ST 661 Note

Frances Lin

1/20/2022

Outline

- Idempotent matrix
- Matrix calculus

Last time

Let A_1 be the 1st subspace and A_2 be the second subspace, then we can write x s.t.

$$x = A_1x + A_2x \quad \forall x \in \mathbb{R}^n$$

$$I = A_1 + A_2 \ \ , \ rank(I) = 2$$
$$rank(A_1) = 1 rank(A_2) = 2$$

Both A_1 and A_2 are idempotent matrices.

Now, we pick $x \in \mathbb{R}^3$ and decompose it s.t.

$$x = A_1x + A_2x,$$

where

$$rank(A_1) = 2 \quad a\ plane rank(A_2) = 1 \quad a\ line,$$

so

$$rank(I) = 3 = 2 + 1 = rank(A_1) + rank(A_2).$$

Thm

\$I: \$ nxn, $I = A_1 + A_2 + ...A_k$ (k can be anything $\leq n$), where each A_i is nxn symmetric of rank r_i.

If $\sum_{i=1}^{k} = n$ (no gain or loss of rank), then

i) A_i (each A_i) is idempotent $i = 1, 2, ...k$

ii) $A_iA_j = 0$ $i \neq j$ (complementary)

Comment. In general, $rank(A + B) \neq rank(A) + rank(B)$. So when it is equal, then all A_i are idempotent.

e.g. $n = 2$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_1 + A_2$$

So, $rank(A_1) + rank(A_2) = 2 = rank(I)$.

=>

i) A_1 A_2 are both idempotent

ii) $A_1 A_2 = 0$ Info does not overlap.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix},$$

projecting to x axis.

$$A_2 x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$

projecting to y axis.
e.g.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

In this case,

$$rank(I) = 2 \neq 2 + 2 = rank(A_1) + rank(A_2)$$

Condition is not met. Info overlap.

ii)

$$A_1 A_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \neq 0,$$

so info overlap.

Vector or matrix calculus (derivatives mostly)

1. vector $\rightarrow$ scalar

$u = f(x)$, where u is a scalar, x is a column vector and

$$x = \begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix},$$

then

$$\frac{\partial u}{\partial x} = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_p} \end{bmatrix}.$$

Thm

Let $u = a^T x$ and $a = (a_1, a_2, ...a_p)^T$ is a constant vector, then

$$\frac{\partial u}{\partial x} = a,$$

i.e.

$$\frac{\partial u}{\partial x_1} = a_1.$$

Thm

page 3