ICFP M2 - Statistical physics 2 – Solution of the exam

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1 Questions on the lectures

1. Q1: We consider the Larkin model for a one-dimensional elastic interface $u(x)$, with $0 \leq x \leq L$ in the presence of a random Gaussian force. The Hamiltonian of this model reads

$$H[\{u(x)\}] = \frac{\gamma}{2}\int_0^L dx \left(\frac{du}{dx}\right)^2 + \int_0^L dx\, f(x)u(x)\ , \tag{1}$$

where γ is the stiffness of the elastic line and $f(x)$ is a Gaussian random force field of zero mean $\langle f(x)\rangle = 0$ and short-range correlations $\langle f(x)f(x')\rangle = \Delta\,\delta(x-x')$. We assumed that the line is pinned at the extremity $x = 0$, i.e. $u(x = 0) = 0$.

a) Recall the physical origin of both contributions in (1).

The first term in (1) corresponds to the elastic energy of the interface, while the second term mimics the effect of impurities than pin the interface. It can be viewed as the linearized approximation (valid for small displacement $u(x)$) of the more general form $\int_0^L dx V(u_x, x) \approx \int_0^L dx\, f(x)u(x)$ where $f(x) = \frac{\delta V(u(x),x)}{\delta u_x}|_{u=0}$.

b) Show that the ground state configuration $u_0(x)$, i.e., which minimises $H[\{u(x)\}]$, satisfies

$$\gamma\frac{d^2u_0(x)}{dx^2} = f(x)\ . \tag{2}$$

The ground state $u_0(x)$ is obtained from the minimisation condition

$$\left.\frac{\delta H[\{u(x)\}]}{\delta u(x)}\right|_{u=u_0} = 0\ . \tag{3}$$

Using

$$\frac{\delta}{\delta u(x)}\left[\int_0^L dx \left(\frac{du}{dx}\right)^2\right] = -2\frac{d^2u(x)}{dx^2} \quad , \quad \frac{\delta}{\delta u(x)}\left[\int_0^L dx f(x)u(x)\right] = f(x) \tag{4}$$

the condition (3) reads

$$-\gamma\frac{d^2u_0(x)}{dx^2} + f(x) = 0\ . \tag{5}$$

c) Deduce that $u_0(x)$ reads

$$u_0(x) = \frac{1}{\gamma}\int_0^x dy \int_0^y dz\, f(z) + A\,x + B \tag{6}$$

and explain how to determine the constants A and B.

Once these constants have been determined, one can show that the optimal profile reads

$$u_0(L) = -\frac{1}{\gamma}\int_0^L dz\, z\, f(z)\ . \tag{7}$$

Integrating (5) leads immediately to (6). The constant B is determined by imposing $u_0(0) = 0$. The constant A is determined by substituting the expression (6) – setting $B = 0$ – in the expression of $H[\{u(x)\}]$ in (1) and minimising the resulting expression with respect to A. This fixes the value of A and leads to the expression for $u_0(L)$ in (7).

d) Compute $\langle u_0(L)\rangle$ and $\langle u_0^2(L)\rangle$. Compare the latter to the fluctuations of a purely elastic line – in the absence of the second term in (1) – which is at thermal equilibrium at temperature T (you are not expected to provide a detailed computations of these thermal elastic fluctuations but to estimate instead their behaviour as a function of L).

Since $\langle f(x)\rangle = 0$, one has $\langle u_0(L)\rangle = 0$. The computation of $\langle u_0^2(L)\rangle$ is as follows:

$$\langle u_0^2(L)\rangle = \frac{1}{\gamma^2}\int_0^L dz_1\, z_1 \int_0^L dz_2\, z_2 \langle f(z_1)f(z_2)\rangle = \frac{\Delta}{\gamma^2}\int_0^L dz_1\, z_1^2 = \frac{\Delta}{3\gamma^2}L^3\,. \tag{8}$$

This should be compared to the mean square displacement of a purely elastic line at temperature T which is $\propto TL^2 \ll L^3$ for large L. Hence the fluctuations due to the disorder are much bigger than the thermal ones (hence the notion of a zero temperature fixed point).

2. Q2: Random Gaussian matrices and the log-gas.

a) Recall what the Gaussian Orthogonal Ensemble (GOE) is. In particular, what does "Orthogonal" refer to?

The Gaussian Orthogonal Ensemble is an ensemble of $N\times N$ real symmetric matrices whose elements $M_{i,j}$ with $i \le j$ are independent Gaussian random variables which are centered and with variances $\mathbb{E}(M_{ii}^2) = 2/N$ while $\mathbb{E}(M_{ij}^2) = 1/N$ for $i < j$. The probability measure can thus be written as

$$P(M) = B_N e^{-\mathrm{Tr}(M^2)}\,, \tag{9}$$

where B_N is a normalization constant. Hence let O be an $N\times N$ orthogonal matrix, then $P(M) = P(OMO^{-1})$: the probability measure is thus invariant under *orthogonal* transformations, hence the name *orthogonal*.

b) The joint distribution of the eigenvalues of a random matrix belonging to the GOE reads

$$P_{\rm joint}(\lambda_1,\cdots,\lambda_N) = \frac{1}{Z_N} e^{-\frac{N}{4}\sum_{i=1}^N \lambda_i^2} \prod_{i\neq j} |\lambda_i - \lambda_j|\,. \tag{10}$$

Explain why Z_N can be interpreted as the partition function of a one-dimensional gas of particles at some inverse temperature $\beta = 1$. What is the corresponding energy function associated to this gas of particle? Why is it called a *log-gas*?

The quantity Z_N is a normalization constant such that $\int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N P_{\rm joint}(\lambda_1,\cdots,\lambda_N) = 1$. Hence it reads:

$$Z_N = \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N\, e^{-\frac{N}{4}\sum_{i=1}^N \lambda_i^2} \prod_{i\neq j} |\lambda_i - \lambda_j|\,. \tag{11}$$

Using the identity $|x| = e^{\ln|x|}$, one can rewrite the Vandermonde term $\prod_{i\neq j}|\lambda_i - \lambda_j|$ as

$$\prod_{i\neq j} |\lambda_i - \lambda_j| = e^{\sum_{i\neq j}\ln|\lambda_i - \lambda_j|}\,. \tag{12}$$

Therefore Z_N in (11) can be written as

$$Z_N = \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N e^{-E(\lambda_1,\cdots,\lambda_N)} \quad \text{, where} \quad E(\lambda_1,\cdots,\lambda_N) = \frac{N}{4}\sum_{i=1}^N \lambda_i^2 - \sum_{i\neq j}\ln|\lambda_i - \lambda_j|\,. \tag{13}$$

It follows from that expression (13) that Z_N can be interpreted as the partition function of a gas of N particles with positions λ_i's on the real axis in the presence of a harmonic potential (the term $\propto \sum_{i=1}^N \lambda_i^2$ in (13)) and interacting via a repulsive *logarithmic* potential (the term $-\sum_{i\neq j}\ln|\lambda_i - \lambda_j|$ in (13)). Hence the name of *log-gas*.

2 Quenched versus annealed disorder

1. Using the chain rule $\rho(\boldsymbol{x},\boldsymbol{y}) = \rho(\boldsymbol{x}|\boldsymbol{y})\rho(\boldsymbol{y})$, we have

$$\begin{aligned}\langle O\rangle &= \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\rho(\boldsymbol{x},\boldsymbol{y})O(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\rho(\boldsymbol{x}|\boldsymbol{y})\rho(\boldsymbol{y})O(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{y}\rho(\boldsymbol{y})\left[\mathrm{d}\boldsymbol{x}\rho(\boldsymbol{x}|\boldsymbol{y})\rho(\boldsymbol{y})O(\boldsymbol{x})\right] \\ &= \int \mathrm{d}\boldsymbol{y}\rho(\boldsymbol{y})\langle O\rangle_{\boldsymbol{y}} = \overline{\langle O\rangle_{\boldsymbol{y}}} \ .\end{aligned} \tag{14}$$

2. A spin glass is a magnetic alloy, such as CuMn with ∼1% Mn. The Cu atoms are metallic and have no magnetic moment, while the Mn impurities carry a magnetic moment (a "spin"). Let us consider the Cu metal as a background matrix, and let our degrees of freedom be the positions of the Mn atoms in the metal, called $\boldsymbol{y}$, and the spins of the Mn atoms, called $\boldsymbol{x}$. Assuming Ising spins for the Mn atoms, a simple model for the magnetic Hamiltonian is that given in Eq.(10) of the text. In addition, there will be interaction terms between the Mn positions, and between the Mn and Cu positions, but these only depend on $\boldsymbol{y}$ and are irrelevant for this discussion.

   The important point is that the alloy is usually prepared by quick cooling from a high-temperature liquid melt, in which atoms can freely diffuse. During the quench process, the positions of the Mn atoms, i.e. the $\boldsymbol{y}$, do not have time to equilibrate with the other degrees of freedom, and remain frozen into out-of-equilibrium positions. At room temperature, Mn atom diffusion in the alloy is so slow that their positions do not change anymore. Hence, the $\boldsymbol{y}$ are not in equilibrium: their probability distribution is not $Z(\boldsymbol{y})/Z$, and the quenched and annealed averages do not always coincide.

   This is why the couplings between the spin of Mn atoms i and j, which is a function of the distance $|y_i - y_j|$ between the two Mn atoms, can be considered as a quenched random variable, whose distribution is however not given by the equilibrium one (and was taken by Edwards-Anderson to be fully random Gaussian for simplicity).

   On the contrary, at high enough temperature (above a few Kelvin), the spin degrees of freedom of the Mn atoms can explore their phase space ergodically (the alloy is paramagnetic), so the $\boldsymbol{x}$ system are in equilibrium, conditioned to the frozen $\boldsymbol{y}$, and are described by $\rho(\boldsymbol{x}|\boldsymbol{y})$. It is only below the spin glass temperature (usually a few Kelvins, depending on the dilution of the Mn atoms) that the magnetic degrees of freedom also freeze in an out-of-equilibrium spin glass state.

3. **Annealed spin glass -** The partition function is

$$\begin{aligned}Z &= \sum_{\boldsymbol{J},\boldsymbol{S}} \prod_{\langle i,j\rangle\in E} e^{\beta J_{ij}S_iS_j} = \sum_{\boldsymbol{S}} \prod_{\langle i,j\rangle\in E} \sum_{J_{ij}} e^{\beta J_{ij}S_iS_j} = \sum_{\boldsymbol{S}} \prod_{\langle i,j\rangle\in E} \left[2\cosh(\beta J_{ij}S_iS_j)\right] \\ &= \sum_{\boldsymbol{S}} \prod_{\langle i,j\rangle\in E} \left[2\cosh(\beta)\right] = 2^N[2\cosh(\beta)]^M \ .\end{aligned} \tag{15}$$

   To obtain this result, we performed first the sum over $\boldsymbol{J}$ noting that it is factorized over the edges, then we used that $\cosh(x)$ is an even function hence $\cosh(\beta J_{ij}S_iS_j) = \cosh(\beta)$.

   The distribution of the disorder is

$$\rho(\boldsymbol{J}) = \frac{Z(\boldsymbol{J})}{Z} = \frac{\sum_{\boldsymbol{S}} e^{\sum_{\langle i,j\rangle\in E}\beta J_{ij}S_iS_j}}{2^N[2\cosh(\beta)]^M} \ . \tag{16}$$

   The sum in the numerator cannot be computed explicitly, because the spins are interacting and there are 2^N terms. The thermal distribution is

$$\rho(\boldsymbol{S}|\boldsymbol{J}) = \frac{e^{\sum_{\langle i,j\rangle\in E}\beta J_{ij}S_iS_j}}{Z(\boldsymbol{J})} \ . \tag{17}$$

   The energy can be computed by noting that the usual formula holds:

$$Ne(\beta) = \frac{\sum_{\boldsymbol{S},\boldsymbol{J}} H(\boldsymbol{S},\boldsymbol{J})e^{-\beta H(\boldsymbol{S},\boldsymbol{J})}}{Z} = -\frac{\partial \log Z}{\partial\beta} \ . \tag{18}$$

   Using Eq. (15), we have

$$e(\beta) = -\frac{1}{N}\frac{\partial}{\partial\beta}\left[N\log 2 + M\log 2 + M\log\cosh(\beta)\right] = -\frac{M}{N}\tanh(\beta) = -\frac{c}{2}\tanh(\beta) \ . \tag{19}$$

Another equivalent way consists in noting that $\langle H(\boldsymbol{S},\boldsymbol{J})\rangle = -\sum_{\langle i,j\rangle\in E}\langle J_{ij}S_iS_j\rangle$. For a single edge, we have

$$\langle J_{ij}S_iS_j\rangle = \frac{\sum_{\boldsymbol{S},\boldsymbol{J}} J_{ij}S_iS_j\prod_{\langle k,l\rangle\in E} e^{\beta J_{kl}S_kS_l}}{2^N[2\cosh(\beta)]^M} = \frac{2^N[2\cosh(\beta)]^{M-1}2\sinh(\beta)}{2^N[2\cosh(\beta)]^M} = \tanh(\beta) \tag{20}$$

In the numerator, we performed the sum over all edges different from $\langle i,j\rangle$ as in Eq. (15), while the sum over J_{ij} has an additional sign that gives a $\sinh(\beta)$ instead of $\cosh(\beta)$. Note that once we average over both $\boldsymbol{S}$ and $\boldsymbol{J}$, all the edges are equivalent, hence

$$e(\beta) = \frac{\langle H(\boldsymbol{S},\boldsymbol{J})\rangle}{N} = -\frac{M\tanh(\beta)}{N} = -\frac{c}{2}\tanh(\beta)\ . \tag{21}$$

4. **Planted spin glass -** Trivially, we have

$$\lim_{\beta\to 0}\frac{e^\beta}{2\cosh(\beta)} = \frac{1}{2}\ , \qquad \lim_{\beta\to\infty}\frac{e^\beta}{2\cosh(\beta)} = 1\ . \tag{22}$$

In the Bayes optimal setting, we have

$$P(\boldsymbol{J}) = \sum_{\boldsymbol{S}^*} P(\boldsymbol{J}|\boldsymbol{S}^*)P(\boldsymbol{S}^*) = \frac{1}{2^N}\sum_{\boldsymbol{S}^*}\prod_{\langle i,j\rangle\in E}\frac{e^{\beta J_{ij}S_i^*S_j^*}}{2\cosh(\beta)} = \frac{\sum_{\boldsymbol{S}^*}\prod_{\langle i,j\rangle\in E}e^{\beta J_{ij}S_i^*S_j^*}}{2^N[2\cosh(\beta)]^M}\ , \tag{23}$$

which coincides with Eq. (16) (the variable in the sum can be equivalently called $\boldsymbol{S}^*$ or $\boldsymbol{S}$). Similarly, we have

$$P(\boldsymbol{S}|\boldsymbol{J}) = \frac{P(\boldsymbol{J}|\boldsymbol{S})P(\boldsymbol{S})}{P(\boldsymbol{J})} = \underbrace{\frac{2^M[2\cosh(\beta)]^M}{\sum_{\boldsymbol{S}^*}\prod_{\langle i,j\rangle\in E}e^{\beta J_{ij}S_i^*S_j^*}}}_{1/P(\boldsymbol{J})}\times\underbrace{\prod_{\langle i,j\rangle\in E}\frac{e^{\beta J_{ij}S_iS_j}}{2\cosh(\beta)}}_{P(\boldsymbol{J}|\boldsymbol{S})}\times\underbrace{\frac{1}{2^N}}_{P(\boldsymbol{S})} = \frac{e^{-\beta H(\boldsymbol{S},\boldsymbol{J})}}{Z(\boldsymbol{J})}\ , \tag{24}$$

which coincides with Eq. (17).

At zero temperature[1], we have that $J_{ij} = S_i^*S_j^*$ with probability $\rho = 1$. Hence, the Hamiltonian is

$$H(\boldsymbol{S},\boldsymbol{J}) = -\sum_{\langle i,j\rangle\in E} S_i^*S_j^*S_iS_j\ . \tag{25}$$

If we choose $\boldsymbol{S} = \boldsymbol{S}^*$, all the terms in the Hamiltonian are equal to -1 and $H = -M$, which is the lowest possible value. Hence, $\boldsymbol{S}^*$ is a ground state.

5. **Overlap of the reconstruction -** Under the "gauge transformation" $\tilde{S}_i = S_iS_i^*$ and $\tilde{J}_{ij} = J_{ij}S_i^*S_j^*$, we have

$$\tilde{H}(\tilde{\boldsymbol{S}},\tilde{\boldsymbol{J}}) = -\sum_{\langle i,j\rangle\in E}\tilde{J}_{ij}\tilde{S}_i\tilde{S}_j = -\sum_{\langle i,j\rangle\in E}J_{ij}S_i^*S_j^*S_iS_i^*S_jS_j^* = -\sum_{\langle i,j\rangle\in E}J_{ij}S_iS_j = H(\boldsymbol{S},\boldsymbol{J})\ , \tag{26}$$

because $(S_i^*)^2 = 1$ for Ising spins. The probability distribution of $\tilde{J}_{ij}$ is

$$P(\tilde{J}_{ij} = \pm 1|S_i^*S_j^*) = P(J_{ij} = \pm S_i^*S_j^*|S_i^*S_j^*) = \frac{e^{\pm\beta(S_i^*)^2(S_j^*)^2}}{2\cosh(\beta)} = \frac{e^{\pm\beta}}{2\cosh(\beta)} = P(\tilde{J}_{ij} = \pm 1)\ , \tag{27}$$

hence the probability of a positive coupling $\tilde{J}_{ij} = 1$ is ρ, independently of $\boldsymbol{S}^*$. The overlap m becomes

$$m = \frac{1}{N}\sum_i S_iS_i^* = \frac{1}{N}\sum_i\tilde{S}_i\ . \tag{28}$$

We conclude that to compute m in the original problem, we can compute the magnetization of the gauge-transformed problem, in which the coupling have a probability ρ to be ferromagnetic and $1-\rho$ to be antiferromagnetic.

6. We know that the free energy is self-averaging, so we expect that this is the case for the problem with a fraction ρ of ferromagnetic coupling. Because the magnetization is the derivative of the free energy with respect to a magnetic field, unless we are at a critical value of the magnetic field, we expect the derivative to exist and as a consequence the magnetization is also self-averaging: in the thermodynamic limit, its typical value coincide with its average. We conclude that the overlap in the original problem is also self-averaging. So, for large N, typical realizations all have the same overlap, coinciding with the average.

[1] In the text of the exam, we forgot to specify that the couplings have to be drawn at zero temperature too. Sorry for the mistake.

7. The annealed average can be written as

$$\overline{Z(\tilde{\boldsymbol{J}})} = \sum_{\tilde{\boldsymbol{J}},\tilde{\boldsymbol{S}}} \prod_{\langle i,j\rangle\in E} \underbrace{\frac{e^{\beta\tilde{J}_{ij}}}{2\cosh(\beta)}}_{P(\tilde{J}_{ij})} \underbrace{e^{\beta\tilde{J}_{ij}\tilde{S}_i\tilde{S}_j}}_{interaction} = \sum_{\tilde{\boldsymbol{S}}} \prod_{\langle i,j\rangle\in E} \frac{\cosh[\beta(1+\tilde{S}_i\tilde{S}_j)]}{\cosh(\beta)} \ , \tag{29}$$

where as before we use that the sum over couplings is factorized. Noting that the variable $\tilde{S}_i\tilde{S}_j$ takes values ± 1, we can use the identity in the text for the function

$$\begin{aligned} f(\tilde{S}_i\tilde{S}_j) &= \log\frac{\cosh[\beta(1+\tilde{S}_i\tilde{S}_j)]}{\cosh(\beta)} = \frac{1}{2}\left[\log\frac{\cosh(2\beta)}{\cosh(\beta)} + \log\frac{1}{\cosh(\beta)}\right] + \frac{\tilde{S}_i\tilde{S}_j}{2}\left[\log\frac{\cosh(2\beta)}{\cosh(\beta)} - \log\frac{1}{\cosh(\beta)}\right] \\ &= \frac{1}{2}\log\frac{\cosh(2\beta)}{\cosh(\beta)^2} + \frac{\tilde{S}_i\tilde{S}_j}{2}\log\cosh(2\beta) = A + J_0\tilde{S}_i\tilde{S}_j \ . \end{aligned} \tag{30}$$

Hence

$$\overline{Z(\tilde{\boldsymbol{J}})} = \sum_{\tilde{\boldsymbol{S}}} \prod_{\langle i,j\rangle\in E} e^{f(\tilde{S}_i\tilde{S}_j)} = \sum_{\tilde{\boldsymbol{S}}} \prod_{\langle i,j\rangle\in E} e^{A+J_0\tilde{S}_i\tilde{S}_j} = e^{MA}\sum_{\tilde{\boldsymbol{S}}} e^{\sum_{\langle i,j\rangle\in E} J_0\tilde{S}_i\tilde{S}_j} \ , \tag{31}$$

which is the desired result.

8. **Reconstruction phase transition -** Eq. (31) is the partition function associated to the Hamiltonian (neglecting the irrelevant constant term associated to A)

$$H_0 = -J_0 \sum_{\langle i,j\rangle\in E} \tilde{S}_i\tilde{S}_j \ , \tag{32}$$

which is a ferromagnetic model on an Erdos-Renyi graph of connectivity c, with exchange coupling J_0. For large c, the number of neighbors on the graph becomes very large, and mean field theory is then justified. Mean field theory gives for the magnetization the self-consistent equation

$$m_i = \tanh(J_0 \sum_{j\in\partial i} m_j) \ , \tag{33}$$

and assuming that $m_i = m$ in a homogeneous phase, we get

$$m = \tanh(J_0 c m) \ , \tag{34}$$

which has non-zero solutions for $J_0 c > 1$. This gives, using the Taylor expansion $\cosh(x) \sim 1 + x^2/2 + O(x^4)$,

$$1 < J_0 c = \frac{c}{2}\log\cosh(2\beta) \sim c\beta^2 + O(\beta^4) \ , \tag{35}$$

hence $\beta > \beta^* = 1/\sqrt{c}$ at leading order for large c. Expanding ρ for small β we get $\rho \sim \frac{(1+\beta)}{2} + O(\beta^2)$, hence inference is possible only if the connectivity of the graph is

$$c > \frac{1}{\beta^2} = \frac{1}{(2\rho-1)^2} \ . \tag{36}$$

This is true at the leading order for small β and large c.

9. The exact critical temperature for the ferromagnetic model on the Erdos-Rényi graph at finite connectivity c is given by the condition $c\tanh(J_0) = 1$, i.e. using $\tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$,

$$\frac{1}{c} = \tanh\left[\frac{1}{2}\log\cosh(2\beta)\right] = \frac{\cosh(2\beta)-1}{\cosh(2\beta)+1} = \frac{\sinh(\beta)^2}{\cosh(\beta)^2} = \tanh(\beta)^2 \ , \tag{37}$$

hence we find a non-zero magnetization for

$$\beta > \beta^* = \operatorname{atanh}\left(\frac{1}{\sqrt{c}}\right) \ , \tag{38}$$

which for large c reproduces the correct asymptotic result $\beta^* \sim c^{-1/2}$. Recalling the definition of $\beta = -\frac{1}{2}\log\frac{1-\rho}{\rho}$, we have

$$\tanh(\beta) = \frac{e^{2\beta}-1}{e^{2\beta}+1} = \frac{\frac{\rho}{1-\rho}-1}{\frac{\rho}{1-\rho}+1} = 2\rho - 1 \qquad \Rightarrow \qquad c > \frac{1}{\tanh(\beta)^2} = \frac{1}{(2\rho-1)^2} \ , \tag{39}$$

which gives the desired result.