Effects of quenched disorder on low T phases and on phase transitions.

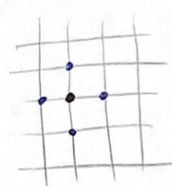
<u>Goal</u>: analyse the effect of infinitesimal disorder on pure models.

→ "opposite" of the strong disorder studied with spin-glass models.

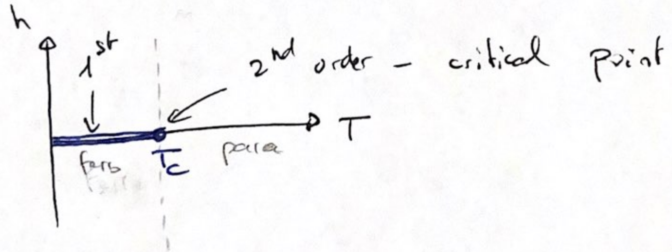
→ opposite style/strategy as well: insted of exact solution of MF models we will use here scaling arguments in finite dimensions d.

I] A reminder on the pure Ising model:

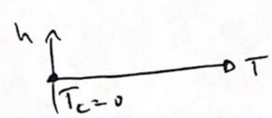
$$H = -\sum_{\langle ij\rangle} J \sigma_i \sigma_j - \sum_i h \sigma_i$$ on a d-dim lattice, $J>0$



Phase diag: (d ⩾ 2)



In $d=1$, $T_c=0$



still a critical point as $\xi =$ '∞'.

At the critical point, critical exponents:

- correlat° length: $\xi \sim |T-T_c|^{-\nu}$, $T \to T_c$
- mag.: $m(T) \sim (T_c - T)^{\beta}$, $T \to T_c^-$
- specific heat: $C_V = \frac{\partial \langle E \rangle}{\partial T} \propto |T - T_c|^{-\alpha}$

Scaling / hyperscaling ~~exponents~~ relations: $\alpha = 2 - d\nu$

↳ 2 independent critical exp.

<u>Questions</u>: upon adding a small amount of disorder in the couplings or in the fields

① does the ordered phase ($T < T_c$, $h = 0$) survive?

② if there is still a critical point, are the critical ~~points~~ exp. modified? If yes on what depends the universality classes? Does it depend on the type of distributions of the disorder?

[<u>Remark</u>: by linear response theory, infinitesimal disorder should produce an infinitesimal modification. But at T_c, $\xi \to \infty$ and this reasoning can be wrong (non-perturbative effects).] → keep it for Harris

Here: present some qd arguments to answer partly these questions, on the Ising model for concreteness but it can be generalized.

II] Imry - Ma argument (1975)

↳ similar in spirit to the domain wall arg. / Peierls arg. to determine the lower critical dim.

→ on the Random Field Ising model (RFIM)

$$H = - \sum_{\langle i,j \rangle} J \sigma_i \sigma_j - \sum_i h_i \sigma_i$$ where h_i's are random

w. $\mathbb{E}(h_i) = 0$

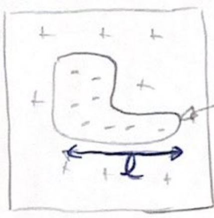
→ RS (Chatterjee 2007) → $\mathbb{E}(h_i^2) = \underline{\Delta^2} > 0$

not a good notation because other Δ's later

At $T=0$: suppose $h_i = 0$, the syst. is in ~~two~~ one of the two G.S.: $\underline{\sigma} = \uparrow$ or $\underline{\sigma} = \downarrow$.

Suppose $\underline{\sigma} = \uparrow$ and turn on the disorder with $\sqrt{\Delta^2} \ll J$.

Can the GS remain magnetized when $\Delta \to 0^+$?



droplet of volume $|\Omega| = \ell^d$ of reversed spins

Energy ~~cost of this~~ associated to this droplet:

$$\Delta E = \underbrace{2 J \ell^{d-1}}_{\substack{\text{energy cost} \\ \text{of breaking} \\ \text{bonds at the} \\ \text{boundary of } \Omega.}} + \underbrace{2 \sum_{i \in \Omega} h_i}_{\substack{\text{effect of the} \\ \text{field.}}}$$

The energy of the field can be estimated by using the central limit theorem: $\sum_{i \in \Omega} h_i \underset{\ell \gg 1}{\sim} \sqrt{\Delta^2 \ell^d}\, Y = O(\ell^{?})$

Y: normal Gaussian $N(0,1)$

Hence: $\Delta E \simeq 2 J l^{d-1} + 2\sqrt{\Delta^2}\, l^{\frac{d}{2}}\, y$

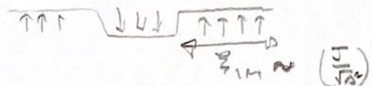
$\Rightarrow$ if $\frac{d}{2} > d-1$: there will be arbitrarily large droplets

$\Leftrightarrow$ $d < 2$ which gain energy to flip ~~[illegible]~~ $(y<0)$

$\Rightarrow$ if $\frac{d}{2} < d-1$ then surface effects will always win

i.e. ~~[illegible]~~ $d > 2$

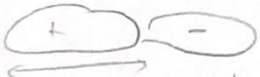
$J + \sqrt{\Delta}\sqrt{l}$



For $d=2$ (more subtle): the two terms are of the same order but since y is unbounded ~~there can be~~ fields eventually wins

Analysis of the roughness of domain wall

Size of the domain:



$\xi \sim \exp\left(c\left(\frac{J}{\sqrt{\Delta^2}}\right)^2\right)$

(cf Binder's paper for more refined argument)

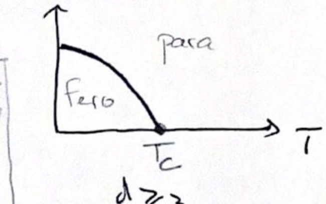
At $T>0$, same reasoning $2J \to \sigma$ (surface tension) by entropy arg. ?

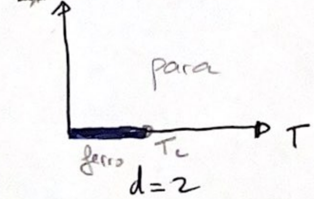
Conclusion (Imry-Ma argument):

* $d=1,2$: no ferro. order can survive infinitesimal random fields

* $d \geq 3$: there is still a magnetization at small Δ but if $\sqrt{\Delta^2} \gg J$ the groundstate will be aligned with the local field $h_i \Rightarrow$ para. phase.

Phase diag.:





Rk: generalized to continuous sym.: $E_{wall} \sim l^{d-2}$ vs l^{d-1}

$d=2 \to d=4$.

This argument ~~too~~, though quite reasonable, has been debated for quite some time (due to contradicting theories like "dimensional reduction" which said that a d-dim disordered system is like $(d-2)$ pure system. RFIM in $d=3$ is like pure Ising in $d=1$: no transition)... until rigorous proofs and a better understanding of the failure of "dimensional reduction":

* Imbrie 84, $\exists$ order in $d=3$, $\Delta \to 0^+$.
  Bricmont / Kupiainen 87
* Aizenman, Wehr '89: no order in $d=2$, $\Delta \to 0^+$.

Q: what about the critical exponents along the line?

III | Harris criterion. $\to$ Autre présentation ? cf Giraud's ? RG point of view?

1974

$\to$ linear response may be invalid here since $\xi \to \infty$.

Question ② above: modification of critical exponents by infinitesimal disorder when the 2^{nd} order phase transition survives.

Discussion on the random-bond Ising model:

$$H(\underline{\sigma}, \underline{J}) = - \sum_{\langle i,j \rangle} J_{ij}\, \sigma_i \sigma_j$$

on a d-dimensional lattice.

. Typically : $J_{ij} = J + \underbrace{\epsilon_{ij}}$, $\mathbb{E}[\epsilon_{ij}] = 0$

$\mathbb{E}[\epsilon_{ij}^2] = \Delta$.

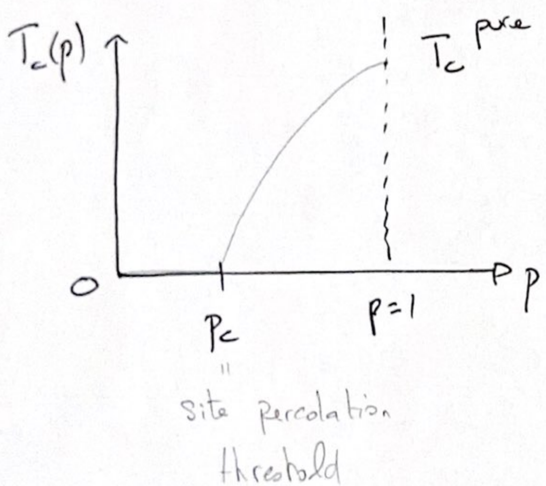
iid random var.

Ex: site-diluted Ising model

$$H[\sigma] = - \sum_{\langle ij \rangle} J n_i n_j \sigma_i \sigma_j \quad , \quad n_i = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$$

$$= - \sum_{\langle ij \rangle} \left(p^2 J + J \underbrace{(n_i n_j - p^2)}_{\epsilon_{ij}} \right) \sigma_i \sigma_j$$

with: $\mathbb{E}(\epsilon_{ij}) = 0$, $\mathbb{E}(\epsilon_{ij}^2) = J^2 p^2 (1-p^2)$

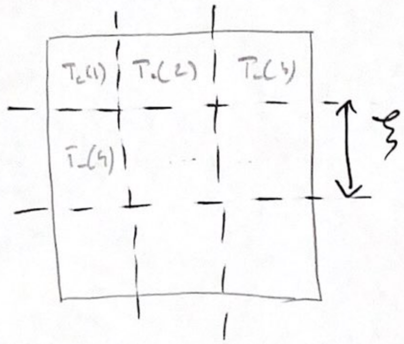


↳ Scaling analysis for $p \simeq 1 - \epsilon$, $\Delta \ll 1$.

Suppose that we consider a large system with $T \gtrsim T_c \simeq T_c^{pure}$:

$$\xi^{pure} \simeq |T - T_c^{pure}|^{-\nu} \quad , \quad \nu \equiv \nu_{pure}$$

Divide the system in independent blocks Ω of size ξ, of volume $|\Omega| = \xi^d$



In the presence of a very small disorder, the effective/average coupling inside a block is:

$$J_{eff} \simeq \frac{1}{N_{bonds}} \sum_{\langle i,j\rangle \in \Omega} J_{ij}$$

$$\simeq J + \frac{\sqrt{\Delta}}{\sqrt{|\Omega|}} Y \leftarrow N(0,1)$$

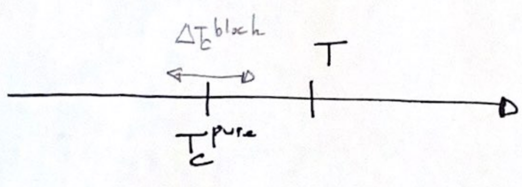
by the central limit theorem

Each block now has its "own" critical temperature such that $\frac{J_{eff}}{T_c(\Omega)} =$ 'c' fixed number.

Therefore: $$|T_c(\Omega) - T_c^{pure}| \sim |\Omega|^{-\frac{1}{2}}$$

$$\sim \xi^{-\frac{d}{2}}$$

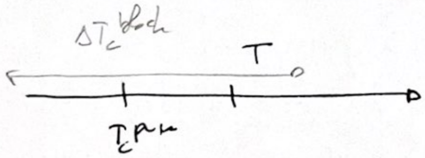
$$\Rightarrow \overbrace{|T_c(\Omega) - T_c^{pure}|}^{\Delta T_c^{block}} \sim |T - T_c^{pure}|^{\frac{\nu d}{2}}.$$



Hence 2 cases might occur:

- If $\Delta T_c^{\text{block}} < T - T_c^{\text{pure}}$: all the blocks behave "uniformly" as if they were all in the high T phase

- If $\Delta T_c^{\text{block}} > T - T_c^{\text{pure}}$



then some blocks are in the disordered phase and some others are in the ordered ferromagnetic phase.

transition is "smeared out" by disorder

$\Rightarrow$ inhomogeneities become important and may change the critical behavior.

Conclusion: This means that for the critical properties of the pure system to remain the same one needs:

$$\Delta T_c^{\text{block}} < T - T_c^{\text{pure}} \iff |T - T_c^{\text{pure}}|^{\frac{\nu d}{2} - 1} < 1$$

as $|T - T_c^{\text{pure}}| \to 0$, i.e.:

$$\nu d > 2 \iff \alpha < 0 .$$

Using hyperscaling relation: $\alpha = 2 - d\nu$

In the language of RG this means that disorder is irrelevant (it flows to zero on large length scale).

IF $\nu d < 2$: disorder is relevant in the RG sense but this criterion does not tell us whether

* phase transit° is completely destroyed
* RG flows to another fixed point
* disorder grows indefinitely under RG $\Rightarrow$ flow to another kind of "phase transition" (zero T, infinite disorder).

Rigorous statement:
Note that one can show that for ~~a large class of disordered~~ disordered systems displaying continuous phase transit° in the presence of disorder one has:

$\nu_{dis} d \geq 2$ (independently of whether or not the critical behavior is the same as in the pure system).

Chayes, Chayes, Fisher, Spencer PRL 86

[IF $\nu d = 2$, the disorder is marginal and one needs more work to know it is marginally relevant or irrelevant.]

Ex: for d-dim Ising:

log. singularity

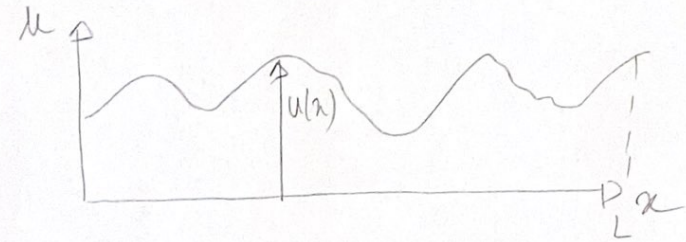
* $d = 2$, $\alpha = 0$ marginally relevant
* $d = 3$, $\alpha \approx 0.1$ relevant but small: new fixed point that can be controlled perturbatively.

IV | An example of zero-temperature fixed-point: elastic manifolds/interfaces in disordered media.

Usually phase transitions are driven by the competition between order / disorder, i.e. energy vs entropy.

In some cases the relevant competition is between 2 different forms of energy, while entropy is irrelevant (to a large extent): hence the name of <u>zero temperature</u> fixed point. In such case, the disorder grows "unboundedly" under RG transformat° $\Rightarrow$ one needs to introduce a peculiar RG ~~transf~~ procedure which amounts to follow an infinite nber of coupling constant — not only $a_2\varphi^2$ and $g\varphi^4$ as in standard RG $\Rightarrow$ function RG (FRG).

Ex: pure elastic interface in 1+1 dimensions



Assuming no overhangs

Elastic line : $E_{el} = \gamma \underbrace{\int dl}_{\text{total length}} = \gamma \int_0^L \sqrt{dx^2 + du^2} = \gamma \int_0^L dx \sqrt{1 + \left(\frac{du}{dx}\right)^2}$

$\simeq \gamma L + \frac{\gamma}{2} \int_0^L dx \left(\frac{du}{dx}\right)^2 + \ldots$

γ surface tension

· At thermal equilibrium, the Boltzmann-Gibbs measure for the <u>pure</u> system is: $P[\{u_x\}_{0\le x\le L}] \propto e^{-\beta E_{el}[\{u\}]}$

$$\propto e^{-\beta \frac{\gamma}{2}\int_0^L dx \left(\frac{du}{dx}\right)^2}$$

<u>Rk</u>: $u(x)$ is a BM w. $u \to x$, $x \to \tau$, i.e. $u(x) \equiv x(\tau)$

$$\frac{\partial x}{\partial \tau} = \underbrace{\eta(\tau)}_{\text{white noise}}$$

$$\Rightarrow \boxed{\langle u(L)^2\rangle_{th} \sim TL} \Rightarrow u(L) \sim \sqrt{L} \quad , T>0$$

What happens if we add disorder?

$$H[\{u_x\}] = \frac{\gamma}{2}\int_0^L dx \left(\frac{du}{dx}\right)^2 + \underbrace{\int_0^L dx\, V(x, u(x))}_{\text{disordered pinning potential}}$$

w. $\underline{u(0) = 0}$

usually $V(x, u(x))$ is a Gaussian random variable typically with ~~correlations~~ short range correlations:

$$\mathbb{E}\left[V(x, u)\, V(x', u')\right] = f\left(\frac{x-x'}{\xi_x}\right) g\left(\frac{u-u'}{\xi_p}\right)$$

<u>At T=0</u>: competition between elasticity and disorder

flat distort

$\Rightarrow$ "frustration"

∴ Minimal model / Larkin model (Sakharov's student)

treat the disorder in perturbation an linearize the potential for small u around a minimum of the potential:

$$H[\{u_x\}] \simeq \frac{\gamma}{2}\int_0^L dx \left(\frac{du}{dx}\right)^2 + \int_0^L dx\, h(x)\, u(x)$$

random force

$\mathbb{E}(h(x)) = 0$

$\mathbb{E}(h(x)\, h(x')) = \Delta\, \delta(x-x')$

Minimisation $\frac{\delta H}{\delta u_x} = 0 \iff \gamma \frac{d^2u}{dx^2} = h(x)$ Euler-Lagrange

$$\Rightarrow \quad u(x) = \frac{1}{\gamma}\int_1^x dy \int_0^y dz\, h(z) + Ax + \underbrace{B}_{=0 \text{ if } u(0)=0}$$

$$\cancel{\Rightarrow \mathbb{E}\{u(x)\} = \mathbb{E}(A)\, x}$$

$\Rightarrow$ inject back in $H[\{u_x\}]$ and minimize with respect to A to give: $A = \frac{-1}{\gamma}\int_0^L dz\, h(z)$

Hence one gets: $\boxed{u(L) = \frac{-1}{\gamma}\int_0^L dz\; z\, h(z)}$

$$\Rightarrow \quad \mathbb{E}[u(L)] = 0$$

$$\mathbb{E}(u(L)^2) = \frac{\Delta}{\gamma^2}\int_0^L dz\; z^2$$

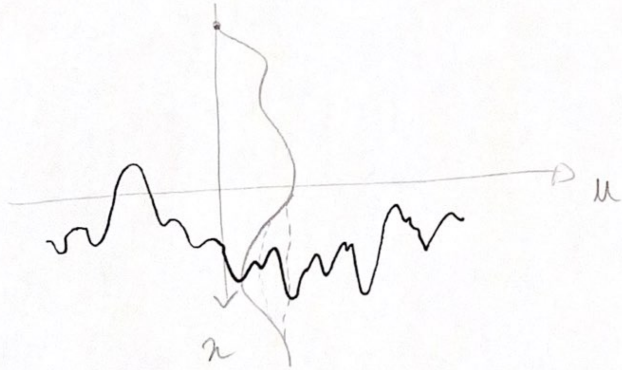
$$\boxed{\mathbb{E}(u(L)^2) = \frac{\Delta}{3\gamma^2} L^3}$$

Several remarks: what do we learn from this computation?

* $\mathbb{E}(u(L)^2) \sim \frac{\Delta}{\gamma^2} L^3 \gg \langle u(L)^2 \rangle_{th} \sim TL$

$\Rightarrow$ fluctuations induced by disorder are much stronger than the thermal ones. This indicates that thermal effects play sub-dominant role $\rightarrow$ "zero-temperature" fixed point.

* This result, indicating that $u(L) \sim L^{3/2}$ ~~is~~ is actually <u>wrong</u> ~~this result~~, the exact one being $u(L) \sim L^{2/3}$. This result $u(L) \sim L^{3/2}$ is also the result of dimensional reduction. Where is the problem? $\rightarrow$ linearizat° of the disorder near <u>one</u> minimum:



But there are actually a lot of minima (nearly degenerate) that the interface might explore. This can not be taken into account by a simple expansion around a <u>single</u> minimum.

In other words, the Larkin approximation holds until L_c such that $\mathbb{E}(u^2(L_c)) \sim r_p^2$ where

r_p is the correlation length of the disorder:

$$\frac{\Delta}{\gamma^2} L_c^3 \sim r_p^2 \Rightarrow L_c \sim \left(\frac{\gamma^2 r_p^2}{\Delta}\right)^{\frac{1}{3}}$$

↳ Larkin length

Beyond that length scale, one needs to treat non-perturbative effects. The way out:

- RSB in mean-field model ($u(x) \in \mathbb{R}^N$, $N \to \infty$)
- non-analytic fixed point in Functional RG.

cf K. Wiese's lectures.