ICFP M2 - Statistical physics 2 – TD n° 9
Dyson Brownian Motion

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We recall that an $N \times N$ symmetric random matrix M is distributed according to the Gaussian Orthogonal Ensemble if its matrix elements M_{ij} are, for $i \leq j$, independent Gaussian random variables with zero mean and variances $\mathbb{E}[M_{ii}^2] = \frac{2}{N}$, $\mathbb{E}[M_{ij}^2] = \frac{1}{N}$ if $i < j$. We shall denote $M \overset{\mathrm{d}}{=} \mathrm{GOE}$ in this case.

The goal of this problem is to prove that if M is a GOE random matrix, the joint density of its eigenvalues is

$$P(\lambda_1, \dots, \lambda_N) = \frac{1}{Z} \exp\left(-N \sum_{\alpha=1}^{N} \frac{\lambda_\alpha^2}{4} \right) \prod_{1 \leq \alpha < \beta \leq N} |\lambda_\alpha - \lambda_\beta| \ , \tag{1}$$

where Z is a normalization constant.

In order to obtain this result we shall study a stochastic process $M(t)$ in the space of matrices, and the process (known as the Dyson Brownian Motion) it induces on the eigenvalues $(\lambda_1(t), \dots, \lambda_N(t))$ of the time-dependent random matrix $M(t)$. More precisely, $M(t)$ will be the solution of the Langevin equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -M(t) + \eta(t) \ , \tag{2}$$

where $\eta(t)$ is a symmetric matrix whose matrix elements are given by independent Gaussian white noises of zero mean and variances

$$\mathbb{E}[\eta_{ii}(t)\eta_{ii}(t')] = \frac{4}{N}\delta(t-t') \ , \qquad \mathbb{E}[\eta_{ij}(t)\eta_{ij}(t')] = \frac{2}{N}\delta(t-t') \quad \text{for} \quad i < j \ . \tag{3}$$

The initial condition $M(t) = M_0$ is deterministic.

1. Describe the distribution of $M(t)$ at a given time t; conclude that in the large-time limit, $M(t) \overset{\mathrm{d}}{\to} \mathrm{GOE}$.
2. Consider two times t and $t+s$ with $s > 0$; show that the matrices at these two times are related by
$$M(t+s) = M(t)e^{-s} + \Delta \ , \tag{4}$$
where Δ is a random matrix, independent of $M(t)$, whose distribution you shall specify.
3. We denote $|v_1\rangle, \dots, |v_N\rangle$ the orthonormal basis of eigenvectors of $M(t)$ associated to the eigenvalues $(\lambda_1(t), \dots, \lambda_N(t))$, and define $\widehat{\Delta}$ the $N \times N$ matrix with elements
$$\widehat{\Delta}_{\alpha\beta} = \langle v_\alpha | \Delta | v_\beta \rangle \ . \tag{5}$$
Explain why $\widehat{\Delta}$ is proportional to a GOE distributed random matrix independent of $M(t)$, and give the proportionality constant. *Hint :* recall what does "O" stand for in "GOE".
4. We now take $s = \mathrm{d}t$, an infinitesimal time-increment. Use second order perturbation theory to show that the variation of an eigenvalue $\lambda_\alpha(t) \to \lambda_\alpha(t+\mathrm{d}t)$ of the matrix $M(t)$ is given by
$$\lambda_\alpha(t+\mathrm{d}t) = \lambda_\alpha(t) - \lambda_\alpha(t)\mathrm{d}t + \widehat{\Delta}_{\alpha,\alpha} + \sum_{\beta \neq \alpha} \frac{(\widehat{\Delta}_{\alpha,\beta})^2}{\lambda_\alpha(t) - \lambda_\beta(t)} + o(\mathrm{d}t) \ . \tag{6}$$

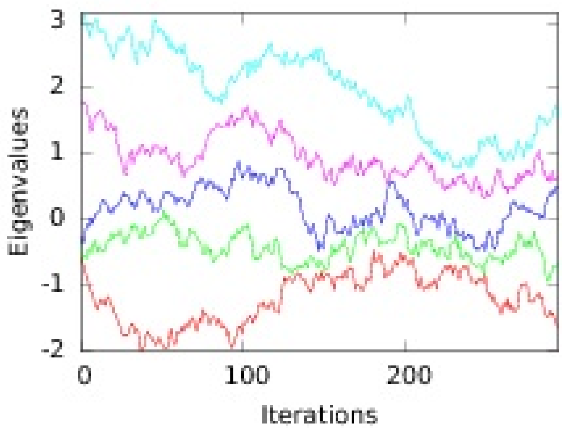
5. Discuss the scalings of the average and the variance (with respect to the randomness in the process in the infinitesimal time interval $[t, t + \mathrm{d}t]$) of the terms in the right hand side, and conclude that the eigenvalues $(\lambda_1(t), \ldots, \lambda_N(t))$ of $M(t)$ obey a set of coupled Langevin equations,

$$\frac{\mathrm{d}\lambda_\alpha(t)}{\mathrm{d}t} = -\lambda_\alpha(t) + \frac{2}{N} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha(t) - \lambda_\beta(t)} + \xi_\alpha(t) \ , \tag{7}$$

where the ξ_α are independent Gaussian white noises of zero average and variance :

$$\mathbb{E}[\xi_\alpha(t)\xi_\beta(t')] = \frac{4}{N}\delta_{\alpha,\beta}\delta(t - t') \ . \tag{8}$$

This stochastic process for the eigenvalues is called the Dyson Brownian Motion, an example of its trajectories is shown in the figure below for $N = 5$:



6. Compute the potential energy $E(\lambda_1, \ldots, \lambda_N)$ from which derives the deterministic force in (7). What is the temperature in the usual interpretation of the Langevin equations ? Write down the associated Gibbs-Boltzmann distribution, and conclude the proof of (1).