ICFP M2 - Statistical physics 2 – TD n° 3
The mean-field p-spin glass model

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In this TD we shall study with the replica method the thermodynamics of the fully connected p-spin glass model, defined by its Hamiltonian

$$H(\underline{\sigma}; \underline{J}) = - \sum_{1 \leq i_1 < i_2 < \ldots i_p \leq N} J_{i_1 i_2 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} \ . \tag{1}$$

The Ising spins σ_i have p-body interactions, the coupling constants $J_{i_1 \ldots i_p}$ are Gaussian i.i.d. random variables of zero mean and variance $\frac{p!}{2N^{p-1}}$. We denote $\mathbb{E}[\bullet]$ the average over these random couplings, $\sigma = \pm 1$ are single spins, and $\underline{\sigma} = \{\sigma_i\}$ are configurations of N spins.

Note that:

- the case $p = 2$ corresponds to the Sherrington-Kirkpatrick model, i.e. a fully connected Ising spin glass.
- for all $p \geq 3$ the model has qualitatively similar behavior (and different from $p = 2$); even though these multi-body interactions do not seem microscopically motivated, the properties of this model has strong similarities with the ones of the structural glasses, and a mean-field theory for the glasses, called Random First Order Transition, was built starting from the p-spin model. Moreover this type of interaction appears naturally in the interdisciplinary applications to computer science.
- for $p \to \infty$ the model converges to the random energy model, as we will show below.

Answer the following questions:

1. Show that the energies $H(\underline{\sigma}; \underline{J})$ are correlated Gaussian random variables with zero mean and covariance
$$\mathbb{E}[H(\underline{\sigma}; \underline{J}) H(\underline{\tau}; \underline{J})] = N \frac{1}{2} q(\underline{\sigma}, \underline{\tau})^p (1 + o(1)) \tag{2}$$
when $N \to \infty$, where $q(\underline{\sigma}, \underline{\tau}) = \frac{1}{N} \sum_{i=1}^N \sigma_i \tau_i$ is the overlap between the two configurations.

2. Explain why this model should become equivalent to the random energy model in the limit $p \to \infty$ (taken after the thermodynamic limit $N \to \infty$).

3. Prove, for a given coupling $J_{i_1 \ldots i_p}$ here called simply J, the relation
$$\mathbb{E}[e^{\beta J A}] = e^{\frac{\beta^2 p!}{4N^{p-1}} A^2} \ , \tag{3}$$
and use it to compute the annealed free-energy $f_{\mathrm{a}}(\beta)$ of the p-spin model. If you are confused by the many indices, you can start by $p = 2$, then do $p = 3$, and finally generalize to arbitrary p.

4. Using again Eq. (3), show that, when n is a positive integer,
$$\mathbb{E}[Z(\beta, \underline{J})^n] = \sum_{\underline{\sigma}^1, \cdots, \underline{\sigma}^n} e^{N \frac{\beta^2}{4} \sum_{ab} q(\underline{\sigma}^a, \underline{\sigma}^b)^p} \ . \tag{4}$$

5. Introduce a $n \times n$ symmetric matrix $Q = \{q_{ab}\}$, with 1 on the diagonal, encoding the overlaps $q(\underline{\sigma}_a, \underline{\sigma}_b)$ between the n replicas of the system. Following similar steps as done for the REM in the lecture, verify that you can write
$$\mathbb{E}[Z(\beta, \underline{J})^n] = \int \mathrm{d}Q e^{N \left[\frac{\beta^2}{4} \sum_{ab} q_{ab}^p + S(Q) \right]} \ , \qquad e^{N S(Q)} = \sum_{\underline{\sigma}^1, \cdots, \underline{\sigma}^n} \prod_{a<b} \delta(q_{ab} - q(\underline{\sigma}^a, \underline{\sigma}^b)) \ . \tag{5}$$

and then

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z(\beta,\underline{J})^n] = \sup_Q A(Q) \ , \qquad \text{with} \quad A(Q) = n\frac{\beta^2}{4} + \frac{\beta^2}{4}\sum_{a\neq b} q_{ab}^p + S(Q) \ . \tag{6}$$

$S(Q)$ is the entropy of n replicas having configurations constrained to satisfy $q(\underline{\sigma}_a, \underline{\sigma}_b) = q_{ab}$. It can be computed (see Ref. [1] for details) to obtain

$$A(Q) = n\frac{\beta^2}{4} + n\log 2 - \frac{\beta^2}{4}(p-1)\sum_{a\neq b} q_{ab}^p + \log\left(\frac{1}{2^n}\sum_{\sigma^1,\cdots,\sigma^n} \exp\left[\frac{\beta^2}{4}p\sum_{a\neq b} q_{ab}^{p-1}\sigma^a\sigma^b\right]\right) \ . \tag{7}$$

Note that the sum over $\sigma^1, \cdots, \sigma^n$ in the last term of Eq. (7) now involves only *one* spin per replica, so we have in total 2^n configurations, and N has disappeared. To determine the quenched free-energy we want to use the replica trick and express

$$f_{\rm q}(\beta) = -\frac{1}{\beta}\lim_{n\to 0}\frac{1}{n}A(Q_*) \ , \tag{8}$$

where Q_* is the saddle-point dominating A. To take the limit $n \to 0$ we have to make an ansatz on the form of Q, as we shall now discuss.

6. We start with the simplest and most natural Replica Symmetric (RS) form of the matrix Q, with $q_{ab} = q \geq 0$ for all $a \neq b$.

   (a) Show that such a saddle-point yields the following free-energy,

   $$f_{\rm RS}(q;\beta) = -\frac{\beta}{4} - T\log 2 + \frac{\beta}{4}(pq^{p-1} - (p-1)q^p) - T\int_{-\infty}^{\infty}\frac{{\rm d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\log\cosh\left(\beta\sqrt{\frac{pq^{p-1}}{2}}z\right) \ ;$$

   to perform this computation you should use the identity

   $$\int_{-\infty}^{\infty}\frac{{\rm d}z}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2 + az} = e^{\frac{1}{2}a^2} \ . \tag{9}$$

   (b) Check that $f_{\rm RS}(q = 0;\beta) = f_{\rm a}(\beta)$, the annealed free-energy.

   (c) Analyze the behavior of the various terms of $f_{\rm RS}$ in the limit $q \to 0$, and conclude that $q = 0$ is always a local maximum for $p \geq 3$, while for $p = 2$ there is a change of concavity, suggesting a phase transition at $T = 1$.

   (d) The best estimate of the quenched free-energy within the RS ansatz is $f_{\rm RS}(\beta) = \max_{q\in[0,1]} f_{\rm RS}(q;\beta)$ (the maximization instead of the usual minimization being a counter-intuitive consequence of the $n \to 0$ limit). Assuming that $q = 0$ is the global maximum, compute the entropy associated to $f_{\rm RS}$ and argue that a phase transition must occur for some $\beta \leq 2\sqrt{\log 2}$.

   (e) Assuming again $f_{\rm RS}(\beta) = \max_{q\in[0,1]} f_{\rm RS}(q;\beta)$ as the best RS estimate, consider the case $p = 2$ and plot $f_{\rm RS}(q;\beta)$ using your favorite numerical software. Show that there is indeed a phase transition at $T = 1$. In the high temperature paramagnetic phase, for $T > 1$, the maximum is in $q = 0$. In the spin glass phase, for $T < 1$, the maximum is in $q^\star > 0$. Extending the analysis of point (c), show that close to the transition, $q^\star \propto 1 - T$.

   (f) *Optional:* still for $p = 2$, show that the entropy can be computed as $s = -\partial_T f_{\rm RS}(q;\beta)|_{q=q^\star}$. Compute $q^\star$ and the entropy numerically in the spin glass phase, and show that it still becomes negative at a finite temperature $T \sim 0.27$.

References

[1] M.Mézard, G.Parisi, and M.Virasoro, *Spin glass theory and beyond: An Introduction to the Replica Method and Its Applications*, World Scientific Publishing Company, 1987.