ICFP M2 - Statistical physics 2 – Correction of the exam

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1 Questions on the lectures

1. An Erdös-Rényi random graph is obtained by considering N sites and adding each possible link (ij) independently at random with probability $p = \alpha/N$. The average connectivity of each site is then $c = \alpha/N \times (N-1) \to \alpha$ when $N \to \infty$. If you sit on a site taken at random and you look at your neighborhood of finite size ℓ, you observe a Galton-Watson tree with probability one. In other words, the probability to observe a loop involving a given site vanishes when $N \to \infty$.

2. A Wigner random matrix is a real symmetric or complex Hermitian matrix with independent entries. The diagonal elements W_{ii} are real and the off-diagonal elements $W_{i<j} = x_{i<j} + iy_{i<j}$ have independent and identically distributed real and imaginary parts.

   The Gaussian Orthogonal Ensemble is the ensemble of real Wigner matrices that is invariant under an orthogonal transformation, i.e. $P(W) = P(OWO^T)$, and with Gaussian statistics of the elements, which leads to (assuming zero mean for simplicity)

$$P(W) \propto e^{-\frac{N}{4}\mathrm{Tr}(W^2)} \ . \tag{1}$$

   The shape of the density of eigenvalues in the limit of large matrices has support on $[-2, +2]$ and is given by the Wigner semicircle,

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2} \ . \tag{2}$$

3. The Harris criterion for a second order phase transition states that if a pure system has critical exponent ν for the correlation length, i.e. $\xi \sim |T - T_c|^{-\nu}$, then an infinitesimal disorder will not alter the critical behavior provided $\nu d > 2$. Equivalently, using hyperscaling relations between the exponents, the disorder is irrelevant if $\alpha < 0$ – where α is the specific heat exponent associated to the specific heat. Instead, the disorder is relevant if $\alpha > 0$. If $\alpha = 0$ (as in the $2d$ Ising model), the disorder is marginal and the Harris criterion is not enough to conclude (in the case of the $2d$ Ising model the disorder is marginally relevant and induces logarithmic corrections to scaling relations).

2 The capacity of a perceptron in classifying random data

1. Defining $\boldsymbol{\eta}^m = \sigma^m \boldsymbol{y}^m$ we have

$$\boldsymbol{\eta}^m \cdot \boldsymbol{J} = \sigma^m \boldsymbol{J} \cdot \boldsymbol{y}^m \geq 0 \qquad \Leftrightarrow \qquad \sigma^m = \mathrm{sign}(\boldsymbol{J} \cdot \boldsymbol{y}^m) \ . \tag{3}$$

   The condition $\boldsymbol{J} \cdot \boldsymbol{\eta} = 0$ defines a hyperplane going through the origin, orthogonal to the vector $\boldsymbol{J}$. The condition $\boldsymbol{\eta} \cdot \boldsymbol{J} > 0$ implies that the vector $\boldsymbol{\eta}$ lies on one side of this hyperplane. Hence, all vectors $\boldsymbol{\eta}^m$ should lie on the same side of this hyperplane.

2. If Eq. (3) of the exam is satisfied, adding the noise we have

$$(\boldsymbol{\eta}^m + \boldsymbol{\epsilon}^m) \cdot \boldsymbol{J} = \boldsymbol{\eta}^m \cdot \boldsymbol{J} + \boldsymbol{\epsilon}^m \cdot \boldsymbol{J} \geq \kappa\sqrt{N} + \boldsymbol{\epsilon}^m \cdot \boldsymbol{J} \geq \kappa\sqrt{N} - |\boldsymbol{\epsilon}^m||\boldsymbol{J}| = (\kappa - \varepsilon)\sqrt{N} \ , \tag{4}$$

   recalling that $|\boldsymbol{J}| = \sqrt{N}$ and $|\boldsymbol{\epsilon}^m| = \varepsilon$. Hence the noisy patterns are still correctly classified. The larger κ, the larger the value of noise that the perceptron can still tolerate.

3. The Hamiltonian introduced in Eq. (4) of the exam is the same that was introduced for other constraint satisfaction problems such as XORSAT or coloring, in the case $v(x) = H(-x)$ where $H(x)$ is the Heaviside function. In fact with this choice of $v(x)$ each misclassified pattern contributes an energy one. Because here variables are continuous, it is interesting to consider other choices of $v(x)$ that still penalize violated constraints (hence $v(x)$ is strictly positive when $x < 0$) but are smoother, i.e. such that the energy cost of a misclassified pattern depends on the distance from the boundary of the inequality in Eq. (3) of the exam.

   The ground state energy $e(\alpha) = \min_{\boldsymbol{J}} H[\boldsymbol{J}]/N$ is written here for a given choice of disorder. In the thermodynamic limit $N \to \infty$ with constant $\alpha = M/N$ we expect that it becomes self-averaging, i.e.

$$e(\alpha) = \lim_{N\to\infty, M=\alpha N} \min_{\boldsymbol{J}} H[\boldsymbol{J}]/N = \lim_{N\to\infty, M=\alpha N} \mathbb{E}\{\min_{\boldsymbol{J}} H[\boldsymbol{J}]\}/N \tag{5}$$

   with probability one.

   Let us consider first the case $\kappa = 0$. We know that if $M < N$ we have a number of patterns smaller than the dimension of space, and we can always find a hyperplane such that all points are on the same side. Hence, for $\alpha \leq 1$, we expect $e(\alpha) = 0$. In this case we also expect that there are many choices of $\boldsymbol{J}$ that satisfy the problem, hence solutions cover a finite fraction of the surface of the sphere $|\boldsymbol{J}|^2 = N$. The partition function is finite, corresponding to a finite entropy of solutions. On the other hand, at large α we expect that no solution exist, hence $e(\alpha) > 0$. It follows that there must be a value $\alpha_c \geq 1$ at which the energy becomes non-zero, corresponding to a SAT-UNSAT transition. We expect the same for any $\kappa > 0$, with a decreasing $\alpha_c(\kappa)$, because the problem becomes more difficult upon increasing κ.

4. We have

$$\begin{aligned}
\mathbb{E}[Z^n] &= \mathbb{E}\left\{\int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N) e^{-\beta H[\boldsymbol{J}_a]}\right]\right\} \\
&= \mathbb{E}\left\{\int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N) e^{-\beta \sum_{m=1}^{M} v\left[\frac{\boldsymbol{\eta}^m\cdot \boldsymbol{J}_a}{\sqrt{N}} - \kappa\right]}\right]\right\} \\
&= \mathbb{E}\left\{\int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N)\right] \int \left[\prod_{m=1}^{M}\prod_{a=1}^{n} \mathrm{d}r_a^m e^{-\beta v(r_a^m - \kappa)} \delta\left(r_a^m - \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}}\right)\right]\right\} \\
&= \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N)\right] \int \left[\prod_{m=1}^{M}\prod_{a=1}^{n} \mathrm{d}r_a^m e^{-\beta v(r_a^m - \kappa)}\right] \mathbb{E}\left[\prod_{m=1}^{M}\prod_{a=1}^{n} \delta\left(r_a^m - \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}}\right)\right] ,
\end{aligned} \tag{6}$$

   We used

$$e^{-\beta v(x-\kappa)} = \int \mathrm{d}r e^{-\beta v(x-\kappa)} \delta(x - r) \tag{7}$$

   and the expectation $\mathbb{E}[\bullet]$ is over the random patterns, which only appear in the last term, hence all the other terms can be brought out of it.

5. Using the integral representation of a delta function we have

$$\delta\left(r_a^m - \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}}\right) = \int \frac{\mathrm{d}\hat{r}_a^m}{2\pi} e^{i\hat{r}_a^m r_a^m - i\hat{r}_a^m \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}}} . \tag{8}$$

   We can insert this into the last term of the last line of Eq. (6) and keep in the expectation only the term that depends on the patterns, hence we get the desired result

$$\mathbb{E}[Z^n] \propto \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N)\right] \int \left[\prod_{m=1}^{M}\prod_{a=1}^{n} \mathrm{d}r_a^m \mathrm{d}\hat{r}_a^m e^{-\beta v(r_a^m - \kappa) + i r_a^m \hat{r}_a^m}\right] \mathbb{E}\left[e^{-\sum_{m=1}^{M}\sum_{a=1}^{n} i\hat{r}_a^m \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}}}\right] . \tag{9}$$

6. The random variable

$$X = \sum_{m=1}^{M}\sum_{a=1}^{n} i\hat{r}_a^m \frac{\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a}{\sqrt{N}} \tag{10}$$

   is a linear combination of Gaussians, hence it is Gaussian too. Its average is zero because $\mathbb{E}[\boldsymbol{\eta}^m] = 0$, and its variance is

$$\mathbb{E}[X^2] = -\frac{1}{N}\sum_{m,m'}^{1,M}\sum_{a,b}^{1,n} \hat{r}_a^m \hat{r}_b^{m'} \mathbb{E}[(\boldsymbol{\eta}^m\cdot\boldsymbol{J}_a)(\boldsymbol{\eta}^{m'}\cdot\boldsymbol{J}_b)] . \tag{11}$$

Using $\mathbb{E}[\eta_i^m \eta_j^{m'}] = \delta_{m,m'}\delta_{i,j}$ we then obtain

$$\mathbb{E}[X^2] = -\frac{1}{N}\sum_{m=1}^{M}\sum_{a,b}^{1,n} \hat{r}_a^m \hat{r}_b^m \boldsymbol{J}_a \cdot \boldsymbol{J}_b = -\sum_{m=1}^{M}\sum_{a,b}^{1,n} \hat{r}_a^m \hat{r}_b^m Q_{ab} \ . \tag{12}$$

Finally, using the relation

$$\mathbb{E}[e^{-X}] = e^{\frac{1}{2}\mathbb{E}[X^2]} \ , \tag{13}$$

which holds for a Gaussian variable with zero mean, we obtain the desired result. (Another possibility is to expand each side of the equality at second order in $\boldsymbol{J}$ as suggested in the text of the exam.)

7. Inserting the previous result into Eq. (9) and reorganizing a bit the products we obtain

$$\begin{aligned}
\mathbb{E}[Z^n] &\propto \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N)\right] \prod_{m=1}^{M} \left\{ \int \left(\prod_{a=1}^{n} \mathrm{d}r_a^m \mathrm{d}\hat{r}_a^m \right) e^{-\sum_{a=1}^{n} \beta v(r_a^m - \kappa) + \sum_{a=1}^{n} i r_a^m \hat{r}_a^m} e^{-\frac{1}{2}\sum_{a,b}^{1,n} \hat{r}_a^m \hat{r}_b^m Q_{ab}} \right\} \\
&= \int \left[\prod_{a=1}^{n} \mathrm{d}\boldsymbol{J}_a \delta(|\boldsymbol{J}_a|^2 - N)\right] \left\{ \int \left(\prod_{a=1}^{n} \mathrm{d}r_a \mathrm{d}\hat{r}_a \right) e^{-\sum_{a=1}^{n} \beta v(r_a - \kappa) + \sum_{a=1}^{n} i r_a \hat{r}_a - \frac{1}{2}\sum_{a,b}^{1,n} \hat{r}_a \hat{r}_b Q_{ab}} \right\}^M
\end{aligned} \tag{14}$$

where we observed that all the M integrals over $\mathrm{d}r_a^m \mathrm{d}\hat{r}_a^m$ are identical, so the index m can be dropped leading to the same integral to the power M.

8. Defining

$$g(Q) = \log \left\{ \int \left(\prod_{a=1}^{n} \mathrm{d}r_a \mathrm{d}\hat{r}_a \right) e^{-\sum_{a=1}^{n} \beta v(r_a - \kappa) + \sum_{a=1}^{n} i r_a \hat{r}_a - \frac{1}{2}\sum_{a,b}^{1,n} \hat{r}_a \hat{r}_b Q_{ab}} \right\} \tag{15}$$

and using Eq. (11) of the exam, we can write the last equation as

$$\mathbb{E}[Z^n] \propto \int \mathrm{d}Q (\det Q)^{N/2} \left[\prod_{a=1}^{n} \delta(Q_{aa} - 1)\right] .e^{M g(Q)} \tag{16}$$

The diagonal elements are then integrated out using the delta functions, which fixes $Q_{aa} = 1$ with a remaining integration over $\mathrm{d}Q = \prod_{a<b} \mathrm{d}Q_{ab}$. Recalling that $M/N = \alpha$, we obtain the result of Eq. (12) with

$$S(Q) = \frac{1}{2}\log\det Q + \alpha g(Q) \ . \tag{17}$$

The function $S(Q)$ is a large deviation function for the overlap matrix. The $\frac{1}{2}\log\det Q$ term is an entropic term that counts how many configurations of the $\boldsymbol{J}_a$ have overlap Q_{ab}, while the function $g(Q)$ takes into account the "energy", i.e. the interaction between replicas induced by averaging over the common disorder, as we have seen for the p-spin model.

9. The replica symmetric ansatz is the simplest ansatz that respect the permutation symmetry of replica indices. We expect it to be exact at high temperatures, and it is the simplest choice that allows one to perform the analytic continuation to $n \to 0$.

   The matrix Q has an eigenvector $v_1 = (1, 1, \cdots, 1)$ with associated eigenvalue $\lambda_1 = 1 + (n-1)q$. All the vectors orthogonal to v_1 are also eigenvectors with associated eigenvalue $1 - q$, which is then $(n-1)$-fold degenerate. This proves Eq. (13) for the determinant. (There are many alternative proofs.)

   For the energy term of $S(Q)$ we note that

$$\sum_{a,b} \hat{r}_a \hat{r}_b Q_{ab} = q\left(\sum_a \hat{r}_a\right)^2 + (1-q)\sum_a \hat{r}_a^2 \ , \tag{18}$$

and we can use Eqs. (14) to write

$$\begin{aligned}
e^{g(Q)} &= \int \left(\prod_{a=1}^{n} \mathrm{d}r_a \mathrm{d}\hat{r}_a \right) e^{-\sum_{a=1}^{n} \beta v(r_a - \kappa) + \sum_{a=1}^{n} i r_a \hat{r}_a - \frac{1}{2} q \left(\sum_a \hat{r}_a\right)^2 - \frac{1}{2}(1-q) \sum_a \hat{r}_a^2} \\
&= \int \mathcal{D}_q z \int \left(\prod_{a=1}^{n} \mathrm{d}r_a \mathrm{d}\hat{r}_a \right) e^{-\sum_{a=1}^{n} \beta v(r_a - \kappa) + \sum_{a=1}^{n} i (r_a - z) \hat{r}_a - \frac{1}{2}(1-q) \sum_a \hat{r}_a^2} \\
&= \int \mathcal{D}_q z \prod_{a=1}^{n} \int \mathrm{d}r_a \, e^{-\beta v(r_a - \kappa)} \sqrt{\frac{2\pi}{1-q}} e^{-\frac{(r_a - z)^2}{2(1-q)}} \\
&\propto \int \mathcal{D}_q z \left(\int \mathcal{D}_{1-q} r \, e^{-\beta v(r + z - \kappa)} \right)^n
\end{aligned} \tag{19}$$

In the last line we used that all the integrals over r_a are identical, we shifted $r \to r + z$, and discared an irrelevant proportionality constant that does not depend on q.

10. Recall that the quenched free energy is

$$f_q = -\lim_{N\to\infty} \frac{T}{N} \mathbb{E}[\log Z] = -\lim_{N\to\infty} \frac{T}{N} \lim_{n\to 0} \frac{1}{n} \log \mathbb{E}[Z^n] = -T \lim_{n\to 0} \frac{1}{n} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z^n] \ , \tag{20}$$

where we are not going to worry about the exchange of limits. We have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z^n] = \lim_{N\to\infty} \frac{1}{N} \log \int \mathrm{d}Q e^{NS(Q)} = S(Q^*) \ , \tag{21}$$

where Q^* is the saddle point matrix. The replica symmetric free energy can then be obtained as follows:

$$f_{\mathrm{RS}}(q) = -T \lim_{n\to 0} \frac{1}{n} S(Q^*) \ . \tag{22}$$

We will exchange the extremization with respect to q with the limit $n \to 0$, so we take the limit $n \to 0$ at fixed q. For the two terms of $S(Q)$, we obtain

$$\begin{aligned}
&\lim_{n\to 0} \frac{1}{n} \left[\frac{n-1}{2} \log(1-q) + \frac{1}{2} \log[1 + (n-1)q] \right] = \lim_{n\to 0} \frac{1}{n} \left[\frac{n}{2} \log(1-q) + \frac{1}{2} \log \left(\frac{1 - q + nq}{1-q} \right) \right] \\
&= \lim_{n\to 0} \frac{1}{n} \left[\frac{n}{2} \log(1-q) + \frac{1}{2} \log \left(1 + n \frac{q}{1-q} \right) \right] = \frac{1}{2} \log(1-q) + \frac{q}{1-q} \ ,
\end{aligned} \tag{23}$$

which gives the first term of $f_{\mathrm{RS}}(q)$.

For the second term, let us define

$$I(z) = \int \mathcal{D}_{1-q} r \, e^{-\beta v(r+z-\kappa)} \ . \tag{24}$$

We have then

$$\log \int \mathcal{D}_q z I(z)^n = \log \int \mathcal{D}_q z e^{n \log I(z)} \sim \log \left[1 + \int \mathcal{D}_q z n \log I(z) \right] \sim n \int \mathcal{D}_q z \log I(z) \ , \tag{25}$$

hence

$$\lim_{n\to 0} \frac{1}{n} \log \left\{ \int \mathcal{D}_q z I(z)^n \right\} = \int \mathcal{D}_q z \log I(z) = \int \mathcal{D}_q z \log \left[\int \mathcal{D}_{1-q} r \, e^{-\beta v(r+z-\kappa)} \right] \ , \tag{26}$$

which gives the second term of $f_{\mathrm{RS}}(q)$.

11. We have seen that the replica symmetric solution corresponds to the assumption that the system has a single thermodynamic state.

In the UNSAT phase, the Hamiltonian is dominated by the smooth part of the potential $v(x)$ for $x < 0$ because many constraints are violated, and it is therefore a smooth function of $\boldsymbol{J}$. Hence, we can expect that the ground state is unique; as a consequence, all replicas are found in the same state at $T = 0$ and $Q_{ab} = q = 1$ for all a, b.

In the SAT phase, instead, most of the constraints are satisfied and $x > 0$, which corresponds to the region where $v(x) = 0$ is flat. Hence we can change a little bit $\boldsymbol{J}$ and we still obtain a solution. The energy landscape has a flat shape with $H = 0$ over a finite region of the space of $\boldsymbol{J}$. The replicas, even at zero temperature, are free to explore this space and find different configurations, leading to a $Q_{ab} = q < 1$.

12. We then consider the UNSAT phase and assume that $q = 1 - \chi T + O(T^2)$ when $T \to 0$. A saddle-point calculation gives

$$\begin{aligned} G(z) &= \lim_{T\to 0} -T \log\left[\int \mathcal{D}_{\chi T} r \, e^{-\beta v(r+z-\kappa)}\right] = \lim_{T\to 0} -T\log\left[\int \frac{\mathrm{d}r}{\sqrt{2\pi\chi T}} e^{-\frac{1}{T}\left[\frac{r^2}{2\chi}+v(r+z-\kappa)\right]}\right] \\ &= \min_r \left[\frac{r^2}{2\chi} + v(r+z-\kappa)\right] . \end{aligned} \tag{27}$$

If $z - \kappa > 0$, the minimum is in $r = 0$ where $v(r + z - \kappa) = 0$ and we obtain $G(z) = 0$. If $z - \kappa < 0$, we look for a minimum in the region where $r + z - \kappa < 0$, and we get the equation for r as

$$\frac{r}{\chi} + r + z - \kappa = 0 \qquad \Rightarrow \qquad r^* = \frac{\kappa - z}{1 + 1/\chi} \qquad \Rightarrow \qquad r^* + z - \kappa = \frac{z-\kappa}{1+\chi} < 0 \ , \tag{28}$$

which is consistent with the initial assumption. Hence

$$G(z) = \frac{(r^*)^2}{2\chi} + \frac{1}{2}(r^* + z - \kappa)^2 = \frac{1}{2}(z-\kappa)^2\left[\frac{1}{\chi(1+1/\chi)^2} + \frac{1}{(1+\chi)^2}\right] = \frac{(z-\kappa)^2}{2(1+\chi)} \ , \tag{29}$$

so we get Eq. (18) of the exam.

Because the free energy reduces to the energy in the limit $T \to 0$, we have

$$e(\alpha) = \lim_{T\to 0} f_{\mathrm{RS}}(q = 1 - \chi T) \ . \tag{30}$$

The first term in $f_{\mathrm{RS}}(q)$ gives for $T \to 0$:

$$-\frac{T}{2}\left[\log(\chi T) + \frac{1-\chi T}{\chi T}\right] \to -\frac{1}{2\chi} \ . \tag{31}$$

The second term, using the result for $G(z)$, gives

$$-T\alpha \int \mathcal{D}_q z \log\left[\int \mathcal{D}_{1-q} r \, e^{-\beta v(r+z-\kappa)}\right] \to \alpha \int \mathcal{D}_q z G(z) = \frac{\alpha}{2(1+\chi)} \int_{-\infty}^{\kappa} \mathrm{d}z \, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} (z-\kappa)^2 \ . \tag{32}$$

Putting together the two terms we obtain Eq. (19) of the text of the exam.

13. The value of χ is determined by stationarity, hence $\partial e/\partial\chi = 0$. This is because the value of q had to be determined by $\partial f_{\mathrm{RS}}(q)/\partial q = 0$ and we took the limit $T \to 0$ leaving χ as a parameter in q. Defining

$$\alpha_c(\kappa) = \left[\int_{-\infty}^{\kappa} \mathrm{d}z \, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} (z-\kappa)^2\right]^{-1} \ , \tag{33}$$

we have

$$e(\alpha) = -\frac{1}{2\chi} + \frac{1}{2(1+\chi)} \frac{\alpha}{\alpha_c(\kappa)} \ . \tag{34}$$

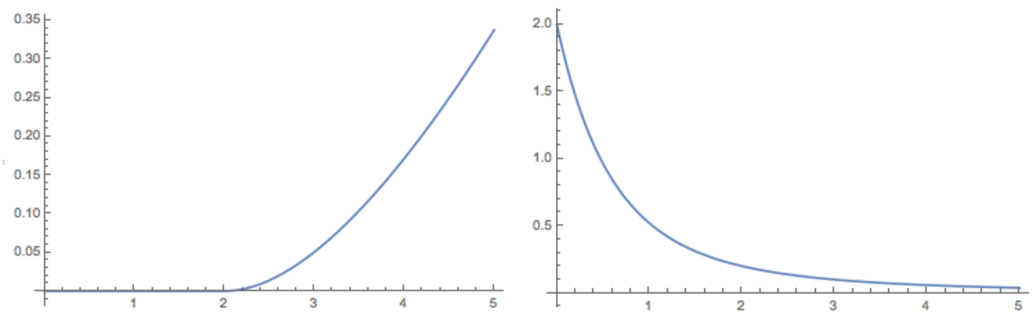

Figure 1: (Left) Plot of $e(\alpha)$ for $\kappa = 0$. (Right) Plot of $\alpha_c(\kappa)$.

Hence the equation for χ is

$$\frac{1}{2\chi^2} - \frac{1}{2(1+\chi)^2}\frac{\alpha}{\alpha_c(\kappa)} = 0 \qquad \Leftrightarrow \qquad \frac{(1+\chi)^2}{\chi^2} = \frac{\alpha}{\alpha_c(\kappa)} \qquad \Rightarrow \qquad \chi = \left(\sqrt{\frac{\alpha}{\alpha_c(\kappa)}} - 1\right)^{-1} \ , \tag{35}$$

which gives Eq. (20) of the exam (note that χ must be positive because $q \leq 1$). Plugging this result for χ in Eq. (34) it is easy to show that

$$e(\alpha) = \frac{1}{2}\left(\sqrt{\frac{\alpha}{\alpha_c(\kappa)}} - 1\right)^2 . \tag{36}$$

The curve of $e(\alpha)$ starts from zero at $\alpha = \alpha_c(\kappa)$ and increases smoothly upon increasing α. For $\alpha < \alpha_c(\kappa)$, we obtain a non-zero energy, but this results is not physical because we expect the ground state energy to be an increasing function of α. In fact, we derived Eq. (36) under the assumption that $q = 1 - \chi T$, which is wrong in the SAT phase where $q < 1$ at $T = 0$. Hence in the SAT phase for $\alpha < \alpha_c(\kappa)$ this result does not apply and $e(\alpha) = 0$. We conclude that there is a phase transition at $\alpha_c(\kappa)$ where the energy goes from zero to strictly positive (see the left panel of Fig. 1).

The critical value $\alpha_c(\kappa)$ is a decreasing function of κ and vanishes for $\kappa \to \infty$, which is consistent with the intuition that the problem becomes more and more difficult upon increasing κ. With larger κ the perceptron can classify less and less patterns (see the right panel of Fig. 1).

To conclude, we note that in this problem it has been shown that the replica symmetric solution is stable (Gardner, 1987). So, the results we derived here are exact. They have recently been proven rigorously (see e.g. Barbier et al, arXiv:1708.03395).