ICFP M2 - Statistical physics 2 – TD n° 1
Extreme values distributions
Solution

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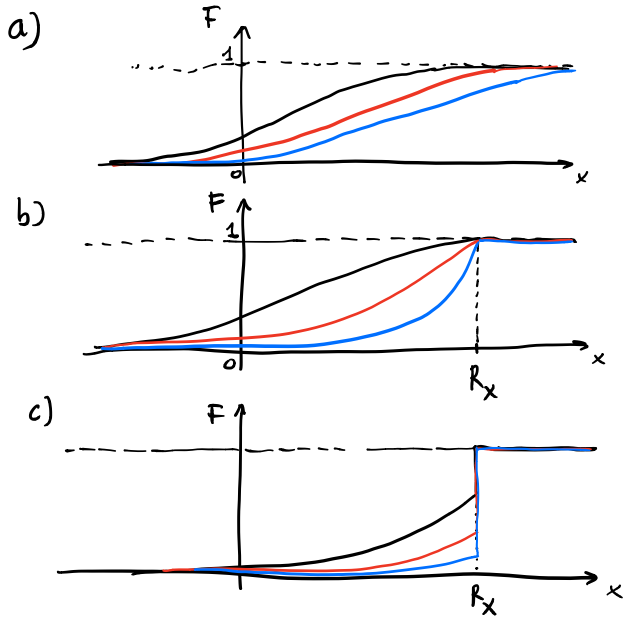
1 Extreme value distributions

1. Because the variables $X_1, \cdots, X_n$ are independent, we have

$$F_{M_n}(x) = \mathbb{P}[M_n < x] = \mathbb{P}[X_1 < x, \cdots, X_n < x] = \prod_{i=1}^{n} \mathbb{P}[X_i < x] = F_X(x)^n \; . \tag{1}$$

2. Recall that $F_X(x) = F_{M_1}(x)$, corresponding to the black curve below. The shape of F_{M_n} for $n > 1$ is then illustrated by the other two curves (red and blue):



3. For $x < R_X$ we have $F_X(x) < 1$, hence $F_{M_n}(x) = F_X(x)^n \to 0$ for $n \to \infty$. Conversely, for $x > R_X$ we have $F_X(x) = 1$, hence $F_{M_n}(x) = F_X(x)^n = 1$ for all n and in particular for $n \to \infty$. We conclude that $F_{M_\infty}(x) = \theta(x - R_X)$, hence $p_{M_\infty}(x) = \delta(x - R_X)$, which implies $M_\infty \overset{\mathrm{d}}{=} R_X$.

4. Recalling that $\widehat{M}_n = \frac{M_n - a_n}{b_n}$, we have

   (a) $F_{\widehat{M}_n}(x) = \mathbb{P}\left[\frac{M_n - a_n}{b_n} < x\right] = \mathbb{P}[M_n < a_n + b_n x] = F_X(a_n + b_n x)^n$,

   (b) If a_n and b_n are chosen in such a way that $F_X(a_n + \widehat{x}\, b_n) = 1 - \frac{\gamma(\widehat{x})}{n} + o\left(\frac{1}{n}\right)$, where $\widehat{x}$ and $\gamma(\widehat{x})$ are finite when $n \to \infty$, then

$$F_{\widehat{M}_n}(\widehat{x}) = F_X(a_n + b_n \widehat{x})^n = \left[1 - \frac{\gamma(\widehat{x})}{n} + o\left(\frac{1}{n}\right)\right]^n \underset{n\to\infty}{\to} e^{-\gamma(\widehat{x})} \; . \tag{2}$$

   Hence, $\widehat{M}_n$ has indeed a non-trivial limit, and its distribution function is $G(x) = e^{-\gamma(x)}$.

5. If $F_X(x) = 1 - e^{-x}$, we can choose $a_n = \log n$ and $b_n = 1$, and we have:

$$F_{\widehat{M}_n}(\widehat{x}) = \left(1 - e^{-a_n - b_n \widehat{x}}\right)^n = \left(1 - \frac{1}{n} e^{-\widehat{x}}\right)^n \underset{n\to\infty}{\to} e^{-e^{-\widehat{x}}} = G_{\mathrm{G}}(\widehat{x}) \ . \tag{3}$$

Note that $x = \log n + \widehat{x} > 0$ for any $\widehat{x} \in \mathbb{R}$ if n is large enough.

6. If $F_X(x) = 1 - (1-x)^\alpha$, we can choose $a_n = 1$ and $b_n = 1/n^{1/\alpha}$, and we have:

$$F_{\widehat{M}_n}(\widehat{x}) = (1 - (1 - a_n - b_n \widehat{x})^\alpha)^n = \left(1 - \frac{1}{n}(-\widehat{x})^\alpha\right)^n \underset{n\to\infty}{\to} e^{-(-\widehat{x})^\alpha} = G_{\mathrm{W}}(\widehat{x}) \ . \tag{4}$$

Note that in order to have $x = 1 + \frac{1}{n^{1/\alpha}}\widehat{x} \in [0,1]$, we need $\widehat{x} \leq 0$ if n is large enough. If $\widehat{x} > 0$, then $x > 1$ and $G_{\mathrm{W}}(\widehat{x}) = 1$.

7. If $F_X(x) = 1 - x^{-\alpha}$, we can choose $a_n = 0$ and $b_n = n^{1/\alpha}$, and we have:

$$F_{\widehat{M}_n}(\widehat{x}) = \left(1 - (a_n + b_n \widehat{x})^{-\alpha}\right)^n = \left(1 - \frac{1}{n}\widehat{x}^{-\alpha}\right)^n \underset{n\to\infty}{\to} e^{-\widehat{x}^{-\alpha}} = G_{\mathrm{F}}(\widehat{x}) \ . \tag{5}$$

Note that in order to have $x = n^{1/\alpha}\widehat{x} > 1$, we need $\widehat{x} > 0$ if n is large enough. If $\widehat{x} \leq 0$, then $x \leq 0$ and $G_{\mathrm{F}}(\widehat{x}) = 0$.

8. Using the asymptotic expression of $F_X(x)$ for a Gaussian and the choices for a_n and b_n given in the text, we have

$$F_{\widehat{M}_n}(\widehat{x}) = \left(1 - \frac{1}{a_n + b_n \widehat{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a_n + b_n \widehat{x})^2}{2}}\right)^n \ . \tag{6}$$

At leading order we can expand:

$$(a_n + b_n \widehat{x})^2 = 2 \log n - \log(4\pi \log n) + 2\widehat{x} + o\left(\frac{1}{\log n}\right) \ , \tag{7}$$

hence, neglecting all terms vanishing for $n \to \infty$, we have

$$F_{\widehat{M}_n}(\widehat{x}) = \left(1 - \frac{1}{\sqrt{4\pi \log n}} e^{-\log n + \frac{1}{2}\log(4\pi \log n) - \widehat{x}}\right)^n = \left(1 - \frac{1}{n} e^{-\widehat{x}}\right)^n \underset{n\to\infty}{\to} e^{-e^{-\widehat{x}}} = G_{\mathrm{G}}(\widehat{x}) \ . \tag{8}$$

2 Proof of the three types theorem

1. We write $M_n \sim a_n + b_n Y$ for large n, consider m independent copies of M_n, which we call $M_n^1 \sim a_n + b_n Y_1, \cdots, M_n^m \sim a_n + b_n Y_m$, and define the random variable $Z = \max(Y_1, \cdots, Y_m)$. The random variables $M_n^1, \cdots, M_n^m$ involve mn independent copies of X. We have then

$$\begin{aligned} Z &= \max\left(\frac{M_n^1 - a_n}{b_n}, \cdots, \frac{M_n^m - a_n}{b_n}\right) = \max\left(\frac{X_1 - a_n}{b_n}, \cdots, \frac{X_{mn} - a_n}{b_n}\right) \\ &= \frac{M_{mn} - a_n}{b_n} = \frac{a_{mn} + b_{mn} Y - a_n}{b_n} \ . \end{aligned} \tag{9}$$

Because both Z and Y are finite for large n, we deduce that

$$Z = \max(Y_1, \dots Y_m) \stackrel{\mathrm{d}}{=} \frac{Y - A_m}{B_m} \quad \text{i.e.} \quad G^m(x) = G(A_m + B_m x) \ , \tag{10}$$

and that the two limits

$$B_m = \lim_{n\to\infty} \frac{b_n}{b_{mn}} \ , \qquad A_m = \lim_{n\to\infty} \frac{a_n - a_{mn}}{b_{mn}} \ , \tag{11}$$

exist and are finite.

2. To generalize Eq. (10) to real s, we invert Eq. (10) to obtain $G(x)^{1/m} = G\left(\frac{x-A_m}{B_m}\right)$. We approximate $s \sim p/q$ with p, q integers, and we write

$$G(x)^{p/q} = G\left(\frac{x-A_q}{B_q}\right)^p = G\left(A_p + B_p\frac{x-A_q}{B_q}\right) = G(A(s)+B(s)x) \ , \tag{12}$$

with, using Eq. (11) for a fixed (arbitrarily large) n,

$$A(s) = A_p - \frac{B_p}{B_q}A_q = \frac{a_{qn}-a_{pn}}{b_{pn}} = \frac{a_k - a_{sk}}{b_{sk}} \ , \qquad B(s) = \frac{B_p}{B_q} = \frac{b_{qn}}{b_{pn}} = \frac{b_k}{b_{sk}} \ , \tag{13}$$

having defined $k = qn$, hence $pn = sk$. Note that the expressions of $A(s)$, $B(s)$ in terms of k are the same as Eqs. (11) and the limit $k \to \infty$ with fixed s is then guaranteed to exist and be finite. We can then take the limit $n \to \infty$, $q \to \infty$ and $p \to \infty$ in such a way that $p/q \to s$ for any real s.

3. By computing $G^{st}(x)$ in two different ways one gets

$$\begin{aligned} G^{st}(x) &= G(A(st)+B(st)x) = (G^s(x))^t = G^t(A(s)+B(s)x) &(14)\\ &= G(A(t)+B(t)(A(s)+B(s)x)) = G(A(t)+B(t)A(s)+B(t)B(s)x) \ . &(15)\end{aligned}$$

As $G(x)$ is the distribution function of a non-trivial random variable, $G(x) = G(\alpha+\beta x)$ for all x implies $\alpha = 0$ and $\beta = 1$, hence the equations satisfied by the functions A and B

$$\begin{cases} B(st) = B(s)B(t) \ , \\ A(st) = A(t)+B(t)A(s) = A(s)+B(s)A(t) \ , \end{cases} \tag{16}$$

for all $s, t > 0$, the last equality being obtained by symmetry between s and t.

4. Taking the derivative with respect to t of the first equation, then setting $t = 1$ yields $sB'(s) = B(s)B'(1)$. This implies $B(1) = 1$, and the differential equation can then be easily integrated to obtain $B(s) = s^\theta$, where θ is an arbitrary real parameter. Actually this is the only type of solution of the equation $B(st) = B(s)B(t)$ with the weaker assumption that $B(s)$ is continuous.

5. If $\theta = 0$ one has $B(s) = 1$ for all s, hence $A(s)$ is solution of the functional equation $A(st) = A(s)+A(t)$. We see that $e^{A(s)}$ is thus solution of the same functional equation than the one on $B(s)$ solved in the previous question, which implies $A(s) = -c\log s$ with c an undetermined constant. We thus have an equation on the distribution function of the limit random variable, $G^s(x) = G(x - c\log s)$. As the left-hand-side is a decreasing function of s one must have $c > 0$. Taking the logarithm of this equation yields $\log G(x) = \frac{\log G(x-c\log s)}{s}$, for all x and $s > 0$. Choosing x_0 such that $G(x_0) = 1/e$, and s such that $x - c\log s = x_0$, yields $G(x) = \exp[-\exp[-\frac{x-x_0}{c}]]$. We have thus proven that if $\theta = 0$ the distribution G is of the Gumbel form, modulo the affine change of variables with parameters x_0 and c.

6. If we assume now that $\theta > 0$, hence $B(s) = s^\theta$, we need to determine the function $A(s)$ from the equation $A(s)+s^\theta A(t) = A(t)+t^\theta A(s)$. Taking an arbitrary value of $t \neq 1$ we rewrite this equation as

$$A(s) = (1-s^\theta)\frac{A(t)}{1-t^\theta} \ . \tag{17}$$

The last fraction being independent of s, we have determined $A(s)$ modulo a multiplicative constant, to be denoted x_0. This yields $G^s(x) = G(x_0(1-s^\theta)+s^\theta x) = G(x_0+s^\theta(x-x_0))$. We need to constrain x to $x < x_0$ as the left-hand-side is decreasing with s. Taking the logarithm of this equation yields $\log G(x) = \frac{\log G(x_0(1-s^\theta)+s^\theta x)}{s}$ for all $x < x_0$ and $s > 0$. This can be solved by choosing s such that $x_0 + s^\theta(x-x_0) = x_1$ independently of x, which yields $G(x) = \exp\left[-\left(\frac{x_0-x}{w}\right)^{\frac{1}{\theta}}\right]$ with w a constant. This is the Weibull distribution with $\alpha = 1/\theta$, up to the affine change of variables with parameters x_0 and w. The case $\theta < 0$ is treated exactly in the same way, but with now the constraint $x > x_0$.