

# Prequel to Hawkes Processes: An Overview of Temporal and Spatio-Temporal Point Processes and Some Simulations

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## Abstract

## 1 Introduction

Hawkes process is also known as a self-exciting point process. More specifically, the original Hawkes processes (e.g. ETAS model) are temporal, whereas the more recently developed spatio-temporal, self-exciting point processes have been extended from temporal Hawkes processes to account for both the spatial and temporal aspects of the data.

The defining characteristic of Hawkes processes is that it self-excites. In other words, the occurrence of an event increases the occurrence of future events. For example, in seismology, an event can be an earthquake occurrence that causes aftershocks. In criminology, an event can be a gang rivalry that triggers retaliations following the gang crime.

## Applications

In addition to modelling earthquake epicenters in seismology and crime patterns in criminology, Hawkes processes have also been used in modelling events such as forest wildfires, insurance claims, financial transactions, social network events, neuron activities, and disease spread or transmission. Thus, it can be found in a wide variety of fields such as emergency and disaster management, insurance, finance, social network, neuroscience, and epidemiology.

More recent work have been focused on modelling..... and leveraging the power the computing power to process and model big data.

## Motivation

Given the flexibility and applicability of Hawkes processes, it is surprising to see that Hawkes processes have not gain enough attentions from the machine learning communities, which would find their predicting capability practical. In addition, understanding Hawkes processes would benefit from knowing some of the relevant point processes (e.g. Cox, cluster processes), which are often left out from graduate-level, introductory spatial statistics and stochastic processes course.

## Objectives

The objective of this project is then to give an overview of various types of point processes so that readers of interest have the necessary background knowledge to understand Hawkes process. The overview includes

1. introducing, defining and discussing properties of counting process, (homogeneous and nonhomogeneous) Poisson process, cluster process, Hawkes process, and spatio-temporal Hawkes process,
2. discussing the algorithms and simulating the processes,
3. extending (temporal) Hawkes process to spatio-temporal Hawkes process, and
4. discussing recent and future work of Hawkes process.

## 2 Introductions, Definitions and Properties

### 2.1 Counting Process

Counting processes count the occurrences (or numbers) of events over time. For example, if we were to count the numbers of events  $N(t)$  such as the numbers of customers arriving at a supermarket or the numbers of phone calls receiving at the help line up to some time  $t$ , we can use counting processes.

Counting processes is independent, stationary, and homogeneous. In addition, . . . . .

Before formally define counting process, first we need to define stochastic process and point process. Stochastic process is a collection of random variables indexed by time  $t$  (or by space  $s$ , but we restrict ourselves to time  $t$  for now). The differences between stochastic process and point process are not clear, but they are included to help defining counting process. The definition of stochastic process and point process are as follow.

**Definition 2.1.1** (Stochastic Process) A stochastic process is a family of random variables indexed by time  $t$  and is defined as

$$\{X_t\}_{t \in T}$$

**Definition 2.1.2** (Point Process) Let  $\{T_i\}_{i \in \mathbb{N}}$  be a sequence of non-negative random variables such that  $T_i < T_{i+1} \forall i \in \mathbb{N}$ , a point process on  $\mathbb{R}^+$  is defined as

$$\{T_i\}_{i \in \mathbb{N}}$$

Next, we define a function  $N(t)$  such that it counts the total number of events up to some time  $t$ . For example, the total number of event from  $t = 0$  up to  $t = 1$  is denoted as  $N(0)$ , the total number of events from  $t = 1$  up to  $t = 2$  is denoted as  $N(1)$ , and so on.

**Definition 2.1.3** (Counting Process) Let  $N(t)$  be the total number of events up to some time  $t$  such that the values are nonnegative, interger and nondecreasing, a stochastic process is said to be a counting process and is defined as

$$\{N(t), t \geq 0\}$$

Here, if we let the first occurrence of event occurs at  $time = T_1$ , the 2nd occurrence of event occurs at  $time = T_2$ , and so on, then with counting process, we again add up the numbers incremently. Thus,  $N(0)$  indicates the total number of events from  $t = 0$  up to  $t = 1$ , which is 0 since there is no event occur in  $(0, 1]$ ,  $N(1)$  is the total number of events from  $t = 1$  up to  $t = 2$ , which is 1 since there is a total of one event occur in  $(1, 2]$ , and so on.

## A PICTURE HERE WOULD HELP

**Definition 2.1.4** (Counting Process) Let  $\{T_i\}_{i \in N}$  be a point process, a counting process associated with  $\{T_i\}_{i \in N}$  is defined as

$$N(t) = \sum_{i \in N} I_{\{T_i \leq t\}}$$

*Corollary 2.1.1* A counting process satisfies that

1.  $N(t) \geq 0$
2.  $N(t)$  is an integer
3. If  $s \leq t$ , then  $N(s) \leq N(t)$
4. If  $s < t$ , then  $N(t) - N(s)$  is the number of events occur in the interval  $(s, t]$

In other words, 1. the total number of events has to  $\geq 0$ , 2. the total number of events is an integer, . . . . .

**Proposition 2.1.1** A counting process has the following properties

1. Independence
2. Stationarity
3. Homogeneity

In other words, 1. the numbers of events  $N(t)$  occur in disjoint interval  $t$  are independent, 2. the distribution of the numbers of events  $N(t)$  depends only on the length of the interval  $t$ , and 3. the transition probability between any two states at two times depends only on the difference between the states.

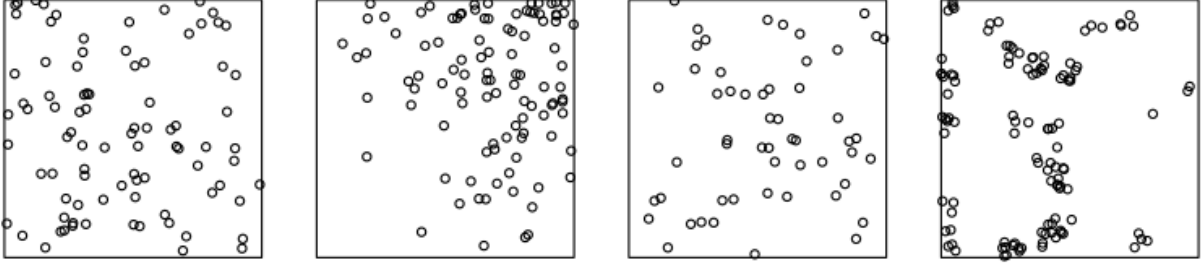


Figure 1: Left: HPP (rate = 100) Middle 1: NPP (intensity =  $400xy$ ) Middle 2: Cox (intensity =  $\exp(n = 1, \text{rate} = 1/100)$ ) Right: Matern (kappa = 20,  $r = 0.05$ ,  $\mu = 5$ )

## A Motivating Example

Before we delve into various point processes, let us first look at a motivating example. The figure on the left is a realization of HPP (homogeneous Poisson process) with constant *rate* = 100 in  $\mathbb{R}^2$ . The figure in the middle 1 is a realization of NPP (nonhomogeneous Poisson process) with *intensity function* =  $400xy$  in  $\mathbb{R}^2$ . The figure in the middle 2 is a realization of Cox process with *intensity function* =  $\exp(1, 1/100)$ . The figure on the right is a realization of Matern cluster process with  $kappa = 20, r = 0.05, \mu = 5$  in  $\mathbb{R}^2$ .

We can see that while HPP points appear randomly spaced and NPP points seem to concentrate at the upper-right corner, Matern cluster process points appear clustered. We elaborate more and discuss in details each type of point process in the following sections.

## 2.2 Poisson Process

Homogeneous Poisson process (HPP) is one of the simplest and most-widely used point processes. For example, if we were to model the numbers of events such as the numbers of bus arrivals at a bus stop, the numbers of car accidents at a site or the requests for documents on a web server, we can use Poisson processes.

HPP is independent, stationary, and homogeneous. In addition, we assume that the numbers of events  $N(t)$  follows a Poisson distribution with a constant rate  $\lambda$  and the interarrival times between events  $W$  are exponentially distributed.

**Definition 2.2.1** (Poisson Process) If the following conditions hold, a counting process  $\{N(t), t \geq 0\}$  is said to be a Poisson Process with constant rate (or intensity)  $\lambda > 0$

1.  $N(0) = 0$
2.  $N(t)$  has independent increments
3.  $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$
4.  $P(N(t+h) - N(t) \geq 2) = o(h)$

where  $o(h)$  function of little o. . . . .

In other words, 1. the process starts at  $t = 0$ , 2. the increments are independent, 3.  $\lambda$  is the rate (or intensity), and 4. no 2 or more events can occur at the same location.

**Proposition 2.2.1** Homogeneous Poisson Process has similar properties as those of counting process; it is independent, stationary, and homogeneous. These properties follow nicely from the above definition.

### A PICTURE HERE WOULD HELP

Again, we denote the total number of events as  $N(t)$ . We also denote the interarrival times between events as  $W$ . For example, while  $T_1$  is the time of the first occurrence of event and  $T_2$  the time of the second occurrence of event, the elapsed time between the none and first event is  $W_1$  and the elapsed time between the first event and second event is  $W_2$ .

We assume that the numbers of events  $N(t)$  follows a Poisson distribution with a constant rate  $\lambda$  and the interarrival times between events  $W$  are exponentially distributed.

**Proposition 2.2.2** Poisson process has additional properties

1. The number of events in any interval  $t$ ,  $N(t)$ ,  $\sim Pos(\lambda t)$ . That is, for all  $s, t \geq 0$

$$P(N(t+s) - N(s) = n) = P(N(t) - N(0) = n) = P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$n = 0, 1, \dots$

2. The interarrival times,  $W$ ,  $\overset{iid}{\sim} exp(\frac{1}{\lambda})$ . That is, for rate  $\lambda > 0$ , the interarrival time  $W_i$   $i = 1, 2, \dots$

$$P(W_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

This is because  $p(1^{st}$  arrival arrives after time  $t$ ) is the same as  $p(\text{no arrival in the interval } [0, t])$ . Similarly,  $W_2$  also  $\sim exp(\frac{1}{\lambda})$  since

$$P(W_2 > t | W_1 = s) = P(N(t+s) - N(s) | N(s) - N(s^-) = 1) = P(N(t+s) - N(s)) = P(N(t) = 0) = e^{-\lambda t}.$$

## 2.3 Nonhomogeneous Poisson Process

Nonhomogeneous Poisson processes (NPP) is a generalization of homogeneous Poisson processes that allow the rate (or intensity)  $\lambda$  to vary with function of time  $t$ .

Previously, we assume that the intensity  $\lambda$  is constant. If we have reasons to believe that the intensity is not constant, we should model using nonhomogeneous Poisson processes. For example, if we were to model the number of customers arriving at a supermarket and we have reasons to believe that the arrival rate of customers is higher during lunch time as compared to say, 2pm, we should model using nonhomogeneous Poisson processes.

In contrast to HPP, NPP is independent but not stationary nor homogeneous. In addition, for NPP, we assume that  $N(t)$  follows a Poisson distribution with an intensity function  $\lambda(t)$ .

**Definition 2.3.1** (Nonhomogeneous Poisson Process) If the following conditions hold, a counting process  $\{N(t), t \geq 0\}$  is said to be a nonhomogeneous Poisson Process with intensity function of time  $\lambda(t), t > 0$

1.  $N(0) = 0$
2.  $N(t)$  has independent increments
3.  $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$
4.  $P(N(t+h) - N(t) \geq 2) = o(h)$

**Proposition 2.3.1** Nonhomogeneous Poisson process is independent but not stationary nor homogeneous. Homogeneous Poisson process has stationary increments since the distribution of the numbers of events  $N(t)$  that occur in any interval of time  $t$  depends only on the length of the interval  $t$  but not the location of the interval  $t$ . In contrast, nonhomogeneous Poisson process does not have stationary increments since the distribution of  $N(t)$  can change when shifted in  $t$ . Since stationarity implies homogeneity, nonhomogeneous Poisson process is nonhomogeneous.

Recall that for HPP, we assume that the numbers of events  $N(t)$  follows a Poisson distribution with a constant intensity  $\lambda$ , here we assume that  $N(t)$  follows a Poisson distribution too but with an intensity function  $\lambda(t)$  such that the intensity vary with function of time.

**Proposition 2.3.2** Nonhomogeneous Poisson process has additional properties

1. The number of events in any interval  $t$ ,  $N(t)$ ,  $\sim \text{Pos}(\Lambda(t) = \int_0^t \lambda(s)ds)$ . That is, for all  $s, t \geq 0$

$$P(N(t) = n) = \frac{(\int_0^t \lambda(s)ds)^n e^{-\int_0^t \lambda(s)ds}}{n!}$$

$n = 0, 1, \dots$

2. The law of occurrence for the next point can be determined by

$$P(N(a, b] = 0) = e^{-\int_a^b \lambda(s)ds}.$$

## Conditional Intensity Function

In the following sections, we discuss conditional intensity models. Examples of such models include Cox, cluster, and Hawkes processes. First we want to define conditional intensity function since it is through the conditional intensity function a point process is fully characterized.

**Definition 2.4.0** (Conditional Intensity Function) Let  $N(t)$  be the numbers of events  $N(t)$  that occur in any interval of time  $t$ , the conditional intensity function  $\lambda(t)$  with respect to  $\mathcal{H}_t$  is defined as

$$\lambda(t) = \lim_{h \rightarrow 0^+} \frac{P(N(t, t+h] > 0 | \mathcal{H}_t)}{h} = \lim_{h \rightarrow 0^+} \frac{E(N(t, t+h) | \mathcal{H}_t)}{h}$$

where  $\mathcal{H}_t$  is the history prior to time  $t$  but can also be... Wiki: Intensity of a point process

(?) Still don't quite see connection to Cox and cluster process

(?) Hazard function

## 2.4 Cox and Cluster Process

Cox and cluster processes are processes that allow dependence between events.

Previously, we assume independence between events. That is, whether events occur at a constant rate  $\lambda$  (e.g. HPP) or depend on an intensity function  $\lambda(t)$  (e.g. NPP), they occur independently. Here, we discuss models that allow dependence between events.

Examples that can be modelled using Cox and cluster processes include seedlings and saplings of California redwood, locations of emergent plants, and locations of trees. In these examples, the patterns appear to be clustered.

We can think of Cox process as a hierarchical model with two levels and cluster process such as Neyman-Scott process a hierarchical model with three levels. **ELABORATE MORE.**

In addition, Cox process conditions on (?) random field, whereas cluster process conditions on..... (?) in bounded region.

In Cox processes (or doubly stochastic Poisson process), the intensity function  $\Lambda(u)$  is treated as random. **PICK ONE TO PLOT THEN TALK IN DETAILS.** Other examples of Cox processes include mixed Poisson process, log Gaussian Cox process, and shot noise Cox process.

In cluster processes, the randomness arises from two steps: First, 'parent' points  $\mathbf{Y}$  is generated. Then, each 'parent' point  $y_i \in \mathbf{Y}$  gives to 'offspring' points  $z_{ij}$ . All the 'offspring' points  $Z_{ij}$  form a cluster process  $\mathbf{X}$  and only  $\mathbf{X}$  is observed.

Specific models of cluster processes depend on the choices of assumptions. Matern cluster process, for example, involves generating homogeneous Poisson parents and each parent gives rise to Poisson number of offspring uniformly distributed in a disc of radius  $r$  centered around the parent. Other examples of cluster processes include Neyman-Scott process and Thomas cluster process.

**Definition 2.4.1** (Cox Process) Let  $\Lambda = (\Lambda(u))_{u \in S \subseteq \mathbb{R}^d}$  be a non-negative random field such that  $\Lambda(u)$  is a locally integrable function. If  $\mathbf{X} | \Lambda \sim \text{Pos}(\Lambda)$ , then  $\mathbf{X}$  is said to be a Cox process driven by  $\Lambda$  with intensity function  $\lambda(u) = E(\Lambda(u))$ . That is,

$$P(N(u) = n) = \frac{(\lambda(u))^n e^{-\lambda(u)}}{n!} = \frac{(E(\Lambda(u)))^n e^{-E(\Lambda(u))}}{n!} =$$

(?)

$$\Lambda(u) \stackrel{a.s.}{=} \int_u \lambda(x) dx$$

(?) How to get here

$$= \int_0^\infty \frac{x^n e^{-x} F_u(dx)}{n!}$$

Note.  $\Lambda$  is a random field means that  $\Lambda(u)$  is a random variable  $\forall u \in S$ .

Note.  $\Lambda(u)$  is a locally integrable function means that  $E(\Lambda(u))$  exists and is locally integrable with probability 1.

**Proposition 2.4.1** Properties of Cox process are as follow

1. Properties of Cox process  $X$  follow immediately from the properties of Poisson process  $X \mid \Lambda$ . For example, if  $\Lambda$  is stationary, then  $X$  is stationary.
2. For bounded  $B \subseteq S$ , the void probabilities are given by

$$\nu(B) = E(P(N(B) = 0) \mid \Lambda) = E(\exp(-\int_B \Lambda(u) du)).$$

Note. The void (or avoidance) probability  $\nu$  is defined as the probability that no points of a point process  $N$  existing in  $B$  where  $B$  is a subset of the underlying space  $R^d$ .

**Definition 2.4.2** (Cluster Process) Let  $x$  be points in a point process  $N$  and replacing every  $x$  with a cluster of points  $N_x$ , then the union of all the clusters forms a cluster process  $N_c$ . That is,

$$N_c = \bigcup_{x \in N} N_x$$

Note. Each  $N_x$  is a finite point process ‘centered’ at  $x$  and it is assumed that each  $N_x$  is independent of one another.

Note.  $N_x$  can be thought of as ‘offspring’??

In addition, condition on  $\dots$ , the cluster process has the intensity function  $\dots$

**Proposition 2.4.2** Cluster process has the following model assumptions

1. Poisson parents
2. Independent clusters
3. Identically distributed clusters
4. Offsprings independent within a cluster
5. Poisson number of offsprings
6. Isotropic clusters



In other words, 1. ‘parent’ points follow a Poisson distribution, 2. clusters are independent of each other, 3. clusters, when shifted, have the same distributions, 4. the locations of ‘offspring’ points of each parent point are independently and identically distributed, 5. the numbers of ‘offspring’ points of each parent point follow a Poisson distribution, and 6. the distribution of ‘offspring’ points for each parent point depends only on the distance between the ‘parent’ and the ‘offspring’.

Under assumption 1 - 4, it is a Neyman-Scott process. Under assumption 1 - 5, the cluster process is a Cox process. Under assumption 1 - 6, they are Matern cluster process and Thomas cluster process.

## 2.5 Hawkes Process

**Definition 2.5.1** (Hawkes Process) A counting process  $\{N(t), t \geq 0\}$  associated with past events  $\{\mathcal{H}_t^N, t > 0\}$  is said to be a Hawkes process with conditional intensity function  $\lambda(t|\mathcal{H}_t^N), t > 0$  and takes the form

$$\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \sum_{i:T_i < t} \phi(t - T_i)$$

where

- $\lambda_0(t)$  is the base intensity function (or  $\mu$  the constant background rate)
- $T_i < t$  are the events time occur before current time  $t$
- $\phi(\cdot)$  is the kernel function (or  $g(\cdot)$  the triggering function) through which intensity function depends on past events
- $\mathcal{H}_t^N$  is the natural filtration (or simply  $\mathcal{H}_t$  the past history) which represents the internal history of  $N$  up to time  $t$

*Corollary 2.5.1* Hawkes process satisfies that

1.  $N(t) = 0$
2.  $\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \int_{-\infty}^t \phi(t - T_i) dN(s) = \lambda_0(t) + \sum_{i:T_i < t} \phi(t - T_i)$
3.  $P(N(t+h) - N(t) = 1 | \mathcal{H}_t^N) = \lambda(t)h + o(h)$
4.  $P(N(t+h) - N(t) \geq 2 | \mathcal{H}_t^N) = o(h)$

**2.5.1** Choices of  $\phi(\cdot)$  include, for example, exponentially decaying function and power-law kernel, and they take the form of

$$\phi(x) = \alpha e^{-\beta x}$$

$$\phi(x) = \frac{\alpha}{(x + \beta)^{\eta+1}}$$

**2.5.2** There are two ways to view Hawkes processes

1. Intensity-based Hawkes Process

Here, Hawkes process is defined through conditional intensity process.

In **Section 3.5**, we discuss algorithm for simulating intensity-based Hawkes process in details.

2. Cluster-based Hawkes Process

Alternatively, Hawkes process can also be defined through marked Poisson cluster process.

## 2.5 Spatio-Temporal Hawkes Process

Spatio-temporal Hawkes processes is an extension of temporal Hawkes processes. Recall that temporal Hawkes processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu + \sum_{i:T_i < t} g(t - t_i)$$

Spatio-temporal Hawkes processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu(s) + \sum_{i:T_i < t} g(s - s_i, t - t_i)$$

where

- $s_i, i = 1, 2, \dots$  are the sequence of locations of events
- $t_i, i = 1, 2, \dots$  are the times of the events

Next, we simulate... through... and packages.

## 3 Algorithms and Simulations

### 3.2 Poisson Process

#### Algorithm 1

### 3.3 Nonhomogeneous Poisson Process

There are multiple ways to simulate nonhomogeneous Poisson process: 1) inversion, 2) order statistics, 3) thinning and 4) hybride (inversion + thinning).

In this example, we use the thinning algorithm (or acceptance-rejection method) to simulate nonhomogeneous Poisson process with the intensity function  $\lambda(t) = \dots$  since it is one of the most popular choices for both temporal and spatio-temporal cases.

Broadly put, thinning algorithm involves randomly deleting points from a point pattern.

#### Algorithm 2

Similarly, there are multiple ways to simulate Hawkes process. Here, we use thinning algorithm (or acceptance-rejection method)

### 3.5 Hawkes Process

#### Algorithm 3

### 3.6 Plots in 2D

All of the corresponding plots can be found in the Appendix section and all of the plots in this section are created using the **spatstat** package in R.

HPP

NPP

Matern cluster process involves generating homogeneous Poisson parents and each parent gives rise to Poisson number of offspring uniformly distributed in a disc of radius  $r$  centered around the parent.  $\kappa \dots r$  specifies  $\dots$ , and  $\mu \dots$

The following functions use thinning algorithm. Simulations of Matern I and Matern II processes are generated using the `rMaternI` and `rMaternII` functions of the **spatstat** package.

## 4 Conclusions and Discussion

## 5 Recent Advancement

## Acknowledgments

## Reference

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## Terminology

Stochastic Process

Counting Processes

Poisson Process

Nonhomogeneous Poisson Process

Cox Process

# Appendix