ST 661 Note

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Outline

Orthogonal matrices (Cont.)

Trace

Eigenvales and eigenvector

- general
- functions of matrices
- symmetric
- pd

Definition

$$C \text{ is orthogonal} <=> C^T C = I <=> CC^T = I <=> C^{-1} = C^T$$

Thm

If C is a pxp orthogonal and A is pxp,

i) $det(C) = \pm 1$

This is because $C^T C = I$, $det(C^T C) = det(C^T)det(C) = det(C)det(C) = 1$.

ii) $det(C^T AC) = det(A)$,

where $C^T AC$ is called conjugate transformation of A.

proof.

$$det(C^T AC) = det(ACC^T) = det(ACC^T) = det(AI) = det(A) \qquad \square$$

iii) $-1 \leq C_{ij} \leq 1$ for all i, j since they were normalized.

Trace (For Square Matrices Only)

For nxn matrix $A = (a_{ij})$, $1 \leq i, j \geq n$, trace is a linear scalar function of A s.t. $tr(A) = \sum_{i=1}^{N} a_{ij}$.

Thm

i) If A, B are both nxn, then
$$tr(A+B) = tr(A) + tr(B).$$

ii) For A: nxp and B: pxn,
$$tr(AB) = tr(BA).$$

Proof. Exercise. Use definition of trace.

Example. If C is orthogonal, is $tr(C^T AC) = tr(A)$?

Proof. Yes, and the proof is similiar to $det(A^T AC) = det(A)$. Here, we have
$$tr(C^T AC) = tr(ACC^T) = tr(AI) = tr(A).$$

This is called the conjugate transformation. There are other transformations. Suppose P is nonsingular, then the similarity transformation is given as $P^{-1}AP$ and
$$tr(P^{-1}AP) = tr(A).$$

Check p.59 of PDF for more.

Taking A Step Back

Saying A is nxp matrix is same as saying that $A \in \mathbb{R}^n$. A can also be thought as a linear transformation. If
$$A_{nxp}X_{px1} = nx1,$$
then $A : \mathbb{R}^p \to \mathbb{R}^n$ for $x \to AX$.

If A is nxn, then $A : \mathbb{R}^n \to \mathbb{R}^n$ for $x \to AX$.

Next, we want to ask

1. For what A, we have $||x||$ (length of x) = $||Ax||$ for all $x \in \mathbb{R}^n$ (i.e. rotation)?

To make
$$||x|| = \sqrt(x^T x) = \sqrt((Ax)^T (Ax)) = \sqrt(x^T A^T Ax),$$
we need
$$x^T x = x^T A^T Ax.$$

Or more like we need
$$x^T Ix = x^T A^T Ax <=> x^T (A^T A - I)x = 0 \ \forall x$$

$$=> A^T A = I$$

<=> A has to be orthogonal (and A is a rotation matrix). I.e., orthogonal matrices preserve length.

2. For what A, do we have x and Ax in the same direction $\forall x$?

x and ax are in the direction ∀ a. For scalar a, we need

$$Ax = ax$$

Then

$$A = aI,$$

if a is the same since

$$(A - aI)x = 0 \ \forall x.$$

Or more flexible A can be

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

Relaxing it a bit, we can ask

3. For a given A, what x guarantee that (nontrivial) x and Ax are in the same direction?

Eigenvalues

A is nxn. A scalar λ is called an eigenvalue of A is $\exists$ x s.t.

$$Ax = \lambda x.$$

Such a vector x is called an eigenvector of A.

To find λ and x, set

$$(A - \lambda I)x = 0.$$

Since $(A - \lambda I)$ has to be singular $<=>$ $det(A - \lambda I) = 0$. This is a degree n polynomial about λ. In addition, root of degree n polynomial has n items and exists, so λ always exists.

Characteristic Equation

$(A - \lambda I)x = 0$ is called the characteristic (or polynomial) equation of A. λ is an eigenvalue of A and x is a corresponding eigenvector.

$Ax = \lambda x$

Claim. $2x$ is also eigenvector of A corresponding to λ.

Proof.

$$A(2x) = 2Ax = 2\lambda x = \lambda(2x)$$

In general, cx is also an eigenvector of A, where c is a scalar and $\neq 0$. Among this class of eigenvectors, we often require $||x^T x|| = 1$ for eigenvector.

Ques. Suppose λ is an eigenvalue of A with corresponding eigenvector x, when does $g(A)$ has the same λ?

i) $g(A) = cA$ where c is a scalar

Proof.

$$Ax = \lambda x => (cA)x = c\lambda x,$$

so $c\lambda$ is an eigenvalue of cA.

ii) $g(A) = cA + bI$

$$g(A)x = (cA + bI)x = cAx + bIx = cAx + bx = c\lambda x + bx = (c\lambda + b)x,$$

so $(c\lambda + b)$ is an eigenvalue of $g(A) = cA + bI$.

iii) Suppose $g(A) = A^2$, is λ^2 an eigenvalue of A^2? Yes.

Proof.

$$(A^2x) = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x \qquad \square$$

In general, A^k has λ^k as an eigenvalue.

iv) Combining ii) and iii), we get that

$$(A^3 + 4A^2 - 3A + 3I)x = A^3x + 4A^2x - 3A + 3x = \lambda^3x + 4\lambda^2x - 3\lambda x + 3x = (\lambda^3 + 4\lambda^2 - 3\lambda + 3)x.$$

v) Suppose A is nonsingular (has inverse), $g(A) = A^{-1}$ has $\frac{1}{\lambda}$ as an eigenvalue.

Claim. If A is nonsingular, then $\lambda \neq 0$.

Proof. If $\lambda = 0$, then $\exists\ x \neq 0$ s.t. $Ax = 0\ x = 0$.

Thm

A is nxn,

i) If P is nonsingular, then $P^{-1}AP$ and A share the same eigenvalue.

ii) If C is orthogonal, then C^TAC and A share the same eigenvalue.

Proof. HW. Remark. Eigenvectors may be different.

(!Important) Thm

A is nxn symmetric (i.e. $A = A^T$ or $a_{ij} = a_{ji}$), then

i) The eigenvalue λ_1, λ_2..., λ_n are real (some of them have the same eigenvetor).

ii) The eigenvector x_1, x_2..., x_k of A corresponding to the distinct eigenvalue λ_1, λ_2..., λ_k are mutually orthogonal and the eigenvector x_{k+1}, x_{k+2}..., x_{k+n} corresponding to the nondistinct eigenvalue can be chosen to be mutually orthogonal, so $x_i^T x_j^T = 0\ \forall\ i \neq j$.

Ques. Why though? C is nxn

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} | & | & \cdots & \cdots & | \end{bmatrix}$$

C is an orthogonal matric becuase using this this thm, there is way to make sure we have orthogonal vector.

$$AC = A(x_1, x_2, ...x_n) = (Ax_1, Ax_2, ...Ax_n) = (\lambda_1 x_1, \lambda_2 x_2, ...\lambda_n x_n) = C \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = (or \quad CD),$$

where x_i are eigenvectors.

$AC = CD$ where C is an orthogonal matrix.

=> $A = CDC^T$ for any symmetric A (This is called spectral decomposition.)

<=> $C^T AC = D$ This is called the diagonization of matrix.

Check p.66 of PDF for more.