Quantum Spin Glasses on the Bethe lattice

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with C.R.Laumann, S. A. Parameswaran, S.L.Sondhi, PRB 81, 174204 (2010)

with G.Carleo and M.Tarzia, PRL 103, 215302 (2009)

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Outline

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Classical spin glasses

Frustration in many-body problems: spin glasses

A classical example: the Edwards-Anderson model on a square lattice:
$H = \sum_{\langle i,j \rangle} J_{ij} S_i S_j$ $\qquad J_{ij} = \pm 1$ with probability $1/2$

Its mean field version: the Sherrington-Kirkpatrick model:
$H = \sum_{i,j} J_{ij} S_i S_j$ $\qquad J_{ij}$ Gaussian with zero mean and $\langle J_{ij}^2 \rangle = 1/N$

High temperature: paramagnetic, $m = 0$
Low temperature: amorphous magnetization profile, $m_i = \langle S_i \rangle \neq 0$
Average magnetization $m = N^{-1} \sum_i m_i \sim N^{-1/2} \to 0$
Edwards-Anderson order parameter $q_{EA} = N^{-1} \sum_i m_i^2$
Linear susceptibility $\chi = N^{-1} \sum_{ij} \langle S_i S_j \rangle$ finite; $\chi_{SG} = N^{-1} \sum_{ij} \langle S_i S_j \rangle^2$

Mean field theory:
Huge degeneracy of classical ground states
Many metastable states, slow relaxation

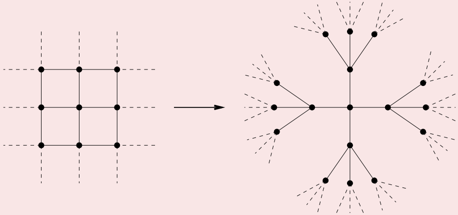
Frustration appears as a key ingredient in many different systems: structural glasses, colloids, spin glasses, electron glasses, optimization problems ...

What happens to the spin glass phase in presence of quantum fluctuations? Tunnelling between different SG states? Speed up the relaxation?

Regular lattices, Bethe lattices and random graphs

A useful tool: Bethe approximation

Discard the small loops of the lattice, graph becomes a tree:



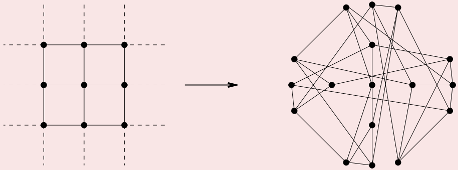
Provides an "improved" mean field theory since the local connectivity remains finite

The tree allows for a simple recursive solution but the frustration is lost: no loops

Regular lattices, Bethe lattices and random graphs

A more useful tool for frustrated systems: Cavity approximation

Replace the lattice by a random graph of the same connectivity:



Regular lattices, Bethe lattices and random graphs

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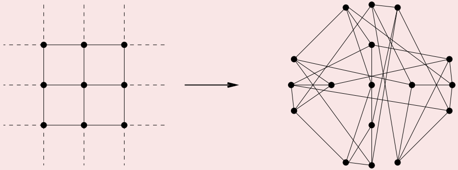

Regular random graphs have many good properties:

- Equal probability to all graphs of N sites such that each site has z neighbors
- Free-energy is self-averaging (a single sample is representative of the average)
- No boundary, all sites play statistically the same role ("periodic boundary conditions")
- Key property: typical graphs are locally tree-like; loops have a length diverging logarithmically with the size N of the system.
- The exact solution on the random graph can still be obtained (cavity method, Mézard-Parisi 2001)

Regular lattices, Bethe lattices and random graphs

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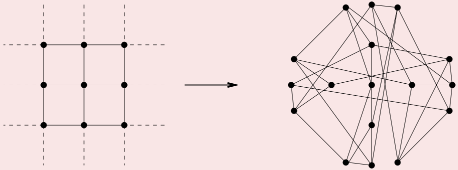

Unfrustrated phase (RS cavity method)

- Correlations decay fast enough
- Long loops can be neglected $\longrightarrow$ back to a tree
- Cavity approximation is equivalent to Bethe approximation

Regular lattices, Bethe lattices and random graphs

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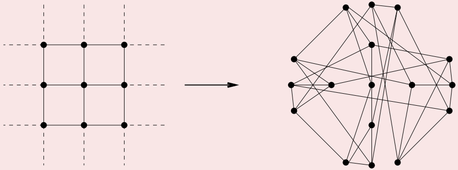


Frustrated phase (1RSB cavity method)

- Correlations do not decay fast enough
- The recursion relation on a tree is initiated from a given boundary condition
- For each boundary condition, a different fixed point is obtained
- One has to sum over boundary conditions in a consistent way
- Cavity approximation describes the glassy phase

Ising antiferromagnet on a random graph

Ising antiferromagnet:

$$H = J \sum_{\langle i,j \rangle} S_i S_j + h \sum_i S_i$$

Equivalent to a model of particles with repulsive interaction:

$$H = V \sum_{\langle i,j \rangle} n_i n_j - \mu \sum_i n_i$$

On the regular random graph:

- Typical graphs characterized by many (large) loops of even and odd length
  Graph is not bipartite
  Frustrates antiferromagnetic ordering: spin glass phase, without disorder in the couplings
- Glassy phase for large J and small h (large V, close to half-filling)
  Same phase diagram of the SK model

`Krzakala and Zdeborova, EPL 81, 57005 (2008)`

Outline

AKLT states

Schwinger bosons: a representation of a spin S

$S_+ = b_\uparrow^\dagger b_\downarrow \quad S_- = b_\downarrow^\dagger b_\uparrow \quad S_z = \frac{1}{2}(b_\uparrow^\dagger b_\uparrow - b_\downarrow^\dagger b_\downarrow)$
Total spin $S = \frac{1}{2}(b_\uparrow^\dagger b_\uparrow + b_\downarrow^\dagger b_\downarrow)$

A general AKLT state can be written in terms of Schwinger bosons as

$$|\Psi(M)\rangle = \prod_{\langle ij \rangle} \left(b_{i\uparrow}^\dagger b_{j\downarrow}^\dagger - b_{i\downarrow}^\dagger b_{j\uparrow}^\dagger \right)^M |\, 0\, \rangle$$

where $\langle ij \rangle$ are the links of a given graph.

Fixed connectivity z: then zM bosons per site, spin $S = zM/2$

Then the maximum possible spin on each link is $2S - M$: the state is annihilated by any Hamiltonian H constructed out of projectors on spin $S' > 2S - M$ on each link (which are polynomials of $\vec{S}_i \cdot \vec{S}_j$)

Affleck, Kennedy, Lieb, Tasaki, 1987

AKLT states: mapping on a classical model

$$|\Psi(M)\rangle = \prod_{\langle ij\rangle} \left(b^\dagger_{i\uparrow} b^\dagger_{j\downarrow} - b^\dagger_{i\downarrow} b^\dagger_{j\uparrow} \right)^M |\, 0\,\rangle$$

Introduce on each site coherent states

$$|\vec{n}\rangle = \frac{1}{\sqrt{(zM)\,!)}} (u_\mu b^\dagger_\mu)^{zM} |\, 0\,\rangle$$

with $u = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\varphi} \end{pmatrix}$ and $\vec{n} = u^\dagger \vec{\sigma} u$

Then

$$\Psi(\{\hat{n}_i\}) = \langle \{\hat{n}_i\} | \Psi \rangle = \prod_{\langle ij\rangle} (u_{i\uparrow} u_{j\downarrow} - u_{i\downarrow} u_{j\uparrow})^M = \prod_{\langle ij\rangle} \left(\frac{1 - \vec{n}_i \cdot \vec{n}_j}{2} \right)^{M/2}$$

$$|\Psi(\{\hat{n}_i\})|^2 \equiv \exp(-M H_{\mathrm{cl}}) \qquad H_{\mathrm{cl}} = -\sum_{\langle ij\rangle} \ln\left(\frac{1 - \vec{n}_i \cdot \vec{n}_j}{2} \right)$$

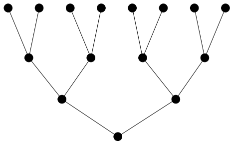
A classical antiferromagnetic model at temperature $T = 1/M$

Arovas et al. 1988

AKLT models on the tree (AKLT, 1987)

$$H_{\rm cl} = -\sum_{\langle ij\rangle} \ln\left(\frac{1-\vec{n}_i\cdot\vec{n}_j}{2}\right)$$

$$T(\vec{n}_0,\vec{n}_1) = \left(\frac{1-\vec{n}_0\cdot\vec{n}_1}{2}\right)^M$$



$$\psi^0(\vec{n}_0) = \frac{1}{Z}\int D\vec{n}_1\cdots D\vec{n}_{z-1}\; T(\vec{n}_0,\vec{n}_1)\psi^1(\vec{n}_1)\cdots T(\vec{n}_0,\vec{n}_{z-1})\psi^{z-1}(\vec{n}_{z-1})$$

High temperature fixed point $\psi(\vec{n})=1$: paramagnetic phase

Stability of the paramagnetic fixed point against AF ordering

- Assume all ψ are equal for a generation, then next generation is $\psi \propto (T\psi)^{z-1}$
- Expand in spherical harmonics $\psi(\vec{n}) = 1+\epsilon\sum_{l\neq 0,m}^{\infty} c_{lm}\psi_{lm}(\vec{n})$
- $(T\psi)(\vec{n}) = \lambda_0 + \epsilon\sum_{l\neq 0,m}\lambda_l c_{lm}\psi_{lm}(-\vec{n})$ with $\lambda_l = \frac{[\Gamma(M+1)]^2}{\Gamma(M-l+1)\Gamma(M+l+2)}$
- $\psi(\vec{n}) = 1+\epsilon(z-1)\lambda_0^{-1}\sum_{l\neq 0,m}\lambda_l c_{lm}\psi_{lm}(-\vec{n}) + \mathcal{O}(\epsilon^2)$
- Instability if $\frac{\lambda_l}{\lambda_0} = \frac{1}{z-1}$: critical point for $l=1$, $M_c = \frac{2}{z-2}$
- Decay of correlations: $\langle\vec{n}_0\cdot\vec{n}_d\rangle \propto \left(\frac{\lambda_1}{\lambda_0}\right)^d = \left(\frac{M}{M+2}\right)^d$
- $\chi_{AF} = \frac{1}{N}\sum_{ij}(-1)^{d(i,j)}\langle\vec{n}_i\cdot\vec{n}_j\rangle \propto \sum_d z(z-1)^{d-1}\left(\frac{M}{M+2}\right)^d$ diverges at M_c

AKLT models on the random graph

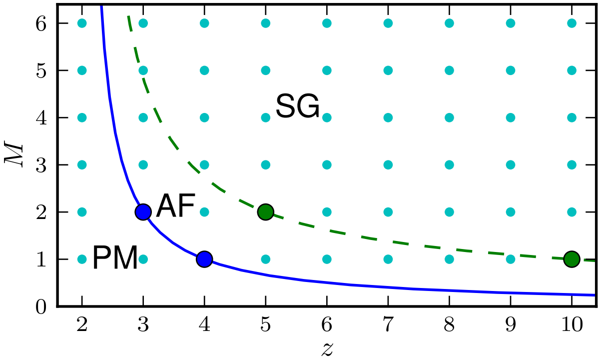
High temperature: paramagnetic phase, $\psi(\vec{n}) = 1$
Lower the temperature: AF instability is avoided, incompatible with odd loops

Continue to lower the temperature: instability to SG phase, $\psi_i(\vec{n}_i)$ is a random variable
Spin frozen in random direction; degenerate states corresponding to different patterns

$$\chi_{SG} = \frac{1}{N} \sum_{ij} [\langle \vec{n}_i \cdot \vec{n}_j \rangle]^2 \propto \sum_d z(z-1)^{d-1} \left(\frac{M}{M+2} \right)^{2d}$$

Finite only if $\frac{1}{z-1} \left(\frac{M}{M+2} \right)^2 < 1$, or $M < M_{SG} = \frac{2}{\sqrt{z-1}-1}$

Phase diagram



- Blue line: AF instability on the tree
- Green line: SG instability on the random graph
- Dots: critical states
- $z = 2$ corresponds to one dimensional chain: no phase transition

C.R.Laumann, S. A. Parameswaran, S.L.Sondhi, FZ, PRB 81, 174204 (2010)

Spectrum of the quantum spin glass phase

AKLT mapping provides a classical *vector* model with a Spin Glass phase

- Classical Gibbs measure decomposes into a collection of clustering pure states $\alpha = 1 \dots \mathcal{N}$ with essentially disjoint support: $P(\{\hat{n}\}) = \sum_{\alpha=1}^{\mathcal{N}} w_\alpha P_\alpha(\{\hat{n}\})$ *Corrections are exponentially small in N*
- In each of these states, the $\psi^i_\alpha(\hat{n}_i)$ – and thus the local magnetizations – are macroscopically different: $q_{EA} \neq 0$
- We want to show that the quantum AKLT ground state $|\Psi\rangle$ itself is a superposition over a collection of degenerate ground states $|\Psi_\alpha\rangle$ each of which corresponds to one of the classical clustering states: $|\Psi\rangle = \sum_{\alpha=1}^{\mathcal{N}} \sqrt{w_\alpha}|\Psi_\alpha\rangle$

Argument

1. Define quantum states $|\Psi_\alpha\rangle$ by $|\langle\{\hat{n}_i\}|\Psi_\alpha\rangle|^2 = P_\alpha(\{\hat{n}_i\})$
2. Since different states have different support (exponentially in N). Therefore $\langle\Psi_\alpha|\mathcal{O}|\Psi_\beta\rangle \sim e^{-N}$ for any operator. Different states are almost orthogonal.
3. Define $|\Phi\rangle = \sum_{\alpha=1}^{\mathcal{N}} \sqrt{w_\alpha}|\Psi_\alpha\rangle$. Interference terms are negligible: $\langle\Phi|\mathcal{O}|\Phi\rangle \sim \sum_{\alpha=1}^{\mathcal{N}} w_\alpha \langle\Psi_\alpha|\mathcal{O}|\Psi_\alpha\rangle = \langle\Psi|\mathcal{O}|\Psi\rangle \quad \Rightarrow \quad |\Phi\rangle = |\Psi\rangle$
4. $0 = \langle\Psi|H|\Psi\rangle = \sum_{\alpha=1}^{\mathcal{N}} w_\alpha \langle\Psi_\alpha|H|\Psi_\alpha\rangle + O(e^{-N}) \quad \Rightarrow \quad \langle\Psi_\alpha|H|\Psi_\alpha\rangle \sim e^{-N}$

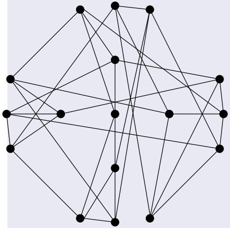
G.Biroli, C.Chamon, FZ, PRB 78, 224306 (2008)

Outline

A lattice model with a superglass phase

Extended Hubbard model on a regular random graph at half-filling and $U = \infty$:
$H = -t \sum_{\langle i,j \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) + V \sum_{\langle i,j \rangle} n_i n_j - \sum_i \mu n_i$

We study the model on a regular random graph of L sites and connectivity $z = 3$



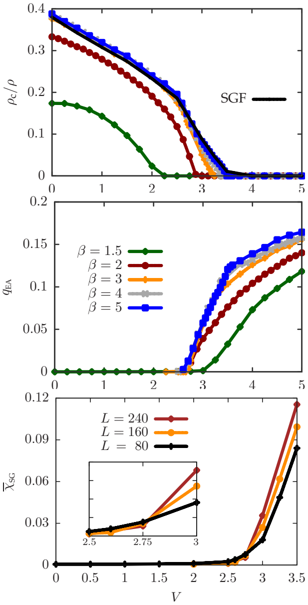
- Frustrated antiferromagnetic interaction
- Solution for $L \to \infty$ possible via the cavity method
- Classical model ($t = 0$): spin glass transition (like Sherrington-Kirkpatrick model)
- RSB, many degenerate glassy states

Methods

- Quantum cavity method: solution for $L \to \infty$
- Canonical Worm Monte Carlo: limited to $L < 240$ by ergodicity problems
- Variational calculation + Green Function Monte Carlo (for illustration)

G. Carleo, M. Tarzia, FZ, PRL 103, 215302 (2009)

Results at half-filling and $t = 1$



Condensate fraction:

$$\rho_c = \lim_{|i-j|\to\infty} \left\langle a_i^{+} a_j \right\rangle = |\left\langle a \right\rangle|^2$$

Edwards-Anderson order parameter:

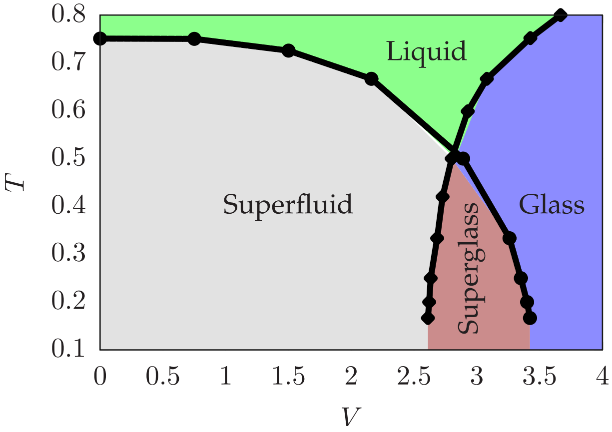
$$q_{EA} = \frac{1}{L} \sum_i \left\langle (\delta n_i)^2 \right\rangle \qquad \delta n_i = n_i - \langle n_i \rangle$$

Spin-Glass susceptibility:

$$\chi_{SG} = \frac{1}{L} \int_0^{\beta} d\tau \sum_{i,j} \langle \delta n_i(\tau) \delta n_j(0) \rangle^2$$

Phase diagram

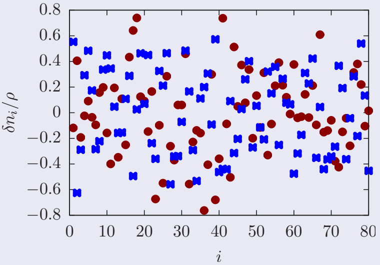
Phase diagram at half-filling and $t = 1$



A variational argument

Many spin glass states

- Each spin glass state breaks translation invariance
- Simplest variational wavefunction: $\langle \underline{n} | \Psi \rangle = \exp(\sum_i \alpha_i n_i)$
- A different set of parameters for each spin glass state
- Optimization of the parameters α_i depends on the initial condition



Stability of the glass state

- Green Function Monte Carlo: $|\Psi(\tau)\rangle = e^{-\tau H}|\Psi\rangle$
- The time τ needed to escape from the initial state $|\Psi\rangle$ increases with system size

Outline

Conclusions

1. AKLT states on regular random graphs provide a model for quantum spin glasses
2. Spin glass phase is characterized by many almost degenerate ground states
3. No tunnelling between SG states, matrix element is exponentially small in N
4. Extended Hubbard model on random graph shows a super-spin glass phase

Future work: identify a more realistic model!