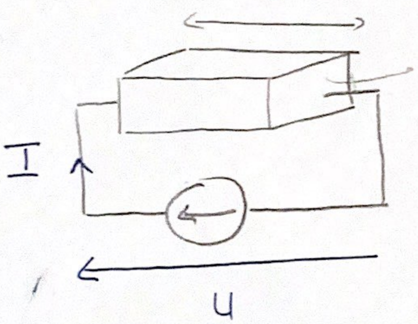
Lecture on Localization.

I] Phenomenology of transport in disordered media.

1.) Conductivity

Ohm's law:



$U = RI$

w. $R = \frac{1}{\sigma} \frac{L}{S}$

conductivity

A microscopic computation of σ is in g^{al} difficult:

because of $\vec{E}$, this is an out-of-equilibrium stat. mech. problem (no general theory as in equilibrium).

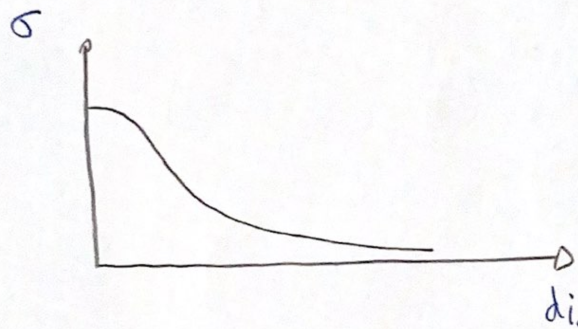
density of electrons

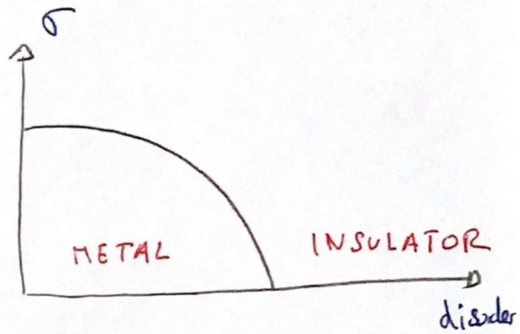
Drude's theory : $\sigma = \frac{n e^2 \tau}{m}$ ← mean free time between collisions

with : * phonons
* impurities

When the density of impurities $\nearrow$, $\sigma \searrow$.

Two possibilities may occur:





Q: is there a phase transition? First studied by Anderson (58)

↳ Since then it has generated a huge amount of works: in physics (theor., exp.), maths...

2.) Diffusion

"Dual" question: at $\vec{t} = \vec{0}$ the particles / electrons diffuse

$$\langle \vec{r}(t)^2 \rangle \underset{t \to +\infty}{\sim} 2Dt$$

diffusion constant

D and σ are related by the Fluctuation-Dissipation-Theorem (Einstein/Kubo formula):

$$D = \frac{kT}{me^2}\,\sigma$$

Here we see that:

* for an insulator: $\sigma = 0 \Rightarrow D = 0$, particles remain <u>localized</u>

* for a metal: $\sigma > 0 \Rightarrow D > 0 \Rightarrow$ <u>diffusion</u>.

3.) <u>Dimension matters.</u>

Usually, fluctuations (thermal or disorder) destroys ordered phases more easily in low dimensions (cf Ising model for instance).

For localization:

* $d = 1$: infinitesimal disorder $\Rightarrow$ localization
* $d = 2$: " (albeit of a weaker form)
* $d = 3$: metal / insulator phase transition as a function of disorder.

4.) Other systems.

Until now we considered e^- in condensed matter systems but Anderson localization occurs and has been studied in other contexts:

* classical waves (light, sound) with disorder

cf lectures by Texier / Cherroret

II] Microscopic models.

1.) Continuum models.

Consider one e^- in a potential $V(\vec{r})$:

Schrödinger: $$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \left(-\frac{\hbar^2}{2m} \Delta + V(\vec{r}) \right) \Psi(\vec{r}, t)$$

w. $\Psi \in L^2(\mathbb{R}^d)$

Random potential:

$$V(\vec{r}) = \sum_n a_n \, \delta^{(d)}(\vec{r} - \vec{x}_n)$$

$\cdot \, \vec{x}_n$

random intensities and positions of scatterers

· Disorder induces multiple scattering.

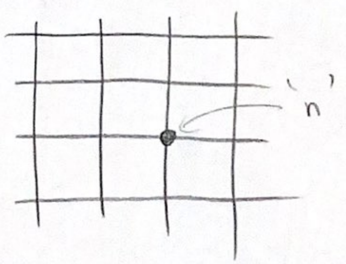
Q: are the interferences constructive or destructive?

Rk: in the periodic case, Bloch theory states that the interferences are constructive. But in the disordered case the answer is not obvious.

2.) Lattice model: Anderson

↳ simpler to work with discrete models.

d=2.



The particle lives on the vertices 'n' of a lattice, with positions $\vec{r}_n \in \mathbb{Z}^d$

→ called the 'tight-binding' approximation

$$\Psi = \{\Psi_n\}_{1 \leq n \leq N}, \quad N \equiv \text{nber of sites}$$

For instance in d=1: $-\frac{\hbar^2}{2m}\Delta\Psi + V(r)\Psi = E\Psi$

becomes: $$-\frac{\hbar^2}{2m}\left[\Psi_{n+1} + \Psi_{n-1} - 2\Psi_n\right] + V_n\Psi_n = E\Psi_n$$

$$\Leftrightarrow \quad -\frac{\hbar^2}{2m}\left(\Psi_{n+1} + \Psi_{n-1}\right) + V_n\Psi_n = \overbrace{\left(E - \frac{2\hbar^2}{2m}\right)}^{\epsilon}\Psi_n$$

which we write : $-t\,(\psi_{n+1} + \psi_{n-1}) + V_n \psi_n = \epsilon \psi_n$

Rk : in the absence of disorder ($V_n = 0\ \forall n$), one has:

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{ikn} , \quad \epsilon_k = -2t \cos k$$
$$k \in [-\pi, \pi] .$$

In g^{al}, V_n can be positive or negative and we write, more formally :

$$(H\psi)_n = \underbrace{J}_{\text{strength of the "hopping"}} \sum_{m \in \partial n} \psi_m + \underbrace{V_n}_{\text{random, for instance iid, unif.} \in [-\frac{W}{2}, \frac{W}{2}]} \psi_n$$

a) The model can then be defined on a finite subset of $\mathbb{Z}^d$ with $N = L^d$ sites. In this case H can be viewed as a $N \times N$ matrix:

$$H_{mn} = \begin{cases} J & \text{if } m, n \text{ are nearest neigh.} \\ V_n & \text{if } m = n \\ 0 & \text{otherwise,} \end{cases}$$

// usually + periodic boundary cond.

If V_n's are random variables, the matrix $H = (H_{mn})$ is a random matrix. But at variance with the GOE:

(i) (H_{mn}) is sparse ("finite dimensional geometry")

(ii) Two energy scales: "J vs ω".

In this case, to observe a phase transition, one needs to take the limit $L \to \infty$ (equiv. $N \to \infty$).

b) One can also work directly in the limit $L \to \infty$ and in this case H is an operator $\Rightarrow$ more difficult mathematically.

c) It can also be defined on other lattices (different from $\mathbb{Z}^d$), like random trees, "Bethe lattice" which allows to give a mean-field version.

3.) Observables

* Linear evolution : $i \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle$ $\quad (\hbar = 1)$

→ characterized by the eigenvectors / values ("states") of H, i.e. $H|v_\alpha\rangle = \omega_\alpha |v_\alpha\rangle$, $\alpha = 1, \dots, N$

$$\langle v_\alpha | v_\beta \rangle = \delta_{\alpha,\beta}$$

We also denote by $\{ |i\rangle \}$ the basis of vectors localized on site 'i'.

* An interesting / natural observable is:

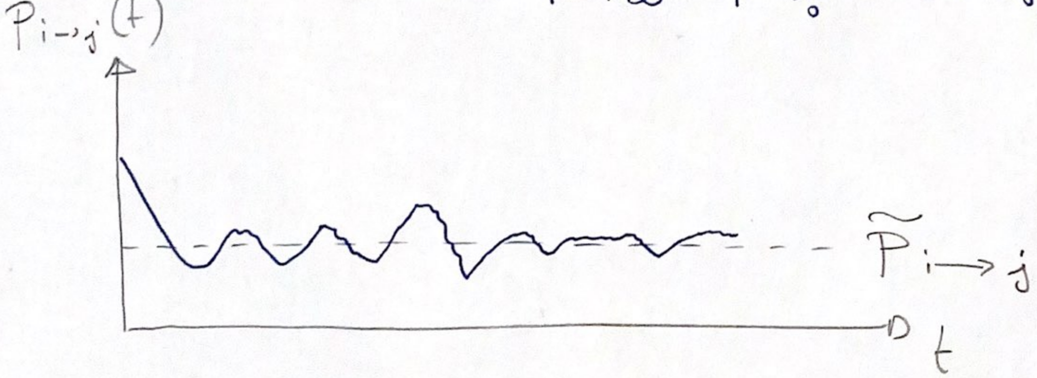
$P_{i\to j}(t)$ = proba. of observing the particle at site 'j' at time t starting from site 'i' at time '0'.

$$= \left| \langle j | e^{-iHt} | i \rangle \right|^2$$

$$= \left| \sum_{\alpha=1}^{N} \langle j | v_\alpha \rangle \langle v_\alpha | i \rangle e^{-i\omega_\alpha t} \right|^2$$

$$= \underbrace{\sum_{\alpha=1}^{N} |\langle j|v_\alpha\rangle|^2 |\langle i|v_\alpha\rangle|^2}_{\tilde{P}_{i\to j}} + \underbrace{\sum_{\alpha\neq\beta} e^{i(\omega_\alpha-\omega_\beta)t} \langle j|v_\alpha\rangle\langle v_\alpha|i\rangle\langle i|v_\beta\rangle\langle v_\beta|j\rangle}_{\text{complicated, oscillating (wildly at large N), quasi-periodic}}$$

One has $\tilde{P}_{i\to j} = \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T dt \; P_{i\to j}(t)$



In particular ("return to the origin"):

$$\tilde{P}_{ii} = \sum_{\alpha=1}^{N} |\langle i|v_\alpha\rangle|^4$$

* Compact representation of the whole eigen vectors / values

. density of states: $\rho(E) = \frac{1}{N}\sum_{\alpha=1}^{N}\delta(E-\omega_\alpha)$
(DOS)

↳ looses the info. on the eigenvectors but tells what are the possible energies.

- local density of states (LDOS)

$$\rho_i(E) = \sum_{\alpha=1}^{N} |\langle i | v_\alpha \rangle|^2 \, \delta(E - \omega_\alpha)$$

↳ tells how much the states of energy E overlap with $|i\rangle$

of course: $\frac{1}{N} \sum_{i=1}^{N} \rho_i(E) = \rho(E)$

Indeed: $\frac{1}{N} \sum_{i=1}^{N} \rho_i(E) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(E - \omega_\alpha) \underbrace{\sum_{i=1}^{N} \langle v_\alpha | i \rangle \langle i | v_\alpha \rangle}_{= \langle v_\alpha | v_\alpha \rangle = 1}$

$= \rho(E)$.

- inverse participation ratio $IPR_\alpha = \sum_{i=1}^{N} |\langle i | v_\alpha \rangle|^4$

Physical interpretation: suppose $|v_\alpha\rangle$ is $\neq 0$ and approximately constant on N_α sites.

Then one has: $\langle i | v_\alpha \rangle \simeq \begin{cases} \frac{1}{\sqrt{N_\alpha}} & \text{on } N_\alpha \text{ sites} \\ 0 & \text{elsewhere} \end{cases}$

(because of normalisation)

$\Rightarrow IPR_\alpha \simeq \frac{1}{N_\alpha} \xrightarrow{N \to \infty} \begin{cases} 0 & \text{delocalized vector} \\ > 0 & \text{localized vector} \end{cases}$

4.) Phase transition in the Anderson model

Recall: $H = J \sum_{\langle i,j \rangle} \left(|i\rangle\langle j| + |j\rangle\langle i| \right) + \sum_i V_i \, |i\rangle\langle i|$

with V_i's are iid, uniform on $\left[-\frac{W}{2}, \frac{W}{2} \right]$.

Let us analyse the eigenstates of the two terms separately:

* if $J = 0$: ~~H~~ H is diagonal in the $|i\rangle$ basis

all states are strictly localized: $IPR_\alpha = 1, \forall \alpha$.

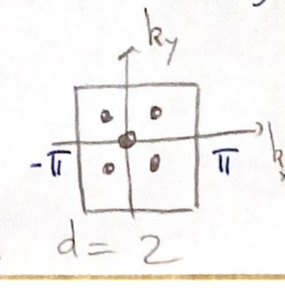
One has $w_\alpha \leftrightarrow V_i \Rightarrow \rho(E) \underset{N \to \infty}{=} \rho(V)$, unif.

One has also $P_{i \to j}(t) = \delta_{i,j} \quad \forall t \geqslant 0$.

* if $w = 0$, the "pure hopping" H can be diagonalized in the Fourier basis:

$$\langle i | v_\alpha \rangle = \frac{1}{\sqrt{N}} e^{i \vec{k}_\alpha \cdot \vec{r}_i}, \quad \vec{k}_\alpha = \left(\overbrace{\frac{2\pi}{L} n_1}^{k_1}, \dots, \overbrace{\frac{2\pi}{L} n_d}^{k_d} \right)$$

in the Brillouin zone

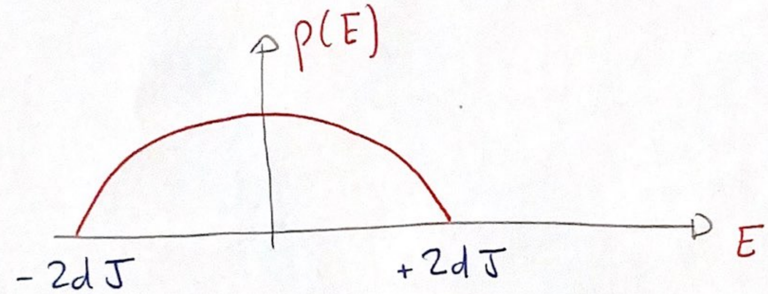


in $d = 2$

· Eigenvalues: $\omega(\vec{k}) = 2J \sum_{\mu=1}^{d} \cos k_\mu$

The density $\rho(E)$ can be computed analytically in the limit $L \to \infty$, yielding (skipping unimportant details)

$d \geq 3$



All eigenvectors are maximally delocalized:

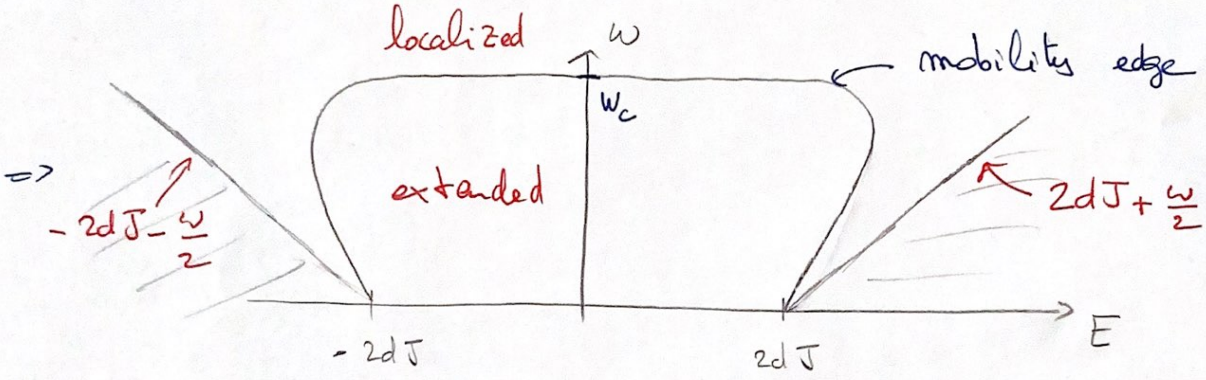
$$IPR_\alpha = \frac{1}{N}$$

One can also show that $\tilde{P}_{i \to j} = \frac{1}{N}$,

$\Rightarrow$ the particle becomes completely delocalized at large time $t \to \infty$.

Finally: $\rho_i(E) = \rho(E) \;\; \forall i$.

<u>Main result</u>: For the Anderson model in $d \geq 3$, there is a competition between these two terms



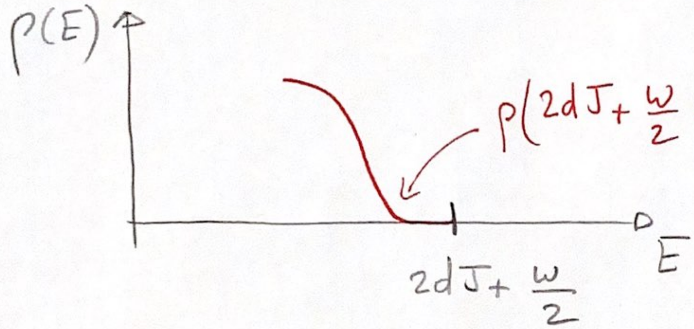
i.e. states exist on $\left[-2dJ - \frac{w}{2}, 2dJ + \frac{w}{2}\right]$ where $\rho(E)$ converges to a non-random limit, $N \to \infty$

. In the (E, w) plane, $\exists$ "mobility edge" separating extended states (i.e. $IPR_\alpha = 0$) from localized states (i.e. $IPR_\alpha > 0$).

This property cannot be seen on $\rho(E)$ since this is a property of the <u>eigenvectors</u>, not of the eigenvalues.

. For $w > w_c$: only localized states.

- Density of states close to the edge:



$\rho\left(2dJ + \frac{W}{2} - \delta\right) \sim A\, e^{-\frac{B}{\delta^{d/2}}}$

"Lifshitz tails"

- Level statistics (e.g. spacing distribution):

  → level repulsion (GOE) for extended states

  → Poissonian statistics for localized states