ICFP M2 - Statistical physics 2 – TD n° 2
The Random Energy Model

Grégory Schehr, Francesco Zamponi

We consider a statistical mechanics model with $M = 2^N$ configurations $\underline{\sigma}$ indexed by N Ising spins, $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N$. The energy of each configuration is denoted $H(\underline{\sigma})$, the system is in equilibrium with an heat bath of inverse temperature β, the probability to find the system in the configuration $\underline{\sigma}$ is thus $p(\underline{\sigma}) = e^{-\beta H(\underline{\sigma})}/Z(\beta)$, with the partition function $Z(\beta) = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}$ normalizing these probabilities.

When the system is disordered the energies $H(\underline{\sigma})$ are random variables, as well as the partition function. We recall the definition of the annealed and quenched free-energies:

$$f_{\rm a}(\beta) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\beta)] \ , \qquad f_{\rm q}(\beta) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \mathbb{E}[\log Z(\beta)] \ , \tag{1}$$

where $\mathbb{E}[\bullet]$ denotes the expectation with respect to the randomness in $H(\underline{\sigma})$.

We consider in this problem the simplest of such disordered system, the Random Energy Model (REM), introduced by Bernard Derrida in 1980, in which the energies $H(\underline{\sigma})$ are M *independent* identically distributed Gaussians of zero mean and variance $\frac{N}{2}$.

1 Preamble: concentration of random variables

1. Prove the Markov inequality: if X is a positive random variable with finite average,
$$\mathbb{P}[X \geq a] \leq \frac{1}{a}\,\mathbb{E}[X] \qquad \forall\, a > 0 \ . \tag{2}$$

2. Deduce from it the Chebychev inequality for a random variable X admitting a variance,
$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t\,\sqrt{\mathrm{Var}[X]}] \leq \frac{1}{t^2} \ . \tag{3}$$

3. One is often interested in integer valued random variables, $X = 0, 1, \dots$. As a consequence of the Markov inequality one has
$$\mathbb{P}[X > 0] \leq \mathbb{E}[X] \ , \tag{4}$$
i.e. if the average of X is very small then X is with high probability equal to 0. On the other hand if the average of X is very large it is not always the case that X is positive with high probability: its variance should not be too large for this to be true. Show as a consequence of the Chebychev inequality that
$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \tag{5}$$
Using the Cauchy-Schwarz inequality obtain a stronger bound:
$$\frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \leq \mathbb{P}[X > 0] \leq \mathbb{E}[X] \ , \tag{6}$$
these two inequalities being called the second and first moment method, respectively.

2 The free-energy of the REM

1. Compute the annealed free-energy of the model.

2. Denote $\mathcal{N}(u, \mathrm{d}u)$ the random variable counting the number of configurations $\underline{\sigma}$ with intensive energies in a small interval of length $\mathrm{d}u$ around u, i.e. with $H(\underline{\sigma}) \in [Nu, N(u + \mathrm{d}u)]$. What is the law of this random variable? Give its average value and its variance.

3. Deduce that for typical realizations of the disorder, $\mathcal{N}$ is close to its typical value, with at the leading exponential order,
$$\mathcal{N}_{\mathrm{typ}}(u, \mathrm{d}u) = \begin{cases} e^{N s_{\mathrm{m}}(u)} \mathrm{d}u & \text{if } \; u \in [-u_{\mathrm{c}}, u_{\mathrm{c}}] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$
where $u_{\mathrm{c}} = \sqrt{\log 2}$ and $s_{\mathrm{m}}(u) = \log 2 - u^2$ is the microcanonical entropy.

4. Conclude that the quenched free-energy reads
$$f_{\mathrm{q}}(\beta) = \min_{u \in [-u_{\mathrm{c}}, u_{\mathrm{c}}]} \left[u - \frac{1}{\beta} s_{\mathrm{m}}(u) \right] = \begin{cases} -\frac{\beta}{4} - \frac{\log 2}{\beta} & \text{if } \; \beta < \beta_{\mathrm{c}} \\ -\sqrt{\log 2} & \text{if } \; \beta > \beta_{\mathrm{c}} \end{cases} , \tag{8}$$
with $\beta_{\mathrm{c}} = 2\sqrt{\log 2}$ the critical inverse temperature of the model. Compare with the annealed result.

5. Give the values of the energy and entropy in the high and low temperature phases. What is the thermodynamic order of the transition?

6. What is the groundstate energy density of the model? Check the agreement of your answer with the results of the TD 1 on the extremes of i.i.d. random variables.

3 (Optional) The structure of $p(\underline{\sigma})$

Let us denote $Y = \sum_{\underline{\sigma}} p(\underline{\sigma})^2$ the probability to find two independent copies of the system in the same configuration.

1. What would be the value of Y if $p(\underline{\sigma})$ were uniform on a subset of M_{eff} configurations?

2. Express Y in terms of $Z(2\beta)$ and $Z(\beta)$. Deduce that in the high-temperature phase Y is typically exponentially small. Explain why in this way we do not get information about the low-temperature phase.

3. To study the low temperature phase we need a more precise approach. We recall that if X is a centered Gaussian random variable and F an arbitrary function, then
$$\mathbb{E}[XF(X)] = \mathbb{E}[X^2] \, \mathbb{E}\left[F'(X)\right] . \tag{9}$$
Use this identity to obtain:
$$\frac{1}{N} \sum_{\underline{\sigma}} \mathbb{E}\left[p(\underline{\sigma}) H(\underline{\sigma})\right] = -\frac{\beta}{2} \mathbb{E}[1 - Y] . \tag{10}$$

4. Conclude that
$$\lim_{N \to \infty} \mathbb{E}[Y] = \begin{cases} 0 & \text{if } \; T > T_{\mathrm{c}} \\ 1 - \frac{T}{T_{\mathrm{c}}} & \text{if } \; T < T_{\mathrm{c}} \end{cases} . \tag{11}$$
The transition at T_{c} is often called a "condensation": in the low-temperature phase the dominant configurations of the Gibbs measure covers a sub-exponential subset of the configuration space.