Lecture on Random Matrices.

I did not have time to talk about section 4).

1) Introduction

M : $N \times N$ matrix w. random entries M_{jk} (real or complex)

Q : stat. of eigenvectors & eigenvalues?

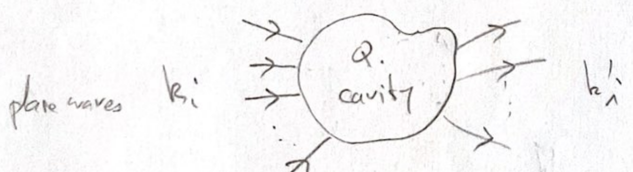
. In physics, initially introduced in the context of nuclear physics by Wigner ('50) to model the energy levels of big complex nuclei (Wigner's surmise of the HW)

c
sity,
cing...)

. Since then, a lot of applications (Oxford handbook of RMT)

* Quantum particle in a random pot, ...

(* mesoscopic physics (quantum scattering)



$\underline{k}' = S \underline{k}$)

[* disordered systems : manifold in random media]

* random graph, $G = (V, E)$

↳ adjacency matrix $A_{ij} = \mathbb{1}(\langle i,j \rangle \in E)$

random graph → random adjacency matrix

* nber theory (Riemann ζ function)

* combinatorics

→ KPZ, directed polymers

1) • First applications actually appeared in stat. w. J. Wishart in 1928 Correlation in time series

$$\vec{X}_t = \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{N,t} \end{pmatrix}$$ price of stock 'i' / daily temperature in city 'i' random, $$\hat{C}_{jk} = \frac{1}{T} \sum_{t=1}^{T} x_{jt}\, x_{kt}$$

time

$1 \leq t \leq T$ random correlation matrix

→ many applications in (financial) data analysis

2) Ensembles of random matrices.

→ Following the initial motivation of Wigner, we consider matrices with <u>real</u> spectrum, either sym. (real) Hermitian (comp.)

→ Two main categories of random matrices :

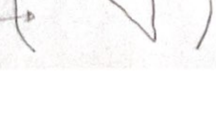
a) <u>Ens. with independent entries</u> : "Wigner" matrices

$$M_{jk} = x_{jk} + i\, y_{jk}$$

Joint ~~PDF~~ Prob. ~~Distribution~~ Density Function (PDF)

$$P(M) = P(\{M_{jk}\}) = \prod_{i=1}^{N} f_i(x_{ii}) \prod_{j<k} f^{(R)}_{jk}(x_{jk})\, f^{(I)}_{jk}(y_{jk})$$

imposed by sym.



Note that for real sym., the number of independent degrees of freedom is $N_{free} = 1 + 2 + \dots + N = \frac{N(N+1)}{2}$.

b) Rotationally invariant ensembles

Suppose that M is real & symmetric

$$\Rightarrow \exists \, (\underline{O}, \underline{\Lambda}) \text{ with } \underline{O}\,\underline{O}^t = \mathbb{1}$$

$$\underline{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_N \end{pmatrix}$$

such that $M = \underline{O}\,\underline{\Lambda}\,\underline{O}^{-1}$

Rotationally inv. ensemble : matrices related via similarity transformat° occur with the same proba.

$$P(M) = P(OMO^{-1}) \quad \forall \, O \mid OO^t = \mathbb{1}.$$

invariance par rotation

↳ in such models the eigenvectors do not play an important role (↳ uniformly distributed on the sphere)

Porter–Rosenzweig '60 thm : the only ensembles that satisfy 1) & 2) are the Gaussian ensembles:

explain why this is Wigner maxl.

$$\boxed{P(M) = \frac{1}{Z_N} e^{-a\,\mathrm{Tr}(M^2) - b\,\mathrm{Tr}(M)}}$$

In the following: focus on these ensembles with $b=0$.

For M real sym : Gaussian Orthogonal Ensemble (GOE)

M complex Hermitian: Gaussian Unitary Ensemble (GUE).

↳ already announce universality

c) Joint law of eigenvalues for GOE/GUE

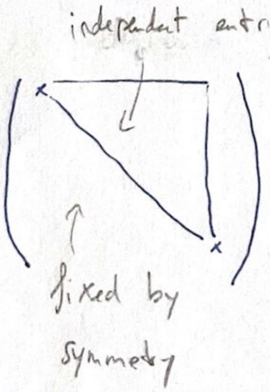
GOE: $M = M^t$

Diagonal form: $M = O \Lambda O^{-1}$, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_N \end{pmatrix}$

Λ ← eigenvalues

O ← eigenvectors

$O O^t = \mathbb{1}$

$M =$ 

$\#$ independent variables $= 1 + 2 + \dots + N = \frac{N(N+1)}{2}$

$= \underbrace{N}_{\text{eigenvalues}} + \underbrace{\frac{N(N-1)}{2}}_{\text{eigenvectors}}$

Change of variables $M = \{M_{ij}\} \longrightarrow \Lambda, O$

$$P(M) \longrightarrow P(\{\lambda_1, \dots, \lambda_N\}, O)$$

$$\Rightarrow P(\{\lambda_1, \dots, \lambda_N\}, O) = P(M)\, |\det J|$$

Where J is the Jacobian of the transformation :

this one is not a bijection (permutation of the indices)

$$M \rightleftarrows \{\Lambda, O\}$$

$$J = \text{"}\left\{ \frac{\partial M_{ij}}{\partial \lambda_k}, \frac{\partial M_{ij}}{\partial O_{kl}} \right\}\text{"}$$

a tedious computation leads to :

$$|\det J| = \prod_{j<k} |\lambda_j - \lambda_k| \quad \text{independantly of } O$$

True also for non-invariant ensembles.

<u>For GOE</u> :

$$P(\{\lambda_1, \dots, \lambda_N\}, O) = \frac{1}{Z_N} e^{-a \sum_{i=1}^{N} \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|$$

$\Rightarrow$ eigenvalues and eigenvectors are independent.
↳ uniformly distributed

Marginal ~~PDF~~ proba. density function of $\{\lambda_1, \dots, \lambda_N\}$ are

$$\boxed{P(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} \prod_{i<j} |\lambda_i - \lambda_j| e^{-a \sum_{i=1}^{N} \lambda_i^2}}$$

level repulsion

<u>For GUE</u> :

A similar computation yields :

$$\boxed{P(\lambda_1, \dots, \lambda_N) = \frac{1}{\hat{Z}_N} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-a' \sum_{i=1}^{N} \lambda_i^2}}$$

Here we choose:

$$P(\lambda_1, \dots, \lambda_N) = B_N \, e^{-\frac{\beta N}{4} \sum_{i=1}^{N} \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

with $\beta = 1$ GOE $\longrightarrow$ corresponds to $\mathbb{E}[M_{ii}^2] = \frac{2}{N}$, $\mathbb{E}[M_{ij}^2] = \frac{1}{N}$ $i<j$

$\beta = 2$ GUE.

3.) Coulomb gas approach. (Dyson '62)

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta = e^{\frac{\beta}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|}$$

$$\Rightarrow P(\lambda_1, \dots, \lambda_N) = B_N \exp\left(-\beta E(\{\lambda_i\}_{1 \le i \le N})\right)$$

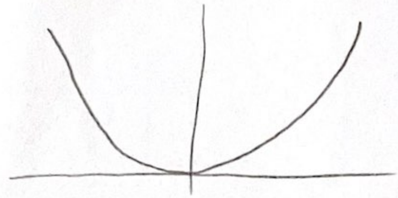
$$E(\{\lambda_i\}) = \frac{N}{4} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$$

repulsive Coulomb interaction in 2d, confined on a line

long-range

Interpretation in terms of a "log-gas" of particles:

λ_i = positions of charged particles interacting via the 2d Coulomb interact° and confined on a line within a quadratic well



$\longrightarrow$ competition between confinement and repulsive interaction

. Typical scale of λ_i's, λ_{typ} :

Potential energy : $\frac{N}{2} \sum_{i=1}^{N} \lambda_i^2 \sim N \times N \times \lambda_{typ}^2 = N^2 \lambda_{typ}^2$

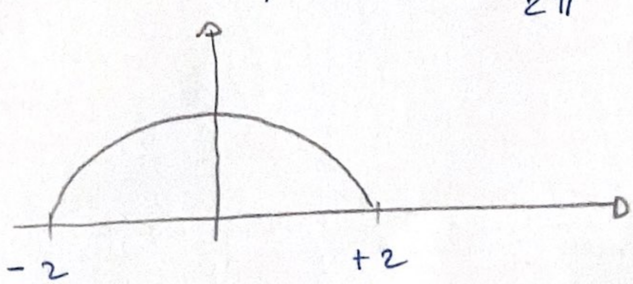
Interactions : $\frac{1}{2} \sum_{i \neq j} \ln|\lambda_i - \lambda_j| \sim N^2$

Balancing the two $\Rightarrow$ $\lambda_{typ} = O(1)$ when $N \to \infty$.

. Empirical eigenvalue distribution

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \quad , \quad \mu_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

$$\mathbb{E}(\mu_N(\lambda)) \longrightarrow \mu_{sc}(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}, \ -2 \leq \lambda \leq 2$$



In the bulk, $\mu_N(\lambda)$ is self-averaging (concentrates around its mean)

$$\mu_N(\lambda) \xrightarrow[N\to\infty]{} \mathbb{E}(\mu_N(\lambda)) .$$

Universality of the Wigner semi-circle

- Wigner semi-circle law holds $\forall$ for Wigner matrices provided the proba. density of the matrix elements ~~have a sub-exponential tail~~,

~~re-check~~

~~i.e.~~ $\cancel{\mathbb{P}(M_{jh} \geq x) \leq C \exp(-x^\theta)}$

~~$\exists C, \theta$~~ <u>provided</u> : $\cancel{\mathbb{E}(x_{ii}^2) = \sigma_i^2 > 0 , \forall i}$ $\quad \mathbb{E}[x_{ii}^2] = \frac{\sigma^2}{N}$

$\cancel{\mathbb{E}(x_{ij}^2) = \sigma^2 \ \forall i \leq j}$ $\quad \cancel{\mathbb{E}[x_{ij}^2] = \frac{1}{N}}$

still holds provided the M_{jh}'s are not "too correlated."

on $[-\sqrt{2}\sigma, +\sqrt{2}\sigma]$

See e.g. L. Erdős arXiv: 1004.0861 for a review

$$\mathbb{E}(M_{jh}^2) = \sigma_{jh}^2 \quad \text{and} \quad \sum_j \sigma_{jh}^2 = 1 \quad \forall h$$

then $\to$ Wigner law with support $[-2, 2]$.

- For invariant ensemble :

$$P(\lambda_1, \dots, \lambda_N) = \tilde{B}_N \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, e^{-N\sum_{i=1}^N V(\lambda_i)}$$

$$\hookrightarrow \quad P(M) \propto e^{-N \operatorname{Tr}(V(M))}$$

If $V(M) = M^2 \Rightarrow$ Wigner semi-circular law

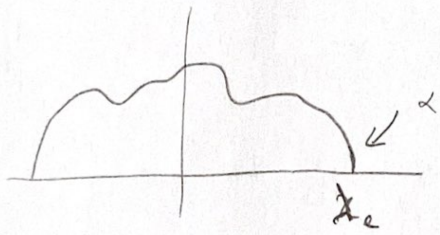
But for $V(M) \neq M^2$, ~~In this case~~ $\mathbb{E}(\mu_N(\lambda))$ is different from Wigner semi-circular

In this case, if $V(x) \gg \ln x \quad , x \to \infty$

$$\frac{V(x)}{\ln x} \xrightarrow[x \to \infty]{} +\infty$$

then $\mathbb{E}(\mu_N(\lambda))$ has a finite support in the limit $N \to \infty$

and



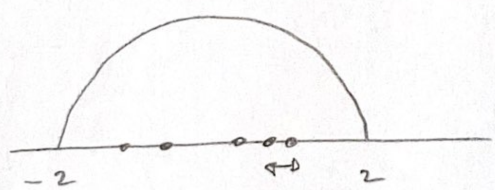
$\propto \sqrt{x_e - \lambda}$ except if $V(x)$ is fine tuned

4.) Local statistics

- Discuss bulk & edge on the figure.

a) Bulk

- Interparticle spacing in the bulk



GOE/GUE

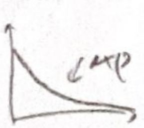
$\lambda_1 < \lambda_2 < \dots < \lambda_N$: $\quad s = \lambda_{i+1} - \lambda_i = O\left(\frac{1}{N}\right)$

(not too close from the edge)

For large N, PDF of s $\quad P_N(s) \underset{N \to \infty}{\sim} \frac{1}{\langle s \rangle_N} P_\beta\left(\frac{s}{\langle s_N \rangle}\right)$

independent of i.

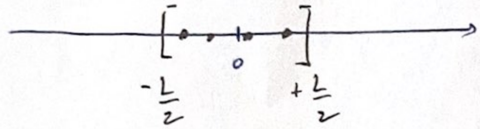
Compare to Poisson for independent random points



where $P_\beta(x)$ is well approximated by the Wigner ~~matrix~~ surmise (result for $N = 2$):

$$P_\beta(x) \simeq P_{WS,\beta}(x) = a_\beta s^\beta e^{-b_\beta s^2} \Rightarrow \text{level repulsion}$$

- Nber variance



$N_L =$ # eigenvalues in $\left[-\frac{L}{2}, \frac{L}{2}\right]$

If the eigenv. are independent, Poisson process:

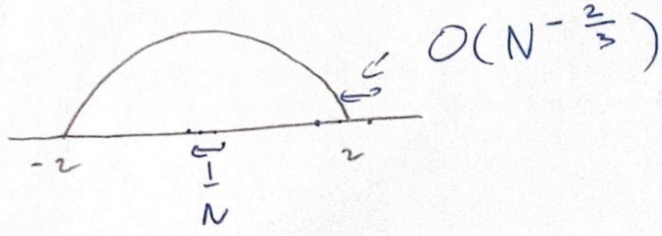
$$\mathbb{E}[N_L] = \mathrm{Var}[N_L] = \mathbb{E}[N_L^2] - (\mathbb{E}[N_L])^2$$
$$\simeq L$$

For eigenvalues: $\mathrm{Var}[N_L] \underset{NL \gg 1}{\sim} \frac{2}{\beta \pi^2} \ln(NL)$

$(L < 2)$.

$\Rightarrow$ more "rigid" than Poisson process.

b) <u>Edge</u>



In particular $\lambda_{max} = 2 + N^{-\frac{2}{3}} \underline{\chi_\beta}$

↳ Tracy-Widom β

→ appeared in many problems like KPZ, LIS, ...