ICFP M2 - Statistical physics 2 – Solution of the exam

Grégory Schehr, Guilhem Semerjian

April 10th, 2020

1 Questions on the lectures

1. (a) The energies $H(\underline{\sigma})$ are linear combinations of the Gaussian random variables J and J', hence they are Gaussian as well.

   (b) The first moment vanishes, as the average of the J and J' is zero :

$$\mathbb{E}[H(\underline{\sigma})] = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} \mathbb{E}[J_{ij}]\sigma_i \sigma_j - \sqrt{\frac{3}{N^2}} \sum_{1 \le i < j < k \le N} \mathbb{E}[J'_{ijk}]\sigma_i \sigma_j \sigma_k = 0 \ .$$

   In the second moment only the terms involving $(J_{ij})^2$ and $(J'_{ijk})^2$ give a non-zero contribution, as these random variables are independent and of zero mean :

$$\mathbb{E}[H(\underline{\sigma})H(\underline{\tau})] = \frac{1}{N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j \tau_i \tau_j + \frac{3}{N^2} \sum_{1 \le i < j < k \le N} \sigma_i \sigma_j \sigma_k \tau_i \tau_j \tau_k \ .$$

   (c) In the large N limit one can replace at the leading order $\sum\limits_{i<j}$ by $\frac{1}{2}\sum\limits_{i,j}$, neglecting the sub-dominant contribution of the diagonal, and similarly $\sum\limits_{i<j<k}$ by $\frac{1}{6}\sum\limits_{i,j,k}$, neglecting the cases where one or two indices are equal. This gives

$$\begin{aligned}
\mathbb{E}[H(\underline{\sigma})H(\underline{\tau})] &= \frac{1}{2N} \sum_{i,j} \sigma_i \sigma_j \tau_i \tau_j + \frac{1}{2N^2} \sum_{i,j,k} \sigma_i \sigma_j \sigma_k \tau_i \tau_j \tau_k + O(1) \\
&= \frac{N}{2} \left(\frac{1}{N} \sum_i \sigma_i \tau_i \right)^2 + \frac{N}{2} \left(\frac{1}{N} \sum_i \sigma_i \tau_i \right)^3 + O(1) \ ,
\end{aligned}$$

   which is of the form of the text with $g(q) = q^2 + q^3$.

2. (a) The average of a binomial random variable is the product of its parameters, the number l of trials and the probability p of success for each trial :

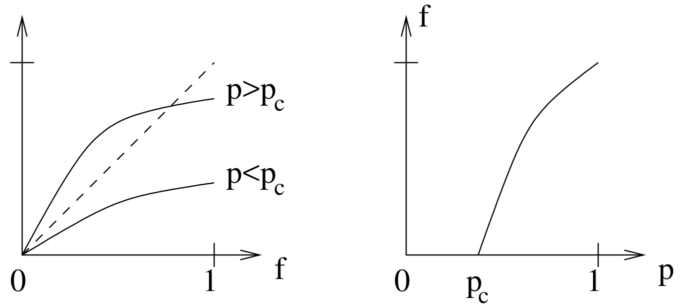
$$m = \mathbb{E}[k] = \sum_{k=0}^{l} q_k k = \sum_{k=0}^{l} \binom{l}{k} k p^k (1-p)^{l-k} = lp \sum_{k=1}^{l} \binom{l-1}{k-1} p^{k-1} (1-p)^{l-k} = lp \ .$$

   (b) One can view a branching process of $g+1$ generations as k copies of branching processes of g generations, where k is the number of offspring of the ancestor. As everything is independent one obtains $\mathbb{E}[\mathcal{N}^{(g+1)}] = \mathbb{E}[k]\mathbb{E}[\mathcal{N}^{(g)}]$, hence by induction $\mathbb{E}[\mathcal{N}^{(g)}] = m^g$. When g goes to infinity this either converges to 0, if the average number of descendent of an individual m is strictly smaller than 1, or diverges to $+\infty$ if $m > 1$. In terms of the parameter p these two behaviors occurs for $p < p_c$ and $p > p_c$ respectively, with the threshold value $p_c = \frac{1}{l}$.

   (c) Suppose the ancestor has k offsprings ; the branching process persists forever if at least one of its k offsprings has an infinite line of descendents. Reciprocally the branching process is finite if all the offsprings have a finite progeny. Hence $f(p,l)$ is solution of

$$1 - f = \sum_{k=0}^{l} q_k (1-f)^k = \sum_{k=0}^{l} \binom{l}{k} p^k (1-p)^{l-k} (1-f)^k = (1-pf)^l \ , \qquad f = 1 - (1-pf)^l \ .$$

   The function $f \mapsto 1 - (1-pf)^l$ is an increasing function on the interval $f \in [0,1]$, from 0 in $f = 0$ to $1 - (1-p)^l$ in $f = 1$, with a derivative lp in $f = 0$. The graphical study of this self-consistent equation is given on the left plot of this figure :



For $p < p_c$ the only solution is $f = 0$, whereas for $p > p_c$ a non-trivial solution appears continuously, hence the shape of $f(p,l)$ plotted on the right plot of the figure. It varies as $(p - p_c)^\beta$ when $p \to p_c^+$, up to a multiplicative constant, with the critical exponent $\beta = 1$, as can be easily established by expanding the self-consistent equation for f and $p - p_c$ small (cf TD 4).

3. (a) Such a domain wall energy is proportional to the area of the surface separating the two phases, hence of order ℓ^{d-1}.
   (b) Flipping the spins inside a domain Ω yields for the second term $\sum_{i\in\Omega} h_i$. This is a sum of $|\Omega|$ (the number of sites in the domain) i.i.d. random variables, its cumulants are thus $|\Omega|$ multiplied by the cumulants of one h_i. As the average of h_i vanishes so does the average of the sum, while the variance of the sum is order $|\Omega| \sim \ell^d$, hence the sum is of order $\ell^{d/2}$.
   (c) Comparing the two terms amount to compare d with $d/2$. For $d > 2$ the surface tension cost is dominant when ℓ is large, hence the ferromagnetic phase is protected from the effect of disorder and does exist in these dimensions if the disorder is not too strong. When $d \leq 2$ on the contrary the energetic gain obtained by flipping the spins will dominate in the large ℓ limit for arbitrarily small disorder, that destroys the possibility of a ferromagnetic phase in these low dimensions.

2 The Wigner semi-circular law via replicas

We recall here the definition of the average density of eigenvalues,

$$\rho(x) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right] \ , \tag{1}$$

and the Gaussian identities :

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}} \ , \tag{2}$$

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}+\vec{b}^T\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}}\mathrm{e}^{\frac{1}{2}\vec{b}^T\mathbf{A}^{-1}\vec{b}} \ , \tag{3}$$

$$\int_{\mathbb{R}^m} \mathrm{d}\vec{x}\ x_k x_l\ \mathrm{e}^{-\frac{1}{2}\vec{x}^T\mathbf{A}\vec{x}} = \frac{(2\pi)^{m/2}}{\sqrt{\det\mathbf{A}}}(\mathbf{A}^{-1})_{kl} \ . \tag{4}$$

2.1 Preamble

1.) Using the distributional identity recalled in the text we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i) = \frac{1}{\pi N}\lim_{\epsilon\to 0^+}\mathrm{Im}\left(\sum_{i=1}^{N}\frac{1}{x-i\epsilon-\lambda_i}\right) = -\frac{1}{\pi N}\lim_{\epsilon\to 0^+}\mathrm{Im}\left(\sum_{i=1}^{N}\frac{1}{\lambda_i - x_\epsilon}\right) \tag{5}$$

where $x_\epsilon = x - i\epsilon$. Therefore, using that

$$\sum_{i=1}^{N}\frac{1}{\lambda_i - x_\epsilon} = \mathrm{Tr}(\mathbf{J} - x_\epsilon\mathbb{I}_N)^{-1} \ , \tag{6}$$

where $\mathbb{I}_N$ denotes the identity matrix of size N, together with the definition of $\rho(x)$ in (1) one obtains

$$\boxed{\rho(x) = -\frac{1}{N\pi} \lim_{\epsilon \to 0^+} \mathrm{Im}\left(\mathbb{E}\left[\mathrm{Tr}(\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1}\right]\right) \;,\; x_\epsilon = x - i\epsilon \;.} \tag{7}$$

2.) The N-dimensional Gaussian integral

$$\begin{aligned} Z_J(x_\epsilon) &= \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2 + \frac{i}{2} \sum_{k,l} J_{kl}\, \phi_k \phi_l\right] &(8)\\ &= \int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\; \exp\left[-\frac{1}{2} \vec{\phi}^T (i x_\epsilon \mathbb{I}_N - i\,\mathbf{J})\vec{\phi}\right] &(9)\end{aligned}$$

is convergent because the absolute value of the integrand is $\exp[-\frac{\epsilon}{2}\sum_k \phi_k^2]$, which is integrable for every $\epsilon > 0$. Taking the derivative of $\ln Z_J(x_\epsilon)$ with respect to x one obtains

$$\begin{aligned} \frac{\mathrm{d} \ln Z_J(x_\epsilon)}{\mathrm{d}x} &= \frac{\int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\, \left(-\frac{i}{2}\sum_{k=1}^N \phi_k^2\right) \exp\left[-\frac{1}{2}\vec{\phi}^T (i x_\epsilon \mathbb{I}_N - i\,\mathbf{J})\vec{\phi}\right]}{\int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\; \exp\left[-\frac{1}{2}\vec{\phi}^T (i x_\epsilon \mathbb{I}_N - i\,\mathbf{J})\vec{\phi}\right]} &(10)\\ &= -\frac{i}{2}\sum_{k=1}^N ((i x_\epsilon \mathbb{I}_N - i\mathbf{J})^{-1})_{kk} = \frac{1}{2}\sum_{k=1}^N ((\mathbf{J} - x_\epsilon \mathbb{I}_N)^{-1})_{kk} &(11)\end{aligned}$$

where, in the second line, we have used the Gaussian identities (2) and (4). This finally leads to

$$\boxed{\mathrm{Tr}(\mathbf{J} - x_\epsilon \mathbb{I})^{-1} = 2\,\frac{\mathrm{d}}{\mathrm{d}x} \ln Z_J(x_\epsilon) \;.} \tag{12}$$

2.2 A simplified computation

3.) The replacement $\mathbb{E}\left[\ln Z_J(x_\epsilon)\right] \approx \ln\left(\mathbb{E}[Z_J(x_\epsilon)]\right)$ is the annealed approximation. In the more general context of disordered systems, this approximation is valid in the large N limit if $Z_J(x_\epsilon)$ does not fluctuate too much, i.e. if it is self-averaging : $Z_J(x_\epsilon) \to \mathbb{E}[Z_J(x_\epsilon)]$ as $N \to \infty$. This is the case when the disorder is weak, or the temperature high.

4.) The average "partition function" $\mathbb{E}[Z_J(x_\epsilon)]$ is given by

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\; \mathrm{e}^{-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2} \prod_{k=1}^N \mathbb{E}\left[\mathrm{e}^{\frac{i}{2} J_{kk} \phi_k^2}\right] \prod_{k<l} \mathbb{E}\left[\mathrm{e}^{i J_{kl} \phi_k \phi_l}\right] \;. \tag{13}$$

Note that in the last factors, the argument of the exponential is indeed $i J_{kl}\phi_k\phi_l$ and not $(i/2) J_{kl}\phi_k\phi_l$ since the product over k and l is restricted to $k < l$, while there is no restriction on these indices in Eq. (8). Using the Gaussian integration formulae (2,3) for $m = 1$, one finds

$$\mathbb{E}\left[\mathrm{e}^{\frac{i}{2} J_{kk}\phi_k^2}\right] = \mathrm{e}^{-\frac{\phi_k^4}{8}\mathbb{E}[J_{kk}^2]} = \mathrm{e}^{-\frac{\phi_k^4}{4N}} \;, \quad k = 1, \cdots, N \;, \tag{14}$$

and similarly

$$\mathbb{E}\left[\mathrm{e}^{i J_{kl}\phi_k\phi_l}\right] = \mathrm{e}^{-\frac{\phi_k^2\phi_l^2}{2}\mathbb{E}[J_{kl}^2]} = \mathrm{e}^{-\frac{\phi_k^2\phi_l^2}{2N}} \;, \quad k < l \;. \tag{15}$$

Injecting these results (14) and (15) in Eq. (13) one finds

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\; \mathrm{e}^{-\frac{i}{2} x_\epsilon \sum_{k=1}^N \phi_k^2}\, \mathrm{e}^{-\frac{1}{4N}\sum_{k=1}^N \phi_k^4 - \frac{1}{2N}\sum_{k<l}\phi_k^2\phi_l^2} \;, \tag{16}$$

which finally yields

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\phi_N \exp\left[-\frac{i}{2}\, x_\epsilon \sum_{k=1}^N \phi_k^2 - \frac{1}{4N}\left(\sum_{k=1}^N \phi_k^2\right)^2\right] \;.} \tag{17}$$

5.) We use the identity (3) with $m = 1$ to write

$$\mathrm{e}^{-\frac{1}{4N}(\sum_{k=1}^{N}\phi_k^2)^2} = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2+iq\sum_{k=1}^{N}\phi_k^2}\,. \tag{18}$$

Hence $\mathbb{E}[Z_J(x_\epsilon)]$ can be written as

$$\begin{aligned}\mathbb{E}[Z_J(x_\epsilon)] &= \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2}\int_{\mathbb{R}^N}\mathrm{d}\vec{\phi}\,\mathrm{e}^{i(q-\frac{x_\epsilon}{2})\sum_{k=1}^{N}\phi_k^2}\\ &= \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2}\left[\int_{-\infty}^{\infty}\mathrm{d}\phi\,\mathrm{e}^{i(q-\frac{x_\epsilon}{2})\phi^2}\right]^N\,. \end{aligned}\tag{19}$$

The integral over ϕ can be evaluated using the relation (2) which, specified for $m = 1$, yields here

$$\int_{-\infty}^{\infty}\mathrm{d}\phi\,\mathrm{e}^{i(q-\frac{x_\epsilon}{2})\phi^2} = \left[\frac{i\pi}{(q-\frac{x_\epsilon}{2})}\right]^{1/2}\,. \tag{20}$$

Therefore, by substituting (20) in (19) one finds that $\mathbb{E}[Z_J(x_\epsilon)]$ can be written as

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-N\,\varphi(q,x_\epsilon)}\ ,\ \varphi(q,x_\epsilon) = q^2 - \frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon-2q)}\right)\ .} \tag{21}$$

6.) By substituting the identity found in (12) in the expression for the density $\rho(x)$ in Eq. (7) one finds

$$\rho(x) = -\frac{2}{N\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left[\ln(Z_J(x_\epsilon))\right] \approx -\frac{2}{N\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\ln(\mathbb{E}[Z_J(x_\epsilon)])\ , \tag{22}$$

where the last relation holds in the annealed approximation considered here. We see that we need to evaluate $N^{-1}\ln(\mathbb{E}[Z_J(x_\epsilon)])$ in the limit of large N and therefore, anticipating that $\rho(x)$ is of order $\mathcal{O}(1)$, we only need to retain the exponentially large terms in (21).

7.) The saddle point equation $\partial_q\varphi(q,x_\epsilon) = 0$ yields the quadratic equation

$$2q + \frac{1}{2q-x_\epsilon} = 0\ , \qquad 4q^2 - 2x_\epsilon q + 1 = 0\ , \tag{23}$$

which admits the two solutions

$$\boxed{q_\pm = \frac{1}{4}\left(x_\epsilon \pm \sqrt{x_\epsilon^2-4}\right)\ .} \tag{24}$$

8.) In the limit of large N the Laplace method applied to (21) yields

$$\lim_{N\to\infty}\frac{\ln\mathbb{E}\left[Z_J(x_\epsilon)\right]}{N} = -\varphi(q_+,x_\epsilon)\ . \tag{25}$$

Therefore, using this result (25) in Eq. (22) one finds

$$\boxed{\lim_{N\to\infty}\rho(x) = \frac{2}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+,x_\epsilon)\ .} \tag{26}$$

9.) To compute $\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+,x_\epsilon)$ we notice that $\varphi(q_+,x_\epsilon)$ depends on x through the dependence of q_+ on x (24) as well as through the x-dependence of the function $\varphi(q,x_\epsilon)$ itself (21). Therefore differentiating using the chain rule one finds

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+,x_\epsilon) = \frac{\partial q_+}{\partial x}\frac{\partial\varphi(q,x_\epsilon)}{\partial q}\Big|_{q=q_+} + \frac{\partial\varphi(q,x_\epsilon)}{\partial x}\Big|_{q=q_+}\ . \tag{27}$$

The first term vanishes since q_+ is a saddle point, solution of $\partial_q\varphi = 0$, and the second term yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+, x_\epsilon) = \frac{\partial\varphi(q, x_\epsilon)}{\partial x}\Big|_{q=q_+} = \frac{1}{2(x_\epsilon - 2q_+)} = q_+ = \frac{1}{4}\left(x_\epsilon + \sqrt{x_\epsilon^2 - 4}\right) , \tag{28}$$

where in the last two equalities we have used (23) and (24). Finally, injecting this result (28) in Eq. (26) one finds

$$\lim_{N\to\infty}\rho(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\left[x_\epsilon + \sqrt{x_\epsilon^2 - 4}\right] , \tag{29}$$

which leads to the Wigner semi-circle law

$$\boxed{\rho(x) \underset{N\to\infty}{\longrightarrow} \begin{cases} \dfrac{1}{2\pi}\sqrt{4 - x^2} \; , \; |x| < 2 \\ \\ 0 \; , \; |x| > 2 \; . \end{cases}} \tag{30}$$

2.3 A more complete computation

10.) By using the replica trick

$$\mathbb{E}[\ln Z_J(x_\epsilon)] = \lim_{n\to 0}\frac{1}{n}\ln\left(\mathbb{E}[Z_J(x_\epsilon)^n]\right) . \tag{31}$$

together with the first equality in (22) one obtains

$$\boxed{\rho(x) = -\frac{2}{N\pi}\lim_{\epsilon\to 0^+}\lim_{n\to 0}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{n}\ln\left(\mathbb{E}[Z_J(x_\epsilon)^n]\right) .} \tag{32}$$

11.) By replicating the system n times to write the n-th power $Z_J(x_\epsilon)^n$ and performing the averages over the random variables J_{kl}, one has

$$\begin{aligned}
\mathbb{E}[Z_J(x_\epsilon)^n] &= \int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a\, \mathrm{e}^{-\frac{ix_\epsilon}{2}\sum_{k=1}^N\sum_{a=1}^n(\phi_k^a)^2}\mathbb{E}[\mathrm{e}^{\frac{i}{2}\sum_{k,l}J_{kl}\sum_{a=1}^n\phi_k^a\phi_l^a}] \qquad (33)\\
&= \int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a\, \mathrm{e}^{-\frac{ix_\epsilon}{2}\sum_{k=1}^N\sum_{a=1}^n(\phi_k^a)^2}\prod_{k=1}^{N}\mathbb{E}[\mathrm{e}^{\frac{i}{2}J_{kk}\sum_{a=1}^n(\phi_k^a)^2}]\prod_{k<l}\mathbb{E}[\mathrm{e}^{iJ_{kl}\sum_{a=1}^n\phi_k^a\phi_l^a}]\\
&= \int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a\, \mathrm{e}^{-\frac{ix_\epsilon}{2}\sum_{k=1}^N\sum_{a=1}^n(\phi_k^a)^2}\mathrm{e}^{-\frac{1}{4N}\sum_{k=1}^N\sum_{a,b=1}^n(\phi_k^a)^2(\phi_k^b)^2} \qquad (34)\\
&\times\ \mathrm{e}^{-\frac{1}{2N}\sum_{k<l}\sum_{a,b=1}^n\phi_k^a\phi_l^a\phi_k^b\phi_l^b} \qquad (35)
\end{aligned}$$

which can be written (by inverting the sum over k and the double sum over a, b in the argument of the last two exponentials) as

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)^n] = \int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a\exp\left[-\frac{i}{2}x_\epsilon\sum_{k=1}^{N}\sum_{a=1}^{n}(\phi_k^a)^2 - \frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^b\right)^2\right] .} \tag{36}$$

12.) We write the exponentials containing quartic terms in Eq. (36) as

$$\exp\left[-\frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^b\right)^2\right] = \exp\left[-\frac{1}{4N}\sum_{a=1}^{n}\left(\sum_{k=1}^{N}(\phi_k^a)^2\right)^2 - \frac{1}{2N}\sum_{1\le a<b\le n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^a\right)^2\right] \tag{37}$$

and we use $n(n+1)/2$ times the Gaussian identity (3) to rewrite it as

$$\begin{aligned}
\exp\left[-\frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^b\right)^2\right] &= \prod_{a=1}^{n}\left(\sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}Q_{aa}\,\mathrm{e}^{-NQ_{aa}^2 + iQ_{aa}\sum_{k=1}^N(\phi_k^a)^2}\right) \qquad (38)\\
&\times\prod_{1\le a<b\le n}\left(\sqrt{\frac{2N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}Q_{ab}\,\mathrm{e}^{-2NQ_{ab}^2 + 2iQ_{ab}\sum_{k=1}^N\phi_k^a\phi_k^b}\right)
\end{aligned}$$

which can be re-written as

$$\exp\left[-\frac{1}{4N}\sum_{a,b=1}^{n}\left(\sum_{k=1}^{N}\phi_k^a\phi_k^b\right)^2\right] = 2^{n(n-1)/4}\left(\frac{N}{\pi}\right)^{n(n+1)/4} \times \int_{-\infty}^{\infty}\prod_{1\le a\le b\le n}\mathrm{d}Q_{ab}\,\exp\left[-N\sum_{a,b=1}^{n}(Q_{ab})^2+i\sum_{a,b=1}^{n}Q_{ab}\sum_{k=1}^{N}\phi_k^a\phi_k^b\right], \tag{39}$$

with the convention, in the argument of the exponential, that $Q_{ab}=Q_{ba}$ if $a>b$. Therefore, substituting this identity (39) in (36) one obtains

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n}\int_{-\infty}^{\infty}\prod_{k=1}^{N}\prod_{a=1}^{n}\mathrm{d}\phi_k^a\int_{-\infty}^{\infty}\prod_{1\le a\le b\le n}\mathrm{d}Q_{ab} \times \exp\left[-N\sum_{a,b=1}^{n}(Q_{ab})^2-\frac{i}{2}x_\epsilon\sum_{k=1}^{N}\sum_{a=1}^{n}(\phi_k^a)^2+i\sum_{a,b=1}^{n}Q_{ab}\sum_{k=1}^{N}\phi_k^a\phi_k^b\right] \tag{40}$$

where $A_{N,n}=2^{n(n-1)/4}\left(\frac{N}{\pi}\right)^{n(n+1)/4}$.

13.) We observe that the n-dimensional vectors $\{\phi_k^1,\phi_k^2,\cdots,\phi_k^n\}$, for different values of $k=1,\cdots,N$, are now independent in (40). Therefore one has

$$\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n}\int_{-\infty}^{\infty}\prod_{1\le a\le b\le n}\mathrm{d}Q_{ab}\,e^{-N\sum_{a,b=1}^{n}(Q_{ab})^2}\left(\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a\mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{n}\phi^a(ix_\epsilon\delta_{ab}-2iQ_{ab})\phi^b}\right)^N . \tag{41}$$

The multiple integral over the ϕ^a's can be explicitly evaluated using (2) with $m=n$, which yields

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)^n] = A_{N,n}\prod_{1\le a\le b=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}Q_{ab}\exp\left[-N\Phi(\mathbf{Q},x_\epsilon)\right]} \tag{42}$$

where $\mathbf{Q}$ is the (symmetric) overlap matrix, of size $n\times n$, with matrix elements Q_{ab} for $a\le b$ (and $Q_{ab}=Q_{ba}$ for $a>b$), and

$$\boxed{\Phi(\mathbf{Q},x_\epsilon)=\mathrm{Tr}[\mathbf{Q}^2]-\frac{1}{2}\ln\left[\det\left(\frac{2\pi}{i}\left(x_\epsilon\,\mathbb{I}_n-2\mathbf{Q}\right)^{-1}\right)\right]} , \tag{43}$$

where $\mathbb{I}_n$ denotes the identity matrix matrix of size n.

14.) Expliciting the independent parameters in $\mathbf{Q}$ we rewrite Φ as

$$\Phi(\mathbf{Q},x_\epsilon)=\sum_{a=1}^{n}Q_{aa}^2+2\sum_{a<b}Q_{ab}^2-\ln\left(\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a\mathrm{e}^{-\frac{i}{2}x_\epsilon\sum_a(\phi^a)^2+i\sum_a Q_{aa}(\phi^a)^2+2i\sum_{a<b}Q_{ab}\phi^a\phi^b}\right) . \tag{44}$$

Taking the derivative with respect to one diagonal element Q_{cc} or one off-diagonal element Q_{cd} with $c<d$ yields

$$\frac{\partial\Phi}{\partial Q_{cc}} = 2Q_{cc}-i\frac{\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a(\phi^c)^2\mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{n}\phi^a(ix_\epsilon\delta_{ab}-2iQ_{ab})\phi^b}}{\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a\mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{n}\phi^a(ix_\epsilon\delta_{ab}-2iQ_{ab})\phi^b}} = 2Q_{cc}-i((ix_\epsilon\mathbb{I}_n-2i\mathbf{Q})^{-1})_{cc} , \tag{45}$$

$$\frac{\partial\Phi}{\partial Q_{cd}} = 4Q_{cd}-2i\frac{\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a\phi^c\phi^d\mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{n}\phi^a(ix_\epsilon\delta_{ab}-2iQ_{ab})\phi^b}}{\int_{-\infty}^{\infty}\prod_{a=1}^{n}\mathrm{d}\phi^a\mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{n}\phi^a(ix_\epsilon\delta_{ab}-2iQ_{ab})\phi^b}} = 4Q_{cd}-2i((ix_\epsilon\mathbb{I}_n-2i\mathbf{Q})^{-1})_{cd} , \tag{46}$$

where we used the Gaussian integrals (2,4). Simplifying the factors i gives the saddle-point equation of the text,

$$\boxed{2\,Q_{ab}-((x_\epsilon\mathbb{I}_n-2\mathbf{Q})^{-1})_{ab}=0\ ,\quad\forall\,a,b=1,\cdots,n\ .} \tag{47}$$

Alternatively one can use the identity $\partial_z(\det\mathbf{Y}) = (\det\mathbf{Y})\,\mathrm{Tr}(\mathbf{Y}^{-1}\partial_z\mathbf{Y})$, hence $\partial_z \ln(\det\mathbf{Y}) = \mathrm{Tr}(\mathbf{Y}^{-1}\partial_z\mathbf{Y})$ applied to $\mathbf{Y} = x_\epsilon\mathbb{I}_n - 2\mathbf{Q}$ and $z = Q_{ab}$; one finds

$$\frac{\partial \ln\det(x_\epsilon\mathbb{I}_n - 2\mathbf{Q})}{\partial Q_{ab}} = \sum_{c,d=1}^{n}((x_\epsilon\mathbb{I}_n - 2\mathbf{Q})^{-1})_{cd}\frac{\partial(x_\epsilon\delta_{cd} - 2Q_{cd})}{\partial Q_{ab}} = -2((x_\epsilon\mathbb{I}_n - 2\mathbf{Q})^{-1})_{ab}\ , \tag{48}$$

which yields the same saddle-point equations (47) when applied to the expression (43) of Φ.

15.) The replica symmetric ansatz $Q_{ab} = q\,\delta_{ab}$ corresponds to $\mathbf{Q} = q\mathbb{I}_n$; the saddle point equations (47) are then equivalent to

$$2q - \frac{1}{x_\epsilon - 2q} = 0\ , \tag{49}$$

which is precisely the equation found in the annealed approximation in (23). It admits the two solutions $q = q_\pm$ as given in (24).

16.) Assuming that $Q_{ab} = q_+\delta_{ab}$ is the correct saddle point solution, one finds, from Eq. (42) that

$$\lim_{N\to\infty}\frac{\ln\left(\mathbb{E}[Z(x_\epsilon)^n]\right)}{N} = -\Phi(\mathbf{Q} = q_+\mathbb{I}_n, x_\epsilon) = -n\varphi(q_+, x_\epsilon)\ , \tag{50}$$

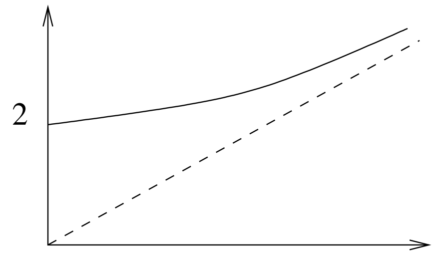
where we have used the explicit expression of $\Phi(\mathbf{Q}, x_\epsilon)$ in (43) together with the expression of the function $\varphi(q, x_\epsilon)$ found before in the annealed approximation and given in (21). Substituting this result (50) in Eq. (32) one obtains (assuming that the limit $N \to \infty$ and $n \to 0$ can be exchanged)

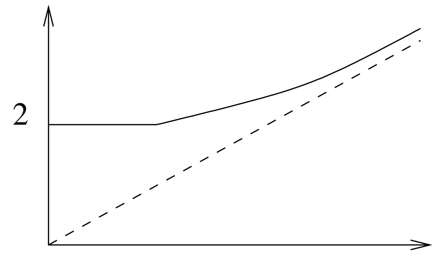
$$\lim_{N\to\infty}\rho(x) = \frac{2}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_+, x_\epsilon)\ , \tag{51}$$

which coincides with the result obtained above in the annealed approximation (26) and therefore this quenched computation also yields the Wigner semi-circle (30) for the average density $\rho(x)$ in the limit of large N.

2.4 The deformed GOE case

17.) It is easy to see that a $N \times N$ matrix with all entries equal to 1 has one eigenvalue equal to N, associated to the constant eigenvector $(1, \dots, 1)^T$, and $N - 1$ eigenvalues equal to 0, the corresponding eigenspace being spanned by the vectors with vanishing sum of their entries. Hence $\mathbf{H}$ has one eigenvalue equal to μ, and $N - 1$ vanishing eigenvalues. When $\mu \to +\infty$ the matrix $\mathbf{J}$ is equivalent to $\mathbf{H}$, hence $\lambda_{\max}(\mu) \sim \mu$ in this limit. On the contrary $\lambda_{\max}(\mu) \to 2$ as $\mu \to 0$, the Wigner semi-circle law being supported on $[-2, 2]$ (actually $\lambda_{\max}(\mu)$ is a random variable, but it concentrates around its mean in the thermodynamic limit). For intermediate values of μ one can imagine two main scenari for the behavior of $\lambda_{\max}$, as depicted on the figure :





In the hypothesis plotted on the left there is a smooth dependency of $\lambda_{\max}$ on μ, whereas in the second scenario there is a phase transition, $\lambda_{\max}$ remains stuck to its GOE value for μ smaller than a critical value. It turns out that the correct scenario is the second one, as will be explained in the following. This comes from the low-rank character of the perturbation introduced by adding $\mathbf{H}$ to the GOE matrix, $\mathbf{H}$ being indeed of rank 1, hence if μ is not large enough it is not able to modify notably the spectrum of the GOE.

18.) The computation of $\mathbb{E}[Z_J(x_\epsilon)]$ can be carried out along the same lines as before, see Eqs. (13)-(16). One obtains

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N}\mathrm{d}\vec{\phi}\ \mathrm{e}^{-\frac{i}{2}x_\epsilon\sum_{k=1}^N\phi_k^2}\prod_{k=1}^{N}\mathbb{E}\left[\mathrm{e}^{\frac{i}{2}J_{kk}\phi_k^2}\right]\prod_{k<l}\mathbb{E}\left[\mathrm{e}^{iJ_{kl}\phi_k\phi_l}\right]\ . \tag{52}$$

Using the Gaussian formulae (2,3) for $m=1$, one finds

$$\mathbb{E}\left[\mathrm{e}^{\frac{i}{2}J_{kk}\phi_k^2}\right] = \mathrm{e}^{-\frac{\phi_k^4}{8}\mathbb{E}[J_{kk}^2]+\frac{i\mu}{2N}\phi_k^2} = \mathrm{e}^{-\frac{\phi_k^4}{4N}+\frac{i\mu}{2N}\phi_k^2}\;, \quad k=1,\cdots,N\;, \tag{53}$$

and similarly

$$\mathbb{E}\left[\mathrm{e}^{iJ_{kl}\phi_k\phi_l}\right] = \mathrm{e}^{-\frac{\phi_k^2\phi_l^2}{2}\mathbb{E}[J_{kl}^2]+\frac{i\mu}{N}\phi_k\phi_l} = \mathrm{e}^{-\frac{\phi_k^2\phi_l^2}{2N}+\frac{i\mu}{N}\phi_k\phi_l}\;, \quad k<l\;. \tag{54}$$

Note the additional term in the argument of the exponentials in (53) and (54) due to the non-zero mean value μ/N of the entries J_{kl}'s. Finally, injecting these results (53) and (54) in Eq. (52) one finds

$$\mathbb{E}[Z_J(x_\epsilon)] = \int_{\mathbb{R}^N} \mathrm{d}\vec{\phi}\; \mathrm{e}^{-\frac{i}{2}x_\epsilon\sum_{k=1}^N\phi_k^2}\, \mathrm{e}^{-\frac{1}{4N}\sum_{k=1}^N\phi_k^4-\frac{1}{2N}\sum_{k<l}\phi_k^2\phi_l^2+\frac{i\mu}{2N}\sum_{k=1}\phi_k^2+\frac{i\mu}{N}\sum_{k<l}\phi_k\phi_l}\;, \tag{55}$$

which finally yields

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)] = \int_{-\infty}^{\infty}\mathrm{d}\phi_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\phi_N \exp\left[-\frac{i}{2}\,x_\epsilon\sum_{k=1}^N\phi_k^2-\frac{1}{4N}\left(\sum_{k=1}^N\phi_k^2\right)^2+i\frac{\mu}{2N}\left(\sum_{k=1}^N\phi_k\right)^2\right]\;.} \tag{56}$$

19.) We use the identity (3) with $m=1$ to write

$$\mathrm{e}^{-\frac{1}{4N}(\sum_{k=1}^N\phi_k^2)^2} = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2+iq\sum_{k=1}^N\phi_k^2}\;. \tag{57}$$

By substituting this identity (57) in (56) one can rewrite $\mathbb{E}[Z_J(x_\epsilon)]$ as

$$\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2}\int_{\mathbb{R}^N}\mathrm{e}^{-\frac{1}{2}\phi^T\,\tilde{\mathbf{A}}\,\vec{\phi}} = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{e}^{-Nq^2}\frac{(2\pi)^{N/2}}{\sqrt{\det\tilde{\mathbf{A}}}}\;, \tag{58}$$

where

$$\tilde{\mathbf{A}} = (ix_\epsilon-2iq)\mathbb{I}_N - i\frac{\mu}{N}\vec{u}\vec{u}^T \qquad \text{where} \qquad \vec{u} = \begin{bmatrix}1\\1\\\cdot\\\cdot\\\cdot\\1\end{bmatrix} \tag{59}$$

Therefore, using the identity $\det(\mathbb{I}_N+\vec{u}\,\vec{v}^T) = 1+\vec{v}^T\vec{u}$ one finds that

$$\det\tilde{\mathbf{A}} = (ix_\epsilon-2iq)^N\,\frac{x_\epsilon-2q-\mu}{x_\epsilon-2q}\;. \tag{60}$$

Finally, injecting this result (60) in (58) one obtains

$$\boxed{\mathbb{E}[Z_J(x_\epsilon)] = \sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}q\,\sqrt{\frac{x_\epsilon-2q}{x_\epsilon-\mu-2q}}\;\mathrm{e}^{-N\,\varphi(q,x_\epsilon)}\;,\;\varphi(q,x_\epsilon) = q^2-\frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon-2q)}\right)\;.} \tag{61}$$

20.) To compute the Taylor expansion of $\varphi(q,x_\epsilon)$ around $q_1=(x_\epsilon-\mu)/2$ up to order $\mathcal{O}((q-q_1)^3)$, we need to compute $\partial_q\varphi(q_1,x_\epsilon)$ and $\partial_q^2\varphi(q_1,x_\epsilon)$. They read

$$\partial_q\varphi(q,x_\epsilon) = 2q-\frac{1}{x_\epsilon-2q} \Longrightarrow \varphi'(q_1,x_\epsilon) = x_\epsilon-\mu-\frac{1}{\mu}\;, \tag{62}$$

and

$$\partial_q^2\varphi(q,x_\epsilon) = 2-\frac{2}{(x_\epsilon-2q)^2} \Longrightarrow \varphi''(q_1,x_\epsilon) = 2-\frac{2}{\mu^2}\;. \tag{63}$$

Therefore the Taylor expansion reads

$$\boxed{\varphi(q,x_\epsilon) = \varphi_x(q_1) + (q-q_1)(x_\epsilon - x_m(\mu)) + (q-q_1)^2\frac{\mu^2-1}{\mu^2} + \mathcal{O}((q-q_1)^3)} \,, \tag{64}$$

with

$$\boxed{x_m(\mu) = \mu + \frac{1}{\mu}} \,. \tag{65}$$

21.) For $\mu < 1$ and $x = x_m(\mu)$, the function $\varphi(q,x)$ admits a local maximum which leads to a local minimum of the integrand, and it is thus of no importance. On the other hand, for $\mu > 1$, $\varphi(q,x)$ admits a local minimum and the contribution for this isolated value $x = x_m(\mu)$ must be included. The function $\mu \mapsto \mu + \frac{1}{\mu}$ can be easily studied ; for $\mu > 0$ it admits a minimum in $\mu = 1$ where it is equal to 2. Hence for $\mu > 1$ one has $x_m(\mu) > 2$, i.e. it is outside the support of the Wigner semi-circle.

22.) By injecting the expression

$$\mathbb{E}[Z_J(x_\epsilon)] \sim B\frac{\mathrm{e}^{-N\varphi(q_1,x_\epsilon)}}{\sqrt{x_\epsilon - x_m(\mu)}} \,, \tag{66}$$

in Eq. (22) one finds, for large N,

$$\rho(x) \sim \frac{-2}{N\pi}\lim_{\epsilon\to 0^+} \mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\left[-N\varphi(q_1,x_\epsilon) - \frac{1}{2}\ln(x_\epsilon - x_m(\mu))\right] \,. \tag{67}$$

Using $q_1 = (x_\epsilon - \mu)/2$ one has

$$\varphi(q_1,x_\epsilon) = q_1^2 - \frac{1}{2}\ln\left(\frac{2\pi}{i(x_\epsilon - 2q_1)}\right) = \frac{1}{4}(x_\epsilon-\mu)^2 - \frac{1}{2}\ln(2\pi/(i\mu)) \,, \tag{68}$$

hence

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(q_1,x_\epsilon) = 0 \,, \tag{69}$$

and the only contribution to $\rho(x)$ in (67) comes from $\ln(x_\epsilon - x_m(\mu))$. It reads

$$\boxed{\rho(x) \sim \frac{1}{N\pi}\lim_{\epsilon\to 0^+} \mathrm{Im}\frac{1}{x_\epsilon - x_m(\mu)} = \frac{1}{N}\delta(x - x_m(\mu))} \,, \tag{70}$$

where we have used the identity

$$\lim_{\epsilon\to 0^+}\frac{1}{x - i\epsilon} = \mathrm{PV}\frac{1}{x} + i\,\pi\delta(x) \,. \tag{71}$$

As $x_m(\mu)$ is outside the support of the Wigner semi-circle law this isolated eigenvalue is the maximal eigenvalue of $\mathbf{J}$, one has thus found

$$\lambda_{\max}(\mu) = \begin{cases} 2 & \text{for } \ 0 \le \mu \le 1 \\ \mu + \frac{1}{\mu} & \text{for } \ \mu \ge 1 \end{cases} \,, \tag{72}$$

i.e. the second scenario plotted above with a phase transition at $\mu = 1$.