QUOTIENT CURVES OF THE GK CURVE

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ABSTRACT. For every $q = \ell^3$ with ℓ a prime power greater than 2, the GK curve $\mathcal{X}$ is an $\mathbb{F}_{q^2}$-maximal curve that is not $\mathbb{F}_{q^2}$-covered by any $\mathbb{F}_{q^2}$-maximal Deligne-Lusztig curve. Interestingly, $\mathcal{X}$ has a very large $\mathbb{F}_{q^2}$-automorphism group with respect to its genus. In this paper we compute the genera of a large variety of curves that are Galois-covered by the GK curve, thus providing several new values in the spectrum of genera of $\mathbb{F}_{q^2}$-maximal curves.

1. INTRODUCTION

Let $\mathbb{F}_{q^2}$ be a finite field with q^2 elements where q is a power of a prime p. An $\mathbb{F}_{q^2}$-rational curve, that is a projective, geometrically absolutely irreducible, non-singular algebraic curve defined over $\mathbb{F}_{q^2}$, is called $\mathbb{F}_{q^2}$-maximal if the number of its $\mathbb{F}_{q^2}$-rational points attains the Hasse-Weil upper bound

$$q^2 + 1 + 2gq,$$

where g is the genus of the curve. Maximal curves have interesting properties and have also been investigated for their applications in Coding theory. Surveys on maximal curves are found in [11, 12, 17, 34, 35] and [25, Chapter 10]; see also [9, 10, 16, 29, 31].

One of the most important problems on maximal curves is the determination of the possible genera of maximal curves over $\mathbb{F}_{q^2}$, see e.g. [11]. For a given q, the highest value of g for which an $\mathbb{F}_{q^2}$-maximal curve of genus g exists is $q(q-1)/2$ [26], and equality holds if and only if the curve is the Hermitian curve with equation $X^{q+1} = Y^q + Y$, up to $\mathbb{F}_{q^2}$-birational equivalence [29].

By a result of Serre, see [27, Prop. 6], any $\mathbb{F}_{q^2}$-rational curve which is $\mathbb{F}_{q^2}$-covered by an $\mathbb{F}_{q^2}$-maximal curve is also $\mathbb{F}_{q^2}$-maximal. This has made it possible to obtain several genera of $\mathbb{F}_{q^2}$-maximal curves by applying the Riemann-Hurwitz formula, especially from the Hermitian curve, see [2, 3, 1, 4, 7, 8, 14, 15, 20, 18, 21, 22]. Others have been obtained from the DLS and DLR curves, see [24, 5, 6, 28].

The problem of the existence of $\mathbb{F}_{q^2}$-maximal curves other than $\mathbb{F}_{q^2}$-subcovers of the Hermitian curve, the DLS curve, and the DLR curve was solved in [23], where for every $q = \ell^3$ with $\ell = p^r > 2$, p prime, an $\mathbb{F}_{q^2}$-maximal curve $\mathcal{X}$ that is not $\mathbb{F}_{q^2}$-covered by any $\mathbb{F}_{q^2}$-maximal Deligne-Lusztig curve was described. Throughout

Keywords: Maximal curves, rational points, quotient curve.
Mathematics Subject Classification (2000): 11G20.

the paper we will refer to $\mathcal{X}$ as to the GK curve. It should be noted that the construction in [23] has been recently generalized in [13]; it is still an open problem to determine whether these generalizations of the GK curve are $\mathbb{F}_{q^2}$-subcovers of a Deligne-Lusztig curve or not.

One of the most interesting features of the GK curve $\mathcal{X}$ is its very large automorphism group with respect to its genus. In this paper we consider quotient curves $\mathcal{X}/L$ under the action of a large variety of subgroups L of $\mathrm{Aut}(\mathcal{X})$. By applying the Riemann-Hurwitz formula to the covering $\mathcal{X} \to \mathcal{X}/L$ a large number of genera of maximal curves is obtained, see Theorems 4.5, 4.6, 5.4, 5.6, 5.7, 6.2, 6.3, 6.4, 6.5, 6.6, 7.2, 7.3, 7.7. It should be noted that when L is tame and contains the centrum Λ of $\mathrm{Aut}(\mathcal{X})$, then the quotient curve $\mathcal{X}/L$ has the same genus of the quotient curve of the Hermitian curve $\mathcal{H}$ over $\mathbb{F}_{\ell^2}$ under the action of the factor group L/Λ, see Corollary 3.4. Apart from these cases, formulas for genera of quotient curves $\mathcal{X}/L$ appear to provide new values in the spectrum of genera of $\mathbb{F}_{q^2}$-maximal curve, cf. Section 8. One of our main tools for the investigation of the tame case is a relationship between the genus of $\mathcal{X}/L$ and that of the quotient curve of $\mathcal{H}$ with respect to the factor group $L/(L \cap \Lambda)$, see Section 3.

2. The GK curve and its automorphism group

Throughout this paper, p is a prime, $\ell = p^h$ and $q = \ell^3$ with $h \geq 1$, $\ell > 2$. Furthermore, $\mathbb{K}$ denotes the algebraic closure of $\mathbb{F}_{q^2}$.

Let

$$h(X) = \sum_{i=0}^{n} (-1)^{i+1} X^{i(n-1)}. \tag{1}$$

In the three–dimensional projective space $\mathrm{PG}(3, q^2)$ over $\mathbb{F}_{q^2}$, consider the algebraic curve $\mathcal{X}$ defined to be the complete intersection of the surface with affine equation

$$Z^{\ell^2-\ell+1} = Xh(Y), \tag{2}$$

and the Hermitian cone with affine equation

$$Y^\ell + Y = X^{\ell+1}. \tag{3}$$

Note that $\mathcal{X}$ is defined over $\mathbb{F}_{q^2}$ but it is viewed as a curve over $\mathbb{K}$. Moreover, $\mathcal{X}$ has degree $\ell^3 + 1$ and possesses a unique infinite point, namely the infinite point P_∞ of the Y-axis.

Theorem 2.1 ([23]). *$\mathcal{X}$ is an $\mathbb{F}_{q^2}$-maximal curve with genus* $g = \frac{1}{2}(\ell^3+1)(\ell^2-2)+1$.

For notation, terminology and basic results on automorphism groups of curves, we refer to [25, Chapter 11].

For every $u \in \mathbb{K}$, with $u \neq 0$, consider the collineation α_u of $\mathrm{PG}(3, \mathbb{F}_{q^2})$ defined by

$$\alpha_u : \ (X, Y, Z, T) \to (uX, uY, Z, uT). \tag{4}$$

For $u^{\ell^2-\ell+1} = 1$, α_u defines an $\mathbb{F}_{q^2}$-automorphism of $\mathcal{X}$. For $u \neq 1$, the fixed points of α_u are exactly the points of the plane π_0 with equation $Z = 0$. Since π_0 contains no tangent to $\mathcal{X}$, the number of fixed points of α_u with $u \neq 1$ is independent from u and equal to $\ell^3 + 1$. Let $\Lambda = \{\alpha_u | u^{\ell^2-\ell+1} = 1\}$.

Theorem 2.2 ([23])**.** *The group Λ is a central subgroup of* $\mathrm{Aut}(\mathcal{X})$*. The quotient curve $\mathcal{X}/\Lambda$ is the Hermitian curve $\mathcal{H}$ over $\mathbb{F}_{\ell^2}$ with equation $X^{\ell+1} = Y^\ell + Y$. The factor group* $\mathrm{Aut}(\mathcal{X})/\Lambda$ *is isomorphic to* $\mathrm{PGU}(3, \ell)$*.*

If the non-degenerate Hermitian form in the three dimensional vector space $V(3, \ell^2)$ over $\mathbb{F}_{\ell^2}$ is given by $Y^\ell T + YT^\ell - X^{\ell+1}$ then the unitary group $\mathrm{U}(3, \ell)$ is the subgroup of $\mathrm{GL}(3, \ell^2)$ whose elements $U = (u_{ij})$ are determined by the condition $U^t D\sigma(U) = D$ where

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\sigma(U) = (u_{ij}^\ell)$. $\mathrm{U}(3, \ell)$ has order $(\ell + 1)(\ell^3 + 1)\ell^2(\ell^2 - 1)$. A diagonal matrix $[u, u, u]$ is in $\mathrm{U}(3, \ell)$ if and only if $u^{\ell+1} = 1$, and such matrices form a cyclic subgroup C of $\mathrm{U}(3, \ell)$.

The (normal) subgroup $\mathrm{SU}(3, \ell)$ is the subgroup of $\mathrm{U}(3, \ell)$ of index $\ell+1$ consisting of all matrices with determinant 1. A set of generators of $\mathrm{SU}(3, \ell)$ are given by the following matrices:

For $b, c \in \mathbb{F}_{\ell^2}$ such that $c^\ell + c - b^{\ell+1} = 0$, and for $a \in \mathbb{F}_{\ell^2}$, $a \neq 0$,

$$Q_{(b,c)} = \begin{pmatrix} 1 & 0 & b \\ b^\ell & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \ R_a = \begin{pmatrix} a^{-n} & 0 & 0 \\ 0 & a^{n-1} & 0 \\ 0 & 0 & a \end{pmatrix}, \ S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$\mathrm{SU}(3, \ell) \cap C$ is either trivial or is a subgroup of order 3, according as $\gcd(3, \ell+1) = 1$ or 3. The center $Z(\mathrm{U}(3, \ell))$ coincides with C, and $Z(\mathrm{SU}(3, \ell)) = \mathrm{SU}(3, \ell) \cap C$. In this context, $\mathrm{PGU}(3, \ell) = U(3, \ell)/C$. A treatise on unitary groups can be found in [33, Section 10].

From each $U \in \mathrm{U}(3, \ell)$ a (4×4)-matrix $\widetilde{U}$ arises by adding $0, 0, 1, 0$ as a third row and as a third column. Obviously, these matrices $\widetilde{U}$ with $U \in \mathrm{SU}(3, \ell)$ form a subgroup Γ of $\mathrm{GL}(4, \ell^2)$ isomorphic to $\mathrm{SU}(3, \ell)$. Since the identity matrix is the only scalar matrix in Γ, we can regard Γ as a projective group in $\mathrm{PGL}(4, \mathbb{F}_{q^2})$.

It is shown in [23] that the group Γ preserves $\mathcal{X}$, Λ centralizes Γ, and $\Gamma \cap \Lambda$ is trivial when $\gcd(3, \ell + 1) = 1$ while it has order 3 when $\gcd(3, \ell + 1) = 3$. Let Λ_3

be the unique subgroup of Λ of order $\frac{\ell^2-\ell+1}{3}$. Then by [23, Lemma 8] $\mathrm{Aut}(\mathcal{X})$ has a subgroup Ξ with

$$\Xi = \begin{cases} \Gamma \times \Lambda, & \text{when} \quad \gcd(3, \ell+1) = 1; \\ \Gamma \times \Lambda_3, & \text{when} \quad \gcd(3, \ell+1) = 3. \end{cases} \tag{5}$$

When $\gcd(3, \ell+1) = 1$, $\mathrm{Aut}(\mathcal{X}) = \Xi$ holds (see [23, Thm. 6 (i)]), whereas for $\gcd(3, \ell+1) = 3$, $\mathcal{X}$ has further $\mathbb{F}_{q^2}$-automorphisms. Let ρ be a primitive (ℓ^3+1)-st root of unity in $\mathbb{F}_{q^2}$, and let $\widetilde{E}$ be the diagonal matrix $[\rho^{-1}, \rho^{\ell^2-\ell}, 1, \rho^{-1}]$. Then $\widetilde{E}$ preserves $\mathcal{X}$, normalizes Γ and commutes with Λ. Moreover, $\widetilde{E} \notin \Xi$ but $\widetilde{E}^3 \in \Xi$. By [23, Thm. 6 (ii)], if $\gcd(3, n+1) = 3$ then $[\mathrm{Aut}(\mathcal{X}) : \Xi] = 3$ and Ξ is a normal subgroup of $\mathrm{Aut}(\mathcal{X})$. Moreover, Γ is a normal subgroup of $\mathrm{Aut}(\mathcal{X})$.

Both Γ and Λ preserve the set of points lying in the plane of equation $Z = 0$.

Theorem 2.3 ([23]). *The set of $\mathbb{F}_{q^2}$-rational points of $\mathcal{X}$ splits into two orbits under the action of* $\mathrm{Aut}(\mathcal{X})$*, one is non-tame, has size ℓ^3+1, and consists of the $\mathbb{F}_{\ell^2}$-rational points on $\mathcal{X}$; the other is tame of size $\ell^3(\ell^3+1)(\ell^2-1)$. Furthermore,* $\mathrm{Aut}(\mathcal{X})$ *acts on the non-tame orbit as* $\mathrm{PGU}(3, \ell)$ *in its doubly transitive permutation representation.*

Henceforth, the orbit of size ℓ^3+1 will be denoted as $\mathcal{O}_1$, whereas the orbit of size $\ell^3(\ell^3+1)(\ell^2-1)$ by $\mathcal{O}_2$. Moreover, the natural projection from $\mathrm{Aut}(\mathcal{X})$ to $PGU(3, \ell)$ will be denoted by π. Let ϕ be the rational map $\phi : \mathcal{X} \to \mathcal{H}$ defined by $\phi(1 : x : y : z) = (1 : x : y)$.

For a subgroup L of $\mathrm{Aut}(\mathcal{X})$, let $\bar{L}$ be the subgroup $\pi(L)$ of $PGU(3, \ell)$.

Throughout the paper we will refer to the following maximal subgroups, defined up to conjugacy, of the group $PGU(3, \ell)$, viewed as the group of the projectivities of $\mathbb{P}^2(\mathbb{K})$ preserving the Hermitian curve $\mathcal{H}$.

(A) The stabilizer of an $\mathbb{F}_{\ell^2}$-rational point, of size $\ell^3(\ell^2-1)$.
(B) The normalizer of a Singer group, of size $3(\ell^2-\ell+1)$. Here a Singer group of $PGU(3, \ell)$ is a cyclic group of size $\ell^2-\ell+1$ stabilizing a point in $\mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2})$.
(C) The self-conjugate triangle stabilizer, of size $6(\ell+1)^2$.
(D) The non-tangent line stabilizer, of size $\ell(\ell+1)(\ell^2-1)$.

3. Preliminary results

Let L be a subgroup of $\mathrm{Aut}(\mathcal{X})$. Let $\mathcal{X}/L$ be the quotient curve of $\mathcal{X}$ with respect to L, and let g_L be its genus. If L is tame, that is p does not divide the order of L, then the Hurwitz genus formula for Galois extensions gives

$$(\ell^3+1)(\ell^2-2) = |L|(2g_L - 2) + e_L \tag{6}$$

with

$$e_L = \sum_{P \in \mathcal{X}} (|L_P| - 1), \tag{7}$$

where L_P is the stabilizer of P in L. The aim of this section is to provide a formula which relates e_L to the action of $\bar{L}$ on the Hermitian curve $\mathcal{H}$, see Proposition 3.1 below. Let $L_\Lambda = L \cap \Lambda$. The factor group L/L_Λ is isomorphic to $\bar{L}$, and the action of L/L_Λ on the orbits of $\mathcal{X}$ under Λ is isomorphic to that of $\bar{L}$ on $\mathcal{H}$.

As to the relation between e_L and the analogous value $\sum_{\bar{P}\in\mathcal{H}}(|\bar{L}_{\bar{P}}|-1)$, for $\bar{L}$, by standard arguments from permutation group theory it is not difficult to prove that

$$\sum_{P\in\mathcal{X}}(|L_P|-1)\cdot m_P = |L_\Lambda|\left(\sum_{\bar{P}\in\mathcal{H}}(|\bar{L}_{\bar{P}}|-1)\cdot|\phi^{-1}(\bar{P})|\right)-\sum_{P\in\mathcal{X}}(m_P-|L_\Lambda|), \tag{8}$$

where m_P denotes the size of the orbit of P under the action of the subgroup of L stabilizing the set $\phi^{-1}(\phi(P))$. However, we will not use (8), as this would require involved computations on $m(P)$. As it has emerged from the literature, a more adequate approach is based on the equality

$$e_L = \sum_{h\in L, h\neq id} N_h, \qquad \text{where } N_h = |\{P\in\mathcal{X}\mid h(P)=P\}| \tag{9}$$

(cf. [20, Eq. 4.7]).

The ramification points of the morphism $\phi:\mathcal{X}\to\mathcal{H}$ are exactly the points in $\mathcal{O}_1$. At these points ϕ is fully ramified. The set $\bar{\mathcal{O}}_1$ of the images of the points in $\mathcal{O}_1$ by ϕ in $\mathcal{H}$ is precisely the set $\mathcal{H}(\mathbb{F}_{\ell^2})$ of the $\mathbb{F}_{\ell^2}$-rational points of $\mathcal{H}$, whereas the image $\bar{\mathcal{O}}_2$ of $\mathcal{O}_2$ by ϕ coincides with $\mathcal{H}(\mathbb{F}_{q^2})\setminus\mathcal{H}(\mathbb{F}_{\ell^2})$. Any point of $\mathcal{H}$ fixed by some non-trivial element in $PGU(3,\ell)$ lies in $\bar{\mathcal{O}}_1\cup\bar{\mathcal{O}}_2$, see e.g. [20, Prop. 2.2].

In order to compute e_L as in (9), it is convenient to write

$$N_h = N_h^{(1)} + N_h^{(2)} \tag{10}$$

with

$$N_h^{(1)} = |\{P\in\mathcal{O}_1\mid h(P)=P\}|,\quad N_h^{(2)} = |\{P\in\mathcal{O}_2\mid h(P)=P\}|.$$

Proposition 3.1. *Let L be a subgroup of* $\mathrm{Aut}(\mathcal{X})$*, and let $L_\Lambda = L\cap\Lambda$. Let $\bar{\mathcal{O}}_1 = \mathcal{H}(\mathbb{F}_{\ell^2})$ and $\bar{\mathcal{O}}_2 = \mathcal{H}(\mathbb{F}_{q^2})\setminus\mathcal{H}(\mathbb{F}_{\ell^2})$. Then*

$$e_L = (|L_\Lambda|-1)(\ell^3+1)+|L_\Lambda|n_1+|L_\Lambda|n_2$$

where

- *n_1 counts the non-trivial relations $\bar{h}(\bar{P})=\bar{P}$ with $\bar{h}\in\bar{L}$ when $\bar{P}$ varies in $\bar{O}_1$, namely*
$$n_1 = \sum_{\bar{h}\in\bar{L},\bar{h}\neq id}|\{\bar{P}\in\bar{\mathcal{O}}_1\mid\bar{h}(\bar{P})=\bar{P}\}|;$$
- *n_2 counts the non-trivial relations $\bar{h}(\bar{P})=\bar{P}$ with $\bar{h}\in\bar{L}$ when $\bar{P}$ varies in $\bar{O}_2$, each counted with a multiplicity $l_{\bar{h},\bar{P}}$ defined as the number of orbits of*

$\phi^{-1}(\bar{P})$ under the action of L_Λ that are fixed by an element $h \in \pi^{-1}(\bar{h})$. That is,

$$n_2 = |L_\Lambda| \sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{\bar{P}\in\bar{\mathcal{O}}_2,\bar{h}(\bar{P})=\bar{P}} l_{\bar{h},\bar{P}}.$$

Proof. Note that

$$\sum_{h\in L,h\neq id} N_h = \sum_{k\in L_\Lambda,k\neq id} N_k + \sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{k\in L_\Lambda} N_{hk}, \tag{11}$$

where $h \in L$ is an element such that $\pi(h) = \bar{h}$. For each non-trivial element $k \in L_\Lambda$, $N_k = |\mathcal{O}_1| = (\ell^3 + 1)$ holds. Therefore

$$\sum_{k\in L_\Lambda,k\neq id} N_k = (|L_\Lambda| - 1)(\ell^3 + 1). \tag{12}$$

As to the second term in the right hand side of (11), write $N_{hk} = N_{hk}^{(1)} + N_{hk}^{(2)}$, with $N_{hk}^{(i)}$ as in (10). As $k(P) = P$ for each $P \in \mathcal{O}_1$, we have

$$\sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{k\in L_\Lambda} N_{hk}^{(1)} = |L_\Lambda| \sum_{\bar{h}\in\bar{L},\bar{h}\neq id} |\{\bar{P} \in \bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P}) = \bar{P}\}|. \tag{13}$$

It remains to compute $\sum_{\bar{h}\in\bar{L},\bar{h}\neq id}\sum_{k\in L_\Lambda} N_{hk}^{(2)}$. Since $\phi(k(P)) = \phi(P)$ for each $P \in \mathcal{X}$, condition $(hk)(P) = P$ yields that $\bar{h}(\phi(P)) = \phi(P)$. Therefore,

$$\sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{k\in L_\Lambda} N_{hk}^{(2)} = \sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{\bar{P}\in\bar{\mathcal{O}}_2,\bar{h}(\bar{P})=\bar{P}} \ \sum_{k\in L_\Lambda} m_{k,\bar{h},\bar{P}}$$

where $m_{k,\bar{h},\bar{P}} = |\{P \in \mathcal{X}, \pi(P) = \bar{P}, (hk)(P) = P\}|$. By the orbit-stabilizer theorem $\sum_{k\in L_\Lambda} m_{k,\bar{h},\bar{P}} = |L_\Lambda| l_{\bar{h},\bar{P}}$, whence

$$\sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{k\in L_\Lambda} N_{hk}^{(2)} = |L_\Lambda| \sum_{\bar{h}\in\bar{L},\bar{h}\neq id} \ \sum_{\bar{P}\in\bar{\mathcal{O}}_2,\bar{h}(\bar{P})=\bar{P}} l_{\bar{h},\bar{P}}.$$

Taking into account (9),(11),(12),(13), this finishes the proof. □

The following corollary to Proposition 3.1 will be useful in the sequel.

Proposition 3.2. *Let L be a tame subgroup of $\mathrm{Aut}(\mathcal{X})$. Assume that no non-trivial element in $\bar{L}$ fixes a point in $\mathcal{H} \setminus \bar{\mathcal{O}}_1$. Then*

$$g_L = g_{\bar{L}} + \frac{(\ell^3 + 1)(\ell^2 - |L_\Lambda| - 1) - |L_\Lambda|(\ell^2 - \ell - 2)}{2|L|},$$

where $g_{\bar{L}}$ is the genus of the quotient curve $\mathcal{H}/\bar{L}$.

Proof. By (6) and Proposition 3.1,

$$(\ell^3+1)(\ell^2-2) = |L|(2g_L-2)+(|L_\Lambda|-1)(\ell^3+1)+|L_\Lambda| \sum_{\bar h \in \bar L, \bar h \neq id} |\{\bar P \in \bar{\mathcal{O}}_1 \mid \bar h(\bar P) = \bar P\}|.$$

On the other hand, by the Hurwitz genus formula applied to the covering $\mathcal{H} \to \mathcal{H}/\bar L$

$$\sum_{\bar h \in \bar L, \bar h \neq id} |\{\bar P \in \bar{\mathcal{O}}_1 \mid \bar h(\bar P) = \bar P\}| = (\ell^2-\ell-2) - |\bar L|(2g_{\bar L}-2),$$

whence

$$(\ell^3+1)(\ell^2-2) = |L|(2g_L-2)+(|L_\Lambda|-1)(\ell^3+1)+|L_\Lambda|(\ell^2-\ell-2)-|L|(2g_{\bar L}-2).$$

Then the claim follows by straightforward computation. □

When $\Lambda \subseteq L$, $l_{\bar h, \bar P} = 1$ for every $\bar h \in \bar L$, and for every $\bar P \in \bar{\mathcal{O}}_2$ with $\bar h(\bar P) = \bar P$. Therefore Proposition 3.1 reads as follows.

Corollary 3.3. *Let L be a subgroup of $\mathrm{Aut}(\mathcal{X})$ containing Λ. Then*

$$e_L = (\ell^2-\ell)(\ell^3+1) + (\ell^2-\ell+1) \sum_{\bar h \in \bar L, \bar h \neq id} |\{\bar P \in \bar{\mathcal{H}}(\mathbb{F}_{q^2}) \mid \bar h(\bar P) = \bar P\}|.$$

We end this section with a result showing that if L is tame and $\Lambda \subset L$, then the genus of g_L is actually the genus of a quotient curve of the Hermitian curve $\mathcal{H}$.

Corollary 3.4. *Let L be a tame subgroup of $\mathrm{Aut}(\mathcal{X})$ containing Λ. Then g_L coincides with the genus of the quotient curve $\mathcal{H}/\bar L$.*

Proof. Let $g_{\mathcal{H}} = \ell(\ell-1)/2$ be the genus of $\mathcal{H}$, and let $g_{\bar L}$ be the genus of the quotient curve $\mathcal{H}/\bar L$. Then by straightforward computation

$$(\ell^3+1)(\ell^2-2) = (\ell^2-\ell+1)(2g_{\mathcal{H}}-2) + (\ell^2-\ell)(\ell^3+1). \tag{14}$$

Since $\bar L$ is tame,

$$2g_{\mathcal{H}}-2 = |\bar L|(2g_{\bar L}-2) + \sum_{\bar h \in \bar L, \bar h \neq id} |\{\bar P \in \bar{\mathcal{H}}(\mathbb{F}_{q^2}) \mid \bar h(\bar P) = \bar P\}|. \tag{15}$$

Note that $|L| = (\ell^2-\ell+1)|\bar L|$. Taking into account (6) and Corollary 3.3, $g_L = g_{\bar L}$ follows by straightforward computation. □

4. Curves $\mathcal{X}/L$ with $\bar L$ subgroup of a group of type (A)

In this section subgroups L of $\mathrm{Aut}(\mathcal{X})$ stabilizing a point $P \in \mathcal{O}_1$ are investigated. Up to conjugacy, we may assume that L is contained in the stabilizer $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ of P_∞ in $\mathrm{Aut}(\mathcal{X})$. By the orbit-stabilizer theorem the size of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ is $\ell^3(\ell^2-1)(\ell^2-\ell+1)$. Since $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ is non-tame, in order to determine the genus of $\mathcal{X}/L$ we will use Hilbert's ramification theory, see [30, Ch. III.8].

Let G be a subgroup of $\mathrm{Aut}(\mathcal{X})$ and let P be a point of $\mathcal{X}$. For an integer $i \geq -1$ the i-th ramification group of G at P is

$$G_i(P) = \{h \in G \mid v_{P_\infty}(h^\star(t) - t) \geq i + 1\},$$

where $h^\star \in \mathrm{Aut}(\mathbb{K}(\mathcal{X}))$ is the pullback of h, v_P is the discrete valuation of $\mathbb{K}(\mathcal{X})$ associated to P, and t is any P-prime element. The group $G_0(P)$ coincides with the stabilizer $\mathrm{Aut}(\mathcal{X})_P$, whereas $G_1(P)$ is the only p-Sylow subgroup of $G_0(P)$, see e.g. [30, Prop. III.8.5]. Moreover, there exists a cyclic group H in $G_0(P)$ such that $G_0(P) = G_1(P) \rtimes H$, see [25, Thm. 11.49]. The Hurwitz genus formula together with the Hilbert different formula (see e.g. [30, Thm. III.8.7]) gives

$$2g - 2 = |G|(2g_G - 2) + \sum_{P \in \mathcal{X}} d_P, \qquad \text{with } d_P = \sum_{i=0}^{\infty} (|G_i(P)| - 1), \tag{16}$$

where g_G denotes the genus of the quotient curve $\mathcal{X}/G$.

Assume that $G = G_0(P_\infty)$, that is every element in G fixes P_∞. Since $\mathcal{X}$ is a maximal curve, no p-element in G can fix a point P different from P_∞, see [25, Thm. 9.76] and [25, Thm. 11.133]. Therefore for any $P \neq P_\infty$ the integer d_P in (16) coincides with $G_0(P) - 1$. The following result then holds.

Lemma 4.1. *For a subgroup L of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$, let g_L be the genus of the quotient curve $\mathcal{X}/L$. Then*

$$(\ell^3 + 1)(\ell^2 - 2) = |L|(2g_L - 2) + e_L + \sum_{i=1}^{\infty} (|L_i(P_\infty)| - 1),$$

with e_L as in (7).

We now provide an explicit description of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$. For $a \in \mathbb{F}_{q^2}$ such that $a^{(\ell^2-\ell+1)(\ell^2-1)} = 1$, for $b, c \in \mathbb{F}_{\ell^2}$ such that $c^\ell + c = b^{\ell+1}$, let $\xi_{a,b,c}$ be the projectivity in $PGL(4, q^2)$ defined by the matrix

$$\begin{pmatrix} a^{\ell^2-\ell+1} & 0 & 0 & b \\ b^\ell a^{\ell^2-\ell+1} & a^{\ell^3+1} & 0 & c \\ 0 & 0 & a^{\ell^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easily seen that $\xi_{a,b,c} = \xi_{a,0,0}\xi_{1,b,c}$, with $\xi_{1,b,c} \in \Gamma \cap \mathrm{Aut}(\mathcal{X})_{P_\infty}$. Also, by straightforward computation $\xi_{a,0,0}$ lies in $\mathrm{Aut}(\mathcal{X})_{P_\infty}$. By a trivial counting argument, we then have

$$\mathrm{Aut}(\mathcal{X})_{P_\infty} = \{\xi_{a,b,c} \mid a^{(\ell^2-\ell+1)(\ell^2-1)} = 1, b, c \in \mathbb{F}_{\ell^2}, b^{\ell+1} = c^\ell + c\}.$$

The elements $\xi_{1,b,c}$ form a subgroup of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ of size ℓ^3; therefore the first ramification group of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ at P_∞ is

$$\{\xi_{1,b,c} \mid b, c \in \mathbb{F}_{\ell^2}, b^{\ell+1} = c^\ell + c\}.$$

In order to determine higher ramification groups at P_∞ we need to compute the integer $v_{P_\infty}(h^\star(t)-t)$, with t a P_∞-prime element, for automorphisms $h=\xi_{1,b,c}$. By [23, Sect. 4]

$$v_{P_\infty}(x)=-(\ell^3-\ell^2+\ell),\quad v_{P_\infty}(y)=-(\ell^3+1),\quad v_{P_\infty}(z)=-\ell^3.$$

Therefore, t can be assumed to be the rational function z/y. Since

$$\xi^\star_{1,b,c}(z)=z,\qquad \xi^\star_{1,b,c}(y)=y+b^\ell x+c$$

we have that

$$\xi^\star_{1,b,c}(z/y)-(z/y)=\frac{c-zxb^\ell}{y(y+b^\ell x+c)},$$

whence

$$v_{P_\infty}(\xi^\star_{1,b,c}(t)-t)=\begin{cases}\ell^2-\ell+2 & \text{if } b\neq 0\\ \ell^3+2 & \text{if } b=0\end{cases}.$$

The following result is then obtained.

Proposition 4.2. *Let L be a subgroup of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$. Then*

$$L_1(P_\infty)=L_2(P_\infty)=\ldots=L_{\ell^2-\ell+1}(P_\infty)=\{\xi_{1,b,c}\mid \xi_{1,b,c}\in L\},$$

and

$$L_{\ell^2-\ell+2}(P_\infty)=L_{\ell^2-\ell+3}(P_\infty)=\ldots=L_{\ell^3+1}(P_\infty)=\{\xi_{1,0,c}\mid \xi_{1,0,c}\in L\}.$$

For $i>\ell^3+1$ the group $L_i(P_\infty)$ is trivial.

As to the computation of e_L in Lemma 4.1, the following fact will be useful.

Lemma 4.3. *Let L be a subgroup of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$. Then any point of $\mathcal{X}$ which is fixed by a non-trivial element in L belongs to $\mathcal{O}_1$.*

Proof. Assume that $\alpha\in L$ fixes a point $P\in\mathcal{X}\setminus\mathcal{O}_1$. Then $\pi(\alpha)$ is an element of $PGU(3,\ell)$ fixing both the infinite point $\bar P_\infty$ of $\mathcal{H}$ and the point $\phi(P)$ in $\mathcal{H}\setminus\mathcal{H}(\mathbb{F}_{\ell^2})$. Then by [20, Sect. 4] $\pi(\alpha)$ is trivial, that is $\alpha\in\Lambda$. Since any non-trivial element in Γ only fixes points in $\mathcal{O}_1$, we obtain $\alpha=id$. □

In Section 4.1 we will deal with the case $L=\Sigma_1\times\Sigma_2$, where Σ_1 is contained in Γ and Σ_2 is a subgroup of Λ, see Section 4.1. To this end, we determine a subgroup Ω of $\Gamma\cap\mathrm{Aut}(\mathcal{X})_{P_\infty}$ such that $\Omega\cap\Lambda=\{id\}$. In Section 4.2 the case $L=\pi^{-1}(\bar G)$ with $\bar G$ a group of type (A) will be dealt with.

Let μ_1 be the highest power of 3 dividing $\ell+1$. Let

$$\Omega=\{\xi_{a,b,c}\mid a^{\frac{\ell^2-1}{\mu_1}}=1\}.$$

Assume that $\alpha=\xi_{a,b,c}\in\Omega\cap\Lambda$. Then clearly $b=c=0$ holds, whence $\alpha=\xi_{a,0,0}$ for some a such that $a^{(\ell^2-1)/\mu_1}=1$. Note that $\xi_{a,0,0}\in\Lambda$ implies $a^{\ell^2-\ell+1}=1$. But then $a=1$ since $\gcd(\frac{\ell^2-1}{\mu_1},\ell^2-\ell+1)=1$. Then the following holds.

Lemma 4.4. *The subgroup generated by Ω and Λ is the direct product $\Omega \times \Lambda$. The projection $\pi(\Omega \times \Lambda)$ is a subgroup of the stabilizer of the infinite point $\bar{P}_\infty$ of $\mathcal{H}$ in $PGU(3,\ell)$ isomorphic to Ω.*

4.1. **$L = \Sigma_1 \times \Sigma_2$, $\Sigma_1 < \Omega$, $\Sigma_2 < \Lambda$.** Let $|\Sigma_1| = mp^{v+w}$, where

$$p^{v+w} = |\Sigma_1 \cap \{\xi_{1,b,c}\}|, \qquad p^v = |\Sigma_1 \cap \{\xi_{1,0,c}\}|,$$

and m is a divisor of $(\ell^2-1)/\mu_1$. Let $|\Sigma_2| = d_2$. Then by Lemma 4.1, together with Proposition 3.1 and Lemma 4.3, it follows that

$$\begin{array}{rl}\ell^5 - 2\ell^3 + \ell^2 - 2 = & md_2p^{v+w}(2g_L-2) + (\ell^2-\ell+1)(p^{v+w}-1) + (\ell^3-\ell^2+\ell)(p^w-1)\\ & +(d_2-1)(\ell^3+1) + d_2\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id}|\{\bar{P}\in\bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P}) = \bar{P}\}|,\end{array}$$

that is

$$\begin{array}{rl}(2g_L-1)md_2p^{v+w} = & (l^3+1)(l^2-d_2) - (\ell^2-\ell+1)p^w(p^v+\ell) + d_2\\ & -d_2\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id}|\{\bar{P}\in\bar{\mathcal{O}}_1, \bar{P}\neq\bar{P}_\infty \mid \bar{h}(\bar{P}) = \bar{P}\}|.\end{array} \tag{17}$$

In order to provide concrete values of genera g_L we are going to consider subgroups of $PGU(3,\ell)_{\bar{P}_\infty}$ that are known in the literature, see [20] and [2]. To this end it is useful to note that in both papers [20] and [2] the group $PGU(3,\ell)_{\bar{P}_\infty}$ is denoted as $\mathcal{A}(P_\infty)$, and that the subgroup $\pi(\Omega)$ of $PGU(3,\ell)$ in the notation of both [20] and [2] is the subgroup of index μ_1 in $\mathcal{A}(P_\infty)$ consisting of elements $[a,b,c]$ with $a^{(\ell^2-1)/\mu_1} = 1$. It should also be noted that for each $\Sigma_1 < \Omega$ the integer $\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id}|\{\bar{P}\in\bar{\mathcal{O}}_1, \bar{P}\neq \bar{P}_\infty \mid \bar{h}(\bar{P}) = \bar{P}\}|$ is computed in [20]. From (17), together with Theorem 4.4 in [20],it follows that

$$g_L = \frac{l^5 + l^2 - (\ell^2-\ell+1)p^w(p^v+\ell) - d_2(l^3 + (d-1)p^v l - dp^{v+w})}{2md_2p^{v+w}},$$

where $d = \gcd(m, \ell+1)$.

The following result then holds.

Theorem 4.5. *Let d_2 be any divisor of $\ell^2-\ell+1$.*

(i) *Let $p \neq 2$ and $m \mid l^2-1$ be such that 3 does not divide m and $m > 1$. Let $d = \gcd(m, l+1)$ and let $s := min\left\{r \geq 1 : p^r \equiv 1 \bmod \frac{m}{d}\right\}$. For each $0 \leq w \leq h$, such that $s \mid w$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*

$$g_L = \frac{(l^3+1)(l^2-p^w) - (l^3-l)d_2 - dd_2(l-p^w)}{2md_2p^w}$$

(cf. [20, Prop. 4.6]*).*

(ii) *Let $p \neq 2$ and $m \mid l-1$. Let $d = \gcd(m, l+1)$. Let s be the order of p in $(\mathbb{Z}/m\mathbb{Z})^*$, and let*

$$r = \begin{cases} \text{order of } p \text{ in } (\mathbb{Z}/\frac{m}{2}\mathbb{Z})^*, & m \text{ even}\\ s, & m \text{ odd}\end{cases}.$$

For each $0 \le v \le h$ *such that* $s \mid v$, *and for each* $0 \le w \le h-1$ *such that* $r \mid w$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - l^3 d_2 - (l^2 - l + 1 - dd_2)p^{v+w} - (l^3 - l^2 + l)p^w - d_2 l p^v (d-1)}{2md_2 p^{v+w}}$$

(cf. [2, Thm. 1]*).*

(iii) *Let* $p \neq 2$ *and* $m \mid l^2 - 1$ *be such that* m *does not divide* $l-1$, *and* 3 *does not divide* m. *Let* $d = \gcd(m, l+1)$. *Let* s *be the order of* p *in* $(\mathbb{Z}/m\mathbb{Z})^*$, *and* r *be the order of* p *in* $(\mathbb{Z}/(m/d)\mathbb{Z})^*$. *For each* $0 \le v \le h$, *such that* $v \mid 2h$, v *does not divide* h *and* $s \mid v$, *and for each* $\frac{v}{2} \le w \le h$, *such that* $r \mid w$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - l^3 d_2 - (l^2 - l + 1 - dd_2)p^{v+w} - (l^3 - l^2 + l)p^w - d_2 l p^v (d-1)}{2md_2 p^{v+w}}$$

(cf. [2, Thm. 2]*).*

(iv) *Let* $p \neq 2$. *For each* $0 \le v \le h$, *and for each* $0 \le w \le h-1$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)p^w(p^v + l) + ld_2(p^v - l^2) - d_2 p^v (l - p^w)}{2d_2 p^{v+w}}$$

(cf. [20, Thm. 3.2]*).*

(v) *Let* $p = 2$. *For all integers* v, w *with* $0 \le v \le w < h$ *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_2 2^v (l - 2^w)}{d_2 2^{v+w+1}}$$

(cf. [20, Cor. 3.4(ii)]*).*

(vi) *Let* $p = 2$. *For all integers* v, w *with* $w \mid h$, $w \mid v$, $v \mid 2h$, $1 \le v < n$, *and* $(2^v - 1)/(2^w - 1) \mid (2^h + 1)$, *there exist subgroups* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^{v'} + l) + ld_2(2^{v'} - l^2) - d_2 2^{v'} (l - 2^w)}{d_2 2^{v'+w+1}}$$

for each v' *with* $0 \le v' \le v$. *(cf.* [20, Cor. 3.4(i), Cor. 3.4(iii)]*).*

(vii) *Let* $p = 2$ *and* h *be odd. Let* $s \mid h$ *and* $0 \le k \le s$. *For each* $1 \le v \le h-1$, *such that* $v = s + k$, *and for each* $s \le w \le h-1$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_2 2^v (l - 2^w)}{d_2 2^{v+w+1}}$$

(cf. [2, Thm. 4]*).*

(viii) *Let $p = 2$ and h be even and such that 4 does not divide h. Let $s \mid h$ be odd and $0 \le k \le s$. For each $1 \le v \le h-1$, such that $v = 2s+k$, and for each $2s \le w \le h-1$, there exists a subgroup L of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^v + l) + ld_2(2^v - l^2) - d_2 2^v(l - 2^w)}{d_2 2^{v+w+1}}$$

(cf. [2, Thm. 5]*).*

(ix) *Let $p = 2$ and write $h = 2^e f$, with $e, f \in \mathbb{N}$ and $f \ge 3$ odd. For each divisor j of f, let k_j be the order of 2 in $(\mathbb{Z}/j\mathbb{Z})^*$ and $r_j = \frac{\Phi(j)}{k_j}$, where Φ is the Euler function. For each $1 \le w \le h-2$, such that $w = 2^e\left[1 + \sum_{\frac{j}{f} \ne 1} l_j k_j\right]$, with $0 \le l_j \le r_j$, there exists a subgroup L of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{l^5 + l^2 - (l^2 - l + 1)2^w(2^{w+1} + l) + ld_2(2^{w+1} - l^2) - d_2 2^{w+1}(l - 2^w)}{d_2 2^{2w+2}}$$

(cf. [2, Thm. 6]*).*

4.2. $L = \pi^{-1}(\bar{G})$, $\bar{G} < PGU(3,\ell)_{\bar{P}_\infty}$. Groups $L = \pi^{-1}(\bar{G})$ that have not already been considered in Section 4.1 are groups $\pi^{-1}(\bar{G})$ with $\bar{G}$ containing elements $[a,b,c]$ with $a^{(\ell^2-1)/\mu_1} \ne 1$ (again the notation of both [20] and [2] is used to describe elements in $PGU(3,\ell)_{\bar{P}_\infty}$). Let $|\bar{G}| = \bar{m}p^{v+w}$, with $\gcd(\bar{m}, p) = 1$, and $p^v = |\{[a,b,c] \in G \mid b = 0\}|$. Then it is easily seen that $|L| = mp^{v+w}$ with $m = (\ell^2 - \ell + 1)\bar{m}$, and

$$p^{v+w} = |L \cap \{\xi_{1,b,c}\}|, \qquad p^v = |L \cap \{\xi_{1,0,c}\}|.$$

Clearly, $L_\Lambda = \Lambda$ holds.

By Lemma 4.1, together with Corollary 3.3 and Lemma 4.3, it follows that

$$\begin{aligned} \ell^5 - 2\ell^3 + \ell^2 - 2 = \;& mp^{v+w}(2g_L - 2) + (\ell^2 - \ell + 1)(p^{v+w} - 1) + (\ell^3 - \ell^2 + \ell)(p^w - 1) \\ & + (\ell^2 - \ell)(\ell^3 + 1) + (\ell^2 - \ell + 1)\sum_{\bar{h} \in G, \bar{h} \ne id} |\{\bar{P} \in \bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P}) = \bar{P}\}|. \end{aligned}$$

In order to provide concrete values of genera g_L we are going to consider subgroups $\bar{G}$ containing elements $[a,b,c]$ with $a^{(\ell^2-1)/\mu_1} \ne 1$ that have been described in either [20] or [2].

Theorem 4.6. (i) *Let $p \ne 2$, $\bar{m}$ be an integer such that $\bar{m} \mid l^2 - 1$ and $3 \mid \bar{m}$. Let $d = \gcd(\bar{m}, l+1)$ and let $s := min\left\{r \ge 1 : p^r \equiv 1 \mod \frac{\bar{m}}{d}\right\}$. For each $0 \le w \le h$, such that $s \mid w$, there exists a subgroup L of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \frac{(l - p^w)(l + 1 - d)}{2\bar{m}p^w}$$

(cf. [20, Prop. 4.6]*).*

(ii) *Let $p \neq 2$ and $\bar{m} \mid l^2-1$ be such that $\bar{m}$ does not divide $l-1$, and $3 \mid \bar{m}$. Let $d = \gcd(\bar{m}, l+1)$. Let s be the order of p in $(\mathbb{Z}/\bar{m}\mathbb{Z})^*$, and let*

$$r = \begin{cases} \text{order of } p \text{ in } (\mathbb{Z}/\frac{\bar{m}}{2}\mathbb{Z})^*, & \bar{m} \text{ even} \\ s, & \bar{m} \text{ odd} \end{cases}.$$

For each $0 \leq v \leq h$, such that $v \mid 2h$, v does not divide h and $s \mid v$, and for each $\frac{v}{2} \leq w \leq h$ such that $r \mid w$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with

$$g_L = \frac{(l^2 - p^{v+w} - lp^w - dlp^v + dp^{v+w} + lp^v)}{2\bar{m}p^{v+w}}$$

(cf. [2, Thm. 2]*).*

5. Curves $\mathcal{X}/L$ with $\bar{L}$ subgroup of a group of type (B)

In this section we investigate subgroups L of $\mathrm{Aut}(\mathcal{X})$ with $L = \Sigma_1 \times \Sigma_2$, where Σ_1 is contained Γ, Σ_2 is a subgroup of Λ, and $\bar{L}$ is a subgroup of a group of type (B). To this end, we determine a subgroup Ω of Γ such that $\pi(\Omega)$ is contained in a group of type (B) and $\Omega \cap \Lambda = \{id\}$.

The construction of Ω requires some technical preliminaries, especially on the Singer groups of $PGU(3,\ell)$. As in [7], we use a representation of a Singer group up to conjugation in $GL(3,q^2)$. This will allow us to deal with diagonal matrices, thus avoiding more involved matrix computation.

By [7, Prop. 4.6] there exists a matrix A_1 in $GL(3,q^2) \setminus GL(3,\ell^2)$ such that $A_1^t D\sigma(A_1) = D_1$, with

$$D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then a matrix M is in $SU(3,\ell)$ if and only if $M_1 = A_1^{-1} M A_1$ is such that

$$M_1^t D_1 \sigma(M_1) = D_1, \quad \det(M_1) = 1. \tag{18}$$

Also, it is straightforward to check that the points of $\mathbb{P}^2(\mathbb{K})$ whose homogeneous coordinates are the columns of A_1 lie in $\mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2})$.

Now we construct a group Ω_1 of $GL(3,q^2)$, contained in the conjugate subgroup of $SU(3,\ell)$ by A_1. Let $\mu_2 = \gcd(\ell^2-\ell+1, 3)$. It is easily seen that either $\mu_2 = 3$ or $\mu_2 = 1$ according to whether $\ell \equiv 2 \pmod 3$ or not. Let $\Pi_{\frac{\ell^2-\ell+1}{\mu_2}}$ be the group of $(\frac{\ell^2-\ell+1}{\mu_2})$-th roots of unity. As $\gcd(\frac{\ell^2-\ell+1}{\mu_2}, \ell+1) = 1$, for each $\lambda \in \Pi_{\frac{\ell^2-\ell+1}{\mu_2}}$ there exists a unique $\tilde{\lambda} \in \Pi_{\frac{\ell^2-\ell+1}{\mu_2}}$ with $\tilde{\lambda}^{\ell+1} = \lambda^{-\ell}$. Let Ω_1 be the group generated by D_1 and

$$T = \begin{pmatrix} \tilde{w}w & 0 & 0 \\ 0 & \tilde{w}w^\ell & 0 \\ 0 & 0 & \tilde{w} \end{pmatrix},$$

where w is a primitive $(\frac{\ell^2-\ell+1}{\mu_2})$-th root of unity.

Let $\Theta =< T >$ and $\Upsilon_1 =< D_1 >$. It is straightforward to check that $\Omega_1 = \Theta \rtimes \Upsilon_1$. Also, every matrix M_1 in Ω_1 satisfies (18), and thus for each $M_1 \in \Omega_1$ the matrix $A_1 M_1 A_1^{-1}$ belongs to $SU(3,\ell)$.

For a matrix $M_1 \in \Omega_1$ let $\epsilon(M_1)$ be the projectivity in $PGL(4,\ell^2)$ defined by the 4×4 matrix obtained from $A_1 M_1 A_1^{-1}$ by adding $0,0,1,0$ as a third row and as a third column. Then $\epsilon : \Omega_1 \to \Gamma$ is an injective group homomorphism. Let $\Omega = \epsilon(\Omega_1)$.

Let $\bar{P}_i$ be the point of $\mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2})$ whose homogeneous coordinates are the elements in the i-th column of A_1. Then the subgroup $\pi(\epsilon(\Theta))$ is contained in the stabilizer of $\bar{P}_i$ in $PGU(3,\ell)$. Also, the group $\pi(\epsilon(\Upsilon_1))$ acts regularly on $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$.

We now prove that $\Omega \cap \Lambda = \{id\}$. Let $\alpha \in \Omega \cap \Lambda$. Since $\pi(\alpha)$ fixes every point in $\mathcal{H}$, $\alpha \in \epsilon(\Theta)$. Taking into account that every non-trivial element in $\Gamma \cap \Lambda$ has order 3, and that 3 does not divide the order of T, we obtain that $\alpha = 1$.

The following result then holds.

Lemma 5.1. *The subgroup generated by Ω and Λ is the direct product $\Omega \times \Lambda$. The projection $\pi(\Omega \times \Lambda)$ is a subgroup of a group of type* (B) *isomorphic to Ω_1.*

Let $\bar{P}_1 = (x_i, y_i, 1)$. As $\bar{P}_1 \in \mathcal{H}(\mathbb{F}_{q^2}) \setminus \mathcal{H}(\mathbb{F}_{\ell^2})$, up to a rearrangement of the indexes we can assume that $x_2 = x_1^{\ell^2}, y_2 = y_1^{\ell^2}$, and $x_3 = x_1^{\ell^4}, y_3 = y_1^{\ell^4}$.

For any point $Q_i = (x_i, y_i, z_0, 1)$ of $\mathcal{X}$ such that $\phi(Q_i) = \bar{P}_i$, the image $\epsilon(T)(Q_i)$ of Q_i by $\epsilon(T)$ is the point $(x_i, y_i, \frac{z_0}{a_{31}x_i + a_{32}y_i + a_{33}}, 1)$, where (a_{31}, a_{32}, a_{33}) is the third row of the matrix $A_1 T A_1^{-1}$. Let $s_i = a_{31}x_i + a_{32}y_i + a_{33}$. Since T has order $(\ell^2-\ell+1)/\mu_2$, we have that $s_i^{(\ell^2-\ell+1)/\mu_2} = 1$. Note that $s_1^{\ell^2} = s_2$ and $s_1^{\ell^4} = s_3$ hold.

Lemma 5.2. *Any s_i is a primitive $\frac{\ell^2-\ell+1}{\mu_2}$-th root of unity.*

Proof. It is enough to prove the assertion for $i = 1$. Assume that $s_1^m = 1$. Then also $s_2^m = 1$ and $s_3^m = 1$ hold. Therefore, $\epsilon(T)^m$ fixes every point Q such that $\phi(Q) = \bar{P}_i$ for some $i = 1, 2, 3$. Since $\ell^2 - \ell + 1 > 3$, the three lines joining P_∞ to the points $(x_i, y_i, 0, 1)$, $i = 1, 2, 3$, are fixed by $\epsilon(T)^m$ pointwise. As these three lines are not coplanar, $\epsilon(T)^m$ is the identical projectivity of $PG(3, \mathbb{K})$. This shows that $\ell^2 - \ell + 1$ divides m, and the proof is complete. □

Subgroups of the normalizers of a Singer group of $PGU(3,n)$ have been classified up to conjugacy in [32, Chapter 4], see also [8, Lemma 4.1]. As a straightforward consequence, the following result holds.

Lemma 5.3. *The following is a complete list of subgroups of Ω, up to conjugacy.*

(a) *For every divisor d of $(\ell^2 - \ell + 1)/\mu_2$, the cyclic subgroup of $\epsilon(\Theta)$ of order d, i.e. the subgroup generated by $\epsilon(T)^{(\ell^2-\ell+1)/d\mu_2}$.*

(b) *For every divisor d of $(\ell^2 - \ell + 1)/\mu_2$, the subgroup of order $3d$ which is the semidirect product of the cyclic subgroup of $\epsilon(\Theta)$ of order d with $\epsilon(\Upsilon_1)$.*

We deal separately with cases (a) and (b) of Lemma 5.3.

5.1. $L = \Sigma_1 \times \Sigma_2$**,** $\Sigma_1 < \epsilon(\Theta)$**,** $\Sigma_2 < \Lambda$**.** Let $\Sigma_1 =< \epsilon(T)^{i_1} >$, with $i_1 = (\ell^2 - \ell + 1)/d\mu_2$, and let Σ_2 be the group generated by the projectivity defined by the diagonal matrix $[1, 1, \beta^{i_2}, 1]$, with β a primitive $(\ell^2 - \ell + 1)$-th root of unity and i_2 a divisor of $\ell^2 - \ell + 1$. Let $d_2 = (\ell^2 - \ell + 1)/i_2$. Without loss of generality assume that $s_1 = \beta^{3^k}$, with $k = 0$ when $\mu_2 = 1$ and $k > 0$ for $\mu_2 = 3$. In order to compute integers $l_{\bar{h},\bar{P}}$ as in Proposition 3.1, we need to investigate the action of Σ_1 on the orbits of $\phi^{-1}(\bar{P}_i)$ under Σ_2. Fix $Q_1 = (x_1, y_1, z_0, 1) \in \phi^{-1}(\bar{P}_1)$. The orbits of $\phi^{-1}(\bar{P}_1)$ under the action of Σ_2 are

$$\Delta_j = \{(x_1, y_1, \beta^j z_0, 1), (x_1, y_1, \beta^{j+i_2} z_0, 1), \ldots, (x_1, y_1, \beta^{j+(d_2-1)i_2} z_0, 1)\},$$

with $j = 0, \ldots, i_2 - 1$. For $1 \le t \le d - 1$, the orbit Δ_j is fixed by $\epsilon(T)^{ti_1}$ if and only if

$$\beta^j s_1^{-ti_1} = \beta^{j+ui_2}, \quad \text{for some } u \in \mathbb{Z},$$

that is,

$$3^k ti_1 = v(\ell^2 - \ell + 1) - ui_2, \quad \text{for some } u, v \in \mathbb{Z}.$$

Equivalently,

$$i_2 | 3^k ti_1. \tag{19}$$

Similarly it can be proved that $\epsilon(T)^{ti_1}$ fixes any orbit of $\phi^{-1}(\bar{P}_2)$ (resp. $\phi^{-1}(\bar{P}_3)$) under Σ_2 if and only if $i_2 | 3^k \ell^2 ti_1$ (resp. $i_2 | 3^k \ell^4 ti_1$). Since $\gcd(i_2, \ell) = 1$, either $\epsilon(T)^{ti_1}$ fixes all the orbits of $\phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\})$ under Σ_2 or none, according to whether (19) holds or not.

If 3 does not divide i_2, then (19) is equivalent to $i_2 | ti_1$. Therefore, the number of non-trivial elements in Σ_1 fixing one (and hence every) orbit of $\phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\})$ under Σ_2 is equal to the number of common multiples of i_1 and i_2 that are strictly less than $(\ell^2 - \ell + 1)/\mu_2$. If $\mathrm{lcm}(i_1, i_2) \le (\ell^2 - \ell + 1)/\mu_2$, then this number is $\frac{\ell^2-\ell+1}{\mu_2 \mathrm{lcm}(i_1,i_2)} - 1$; if $\mathrm{lcm}(i_1, i_2) = \ell^2 - \ell + 1$, then no orbit is fixed by a non-trivial element in Σ_1.

If 3 divides i_2, then (19) is equivalent to $\frac{i_2}{3} | ti_1$. Arguing as in the previous case, it can be deduced that the number of non-trivial elements in Σ_1 fixing one (and hence every) orbit of $\phi^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\})$ under Σ_2 is $\frac{\ell^2-\ell+1}{\mu_2 \mathrm{lcm}(i_1,i_2/3)} - 1$.

The last term of e_L as in Proposition 3.1, can be written as

$$|L_\Lambda| n_2 = |\Sigma_2| \sum_{\bar{h} \in \bar{\Sigma}_1, \bar{h} \ne id} \; \sum_{\bar{P} \in \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}, \bar{h}(\bar{P}) = \bar{P}} l_{\bar{h},\bar{P}};$$

then, it is equal to

$$\begin{cases} 3(\ell^2 - \ell + 1)(\frac{\ell^2-\ell+1}{\mu_2 \mathrm{lcm}(i_1,i_2)} - 1) & \text{if } 3 \nmid i_2,\ \mathrm{lcm}(i_1, i_2) \le (\ell^2 - \ell + 1)/\mu_2, \\ 0 & \text{if } 3 \nmid i_2,\ \mathrm{lcm}(i_1, i_2) = \ell^2 - \ell + 1, \\ 3(\ell^2 - \ell + 1)(\frac{\ell^2-\ell+1}{\mu_2 \mathrm{lcm}(i_1,i_2/3)} - 1) & \text{if } 3 \mid i_2. \end{cases}$$

By (6) and Proposition 3.1 the following result holds.

Theorem 5.4. *Let $\mu_2 = \gcd(\ell^2 - \ell + 1, 3)$.*

- *For i_1 divisor of $(\ell^2 - \ell + 1)/\mu_2$, i_2 divisor of $\ell^2 - \ell + 1$ such that $3 \nmid i_2$ and $\mathrm{lcm}(i_1, i_2) \le (\ell^2 - \ell + 1)/\mu_2$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{2}\left((\ell+2)\mu_2 i_1 i_2 - (\ell+1)\mu_2 i_1 - \frac{3i_1i_2}{\mathrm{lcm}(i_1, i_2)}\right) + 1.$$
- *For i_1 divisor of $(\ell^2 - \ell + 1)/\mu_2$, i_2 divisor of $\ell^2 - \ell + 1$ such that $3 \nmid i_2$ and $\mathrm{lcm}(i_1, i_2) = (\ell^2 - \ell + 1)$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{2}\left((\ell+2)\mu_2 i_1 i_2 - (\ell+1)\mu_2 i_1 - \frac{3\mu_2 i_1i_2}{\ell^2 - \ell + 1}\right) + 1.$$
- *For i_1 divisor of $(\ell^2 - \ell + 1)/\mu_2$, i_2 divisor of $\ell^2 - \ell + 1$ such that $3 \mid i_2$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{2}\left((\ell+2)\mu_2 i_1 i_2 - (\ell+1)\mu_2 i_1 - \frac{3i_1i_2}{\mathrm{lcm}(i_1, i_2/3)}\right) + 1.$$

5.2. $L = (\Sigma_1 \rtimes \epsilon(\Upsilon_1)) \times \Sigma_2)$, $\Sigma_1 < \epsilon(\Theta)$, $\Sigma_2 < \Lambda$**.** Here we assume that $p \neq 3$. Let $\Sigma_1 =< \epsilon(T)^{i_1} >$, with $i_1 = (\ell^2 - \ell + 1)/d\mu_2$. Let Σ_2 be the group generated by the projectivity defined by the diagonal matrix $[1, 1, \beta^{i_2}, 1]$, with β a primitive $(\ell^2 - \ell + 1)$-th root of unity and i_2 a divisor of $\ell^2 - \ell + 1$; let $d_2 = (\ell^2 - \ell + 1)/i_2$.

The action of $\pi(L)$ on $\mathcal{H}$ is described in [8].

Lemma 5.5 (Proposition 4.2 in [8])**.** *If $\mu_2 = 1$, then the group $\pi(L)$ has 3 short orbits, namely $\{\bar P_1, \bar P_2, \bar P_3\}$ and 2 short orbits of size d consisting of $\mathbb{F}_{\ell^2}$-rational points of $\mathcal{H}$. If $\mu_2 = 3$, then the only short orbit of $\pi(L)$ is $\{\bar P_1, \bar P_2, \bar P_3\}$.*

As a consequence, the last term $|L_\Lambda|n_2$ of e_L as in Proposition 3.1 is just
$$|\Sigma_2| \sum_{\bar h \in \bar\Sigma_1, \bar h \neq id} \quad \sum_{\bar P \in \{\bar P_1, \bar P_2, \bar P_3\}, \bar h(\bar P) = \bar P} l_{\bar h, \bar P},$$
which has already been computed in Section 5.1. We will distinguish the cases $\mu_2 = 1$ and $\mu_2 = 3$.

5.2.1. $\mu_2 = 1$. Apart from $\{\bar P_1, \bar P_2, \bar P_3\}$, the group $\pi(L)$ has further 2 short orbits of size d consisting of $\mathbb{F}_{\ell^2}$-rational points of $\mathcal{H}$. By (6) and Proposition 3.1 the following result holds.

Theorem 5.6. *Let $p \neq 3$, $\gcd(\ell^2 - \ell + 1, 3) = 1$. For i_1, i_2 divisors of $(\ell^2 - \ell + 1)$ there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{3}\left(\frac{1}{2}\left((\ell+2)i_1i_2 - (\ell+1)i_1 - \frac{3i_1i_2}{\mathrm{lcm}(i_1, i_2)}\right) + 1\right).$$

5.2.2. $\mu_2 = 3$. The only short orbit of $\pi(L)$ is $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$. By (6) and Proposition 3.1 the following result holds.

Theorem 5.7. *Let* $\gcd(\ell^2 - \ell + 1, 3) = 3$.

- *For* i_1 *divisor of* $(\ell^2 - \ell + 1)/3$, i_2 *divisor of* $\ell^2 - \ell + 1$ *such that* $3 \nmid i_2$ *and* $\mathrm{lcm}(i_1, i_2) \leq (\ell^2 - \ell + 1)/3$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*
$$g_L = \frac{1}{2}\left((\ell+2)i_1 i_2 - (\ell+1)i_1 - \frac{i_1 i_2}{\mathrm{lcm}(i_1, i_2)}\right) + 1.$$
- *For* i_1 *divisor of* $(\ell^2 - \ell + 1)/3$, i_2 *divisor of* $\ell^2 - \ell + 1$ *such that* $3 \nmid i_2$ *and* $\mathrm{lcm}(i_1, i_2) = (\ell^2 - \ell + 1)$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*
$$g_L = \frac{1}{2}\left((\ell+2)i_1 i_2 - (\ell+1)i_1 - \frac{3 i_1 i_2}{\ell^2 - \ell + 1}\right) + 1.$$
- *For* i_1 *divisor of* $(\ell^2 - \ell + 1)/3$, i_2 *divisor of* $\ell^2 - \ell + 1$ *such that* $3 \mid i_2$, *there exists a subgroup* L *of* $\mathrm{Aut}(\mathcal{X})$ *with*
$$g_L = \frac{1}{2}\left((\ell+2)i_1 i_2 - (\ell+1)i_1 - \frac{i_1 i_2}{\mathrm{lcm}(i_1, i_2/3)}\right) + 1.$$

Remark 5.8. When $L = \pi^{-1}(\bar{G})$ with $\bar{G}$ a group of type (B), then by Corollary 3.4 the genus g_L coincides with $g_{\bar{L}}$. All the possibilities for $g_{\bar{L}}$ are determined in [8]. It should be noted that the statement of Proposition 4.2(3) in [8] contains a misprint, as $(q^2 - q + 1 - 3n)/6n$ should read $(q^2 - q + 1 + 3n)/6n$.

6. Curves $\mathcal{X}/L$ with $\bar{L}$ subgroup of a group of type (C)

In this section we investigate subgroups L of $\mathrm{Aut}(\mathcal{X})$ with $L = \Sigma_1 \times \Sigma_2$, where Σ_1 is contained Γ, Σ_2 is a subgroup of Λ, and $\bar{L}$ is a subgroup of a group of type (C). To this end, we determine a subgroup Ω of Γ such that $\pi(\Omega)$ is contained in a group of type (C) and $\Omega \cap \Lambda = \{id\}$. The construction of Ω requires some technical preliminaries. In particular, a group conjugate to $SU(3, \ell)$ in $GL(3, q^2)$ needs to be considered.

Let $b, c \in \mathbb{F}_{\ell^2}$ be such that $b^{\ell+1} = c^\ell + c = -1$, and let
$$A_2 = \begin{pmatrix} 0 & 0 & \frac{1}{b} \\ \frac{c+1}{b} & \frac{-c}{b^2} & 0 \\ \frac{-1}{b} & \frac{1}{b^2} & 0 \end{pmatrix}.$$
Then A_2 is a matrix in $GL(3, \ell^2)$ such that $A_2^t D \sigma(A_2) = I_3$. A matrix M is in $SU(3, \ell)$ if and only if $M_2 = A_2^{-1} M A_2$ is such that
$$M_2^t \sigma(M_2) = I_3, \quad \det(M_2) = 1. \tag{20}$$

Let μ_1 be the largest power of 3 dividing $\ell+1$, and let $\Pi_{\frac{\ell+1}{\mu_1}}$ be the group of $(\frac{\ell+1}{\mu_1})$-th roots of unity. As $\gcd(\frac{\ell+1}{\mu_1}, 3) = 1$, for each $\lambda \in \Pi_{\frac{\ell+1}{\mu_1}}$ there exists a unique $\tilde{\lambda} \in \Pi_{\frac{\ell+1}{\mu_1}}$ with $\tilde{\lambda}^3 = \lambda$. Let Ω_2 be the the group generated by the matrices

$$T_1 = \begin{pmatrix} \frac{w}{\tilde{w}} & 0 & 0 \\ 0 & \frac{1}{\tilde{w}} & 0 \\ 0 & 0 & \frac{1}{\tilde{w}} \end{pmatrix}, T_2 = \begin{pmatrix} \frac{1}{\tilde{w}} & 0 & 0 \\ 0 & \frac{w}{\tilde{w}} & 0 \\ 0 & 0 & \frac{1}{\tilde{w}} \end{pmatrix}, U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where w is a primitive $(\frac{\ell+1}{\mu_1})$-th root of unity. Let $\Theta_1 =< T_1 >$, $\Theta_2 =< T_2 >$, $\Upsilon_1 =< U_1 >$, $\Upsilon_2 =< U_2 >$, and $\Upsilon =< U_1, U_2 >$.

It is straightforward to check that every matrix in Ω_2 satisfies (20), and that $\Omega_2 = (\Theta_1 \times \Theta_2) \rtimes \Upsilon$. Moreover, Υ is isomorphic to Sym_3. For a matrix $M_2 \in \Omega_2$ let $\epsilon(M_2)$ be the projectivity in $PGL(4, \ell^2)$ defined by the 4×4 matrix obtained from $A_2 M_2 A_2^{-1}$ by adding $0, 0, 1, 0$ as a third row and as a third column. Then $\epsilon : \Omega_2 \to \Gamma$ is an injective group homomorphism. Let $\Omega = \epsilon(\Omega_2)$.

Let $\bar{P}_i$ be the point in $PG(2, \ell^2)$ whose homogeneous coordinates are the elements in the i-th column of A_2. Then the subgroup $\pi(\Omega)$ is contained in the stabilizer of the triangle $\bar{P}_1\bar{P}_2\bar{P}_3$ in $PGU(3, \ell)$.

We now prove that $\Omega \cap \Lambda = \{id\}$. Let $\alpha \in \Omega \cap \Lambda$. Since $\pi(\alpha)$ fixes pointwise the set of points in $\mathcal{H}$ belonging to the triangle $\bar{P}_1\bar{P}_2\bar{P}_3$ in $PGU(3, \ell)$, $\alpha \in \epsilon(\Theta_1 \times \Theta_2)$. Taking into account that every non-trivial element in $\Gamma \cap \Lambda$ has order 3, and that 3 does not divide the order of $\Theta_1 \times \Theta_2$ by construction, we obtain that $\alpha = 1$.

Therefore, the following lemma holds.

Lemma 6.1. *The subgroup generated by Ω and Λ is the direct product $\Omega \times \Lambda$. The projection $\pi(\Omega \times \Lambda)$ is a subgroup of a group of type* (C) *isomorphic to Ω_2.*

Note that the action of $\pi(\Omega)$ on $\mathcal{H}$ can be viewed as the action of the group of projectivities defined by the matrices in Ω_2 on the set of points of the plane curve with equation $X^{\ell+1} + Y^{\ell+1} + T^{\ell+1} = 0$.

For a divisor d of $(\ell+1)/\mu_1$ and for $i = 1, 2$, let $C_d^{(i)}$ be the subgroup of $\epsilon(\Theta_i)$ of order d. We consider subgroups Σ_1 of Ω of the following types:

(a) $C_{d_1}^{(1)} \times C_{d_2}^{(2)}$;
(b) the cyclic subgroup of order $(\ell+1)/(\mu_1 d_1)$ generated by $\epsilon(T_1)^{d_1}\epsilon(T_2)^{2d_1}$, with d_1 a divisor of $(\ell+1)/\mu_1$;
(c) $(C_{d_1}^{(1)} \times C_{d_1}^{(2)}) \rtimes \Upsilon_2$;
(d) $(C_{d_1}^{(1)} \times C_{d_1}^{(2)}) \rtimes \Upsilon_1$;
(e) $(C_{d_1}^{(1)} \times C_{d_1}^{(2)}) \rtimes \Upsilon$.

Cases (a)-(e) are dealt with separately.

6.1. $L = \Sigma_1 \times \Sigma_2$, **with** Σ_1 **as in (a),** $\Sigma_2 < \Lambda$, $|\Sigma_2| = d$**.** The action of $\pi(\Sigma_1)$ on $\mathcal{H}$ was investigated in [20, Example 5.11]. Any non-trivial element in $\pi(\Sigma_1)$ fixes no point in $\bar{\mathcal{O}}_2$. Moreover,

$$\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id} |\{\bar{P}\in\bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P})=\bar{P}\}| = (\ell+1)(d_1+d_2+\gcd(d_1,d_2)-3).$$

By (6) and Proposition 3.1 the following result holds.

Theorem 6.2. *Let μ_1 be the highest power of 3 which divides $\ell+1$. For any two divisors d_1, d_2 of $(\ell+1)/\mu_1$, and for any d divisor of $\ell^2-\ell+1$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*

$$g_L = 1 + \frac{(l^3+1)(l^2-d-1) - d(l+1)(d_1+d_2+\gcd(d_1,d_2)-3)}{2dd_1d_2}.$$

6.2. $L=\Sigma_1\times\Sigma_2$**, with** Σ_1 **as in (b),** $\Sigma_2<\Lambda$, $|\Sigma_2|=d$**.** By [20, Example 5.10] any non-trivial element in $\pi(\Sigma_1)$ fixes no point in $\bar{\mathcal{O}}_2$. Moreover,

$$g_{\bar{L}} = 1 + \frac{\mu_1 d_1 (l-2)(l+1)}{2(l+1)}$$

when $(\ell+1)/(\mu_1 d_1)$ is odd, whereas

$$g_{\bar{L}} = 1 + \frac{\mu_1 d_1 (l-3)(l+1)}{2(l+1)}$$

when $(\ell+1)/(\mu_1 d_1)$ is even. By Proposition 3.2 the following result holds.

Theorem 6.3. *Let μ_1 be the highest power of 3 which divides $\ell+1$.*

- *For any divisor d_1 of $(\ell+1)/\mu_1$ such that $(\ell+1)/(\mu_1 d_1)$ is odd, and for every divisor d of $\ell^2-\ell+1$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{2}\left(\mu_1 d_1 \left(\frac{\ell^2-\ell+1}{d}\right)(\ell^2-d-1)+2\right).$$
- *For any divisor d_1 of $(\ell+1)/\mu_1$ such that $(\ell+1)/(\mu_1 d_1)$ is even, and for every divisor d of $\ell^2-\ell+1$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*
$$g_L = \frac{1}{2}\left(\mu_1 d_1 \left(\frac{\ell^2-\ell+1}{d}\right)(\ell^2-d-1)-\mu_1 d_1+2\right).$$

6.3. $p\neq 2$**,** $L=\Sigma_1\times\Sigma_2$**, with** Σ_1 **as in (c),** $\Sigma_2<\Lambda$**,** $|\Sigma_2|=d$**.** As $C_{d_1}^{(1)}\times C_{d_1}^{(2)}$ is a normal subgroup of Σ_1, the subgroup $\pi(\Upsilon_2)$ acts on the quotient curve $\mathcal{H}/\pi(C_{d_1}^{(1)}\times C_{d_1}^{(1)})$. Such action is equivalent to the action of the involutory projectivity defined by U_2 on the plane curve $\mathcal{E}$ with equation $X^{\frac{\ell+1}{d_1}}+Y^{\frac{\ell+1}{d_1}}+T^{\frac{\ell+1}{d_1}}=0$. The fixed points of U_2 on $\mathcal{E}$ are the points on the line $X=T$, together with $(-1,0,1)$ if $(\ell+1)/d_1$ is odd. It is straightforward to check that any point of $\mathcal{H}$ lying over one of those fixed points of $\mathcal{X}$ is $\mathbb{F}_\ell$-rational. As $\mathcal{E}$ is non-singular, the genus $\bar{g}$ of $\mathcal{E}/U_2$ is given by

$$\left(\frac{\ell+1}{d_1}-1\right)\left(\frac{\ell+1}{d_1}-2\right)-2 = 2(2\bar{g}-2)+2\lceil(\ell+1)/(2d_1)\rceil,$$

that is

$$\bar{g} = \frac{((\ell+1)/(d_1)-2)^2}{4}, \quad \text{if } (\ell+1)/d_1 \text{ is even,} \tag{21}$$

and

$$\bar{g} = \frac{((\ell+1)/(d_1)-3)((\ell+1)/(d_1)-1)}{4}, \quad \text{if } (\ell+1)/d_1 \text{ is odd.} \tag{22}$$

Note that no point in $\bar{\mathcal{O}}_2$ is fixed by a non trivial element in $\pi(\Sigma_1)$. Also, since $\mathcal{E}/U_2$ is isomorphic to $\mathcal{H}/\pi(\Sigma_1)$, from Hurwitz's genus formula it follows

$$\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id} |\{\bar{P}\in\bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P})=\bar{P}\}| = \ell^2-\ell-2-2d_1^2(2\bar{g}-2).$$

By (6) and Proposition 3.2 the following result holds.

Theorem 6.4. *Assume that $p\neq 2$. Let μ_1 be the highest power of 3 dividing $\ell+1$. For any divisor d_1 of $(\ell+1)/\mu_1$, and for every divisor d of $\ell^2-\ell+1$, there exists a subgroup L of $\mathrm{Aut}(\mathcal{X})$ with*

$$g_L = \bar{g} + \frac{(\ell+1)(\ell^2-1)}{4d_1^2}\left(\frac{\ell^2-\ell+1}{d}-1\right),$$

with $\bar{g}$ as in (21) for $(\ell+1)/d_1$ even, and with $\bar{g}$ as in (22) for $(\ell+1)/d_1$ odd.

6.4. $p\neq 3$**,** $L=\Sigma_1\times\Sigma_2$**, with** Σ_1 **as in (d),** $\Sigma_2<\Lambda$**,** $|\Sigma_2|=d$**.** As $C_{d_1}^{(1)}\times C_{d_1}^{(2)}$ is a normal subgroup of Σ_1, the subgroup $\pi(\Upsilon_1)$ acts on the quotient curve $\mathcal{H}/\pi(C_{d_1}^{(1)}\times C_{d_1}^{(2)})$. Such action is equivalent to the action of the projectivity defined by U_1 on the plane curve $\mathcal{E}$ with equation $X^{\frac{\ell+1}{d_1}}+Y^{\frac{\ell+1}{d_1}}+T^{\frac{\ell+1}{d_1}}=0$. Let f be a primitive third root of unity in $\mathbb{F}_{\ell^2}$. If 3 does not divide $(\ell+1)/d_1$, then the fixed points of U_1 on $\mathcal{E}$ are precisely $(f,f^2,1)$ and $(f^2,f,1)$; otherwise no point on $\mathcal{E}$ is fixed by U_1. Arguing as in Section 6.3 we get that

$$\sum_{\bar{h}\in\pi(\Sigma_1),\bar{h}\neq id} |\{\bar{P}\in\bar{\mathcal{O}}_1 \mid \bar{h}(\bar{P})=\bar{P}\}| = \ell^2-\ell-2-3d_1^2(2\bar{g}-2),$$

where

$$\bar{g} = \frac{((\ell+1)/(d_1)-1)((\ell+1)/(d_1)-2)}{6}, \quad \text{if } 3\nmid(\ell+1)/d_1, \tag{23}$$

and

$$\bar{g} = \frac{((\ell+1)/(d_1)-1)((\ell+1)/(d_1)-2)+4}{6}, \quad \text{if } 3\mid(\ell+1)/d_1. \tag{24}$$

By (6) and Proposition 3.2 the following result holds.

Theorem 6.5. *Assume that $p \neq 3$. Let μ_1 be the highest power of 3 dividing $\ell+1$. For any divisor d_1 of $(\ell+1)/\mu_1$, and for every divisor d of $\ell^2-\ell+1$, there exists a subgroup L of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \bar{g} + \frac{(\ell+1)(\ell^2-1)}{6d_1^2}\left(\frac{\ell^2-\ell+1}{d}-1\right),$$

with $\bar{g}$ as in (23) *when* 3 *does not divide* $(\ell+1)/d_1$, *and with $\bar{g}$ as in* (24) *if* $3 \mid (\ell+1)/d_1$.

6.5. $p \neq 2,3$, $L = \Sigma_1 \times \Sigma_2$, **with** Σ_1 **as in (e),** $\Sigma_2 < \Lambda$, $|\Sigma_2| = d$**.** As $C_{d_1}^{(1)} \times C_{d_1}^{(2)}$ is a normal subgroup of Σ_1, the subgroup $\pi(\Upsilon)$ acts on the quotient curve $\mathcal{H}/\pi(C_{d_1}^{(1)} \times C_{d_1}^{(2)})$. Such action is equivalent to that of all permutations of the coordinates (X,Y,T) on the plane curve $\mathcal{E}$ with equation $X^{\frac{\ell+1}{d_1}} + Y^{\frac{\ell+1}{d_1}} + T^{\frac{\ell+1}{d_1}} = 0$. It is straightforward to check that:

- $(X,Y,T) \mapsto (T,Y,X)$ fixes the points on the line $X = T$, together with $(-1,0,1)$ if $(\ell+1)/d_1$ is odd;
- $(X,Y,T) \mapsto (X,T,Y)$ fixes the points on the line $Y = T$, together with $(0,-1,1)$ if $(\ell+1)/d_1$ is odd;
- $(X,Y,T) \mapsto (Y,X,T)$ fixes the points on the line $X = Y$, together with $(-1,1,0)$ if $(\ell+1)/d_1$ is odd;
- $(X,Y,T) \mapsto (Y,T,X)$ fixes the points $(f,f^2,1)$ and $(f^2,f,1)$ if 3 does not divide $(\ell+1)/d_1$, and fixes no point if $3 \mid (\ell+1)/d_1$;
- $(X,Y,T) \mapsto (T,X,Y)$ fixes the points $(f,f^2,1)$ and $(f^2,f,1)$ if 3 does not divide $(\ell+1)/d_1$, and fixes no point if $3 \mid (\ell+1)/d_1$.

Therefore, for the genus $\bar{g}$ of the quotient curve of $\mathcal{H}/\pi(C_{d_1}^{(1)} \times C_{d_1}^{(1)})$ with respect to $\pi(\Upsilon)$ it turns out that

$$\bar{g} = \frac{1}{12}\left(\left(\frac{\ell+1}{d_1}\right)^2 - 6\frac{\ell+1}{d_1} + o\right), \tag{25}$$

where

$$o = \begin{cases} 12 \text{ if } 3 \mid (\ell+1)/d_1, 2 \mid (\ell+1)/d_1, \\ 9 \text{ if } 3 \mid (\ell+1)/d_1, 2 \nmid (\ell+1)/d_1, \\ 8 \text{ if } 3 \nmid (\ell+1)/d_1, 2 \mid (\ell+1)/d_1, \\ 5 \text{ if } 3 \nmid (\ell+1)/d_1, 2 \nmid (\ell+1)/d_1 \end{cases}.$$

Arguing as in the preceedings Sections, we obtain the following result as a consequence of (6) and Proposition 3.2.

Theorem 6.6. *Assume that $p \neq 3$. Let μ_1 be the highest power of 3 dividing $\ell+1$. For any divisor d_1 of $(\ell+1)/\mu_1$, and for every divisor d of $\ell^2-\ell+1$, there exists a* *subgroup L of* $\mathrm{Aut}(\mathcal{X})$ *with*

$$g_L = \bar{g} + \frac{(\ell+1)(\ell^2-1)}{12d_1^2}\left(\frac{\ell^2-\ell+1}{d} - 1\right),$$

with $\bar{g}$ as in (25).

Remark 6.7. When $L = \pi^{-1}(\bar{G})$ with $\bar{G}$ a group of type (C), then by Corollary 3.4 the genus g_L coincides with $g_{\bar{L}}$. Some possibilities for $g_{\bar{L}}$ are determined in [20] when either $\bar{G}$ is isomorphic to Σ_1 as in cases (a)-(b), or $\bar{G}$ is isomorphic to Υ. It should be noted that other possibilities for $g_{\bar{L}}$ are computed here, namely the integers $\bar{g}$ as in (21),(22),(23),(24),(25). To our knowledge these integers provide genera of quotient curves of the Hermitian curve that have not appeared in the literature so far.

7. Curves $\mathcal{X}/L$ with $\bar{L}$ subgroup of a group of type (D)

In this section we will consider the case where $\bar{L}$ is contained in one of the following subgroups $\bar{G}_1$ and $\bar{G}_2$ of $PGU(3,\ell)$ stabilizing the line with equation $X = 0$.

- Let Ψ_1 be the subgroup of $GL(3,\ell^2)$ consisting of matrices

$$M_{a_1,a_2,a_3,a_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

  with $a_1^\ell = a_1$, $a_4^\ell = a_4$, $a_3^\ell = -a_3$, $a_2^\ell = -a_2$, $a_1a_4 - a_2a_3 = 1$. Let $\bar{G}_1$ be the subgroup of $PGL(3,\ell^2)$ of the projectivities defined by the matrices in Ψ_1. Clearly $\bar{G}_1$ is isomorphic to Ψ_1. It is straightforward to check that Ψ_1 is a subgroup of $SU(3,\ell)$; in particular, $\bar{G}_1$ is a subgroup of $PGU(3,\ell)$ preserving the line $X = 0$. Also, it is easily seen that the map

$$M_{a_1,a_2,a_3,a_4} \mapsto \begin{pmatrix} a_1 & \lambda a_2 \\ a_3\lambda^{-1} & a_4 \end{pmatrix}$$

  with $\lambda^{\ell-1} = -1$, defines a isomorphism of Ψ_1 and $SL(2,\ell)$, and the action of $\bar{G}_1$ on the points of $\mathcal{H}$ on the line $X = 0$ is isomorphic to the action of $SL(2,\ell)$ on the projective line $PG(1,\ell)$.
- Let $\bar{G}_2$ be the subgroup of $GL(3,\ell^2)$ generated by the projectivities defined by the matrices

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_a = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{\ell+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  with a ranging over the set of non-zero elements in $\mathbb{F}_{\ell^2}$. It is easily seen that $\bar{G}_2$ is a subgroup of $PGU(3,\ell)$ preserving the line with equation $X = 0$, and that $\bar{G}_2$ is isomorphic to the diedral group of order $2(\ell^2-1)$.

7.1. $\bar{L}$ **subgroup of** $\bar{G}_1$. Let Ω be the subgroup of $PGL(4,\ell^2)$ consisting of matrices $\tilde{M}_{a_1,a_2,a_3,a_4}$, with $M_{a_1,a_2,a_3,a_4} \in \Psi_1$. As Ψ_1 is contained in $SU(3,\ell)$, we have that Ω is actually a subgroup of Γ. It is straightforward to check that $\Omega \cap \Lambda$ is trivial. Also, $\bar{G}_1 = \pi(\Omega)$ clearly holds.

Lemma 7.1. *Any non-trivial element in $\bar{G}_1$ fixes no point of $\mathcal{H}$ outside the line with equation $X = 0$.*

Proof. Assume that $M_{a_1,a_2,a_3,a_4}(x_1,y_1,t_1)^t = \varrho(x_1,y_1,t_1)^t$ for some $\varrho \in \mathbb{K}$, $\varrho \neq 0$, and for some $(x_1,y_1,t_1) \in \mathbb{K}^3$ with $x_1 \neq 0$ and $x_1^{\ell+1} = y_1^{\ell} t_1 + y_1 t_1^{\ell}$. Clearly this can only happen for $\varrho = 1$ and $y_1 t_1 \neq 0$, whence we can assume that $t_1 = 1$ and that

$$(a_1-1)y_1 + a_2 = 0, \qquad a_3 y_1 + (a_4-1) = 0.$$

Taking ℓ-th powers in both equalities we then have

$$(a_1-1)y_1^{\ell} - a_2 = 0, \qquad -a_3 y_1^{\ell} + (a_4-1) = 0.$$

It is then straightforward to deduce that $a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, that is $M_{a_1,a_2,a_3,a_4} = I_3$. □

Assume that L is a tame subgroup of $\mathrm{Aut}(\mathcal{X})$ with $L = \Sigma_1 \times \Sigma_2$, where $\Sigma_1 < \Omega$ and $\Sigma_2 < \Lambda$. Let $d = |\Sigma_2|$. By Proposition 3.2 we then have that

$$g_L = g_{\bar{L}} + \frac{(\ell^3+1)(\ell^2-d-1) - d(\ell^2-d-2)}{2d|\Sigma_1|} \tag{26}$$

where $g_{\bar{L}}$ is the genus of the quotient curve $\mathcal{H}/\pi(\Sigma_1)$.

In [8] a number of genera of quotient curves $\mathcal{H}/\pi(\Sigma_1)$ with Σ_1 a subgroup of Ω are computed. The following results are then straightforward consequences of (26) together with Proposition 3.3 in [8].

Theorem 7.2. *Assume that $p = 2$.*

Let d be any divisor of $\ell^2-\ell+1$. Then there exist subgroups L of $\mathrm{Aut}(\mathcal{X})$ with the following properties.

- *For any $e \mid \ell+1$, $|L| = de$ and*

$$g_L = \frac{1}{2e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)\right) + 1.$$

- *For any $e \mid \ell-1$, $|L| = de$ and*

$$g_L = \frac{1}{2e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1) - 2(e-1)\right) + 1.$$

Theorem 7.3. *Assume that $p \neq 2$. Let d be any divisor of $\ell^2-\ell+1$. Then there exist subgroups L of $\mathrm{Aut}(\mathcal{X})$ with the following properties.*

- *For any divisor e of $(\ell+1)/2$, $|L|=2de$ and*
$$g_L=\frac{1}{4e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)\right)+1.$$
- *For any divisor e of $(\ell+1)/2$, $|L|=4de$ and*
$$g_L=\frac{1}{8e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-2o\right)+1,$$
with
$$o=\begin{cases}2e & \text{if } \ell\equiv 1 \pmod 4\\ 0 & \text{if } \ell\equiv 3 \pmod 4\end{cases}.$$
- *For any divisor e or $(\ell-1)/2$, $|L|=2de$ and*
$$g_L=\frac{1}{4e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-4e+4\right)+1.$$
- *For any divisor e or $(\ell-1)/2$, $|L|=4de$ and*
$$g_L=\frac{1}{8e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-2o\right)+1$$
with
$$o=\begin{cases}4e-2 & \text{if } \ell\equiv 1 \pmod 4\\ 2e-2 & \text{if } \ell\equiv 3 \pmod 4\end{cases}.$$
- *When $p\geq 5$, $\ell^2\equiv 1 \pmod{16}$, $|L|=48d$ and*
$$g_L=\frac{1}{96}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-2o\right)+1,$$
with
$$o=\begin{cases}46 & \text{if } \ell\equiv 1 \pmod 4 \text{ and } \ell\equiv 1 \pmod 3\\ 30 & \text{if } \ell\equiv 1 \pmod 4 \text{ and } \ell\equiv 2 \pmod 3\\ 16 & \text{if } \ell\equiv 3 \pmod 4 \text{ and } \ell\equiv 1 \pmod 3\\ 0 & \text{if } \ell\equiv 3 \pmod 4 \text{ and } \ell\equiv 2 \pmod 3\end{cases}.$$
- *When $p\geq 5$, $|L|=24d$ and*
$$g_L=\frac{1}{48}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-2o\right)+1,$$
with
$$o=\begin{cases}22 & \text{if } \ell\equiv 1 \pmod 4 \text{ and } \ell\equiv 1 \pmod 3\\ 6 & \text{if } \ell\equiv 1 \pmod 4 \text{ and } \ell\equiv 2 \pmod 3\\ 16 & \text{if } \ell\equiv 3 \pmod 4 \text{ and } \ell\equiv 1 \pmod 3\\ 0 & \text{if } \ell\equiv 3 \pmod 4 \text{ and } \ell\equiv 2 \pmod 3\end{cases}.$$

- *When* $p \geq 7$, $\ell^2 \equiv 1 \pmod 5$, $|L| = 120d$ *and*
$$g_L = \frac{1}{240}\left(\frac{\ell^3+1}{d}(\ell^2-d-1)-(\ell+1)-2o\right)+1$$
with
$$o = \begin{cases} 118 & \textit{if } \ell \equiv 1 \pmod 3,\ \ell \equiv 1 \pmod 4 \textit{ and } \ell \equiv 1 \pmod 5 \\ 78 & \textit{if } \ell \equiv 2 \pmod 3,\ \ell \equiv 1 \pmod 4 \textit{ and } \ell \equiv 1 \pmod 5 \\ 78 & \textit{if } \ell \equiv 1 \pmod 3,\ \ell \equiv 3 \pmod 4 \textit{ and } \ell \equiv 1 \pmod 5 \\ 48 & \textit{if } \ell \equiv 2 \pmod 3,\ \ell \equiv 3 \pmod 4 \textit{ and } \ell \equiv 1 \pmod 5 \\ 70 & \textit{if } \ell \equiv 1 \pmod 3,\ \ell \equiv 1 \pmod 4 \textit{ and } \ell \equiv 4 \pmod 5 \\ 30 & \textit{if } \ell \equiv 2 \pmod 3,\ \ell \equiv 1 \pmod 4 \textit{ and } \ell \equiv 4 \pmod 5 \\ 40 & \textit{if } \ell \equiv 1 \pmod 3,\ \ell \equiv 3 \pmod 4 \textit{ and } \ell \equiv 4 \pmod 5 \\ 0 & \textit{if } \ell \equiv 2 \pmod 3,\ \ell \equiv 3 \pmod 4 \textit{ and } \ell \equiv 4 \pmod 5 \end{cases}.$$

Remark 7.4. It should be noted that by the discussion in [8, Sect. 3], Theorems 7.2 and 7.3 describe genera g_L for all tame subgroups $L = \Sigma_1 \times \Sigma_2$ such that Σ_1 is a subgroup of Ω containing the diagonal matrix $J = [1,-1,1,-1]$, and $\Sigma_2 < \Lambda$.

7.2. $\bar{L}$ **subgroup of** $\bar{G}_2$**.** We determine a subgroup Ω of Γ such that $\pi(\Omega)$ is contained in $\bar{G}_2$ and $\Omega \cap \Lambda = \{id\}$ holds. Let μ_1 be the maximum power of 3 dividing $\ell+1$, and let γ be a primitive $((\ell^2-1)/\mu_1)$-th root of unity in $\mathbb{F}_{\ell^2}$. Let Ω the group generated by the projectivities defined by the matrices $V_\gamma = [\gamma^{2-\ell}, \gamma^{\ell+1}, \gamma, 1]$ and
$$\bar{W} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Lemma 7.5. *The group* $\pi(\Omega)$ *is contained in* $\bar{G}_2$*.*

Proof. The image by π of the projectivity defined by V_γ is associated to the matrix
$$\begin{pmatrix} \gamma^{2-\ell} & 0 & 0 \\ 0 & \gamma^{\ell+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
This is the matrix M_a for $a = \gamma^{2-\ell}$, as $(\gamma^{2-\ell})^{\ell+1} = \gamma^{-(\ell^2-1-\ell-1)} = \gamma^{\ell+1}$.

The image by π of the projectivity defined by $\bar{W}$ is associated to the matrix
$$W' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
Since for $a = -1$ we have $M_a W' = W$, this projectivity belong to $\bar{G}_2$, and the proof is complete. □

Lemma 7.6. *The intersection $\Omega \cap \Lambda$ is trivial.*

Proof. Any element α in $\Omega \cap \Lambda$ is defined by a diagonal matrix of Ω, that is $[\gamma^{i(2-\ell)}, \gamma^{i(\ell+1)}, \gamma^i, 1]$ for some i. Moreover, $\gamma^{i(\ell^2-\ell+1)} = 1$. Since $\gamma^{i(\ell^2-1)/\mu_1} = 1$ and $\gcd(\ell^2-\ell+1, (\ell^2-1)/\mu_1) = 1$, this can only happen $\gamma^i = 1$. Then the assertion is proved. $\square$

By the above lemmas, the map π defines an isomorphism of Ω onto the subgroup of $\bar{G}_2$ generated by the projectivities defined by W and M_a, with a an $((\ell^2-1)/\mu_1)$-th root of unity. A number of genera of quotient curves $\mathcal{H}/\pi(\Sigma_1)$ for Σ_1 tame subgroup of $\pi(\Omega)$ have been established in [20, Thm. 5.4, Ex. 5.6]. Taking into accout Proposition 3.2, we are in a position to compute genera g_L for tame subgroups $L = \Sigma_1 \times \Sigma_2$ with $\Sigma_1 < \Omega$ and $\Sigma_2 < \Lambda$.

Theorem 7.7. *Assume that $p \neq 2$. Let μ_1 be the highest power of 3 dividing $\ell+1$. Let d be any divisor of $\ell^2-\ell+1$. Then there exist subgroups L of $\mathrm{Aut}(\mathcal{X})$ with the following properties.*

- *For any divisor e of $(\ell^2-1)/\mu_1$, $|L| = 2ed$ and*
$$g_L = \frac{1}{4e}\left(\frac{\ell^3+1}{d}(\ell^2-d-1) + (\ell+1)(1-u-\tilde{u}) + 2(e+u) - \delta\right),$$
where $u = \gcd(e, \ell+1)$, $\tilde{u} = \gcd(e, \ell-1)$,
$$\delta = \begin{cases} 0 & \text{if } e \text{ divides } (\ell^2-1)/2 \\ e & \text{otherwise} \end{cases}.$$
- *When $\ell \equiv 1 \pmod 4$, for any even divisor e of $\ell-1$, $|L| = 2ed$ and*
$$g_L = \frac{1}{4e}\left(\frac{(\ell^3+1)}{d}(\ell^2-d-1) - \ell + 3\right).$$
- *When $\ell \equiv 3 \pmod 4$, for any even divisor e of $\ell-1$, $|L| = 2ed$ and*
$$g_L = \frac{1}{4e}\left(\frac{(\ell^3+1)}{d}(\ell^2-d-1) - \ell + 2e + 3\right).$$
- *When $\ell \equiv 1 \pmod 4$, for any odd divisor e of $\ell-1$, $|L| = 2ed$ and*
$$g_L = \frac{1}{4e}\left(\frac{(\ell^3+1)}{d}(\ell^2-d-1) + 2\right).$$
- *When $\ell \equiv 3 \pmod 4$, for any odd divisor e of $\ell-1$, $|L| = 2ed$ and*
$$g_L = \frac{1}{4e}\left(\frac{(\ell^3+1)}{d}(\ell^2-d-1) + 2e + 2\right).$$

Remark 7.8. When $L = \pi^{-1}(\bar{G})$ with $\bar{G}$ a group of type (D), then by Corollary 3.4 the genus g_L coincides with $g_{\bar{L}}$. As already mentioned in the present section, a number of possibilities for $g_{\bar{L}}$ are determined in [8, Prop. 3.3], and in [20, Thm. 5.4].

8. The case $\ell = 5$

To exemplify how the results of this paper provide many new genera for $\mathbb{F}_{q^2}$-maximal curves we consider in this section the case $q = 5^3$. Up to our knowledge, the 32 integers listed in the following table are new values in the spectrum of genera of $\mathbb{F}_{5^6}$-maximal curves.

New genera of $\mathbb{F}_{5^6}$-maximal curves

g	Ref.	g	Ref.
5	7.7	9	4.5(i), 7.7
14	4.5(ii)	21	4.5(ii)
22	5.4, 5.7	25	6.6
27	7.7	37	6.2, 6.5, 7.3, 7.7
38	4.5(i), 4.5(ii), 7.3, 7.7	70	4.5(ii), 5.7
73	6.6	74	7.3
76	4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7	77	7.7
86	4.5(iv)	109	6.2, 7.7
121	6.5, 7.3	140	4.5(ii)
148	5.7, 6.5	180	6.4, 7.7
208	5.4	220	4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7
221	7.7	241	6.6
242	7.3	282	4.5(iv)
361	6.2, 7.7	362	4.5(i), 4.5(ii), 7.3, 7.7
442	4.5(iv), 5.4, 6.2, 6.3	484	5.7, 6.5
724	4.5(i), 4.5(ii), 6.2, 6.3, 6.4, 7.3, 7.7	725	7.7

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