element  $X_{ij}(t)$  in (2) is also a Gaussian random variable, being a linear combination of Gaussians. It has zero mean,  $\mathbb{E}[X_{ij}(t)] = 0$  and the variance is given by

$$\mathbb{E}[X_{ij}^2(t)] = \int_0^t dt_1 \int_0^t dt_2 e^{-(2t-t_1-t_2)} \mathbb{E}[\eta_{ij}(t_1)\eta_{ij}(t_2)] \quad (3)$$

$$= (1 - e^{-2t}) \times \begin{cases} \frac{2}{N}, & i = j \\ \frac{1}{N}, & i < j. \end{cases} \quad (4)$$

Besides, it is clear that  $\mathbb{E}[X_{ij}(t)X_{kl}(t)] = 0$  if  $i \neq k$  or  $j \neq l$ . And, therefore, from the definition of a GOE matrix one has

$$X(t) \stackrel{d}{=} \sqrt{(1 - e^{-2t})} \text{GOE}, \quad (5)$$

where  $\stackrel{d}{=}$  means an equality in distribution. Hence

$$M(t) \stackrel{d}{=} M_0 e^{-t} + \sqrt{(1 - e^{-2t})} \text{GOE} \xrightarrow{d} \text{GOE} \quad (6)$$

when  $t \rightarrow \infty$ , as the initial condition disappears and the prefactor in front of the GOE matrix tends to 1.

2. Here, we can repeat the same calculation as in the previous question but in the interval  $[t, t + s]$  instead of the interval  $[0, t]$ . This yields straightforwardly

$$M(t + s) = M(t)e^{-s} + \Delta, \quad \Delta = \int_t^{t+s} e^{-(t+s-t')} \eta(t') dt'. \quad (7)$$

Performing the same computation as before leading to (5), one obtains here

$$\Delta \stackrel{d}{=} \sqrt{(1 - e^{-2s})} \text{GOE}. \quad (8)$$

Besides, in Eq. (7) we see that  $M(t)$  depends on the noise variables  $\eta(t')$  only for  $t' \in [0, t]$  while  $\Delta$  depends on the noise variables  $\eta(t')$  only for  $t' \in [t, t + s]$ . Since the noise variables are not correlated in time this means that  $\Delta$  and  $M(t)$  are independent matrices.

3. We denote by  $R$  the *orthogonal* matrix encoding the change of basis from the canonical basis to the orthogonal basis of eigenvectors of  $M(t)$ ,  $|v_1\rangle, |v_2\rangle, \dots, |v_N\rangle$ . The relation between  $\hat{\Delta}$  and  $\Delta$

thus simply reads  $\hat{\Delta} = R \Delta R^{-1}$  where  $R$  and  $\Delta$  are independent (since we have shown that  $\Delta$  is independent of  $M(t)$ ). Recalling that the law of GOE is invariant under the orthogonal group, which means in particular that  $\hat{\Delta}$  and  $\Delta$  have the same statistical properties, one deduces that  $\hat{\Delta}$  and  $\Delta$  are identical in distribution, i.e.,

$$\hat{\Delta} \stackrel{d}{=} \sqrt{(1 - e^{-2s})} \text{GOE} , \quad (9)$$

and both are independent of  $M(t)$ .

4. We now use the relation in (7) with  $s = dt \ll 1$  and write

$$M(t + dt) = M(t) e^{-dt} + \Delta , \quad \Delta \stackrel{d}{=} \sqrt{(1 - e^{-2dt})} \text{GOE} \quad (10)$$

$$M(t + dt) = M(t) + \epsilon + o(dt) , \quad \epsilon = \Delta - dt M(t) , \quad (11)$$

where  $\epsilon$  is a “small” matrix that we will treat in perturbation theory. Using standard perturbation theory (e.g. used in quantum mechanics) up to second order in  $\epsilon$ , one finds

$$\lambda_\alpha(t + dt) = \lambda_\alpha(t) + \langle v_\alpha | \epsilon | v_\alpha \rangle + \sum_{\beta \neq \alpha} \frac{|\langle v_\alpha | \epsilon | v_\beta \rangle|^2}{\lambda_\alpha(t) - \lambda_\beta(t)} + o(\epsilon^2) . \quad (12)$$

Let us now evaluate the different matrix elements entering this formula (12). Let us start with the diagonal elements

$$\langle v_\alpha | \epsilon | v_\alpha \rangle = -dt \langle v_\alpha | M(t) | v_\alpha \rangle + \langle v_\alpha | \Delta | v_\alpha \rangle \quad (13)$$

$$= -dt \lambda_\alpha(t) + \hat{\Delta}_{\alpha\alpha} , \quad (14)$$

since  $|v_\alpha\rangle$  is an eigenvector of  $M(t)$  with eigenvalue  $\lambda_\alpha$  and where we have used the definition of the matrix  $\hat{\Delta}$ . Similarly, the off-diagonal elements read

$$\langle v_\alpha | \epsilon | v_\beta \rangle = \hat{\Delta}_{\alpha\beta} , \quad \alpha \neq \beta . \quad (15)$$

Hence, we get finally

$$\lambda_\alpha(t + dt) = \lambda_\alpha(t) - dt \lambda_\alpha(t) + \hat{\Delta}_{\alpha\alpha} + \sum_{\beta \neq \alpha} \frac{\hat{\Delta}_{\alpha\beta}^2}{\lambda_\alpha(t) - \lambda_\beta(t)} + o(dt) . \quad (16)$$

Note that the left over terms in (16) are indeed of order  $o(dt)$  since  $\epsilon = O(\sqrt{dt})$  such that  $o(\epsilon^2) = o(dt)$  in Eq. (12).

5. From Eq. (9), with the substitution  $s = dt$ , one has that  $\hat{\Delta}_{\alpha\alpha}$  is a Gaussian random variable of zero mean and variance

$$\mathbb{E}[\hat{\Delta}_{\alpha\alpha}^2] = \frac{4}{N} dt , \quad \alpha = 1, \dots, N , \quad (17)$$

while the off-diagonal terms are Gaussian random variables of zero mean and variance

$$\mathbb{E}[\hat{\Delta}_{\alpha\beta}^2] = \frac{2}{N} dt , \quad \alpha \neq \beta . \quad (18)$$

In fact it is useful to write this off-diagonal term in Eq. (9) as

$$\hat{\Delta}_{\alpha\beta}^2 \stackrel{d}{=} \frac{2 dt}{N} X^2 , \quad (19)$$

where  $X$  is a Gaussian random variable with zero mean, i.e.  $\mathbb{E}[X] = 0$ , and unit variance, i.e.  $\mathbb{E}[X^2] = 1$ . It is convenient to think about the right hand side of Eq. (16) as the sum of a deterministic

contribution (given the  $\lambda_\alpha(t)$ ) plus a noise part, which has zero mean, and rewrite this Eq. (16) as follows

$$\lambda_\alpha(t + dt) - \lambda_\alpha(t) = -dt\lambda_\alpha(t) + \sum_{\beta \neq \alpha} \frac{\mathbb{E}[\hat{\Delta}_{\alpha\beta}^2]}{\lambda_\alpha(t) - \lambda_\beta(t)} + \hat{\Delta}_{\alpha\alpha} + \sum_{\beta \neq \alpha} \frac{\hat{\Delta}_{\alpha\beta}^2 - \mathbb{E}[\hat{\Delta}_{\alpha\beta}^2]}{\lambda_\alpha(t) - \lambda_\beta(t)} + o(dt). \quad (20)$$

The blue part is the deterministic contribution and all the terms there are actually of the same order  $\mathcal{O}(dt)$  – see Eq. (19). The red part is the noise (of zero mean): there the term  $\hat{\Delta}_{\alpha\alpha}$  is of order  $\mathcal{O}(\sqrt{dt})$  [see Eq. (17)] while the other ones (the sum over  $\beta \neq \alpha$ ) are actually of order  $\mathcal{O}(dt)$  [see Eq. (19)] and thus sub-leading compared to  $\hat{\Delta}_{\alpha\alpha}$ . They can be discarded such that, to leading order, one has

$$\lambda_\alpha(t + dt) - \lambda_\alpha(t) = -dt\lambda_\alpha(t) + \frac{2}{N} \sum_{\beta \neq \alpha} \frac{dt}{\lambda_\alpha(t) - \lambda_\beta(t)} + \hat{\Delta}_{\alpha\alpha} \quad (21)$$

which indeed, in the limit  $dt \rightarrow 0$ , corresponds to the Langevin equation

$$\frac{d\lambda_\alpha(t)}{dt} = -\lambda_\alpha(t) + \frac{2}{N} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha(t) - \lambda_\beta(t)} + \xi_\alpha(t) \quad (22)$$

where  $\xi_\alpha$  are independent Gaussian white noises of zero mean and variance:

$$\mathbb{E}[\xi_\alpha(t)\xi_\beta(t')] = \frac{4}{N}\delta_{\alpha,\beta}\delta(t-t'). \quad (23)$$

Indeed, by integrating the Langevin equation (22) over the infinitesimal interval  $[t, t + dt]$  one finds

$$\lambda_\alpha(t + dt) - \lambda_\alpha(t) = -dt\lambda_\alpha(t) + \frac{2}{N} \sum_{\beta \neq \alpha} \frac{dt}{\lambda_\alpha(t) - \lambda_\beta(t)} + \int_t^{t+dt} \xi_\alpha(t')dt'. \quad (24)$$

We need to show that the noise in the right hand side of (24) has the same characteristics as  $\hat{\Delta}_{\alpha\alpha}$  in Eq. (21). First, the noise term in (24) is a Gaussian random variable of zero mean, since it is the sum of zero mean Gaussian random variables. Let us compute the covariance matrix. From the relation (23) one can easily check that

$$\mathbb{E}\left[\int_t^{t+dt} \xi_\alpha(t_1)dt_1 \int_t^{t+dt} \xi_\beta(t_2)dt_2\right] = \frac{4}{N}\delta_{\alpha,\beta}dt \quad (25)$$

which coincides with  $\mathbb{E}[\hat{\Delta}_{\alpha\alpha}\hat{\Delta}_{\beta\beta}] = (4/N)\delta_{\alpha,\beta}$  which follows from the previous result in Eq. (17) together with the fact that the matrix  $\hat{\Delta}$  is a GOE matrix – and hence its elements are independent Gaussian random variables. We also note that the noise variables  $\hat{\Delta}_{\alpha\alpha}$  in Eq. (21) are independent of the eigenvalues  $\lambda_\alpha(t)$  since  $\hat{\Delta}$  is independent of the matrix  $M(t)$ .

6. The potential energy  $E(\lambda_1, \dots, \lambda_N)$  and the temperature  $T$  are such that the Langevin equation in (22) reads

$$\frac{d\lambda_\alpha(t)}{dt} = -\frac{\partial}{\partial \lambda_\alpha} E(\lambda_1, \dots, \lambda_N) + \xi_\alpha(t) \quad (26)$$

where

$$\mathbb{E}[\xi_\alpha(t)\xi_\beta(t')] = 2T\delta_{\alpha,\beta}\delta(t-t'). \quad (27)$$

Therefore, by identification with the right hand side of (22) together with (23) one finds

$$E(\lambda_1, \dots, \lambda_N) = \frac{1}{2} \sum_{\alpha=1}^N \lambda_\alpha^2 - \frac{2}{N} \sum_{1 \leq \alpha < \beta \leq N} \ln(|\lambda_\alpha - \lambda_\beta|) \quad (28)$$

and  $T = 2/N$ . Finally, in the limit  $t \rightarrow \infty$ , the law of the eigenvalues converges to the Gibbs-Boltzmann distribution which is given by

$$P_{GB}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z} \exp\left(-\frac{1}{T}E(\lambda_1, \dots, \lambda_N)\right) = \frac{1}{Z} \exp\left(-N \sum_{\alpha=1}^N \frac{\lambda_\alpha^2}{4}\right) \prod_{1 \leq \alpha < \beta \leq N} |\lambda_\alpha - \lambda_\beta|. \quad (29)$$