# ON LARGE COMPLETE ARCS: ODD CASE

M. GIULIETTI, F. PAMBIANCO, F. TORRES, AND E. UGHI

Abstract. An approach for the computations of upper bounds on the size of large complete arcs is presented. We obtain in particular geometrical properties of irreducible envelopes associated to a second largest complete arc provided that the order of the underlying field is large enough.

## 1. Introduction

A *k-arc* in the projective plane $\mathbf{P}^2(\mathbf{F}_q)$, where $\mathbf{F}_q$ is the finite field with $q$ elements, is a set of $k$ points no three of which are collinear. An arc is *complete* if it is not properly contained in another arc. For a given $q$, a basic problem in Finite Geometry is to find the values of $k$ for which a complete $k$-arc exists. For a $k$-arc $\mathcal{K}$ in $\mathbf{P}^2(\mathbf{F}_q)$, Bose [3] showed that

$$k \le m_2(q) := \begin{cases} q+1 & \text{if } q \text{ is odd}\,, \\ q+2 & \text{otherwise}\,. \end{cases}$$

For $q$ odd the bound $m_2(q)$ is attained if and only if $\mathcal{K}$ is an irreducible conic [17], [11, Thm. 8.2.4]. For $q$ even the bound is attained by the union of an irreducible conic and its nucleus, and not every $(q+2)$-arc arises in this way; see [11, §8.4]. Let $m_2'(q)$ denote the second largest size that a complete arc in $\mathbf{P}^2(\mathbf{F}_q)$ can have. Segre [17], [11, §10.4] showed that

$$m_2'(q) \le \begin{cases} q - \frac{1}{4}\sqrt{q} + \frac{7}{4} & \text{if } q \text{ is odd,} \\ q - \sqrt{q} + 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

Besides small $q$, namely $q \le 29$ [4], [11], [15], the only case where $m_2'(q)$ has been determinated is for $q$ an even square. Indeed, for $q$ square, examples of complete $(q-\sqrt{q}+1)$-arcs [2], [5], [6], [7], [16] show that

$$m_2'(q) \ge q - \sqrt{q} + 1\,, \tag{1.2}$$

and so the bound (1.1) for $q$ an even square is sharp. This result has been recently extended by Hirschfeld and Korchmáros [14] who showed that the third largest size that a complete arc can have is bounded from above by $q - 2\sqrt{q} + 6$.

If $q$ is not a square, Segre's bounds were notably improved by Voloch [20], [21] (see §3 here).

1991 Math. Subj. Class.: Primary 05B, Secondary 14G.

If $q$ is odd, Segre's bound was slightly improved to $m_2'(q) \leq q - \sqrt{q}/4 + 25/16$ by Thas [19]. If $q$ is an odd square and large enough, Hirschfeld and Korchmáros [13] significantly improved the bound to

$$m_2'(q) \leq q - \frac{1}{2}\sqrt{q} + \frac{5}{2}\,. \tag{1.3}$$

The two last bounds suggest the following problem, which seems to be difficult and has remained open since the 60's.

**Problem 1.1.** For $q$ an odd square, is it true that $m_2'(q) = q - \sqrt{q} + 1$?

The answer is no for $q = 9$ and yes for $q = 25$ [4], [11], [15]. So Problem 1.1 is indeed open for $q \geq 49$.

In this paper we investigate irreducible components of the envelope associated to large arcs in $\mathbf{P}^2(\mathbf{F}_q)$. Such components will be called *irreducible envelopes* and their existence is related to the existence of certain rational points which will be called *special points*, see §2. This set up allows us to prove a general bound for the size of a complete arc (Proposition 3.1) which depends on $q$ and the 4th positive $\mathbf{F}_q$-Frobenius order of the linear series obtained from quadrics in $\mathbf{P}^2(\bar{\mathbf{F}}_q)$ defined on any irreducible envelope. From this result, for $q$ odd and not a square, we recover the bounds on the size of arcs that were established so far in the literature (Lemma 3.2, Lemma 3.9). For $q$ an odd square the best that our approach gives is another proof of Segre's bound.

Our research was inspired and motivated by the papers of Voloch [21, §4] and Hirschfeld–Korchmáros [12], [13]. In fact our results are implicitly contained in such works and this paper can be considered as a set of footnotes to those. Nonetheless, the main contribution of this paper are the following.

(I) We explicitly determinate the type of curves (see (III) below) associated to complete large arcs (Proposition 4.1 and Proposition 4.2) whenever the underlying field is large enough;

(II) We give a systematic account of how to bound the size of complete arcs by means of Stöhr-Voloch's approach to the Hasse-Weil bound [18];

(III) We provide motivation for the study of irreducible plane curves over $\mathbf{F}_q$ whose $\mathbf{F}_q$ non-singular model is classical for the linear series $\Sigma_1$ obtained from lines and whose $\Sigma_2 := 2\Sigma_1$ orders are $0, 1, 2, 3, 4, \epsilon_5$ and whose $\mathbf{F}_q$-Frobenius orders for $\Sigma_2$ are $0, 1, 2, 3, \nu_4$, where $\epsilon_5 = \nu_4 \in \{\sqrt{q/p}, \sqrt{q}, \sqrt{q}/3, 3\sqrt{q}\}$. See §4 here.

Finally, for the convenience of the reader, we include an appendix containing basic facts from Weierstrass points and Frobenius orders based on Stöhr-Voloch's paper [18].

## 2. Special points and irreducible envelopes

Throughout this section $\mathcal{K}$ will be an arc in $\mathbf{P}^2(\mathbf{F}_q)$. Segre associates to $\mathcal{K}$ a plane curve $\mathcal{C}$ in the dual plane of $\mathbf{P}^2(\bar{\mathbf{F}}_q)$, where $\bar{\mathbf{F}}_q$ denotes the algebraic closure of $\mathbf{F}_q$. This curve is defined over $\mathbf{F}_q$ and it is called *the envelope of* $\mathcal{K}$. For $P \in \mathbf{P}^2(\bar{\mathbf{F}}_q)$, let $\ell_P$ denote the corresponding line in the dual plane. The following result summarize the main properties of $\mathcal{C}$ for the odd case.

**Theorem 2.1.** *If $q$ is odd, then the following statements hold:*

1. *The degree of $\mathcal{C}$ is $2t$, with $t = q - k + 2$ being the number of 1-secants through a point of $\mathcal{K}$.*
2. *All $kt$ of the 1-secants of $\mathcal{K}$ belong to $\mathcal{C}$.*
3. *Each 1-secant $\ell$ of $\mathcal{K}$ through a point $P \in \mathcal{K}$ is counted twice in the intersection of $\mathcal{C}$ with $\ell_P$, i.e. $I(\ell, \mathcal{C} \cap \ell_P) = 2$.*
4. *The curve $\mathcal{C}$ contains no 2-secant of $\mathcal{K}$.*
5. *The irreducible components of $\mathcal{C}$ have multiplicity at most two, and $\mathcal{C}$ has at least one component of multiplicity one.*
6. *The arc $\mathcal{K}$ is incomplete if and only if $\mathcal{C}$ admits a linear component over $\mathbf{F}_q$. The arc $\mathcal{K}$ is a conic if and only it is complete and $\mathcal{C}$ admits a quadratic component over $\mathbf{F}_q$.*

*Proof.* See [17], [11, §10]. □

We recall that a non-singular point $P$ of a plane curve $\mathcal{A}$ is called an *inflexion point of* $\mathcal{A}$ if $I(P, \mathcal{A} \cap \ell) > 2$, with $\ell$ being the tangent line of $\mathcal{A}$ at $P$. We introduce the following terminology:

**Definition 2.2.** A point $P_0$ of $\mathcal{C}$ is called *special* if the following conditions hold:

(i) it is non-singular;
(ii) it is $\mathbf{F}_q$-rational;
(iii) it is not an inflexion point of $\mathcal{C}$.

Then, by (i), a special point $P_0$ belongs to an unique irreducible component of the envelope which will be called *the irreducible envelope* associated to $P_0$ or *an irreducible envelope of* $\mathcal{K}$.

**Lemma 2.3.** *Let $\mathcal{C}_1$ be an irreducible envelope of $\mathcal{K}$. Then*

1. *$\mathcal{C}_1$ is defined over $\mathbf{F}_q$;*
2. *if $q$ is odd and the arc is not a conic and complete, then the degree of $\mathcal{C}_1$ is at least three.*

*Proof.* (1) Let $\mathcal{C}_1$ be associated to $P_0$, let $\Phi$ be the Frobenius morphism (relative to $\mathbf{F}_q$) on the dual plane of $\mathbf{P}^2(\bar{\mathbf{F}}_q)$, and suppose that $\mathcal{C}_1$ is not defined over $\mathbf{F}_q$. Then, since the envelope is defined over $\mathbf{F}_q$ and $P_0$ is $\mathbf{F}_q$-rational, $P_0$ would belong to two different components of the envelope, namely $\mathcal{C}_1$ and $\Phi(\mathcal{C}_1)$. This is a contradiction because the point is non-singular.

(2) This follows from Theorem 2.1(6). □

The next result will show that special points do exist provided that $q$ is odd and the arc is large enough.

**Proposition 2.4.** *Let $\mathcal{K}$ be an arc in $\mathbf{P}^2(\mathbf{F}_q)$ of size $k$ such that $k > (2q+4)/3$. If $q$ is odd, then the envelope $\mathcal{C}$ of $\mathcal{K}$ has special points.*

**Remark 2.5.** The hypothesis $k > (2q+4)/3$ in the proposition is equivalent to $k > 2t$, with $t = q - k + 2$. Also, under this hypothesis, the envelope $\mathcal{C}$ is uniquely determined by $\mathcal{K}$, see [11, Thm. 10.4.1(i)].

To prove Proposition 2.4 we need the following lemma, for which we could not find a reference.

**Lemma 2.6.** *Let $\mathcal{A}$ be a plane curve defined over $\bar{\mathbf{F}}_q$ and suppose that it has no multiple components. Let $\alpha$ be the degree of $\mathcal{A}$ and $s$ the number of its singular points. Then,*

$$s \le \binom{\alpha}{2},$$

*and equality holds if $\mathcal{A}$ consists of $\alpha$ lines no three concurrent.*

*Proof.* That a set of $\alpha$ lines no three concurrent satisfies the bound is trivial. Let $G = 0$ be the equation of $\mathcal{A}$, let $G = G_1 \ldots G_r$ be the factorization of $G$ in $\bar{\mathbf{F}}_q[X,Y]$, and let $\mathcal{A}_i$ be the curve given by $G_i = 0$. For simplicity we assume $\alpha$ even, say $\alpha = 2M$. Setting $\alpha_i := \deg(G_i)$, $i = 1, \ldots, r$ and $I := \sum_{i=1}^{r-1} \alpha_i$ we have $\alpha_r = 2M - I$. The singular points of $\mathcal{A}$ arise from the singular points of each component or from the points in $\mathcal{A}_i \cap \mathcal{A}_j$, $i \neq j$. Recall that an irreducible plane curve of degree $d$ has at most $\binom{d-1}{2}$ singular points, and that $\#\mathcal{A}_i \cap \mathcal{A}_j \le a_i a_j$, $i \neq j$ (Bézout's Theorem). So

$$\begin{aligned}
s &\le \sum_{i=1}^{r-1} \binom{\alpha_i - 1}{2} + \binom{2M - I - 1}{2} + \sum_{1 \le i_1 < i_2 \le r-1} \alpha_{i_1}\alpha_{i_2} + \sum_{i=1}^{r-1} (2M - I)\alpha_i \\
&= \sum_{i=1}^{r-1} \frac{\alpha_i^2 - 3\alpha_i + 2}{2} + \frac{4M^2 - 4MI + I^2 - 6M + 3I + 2}{2} + \sum_{1 \le i_1 < i_2 \le r-1} \alpha_{i_1}\alpha_{i_2} + (2M - I)I \\
&= \frac{1}{2}[\sum_{i=1}^{r-1} \alpha_i^2 - 3I + 2(r-1) + 4M^2 - 4MI + I^2 - 6M + 3I + 2 + \\
&\quad 2 \sum_{1 \le i_1 < i_2 \le r-1} \alpha_{i_1}\alpha_{i_2} + 4MI - 2I^2] \\
&\le 2M^2 - 3M + \alpha = 2M^2 - M\,.
\end{aligned}$$

□

*Proof.* (*Proposition 2.4*) Let $F = 0$ be the equation of $\mathcal{C}$ over $\mathbf{F}_q$. By Theorem 2.1(5), $F$ admits a factorization in $\bar{\mathbf{F}}_q[X, Y, Z]$ of type

$$G_1 \dots G_r H_1^2 \dots H_s^2 ,$$

with $r \geq 1$ and $s \geq 0$. Let $\mathcal{A}$ be the plane curve given by

$$G := G_1 \dots G_r = 0 .$$

Then $\mathcal{A}$ satisfies the hypothesis of Lemma 2.3 and it has even degree by Theorem 2.1(1). From Theorem 2.1(3) and Bézout's theorem, for each line $\ell_P$ (in the dual plane) corresponding to a point $P \in \mathcal{K}$, we have

$$\#(\mathcal{A} \cap \ell_P) \geq M ,$$

where $2M = \deg(G)$, and so at least $kM$ points corresponding to unisecants of $\mathcal{K}$ belong to $\mathcal{A}$. Since $k > 2t$ (see Remark 2.5) and $2t \geq 2M$, then $kM > 2M^2$ and from Lemma 2.3 we have that at least one of the unisecant points in $\mathcal{A}$, says $P_0$, is non-singular. Suppose that $P_0$ goes through $P \in \mathcal{K}$. The point $P_0$ is clearly $\mathbf{F}_q$-rational and $P_0$ is not a point of the curve of equation $H = 0$: otherwise $I(P_0, \mathcal{C} \cap \ell_P) > 2$ (see Theorem 2.1(3)). Then, $I(P_0, \mathcal{C} \cap \ell_P) = I(P_0, \mathcal{A} \cap \ell_P) = 2$ and so $\ell_P$ is the tangent of $\mathcal{C}$ at $P_0$. Therefore $P_0$ is not an inflexion point of $\mathcal{C}$, and the proof of Proposition 2.4 is complete. $\square$

Let $\mathcal{C}_1$ be an irreducible envelope associated to a special point $P_0$, and

$$\pi : \mathcal{X} \to \mathcal{C}_1 ,$$

the normalization of $\mathcal{C}_1$. Then by Lemma 2.3(1) we can assume that $\mathcal{X}$ and $\pi$ are defined over $\mathbf{F}_q$. In particular, the linear series $\Sigma_1$ on $\mathcal{X}$ obtained by the pullback of lines of $\mathbf{P}^2(\bar{\mathbf{F}}_q)^*$, the dual of $\mathbf{P}^2(\bar{\mathbf{F}}_q)$, is $\mathbf{F}_q$-rational. Also, there is just one point $\tilde{P}_0 \in \mathcal{X}$ such that $\pi(\tilde{P}_0) = P_0$. For basic facts on orders and Frobenius orders the reader is referred to [18] or the appendix here.

**Lemma 2.7.** *Let $q$ be odd. Then,*

1. *the $(\Sigma_1, \tilde{P}_0)$-orders are $0, 1, 2$;*
2. *the curve $\mathcal{X}$ is classical with respect to $\Sigma_1$.*

*Proof.* (1) This follows from the proof of Proposition 2.4.

(2) This follows from Item (1) and (W1) in the appendix. $\square$

**Remark 2.8.** The hypothesis $q$ odd in Lemma 2.7 (as well as in Proposition 2.4) is necessary. In fact, from [7] and [19] follow that the envelope associated to the cyclic $(q - \sqrt{q} + 1)$-arc, with $q$ an even square, is irreducible and $\mathbf{F}_q$-isomorphic to the plane curve $XY^{\sqrt{q}} + X^{\sqrt{q}}Z + YZ^{\sqrt{q}} = 0$ which is not $\Sigma_1$-classical.

Next consider the following sets:

$$\begin{aligned}
\mathcal{X}_1(\mathbf{F}_q) &:= \{P \in \mathcal{X} : \pi(P) \in \mathcal{C}_1(\mathbf{F}_q)\}\,,\\
\mathcal{X}_{11}(\mathbf{F}_q) &:= \{P \in \mathcal{X}_1(\mathbf{F}_q) : j_2^1(P) = 2j_1^1(P)\}\,,\\
\mathcal{X}_{12}(\mathbf{F}_q) &:= \{P \in \mathcal{X}_1(\mathbf{F}_q) : j_2^1(P) \neq 2j_1^1(P)\}\,,
\end{aligned}$$

and the following numbers:

$$M_q = M_q(\mathcal{C}_1) := \sum_{P \in \mathcal{X}_{11}(\mathbf{F}_q)} j_1^1(P)\,, \qquad M'_q = M'_q(\mathcal{C}_1) := \sum_{P \in \mathcal{X}_{12}(\mathbf{F}_q)} j_1^1(P)\,, \tag{2.1}$$

where $0 < j_1^1(P) < j_2^1(P)$ denotes the $(\Sigma_1, P)$-order sequence. We have that

$$M_q + M'_q \geq \#\mathcal{X}_1(\mathbf{F}_q) \geq \#\mathcal{X}(\mathbf{F}_q) \qquad \text{and} \qquad \#\mathcal{X}_1(\mathbf{F}_q) \geq \#\mathcal{C}_1(\mathbf{F}_q)\,.$$

**Proposition 2.9.** *Let $\mathcal{K}$ be an arc of size $k$ and $d$ the degree of an irreducible envelope of $\mathcal{K}$. For $M_q$ and $M'_q$ as above we have*

$$2M_q + M'_q \geq kd\,.$$

To prove the proposition we first prove the following

**Lemma 2.10.** *Let $\mathcal{K}$ be an arc and $\mathcal{C}_1$ an irreducible envelope of $\mathcal{K}$. Let $Q \in \mathcal{K}$ and $\mathcal{A}_Q$ be the set of points of $\mathcal{C}_1$ corresponding to unisecants of $\mathcal{K}$ passing through $Q$. Let $u := \#\mathcal{A}_Q$ and $v$ be the number of points in $\mathcal{A}_Q$ which are non-singular and inflexion points of $\mathcal{C}_1$. Then*

$$2(u - v) + v \geq d\,,$$

*where $d$ is the degree of $\mathcal{C}_1$.*

*Proof.* Let $P' \in \mathcal{A}_Q$. Suppose that it is non-singular and an inflexion point of $\mathcal{C}_1$. Then, from Theorem 2.1(3) and the definition of $\mathcal{A}_Q$, we have that $\ell_Q$ is not the tangent line of $\mathcal{C}_1$ at $P'$, i.e. we have that $I(P', \mathcal{C}_1 \cap \ell_Q) = 1$. Now suppose that $P'$ is either singular or a non-inflexion point of $\mathcal{C}_1$. Then from Theorem 2.1(3) we have $I(P', \mathcal{C}_1 \cap \ell_Q) \leq 2$ and the result follows from Bézout's theorem applied to $\mathcal{C}_1$ and $\ell_Q$. □

*Proof of Proposition 2.9.* For $Q \in \mathcal{K}$ let $\mathcal{A}_Q$ be as in Lemma 2.10 and set

$$\mathcal{Y}_Q := \{P \in \mathcal{X}_1(\mathbf{F}_q) : \pi(P) \in \mathcal{A}_Q\}\,.$$

We claim that

$$m(Q) := 2 \sum_{P \in \mathcal{X}_{11}(\mathbf{F}_q) \cap \mathcal{Y}_Q} j_1^1(P) + \sum_{P \in \mathcal{X}_{12}(\mathbf{F}_q) \cap \mathcal{Y}_Q} j_1^1(P) \geq d\,.$$

This claim implies the proposition since, from Theorem 2.1(4),

$$\mathcal{Y}_Q \cap \mathcal{Y}_{Q_1} = \emptyset \qquad \text{whenever} \qquad Q \neq Q_1\,.$$

To prove the claim we distinguish four types of points in $\mathcal{Y}_Q$, namely

$$\begin{aligned}
\mathcal{Y}_Q^1 :=&\{P \in \mathcal{Y}_Q : \pi(P) \text{ is non-singular and non- inflexion point of } \mathcal{C}_1\}\,,\\
\mathcal{Y}_Q^2 :=&\{P \in \mathcal{Y}_Q : \pi(P) \text{ is a non-singular inflexion point of } \mathcal{C}_1\}\,,\\
\mathcal{Y}_Q^3 :=&\{P \in \mathcal{Y}_Q : \pi(P) \text{ is a singular point of } \mathcal{C}_1 \text{ such that } \#\pi^{-1}(\pi(P)) = 1\}\,,\\
\mathcal{Y}_Q^4 :=&\{P \in \mathcal{Y}_Q : \pi(P) \text{ is a singular point of } \mathcal{C}_1 \text{ such that } \#\pi^{-1}(\pi(P)) > 1\}\,.
\end{aligned}$$

Observe that $\mathcal{Y}_Q^1 \subseteq \mathcal{X}_{11}(\mathbf{F}_q)$ and so

$$m(Q) \geq 2 \sum_{P \in \mathcal{Y}_Q^1} j_1^1(P) + \sum_{P \in \mathcal{Y}_Q^2} j_1^1(P) + \sum_{P \in \mathcal{Y}_Q^3} j_1^1(P) + \sum_{P \in \mathcal{Y}_Q^4} j_1^1(P)\,.$$

Since $j_1^1(P) > 1$ for all $P \in \mathcal{Y}_Q^4$, the above inequality becomes

$$m(Q) \geq 2\#\mathcal{Y}_Q^1 + 2\#\mathcal{Y}_Q^4 + \#\mathcal{Y}_Q^3 + \#\mathcal{Y}_Q^2\,.$$

Therefore, as to each singular non-cuspidal point of $\mathcal{C}_1$ in $\mathcal{A}_Q$ corresponds at least two points in $\mathcal{Y}_Q^3$, it follows that

$$\begin{aligned}
m(Q) \geq\ & 2\#\{P' \in \mathcal{A}_Q : P' \text{ is either singular or not an inflexion point of } \mathcal{C}_1\} +\\
& \#\{P' \in \mathcal{A}_Q : P' \text{ is a nonsingular inflexion point of } \mathcal{C}_1\}\,.
\end{aligned}$$

Then the claim follows from Lemma 2.10 and the proof of Proposition 2.9 is complete.

## 3. Bounding the size of an arc

Throughout the whole section we fix the following notation:

- $q$ is a power of an odd prime $p$;
- $\mathcal{K}$ is a complete arc of size $k$ such that $(2q+4)/3 < k \leq m_2'(q)$; therefore the degree of any irreducible envelope of $\mathcal{K}$ has at least degree three;
- $P_0$ is an special point of the envelope $\mathcal{C}$ of $\mathcal{K}$ and the plane curve $\mathcal{C}_1$ of degree $d$ is an irreducible envelope associated to $P_0$;
- $\pi : \mathcal{X} \to \mathcal{C}_1$ is the normalization of $\mathcal{C}_1$ which is defined over $\mathbf{F}_q$; as a matter of terminology, $\mathcal{X}$ will be also called an irreducible envelope of $\mathcal{K}$.
- $\tilde{P}_0$ is the only point in $\mathcal{X}$ such that $\pi(\tilde{P}_0) = P_0$; $g$ is the genus of $\mathcal{X}$ (so that $g \leq (d-1)(d-2)/2$);
- The symbols $\mathcal{X}_1(\mathbf{F}_q)$, $M_q$ and $M_q'$ are as in §2;
- $\Sigma_1$ is the linear series $g_d^2$ on $\mathcal{X}$ obtained from the pullback of lines of $\mathbf{P}^2(\bar{\mathbf{F}}_q)^*$; $\Sigma_2$ is the linear series $g_{2d}^5$ on $\mathcal{X}$ obtained from the pullback of conics of $\mathbf{P}^2(\bar{\mathbf{F}}_q)^*$, i.e. $\Sigma_2 = 2\Sigma_1$ (notice that $\dim(\Sigma_2) = 5$ because $d \geq 3$);
- $S$ is the $\mathbf{F}_q$-Frobenius divisor associated to $\Sigma_2$;
- $j_5(\tilde{P}_0)$ is the 5th positive $(\Sigma_2, \tilde{P}_0)$-order; $\epsilon_5$ is the 5th positive $\Sigma_2$-order; $\nu_4$ is the 4th positive $\mathbf{F}_q$-Frobenius order of $\Sigma_2$.

We apply the appendix to both $\Sigma_1$ and $\Sigma_2$. We have already noticed that the $(\Sigma_1, \tilde{P}_0)$-orders, as well as the $\Sigma_1$-orders, are 0,1 and 2; see Lemma 2.7. Then, the $(\Sigma_2, \tilde{P}_0)$-orders are 0,1,2,3,4 and $j_5(\tilde{P}_0)$, with $5 \le j_5(\tilde{P}_0) \le 2d$, and the $\Sigma_2$-orders are 0,1,2,3,4 and $\epsilon_5$ with $5 \le \epsilon_5 \le j_5(\tilde{P}_0)$; cf. [9, p. 464].

Then, we compute the $\mathbf{F}_q$-Frobenius orders of $\Sigma_2$. We apply (F3) in the appendix to $\tilde{P}_0$ and conclude that this sequence is 0,1,2,3 and $\nu_4$, with

$$\nu_4 \in \{4, \epsilon_5\}\,.$$

Therefore (see appendix)

$$\deg(S) = (6+\nu_4)(2g-2) + (q+5)2d\,,$$

and

$$v_P(S) \ge 5j_1^2(P), \qquad \text{for each } P \in \mathcal{X}_1(\mathbf{F}_q)\,,$$

where $j_1^2(P)$ stands for the first positive $(\Sigma_2, P)$-order. Since $j_1^2(P)$ is equal to the first positive $(\Sigma_1, P)$-order (cf. [9, p. 464]), we then have

$$\deg(S) \ge 5(M_q + M_q')\,,$$

where $M_q$ and $M_q'$ were defined in (2.1). Then, taking into consideration the following facts:

1. $2M_q + M_q' \ge kd$ (Proposition 2.9),
2. $2g-2 \le d(d-3)$,
3. $\nu_4 \le j_5(\tilde{P}_0) - 1 \le 2d-1$ ((F3) appendix), and
4. $d \le 2t = 2(q+2-k)$ (Theorem 2.1(1)),

we obtain the following.

**Proposition 3.1.** *Let $\mathcal{K}$ be a complete arc of size $k$ such that $(2q+4)/3 < k \le m_2'(q)$. Then*

$$k \le \min\{q - \frac{1}{4}\nu_4 + \frac{7}{4}\,,\ \frac{28+4\nu_4}{29+4\nu_4}q + \frac{32+2\nu_4}{29+4\nu_4}\}\,,$$

*where $\nu_4$ is the 4th positive $\mathbf{F}_q$-Frobenius order of the linear series $\Sigma_2$ defined on an irreducible envelope of $\mathcal{K}$.*

Now consider separately the cases $\nu_4 = 4$ and $\nu_4 = \epsilon_5$.

**1. $\nu_4 = 4$.**

In this case, the corresponding irreducible envelope will be called *Frobenius classical.* Proposition 3.1 becomes the following.

**Lemma 3.2.** *Let $\mathcal{K}$ be a complete arc of size $k$ such that $(2q+4)/3 < k \le m_2'(q)$. Suppose that $\mathcal{K}$ admits a Frobenius classical irreducible envelope. Then*

$$k \le \frac{44}{45}q + \frac{40}{45}\,.$$

This lemma holds in the following cases:

(3.1.1) Whenever $q = p$ is an odd prime: Voloch's bound [21];

(3.1.2) The arc is cyclic of Singer type whose size $k$ satisfies $2k \not\equiv -2, 1, 2, 4 \pmod{p}$, where $p > 5$; see Giulietti's paper [10].

For the sake of completeness we prove (3.1.1)

*Proof.* (*Item (3.1.1)*) Let $\mathcal{C}_1$ be an irreducible envelope of $\mathcal{K}$ and $d$ the degree of $\mathcal{C}_1$. If $p < 2d$, then $p < 4t = 4(p + 2 - k)$ so that $k < (3p + 8)/4$ and the result follows. So let $p \ge 2d$. Then from [18, Corollary 2.7] we have that $\mathcal{C}_1$ is Frobenius classical and (3.1.1) follows from Proposition 3.1. $\square$

Next we show that, for $q$ square and $k = m_2'(q)$, Lemma 3.2 is possible only for $q$ small.

**Corollary 3.3.** *Let $\mathcal{K}$ be an arc of size $m_2'(q)$ and suppose that $q$ is a square. Then,*

1. *if $q > 9$, $\mathcal{K}$ has irreducible envelopes;*
2. *if $q > 43^2$, any irreducible envelope of $\mathcal{K}$ is Frobenius non-classical.*

*Proof.* (1) As we mentioned in (1.2), $m_2'(q) \ge q - \sqrt{q} + 1$. Since $q - \sqrt{q} + 1 > (2q + 4)/3$ for $q > 9$, Item (1) follows from Proposition 2.4.

(2) If would exist a Frobenius classical irreducible envelope of $\mathcal{K}$, then from Lemma 3.2 and (1.2) we would have

$$q - \sqrt{q} + 1 \le m_2'(q) \le 44q/45 + 40/45\,.$$

so that $q \le 43^2$. $\square$

**2. $\nu_4 = \epsilon_5$.**

Here, from [8, Corollary 3], we have that $p$ divides $\epsilon_5$. More precisely we have the following.

**Lemma 3.4.** *Either $\epsilon_5$ is a power of $p$ or $p = 3$ and $\epsilon_5 = 6$.*

*Proof.* We can assume $\epsilon_5 > 5$. If $\epsilon_5$ is not a power of $p$, by the $p$-adic criterion [18, Corollary 1.9] we have $p \le 3$ and $\epsilon = 6$. $\square$

From Proposition 3.1, the case $\nu_4 = \epsilon_5 = 6$ provides the following bound:

**Lemma 3.5.** *Let $\mathcal{K}$ be a complete arc of size $k$ such that $(2q+3)/4 < k \le m_2'(q)$. Suppose that $\mathcal{K}$ admits an irreducible envelope such that $\nu_4 = \epsilon_5 = 6$. Then $p = 3$ and*

$$k \le \frac{52}{53}q + \frac{44}{53}\,.$$

As in the previous case, for $q$ an even power of 3 and $k = m_2'(q)$ the case $\nu_4 = \epsilon_5 = 6$ occur only for $q$ small. More precisely, we have the following.

**Corollary 3.6.** *Let $\mathcal{K}$ be an arc of size $m_2'(q)$. Suppose that $q$ is an even power of $p$ and that $\mathcal{K}$ admits an irreducible envelope with $\nu_4 = \epsilon_5 = 6$. Then $p = 3$ and $q \le 3^6$.*

*Proof.* From the $p$-adic criterion [18, Corollary 1.9], $p = 3$. Then from Proposition 3.1 and (1.2) we have

$$q - \sqrt{q} + 1 \le m_2'(q) \le 52q/53 + 44/53\,,$$

and the result follows. □

From now on we assume

$$\nu_4 = \epsilon_5 = \text{a power of } p\,.$$

Then, the bound

$$k \le q - \frac{1}{4}\nu_4 + \frac{7}{4} \tag{3.1}$$

in Proposition 3.1 and Segre's bound (1.1) provide motivation to consider three cases according as $\nu_4 > \sqrt{q}$, $\nu_4 < \sqrt{q}$, or $\nu_4 = \sqrt{q}$.

**3.2.1.** $\nu_4 > \sqrt{q}$**.**

Since $\nu_4$ is a power of $p$, then we have that $\nu^2 \ge pq$ and so from (3.1) the following holds:

**Lemma 3.7.** *Let $\mathcal{K}$ be a complete arc of size $k$ such that $(2q+4)/3 < k \le m_2'(q)$. Suppose that $\mathcal{K}$ admits an irreducible envelope such that $\nu_4$ is a power of $p$ and that $\nu_4 > \sqrt{q}$. Then*

$$k \le \begin{cases} q - \frac{1}{4}\sqrt{pq} + \frac{7}{4} & \text{if } q \text{ is not a square}\,, \\ q - \frac{1}{4}p\sqrt{q} + \frac{7}{4} & \text{otherwise}\,. \end{cases}$$

If $q$ is a square and $k = m_2'(q)$, then $\nu_4 > \sqrt{q}$ can only occur in characteristic 3:

**Corollary 3.8.** *Let $\mathcal{K}$ be an arc of size $m_2'(q)$. Suppose that $q$ is an even power of $p$ and that $\mathcal{K}$ admits an irreducible envelope with $\nu_4$ a power of $p$ and $\nu_4 > \sqrt{q}$. Then $p = 3$, $\nu_4 = 3\sqrt{q}$, and*

$$k \le q - \frac{3}{4}\sqrt{q} + \frac{7}{4}\,.$$

*Proof.* From Lemma 3.7 and $m_2'(q) \ge q - \sqrt{q} + 1$ follow that $\sqrt{q}(p-4) \le 3$ and so that $p = 3$. From $\nu_4 \le 2d - 1$ and $2d \le 4t = 4(q + 2 - m_2'(q)) \le 4\sqrt{q} + 4$ we have that $\nu_4 \le 4\sqrt{q} + 3$ and it follows the assertion on $\nu_4$. The bound on $k$ follows from Lemma 3.7. □

**3.2.2.** $\nu_4 < \sqrt{q}$**.**

Let
$$F(x) := (2x + 32 - q)/(4x + 29)\,.$$
Then the bound
$$k \le \frac{28 + 4\nu_4}{29 + 4\nu_4} q + \frac{32 + 2\nu_4}{29 + 4\nu_4}$$
in Proposition 3.1 can be written as
$$k \le q + F(\nu_4)\,. \tag{3.2}$$
For $x > 0$, $F(x)$ is an increasing function so that
$$F(\nu_4) \le \begin{cases} F(\sqrt{q/p}) = -\frac{1}{4}\sqrt{pq} + \frac{29}{16}p + \frac{1}{2} + R & \text{if } q \text{ is not a square}\,, \\ F(\sqrt{q}/p) = -\frac{1}{4}p\sqrt{q} + \frac{29}{16}p^2 + \frac{1}{2} + R & \text{otherwise}\,, \end{cases}$$
where
$$R = \begin{cases} -\frac{841p - 280}{16(4\sqrt{q/p} + 29)} & \text{if } q \text{ is not a square}\,, \\ -\frac{841p^2 - 280}{16(4\sqrt{q}/p + 29)} & \text{otherwise}\,. \end{cases}$$
Then from (3.2) and since $R < 0$ we have the following.

**Lemma 3.9.** *Let $\mathcal{K}$ be a complete arc of size $k$ such that $(2q+3)/4 < k \le m_2'(q)$. Suppose that $\mathcal{K}$ admits an irreducible envelope such that $\nu_4$ is a power of $p$ and that $\nu_4 < \sqrt{q}$. Then*
$$k < \begin{cases} q - \frac{1}{4}\sqrt{pq} + \frac{29}{16}p + \frac{1}{2} & \textit{if } q \textit{ is not a square}\,, \\ q - \frac{1}{4}p\sqrt{q} + \frac{29}{16}p^2 + \frac{1}{2} & \textit{otherwise}\,. \end{cases}$$

**Corollary 3.10.** *Let $\mathcal{K}$ be a complete arc of size $m_2'(q)$. Suppose that $q$ is an even power of $p$ and that $\mathcal{K}$ admits an irreducible envelope with $\nu_4$ a power of $p$ and $\nu_4 < \sqrt{q}$. Then one of the following statements holds:*

1. *$p = 3$, $\nu_4 = \sqrt{q}/3$, and $m_2'(q)$ satisfies Lemma 3.9.*
2. *$p = 5$, $q = 5^4$, $\nu_4 = 5$, and $m_2'(5^4) \le 613$;*
3. *$p = 5$, $q = 5^6$, $\nu_4 = 5^2$, and $m_2'(5^6) \le 15504$;*
4. *$p = 7$, $q = 7^4$, $\nu_4 = 7$, and $m_2'(7^4) \le 2359$.*

*Proof.* Let $q = p^{2e}$; so $e \ge 2$ as $p \le \nu_4 < p^e$. From (1.2) and Lemma 3.9 we have that
$$(p - 4)p^e/4 < 29p^2/16 - 0.5\,,$$
so that $p \in \{3, 5, 7, 11\}$.

Let $p = 3$. If $\nu_4 \le \sqrt{q}/9$ (so $e \ge 4$), then from (1.2) and $m_2'(q) \le q + F(\sqrt{q}/9)$ we would have that
$$q - \sqrt{q} + 1 \le q - 9\sqrt{q}/4 + 2357/16 - 67841/16(43^{e-2} + 29)\,,$$
which is a contradiction for $e \ge 4$.

Let $p = 11$. Then $p^e \le 125$ and $e = 2$ and $\nu_4 = 11$. Thus from Proposition 3.1 we have $m_2'(11^4) \le 11^4 + F(11)$, i.e. $m_2'(11^4) \le 14441$. This is a contradiction since by (1.2) we must have $m_2'(11^4) \ge 14521$. This eliminates the possibility $p = 11$.

The other cases can be handled in an analogous way. □

**3.2.3.** $\nu_4 = \sqrt{q}$**.**

In this case, according to (3.1), we just obtain Segre's bound (1.1).

## 4. Irreducible envelopes of large complete arcs

Throughout this section we keep the notations of the previous section. Here we study geometrical properties of irreducible envelopes associated to large complete arcs in $\mathbf{P}^2(\mathbf{F}_q)$, $q$ odd. To do so we use the bounds obtained in §3 and divide our study in two cases according as $q$ is a square or not.

**1.** $q$ **square.**

Let $\mathcal{X}$ be an irreducible envelope associated to an arc of size $m_2'(q)$. Then from Lemma 2.7, and Corollaries 3.3, 3.6, 3.8, 3.10, we have the following

**Proposition 4.1.** *If $q$ is an odd square and $q > 43^2$, then $\mathcal{X}$ is $\Sigma_1$-classical. The $\Sigma_2$-orders are $0, 1, 2, 3, 4, \epsilon_5$ and the $\mathbf{F}_q$-Frobenius $\Sigma_2$-orders are $0, 1, 2, 3, \nu_4$, with $\epsilon_5 = \nu_4$, where also one of the following holds:*

1. $\nu_4 \in \{\sqrt{q}/3, 3\sqrt{q}\}$ *for* $p = 3$*;*
2. $(\nu_4, q) \in \{(5, 5^4), (5^2, 5^6), (7, 7^4)\}$*;*
3. $\nu_4 = \sqrt{q}$ *for* $p \ge 5$.

**2.** $q$ **non-square.** In this case there is no analogue to bound (1.2). From Lemmas 3.2, 3.5, 3.7, 3.9 and taking into consideration (3.2) we have the following.

**Proposition 4.2.** *Let $q > 43^2$ and $q = p^{2e+1}$, $e \ge 1$. Then, apart from the values on $\nu_4$, the curve $\mathcal{X}$, $\nu_4$ and $\epsilon_5$ are as in Proposition 4.1. In this case*

$$m_2'(q) > q - 3\sqrt{pq}/4 + 7/4$$

*implies*

1. $\nu_4 = \sqrt{q/p}$;
2. $m_2'(q) < q - \sqrt{pq}/4 + 29p/16 + 1/2$.

In particular our approach just gives a proof of Segre's bound (1.1) and Voloch's bound [21]. However, both propositions above show the type of curves associated to large complete arcs. The study of such curves, for $q$ square and large enough, allowed Hirschfeld and Korchmáros [12], [13] to improve Segre's bound (1.1) to the bound in (1.3). For the sake of completeness we stress here the main ideas from [13] necessary to deal with Problem 1.1. Due to Proposition 2.9, the main strategy is to bound from above the number $2M_q + M'_q$ (which is defined via (2.1)). For instance, if one could prove that

$$2M_q + M'_q \leq d(q - \sqrt{q} + 1)\,, \tag{4.1}$$

where $d$ is the degree of the irreducible envelope whose normalization is $\mathcal{X}$, then from Proposition 2.9 would follow immediately an affirmative answer to Problem 1.1. However, since we know the answer to be negative for $q = 9$ and $d \leq 2t = 2(q + 2 - m'_2(q))$, then one can assume that $d$ is bounded by a linear function on $\sqrt{q}$ and should expect to prove (4.1) only under certain conditions on $q$.

**Lemma 4.3.** *Let $q$ be an odd square. If (4.1) holds true for $d \leq 2\sqrt{q} - \alpha$ with $\alpha \geq 0$, then $m'_2(q) < q - \sqrt{q} + 2 + \alpha/2$. In particular, if (4.1) holds true for $d \leq 2\sqrt{q}$, then the answer to Problem 1.1 is positive; that is, $m'_2(q) = q - \sqrt{q} + 1$.*

*Proof.* If $m'_2(q) \geq q - \sqrt{q} + 2 + \alpha/2$, then from $d \geq 2(q + 2 - m'_2(q)$ we would have that $d \leq 2\sqrt{q} - \alpha$ and so, from Proposition 2.9 and (4.1), that $m'_2(q) \leq q - \sqrt{q} + 1$, a contradiction. □

Now, in [12], Lemma 4.3 is proved for $\alpha \geq \sqrt{q} + 3$, i.e. whenever $d \leq \sqrt{q} - 3$, and so (1.3) follows. Recently, Aguglia and Korchmáros [1] proved a weaker version of (4.1) for $d = \sqrt{q} - 2$ and $q$ large enough, namely $2M_q + M'_q < d(q - \sqrt{q}/2 - 2)$. From this inequality and Proposition 2.9 one slightly improves (1.3) to $m'_2(q) \leq q - \sqrt{q}/2 - 5/2$ whenever $d = \sqrt{q} - 2$ and $q$ is large enough. Therefore the paper [1], as well as [12] or [13], is a good guide toward the proof of (4.1) for $\sqrt{q} - 2 \leq d \leq 2\sqrt{q}$.

## APPENDIX: Background on Weierstrass points and Frobenius orders

In this section we summarize relevant material from Stöhr-Voloch's paper [18] concerning Weierstrass points and Frobenius orders.

Let $\mathcal{X}$ be a projective geometrically irreducible non-singular algebraic curve defined over $\bar{\mathbf{F}}_q$ equipped with the action of the Frobenius morphism $\mathbf{F}r_{\mathcal{X}}$ over $\mathbf{F}_q$. Let $p := \mathrm{char}(\mathbf{F}_q)$. Let $\mathcal{D}$ be a base-point-free linear series $g^r_d$ on $\mathcal{X}$ and assume that it is defined over $\mathbf{F}_q$. Let $\pi : \mathcal{X} \to \mathbf{P}^r(\bar{\mathbf{F}}_q)$ be the $\mathbf{F}_q$-morphism associated to $\mathcal{D}$. Then by considering the pullback of hyperplanes in $\mathbf{P}^r(\bar{\mathbf{F}}_q)$ (via $\pi$) one can define, for each $P \in \mathcal{X}$, a sequence of numbers $j_0(P) = 0 < \ldots < j_r(P)$, called the $(\mathcal{D}, P)$-*order sequence*. It turns out that this sequence is the same, say $\epsilon_0 < \ldots < \epsilon_r$, for all but a finitely many points. This constant sequence is called the *order sequence of* $\mathcal{D}$. There exists a divisor $R = R^{\mathcal{D}}$, the so called *ramification divisor of* $\mathcal{D}$, such that the Supp$(R)$ is the set of points whose $(\mathcal{D}, P)$-orders are different from $(\epsilon_0, \ldots, \epsilon_r)$. The curve is called $\mathcal{D}$-*classical* with if $\epsilon_i = i$ for each $i$. The following are the main properties of these invariants.

(W1) $j_i(P) \geq \epsilon_i$ for each $P$ and each $i$;

(W2) $v_P(R) \geq \sum_i (j_i(P) - \epsilon_i)$; equality holds iff $\det(\binom{j_i(P)}{\epsilon_j}) \not\equiv 0 \pmod{p}$;
(W3) $\deg(R) = (2g-2)\sum_i \epsilon_i + (r+1)d$.

Now to count $\mathbf{F}_q$-rational points one looks for those points $P$ such that $\pi(\mathbf{F}r_{\mathcal{X}}(P))$ belongs to the osculating hyperplane at $P$. This led to the construction of a divisor $S = S^{\mathcal{D},q}$, the so called $\mathbf{F}_q$-*Frobenius divisor associated to* $\mathcal{D}$, such that

(F1) $\mathcal{X}(\mathbf{F}_q) \subseteq \mathrm{Supp}(S)$;
(F2) $\deg(S) = (2g-2)\sum_{i=0}^{r-1} \nu_i + (q+r)d$, where $\nu_0 = 0$ and $(\nu_1, \ldots, \nu_{r-1})$, called the $\mathbf{F}_q$-*Frobenius orders of* $\mathcal{D}$, is a subsequence of $(\epsilon_1, \ldots, \epsilon_r)$.

The curve is called $\mathbf{F}_q$-*Frobenius classical with respect to* $\mathcal{D}$ if $\nu_i = i$ for each $i$. In addition, for each $P \in \mathcal{X}(\mathbf{F}_q)$ holds

(F3) $\nu_i \leq j_{i+1}(P) - j_1(P)$, $i = 0, \ldots, r-1$;
(F4) $v_P(S) \geq \sum_{i=0}^{r-1} (j_{i+1}(P) - \nu_i)$.

Hirschfeld and Korchmáros [13] noticed that (F3) and (F4) even holds for points in the set

$$\mathcal{X}_1(\mathbf{F}_q) := \{P \in \mathcal{X} : \pi(P) \in \pi(\mathcal{X})(\mathbf{F}_q)\}\,.$$

Therefore from (F3) and (F4) we have $v_P(S) \geq r j_1(P)$ for each $P \in \mathcal{X}_1(\mathbf{F}_q)$ and hence we obtain the main result in [18]:

$$\deg(S)/r \geq \#\mathcal{X}_1(\mathbf{F}_q) \geq \#\mathcal{X}(\mathbf{F}_q)\,.$$

**Acknowledgments.** The authors wish to thank James W.P. Hirschfeld and G. Korchmáros for useful comments. This research was carried out with the support of the Italian Ministry for Research and Technology (project 40% "Strutture geometriche, combinatorie e loro applicazioni"). Part of this paper was written while Torres was visiting ICTP, Trieste-Italy (June-July 1998) supported by IMPA/Cnpq-Brazil and ICTP.

## References

[1] A. Aguglia and G. Korchmáros, On the number of rational points of an algebraic curve over finite fields, II, preprint.
[2] E. Boros and T. Szönyi, On the sharpeness of a theorem of B. Segre, *Combinatorica* **6** (1986), 261–268.
[3] R.C. Bose, Mathematical theory of the symmetrical factorial design, *Sankhya* **8** (1947), 107–166.
[4] J.M. Chao and H. Kaneta, Classical arcs in $PG(2,q)$ for $23 \leq q \leq 27$, preprint.
[5] A. Cossidente, New proof of the existence of $(q^2-q+1)$-arcs in $PG(2,q^2)$, *J. Geometry* **53** (1995), 37–40.
[6] G.L. Ebert, Partitioning projective geometries into caps, *Canad. J. Math.* **37** (1985), 1163–1175.
[7] J.C. Fisher, J.W.P. Hirschfeld and J.A. Thas, Complete arcs in planes of square order, *Ann. Discrete Math.* **30**, North Holland, 243–250, 1986.

[8] A. Garcia and M. Homma, Frobenius order-sequences of curves, *Algebra and number theory* (Eds. G. Frey; J. Ritter), Walter de Gruyter Co., Berlin, 27–41, 1994.
[9] A. Garcia and J.F. Voloch, Wronskians and linear independence in fields of prime characteristic, *Manuscripta Math.* **59** (1987), 457–469.
[10] M. Giulietti, On cyclic $k$-arcs of Singer type in $PG(2, q)$, in preparation.
[11] J.W.P. Hirschfeld, *Projective Geometries Over Finite Fields*, second edition, Oxford University Press, Oxford, 1998.
[12] J.W.P. Hirschfeld and G. Korchmáros, Embedding an arc into a conic in a finite plane, *Finite Fields Appl.* **2** (1996), 274–292.
[13] J.W.P. Hirschfeld and G. Korchmáros, On the number of rational points on an algebraic curve over a finite field, *Bull. Belg. Math. Soc.* **5**(2–3) (1998), 313–340.
[14] J.W.P. Hirschfeld and G. Korchmáros, Arcs and curves over finite fields, preprint.
[15] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, preprint.
[16] B. Kestenband, Unital intersections in finite projective planes, *Geom. Dedicata* **11** (1981), 107–117.
[17] B. Segre, Ovals in a finite projective plane, *Canad. J. Math.* **7** (1955), 414–416.
[18] K.O. Stöhr and J.F. Voloch, Weierstrass points and curves over finite fields, *Proc. London Math. Soc.* **52** (1986), 1–19.
[19] J.A. Thas, Complete arcs and algebraic curves in $PG(2, q)$, *J. Algebra* **106** (1987), 451–464.
[20] J.F. Voloch, Arcs in projective planes over prime fields, *J. Geometry* **38** (1990), 198–200.
[21] J.F. Voloch, Complete arcs in Galois plane of non-square order, "Advances in Finite Geometries and Designs," J.W.P. Hirschfeld et al. (eds.), Isle of Thorns 1990, Oxford University Press, Oxford, 1991, pp. 401–405.

Dipartimento di Matematica - Università degli Studi di Perugia, Via Vanvitelli 1 - 06123 Perugia (Italy)

*E-mail address*: giuliet@dipmat.unipg.it

*E-mail address*: fernanda@dipmat.unipg.it

IMECC-UNICAMP, Cx. P. 6065, Campinas, 13083-970-SP, Brazil

*E-mail address*: ftorres@ime.unicamp.br

Dipartimento di Matematica - Università degli Studi di Perugia, Via Vanvitelli 1 - 06123 Perugia (Italy)

*E-mail address*: ughi@dipmat.unipg.it