On upper bounds on the smallest size of a saturating set in a projective plane*

Daniele Bartoli[1], **Alexander A. Davydov**[2], **Massimo Giulietti**[3], **Stefano Marcugini**[4], **and Fernanda Pambianco**[5]

Abstract: Using probabilistic methods the following upper bound on the smallest size $s(2,q)$ of a saturating set in a projective plane Π_q (not necessary Desarguesian) of order q is proved:

$$s(2,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}.$$

As byproduct, we obtain that in Π_q a random point k-set with

$$k \leq 2c\sqrt{(q+1)\ln(q+1)} + 2 < \frac{q^2-1}{q+2}, \quad c \geq 1 \text{ independent of } q,$$

is a saturating set with probability

$$p > 1 - \frac{1}{q^{2c^2-2}}.$$

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[1]Department of Mathematics Ghent University, Ghent, 9000, Belgium, E-mail: dbartoli@cage.ugent.be

[2]Institute for Information Transmission Problems (Kharkevich institute), Russian Academy of Sciences, GSP-4, Moscow, 127994, Russian Federation, E-mail: adav@iitp.ru

[3]Department of Mathematics and Informatics, Perugia University, Perugia, 06123, Italy, E-mail: massimo.giulietti@unipg.it

[4]Department of Mathematics and Informatics, Perugia University, Perugia, 06123, Italy, E-mail: stefano.marcugini@unipg.it

[5]Department of Mathematics and Informatics, Perugia University, Perugia, 06123, Italy, E-mail: fernanda.pambianco@unipg.it

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1 Introduction

We denote by Π_q a projective plane (not necessary Desarguesian) of order q and by $PG(2,q)$ the projective plane over the Galois field of q elements. For an introduction to projective geometries over finite fields see [13].

A point set $S \subset \Pi_q$ is *saturating* if any point of $\Pi_q \setminus S$ is collinear with two points in S, see [8, 9, 12, 17] and references therein. It should be noted that saturating sets are also called "saturated sets" [15, 17], "spanning sets" [5], and "dense sets" [3, 10, 11].

The points of a saturating set in $PG(2,q)$ form a parity check matrix of a q-ary linear covering code with codimension 3, covering radius 2, and minimum distance 3 or 4. For an introduction to covering codes see [4, 6]. An online bibliography on covering codes is given in [16].

The main problem in this context is to find small saturating sets (i.e. short covering codes).

Let $s(2,q)$ be the smallest size of a saturating set in Π_q.

In [3] using the approach of [15] by probabilistic methods the following upper bound is proved:

$$s(2,q) < 3\sqrt{2}\sqrt{q \ln q} < 5\sqrt{q \ln q}. \tag{1.1}$$

Surveys on random constructions for geometrical objects can be found in [3, 10, 14, 15], see also references therein. Saturating sets in $PG(2,q)$ obtained by algebraic constructions and computer search can be found in [2, 5, 7–9, 11, 12, 17].

In this paper, we use probabilistic methods to obtain upper bounds on $s(2,q)$. The main results are given by the following theorems.

Theorem 1. *For the smallest size $s(2,q)$ of a saturating set in the projective plane (not necessary Desarguesian) of order q the following upper bound holds.*

$$s(2,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}. \tag{1.2}$$

Theorem 2. *Let*

$$k \leq 2c\sqrt{(q+1)\ln(q+1)} + 2 < \frac{q^2-1}{q+2}, \quad c \geq 1 \text{ independent of } q.$$

Then in the projective plane (not necessary Desarguesian) of order q, a random point k-set is a saturating set with probability

$$p > 1 - \frac{1}{q^{2c^2-2}}. \tag{1.3}$$

It is worth noting that, even if the bounds (1.1) and (1.2) have the same shape, the constant in (1.2) is slightly smaller than the one in (1.1). Also, we use a different approach from those in [3,15] where random sets placed on two or three lines are constructed. In this paper we consider arbitrary random sets.

The *length function* $\ell(2,3,q)$ denotes the smallest length of a q-ary linear code with covering radius 2 and codimension 3; see [4–6]. The result of Theorem 1 means that

$$\ell(2,3,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q\ln q}. \tag{1.4}$$

2 Upper bound on the smallest size of a saturating set

Let w be a fixed integer and consider a random $(w+1)$-subset $\mathcal{K}_{w+1}$ of Π_q. The total number of such subsets is $\binom{q^2+q+1}{w+1}$. A fixed point A of Π_q is *covered* by $\mathcal{K}_{w+1}$ if it belongs to an r-secant of $\mathcal{K}_{w+1}$ with $r \geq 2$. We denote by $\mathrm{Prob}(\diamond)$ the probability of some event $\diamond$.

We estimate

$$\pi := \mathrm{Prob}(A \text{ not covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w+1)$-subsets not covering A to the total number of subsets of size $(w+1)$. Since a set $\mathcal{K}_{w+1}$ does not cover A if and only if every line through A contains at most one point of $\mathcal{K}_{w+1}$, we have

$$\pi = \frac{q^{w+1}\binom{q+1}{w+1}}{\binom{q^2+q+1}{w+1}}.$$

By straightforward calculations,

$$\pi = \frac{(q^2+q)(q^2)\cdots(q^2+q-iq)\cdots(q^2+q-wq)}{(q^2+q+1)(q^2+q)\cdots(q^2+q-i)\cdots(q^2+q+1-w)} =$$

$$= \prod_{i=0}^{w} \frac{q^2+q-iq}{q^2+q+1-i} = \prod_{i=0}^{w}\left(1 - \frac{iq-i+1}{q^2+q+1-i}\right) < \prod_{i=0}^{w}\left(1 - \frac{i(q-1)}{q^2+q+1}\right).$$

Using the classical inequality $1 - x \leq e^{-x}$ we obtain that

$$\pi < e^{-\sum_{i=0}^{w} \frac{i(q-1)}{q^2+q+1}} = e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}},$$

which implies

$$\pi < e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}} < e^{-\frac{w^2}{2q+2}}, \tag{2.1}$$

provided that

$$\frac{(w+1)(q^2-1)}{(q^2+q+1)} > w$$

that is

$$w < \frac{q^2 - 1}{q + 2} \sim q.$$

The set $\mathcal{K}_{w+1}$ is not saturating if at least one point $A \in \Pi_q$ is not covered by $\mathcal{K}_{w+1}$. Similarly to [3, Proposition 4.1], we note that

$$\text{Prob}\left(\mathcal{K}_{w+1} \text{ is not saturating set}\right) \leq \sum_{A \in \Pi_q} \text{Prob}(A \text{ is not covered}).$$

Now, using (2.1), we obtain that

$$\text{Prob}\left(\mathcal{K}_{w+1} \text{ is not saturating set}\right) \leq (q^2 + q + 1)\pi < (q + 1)^2 e^{-\frac{w^2}{2q+2}}.$$

Therefore, the probability that all the points of Π_q are covered is

$$\text{Prob}\left(\mathcal{K}_{w+1} \text{ is saturating set}\right) > 1 - (q + 1)^2 e^{-w^2/(2q+2)}. \tag{2.2}$$

This quantity is larger than 0 taking for instance

$$w = \left\lceil \sqrt{(2q + 2) \ln\left((q + 1)^2\right)} \right\rceil.$$

This shows that in Π_q there exists a saturating k-set with

$$k \leq 2\sqrt{(q + 1)\ln(q + 1)} + 2 \sim 2\sqrt{q \ln q}$$

and therefore Theorem 1 is proved.

In conclusion, we note that any

$$w = \left\lceil c\sqrt{(2q + 2) \ln\left((q + 1)^2\right)} \right\rceil < \frac{q^2 - 1}{q + 2}$$

with the parameter $c \geq 1$ independent of q, is such that (2.2) is positive; therefore Theorem 2 holds true.

It is worth noting that in (1.3) for q large enough, choosing $c = 1+\varepsilon$, with $\varepsilon = o(1) > 0$, the probability p is close to 1.

Remark 3. Let an $[n, n - r]_q 2$ code be a linear q-ary code of length n, codimension r, and covering radius 2. Columns of a parity check matrix of the code can be treated as points of a saturating n-set in the space $PG(r - 1, q)$ [5–8, 12]. Let

$$n_q = 2\sqrt{(q + 1)\ln(q + 1)} + 2 \sim 2\sqrt{q \ln q}. \tag{2.3}$$

By above, there exists a saturating n_q-set in $PG(2, q)$; hence there is the corresponding $[n_q, n_q - 3]_q 2$ code. From this code, using the construction of [7, Ex. 6], see also [8, Th. 4.4], one can obtain an $[n,n-r]_q2$ code with $q\geq 7,\ r=2t-1\geq 7,\ r\neq 9,13,\ n=n_qq^{t-2}+2q^{t-3}$. It means that in $PG(r-1,q)$ there exists a saturating $(n_qq^{t-2}+2q^{t-3})$-set. This implies the following upper bound on the smallest size $s(N,q)$ of a saturating set in $PG(N,q)$:

$$s(N,q)\leq n_qq^{t-2}+2q^{t-3},\ q\geq 7,\ N=2t-2\geq 6,\ N\neq 8,12. \tag{2.4}$$

Surveys of the known $[n,n-r]_q2$ codes and saturating sets in $PG(N,q)$ can be found in [8, 12]. In a number cases the bound (2.4) is better than the known one.

Remark 4. In [1], the concept of multiple saturating sets is introduced. In particular, a point set $S\subset PG(2,q)$ is $(1,\mu)$-saturating if every point Q in $PG(2,q)$ not belonging to S is such that the number of secants of S through Q is at least μ, counted with multiplicity. The multiplicity m_ℓ of a secant ℓ is computed as $m_\ell=\binom{\#(\ell\cap S)}{2}$. A probabilistic upper bound $66\sqrt{\mu q\ln q}$ on the smallest size of a $(1,\mu)$-saturating set in $PG(2,q)$ is given by [1, Prop. 5.2(ii)]. If one can found μ *disjoint* saturating n_q-sets in $PG(2,q)$, see (2.3), then the upper bound is roughly $\sim 2\mu\sqrt{q\ln q}$ that is smaller than the known one for $\mu<33\sqrt{\mu}$.

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