Statistical Physics 2: Disordered Systems and Interdisciplinary Applications

15.01.2021

I Defs

X random variable

1) $p_X(x)$ $\qquad \mathbb{P}(X \in [a,b]) = \int_a^b dx \, p_X(x)$

2) expectation $\qquad \mathbb{E}[g(x)] = \int dx \, g(x) \, p(x)$

3) (cumulative) distribution f. $\quad F_X(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} dx' \, p(x')$

4) characteristic f. $\quad \varphi(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} dx \, e^{itx} \, p(x)$

- absolutely convergent $\quad |e^{itx}| = 1 \qquad |\varphi(t)| \leq \int dx \, p(x) = 1$
- $\varphi(0) = 1$

If X has p moments

$$\varphi(t) = 1 + it \, \mathbb{E}[x] - \frac{t^2}{2} \mathbb{E}(x^2) + \cdots + \frac{(it)^p}{p!} \mathbb{E}(x^p) + o(t^p)$$

$$\mathbb{E}(x^p) = \frac{1}{i^p} \frac{d^p}{dt^p} \varphi(t) \Big|_{t=0}$$

5) cumulants

$$\log \varphi(t) = it \, c_1 + \cdots + \frac{(it)^p}{p!} c_p + o(t^p)$$

$$c_p = \frac{1}{i^p} \frac{d^p}{dt^p} \log \varphi(t) \Big|_{t=0}$$

6) moment generating function $\quad M(t) = \mathbb{E}(e^{tX}) = \varphi(-it)$

$$M(t) = 1 + t \, \mathbb{E}(x) + \cdots + \frac{t^p}{p!} \mathbb{E}(x^p) + \cdots$$

$$\log M(t) = t c_1 + \cdots + \frac{t^p}{p!} c_p + \cdots$$

7) $X \overset{d}{=} Y$ have the same distribution

$X_n \xrightarrow[n\to\infty]{d} Y$ the distr. of X_n tends to that of Y

II Sums of random variables

$X_1\, X_2 \dots X_n$ i.i.d. and $S_n = X_1 + X_2 + \dots + X_n$

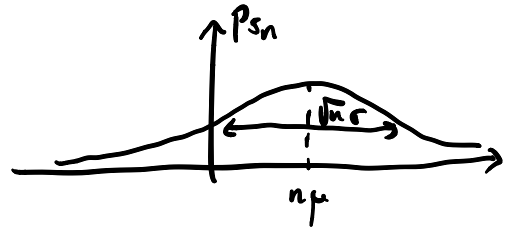
What happens when $n \to \infty$?

1. Central limit theorem

Suppose $\overset{=\sigma^2}{Var(X)} = \mathbb{E}(X^2) - \mathbb{E}(X)^2 < \infty \Rightarrow \overset{=\mu}{\mathbb{E}(X)} < \infty$

$$p(x) \le \frac{1}{|x|^{3+\delta}} \quad \delta > 0 \quad |x| \to \infty$$

$$\mathbb{E}(S_n) = n\mu$$
$$Var(S_n) = n\sigma^2$$



Rescale: $\hat{S}_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $\qquad \mathbb{E}(\hat{S}_n) = 0$, $\quad Var(\hat{S}_n) = 1$

$$\varphi_{\hat{S}_n}(t) = \mathbb{E}\left(e^{it\frac{1}{\sigma\sqrt{n}}(X_1 + X_2 + \dots X_n - n\mu)}\right) = \mathbb{E}\left[e^{\frac{it}{\sigma\sqrt{n}}(X-\mu)}\right]^n$$

$$= e^{n\left(-\cancel{\frac{it\mu}{\sigma\sqrt{n}}} + \log \varphi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)} \xrightarrow[n\to\infty]{} e^{-t^2/2}$$

$$\log \varphi_X\left(\frac{t}{\sigma\sqrt{n}}\right) = \cancel{\frac{it}{\sigma\sqrt{n}}\mu} - \frac{t^2}{2\sigma^2 n}\sigma^2 + o\left(\frac{t^2}{n}\right)$$

It follows that $\hat{S}_n \xrightarrow[n\to\infty]{d} Z \sim \mathcal{N}(0,1)$

$$S_n \approx \mu n + \sqrt{n}\,\sigma z \qquad z \sim \mathcal{N}(0,1)$$

What can go wrong with the CLT?

1) Large deviations : CLT holds when $\hat{S}_n \sim O(1)$

$$u_n = \frac{1}{n} S_n = \mu + \frac{\sigma}{\sqrt{n}} \hat{S}_n$$

2) Correlated variables – CLT holds if you have "short range" correlations

Ex. block variables $\underbrace{x_1 + x_2 + x_3} + \underbrace{x_4 + x_5 + x_6} + \dots$

3) The $\mathrm{Var}(x)$ could be infinite – fat tails

2. Large deviations

$$u_n = \frac{1}{n} S_n = \frac{1}{n}(x_1 + \dots + x_n) \qquad \mathbb{E}(u_n) = \mu = \mathbb{E}(x)$$

choose $\lambda \geq 0$, $a > 0$

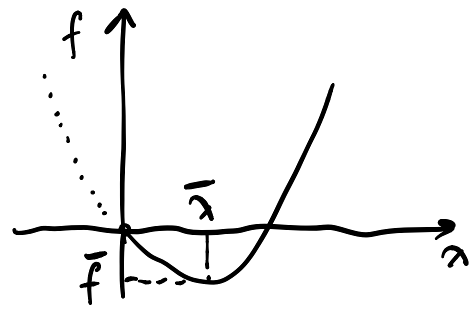
$$\mathbb{P}(u_n - \mu \geq a) = \int_{u_n - \mu \geq a} \prod_{i=1}^{n} \overset{p(x_i)\,dx_i}{dp(x_i)} \leq \int_{u_n - \mu \geq a} \prod_i dp(x_i)\, e^{\overbrace{n\lambda(u_n - \mu - a)}^{\geq 1}}$$

$$\leq \int \prod_i dp(x_i)\, e^{n\lambda(u_n - \mu - a)} = e^{n[\log M(\lambda) - \lambda(\mu + a)]}$$

$$M(\lambda) = \int dx\, e^{\lambda x} p(x) \qquad \frac{d}{d\lambda} \log M(\lambda) = \frac{\int dx\, x e^{\lambda x} p(x)}{\int dx\, e^{\lambda x} p(x)} = \mathbb{E}_\lambda(x)$$

$$\frac{d^2\lambda}{d\lambda^2}\log M(\lambda) = \mathbb{E}_\lambda(x^2) - \mathbb{E}_\lambda(x)^2 \geq 0$$

Hence:
$$\begin{cases} f(\lambda) = \log M(\lambda) - \lambda(\mu+a) \\ f'(\lambda) = \frac{d}{d\lambda}\log M(\lambda) - (\mu+a) \underset{\lambda=0}{=} -a = f'(0) \\ f''(\lambda) = \frac{d^2}{d\lambda^2}\log M(\lambda) \geq 0 \end{cases}$$



$$\mathbb{P}\left[u_n - \mu \geq a\right] \leq e^{n \min_\lambda f(\lambda)}$$
$$= e^{-n|\bar f|}$$

Final result:

$$\mathbb{P}\left[u_n \geq u\right] \leq e^{n\Omega(u)} \qquad u > \mu$$
$$\mathbb{P}\left[u_n \leq u\right] \leq e^{n\Omega(u)} \qquad u < \mu$$

$$\Omega(u) = \min_\lambda \left[m(\lambda) - \lambda u\right] \qquad m(\lambda) = \log M(\lambda)$$
$$= m(\bar\lambda(u)) - \bar\lambda(u)\,u$$

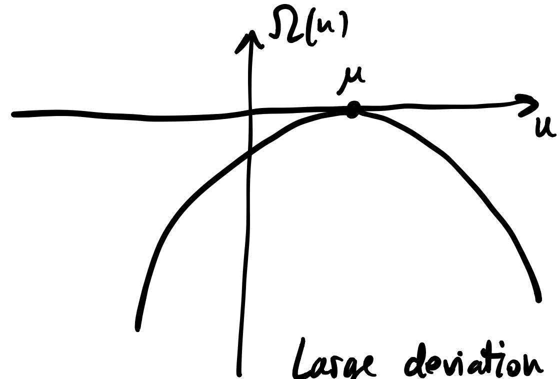
$$m'(\bar\lambda) = +u \qquad \frac{d}{du}m'(\bar\lambda) = m''(\bar\lambda)\frac{d\bar\lambda}{du} = 1$$
$$\frac{d\bar\lambda}{du} = \frac{1}{m''(\bar\lambda)} \geq 0$$

$$p(u) \sim e^{n\Omega(u)}$$

a) $\Omega(u=\mu)=0$

b) $\dfrac{d\Omega}{du} = -\bar{\lambda}(u)$

c) $\dfrac{d^2\Omega}{du^2} = -\dfrac{d\bar{\lambda}}{du} \leq 0$



Large deviation function

3. Lévy stable laws

What happens when $\sigma = \infty$? Fat tail - power laws

Suppose $p(x) \sim |x|^{-(\alpha+1)}$ when $x \to \infty$ or $x \to -\infty$

$\alpha > 2$ Var(X) exists $\to$ CLT

$2 > \alpha > 1$ μ exists but not σ

$1 > \alpha > 0$ neither μ nor σ exist

Large deviations have much higher prob. than CLT

[arXiv: 0706.1062]

Lévy stable distribution

Stable under sum $aX_1 + bX_2 \stackrel{d}{=} cX + d$

Sol. parametrized by four params:

$$\varphi(t;\alpha,\beta,c,\mu) = e^{it\mu - |ct|^\alpha \left[1 - i\beta \,\mathrm{sgn}(t)\, \Phi(t)\right]}$$

$$\Phi(t) = \begin{cases} \tan \frac{\pi\alpha}{2} & \alpha \neq 1 \\ -\frac{2}{\pi} \log t & \alpha = 1 \end{cases} \qquad \begin{matrix} \mu \in \mathbb{R} & \beta \in [-1,1] \\ c \in \mathbb{R}^+ & \alpha \in (0,2] \end{matrix}$$

Note: μ, c are trivial

β controls skewness (asymmetry)

α controls tails $\begin{cases} \alpha = 2 \text{ is the Gaussian} \\ \alpha = 1 \text{ is the Cauchy-Lorentz} \end{cases}$

Add two independent Lévy-stable var. preserves α, β

$\varphi(t)$ is the Fourier transform of $p(y)$

Ex. $\alpha < 1$ $\quad \varphi(t) \sim 1 - \text{const.}\, |t|^\alpha \longrightarrow p(y) \sim |y|^{-(\alpha+1)}$

<u>Result:</u> $\quad X$ s.t. $p(x) \sim |x|^{-(\alpha+1)} \quad x \to \pm\infty$

$$2 > \alpha > 1 \qquad \frac{S_n - n\,\mathbb{E}(X)}{n^{1/\alpha}} \xrightarrow[n\to\infty]{d} Y_\alpha$$

$$1 > \alpha > 0 \qquad \frac{S_n}{n^{1/\alpha}} \xrightarrow[n\to\infty]{d} Y_\alpha$$

<u>Idea:</u> why $n^{1/\alpha}$? $\quad$ case $\alpha < 1$

$$\hat{S}_n = \frac{X_1 + \dots + X_n}{b_n} \qquad \varphi_{\hat{S}_n}(t) = \left[\varphi_X\left(\frac{t}{b_n}\right)\right]^n \sim \left[1 + c\left|\frac{t}{b_n}\right|^\alpha\right]^n$$

Choose $b_n \sim n^{1/\alpha}$ $\qquad = \left[1 + \frac{c|t|^\alpha}{n}\right]^n \sim e^{c|t|^\alpha}$

III. Extremes of random variables

$X_1 \cdots X_n$ i.i.d. copies of X, $M_n = \max(X_1 \cdots X_n)$

What is the behavior of M_n when $n \to \infty$?

Note: $\min(x_1 \cdots x_n) = -\max(-x_1 \cdots -x_n)$ Same theory

Exercise: discuss that M_n has only three possible universal limit distributions

IV Conclusion

a) If $Var[X]$ exists:

$$S_n \sim n\,\mathbb{E}(X) + \sqrt{n}\,\sqrt{Var X}\, z \quad \sim \mathcal{N}(0,1)$$

$$M_n \sim \sqrt{\log n} \quad \text{(Gaussian)}$$

$M_n \ll S_n$
No single X dominates the sum.

b) If $\mathbb{E}(X)$ does not exist, $p(x) \sim |x|^{-(1+\alpha)}$ $\alpha < 1$

$$S_n \sim n^{1/\alpha}\, Y_\alpha$$

stable Lévy

$$M_n \sim n^{1/\alpha}\, \hat{M}_n \leftarrow \text{Fréchet}$$

$S_n \sim M_n$
A few X dominate the sum

$n^{1/\alpha} \gg n$ the sum scales faster than its number of elements

c) Intermediate $1 < \alpha < 2$ $E(x) = \mu$ $Var(x) = \infty$

$$S_n \sim n\mu + n^{1/\alpha} Y_\alpha \leftarrow \text{Stable Lévy}$$

$$M_n \sim n^{1/\alpha} \hat{M}_n \leftarrow \text{Fréchet}$$

But $n^{1/\alpha} \ll n$, large events contribute to fluctuations

<u>Universality</u> RG view

$$S_n = \underbrace{x_1 + x_2}_{Y_1} + \underbrace{x_3 + x_4}_{Y_2} + \dots + \underbrace{x_{n-1} + x_n}_{Y_{n/2}}$$

$$S_n(x) = S_{\frac{n}{2}}(Y)$$

$$M_n = \max(\underbrace{x_1\ x_2}_{Y_1 = \max(x_1, x_2)}\ \underbrace{x_3\ x_4}\ \dots\ \underbrace{x_{n-1}, x_n}_{Y_{\frac{n}{2}} = \max(x_{n-1}, x_n)})$$