Nakajima's remark on Henn's proof

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Abstract

We fill up a gap in Henn's proof concerning large automorphism groups of function fields of degree 1 over an algebraically closed field of positive characteristic.

1 Introduction

In 1973, Stichtenoth [4] showed that the Hermitian function fields are the unique function fields K of transcendency degree 1 over an algebraically closed ground field Ω of characteristic p whose automorphism group $G = \text{Aut}(K/\Omega)$ has order at least $16g^4$ where $g \geq 2$ is the genus of K.

In 1978, Henn [1] gave a complete classification of such function fields under the weaker hypothesis $|G| \geq 8g^3$. Later Nakajima [3] improved Henn's result for ordinary curves. In a footnote of his paper, Nakajima claimed: "Stichtenoth's result was improved by Henn. But his proof contains a gap (last paragraph of pg. 104). I do not know if the gap can be covered." The gap appears to be in the penultimate line on pg. 104 when Henn claims "und ersichtlich $z \cong \zeta(z)$ gilt, folgt hieraus $E \leq 2$". The purpose of the present note is to fill up this gap. Actually, we are going to show the missing details in Henn's proof.

We keep notation and terminology from [1]. In the last paragraph on pg. 104, the following case is investigated: K has two places $\mathfrak{B}$ and $\mathfrak{B}_2$ with $G_0(\mathfrak{B}) = G_0(\mathfrak{B}_2)$ but none of the hypotheses in Proposition 1 holds.

For this case, Henn explicitly proves the following claims:

(i) $\nu_1 = 2$;

(ii) the genus g_2 of $K^{G_2(\mathfrak{B})}$ is equal to $q_1 - 1$ with $q_1 = |G_1(\mathfrak{B})/G_2(\mathfrak{B})|$.

(iii) the smallest non-gap at $\widetilde{\mathfrak{B}}$ is q_1.

Then he observes that $q_1 + 1$ must also be a pole number at $\widetilde{\mathfrak{B}}$, and chooses an element $z \in K$ such that

$$z \cong \frac{\widetilde{\mathfrak{B}}_2^a \mathsf{a}}{\widetilde{\mathfrak{B}}^{q_1+1}}, \quad a > 1, \ \widetilde{\mathfrak{B}}_2 \nmid \mathsf{a}.$$

For $\sigma \in \mathrm{Gal}(K^{G_2(\mathfrak{B})}/K^{G_1(\mathfrak{B})})$, he shows that $\sigma(z) = z + \alpha$ with $\alpha \in \Omega$ and $\alpha \neq 0$. From this and Lemma 2, he deduces that

$$2 = \nu_1 = 1 - a + q_1,$$

whence $a = q_1 - 1$ and $\deg(\mathsf{a}) = 2$ follow. At this point Henn takes an element ζ of order $|\zeta| = E$ from $G_0(\mathfrak{B})$. To end the proof it is sufficient to show that ζ has order at most 2, as the hypothesis $1 < e < E$ will give then a contradiction. Henn's idea is to show first that

$$\zeta(z) = cz \text{ with } c \in \Omega \setminus \{0\}, \tag{1}$$

and then to deduce $E = 2$ from it. He claims that (1) follows from the fact that "jeder Punkt $\neq \mathfrak{B}, \mathfrak{B}_2$ unter ζ genau E Konjugierte hat."

We are going to show that z may be chosen such a way that (1) holds indeed.

Since q_1 and $q_1 + 1$ are coprime, we have $K^{G_2(\mathfrak{B})} = \Omega(y, z)$ where y as in Henn's paper has the property:

$$y \cong \frac{\widetilde{\mathfrak{B}}_2^{q_1}}{\widetilde{\mathfrak{B}}^{q_1}}.$$

Let $f \in \Omega[Z, Y]$ be an irreducible polynomial over Ω such that $f(z, y) = 0$. It may be noted that the plane irreducible curve C with affine equation $f(Z, Y) = 0$ is in its Weierstrass normal form:

$$f(Z, Y) = Y^{q_1+1} + \gamma Z^{q_1} + U_1(Z)Y^{q_1} + \ldots + U_{q_1}(Z)Y + U_{q_1+1}(Z),$$

where $\gamma \in \Omega \setminus \{0\}$ and $\deg U_i(Z) \leq i q_1/(q_1 + 1)$ for $i = 1, \ldots, q_1$ and $\deg U_{q_1+1}(Z) < q_1$.

2

In particular, C has only one point at infinity, namely the infinity point Z_∞ of the Z-axis which is the center of the place $\widetilde{\mathfrak{B}}$. Furthermore, the origin O is the center of $\widetilde{\mathfrak{B}}_2$, and no other place of $K^{G_2(\mathfrak{B})}$ is centered either at Z_∞ or O.

Since ζ belongs to the normalizer of $G_2(\mathfrak{B})$, ζ may be viewed as an Ω-automorphism of $K^{G_2(\mathfrak{B})}$ of order E. Moreover, ζ is a linear collineation that preserves C. Since ζ fixes both $\widetilde{\mathfrak{B}}$ and $\widetilde{\mathfrak{B}}_2$, there exist $\alpha, \beta, \gamma \in \Omega$ with $\alpha, \gamma \neq 0$, such that
$$\begin{aligned} \zeta(y) &= \alpha y; \\ \zeta(z) &= \beta y + \gamma z. \end{aligned}$$
If $\beta = 0$, then (1) holds. Now assume that $\beta \neq 0$. Then $\alpha \neq \gamma$. In fact, if $\alpha = \gamma$ then $\zeta^j(y) = \alpha^j(y)$ and $\zeta^j(z) = j\alpha^{j-1}\beta y + \alpha^j z$ for every positive integer j. Now, for $j = E$ this implies that $\alpha^E = 1$ and hence that $E \equiv 0 \pmod{p}$, a contradiction.

Let $u = \beta/(\alpha - \gamma)$, and $z' = z - uy$. Then $\zeta(z') = \zeta(z) - u\zeta(y) = \gamma z + \beta y - u\alpha y = \gamma(z - uy)$. Replacing z by z' we obtain
$$\zeta(z') = \gamma z'.$$

Actually z may be replaced by z' from the very beginning of the argument, therefore Henn's claim (1) may be assumed to be true.

Henn's also claims without proof that (1) implies $E \leq 2$. This can be shown as follows. From $\operatorname{div}\zeta(z) = \operatorname{div}z$ it follows that ζ must preserve the above divisor a. Since $\deg\mathsf{a} = 2$, this implies that ζ^2 fixes each place in the support of a. Therefore, ζ^2 as an Ω-automorphism of the rational function field $K^{G_1(\mathfrak{B})}$ fixes at least three distinct places, namely, $\mathfrak{B}', \mathfrak{B}'_2$ and each place lying under those in the support of a in the covering $K^{G_2(\mathfrak{B})} \to K^{G_1(\mathfrak{B})}$. But then ζ^2 is the identity, and hence $E = 2$.

Remark

A revised proof of Henn's classification is found in [2, Chapter 11.12].

Acknowledgements

References

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