ICFP M2 - Statistical physics 2
Homework n^o 4
Langevin and Fokker-Planck equations

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This exercise is a preparation for the next TD on the Dyson Brownian Motion for random matrices ; its goal is to recall you basic facts on the Langevin and Fokker-Planck equations.

Consider a particle that moves in one dimension according to the (overdamped) Langevin equation :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -V'(x(t)) + \eta(t) \ , \tag{1}$$

where the first term is a deterministic conservative force deriving from the potential energy $V(x)$, and the second is a random force. We assume η to be a Gaussian white noise characterized by its first two moments, $\mathbb{E}[\eta(t)] = 0$, $\mathbb{E}[\eta(t)\eta(t')] = 2T\delta(t-t')$ with T the temperature of the environment in contact with the particle.

As a consequence of the Langevin equation the probability density for the position of the particle, $P(x,t)$, evolves according to the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial J}{\partial x} \ , \qquad \text{with} \quad J(x,t) = -V'(x)P(x,t) - T\frac{\partial P}{\partial x} \ . \tag{2}$$

1. Interpret the Fokker-Planck equation as a conservation law, and specify the origin of the two terms in J.

2. Describe the random variable
$$\Delta x = \int_t^{t+\Delta t} \mathrm{d}t' \, \eta(t') \ , \tag{3}$$
for a given time-interval Δt.

3. Give the solution of the Langevin and Fokker-Planck equations, with the initial condition $x(t=0) = x_0$, hence $P(x, t=0) = \delta(x - x_0)$, in the two extreme cases :
   (a) $T = 0$.
   (b) $V(x)$ independent of x.

4. Check that the Gibbs-Boltzman distribution $P_{\mathrm{GB}}(x) = \frac{1}{Z}e^{-\beta V(x)}$, with $\beta = \frac{1}{T}$, is a stationary solution of (2).

5. When the potential is quadratic, $V(x) = \frac{1}{2}x^2$, the random trajectory $x(t)$ is called an Ornstein-Uhlenbeck stochastic process. Give an explicit solution of $x(t)$ as a function of the trajectory of the noise η (taking the initial condition $x(t) = x_0$ deterministic). Conclude that, for a given time t, $x(t)$ is a Gaussian random variable ; specifiy its mean and variance.