

ST 661 Note

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1/11/2022

Outline

pd matrices

Determinant

Orthogonal matrices

Thm (*)

Suppose A is symmetric, A is pd iff \exists a nonsingular matrix P s.t. $A = P^T P$.

Proof. \Leftarrow

Suppose $A = P^T P$ for some nonsingular P , then

$$y^T A y = y^T P^T P y = (P y)^T P y \geq 0$$

The last step sort of works like a sum of square. In addition,

$$(P y)^T P y = 0$$

only if $P y = 0$.

$\Rightarrow y = 0$ because P is nonsingular (or full rank)

$\Rightarrow A$ is pd since whenever $y \neq 0$, $y^T A y > 0$ \square

Proof. \Rightarrow is included in the HW and requires eigenvalues.

If we relax the assumption such that P is nonsingular, then we get psd.

Comment. P is not unique, unless... see the following

Cholesky Decomposition

Suppose that A is pd, then it is possible to find a nonsingular upper triangular matrix

$$T = \begin{bmatrix} \cdot & * & * \\ 0 & \cdot & * \\ 0 & 0 & \cdot \end{bmatrix}$$

s.t. $A = T^T T$.

Thm

Let B be a $n \times p$ matrix,

- i) If $\text{rank}(B) = p$, then $B^T B$ is pd.
- ii) If $\text{rank}(B) < p$, then $B^T B$ is psd, not pd. (psd can be pd but not in this case.)

Proof. Exercise or check p.42 of PDF (textbook). Use quadratic form from previous proof and rank.

Thm

If A is pd, A^{-1} is also pd.

Proof. Use Thm (*),

A is pd $\Rightarrow A = P^T P$ for nonsingular P , then

$$A^{-1} = (P^T P)^{-1} = P^{-1} (P^{-1})^T = (\text{new } P)(\text{new } P)^T$$

$\Rightarrow A^{-1}$ is pd.

Determinant

Check p.52 of PDF (textbook) for full definition. As a demonstration, suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ is a degree 2 polynomial function, for example. In general, let A be $n \times n$, then $\det(A)$ (or $|A|$) is a scalar degree n polynomial function of A .

Thm

- i) If

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & d_n \end{bmatrix},$$

is diagonal, then $\det(D) = d_1 d_2 \dots d_n$.

If D_2 is upper (or lower) triangular matrix such as

$$D_2 = \begin{bmatrix} d_1 & * & * \\ 0 & \cdots & * \\ 0 & 0 & d_n \end{bmatrix},$$

then $\det(D) = d_1 d_2 \dots d_n$.

- ii) A is square matrix and A is singular $\Leftrightarrow \det(A) = 0$.
- iii) If A is pd, then $\det(A) > 0$. (Not the inverse.)

If A is psd, then $\det(A) \leq 0$. (Again, not the inverse.)

iv) $\det(A) = \det(A^T)$

v) If A is singular ($\det(A) \neq 0$), then $\det(A^{-1}) = \frac{1}{\det(A)}$.

vi) If cA where c is scalar and A is $n \times n$, then $\det(cA) = c^n \det(A)$.

e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$2A = \begin{bmatrix} 2a_{11} & 2a_{12} \\ 2a_{21} & 2a_{22} \end{bmatrix}.$$

Thm

If A and B are both square $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Corollary

i)

$$\det(BA) = \det(AB)$$

ii) (special case of thm)

$$\det(A^k) = (\det(A))^k$$

e.g.

$$\det(A^2) = (\det(A))^2$$

Thm

If A is square and partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and A_{11} and A_{22} are both square and nonsingular (can be of different size), then

$$\det(A) = \det(A_{11})\det(A_{22} - A_{21}(A_{11})^{-1}A_{12}) = \det(A_{22})\det(A_{11} - A_{12}(A_{22})^{-1}A_{21}).$$

Check p.54 of PDF for more.

Intuition.

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21} = a_{11}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) = a_{22}\left(a_{11} - \frac{a_{12}a_{21}}{a_{22}}\right).$$

Corollary

Suppose

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

or

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

and assume A_{11} and A_{22} are square, then

$$\det(A) = \det(A_{11})\det(A_{22}).$$

This is the extension of the previous thm where

i) If

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & d_n \end{bmatrix},$$

is diagonal, then $\det(D) = d_1 d_2 \dots d_n$.

Thm

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and assume

- 1) A_{11} & A_{12} are square
- 2) A_{11} is nonsingular
- 3) $B = A_{22} - A_{21}(A_{11})^{-1}A_{12}$ is nonsingular, then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B^{-1} \\ -B^{-1}A_{21}A_{11}^{-1} & B^{-1} \end{bmatrix}.$$

? $\dim(A_{11}) = \dim(3 \text{ terms})$

Check. p.38 of PDF for more.

Note. We only need B^{-1} and A_{11}^{-1} and the rest are matrix multiplication.

In addition, this is helpful when we need to compare $A_{11}^{-1}(-B^{-1}A_{21}A_{11}^{-1}) + A_{11}^{-1}A_{12}B^{-1}A_{21}A_{11}^{-1}$ to A_{11}^{-1} . Remove $(-B^{-1}A_{21}A_{11}^{-1})$ after comparing to p.38 of PDF.

e.g.

$$X_{new} = \begin{bmatrix} X & Z \end{bmatrix}$$

$$X_{new}^T X_{new} = \begin{bmatrix} X^T \\ Z^T \end{bmatrix} \begin{bmatrix} X & Z \end{bmatrix} = \begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z \end{bmatrix}$$

Orthogonal Matrices

Suppose

$$a = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T$$

length of a (or $|a|$) = $\sqrt{(a_1^2 + a_2^2)} = \sqrt{(a^T a)}$.

a is a unit vector iff length of a (or $|a|$) = 1.

Suppose

$$b = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T,$$

if a and b are both unit vectors, then

$$\cos(\theta) = a_1 b_1 + a_2 b_2 = a^T b = b^T a = \langle a, b \rangle.$$

The last term is called an inner product.

In general, if a and b are not unit vectors, then

$$\cos(\theta) = \frac{a^T b}{\sqrt{(a^T a)} \sqrt{(b^T b)}},$$

where $\sqrt{(a^T a)}$ is length of a and $\sqrt{(b^T b)}$ is length of b.

a and b are perpendicular (i.e. $\theta = 90^\circ$) iff

$$\cos(\theta) = 0 = a^T b \quad (\text{or} \quad b^T a = 0)$$

Definition

In general, if a is an n -dim vector, then length of a

$$= \sqrt{(a^T a)} = \sqrt{(a_1^2 + a_2^2 + \dots + a_n^2)}.$$

Definition

Suppose $a, b \in \mathbb{R}^n$, a and b are said to be orthogonal ("perpendicular" in 2D) iff

$$a^T b \quad (\text{or} \quad b^T a) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = 0.$$

Definition

If $a^T a = 1 = |a|$, then a is said to be normalized. (Any vector b can be normalized by

$$c = \frac{b}{\sqrt{(b^T b)}}.$$

Proof. $c^T c = \dots = 1$.

Definition

A set of p -dim vectors c_1, c_2, \dots, c_p that are normalized and mutually orthogonal are said to be an orthonormal set of vectors.

i.e.

i) $c_i^T c_i = 1$ for any i

ii) $c_i^T c_j = 0$ for any $i \neq j$

Definition

If a $p \times p$ matrix $C = (c_1, c_2, \dots, c_p)$ has orthonormal columns, then C is called an orthogonal matrix.

Corollary

If C is orthogonal, then $C^T C = I$.

$$C^T C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_p^T \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} =$$

(i, j) entry of $C^T C = c_i^T c_j = 0$ when $i \neq j$. $C^T C = c_i^T c_j = 1$ when $i = j$, so

$$C^T C = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I.$$

Remark.

$$C^T C = I \iff C C^T = I$$

$$C \text{ is orthogonal} \iff C C^T = I$$

$C C^T = I$ means that the rows of C are orthonormal.

End. Verify A^{-1} and review the final definition.