Reduced-Rank Regression Model: A Review

Frances Lin

March 2022

Background and Introduction

A classical multivariate linear model, which is given as

$$Y_k = CX_k + \varepsilon_k, \ \ k = 1, ..., T$$

where $Y_i = (y_{1k}, ...y_{mk})^T$ is a $mx1$ response vector, C is a mxn regression coefficient matrix, $X_k = (x_{1k}, ...x_{mk})^T$ is a $nx1$ predictor vector, and $\varepsilon_k = (\varepsilon_{1k}, ...\varepsilon_{mk})^T$ is a $mx1$ error vector with $E(\varepsilon_k) = 0$ and $cov(\varepsilon_k) = \Sigma_{\varepsilon\varepsilon}$, does not make use of the fact that the response variables are likely correlated.

In many practical situations, there is also often a need to reduce the number of parameters in the model.

Introduction

Further assuming reduced rank of the matrix C such that

$$rank(C) = r \leq min(m, n)$$

leads us to two implications.

1. Let l be the constraint vector, the linear combination, $l^T Y_k$, $i = 1, ..., (m - r)$, can be modeled through the distribution of the error term ε_k (w/o referencing to the predictors X_k).
2. C can be expressed as $C = AB$, where A is of dimension mxr and B is of dimension rxn. Then, the multivariate linear model can be rewritten as

$$Y_k = A(BX_k) + \varepsilon_k, \ k = 1, ..., T,$$

where BX_k is of reduced dimension with *only* r components, and as a result, there is a gain in simplicity and interpretation.

Introduction

The first application of reduced-rank regression model appeared in an initial work of Anderson (1951) in the field of economics. The model and its statistical properties were further examined by a few other authors.

Subsequent but separate work that were studied using related concepts were

- principle components (Rao, 1964),
- simultaneous linear prediction modeling (Fortier, 1966),
- redundancy analysis, an alternative to canonical correlation analysis (van den Wollenberg, 1977), etc.

More complex models have also been developed ever since.

Applications

Applications of the reduced-rank regression model include

(1) the experimental properties of hydrocarbon fuel mixtures in relating response to composition (Davies and Tso, 1982),

(2) an econometric model of the United Kingdom from 1948 to 1956 (Gudmundsson, 1977), which consists of 37 time series of response variables and 32 time series of predictors,

(3) the relationship between measurements on solar radiation taken over various sites in Scotland and the physical characteristics of the sites (Glasbey, 1992), etc,

Estimation

The parameters that are to be estimated are the matrix A of dimension mxr, B of dimension rxn and $\Sigma_{\varepsilon\varepsilon}$ of the error term.

To estimate A and B, we need the Brillinger's theorem (Brillinger, 1981, Section 10.2), which can be proven by the Eckart-Young theorem (Eckart and Young, 1936).

Brillinger's Theorem

Suppose the random vector (Y, X) has mean vector 0 and covariance matrix $Cov(Y, X) = \Sigma_{yx} = {\Sigma_{xy}}^T$ and $Cov(X) = \Sigma_{xx}$ is nonsingular, then for any positive-definite matrix Γ, the mxr matrix A and rxn matrix B, for $r \leq min(m, n)$ that minimize

$$tr(E(\Gamma^{1/2}(Y - ABX)(Y - ABX)^T \Gamma^{1/2}))$$

are given by

$$A = \Gamma^{1/2} V, \qquad B = V^T \Gamma^{1/2} \Sigma_{yx} {\Sigma_{xx}}^{-1},$$

where $V = (V_1, ..., V_r)$ and V_j is the (normalized) eigenvector that corresponds to the jth largest eigenvalue λ_j^2 of the matrix $\Gamma^{1/2} \Sigma_{yx} {\Sigma_{xx}}^{-1} \Sigma_{xy} \Gamma^{1/2}$, $j = 1, ..., r$.

Remark - SVD (Singular Value Decomposition)

Remark. Brillinger's theorem is proved by setting $S^* = \Gamma^{1/2}\Sigma_{yx}{\Sigma_{xx}}^{-1}$, where the positive square roots of SS^T are called the singular values of the matrix S.

In general, a *mxn* matrix S of rank s can be expressed in the singular value decomposition as $S = V\Lambda U^T$, where $\Lambda = diag(\lambda_1, ...\lambda_s)$ with ${\lambda_i}^2 > 0$ and V is a *mxs* matrix s.t. $V^T V = I_s$ and U is a *nxs* matrix s.t. $U^T U = I_s$.

Estimation - $\hat{\beta}_{RR}$ (or $\hat{A}$ and $\hat{B}$)

Back to the model, recall that in the full-rank case, the OLS estimator is given as

$$\hat{\beta}_{OLS} = C^T = (X^T X)^{-1} X^T Y,$$

in the reduced-rank case and at the simple level, the estimator can be written as

$$\hat{\beta}_{RR} = B^T A^T = (X^T X)^{-1} X^T Y V V^T = \hat{\beta}_{OLS} V V^T.$$

Note.

1. $C = AB = \Gamma^{1/2} V V^T \Gamma^{1/2} Y^T X (X^T X)^{-1} = P_\Gamma Y^T X (X^T X)^{-1}$, where P_Γ is an idempotent matrix for any Γ (Brillinger's theorem).
2. In the reduced-rank regression, Γ is typically set to be the identity matrix I.

Estimation - $\hat{\Sigma_{\varepsilon\varepsilon}}$

To maximize

$$L(C, \Sigma_{\varepsilon\varepsilon}) = (\frac{T}{2})(log|\Sigma_{\varepsilon\varepsilon}{}^{-1}| - tr(\Sigma_{\varepsilon\varepsilon}{}^{-1}W))$$

is the same as minimizing $|W|$ (the determinant of W), where

$$W = \tilde{\Sigma_{\varepsilon\varepsilon}} + (\tilde{C} - AB)\hat{\Sigma}_{xx}(\tilde{C} - AB)^T.$$

It has been shown that the solutions

$$A = \Gamma^{1/2}V, \qquad B = V^T\Gamma^{1/2}\Sigma_{yx}\Sigma_{xx}{}^{-1},$$

with the choice of $\Gamma = \Sigma_{\varepsilon\varepsilon}{}^{-1}$, minimizes simultaneously all the eigenvalues of $\Sigma_{\varepsilon\varepsilon}{}^{-1}W$ and hence minimizes $|W|$ (Robinson, 1974).

Estimation

So, $\hat{A}_r$ and $\hat{B}_r$ are ML estimates for A and B.

However,

$$\tilde{\Sigma}_{\varepsilon\varepsilon} = (1/T)(Y - \tilde{C}X)(Y - \tilde{C}X)^T$$

is the ML estimate in the full-rank regression model.

$$\hat{\Sigma}_{\varepsilon\varepsilon} = (1/T)(Y - C_r X)(Y - C_r X)^T$$

is the ML estimate under the reduced-rank structure.

Discussion

An explicit solution for the matrices A and B can be obtained by computing eigenvalues and eigenvectors of SS^T or S^TS.

However, when an explicit solution is not possible to find, iterative procedures need to be considered.

Reference

Velu, R., & Reinsel, G. C. (2013). Multivariate reduced-rank regression: theory and applications (Vol. 136). Springer Science & Business Media.

Turgeon, M. (2021, January 19). Reduced Rank Regression [Video]. YouTube. https://www.youtube.com/watch?v=AYKmzhbsTwg&list=PLYpxJrn6DQgj3bCPRQDogPKkgzILomIAN&index=4&t=4s

Thank you!

Reduced-Rank Regression Model: A Review

Frances Lin

PhD student, Dept. of Statistics, Oregon State University