ICFP M2 - Statistical physics 2
Homework n° 2
Some simple applications of the replica method

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These two exercises are an illustration of the replica method, which is a very powerful technique used in disordered systems. This theoretical framework seems at first quite acrobatic–to say the least; putting it on a firm mathematical basis has been and still is an important research problem in mathematical physics and probability theory.

The starting point of the replica method is one of the simple equalities :

$$\mathbb{E}[\log Z] = \lim_{n\to 0} \frac{\mathbb{E}[Z^n] - 1}{n} = \lim_{n\to 0} \frac{\log \mathbb{E}[Z^n]}{n} \tag{1}$$

where $\mathbb{E}[\bullet]$ denotes the average over the quenched disorder. These identities are certainly true when n is a real parameter. However in complicated models one can easily compute $\mathbb{E}[Z^n]$ only when n is an integer. What makes the replica method acrobatic is the "analytic continuation" from integer values to 0, which in some cases is not unique.

1 A toy model

Consider a "toy" disordered partition function $Z = e^{xN}$ where x plays the role of the quenched disorder; x is equal to a with probability $1 - e^{-cN}$ and to b with probability e^{-cN}, with $c > 0$. We are interested in the thermodynamic limit where the size of the system, N, goes to infinity.

- Compute $\lim\limits_{N\to\infty} (\log \mathbb{E}[Z])/N$ and $\lim\limits_{N\to\infty} (\mathbb{E}[\log Z])/N$. Give a condition on a, b, c under which these two limits are distinct; where does the difference among the two results come from?

- Use now the replica trick (1) to get the result doing the thermodynamic limit first and the $n \to 0$ limit later (the reverse order must trivially give the correct result). Why does the $n \to 0$ limit help compared to the previous computation?

2 A slightly more involved case

We now consider another very simple disordered system: an Ising spin in a random quenched magnetic field h, where h is a Gaussian random variable of mean zero and variance one.

- Show that
$$\mathbb{E}[\log Z] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}h}{\sqrt{2\pi}} e^{-h^2/2} \log\left(1 + e^{2\beta h}\right) \ . \tag{2}$$

- In order to use the replica trick, you have to obtain $\mathbb{E}[Z^n]$. Show that for n integer,
$$\mathbb{E}[Z^n] - 1 = \sum_{k=1}^{n} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{n!}{(n-k)!k!} + e^{\beta^2 n^2/2} - 1 \ ;$$
you need to know the value of $\mathbb{E}[e^{ah}]$ when h is a centered Gaussian variable.

- How to continue for real values of n the above expression? A natural idea is to use the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. Recall, or prove if you have never seen them, the identities $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma(p+1) = p!$ for positive integer p (we will use the standard definition $0! = 1$). This allows you to rewrite the previous equation as

$$\mathbb{E}[Z^n] - 1 = \sum_{k=1}^{n} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} + e^{\beta^2 n^2/2} - 1 \ .$$

  Explain why one can extend the sum over k up to infinity, thus obtaining the expression

$$\mathbb{E}[Z^n] - 1 = \sum_{k=1}^{\infty} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} + e^{\beta^2 n^2/2} - 1 \ ,$$

  where n can be continued to real values.

- We are now ready to perform the $n \to 0$ limit. In order to do that you have to establish (using the equality $\Gamma(z+1) = z\Gamma(z)$) a few properties of the Gamma function. Show that for $n \to 0$ and l a positive integer

$$\Gamma(l+n) \to (l-1)!$$

  whereas for $n \to 0$ and l a negative (or zero) integer:

$$\Gamma(l+n) = \frac{1}{n}\frac{(-1)^{-l}}{(-l)!}(1+o(1)) \ .$$

- Using this result and the replica trick recover that

$$\mathbb{E}[\log Z] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}h}{\sqrt{2\pi}} e^{-h^2/2} \log\left(1 + e^{2\beta h}\right) \ .$$

The aim of this exercise was to make you acquainted with the acrobatic procedures one has to do in order to use the replica trick. You could certainly claim that there was some arbitrariness in the procedure–and you would be right!–but the final result is the correct one. The lecture and problem class show the replica method at work in much more challenging models, for some of them no direct average of $\log Z$ is doable.