# ICFP M2 - Statistical physics 2
# Homework n° 5
# Effect of Quenched Disorder on Fluctuating Interfaces

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This homework is a preamble to the study of the effect of quenched disorder on low temperature phases (Lecture 7 and TD 7). In the following, as an introduction, we shall study the effect of quenched disorder on fluctuating interfaces. This is a central issue arising in several different physical problems (super-conductors, domain-walls, polymers, etc.).

## 1 One dimensional interfaces

The energy of an interface is proportional to its length. The proportionality coefficient is called the surface tension. In the following we consider a one dimensional interface, i.e. a line $x(t)$ in the $(x,t)$ plane, that starts at $(0,0)$ and ends at $(x,L)$. Its energy reads :

$$E = \sigma \int dl = \sigma \int_0^L dt \sqrt{1 + \left(\frac{dx}{dt}\right)^2} \simeq \sigma L + \frac{\sigma}{2} \int_0^L dt \left(\frac{dx}{dt}\right)^2 \qquad ,$$

where $x(t)$ is the height of the interface. We consider the temperature low enough so that the interface does not fluctuate too much, i.e. it has no overhangs (so $x(t)$ is a single valued function) and the expansion of the square root above to first order is justified.

The partition function of the interface is obtained by performing a functional integral :

$$\begin{aligned} Z &= \int dx Z(x,L;0,0) \qquad (1) \\ Z(x,L;0,0) &= \int_{x(0)=0}^{x(L)=x} \mathcal{D}[x(t)] \exp\left(-\beta\sigma L - \beta\frac{\sigma}{2}\int_0^L dt \left(\frac{dx}{dt}\right)^2\right) \end{aligned}$$

In order to analyze this partition function and the fluctuations of the interface, we now consider the $t$ coordinate as time. Consider the Langevin equation

$$\frac{dx}{dt} = \xi(t) \qquad \langle \xi(t)\xi(t')\rangle = D\delta(t-t')$$

where $\xi(t)$ is a Gaussian random white noise with zero average.

1. Write the probability functional for the function $\xi(t)$.
2. Using the Langevin equation show that up to a normalization constant the probability functional for the trajectory $x(t)$ is the same of eq.(1). What is the expression of $D$ in terms of $\beta\sigma$ ?
3. Given that the Langevin process leads to diffusion of the particle, we know that the probability that the particle starts in 0 at time 0 and reaches $x$ at time $t = L$ is equal to $P(x,L;0,0) = \frac{e^{-\frac{x^2}{2DL}}}{\sqrt{2\pi DL}}$. Use this result and the mapping above to obtain that

   $$Z(x,L;0,0) = \mathcal{N} \exp\left(-\beta\sigma L - \beta\sigma\frac{x^2}{2L}\right)$$

   where $\mathcal{N}$ is a numerical constant which depends on the normalization of the functional integral (do not try to compute, it is irrelevant for our purposes).
4. From the expression above obtain how the average fluctuations of the interface $\langle (x(L) - x(0))^2\rangle$, scales with $L$.

## 2 The effect of quenched disorder : zero temperature case

We now consider the effect of quenched disorder on the physics of the interface in a perturbative way, i.e. for arbitrary small disorder, and in the thermodynamic limit $L \to \infty$. The partition function in presence of quenched disorder reads :

$$Z = \int dx Z(x, L; 0, 0) = \int dx \int_{x(0)=0}^{x(L)=x} \mathcal{D}[x(t)] \exp\left(-\beta\sigma L - \beta \int_0^L dt \left[\frac{\sigma}{2}\left(\frac{dx}{dt}\right)^2 + V(x(t), t)\right]\right) \quad (2)$$

where $V(x(t), t)$ is the quenched random potential acting on the interface.

1. Show that without quenched disorder the zero-temperature profile of the interface is $x_{GS}(t) = 0 \; \forall t$.
2. Assuming that the disorder is small, and hence that $x(t)$ is close to 0, we expand $V(x(t), t)$ to linear order, and obtain that besides an irrelevant constant its contribution to the energy of the interface can be written as $\int dt h(t) x(t)$ where $h(t) = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=0}$. For simplicity we assume a simple Gaussian statistics for $h(t)$ : $\overline{h(t)} = 0$ and $\overline{h(t)h(t')} = \Delta^2 \delta(t - t')$ with $\Delta \ll 1$.
   Obtain in Fourier space the ground state profile $\hat{x}_{GS}(k)$ as a function of $\hat{h}(k)$ where $k$ is the Fourier wave-vector.
3. Show that $\overline{\hat{h}(k)\hat{h}(k')} = 2\pi\delta(k + k')$ and then use this result to obtain

$$\overline{(x_{GS}(t) - x_{GS}(t'))^2} = C \frac{\Delta^2}{\sigma^2} |t - t'|^3$$

   where $C$ is a numerical constant.

The result above, taken literally, would imply that in presence of quenched disorder the interface strongly fluctuates at zero temperature on large length-scales—actually even stronger than at finite temperature without disorder.

This conclusion is incorrect since we assumed from the start that the effect of disorder is perturbative and hence expanded $V(x, t)$ to linear order. This is not allowed if $x(t)$ has very large fluctuations as we found.

The correct conclusion is that no matter how small is $\Delta$, the disorder becomes overwhelming beyond a length-scale $\ell \sim \Delta^{2/3}$, which is called the Larkin length. In consequence the effect of quenched disorder cannot be considered a small perturbation around the non-disordered case at large length-scales : no matter how small is $\Delta$ the physics of fluctuating interfaces in presence of quenched disorder is altered and, as we shall explain in lecture 7 and TD 7, is drastically different from its no-disorder counterpart.