ST 661 Note

Frances Lin

1/18/2022

Outline

- Eigenvalue, eigvenvector, eigenspace
- Idempotent matrix

Last time

Whenever we have 2 eigenvalues, we would like to find eigenvalues s.t. they are perpendicular, so they span the entire plane.

Thm

If A is nxn symmetric with eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$ and corresponding eigenvectors (orthogonalized & normalized) $x_1, x_2, ...x_n$, then A can be expressed as

$$A = CDC^T,$$

where

$$C = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is an orthogonal (w/ eigenvectors) and

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is diagonal (w/ eigenvalues). This is called spectral decomposition (or eigendecomposition).

Corollary

If A is symmetric and C and D are defined as before, then

$$C^T AC = D <=> A = CDC^T.$$

This is called diagonalization.

e.g.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

then what this transformation does is that it doubles x_1 and triples x_2.

Say, for example, $x = \begin{bmatrix} 0.5, 1.5 \end{bmatrix}$. In order to quantify it, we set bases to be $\begin{bmatrix} 0, 1 \end{bmatrix}$ and $\begin{bmatrix} 1, 0 \end{bmatrix}$ to construct x and y axes. In general, choices of bases are arbitrary:

$C^T b$

$DC^T b$ gets matrix amplification

$DC^T b$ gets to the new system

$CDC^T b$ transforms it back.

Note. Double check geometric of spectral decomposition.

Thm

If A is nxn with eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$, then

i) $det(A) = \lambda_1 \lambda_2 ... \lambda_n$

ii) $trace(A) = \lambda_1 + \lambda_2 ... + \lambda_n$

Proof. HW. Assume A is symmetric and then use spectral decomposition. In general, A does not need to be symmetric.

Remark.

i) A is singular $<=>$ $det(A) = 0$ $<=>$ at least one $\lambda_i = 0$ since Thm i)

ii) $rank(A) = \#$ of nonzero λ_i

In general, $A = CDC^T$ and check D.

Proof. $A = CDC^T$ where C is orthogonal, then

$$rank(A) = rank(CDC^T) = rank(D)$$

since $*$ orthogonal does not change rank

$$= rank(\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix})$$

$= \#$ of nonzero λ_i

e.g.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

with $\lambda_1 = 3$, $\lambda_2 = 2$. Vertify that

i) $det(A) = 1x4 - 2x(-1) = 6 = 3x2 = \lambda_1 \lambda_2$

ii) $tr(A) = 1 + 4 = 5 = 3 + 2 = \lambda_1 + \lambda_2$

Thm

Let A be nxn symmetric with eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$,

i) A is pd <=> $\lambda_i > 0 \; \forall i$
ii) A is psd <=> $\lambda_i \geq 0 \; \forall i$

Proof. HW.

i) makes sense. Full rank.
ii) Consider $A^{1/2}$, if A is psd, then $A = CDC^T$ where D is diagonal with $\lambda_i \geq 0$. Define

$$A^{1/2} = CD^{1/2}C,$$

where

$$D = \begin{bmatrix} \lambda_1^{1/2} & & & \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ & & & \lambda_n^{1/2} \end{bmatrix}$$

Ques. Why does it make sense to take $D^{1/2}$?

$$A^{1/2}A^{1/2} = (CD^{1/2}C^T)(CD^{1/2}C^T) = CD^{1/2}ID^{1/2}C^T = CD^{1/2}D^{1/2}C^T = CDC^T = A$$

A is nxn, $x \in \mathbb{R}^n$ $A : \mathbb{R}^n \to \mathbb{R}^n$ for $x \to Ax$

For some A, the mapping projects any vector x into a linear subspace of $\mathbb{R}^n$.

e.g. For $n = 2$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$Ax = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{bmatrix}$$

A is a projection matrix that projects x to the subspace $x_1 = x_2$.

Ques. Any restriction?

We need $A^2x = Ax \; \forall x$. Therefore, we need $A^2 = A$ in order for A to be a projection. (In general, $A^n = A$.) This is also why A is also called idempotent matrix.

Def

A is a square matrix and A is said to be idempotent if $A^2 = A$.

Observation. If A is nonsingular and idempotent, then $A = I$. We will focus on idempotent matrices that are also symmetric. This class of matrices are easy to deal with because of the decomposition.

Thm

If A is symmetric and idempotent, then A is psd.

Proof.

$$A = A^2 = AA^T,$$

so A is psd.

If A is symmetric, then A can be written as $A = CDC^T$.

A is idempotent <=> $A^2 = A$

$$\begin{aligned} &<=> (CDC^T)(CDC^T) = CDC^T \\ &<=> CD^2C^T = CDC^T \\ &<=> D^2 = D \end{aligned}$$

Noice that if C is nonsingular, then C^T is nonsingular.

Recall that

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$<=> \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$<=> \lambda_i^2 = \lambda_i \qquad \forall i = 1, 2, ...n$$

$$<=> \lambda_i = 0 \; or \; 1 \qquad \forall i = 1, 2, ...n$$

Thm

If A is nxn symmetric, idempotent, and of $rank(A) = r \leq n$, then A has r eigenvalues $= 1$ and $(n - r)$ eigenvalues $= 0$.

Remark. $tr(A) = r = \sum_{i=1}^{n} \lambda_i$

e.g. before

$$Ax = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A is 2D, but $rank(A) = 1$. Dimension of subspace has $rank(A) = 1$, which is a line $x_1 = x_2$. I.e. Original x lives in $dim = 2$, A projects x to the subspace $x_1 = x_2$. Some information may be lost. However, we can find a perpendicular subspace Bx that perserves the information. Here,

$$x = Ax + Bx.$$

x can be projcted to two subspaces Ax and Bx. Suppose we know Ax, then Bx is easy to recover since

$$Bx = x - Ax = (I - A)x \qquad \forall x$$

$$<=> B = I - A$$

can also be shown to be idempotent.

Thm

If A is nxn idempotent,

i) $(I - A)$ is also idempotent. (Projection into the complementary space.)

Proof. Verify that $(I - A)^2 = I - A$ and use A is idempotent.

ii) $A(I - A) = A - A^2 = 0$ or $(I - A)A = 0$

Idea. Once we project to Ax, is there anything that is in Ax that is informative? No.

iii) If P is nxn nonsingular, then $P^{-1}AP$ is idempotent.

iv) If C is orthogonal, then C^TAC is idempotent.

Summary

Suppose A is nxn, P is nxn nonsingular, and C is nxn orthogonal and let $P^{-1}AP$ be similarity transformation and C^TAC be conjugate transformation, then list of properties include:

	$P^{-1}AP$	C^TAC
rank	yes	yes
determinant	yes	yes
trace	yes	yes
eigenvalue	yes	yes
eigenvector	no	no
symmetric	no	yes
pd/psd or not	N/A	yes
idempotent	yes	yes