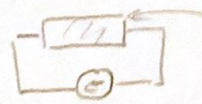
Intro. to statistical mechanics of disordered systems.

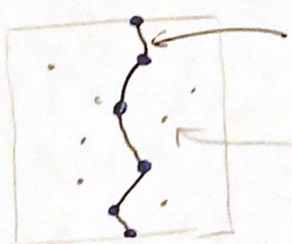
<u>Preliminary remark</u>: all real materials are (more or less) disordered → important to understand the effects of disorder and see to which extent pure models are relevant.

I] Some examples of disordered systems.

- Disordered conductors (see lectures by C. Texier)
  ↳ localization



- Elastic systems in the presence of disorder (see K. Wiese)

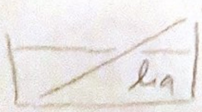


competition between elasticity and disorder

<u>Applications</u>: - domain walls in disordered ferro.

- mixed phase of superconductors
  (pinned vortex lines)

- contact lines in wetting exp.



- Structural glasses : Some liquids, when cooled "rapidly", become "amorphous solids"
  $\to$ no lattice structure any more

- Spin systems ; pure ex Ising model

$$H = - \sum_{\langle i,j \rangle} J \sigma_i \sigma_j - \sum_i h \sigma_i$$

  * site disorder : $h \to h_i$ random field
  * bond disorder : $J \to J_{ij}$ random bond

- heteropolymers : A A T C G A
  T T A G C T

  The interactions depend on the bases $\Rightarrow$ no more translational invariant

In all these cases the generic situation is that the energy / Hamiltonian $H(\underline{\sigma}, \underline{J})$ with <u>two different</u> type of variables: e.g. $\underline{\sigma}$ spins vs. $\underline{J}$ couplings

or posit° of interface vs position of the pinning centers

On the time scales of the experiment:

- $\underline{\sigma}$ evolve / dynamic / thermal variables
- $\underline{J}$ are <u>frozen</u> or <u>quenched</u>

. <u>Remark</u>: in glasses, the disorder is self induced (no real "quenched" disorder).

There are thus <u>two levels</u> of randomness / proba. measures

. At fixed $\underline{J}$, $\underline{S}$ has proba $\propto \dfrac{e^{-\beta H[\underline{S},\underline{J}]}}{Z(\beta,\underline{J})}$

↳ compute averages at fixed $\underline{J}$, i.e. $\langle \ldots \rangle_{\underline{J}}$

. Since $\underline{J}$ is random, one has to average over the realizations of $\underline{J}$: $\mathbb{E}[\cdot]$.

II] Spin-glasses.

1.) Phenomenology.

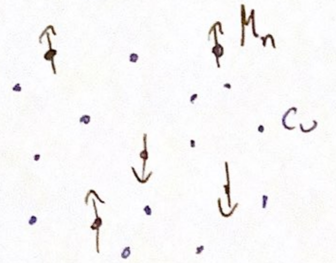
. 1^{st} experiments in the 70's

A few percents of Mn (magnetic) in Cu (non-mag.)

Mn = "manganese" Mixed at high T (liquid) and then suddenly

Cu = "copper" cooled down → solid where the positions of the Mn are <u>frozen</u>, i.e. do not evolve.

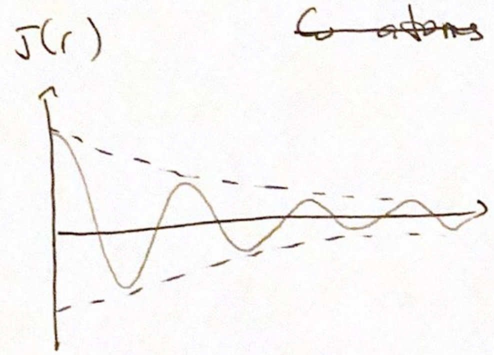
On a donc :



→ effective interactions

$$J_{ij} = J(\vec{r}_i - \vec{r}_j)$$

mediated by the conduction e^- ~~carried by the Cu atoms~~ : RKKY interaction



$$H = -\sum_{ij} J_{ij}\, \sigma_i \sigma_j$$

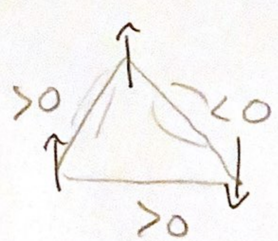
Ruderman - Kittel - Kasuya - Yoshida

It decays at large r but more importantly J changes sign ⟹ frustration .

- If J_{ij} are disordered but all ≥ 0 , the ground-state is still $(\uparrow\uparrow\uparrow \ldots)$ or $(\downarrow\downarrow\downarrow \ldots)$

→ If all $J_{ij} < 0$ $\uparrow\downarrow\uparrow\downarrow$ $\downarrow\uparrow\downarrow\uparrow \ldots$

- But if some $J_{ij} \geq 0$ and some others ≤ 0 ?



⟹ not all interactions can be minimized simultaneously

- a compromise has to be found
- a lot of degeneracies appear

Spin-glass phase transition

- High T : entropy wins, paramagnetic phase where $\langle \sigma_i \rangle_{\underline{J}} = 0$
- Low T : freezing of the spins

$\langle \sigma_i \rangle_J > 0$ for some i's

$\langle \sigma_j \rangle_J < 0$ for some j's.

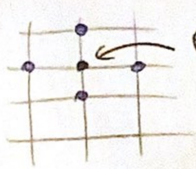
$\Rightarrow$ Some kind of order appears at low T but it is different from a simple ferromagnet.

It is thus similar to a glass which is a solid at low T but not a regular crystal with lattice translation symmetry.

↳ hence the name "spin glass".

2.) Models.

- Edwards - Anderson (75) random positions are too complicated $\Rightarrow \mathbb{Z}^d$



$\sigma_i = \pm 1$

$$H(\underline{\sigma}, \underline{J}) = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j$$

$J(r) \xrightarrow[r \to \infty]{} 0$ in RKKY where J_{ij} are iid random variables

for ex: • $J_{ij} = \begin{cases} +J & \text{w. proba } 1/2 \\ -J & \text{'' } 1/2 . \end{cases}$

• J_{ij} are Gaussian rand. var.

$$\mathbb{E}[J_{ij}] = 0 ; \quad \mathbb{E}[J_{ij}^2] = J^2$$

• Sherrington - Kirkpatrick (75)

Mean-field version of the EA, replace $\mathbb{Z}^d$ by the fully connected graph,

$$H[\underline{\sigma}, \underline{J}] = - \sum_{i<j} J_{ij} \sigma_i \sigma_j$$

where J_{ij} are iid, $\mathbb{E}[J_{ij}] = 0$

$$\mathbb{E}[J^2{}_{ij}] = \frac{J^2}{N}$$

The $\frac{1}{N}$ is necessary to have a good limit $N \to \infty$.

<u>Rk</u>: one can show that in the high temp. expansion the free energy is extensive. <u>Q</u>: how do we show that?

It is a MF model because there is no geometry.

One can also consider the model ~~ofer~~ on a random graph, like Erdős-Rényi (called Viana-Bray, still MF).

- The SK model was solved in 1980 by Parisi using the replica method. Rigorous proof in 2006 (Guerra, Toninelli, Panchenko, Talagrand).

- EA model in finite dim. is still much debated "droplet model" from Fisher-Huse vs MF?

3.) Order parameter?

How to characterize the low-T phase?

- Recall that for a ferromagnet:

$$m = \lim_{h \to 0^+} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle$$

where h is a unif. mag. field to break the +/- symmetry.

- For a spin-glass, $m = 0$ and the good order parameter is the EA overlap, q_{EA}:

$$q_{EA} = \lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle^2 \right]$$

(ε: symmetry breaking field / subtle point not discussed here)

Note that $\langle \sigma_i \rangle^2 = \langle \sigma_i \tau_i \rangle$ where $\underline{\sigma}$ and $\underline{\tau}$ are 2 independent copies or <u>replicas</u> drawn from the Boltzmann-Gibbs weight.

One can define the overlap between two configurations $\underline{\sigma}$, $\underline{\tau}$ for fixed $\underline{J}$:

$$q(\underline{\sigma}, \underline{\tau}) = \frac{1}{N} \sum_i \sigma_i \tau_i = \frac{1}{N} \sum_{i=1}^{N} \left(1 - 2\, \mathbb{1}(\sigma_i \neq \tau_i)\right)$$

$$= 1 - 2 \times \underbrace{\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(\sigma_i \neq \tau_i)}$$

$d(\underline{\sigma}, \underline{\tau})$: Hamming distance between $\underline{\sigma}$ and $\underline{\tau}$; it counts the fraction of spins which are different in the 2 config.

4.) Other mean-field models.

SK: $H(\underline{\sigma}, \underline{J}) = - \sum_{i<j} J_{ij} \sigma_i \sigma_j$

p-spin: $H(\underline{\sigma}, \underline{J}) = - \sum_{i_1 < i_2 < \dots < i_p} J_{i_1, i_2, \dots, i_p} \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_p}$

On the complete graph: $J_{i_1, \dots, i_p}$ Gaussian with zero mean and variance: $\frac{J^2}{N^{p-1}}$.

For all these models, one can view $\{H(\underline{\sigma})\}_{\underline{\sigma}}$ as a vector of 2^N components which are random variables. In g^{al} they are correlated.

Ex: for SK and p-spin show that

To do

$$\mathbb{E}[\, H(\underline{\sigma}) H(\underline{\tau}) \,] = N g(\, q(\underline{\sigma}, \underline{\tau}) \,)$$

function - dependent model

and that's why they are MF.

Simplest model of this kind: Random Energy Model

where $\mathbb{E}[\, H(\underline{\sigma}) H(\underline{\tau}) \,] = \frac{N}{2} \delta_{\underline{\sigma}, \underline{\tau}}$ of TD.

III] Self-averaging.

1.) Self-averaging of the free energy.

In a disordered system, micoscopic details differ from sample to sample: does one get different physical quantities at each repetition of the experiment? If yes, great trouble: how could one define the specific heat of copper for instance, never perfectly pure?

Fortunately:
- all $\forall$ obs. are derivatives of the free en.
- $\forall$ systems are in the thermo. limit
- free energy does not fluctuate in the thermo. limit.

More precisely: $Z_N(\beta, J)$ the partition function

$$F_N = -\frac{1}{\beta} \ln Z_N(\beta, J)$$

then when $N \gg 1$, $F_N(\beta, J) \simeq \mathbb{E}\left[F_N(\beta, J)\right]$

in the sense that: $f_N(\beta, J) = \frac{1}{N} F_N(\beta, J)$

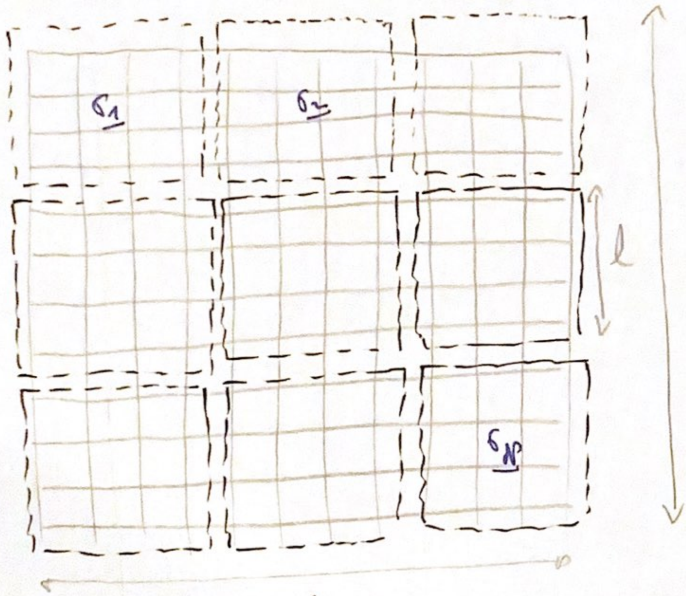
$$\Big\downarrow \; N \to \infty$$

$$f_q(\beta) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[F_N(\beta, J)\right].$$

↳ "quenched" free energy density.

→ "Self averaging": concentration of a quantity around its average in the large size limit. Hence no more sample dependent.

Q: Why is it so? In finite d, $N = L^d$



12 lines

→ Divide the system in blocks of size $1 \ll \ell \ll L$.

$$H_L(\underline{\sigma}, \underline{J}) = \sum_{\alpha=1}^{\mathcal{N}} H_\ell(\underline{\sigma}_\alpha, \underline{J}_\alpha) + R(\underline{\sigma}, \underline{J})$$

with: $\mathcal{N} = \left(\frac{L}{\ell}\right)^d$ nber of blocks

$H_\ell(\underline{\sigma}_\alpha, \underline{J}_\alpha)$ contains the interaction inside the block α

R contains the interaction between the blocks

$$|R(\underline{\sigma}, \underline{J})| \leq \underset{\text{largest coupling}}{c} \times \mathcal{N} \times \ell^{d-1}.$$

One has:

$$e^{-\beta c \mathcal{N} \ell^{d-1}} \prod_{\alpha=1}^{\mathcal{N}} Z_\ell(\beta, \underline{J}_\alpha) \leq Z_L(\beta, \underline{J}) \leq \prod_{\alpha=1}^{\mathcal{N}} Z_\ell(\beta, \underline{J}_\alpha) e^{\beta c \mathcal{N} \ell^{d-1}}$$

$$-c\mathcal{N}\ell^{d-1} + \sum_\alpha F_\ell(\beta, \underline{J}_\alpha) \leq F_L(\beta, \underline{J}) \leq \sum_\alpha F_\ell(\beta, \underline{J}_\alpha) + c\mathcal{N}\ell^{d-1}$$

$$\frac{c}{\ell} - \frac{\ell^d}{L^d} \sum_\alpha \frac{1}{\ell^d} F_\ell(\beta, \underline{J}_\alpha) \leq \frac{1}{L^d} F_L(\beta, \underline{J}) \leq \underbrace{\underbrace{\frac{\ell^d}{L^d}}_{\frac{1}{\mathcal{N}}} \sum_\alpha \frac{1}{\ell^d} F_\ell(\beta, \underline{J}_\alpha)}_{\xrightarrow{L \to \infty \ (\mathcal{N} \to \infty)} \ \mathbb{E}[f_\ell(\beta, \underline{J})]} + \frac{c}{\ell}$$

by the law of large numbers (sum of iid random variables)

Then $\ell \to \infty$ to get rid of the surface term.

<u>Rk</u>: self-averaging of the free energy is also true for mean-field models but one needs another type of proof (concentration inequality).