Prequel to Hawkes Processes: An Overview of Temporal and Spatio-Temporal Point Processes and Some Simulations

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Abstract

1 Introduction

Hawkes process is also known as a self-exciting point process.

Motivation

Applications

Applications of Hawkes processes can be found in a wide variety of fields such as seismology, criminology, insurance, finance, social network, and neuroscience.

The defining characteristic of Hawkes processes is that it self-excites. In other words, the occurrence of an event increases the occurrence of future events. For example, in seismology, an event can be an earthquake occurrence that causes aftershocks. In criminology, an event can be a gang rivalry that triggers retaliations following the gang crime. In insurance, an event can be a standard claim that increases claims. In finance, an event can be a transcation that influences future prices or volumes of transcations or a news that leads to movements in stock prices or trading behaviors. In social network, an event can be a tweet about an event on Twitter that follows a cascade of retweets from other users on the same social networking platform. In neuroscience, an event can be firing of a neuron that triggers spikes (or action potentials) of other neurons.

Objectives

The objectives of this project is to give an overview of various types of point processes so that readers of interest have the necessary background knowledge to understand Hawkes process. The overview includes

1. defining and discussing properties of counting process, (homogeneous) Poisson process, nonhomogeneous Poisson process, cluster process, Hawkes process, and spatio-temporal Hawkes process,
2. discussing the algorithms and simulating all of the above processes,
3. extending (temporal) Hawkes process to spatio-temporal Hawkes process, and
4. discussing recent and future work of Hawkes process.

2 Definitions and Properties

2.1 Counting Process

Definition 2.1.1 (Stochastic Process) A stochastic process is a family of random variables indexed by time t and is defined as

$$\{X(t), t \in T\}$$

Definition 2.1.2 (Counting Process) Let *N(t)* be the total number of events up to some time t such that the values are nonnegative, interger and nondecreasing, a stocastic process is said to be a counting process and is defined as

$$\{N(t), t \geq 0\}$$

Definition 2.1.3 (Point Process) Let $\{T_i, i \in N\}$ be a sequence of non-negative random variables such that $T_i < T_{i+1}$ $\forall i \in N$, a point process on R^+ is defined as

$$\{T_i, i \in N\}$$

Definition 2.1.4 (Counting Process) Let $\{T_i, i \in N\}$ be a point process, a counting process associated with $\{T_i, i \in N\}$ is defined as

$$N(t) = \sum_{i \in N} I_{\{T_i \leq t\}}$$

Corollary **2.1.1** A counting process satisfies that

1. $N(t) \geq 0$
2. $N(t)$ is an integer
3. If $s \leq t$, then $N(s) \leq N(t)$
4. If $s < t$, then $N(t) - N(s)$ is the number of events occur in the interval (s, t]

Proposition 2.1.1 A counting process has the following properties

1. (Independence) If the numbers of events $N(t)$ occur in disjoint interval t are independent, a counting process is said to have independent increments.
2. (Stationarity) If the distribution of the numbers of events $N(t)$ depends only on the length of the interval t, a counting process is said to have stationary increments.
3. (Homogeneity) If the transition probability between any two states at two times depends only on the difference between the states, a stochastic process is said to be homogeneous. Stationarity implies homogeneity.

2.2 Poisson Process

Definition 2.2.1 (Poisson Process) If the following conditions hold, a counting process $\{N(t), t \geq 0\}$ is said to be a Poisson Process with constant rate (or intensity) $\lambda > 0$

1. $N(0) = 0$
2. $N(t)$ has independent increments
3. $P(N(t+h)) - N(t) = 1)) = \lambda h + o(h)$
4. $P(N(t+h)) - N(t) \geq 2)) = o(h)$

In other words, 1. the process starts at $t = 0$, 2. the increments are independent, 3. λ is the rate (or intensity), and 4. no 2 or more events can occur at the same location.

Proposition 2.1.1 Poisson process has the same (?) properties as those of counting process.

Proposition 2.2.2 Poisson process has additional properties

1. The number of events in any interval t, $N(t)$, $\sim Pos(\lambda t)$. That is, for all $s, t \geq 0$

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

or

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$n = 0, 1, ...$

2. The interarrival times, W, $\overset{iid}{\sim} exp(\frac{1}{\lambda})$. That is, for rate $\lambda > 0$, the interarrival time W_i $i = 1, 2, ...$

$$P(W_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

This is because p(1^{st} arrivial arrives after time t) is the same as p(no arrivial in the interval [0,t]). Similarly, W_2 also $\sim exp(\frac{1}{\lambda})$ since

$$P(W_2 > t | W_1 = s) = P(N(t+s) - N(s)) = P(N(t) = 0) = e^{-\lambda t}.$$

2.3 Nonhomogeneous Poisson Process

Definition 2.3.1 (Nonhomogeneous Poisson Process) If the following conditions hold, a counting process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson Process with intensity function of time $\lambda(t), t > 0$

1. $N(0) = 0$
2. $N(t)$ has independent increments
3. $P(N(t+h)) - N(t) = 1)) = \lambda(t)h + o(h)$
4. $P(N(t+h)) - N(t) \geq 2)) = o(h)$

Proposition 2.2.1 Nonhomogeneous Poisson process has the following properties

1. Independence

(Homogeneous) Poisson process has stationary increments since the distribution of the numbers of events $N(t)$ that occur in any interval of time t depends only on the length of the interval t but not the location of the interval t. In contrast, nonhomogeneous Poisson process does not have stationary increments since the distribution of $N(t)$ can change when shifted in t.

Proposition 2.2.2 Nonhomogeneous Poisson process has the following properties

1. The number of events in any interval t, $N(t)$, $\sim Pos(\Lambda(t) = \int_0^t \lambda(s)ds)$. That is, for all $s, t \geq 0$

$$P(N(t) = n) = \frac{(\int_0^t \lambda(s)ds)^n e^{-\int_0^t \lambda(s)ds}}{n!}$$

or if we let n to be the numbers of events occur in the interval $[a, b]$, then

$$P(N(b) - N(a) = n) = \frac{(\int_a^b \lambda(s)ds)^n e^{-\int_a^b \lambda(s)ds}}{n!}$$

$n = 0, 1, ...$

In the following sections, we discuss conditional intensity models. Such models include Cox, cluster, and Hawkes processes.

2.4 Cox and Cluster Process

Definition 2.4.1 (Cox Process) Let $\Lambda = (\Lambda(u))_{u \in S \subseteq R^d}$ be a non-negative random field such that $\Lambda(u)$ is a locally integrable function. If X | $\Lambda \sim Pos(\Lambda)$, then X is said to be a Cox process driven by Λ with intensity function $\lambda(u) = E(\Lambda(u))$. That is,

$$P(N(u) = n) = \frac{(\lambda(u))^n e^{-\lambda(u)}}{n!} = \frac{(E(\Lambda(u)))^n e^{-E(\Lambda(u))}}{n!} =$$

? How to get here

$$= \int_0^\infty \frac{x^n e^{-x} F_u(dx)}{n!}$$

Note. Λ is a random field means that $\Lambda(u)$ is a random variable $\forall u \in S$.

Note. $\Lambda(u)$ is a locally integrable function means that $E(\Lambda(u))$ exists and is locally intergrable with probabiliy 1.

Proposition 2.4.1 Propertities of Cox process are as follow

1. Propertities of Cox process X follow immediately from the properties of Poisson process X | Λ. For example, if Λ is stationary, then X is stationary.

2. For bounded $B \subseteq S$, the void probabilities are given by

$$\nu(B) = E(P(N(B) = 0)|\Lambda) = E(exp(-\int_B Z(u)du)).$$

Note. The void (or avoidance) probability ν is defined as the probability that no points of a point process N existing in B where B is a subset of the underlying space R^d.

Definition 2.4.2 (Cluster Process)

Proposition 2.4.2 Cluster Process has the following model assumptions

1. Poisson parents
2. Independent clusters
3. Identically distributed clusters
4. Offsprings independent within a cluster
5. Poisson number of offsprings
6. Isotropic clusters

In other words, 1. 'parent' points follow a Poisson distribution, 2. clusters are independent of each other, 3. clusters, when shifted, have the same distributions, 4. the locations of 'offspring' points of each parent point are independently and identically distributed, 5. the numbers of 'offspring' points of each parent point follow a Poisson distribution, and 6. the distribution of 'offspring' points for each parent point depends only on the distance between the 'parent' and the 'offspring'.

Under assumption 1 - 4, it is a Neyman-Scott process. Under assumption 1 - 5, the cluster process is a Cox process. Under assumption 1 - 6, we have Matern cluster process and Thomas cluster process.

2.5 Hawkes Process

Definition 2.5.1 (Hawkes Process) A counting process $\{N(t), t \geq 0\}$ associated with past events $\{\mathcal{H}_t^N, t > 0\}$ is said to be a Hawkes process with conditional intensity function $\lambda(t|\mathcal{H}_t^N), t > 0$ and takes the form

$$\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \sum_{i:T_i<t} \phi(t - T_i)$$

where

- $\lambda_0(t)$ is the base intensity function (or μ the constant background rate)
- $T_i < t$ are the events time occur before current time t
- $\phi(\cdot)$ is the kernel function (or $g(\cdot)$ the triggering function) through which intensity function depends on past events
- $\mathcal{H}_t^N$ is the natural filration (or simply $\mathcal{H}_t$ the past history) which represents the internal history of N up to time t

Corollary **2.5.1** Hawkes process satisfies that

1. $N(t) = 0$

2. $\lambda(t|\mathcal{H}_t^N) = \lambda_0(t) + \int_{-\infty}^t \phi(t - T_i)dN(s) = \lambda_0(t) + \sum_{i:T_i<t} \phi(t - T_i)$
3. $P(N(t+h)) - N(t) = 1)|\mathcal{H}_t^N) = \lambda(t)h + o(h)$
4. $P(N(t+h)) - N(t) \geq 2)|\mathcal{H}_t^N) = o(h)$

2.5.1 Choices of $\phi(\cdot)$ include, for example, exponentially decaying function and power-law kernel, and they take the form of

$$\phi(x) = \alpha e^{-\beta x}$$

$$\phi(x) = \frac{\alpha}{(x+\beta)^{\eta+1}}$$

2.5.2 There are two ways to view Hawkes processes

1. Intensity-based Hawkes Process
2. Cluster-based Hawkes Process

2.5 Spatio-Temporal Hawkes Process

Spatio-temporal Hawkes processes is an extention of temporal Hawkes processes. Recall that temporal Hawkes processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu + \sum_{i:T_i<t} g(t - t_i)$$

Spatio-temporal Hawkes processes take the form of

$$\lambda(t|\mathcal{H}_t) = \mu(s) + \sum_{i:T_i<t} g(s - s_i, t - t_i)$$

where

- $s_i, i = 1, 2, ...$ are the sequence of locations of events
- $t_i, i = 1, 2, ...$ are the times of the events

Next, we

3 Algorithms and Simulations

3.2 Poisson Process

Algorithm 1

3.3 Nonhomogeneous Poisson Process

There are multiple ways to simulate nonhomogeneous Poisson process: 1) inversion, 2) order statistics, 3) thinning and 4) hybride (inversion + thinning).

In this example, we use the thinning algorithm (or acceptance-rejection method) to simulate nonhomogeneous Poisson process with the intensity function $\lambda(t) = \dots$ since it is one of the most popular choices for both temporal and spatio-temporal cases.

Broadly put, thinning algorithm involves randomly deleting points from a point pattern.

Algorithm 2

3.5 Hawkes Process

Algorithm 3

3.5 Plots in 2D

All the following plots are created using the **spatstat** package in `R`.

HPP

NPP

Matern process involves generating homogeneous Poisson parents and each parent gives rise to Poisson number of offspring uniformmly distributed in a disc of radius r centered around the parent.

The following functions use thinning algorithm. Simulations of Matern I and Matern II processes are generated using the `rMaternI` and `rMaternII` functions of the **spatstat** package.

4 Conclusions and Discussion

Acknowledgments

Reference

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Terminology

Counting Processes

Poisson Process

Nonhomogeneous Poisson Processes