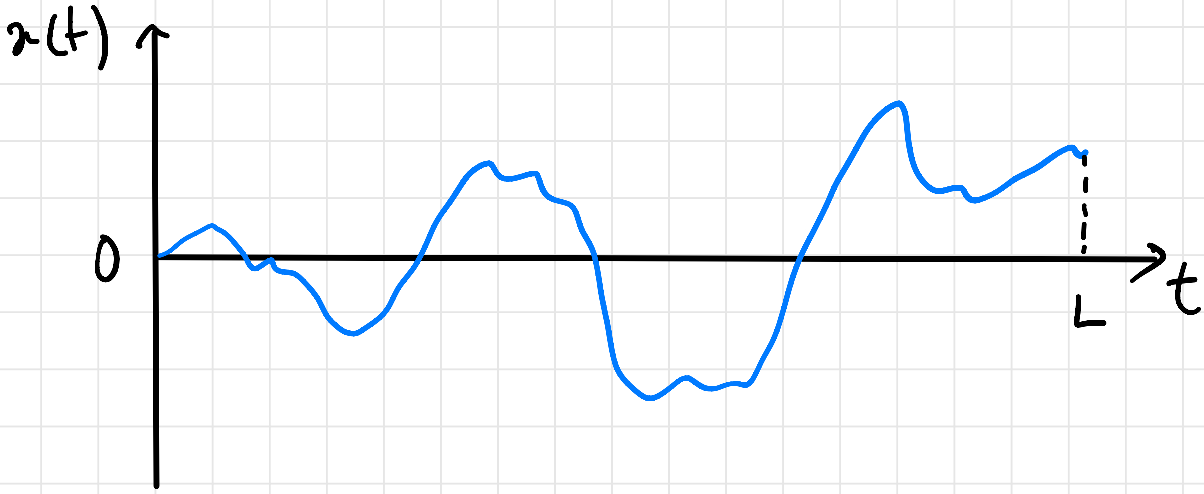
TD 6: Directed polymers and interfaces in random media



$$E[\{x(t)\}_{0\le t\le L}] = \sigma L + \underbrace{\frac{\sigma}{2}\int_0^L dt\left(\frac{dx}{dt}\right)^2}_{\text{elasticity}} + \underbrace{\int_0^L dt\, V(x(t),t)}_{\text{disorder}}$$

elasticity vs disorder

where $V(x(t),t)$ is a quenched Gaussian rand. var.:

$$\mathbb{E}[V(x,t)] = \overline{V(x,t)} = 0$$

$$\mathbb{E}[V(x,t)V(x',t')] = \overline{V(x,t)\,V(x',t')} = \Delta^2\,\delta(x-x')\,\delta(t-t')$$

I] From 1d interfaces in random media to quantum interacting particles via replicas

1.) $$Z^n = \int_{x^1(0)=0}^{x^1(L)=x^1} \mathcal{D}x^1 \cdots \int_{x^n(0)=0}^{x^n(L)=x^n} \mathcal{D}x^n \, e^{-\beta n \sigma L - \beta \frac{\sigma}{2} \sum_{a=1}^{n} \int_0^L dt \left(\frac{dx^a}{dt}\right)^2 - \underbrace{\beta \sum_{a=1}^{n} \int_0^L dt \, V(x^a(t), t)}_{-\beta \int_{-\infty}^{+\infty} dx \int_0^L dt \, \rho(x,t) V(x,t)}}$$

$\to Z^n(x^1, \dots, x^n, L; 0, \dots 0, 0)$

where $$\rho(x,t) = \sum_{a=1}^{n} \delta(x - x^a(t))$$

2.) We now perform the average over the disorder:

$$\underbrace{\overline{Z^n} = \mathbb{E}(Z^n)}_{\to Z^n(x^1, \dots, x^n, L; 0, 0, \dots 0, 0)} = \int \mathcal{D}x^1 \cdots \int \mathcal{D}x^n \, e^{-\beta n \sigma L - \beta \frac{\sigma}{2} \sum_{a=1}^{n} \int_0^L dt \left(\frac{dx^a}{dt}\right)^2}$$

$$\times \underbrace{\mathbb{E}\left[e^{-\beta \int_{-\infty}^{+\infty} dx \int_0^L dt \, \rho(x,t) V(x,t)} \right]}$$

$$= e^{\frac{\beta^2}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \int_0^L dt \int_0^L dt' \, \rho(x,t) \rho(x',t') \mathbb{E}[V(x,t) V(x',t')]}$$

$$= e^{\frac{\beta^2}{2} \Delta^2 \int_{-\infty}^{+\infty} dx \int_0^L dt \, \rho^2(x,t)}$$

$$= e^{\frac{\beta^2}{2} \Delta^2 \int_{-\infty}^{+\infty} dx \int_0^L dt \sum_{a=1}^{n} \sum_{b=1}^{n} \delta(x - x^a(t)) \delta(x - x^b(t))}$$

$$= e^{\beta^2 \frac{\Delta^2}{2} \sum_{a,b=1}^{n} \int_0^L dt \, \delta\left(x^a(t) - x^b(t) \right)}$$

Hence we obtain the desired formula.

3.) Using the so-called "Feynman-Kac" formula, one can show that:

$$\overline{Z^n(x^1, \dots, x^n, L; 0, \dots, 0, 0)} = \langle x^1, \dots, x^n | e^{-\hat{H} L} | 0, \dots, 0 \rangle$$

where $\hat{H} = \frac{n}{\xi} - \frac{1}{2\beta\sigma} \sum_a \left(\frac{\partial}{\partial x^a} \right)^2 - \beta^2 \frac{\Delta^2}{2} \sum_{a \neq b} \delta(x^a - x^b)$

which is the Hamiltonian of n quantum particles interacting attractively via a $\delta(x-x')$ potential (attractive Lieb-Liniger model).

Integrating over the final position of the polymer

$$\ln \mathbb{E}[Z^n] = \ln \left(\int_{-\infty}^{\infty} dx^1 \dots \int_{-\infty}^{\infty} dx^n \langle x^1, \dots, x^n | e^{-\hat{H} L} | 0, \dots 0 \rangle \right)$$

One can show that the spectrum of $\hat{H}$ is discrete, with many-body eigenstates $|\alpha\rangle$ and corresponding eigenvalue E_α, such that:

$$\hat{H} = \sum_{\alpha=0}^{\infty} E_\alpha |\alpha\rangle\langle\alpha|$$

with $E_{GS} = E_0 < E_1 < \dots$

Hence, as $L \to \infty$

$$\ln \mathbb{E}(Z^n) \underset{L\to\infty}{\sim} \ln \left[\int_{-\infty}^{\infty} dx^1 \dots \int_{-\infty}^{\infty} dx^n \, e^{-E_{GS} L} \langle x^1, \dots, x^n | \alpha\rangle\langle\alpha | 0 \dots 0\rangle \right]$$

$$\boxed{\ln \mathbb{E}(Z^n) \underset{L\to\infty}{\sim} - E_{GS} L}$$

II) Solution of the quantum problem and average free energy.

1.) The exact ground state wave function reads:

$$\Psi_{GS}(x^1, \dots, x^n) = \mathcal{N} e^{-\frac{k}{2}\sum_{a \neq b} |x^a - x^b|}$$

Note that apart from the contact interactions in $\delta(x^a - x^b)$, the particles are free. Therefore the eigenfunctions are plane waves + boundary conditions due to the $\delta(x^a - x^b)$.

In a given sector (called Weyl chamber) $x^1 < x^2 < \dots < x^n$:

$$\frac{1}{2}\sum_{a \neq b} |x^a - x^b| = \frac{1}{2}\sum_{a<b}\left(|x^a - x^b| + |x^b - x^a|\right)$$

$$= \sum_{a<b} (x^b - x^a)$$

$$= \sum_{b=1}^{n} x^b (b-1) - \sum_{a=1}^{n} x^a (n-a)$$

$$= \sum_{a=1}^{n} x^a (2a - n - 1)$$

Hence, for $x^1 < x^2 < \dots < x^n$:

$$\Psi_{GS}(x^1, \dots, x^n) = \mathcal{N} e^{-\sum_{a=1}^{n} k_a x^a}, \quad k_a = k(2a - (n+1))$$

2.) In the sector $x^1 < x^2 < \dots < x^n$:

$$\hat{H} = \frac{n}{\xi} - \frac{1}{2\beta\sigma} \sum_{a=1}^{n} \left(\frac{\partial}{\partial x^a}\right)^2$$

therefore $E_{GS} = \frac{n}{\xi} - \frac{1}{2\beta\sigma} \sum_{a=1}^{n} (k_a)^2$

$$E_{GS} = \frac{n}{\xi} - \frac{k^2}{2\beta\sigma} \sum_{a=1}^{n} \left[2a - (n+1)\right]^2$$

Rk: In the other sector, e.g. $x^2 < x^1 < x^3 < \dots < x^n$ the wave function as of course a similar structure and the same E_{GS} (by symmetry).

3.) One thus obtains: $\ln \mathbb{E}(Z^n) \sim -L E_{GS}$

$$= -Ln\left[\frac{1}{\xi} - \frac{k^2}{6\beta\sigma}(n^2-1)\right]$$

Remembering that the replica trick yields:

$$\log \overline{Z^n} = n \overline{\ln Z} + O(n^2), \quad n \to 0$$

One finds: $\ln Z \underset{L\to\infty}{\sim} -L\left[\frac{1}{\xi} + \frac{k^2}{6\beta\sigma}\right]$

and therefore: $\boxed{-\beta f = \lim_{L\to\infty} \frac{\overline{\ln Z}}{L} = -\left(\frac{1}{\xi} + \frac{k^2}{6\beta\sigma}\right)}$

III) A paradox, its solution and the wandering of the interface.

1.) From the above results we have, for $L\to\infty$:

$$\ln \overline{Z^n} = -nL\left[\frac{1}{\xi} + \frac{k^2}{6\beta\sigma}\right] + n^3 L \frac{k^2}{6\beta\sigma}$$

$$= n\,\overline{\ln Z} + \frac{n^2}{2}\overline{(\ln Z)^2_c} + \frac{n^3}{3!}\overline{(\ln Z)^3_c} + \dots$$

where $\overline{(\ln Z)^k_c}$ is the k^{th} cumulant of $\ln Z$.

Hence by identifying the powers of 'n', one gets:

$$\overline{(\ln Z)^2_c} = \overline{(\beta F)^2_c} = 0$$

$$\frac{1}{3!}\overline{(\ln Z)^3_c} = L\frac{k^2}{6\beta\sigma} = \frac{1}{3!}\overline{(-\beta F)^3_c}$$

$\Rightarrow$ that $\boxed{\delta F \sim L^{1/3}}$

typical fluctuations of F.

2.) But this result contains a paradox.

Indeed for X a random variable:

$$\mathrm{Var}(X) = 0 \iff \mathbb{E}\left(X - \mathbb{E}(X)\right)^2 = 0$$

$$\Rightarrow X - \mathbb{E}(X).$$

And therefore the higher cumulants of X are <u>zero</u>.

However here: $\mathrm{Var}\left((\beta F)^2\right) = \overline{(\beta F)^2_c} = 0$

but $\overline{(\beta F)^3_c} \neq 0 \Rightarrow$ **PARADOX**

3.) This paradox indicates that the two limits:

(i) $L \longrightarrow \infty$

(ii) $n \longrightarrow 0$ do not commute.

Here we did (i) AND (ii). However to obtain correctly the cumulants one needs to take (ii) AND THEN (i).

As is often the case in such situations where two limits do not commute, there is actually a scaling function of n and L (or of some powers of n and L).

Here an exact computation of E_{GS} shows:

$$(*) \quad -\frac{1}{L}\ln \overline{Z^n} = \left(\frac{1}{\xi}+\frac{k^2}{6\beta\sigma}\right)n + \frac{1}{L} f(nL^{1/3})$$

Note that the extensivity of $-\frac{1}{L}\ln\overline{Z^n}$ implies

that $$\frac{1}{L} f(nL^{1/3}) \underset{L\to\infty}{\sim} c^{st}$$
(n fixed)

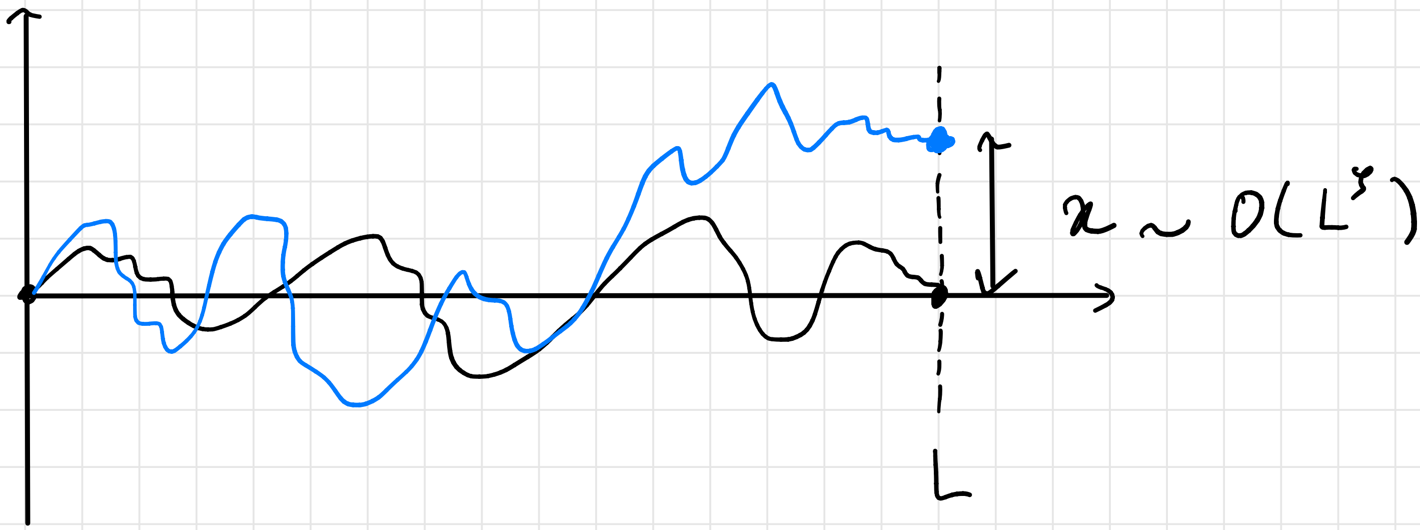
and therefore $f(z) \propto z^3, \; z\to\infty$.

But one can then expand the right hand side of the Eq. (*) above to get:

$$\left(\frac{1}{\xi}+\frac{k^2}{6\beta\sigma}\right)n + \frac{1}{L} f(nL^{1/3}) = \frac{1}{L} f(0) + n\left[\frac{1}{\xi}+\frac{k^2}{6\beta\sigma}+\frac{1}{L^{2/3}} f'(0)\right] + \frac{1}{2} n^2 \frac{1}{L^{1/3}} f''(0) + \frac{1}{3!} n^3 f'''(0)$$

Hence there is no real paradox: there is a $O(n^2)$ term but it is much smaller than $O(n^3)$, in the limit $L \to \infty$.

4.) Scaling argument for the transverse fluctuations :



Assuming $^{(**)}$ $$-\beta \delta F = L^{1/3} g_\beta\left(\frac{x}{L^\zeta}\right)$$

From the previous computation

Q: what is the value of ζ ?

When $\frac{x}{L^\zeta} \gg 1$ one expects an elastic behavior of the line, i.e.

$$-\beta\delta F \sim \frac{x^2}{L} \quad \text{(elasticity theory)}$$

On the other hand, from the scaling form in (**):

$$-\beta\delta F \sim L^{1/3}\left(\frac{x}{L^{\zeta}}\right)^2 \sim \frac{x^2}{L^{2\zeta-\frac{1}{3}}}$$

Hence by matching with the elastic behavior:

$$L = L^{2\zeta - 1/3}$$

$$\Rightarrow \quad \zeta = \frac{2}{3}$$

which is indeed the correct result.

<u>Conclusion</u>: $x = O(L^{2/3}) \gg O(L^{1/2})$