Lin_Masters

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Abstract

1 Introduction

Motivation

Applications

Applications of Hawkes Processes can be found in a wide variety of fields such as seismology, criminology, insurance, finance, social media, and neuroscience.

In seismology, an event can be an earthquake occurrence that causes aftershocks. In criminology, an event can be a gang rivalry that triggers retaliations following the gang crime. In insurance, an event can be a standard claim that increases claims. In finance, an event can be a transcation that influences future prices or volumes of transcations or a news that leads to movements in stock prices or trading behaviors. In social media, an event can be a tweet about an event on Twitter that follows a cascade of retweets from other users on the same social networking platform. In neuroscience, an event can be firing of a neuron that triggers spikes (or action potentials) of other neurons.

Objectives

The objectives of this project is to..., which include

1. defining and discussing properties of counting process, (homogeneous) Poisson process, nonhomogeneous Poisson process, cluster process, Hawkes process, and spatio-temporal Hawkes process,
2. discussing the algorithms and simulating all of the above processes,
3. extending (temporal) Hawkes process to spatio-temporal Hawkes process, and
4. discussing recent and future work of Hawkes (self-exciting) process.

2 Definitions and Properties

2.1 Counting Process

SAY SOMETHING HERE

For example,

Definition 2.1.1 (Stochastic Process) A stochastic process is a family of random variables indexed by time t and is defined as

$$\{X(t), t \in T\}$$

Definition 2.1.2 (Counting Process) Let *N(t)* be the total number of events up to some time t such that the values are nonnegative, interger and nondecreasing, a stocastic process is said to be a counting process and is defined as

$$\{N(t), t \geq 0\}$$

Definition 2.1.3 (Point Process) Let $\{T_i, i \in N\}$ be a sequence of non-negative random variables such that $T_i < T_{i+1}$ $\forall i \in N$, a point process on R^+ is defined as

$$\{T_i, i \in N\}$$

Definition 2.1.4 (Counting Process) Let $\{T_i, i \in N\}$ be a point process, a counting process associated with $\{T_i, i \in N\}$ is defined as

$$N(t) = \sum_{i \in N} I_{\{T_i \leq t\}}$$

Corollary **2.1.1** A counting process satisfies that

1. $N(t) \geq 0$
2. $N(t)$ is an integer
3. If $s \leq t$, then $N(s) \leq N(t)$
4. If $s < t$, then $N(t) - N(s)$ is the number of events occur in the interval (s, t]

Proposition 2.1.1 A counting process has the following properties

1. (Independence) If... , a counting process is said to have independent increments.

e.g. Poisson process, nonhomogeneous Poisson process

2. (Stationarity) If... , a counting process is said to have stationary increments.

e.g. Poisson process

3. (Homogeneity)

Stationarity implies homogeneity.

2.2 Poisson Process

Poisson (point) process is the simplest form of point process.

For example, if we were to model..., we can consider modelling using Poisson process.

Definition 2.2.1 (Poisson Process) If the following conditions hold, a counting process $\{N(t), t \geq 0\}$ is said to be a Poisson Process with constant rate (or intensity) $\lambda > 0$

1. $N(0) = 0$
2. $N(t)$ has independent increments
3. $P(N(t+h)) - N(t) = 1)) = \lambda h + o(h)$
4. $P(N(t+h)) - N(t) \geq 2)) = o(h)$

In other words, 1. the process starts at $t = 0$, 2. the increments are independent, 3. λ is the rate (or intensity), and 4. no 2 or more events can occur at the same location.

Proposition 2.2.1 Poisson process has the following properties

1. $N(t)$, the number of events in any interval t, $\sim Pos(\lambda t)$. That is, for all $s, t \geq 0$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$n = 0, 1, ...$

2. W, the interarrival times, $\overset{iid}{\sim} exp(\frac{1}{\lambda})$. That is, for rate $\lambda > 0$, the interarrival time W_i $i = 1, 2, ...$

$$P(W_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

This is because p(1^{st} arrivial arrives after time t) is the same as p(no arrivial in the interval [0,t]),

$$F(t) = P(W_1 \leq t) = 1 - e^{-\lambda t}$$

and

$$f(t) = \lambda e^{-\lambda t}.$$

Similarly,

$$P(W_2 > t | W_1 = s) = P(N(t+s) - N(s)) = P(N(t) = 0) = e^{-\lambda t}.$$

2.3 Nonhomogeneous Poisson Process

Since the distribution of the number of events $N(t)$ that occur in any interval of time t depends only on the length of the time interval but not the location of the interval, (homogeneous) Poisson process has stationary increments. In contrast, nonhomogeneous Poisson process does not have stationary increments since the distribution of $N(t)$ can change when shifted in t.

For example, if we were to model. . . , we can consider modelling using nonhomogeneous Poisson process.

Definition 2.3.1 (Nonhomogeneous Poisson Process) If the following conditions hold, a counting process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson Process with intensity function of time $\lambda(t), t > 0$

1. $N(0) = 0$
2. $N(t)$ has independent increments
3. $P(N(t+h)) - N(t) = 1)) = \lambda(t)h + o(h)$
4. $P(N(t+h)) - N(t) \geq 2)) = o(h)$

Proposition 2.2.1 Nonhomogeneous Poisson process has the following properties

1. $N(t)$, the number of events in any interval t, $\sim Pos(\int_0^t \lambda(s)ds)$. That is, for all $s, t \geq 0$

$$P(N(t) = n) = \frac{(\int_0^t \lambda(s)ds)^n e^{\int_0^t \lambda(s)ds}}{n!}$$

or

$$P(N(b) - N(a) = n) = \frac{(\int_a^b \lambda(s)ds)^n e^{\int_a^b \lambda(s)ds}}{n!}$$

$n = 0, 1, ...$

2.4 Cluster Process

In previous cases, whether events arrive at a constant rate λ (e.g. Poisson process) or depend on an intensity function $\lambda(t)$ (e.g. nonhomogeneous Poisson Process), they arrive independently. Here, we want to discuss cases in which events do not arrive independently.

Examples of cluster process include Cox process (or doubly stochastic Poisson process), Matérn I and Matérn II process.

2.5 Hawkes Process

Hawkes Process is also known as a self-exciting point processes.

In the case of Hawkes process, events also do not arrive independently. The arrivial rate of the events depends not only on time t but also past events $\mathcal{H}_t^N$.

Examples of Hawkes process include. . . In these examples, the arrivial of an event increases the occurrence of future events.

Definition 2.5.1 (Hawkes Process) A counting process $\{N(t), t \geq 0\}$ associated with past events $\{\mathcal{H}_t^N, t > 0\}$ is said to be a Hawkes process with conditional intensity function $\lambda(t|\mathcal{H}_t^N), t > 0$ and takes the form

$$\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \sum_{i:T_i<t} \phi(t - T_i)$$

where

- $\lambda_0(t)$ is the base intensity function (or μ the constant background rate)
- $T_i < t$ are the events time occur before current time t
- $\phi(\cdot)$ is the kernel function (or $g(\cdot)$ the triggering function) through which intensity function depends on past events
- $\mathcal{H}_t^N$ is the natural filration (or simply $\mathcal{H}_t$ the past history) which represents the internal history of N up to time t

Corollary **2.5.1** Hawkes process satisfies that

1. $N(t) = 0$
2. $\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \int_{-\infty}^{t} \phi(t - T_i) dN(s) = \lambda_0(t) + \sum_{i:T_i<t} \phi(t - T_i)$
3. $P(N(t + h)) - N(t) = 1)|\mathcal{H}_t^N) = \lambda(t)h + o(h)$
4. $P(N(t + h)) - N(t) \geq 2)|\mathcal{H}_t^N) = o(h)$

2.5.1 Choices of $\phi(\cdot)$ include, for example, exponentially decaying function and power-law kernel, and they take the form of

$$\phi(x) = \alpha e^{-\beta x}$$

$$\phi(x) = \frac{\alpha}{(x + \beta)^{\eta+1}}$$

2.5.2 There are two ways to view Hawkes processes

1. Intensity-based Hawkes Process
2. Cluster-based Hawkes Process

2.5 Spatio-Temporal Hawkes Process

Spatio-temporal Hawkes processes is an extention of temporal Hawkes processes. Recall that temporal Hawkes point processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu + \sum_{i:T_i<t} g(t - t_i)$$

Spatio-temporal self-exciting point processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu(s) + \sum_{i:T_i<t} g(s - s_i, t - t_i)$$

where

- $s_i, i = 1, 2, ...$ are the sequence of locations of events
- $t_i, i = 1, 2, ...$ are the times of the events

For this project, we focus on

3 Algorithms and Simulations

3.1 Counting Process

3.2 Poisson Process

3.3 Nonhomogeneous Poisson Process

3.4 Cluster Process

3.5 Hawkes Process

4 Conclusions and Discussion

Acknowledgments

Reference

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