

# ST 661 Note

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## Outline

- Eigenvalue, eigenvector, eigenspace
- Idempotent matrix

## Last time

Whenever we have 2 eigenvalues, we would like to find eigenvalues s.t. they are perpendicular, so they span the entire plane.

## Thm

If  $A$  is  $n \times n$  symmetric with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  and corresponding eigenvectors (orthogonalized & normalized)  $x_1, x_2, \dots, x_n$ , then  $A$  can be expressed as

$$A = CDC^T,$$

where

$$C = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is an orthogonal (w/ eigenvectors) and

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is diagonal (w/ eigenvalues). This is called spectral decomposition (or eigendecomposition).

## Corollary

If  $A$  is symmetric and  $C$  and  $D$  are defined as before, then

$$C^T A C = D \iff A = CDC^T.$$

This is called diagonalization.

e.g.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

then what this transformation does is that it doubles  $x_1$  and triples  $x_2$ .

Say, for example,  $x = [0.5, 1.5]$ . In order to quantify it, we set bases to be  $[0, 1]$  and  $[1, 0]$  to construct x and y axes. In general, choices of bases are arbitrary:

$C^T b$

$DC^T b$  gets matrix amplification

$DC^T b$  gets to the new system

$CDC^T b$  transforms it back.

Note. Double check geometric of spectral decomposition.

## Thm

If  $A$  is  $n \times n$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then

i)  $\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$

ii)  $\text{trace}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$

Proof. HW. Assume  $A$  is symmetric and then use spectral decomposition. In general,  $A$  does not need to be symmetric.

Remark.

i)  $A$  is singular  $\iff \det(A) = 0 \iff$  at least one  $\lambda_i = 0$  since Thm i)

ii)  $\text{rank}(A) = \#$  of nonzero  $\lambda_i$

In general,  $A = CDC^T$  and check  $D$ .

Proof.  $A = CDC^T$  where  $C$  is orthogonal, then

$$\text{rank}(A) = \text{rank}(CDC^T) = \text{rank}(D)$$

since \* orthogonal does not change rank

$$= \text{rank}\left(\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\right)$$

$$= \# \text{ of nonzero } \lambda_i$$

e.g.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

with  $\lambda_1 = 3, \lambda_2 = 2$ . Verify that

i)  $\det(A) = 1 \cdot 4 - 2 \cdot (-1) = 6 = 3 \cdot 2 = \lambda_1 \lambda_2$

ii)  $\text{tr}(A) = 1 + 4 = 5 = 3 + 2 = \lambda_1 + \lambda_2$

## Thm

Let  $A$  be  $n \times n$  symmetric with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ ,

- i)  $A$  is pd  $\iff \lambda_i > 0 \ \forall i$
- ii)  $A$  is psd  $\iff \lambda_i \geq 0 \ \forall i$

Proof. HW.

i) makes sense. Full rank.

ii) Consider  $A^{1/2}$ , if  $A$  is psd, then  $A = CDC^T$  where  $D$  is diagonal with  $\lambda_i \geq 0$ . Define

$$A^{1/2} = CD^{1/2}C,$$

where

$$D = \begin{bmatrix} \lambda_1^{1/2} & & & \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ & & & \lambda_n^{1/2} \end{bmatrix}$$

Ques. Why does it make sense to take  $D^{1/2}$ ?

$$A^{1/2}A^{1/2} = (CD^{1/2}C^T)(CD^{1/2}C^T) = CD^{1/2}ID^{1/2}C^T = CD^{1/2}D^{1/2}C^T = CDC^T = A$$

$A$  is  $n \times n$ ,  $x \in \mathbb{R}^n$   $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $x \rightarrow Ax$

For some  $A$ , the mapping projects any vector  $x$  into a linear subspace of  $\mathbb{R}^n$ .

e.g. For  $n = 2$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$Ax = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{bmatrix}$$

$A$  is a projection matrix that projects  $x$  to the subspace  $x_1 = x_2$ .

Ques. Any restriction?

We need  $A^2x = Ax \ \forall x$ . Therefore, we need  $A^2 = A$  in order for  $A$  to be a projection. (In general,  $A^n = A$ .) This is also why  $A$  is also called idempotent matrix.

## Def

$A$  is a square matrix and  $A$  is said to be idempotent if  $A^2 = A$ .

Observation. If  $A$  is nonsingular and idempotent, then  $A = I$ . We will focus on idempotent matrices that are also symmetric. This class of matrices are easy to deal with because of the decomposition.

## Thm

If  $A$  is symmetric and idempotent, then  $A$  is psd.

Proof.

$$A = A^2 = AA^T,$$

so  $A$  is psd.

If  $A$  is symmetric, then  $A$  can be written as  $A = CDC^T$ .

$A$  is idempotent  $\Leftrightarrow A^2 = A$

$$\Leftrightarrow (CDC^T)(CDC^T) = CDC^T$$

$$\Leftrightarrow CD^2C^T = CDC^T$$

$$\Leftrightarrow D^2 = D$$

Noice that if  $C$  is nonsingular, then  $C^T$  is nonsingular.

Recall that

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Leftrightarrow \lambda_i^2 = \lambda_i \quad \forall i = 1, 2, \dots, n$$

$$\Leftrightarrow \lambda_i = 0 \text{ or } 1 \quad \forall i = 1, 2, \dots, n$$

## Thm

If  $A$  is  $n \times n$  symmetric, idempotent, and of  $\text{rank}(A) = r \leq n$ , then  $A$  has  $r$  eigenvalues = 1 and  $(n - r)$  eigenvalues = 0.

Remark.  $\text{tr}(A) = r = \sum_{i=1}^n \lambda_i$

e.g. before

$$Ax = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A$  is 2D, but  $\text{rank}(A) = 1$ . Dimension of subspace has  $\text{rank}(A) = 1$ , which is a line  $x_1 = x_2$ . I.e. Original  $x$  lives in  $\dim = 2$ ,  $A$  projects  $x$  to the subspace  $x_1 = x_2$ . Some information may be lost. However, we can find a perpendicular subspace  $Bx$  that perserves the information. Here,

$$x = Ax + Bx.$$

$x$  can be projected to two subspaces  $Ax$  and  $Bx$ . Suppose we know  $Ax$ , then  $Bx$  is easy to recover since

$$Bx = x - Ax = (I - A)x \quad \forall x$$

$$\Leftrightarrow B = I - A$$

can also be shown to be idempotent.

## Thm

If  $A$  is  $n \times n$  idempotent,

- i)  $(I - A)$  is also idempotent. (Projection into the complementary space.)

Proof. Verify that  $(I - A)^2 = I - A$  and use  $A$  is idempotent.

- ii)  $A(I - A) = A - A^2 = 0$  or  $(I - A)A = 0$

Idea. Once we project to  $Ax$ , is there anything that is in  $Ax$  that is informative? No.

- iii) If  $P$  is  $n \times n$  nonsingular, then  $P^{-1}AP$  is idempotent.

- iv) If  $C$  is orthogonal, then  $C^TAC$  is idempotent.

## Summary

Suppose  $A$  is  $n \times n$ ,  $P$  is  $n \times n$  nonsingular, and  $C$  is  $n \times n$  orthogonal and let  $P^{-1}AP$  be similarity transformation and  $C^TAC$  be conjugate transformation, then list of properties include:

	$P^{-1}AP$	$C^TAC$
rank	yes	yes
determinant	yes	yes
trace	yes	yes
eigenvalue	yes	yes
eigenvector	no	no
symmetric	no	yes
pd/psd or not	N/A	yes
idempotent	yes	yes