ICFP M2 - Statistical physics 2 – Exam

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The exam is made of two parts. The first one is a series of short independent questions to check your knowledge of the contents of some of the lectures, the second one is a longer problem with partially independent subparts.

No document, calculator nor phone is allowed.

You can write your answers in English or French.

1 Questions on the lectures

1. Q1: We consider the Larkin model for a one-dimensional elastic interface $u(x)$, with $0 \leq x \leq L$ in the presence of a random Gaussian force. The Hamiltoian of this model reads

$$H[\{u(x)\}] = \frac{\gamma}{2} \int_0^L dx \left(\frac{du}{dx}\right)^2 + \int_0^L dx\, f(x) u(x) \;, \tag{1}$$

where γ is the stiffness of the elastic line and $f(x)$ is a Gaussian random force field of zero mean $\langle f(x) \rangle = 0$ and short-range correlations $\langle f(x) f(x') \rangle = \Delta\, \delta(x-x')$. We assumed that the line is pinned at the extremity $x = 0$, i.e. $u(x = 0) = 0$.

a) Recall the physical origin of both contributions in (1).

b) Show that the ground state configuration $u_0(x)$, i.e., which minimises $H[\{u(x)\}]$, satisfies

$$\gamma \frac{d^2 u_0(x)}{dx^2} = f(x) \;. \tag{2}$$

c) Deduce that $u_0(x)$ reads

$$u_0(x) = \frac{1}{\gamma} \int_0^x dy \int_0^y dz\, f(z) + A\, x + B \tag{3}$$

and explain how to determine the constants A and B.

Once these constants have been determined, one can show that the optimal profile reads

$$u_0(L) = -\frac{1}{\gamma} \int_0^L dz\, z\, f(z) \;. \tag{4}$$

d) Compute $\langle u_0(L) \rangle$ and $\langle u_0^2(L) \rangle$. Compare the latter to the fluctuations of a purely elastic line – in the absence of the second term in (1) – which is at thermal equilibrium at temperature T (you are not expected to provide a detailed computations of these thermal elastic fluctuations but to estimate instead their behaviour as a function of L).

2. Q2: Random Gaussian matrices and the log-gas.

a) Recall what the Gaussian Orthogonal Ensemble (GOE) is. In particular, what does "Orthogonal" refer to?

b) The joint distribution of the eigenvalues of a random matrix belonging to the GOE reads

$$P_{\rm joint}(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} e^{-\frac{N}{4} \sum_{i=1}^N \lambda_i^2} \prod_{i \neq j} |\lambda_i - \lambda_j| \;. \tag{5}$$

Explain why Z_N can be interpreted as the partition function of a one-dimensional gas of particles at some inverse temperature $\beta = 1$. What is the corresponding energy function associated to this gas of particle? Why is it called a *log-gas*?

2 Quenched versus annealed disorder

Consider a system whose degrees of freedom are arbitrarily split into two distinct vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with Hamiltonian $H(\boldsymbol{x},\boldsymbol{y})$. The equilibrium probability distribution of the total system is

$$\rho(\boldsymbol{x},\boldsymbol{y}) = \exp[-\beta H(\boldsymbol{x},\boldsymbol{y})]/Z \ , \tag{6}$$

where Z is the total partition function. An equilibrium configuration of the system can be generated by sampling from $\rho(\boldsymbol{x},\boldsymbol{y})$. Consider then the degrees of freedom $\boldsymbol{y}$ alone: their statistics is described by the marginal probability distribution

$$\rho(\boldsymbol{y}) = \int \mathrm{d}\boldsymbol{x} \rho(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{Z}\int \mathrm{d}\boldsymbol{x} e^{-\beta H(\boldsymbol{x},\boldsymbol{y})} = \frac{Z(\boldsymbol{y})}{Z} \ , \tag{7}$$

where $Z(\boldsymbol{y}) = \int \mathrm{d}\boldsymbol{x} e^{-\beta H(\boldsymbol{x},\boldsymbol{y})}$ is the partition function of $\boldsymbol{x}$ at fixed $\boldsymbol{y}$. Next, consider a different thought experiment, in which one first extracts the $\boldsymbol{y}$ degrees of freedom from $\rho(\boldsymbol{y})$, and then considers the $\boldsymbol{x}$ degrees of freedom as dynamical variables with equilibrium probability $\rho(\boldsymbol{x}|\boldsymbol{y})$, at fixed $\boldsymbol{y}$ playing the role of a quenched disorder. We then have

$$\rho(\boldsymbol{x}|\boldsymbol{y}) = \frac{\rho(\boldsymbol{x},\boldsymbol{y})}{\rho(\boldsymbol{y})} = \frac{e^{-\beta H(\boldsymbol{x},\boldsymbol{y})}}{Z(\boldsymbol{y})} \ . \tag{8}$$

Given these definitions, answer the following questions.

1. For any observable $O(\boldsymbol{x})$, we can define an "annealed" average $\langle O \rangle$, a "thermal" average $\langle O \rangle_{\boldsymbol{y}}$, and a "quenched" average $\overline{\langle O \rangle_{\boldsymbol{y}}}$ from

$$\langle O \rangle = \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \rho(\boldsymbol{x},\boldsymbol{y}) O(\boldsymbol{x}) \ , \qquad \langle O \rangle_{\boldsymbol{y}} = \int \mathrm{d}\boldsymbol{x} \rho(\boldsymbol{x}|\boldsymbol{y}) O(\boldsymbol{x}) \ , \qquad \overline{\langle O \rangle_{\boldsymbol{y}}} = \int \mathrm{d}\boldsymbol{y} \rho(\boldsymbol{y}) \langle O \rangle_{\boldsymbol{y}} \ . \tag{9}$$

   Show that the annealed and quenched averages coincide.

2. Consider a spin glass alloy, in which the magnetic degrees of freedom are denoted by $\boldsymbol{x}$. The position of the magnetic atoms is denoted by $\boldsymbol{y}$. The total magnetic Hamiltonian, e.g. of the form

$$H(\boldsymbol{x},\boldsymbol{y}) = -\sum_{i,j} J(|y_i - y_j|) x_i x_j \ , \tag{10}$$

   depends on both the spins and positional degrees of freedom of the atoms. Explain why, in a typical realization of a spin glass, the quenched and annealed averages do not coincide.

Bayes optimal inference - Consider a signal $\boldsymbol{x}$, generated with probability distribution $P(\boldsymbol{x})$, which is transmitted through a noisy channel, resulting in a corrupted signal $\boldsymbol{y}$ whose distribution is $P(\boldsymbol{y}|\boldsymbol{x})$. We assume that the receiver knows both $P(\boldsymbol{x})$ and $P(\boldsymbol{y}|\boldsymbol{x})$ (this is called *Bayes optimal* inference), and they want to reconstruct $\boldsymbol{x}$ given $\boldsymbol{y}$ using Bayes' formula,

$$P(\boldsymbol{x}|\boldsymbol{y}) = \frac{P(\boldsymbol{y}|\boldsymbol{x})P(\boldsymbol{x})}{P(\boldsymbol{y})} = \frac{P(\boldsymbol{x},\boldsymbol{y})}{\int \mathrm{d}\boldsymbol{x} P(\boldsymbol{x},\boldsymbol{y})} \ . \tag{11}$$

The receiver estimates $\boldsymbol{x}$ by sampling from $P(\boldsymbol{x}|\boldsymbol{y})$, which is then considered as a "thermal" probability distribution for the estimated $\boldsymbol{x}$ given the "quenched disorder" $\boldsymbol{y}$. Defining a Hamiltonian $H(\boldsymbol{x},\boldsymbol{y}) = -\log P(\boldsymbol{x},\boldsymbol{y})$ (the temperature can be conventionally fixed to $\beta = 1$), this setting corresponds to Eq. (8). We conclude that in Bayes optimal inference the disorder is annealed, so that the quenched and annealed averages coincide.

3. **Annealed spin glass -** Consider an Ising spin glass defined on an Erdös-Rényi random graph $G = (V,E)$ with N nodes and $M = cN/2$ edges. On each node $i \in V$ there is an Ising variable $S_i = \pm 1$, and on each edge $\langle i,j \rangle \in E$ there is a coupling $J_{ij} = \pm 1$. The thermal degrees of freedom are $\boldsymbol{x} \to \boldsymbol{S} = \{S_i\}$ and the disorder is $\boldsymbol{y} \to \boldsymbol{J} = \{J_{ij}\}$ (we consider here the graph as fixed). The Hamiltonian is

$$H(\boldsymbol{S},\boldsymbol{J}) = -\sum_{\langle i,j \rangle \in E} J_{ij} S_i S_j \ . \tag{12}$$

   In the setting discussed above, where all degrees of freedom $(\boldsymbol{S},\boldsymbol{J})$ are in equilibrium at temperature $1/\beta$, give the expression of the total partition function Z, of the distribution of the disorder $\rho(\boldsymbol{J})$ and of the thermal distribution $\rho(\boldsymbol{S}|\boldsymbol{J})$ in terms of the Hamiltonian. Show that the average energy per spin is

$$e(\beta) = -\frac{c}{2}\tanh(\beta) \ . \tag{13}$$

4. **Planted spin glass -** Consider the following inference problem. We have N agents, and each agent is assigned to one of two groups: agents with $S_i^* = 1$ are in the first group, and agents with $S_i^* = -1$ are in the second group. The assignment $S_i^* = \pm 1$ is chosen independently with equal probability for the two groups. An Erdös-Renyi random graph $G = (V, E)$ with N nodes and $M = cN/2$ edges is then generated. For each pair of agents $\langle i, j \rangle \in E$, a coupling $J_{ij} = S_i^* S_j^*$ is chosen with probability ρ, and $J_{ij} = -S_i^* S_j^*$ with probability $1 - \rho$. Hence, with probability ρ the coupling tells you the truth about whether the two agents are in the same group ($J_{ij} = 1$) or not ($J_{ij} = -1$), while with probability $1 - \rho$ the coupling has the "wrong" sign. We introduce a temperature by the change of variable

$$\rho = \frac{e^\beta}{2\cosh(\beta)} \qquad \Rightarrow \qquad P(J_{ij}|S_i^* S_j^*) = \frac{e^{\beta J_{ij} S_i^* S_j^*}}{2\cosh(\beta)} \ . \tag{14}$$

Consider the problem in which the true assignment $\boldsymbol{S}^*$ has been hidden, and one has to infer it from $\boldsymbol{J}$ (again, at fixed graph G). Show that the zero temperature limit corresponds to $\rho = 1$, hence perfect information, while the infinite temperature limit corresponds to $\rho = 1/2$, hence no information. In the Bayes optimal setting, compute $P(\boldsymbol{J})$ and $P(\boldsymbol{S}|\boldsymbol{J})$ and show that they coincide with $\rho(\boldsymbol{J})$ and $\rho(\boldsymbol{S}|\boldsymbol{J})$ introduced in Eqs. (6,7,8), with the Hamiltonian given in Eq. (12). Show that $\boldsymbol{S}^*$ is a ground state of the Hamiltonian in Eq. (12). This is called a "planted spin glass" because a ground state has been planted "by hand" in the system.

5. **Overlap of the reconstruction -** Consider again the inference problem in which one first generates $\boldsymbol{S}^*$ uniformly among the 2^N possible assignments, then generates $\boldsymbol{J}$ according to Eq. (14), and finally tries to infer $\boldsymbol{S}$ from $P(\boldsymbol{S}|\boldsymbol{J})$ in the Bayes optimal setting. We want to compute the typical overlap $m = N^{-1} \sum_i S_i S_i^*$ between the true assignement and the inferred one, in the thermodynamic limit $N \to \infty$. If $m = 0$, then no information can be obtained. If $m > 0$, then the assignment can be partially recovered. Introduce the "gauge transformation"

$$\tilde{S}_i = S_i S_i^* \ , \qquad \tilde{J}_{ij} = J_{ij} S_i^* S_j^* \ . \tag{15}$$

Show that the Hamiltonian in Eq. (12) is invariant under this transformation, and that

$$P(\tilde{J}_{ij} = \pm 1 | S_i^* S_j^*) = \frac{e^{\pm\beta}}{2\cosh(\beta)} \tag{16}$$

is now independent of $\boldsymbol{S}^*$. Deduce that the problem of computing the typical value of m is equivalent to the problem of computing the typical magnetization of an Ising spin glass with Hamiltonian Eq. (12) and random couplings distributed according to Eq. (16)

6. *Optional:* Comment on the notion of "typical" versus "average", and on whether you expect m to be self-averaging.

7. Because we are working in the Bayes optimal setting, the quenched and annealed averages coincide. Show that the annealed average of the partition function of the model defined by Eqs. (12) and (16) can be written as

$$\overline{Z(\tilde{\boldsymbol{J}})} = e^{\frac{NcA}{2}} \sum_{\tilde{\boldsymbol{S}}} e^{J_0 \sum_{\langle i,j \rangle \in E} \tilde{S}_i \tilde{S}_j} \ , \qquad A = \frac{1}{2} \log \frac{\cosh(2\beta)}{\cosh(\beta)^2} \ , \qquad J_0 = \frac{1}{2} \log \cosh(2\beta) \ . \tag{17}$$

Hint: recall that any function of an Ising spin variable $S = \pm 1$ can be rewritten as

$$f(S) = [f(1) + f(-1)]/2 + S[f(1) - f(-1)]/2 \ . \tag{18}$$

8. **Reconstruction phase transition -** Observe that Eq. (17) is the partition function of a ferromagnetic model on an Erdos-Renyi graph of connectivity c, with exchange coupling J_0. Recall why the standard mean-field approximation is justified for large c, and deduce that the critical temperature β^* of the model is defined by the condition

$$J_0 c = 1 \ , \qquad c \to \infty \ . \tag{19}$$

Deduce that $\beta^* = 1/\sqrt{c}$ at leading order for large c. Conclude that inference is possible only if the connectivity of the graph is

$$c > \frac{1}{(2\rho - 1)^2} \ , \qquad c \to \infty \ , \quad \rho \to 1/2 \ . \tag{20}$$

9. *Optional:* The exact critical temperature for the ferromagnetic model on the Erdos-Rényi graph at finite connectivity c is given by the condition $c \tanh(J_0) = 1$ (this can be derived by a slightly more complex replica calculation or, more easily, by the cavity method). Show that this gives $\beta^* = \operatorname{atanh}(1/\sqrt{c})$ and leads to the same condition in Eq. (20), which then holds at any finite c and ρ.