COMP 557 Study guide

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Introduction

1 Disclaimer

These notes are curated from Professor Paul Kry COMP557 lectures at McGill University. They are for study purposes only. They are not to be used for monetary gain.

2 About This Guide

I make my notes freely available for other students as a way to stay accountable for taking good notes. If you spot anything incorrect or unclear, don't hesitate to contact me via Facebook or e-mail at `http://francispiche.ca/contact/`

Part II
Math Review

3 Notation

3.1 Implicit Representation

$$(1)\{\vec{v}|f(\vec{v}) = 0\}$$

In English, this means the set of all vectors v, such that some function f of v gives zero. For example, if $f = \vec{v} \cdot \vec{u} + k$. Then the set of all vectors which satisfy (1) would be some line (since the dot product gives a scalar, and the k is a scalar. So we need to dot product to be $-k$). If k is 0, then the set is just the set of vectors orthogonal to $\vec{u}$, which is a line.

Another example:

$$\{\vec{v}|(\vec{v} - \vec{p}) \cdot (\vec{v} - \vec{p}) - \vec{r}^2 = 0\}$$

This is just a fancy way of expressing the equation of a circle centered at $\vec{p}$. (Plug some sample vectors in if you don't believe it)

3.2 Explicit (Parametric) Representation

A parameter is given with a specified domain to describe the equation. For example:

$$\{\vec{p} + r\begin{pmatrix} cos(t) \\ sin(t) \end{pmatrix} |t \in [0, 2\pi]\}$$

describes a circle.

4 Linear Transformations

A great video (and series) to visualize the linear algebra used in graphics is this from 3Blue1Brown on YouTube.

Transformations are like "functions" that operate on a set of points.

Parametric form of a mapping from one set to another using some transform T:

$$\{f(t)|t \in D\} \rightarrow \{T(f(t))|t \in D\}$$

Implicit form:

$$\{\vec{v}|f(\vec{v}=0)\} \rightarrow \{T(\vec{v})|f(\vec{v})=0\}$$
$$=\{\vec{v}|f(T^{-1}(\vec{v}))=0\}$$

To convince yourself of that last equality, try it on few examples.

Translation:

$$T(\vec{v}) = \vec{v} + \vec{u}$$

See the slides for visual representations of the common linear transformations: (translation, sheer, scale, rotation, reflection etc.) The video mentioned above is also nice for getting a feel of how it works.

4.1 Combining Linear Transforms with Translation

Could do it this way:

$$T(\vec{p}) = M\vec{p} + \vec{u}$$

for some matrix M, but if you try to do that with a composition:

$$T(\vec{p}) = M_T\vec{p} + \vec{u}_T$$

$$S(\vec{p}) = M_S\vec{p} + \vec{u}_S$$

then

$$(S \circ T) = M_S(M_T\vec{p} + \vec{u}_T) + \vec{u}_S$$

which is honestly pretty gross to look at. We can do better using *homogeneous coordinates*.

4.1.1 Homogeneous Coordinates

This is the use of a 3x3 matrix to perform translation with a linear transformation. We add an extra component w, to our 2x2 vectors, and an extra row $(0,0,w)$ and column $\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}$ For points in an affine space, $w = 1$.

Linear transformations:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \\ 1 \end{pmatrix}$$

Translation (uses the extra column):

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+t \\ y+s \\ 1 \end{pmatrix}$$

If we now do composition, we can do it like this (using block notation):

$$\begin{pmatrix} M_S & \vec{u}_S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_T & \vec{u_T} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} M_S M_T & M_S \vec{u_T} + \vec{u_S} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

Which is essentially the same, but cleaner and will be more useful later.

4.2 Affine Transformations

These are transformations in which lines that were straight, and lines that were parallel to each other are still straight and parallel to each other. Also, the ratios of lengths along lines are preserved.

Common transforms:

Translation:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

is a translation of t_x in the x direction and t_y in the y.

Scale:

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to scale s_x in the x-axis and s_y in the y-axis.

4.3 Rigid Motions

A transformation made up of only translation and rotation is a rigid motion.

Note that for rotations, the inverse is the transpose (rotation matrices are orthogonal), and so the inverse of a rigid motion E is:

$$E = \begin{pmatrix} R & \vec{u} \\ 0 & 1 \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} R^T & -R^T\vec{u} \\ 0 & 1 \end{pmatrix}$$

4.4 Composing to change axes

To rotate about a point other than the origin, first we translate, then rotate, and translate back.

$$M = T^{-1}RT$$

To scale along a particular axis and point, you would move it to the point, rotate so that the axis lines up with the x or y axis, scale, then undo all the operations.

$$M = T^{-1}R^{-1}SRT$$

4.5 Points vs. Vectors

Points and vectors are NOT the same. Points are locations in space, whereas vectors can be thought of displacements in space, or a tuple of distance and direction between points.

In homogeneous coordinates, vectors have $w = 0$.

Translations do not affect vectors:

$$\begin{pmatrix} M & t \\ 00 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} Mp + t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} M & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} Mv \\ 0 \end{pmatrix}$$

Part III