ICFP M2 - Statistical physics 2 – Exam

Grégory Schehr, Guilhem Semerjian

April 10th, 2020

The exam is made of two parts. The first one is a series of short independent exercices to check your knowledge and understanding of the contents of some of the lectures, the second one is a longer problem with partially independent subparts.

No document, calculator nor phone is allowed.

You can write your answers in English or French.

1 Questions on the lectures

1. We consider a disordered system whose microscopic degrees of freedom are N Ising spins, their configurations being denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1,1\}^N$. The energies of these configurations are given by

$$H(\underline{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - \sqrt{\frac{3}{N^2}} \sum_{1 \le i < j < k \le N} J'_{ijk} \sigma_i \sigma_j \sigma_k \ ,$$

where the various coupling constants J_{ij} and J'_{ijk} are independent identically distributed Gaussian random variables with zero average and variance 1.

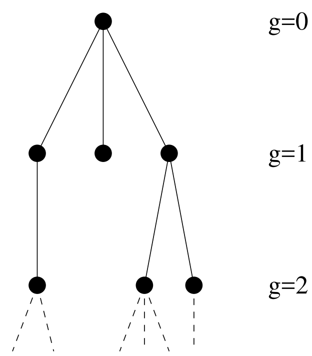
   (a) Explain why the energies $H(\underline{\sigma})$ are Gaussian random variables.

   (b) Compute their first two moments, $\mathbb{E}[H(\underline{\sigma})]$ and $\mathbb{E}[H(\underline{\sigma})H(\underline{\tau})]$, for two arbitrary configurations $\underline{\sigma}$ and $\underline{\tau}$.

   (c) Show that in the large N limit one can write the latter as $\mathbb{E}[H(\underline{\sigma})H(\underline{\tau})] = \frac{N}{2} g(q(\underline{\sigma}, \underline{\tau})) + O(1)$, where $q(\underline{\sigma}, \underline{\tau}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tau_i$ is the overlap between the two configurations. Specify the function $g(q)$.

2. This exercice is about Galton-Watson branching processes.

In such a process one considers an individual (the ancestor) that has a random number of offsprings (its descendents), who themselves have offsprings, and so on and so forth. One assumes that the numbers of descendents of each individual are i.i.d. random variables, the probability of having k descendents being q_k. We denote $\mathcal{N}^{(g)}$ the number of individuals in the g-th generation (with $g = 0$ for the ancestor, hence $\mathcal{N}^{(0)} = 1$).



We assume that the number of descendents k is a binomial random variable of parameters l, an integer ≥ 2, and $p \in [0,1]$, in other words that $q_k = \binom{l}{k} p^k (1-p)^{l-k}$ for $k = 0, 1, \dots, l$.

   (a) Compute the average number of descendents of one individual, $m = \mathbb{E}[k]$.

   (b) Compute the average number of individuals at the g-th generation, $\mathbb{E}[\mathcal{N}^{(g)}]$. Discuss its behavior when $g \to \infty$; is there a phase transition on this quantity when the parameter p is varied ? At which critical value p_c ?

   (c) We denote $f(p,l)$ the probability that the branching process persists during an infinite number of generations. Show that $f(p,l)$ is solution of the self-consistent equation $f = 1 - (1-pf)^l$. Study graphically this equation, draw the shape of $f(p,l)$ as a function of p, and give the value of the critical exponent that describes its behavior around p_c.

3. We come back in this exercise on the Imry-Ma criterion for the possibility of a ferromagnetically ordered phase in presence of random fields. We thus consider an Hamiltonian of the form

$$-J\sum_{\langle i,j\rangle}\sigma_i\sigma_j-\sum_i h_i\sigma_i\ ,$$

where the $\sigma_i=\pm1$ are Ising spins, the first sum is over the edges of the d-dimensional lattice $\mathbb{Z}^d$, $J>0$ is a ferromagnetic coupling constant, and the fields h_i are independent identically distributed random variables with zero average and some strictly positive variance.

(a) What is the order of magnitude of the energy cost arising from the first term when creating a domain of linear size ℓ of negative magnetization inside a configuration of positive magnetization ?

(b) What is the order of magnitude of the second term for such a transformation ?

(c) Compare the two and recall the conclusion of the Imry-Ma argument concerning the existence of a ferromagnetically ordered phase at low temperature, as a function of the space dimension d.

2 The Wigner semi-circular law via replicas

We consider an $N\times N$ random matrix $\mathbf{J}$ drawn from the Gaussian Orthogonal Ensemble (GOE), i.e. $\mathbf{J}$ is a real symmetric matrix such that, for $k\le l$, the entries J_{kl} are independent Gaussian random variables of zero mean and variances $\mathbb{E}[J_{kk}^2]=2/N$ and $\mathbb{E}[J_{kl}^2]=1/N$ for $k<l$. We denote $\lambda_1,\lambda_2,\cdots,\lambda_N$ the N real eigenvalues of $\mathbf{J}$ and the goal is to compute their average density

$$\rho(x)=\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right]\ , \tag{1}$$

in the limit $N\to\infty$.

In the following ϵ denotes a strictly positive real number, $\epsilon>0$.

We recall the following Gaussian integrals which hold for an invertible symmetric square matrix $\mathbf{A}$ of size $m\times m$ (not necessarily real, provided the corresponding integrals are well defined) and a complex vector $\vec{b}\in\mathbb{C}^m$:

$$\int_{\mathbb{R}^m}\mathrm{d}\vec{x}\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}}=\frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}}\ , \tag{2}$$

$$\int_{\mathbb{R}^m}\mathrm{d}\vec{x}\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}+\vec{b}^T\vec{x}}=\frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}}\mathrm{e}^{\frac{1}{2}\vec{b}^T\mathbf{A}^{-1}\vec{b}}\ , \tag{3}$$

$$\int_{\mathbb{R}^m}\mathrm{d}\vec{x}\ x_kx_l\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}}=\frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}}(\mathbf{A}^{-1})_{kl}\ . \tag{4}$$

2.1 Preamble

1.) Using the identity, valid for any real x in the sense of distribution,

$$\lim_{\epsilon\to0^+}\frac{1}{x-i\epsilon}=\mathrm{PV}\frac{1}{x}+i\,\pi\delta(x)\ , \tag{5}$$

where PV denotes the Cauchy principal value, show that

$$\rho(x)=-\frac{1}{N\pi}\lim_{\epsilon\to0^+}\mathrm{Im}\left(\mathbb{E}\left[\mathrm{Tr}(\mathbf{J}-x_\epsilon\mathbb{I}_N)^{-1}\right]\right)\ , \tag{6}$$

where here and in the following x_ϵ is defined as $x_\epsilon=x-i\epsilon$, and $\mathbb{I}_N$ denotes the identity matrix of size N.

2.) Consider the N-dimensional Gaussian integral

$$\begin{aligned} Z_J(x_\epsilon) &= \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2 + \frac{i}{2} \sum_{k,l} J_{kl}\, \phi_k \phi_l\right] && (7) \\ &= \int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\, \exp\left[-\frac{1}{2}\vec{\phi}^T (i x_\epsilon \mathbb{I}_N - i\,\mathbf{J})\vec{\phi}\right] . && (8) \end{aligned}$$

Justify that $Z_J(x_\epsilon)$ is well-defined as a convergent integral, and show, using the results for Gaussian integrals recalled above, that

$$\mathrm{Tr}(\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1} = 2\,\frac{\mathrm{d}}{\mathrm{d}x} \ln Z_J(x_\epsilon) . \tag{9}$$

2.2 A simplified computation

3.) To compute $\rho(x)$ we will first make the approximation $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$. In the context of disordered systems, how is such an approximation called ? In the general context of disordered systems, when do you expect this approximation to be valid ?

4.) Show that the average "partition function" $\mathbb{E}[Z_J(x_\epsilon)]$ can be written as

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^N \phi_k^2 - \frac{1}{4N}\left(\sum_{k=1}^N \phi_k^2\right)^2\right] . \tag{10}$$

5.) Using the standard Gaussian identity (3) with $m = 1$ to disentangle the ϕ_k variables show that

$$\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{-N\varphi(q,x_\epsilon)} , \quad \text{where} \quad \varphi(q, x_\epsilon) = q^2 - \frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon - 2q)}\right) . \tag{11}$$

6.) Explain why, to compute the density $\rho(x)$ in the large N limit, we only need to retain the leading exponential behavior in (11).

7.) At this order of accuracy, the integral over q in (11) can be evaluated, for large N, by the Laplace (or saddle-point) method. To find this saddle point, we need to solve $\partial_q \varphi(q, x_\epsilon) = 0$. Show that this equation has two solutions $q = q_+$ and $q = q_-$, with

$$q_\pm = \frac{1}{4}\left(x_\epsilon \pm \sqrt{x_\epsilon^2 - 4}\right) . \tag{12}$$

8.) It can be shown that the correct saddle point is $q = q_+$, if one chooses the branch of the square root such that $\sqrt{-u} = i\sqrt{u}$ for $u > 0$. Deduce then that in the limit $N \to \infty$ one has

$$\lim_{N\to\infty} \rho(x) = \frac{2}{\pi} \lim_{\epsilon\to 0^+} \mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+, x_\epsilon) . \tag{13}$$

9.) Obtain finally that, when $N \to \infty$, $\rho(x) = \sqrt{4 - x^2}/(2\pi)$ if $|x| \leq 2$ while $\rho(x) = 0$ if $|x| > 2$.

Even if we have made the approximation $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$ this computation has given the correct result, namely the Wigner semi-circle law. In the next part of the problem we shall see how to avoid this approximation.

2.3 A more complete computation

To compute the average $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right]$, going beyond the approximation $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$, we will use the replica trick under the form

$$\mathbb{E}[\ln Z_J(x_\epsilon)] = \lim_{n\to 0} \frac{1}{n} \ln\left(\mathbb{E}[Z_J(x_\epsilon)^n]\right) \ . \tag{14}$$

10.) Use this replica trick (14) to write $\rho(x)$ in terms of $\mathbb{E}\left[Z_J(x_\epsilon)^n\right]$.

11.) By performing the averages over the random variables J_{kl}, show that when n is a positive integer,

$$\mathbb{E}[Z_J(x_\epsilon)^n] = \int_{-\infty}^{\infty} \prod_{k=1}^{N}\prod_{a=1}^{n} \mathrm{d}\phi_k^a \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^{N}\sum_{a=1}^{n}(\phi_k^a)^2 - \frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^b\right)^2\right] . \tag{15}$$

12.) By using $n(n+1)/2$ times the Gaussian identity (3) with $m=1$ show that

$$\begin{aligned}\mathbb{E}[Z_J(x_\epsilon)^n] \quad &= \quad A_{N,n}\int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a \prod_{1\le a\le b\le n}\mathrm{d}Q_{ab} \\ &\times \quad \exp\left[-N\sum_{a,b=1}^{n}(Q_{ab})^2 - \frac{i}{2}\,x_\epsilon\sum_{k=1}^{N}\sum_{a=1}^{n}(\phi_k^a)^2 + i\sum_{a,b=1}^{n} Q_{ab}\sum_{k=1}^{N}\phi_k^a\phi_k^b\right] ,\end{aligned} \tag{16}$$

where $A_{N,n} = 2^{n(n-1)/4}\left(\frac{N}{\pi}\right)^{n(n+1)/4}$ and with the convention, in (16), that $Q_{ab} = Q_{ba}$ when $a > b$.

13.) Observe that the n-dimensional vectors $\{\phi_k^1, \phi_k^2, \cdots, \phi_k^n\}$, for different values of $k = 1, \cdots, N$, are now independent in this expression (16), and show that

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n}\int_{-\infty}^{\infty}\prod_{1\le a\le b\le n}\mathrm{d}Q_{ab}\exp\left[-N\Phi(\mathbf{Q}, x_\epsilon)\right] , \tag{17}$$

where $\mathbf{Q}$ is the (symmetric) overlap matrix, of size $n\times n$, with matrix elements Q_{ab} for $a \le b$ (and thus $Q_{ab} = Q_{ba}$ if $a > b$), while $\Phi(\mathbf{Q}, x_\epsilon)$ is given by

$$\Phi(\mathbf{Q}, x_\epsilon) = \mathrm{Tr}[\mathbf{Q}^2] - \frac{1}{2}\ln\left[\det\left(\frac{2\pi}{i}\left(x_\epsilon\,\mathbb{I}_n - 2\mathbf{Q}\right)^{-1}\right)\right] , \tag{18}$$

with $\mathbb{I}_n$ the identity matrix of size n.

14.) In the large N limit, this multiple integral (17) can be evaluated by the Laplace (or saddle-point) method. Show that the saddle point equations read

$$2\,Q_{ab} - ((x_\epsilon\mathbb{I}_n - 2\mathbf{Q})^{-1})_{ab} = 0\ , \quad \forall\, a, b = 1, \cdots, n\ . \tag{19}$$

Hint : to compute the derivative of the second term in Φ you can either exploit its expression in terms of a Gaussian integral, or use the fact that, for an invertible matrix $\mathbf{Y}$ depending on a parameter z, one has $\partial_z(\det\mathbf{Y}) = (\det\mathbf{Y})\,\mathrm{Tr}(\mathbf{Y}^{-1}\partial_z\mathbf{Y})$.

15.) Show that these equations (19) admit the (replica symmetric) solution $Q_{ab} = q\,\delta_{ab}$ with $q = q_\pm$ as defined in (12).

16.) Using the ansatz $Q_{ab} = q_+\delta_{ab}$, assuming that this is the correct saddle-point solution to (19), compute the density $\rho(\lambda)$ in the large N limit and show that it gives back the Wigner semi-circle.

2.4 The deformed GOE case

We now consider the case of a symmetric Gaussian random matrix where the matrix elements J_{kl} have all the same mean value $\mathbb{E}[J_{kl}] = \mu/N$, with $\mu > 0$, their variances being the same as before, i.e. $\mathbb{E}[(J_{kk}-\mu/N)^2] = 2/N$, while $\mathbb{E}[(J_{kl}-\mu/N)^2] = 1/N$ for $k < l$. Equivalently one can view this matrix $\mathbf{J}$ as the sum of an usual GOE random matrix and a matrix $\mathbf{H}$ with $H_{ij} = \mu/N$ for all its matrix elements.

17.) What is the spectrum of the matrix $\mathbf{H}$? We denote $\lambda_{\max}(\mu)$ the largest eigenvalue of $\mathbf{J}$. Discuss the behavior of this quantity when $\mu \to +\infty$ (you can then neglect the GOE contribution to $\mathbf{J}$) and when $\mu \to 0$ (where one recovers the GOE case). Which behavior can one imagine for intermediate values of μ ?

In the following we investigate this question by adapting the simplified computation to the $\mu > 0$ case (one could also show that the more complete one yields the same result).

18.) Show that the average "partition function" $\mathbb{E}[Z_J(x_\epsilon)]$ becomes

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^{N} \phi_k^2 - \frac{1}{4N}\left(\sum_{k=1}^{N}\phi_k^2\right)^2 + i\frac{\mu}{2N}\left(\sum_{k=1}^{N}\phi_k\right)^2\right] . \tag{20}$$

19.) By using the standard Gaussian identity (3) for $m = 1$ to treat the quartic terms in Eq. (20) and by subsequently performing the integrals over the ϕ_k's [using (2)] show that

$$\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q \sqrt{\frac{x_\epsilon - 2q}{x_\epsilon - \mu - 2q}}\; \mathrm{e}^{-N\,\varphi(q,x_\epsilon)} \;,\; \varphi(q,x_\epsilon) = q^2 - \frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon - 2q)}\right) . \tag{21}$$

<u>Hint :</u> we recall that, given two vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{C}^N$, one has $\det(\mathbb{I}_N + \vec{u}\,\vec{v}^T) = 1 + \vec{v}^T\vec{u}$.

We thus see on (21) that, if $\mu > 0$, the integrand in the integral over q has now an additional square-root singularity at $q_1 = (x_\epsilon - \mu)/2$. We thus deform the contour of integration over q in the complex q-plane such that it passes near q_1 and write the contour integral as the sum of two contributions : a first contribution, away from q_1, where the square-root singularity can be neglected because of the factor N in the exponential, namely $\mathrm{e}^{-N\varphi(q,x_\epsilon)}$, which gives the Wigner semi-circle law studied previously, and a second contribution when q is close to q_1 which we now analyse in detail.

20.) Show that the Taylor expansion of $\varphi(q, x_\epsilon)$ close to q_1, for a fixed value of x_ϵ, can be written as

$$\varphi(q,x_\epsilon) = \varphi(q_1,x_\epsilon) + (q-q_1)(x_\epsilon - x_m(\mu)) + (q-q_1)^2\frac{\mu^2-1}{\mu^2} + \mathcal{O}((q-q_1)^3) \;, \tag{22}$$

and determine $x_m(\mu)$.

21.) Hence if $x = x_m(\mu)$ exactly, and in the limit $\epsilon \to 0^+$, $\varphi(q, x_\epsilon)$ has a stationary point at q_1. Explain why the contribution of this stationary point to $\mathbb{E}[Z_J(x_\epsilon)]$ needs to be included only for $\mu > 1$. Show that in this case $x_m(\mu)$ is *outside* the support of the Wigner semi-circle.

22.) A more detailed analysis of (21) reveals that for $\mu > 1$ the additional contribution to $\mathbb{E}[Z_J(x_\epsilon)]$ that arises from the neighborhood of q_1 is, for small ϵ, large N and x close to $x_m(\mu)$:

$$\mathbb{E}[Z_J(x_\epsilon)] \sim B\frac{\mathrm{e}^{-N\varphi(q_1,x_\epsilon)}}{\sqrt{x_\epsilon - x_m(\mu)}} \;, \tag{23}$$

where B is an unimportant constant, independent of N and x_ϵ. Deduce from the expression (23) that, for $\mu > 1$, this yields an additional contribution to the average distribution that reads

$$\rho(\lambda) \sim \frac{1}{N}\delta(\lambda - x_m(\mu)) \;, \tag{24}$$

and hence that one can identify $x_m(\mu)$ with $\lambda_{\max}(\mu)$. The appearance, for $\mu > 1$, of this "outlier" maximal eigenvalue outside of the support of the Wigner semi-circle, is known under the name of the Baik-Ben Arous-Péché (BBP) transition.