ICFP M2 - Statistical physics 2
Solution of the homework n° 4
Langevin and Fokker-Planck equations

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1. The Fokker-Planck equation is indeed written here as a continuity equation that traduces the local conservation of the probability (or of the number of particles if we don't normalize P to 1). The probability $P(x,t)\mathrm{d}x$ of finding the particle in the infinitesimal interval $[x, x+\mathrm{d}x]$ evolves during the infinitesimal time $\mathrm{d}t$ because of the incoming flux $J(x,t)\mathrm{d}t$ and of the outcoming flux $-J(x+\mathrm{d}x,t)\mathrm{d}t$. One can thus interpret $J(x,t)$ as the current of particles flowing through the position x at time t, counted in the increasing x direction. The first contribution to J arises from the deterministic force $-V'(x)$, and counts the number of particles that crosses x with this velocity. The second term is a diffusion effect, the random force η tends to equalize the density of presence of the particle.

2. Δx is a sum of Gaussian random variables, it is thus Gaussian. One can characterize it by its two first moments,

$$\mathbb{E}[\Delta x] = \int_t^{t+\Delta t} \mathrm{d}t' \, \mathbb{E}[\eta(t')] = 0 \, , \tag{1}$$

$$\mathbb{E}[(\Delta x)^2] = \int_t^{t+\Delta t} \mathrm{d}t_1' \int_t^{t+\Delta t} \mathrm{d}t_2' \, \mathbb{E}[\eta(t_1')\eta(t_2')] = 2T\Delta t \, . \tag{2}$$

3. (a) When $T = 0$ the random term disappears from the Langevin equation, the deterministic trajectory $x_*(t)$ of the particle is thus the solution of the ordinary differential equation $x_*'(t) = -V'(x_*(t))$ with the initial condition $x_*(t=0) = x_0$. The solution of the Fokker-Planck equation is then $P(x,t) = \delta(x - x_*(t))$.

   (b) When $V(x)$ is independent of x there is no deterministic force and the Fokker-Planck equation reduces to the diffusion equation $\frac{\partial P}{\partial t} = T \frac{\partial^2 P}{\partial x^2}$. The solution of this equation with the initial condition peaked in x_0 is the Gaussian distribution

$$P(x,t) = \frac{1}{\sqrt{4\pi T t}} e^{-\frac{(x-x_0)^2}{4Tt}} \, . \tag{3}$$

   The solution of the Langevin equation is

$$x(t) = x_0 + \int_0^t \mathrm{d}t' \, \eta(t') \, , \tag{4}$$

   which is indeed a Gaussian random variable centered in x_0 with variance $2Tt$.

4. One has $T\frac{\mathrm{d}P_{\mathrm{GB}}}{\mathrm{d}x} = -V'(x)P_{\mathrm{GB}}(x)$, hence the current J vanishes for this choice of P, which makes $P_{\mathrm{GB}}(x)$ a stationary solution of the Fokker-Planck distribution. If the potential $V(x)$ is confining, in such a way that the Gibbs-Boltzman distribution is normalizable, one has $P(x,t) \to P_{\mathrm{GB}}(x)$ at large times, for all initial conditions.

5. In this case the Langevin equation $\frac{\mathrm{d}x}{\mathrm{d}t} = -x(t) + \eta(t)$ can be integrated with, for instance, the variation of constant method, to give

$$x(t) = x_0 \, e^{-t} + \int_0^t \mathrm{d}t' \, e^{-(t-t')} \, \eta(t') \, . \tag{5}$$

The mean of this Gaussian random variable is $x_0\, e^{-t}$, that decays to zero at large time : the particle forgets the initial condition and relaxes on average to the bottom of the potential well. The variance of $x(t)$ reads

$$\int_0^t \mathrm{d}t_1' \int_0^t \mathrm{d}t_2'\, e^{-(2t-t_1'-t_2)}\, \mathbb{E}[\eta(t_1')\eta(t_2')] = 2T \int_0^t \mathrm{d}t'\, e^{-2(t-t')} = T(1-e^{-2t}) \ , \tag{6}$$

it grows with time towards the equilibrium one.