# ICFP M2 - Statistical physics 2
## Solution of the homework n° 6
## Random Matrices

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The two eigenvalues of $M$ are the solutions of its characteristic equation

$$\lambda^2 - (M_{11} + M_{22})\lambda + (M_{11}M_{22} - M_{12}^2) = 0 \ , \tag{1}$$

which read

$$\frac{M_{11} + M_{22}}{2} \pm \frac{1}{2}\sqrt{(M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}^2)} \ , \tag{2}$$

hence

$$\Delta = \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2} \ . \tag{3}$$

Denoting $X = M_{11} - M_{22}$ and $Y = 2M_{12}$, we realize that $X$ and $Y$ are two independent Gaussian random variables, both of variance 2, and that $\Delta = \sqrt{X^2 + Y^2}$ can be seen as the distance from the origin of a point drawn in the plane with this distribution. Hence the density of $\Delta$ is

$$\widehat{P}(\Delta) = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y \frac{1}{4\pi} e^{-\frac{x^2+y^2}{4}} \delta(\Delta - \sqrt{x^2 + y^2}) = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r\, e^{-\frac{r^2}{4}} \delta(\Delta - r) = \frac{1}{2}\Delta e^{-\frac{\Delta^2}{4}} \ , \tag{4}$$

after a change of variable towards polar coordinates. The average value of $\Delta$ is thus

$$\mathbb{E}[\Delta] = \int_0^\infty \mathrm{d}\Delta \frac{1}{2}\Delta e^{-\frac{\Delta^2}{4}} \Delta = \sqrt{\pi} \ . \tag{5}$$

Changing variables from $\Delta$ to $s = \Delta/\mathbb{E}[\Delta]$ yields the probability density

$$P(s) = \widehat{P}(\Delta = s\sqrt{\pi})\sqrt{\pi} = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \ .$$