# 7.5 Homogenous Linear Systems

**Problem 2**
(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.
(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$$

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**(a)** Solving this system we assume that $\mathbf{x} = \xi e^{rt}$ and substitute into our system to obtain

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

After setting the determinant equal to zero we end up with

$$(r+1)(r+2) = 0 \quad \Rightarrow r_1 = -1, r_2 = -2$$

For $r_1 = -1$ our system becomes

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

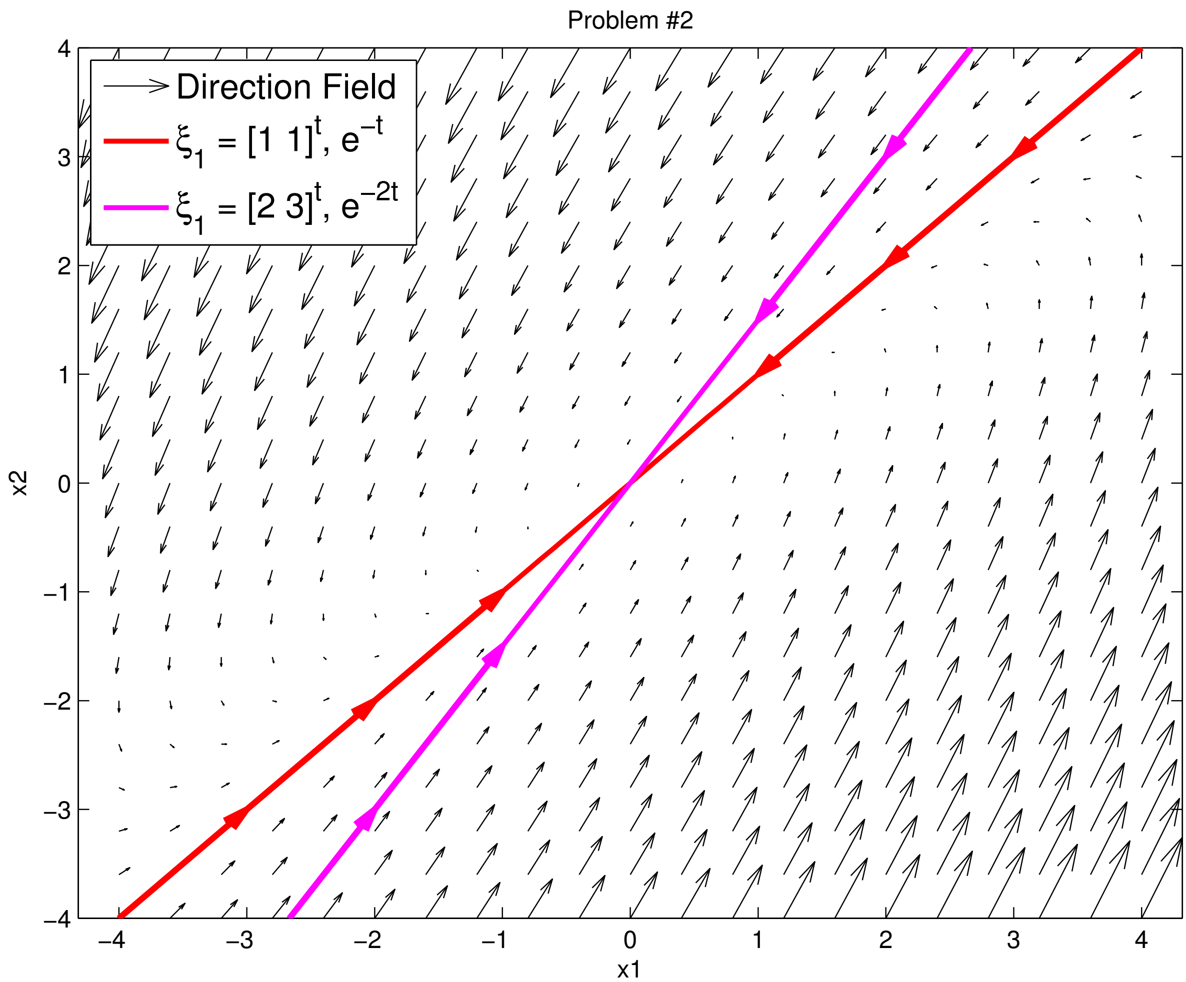
Repeating this process for $r_2 = -2$ our second eigenvector is

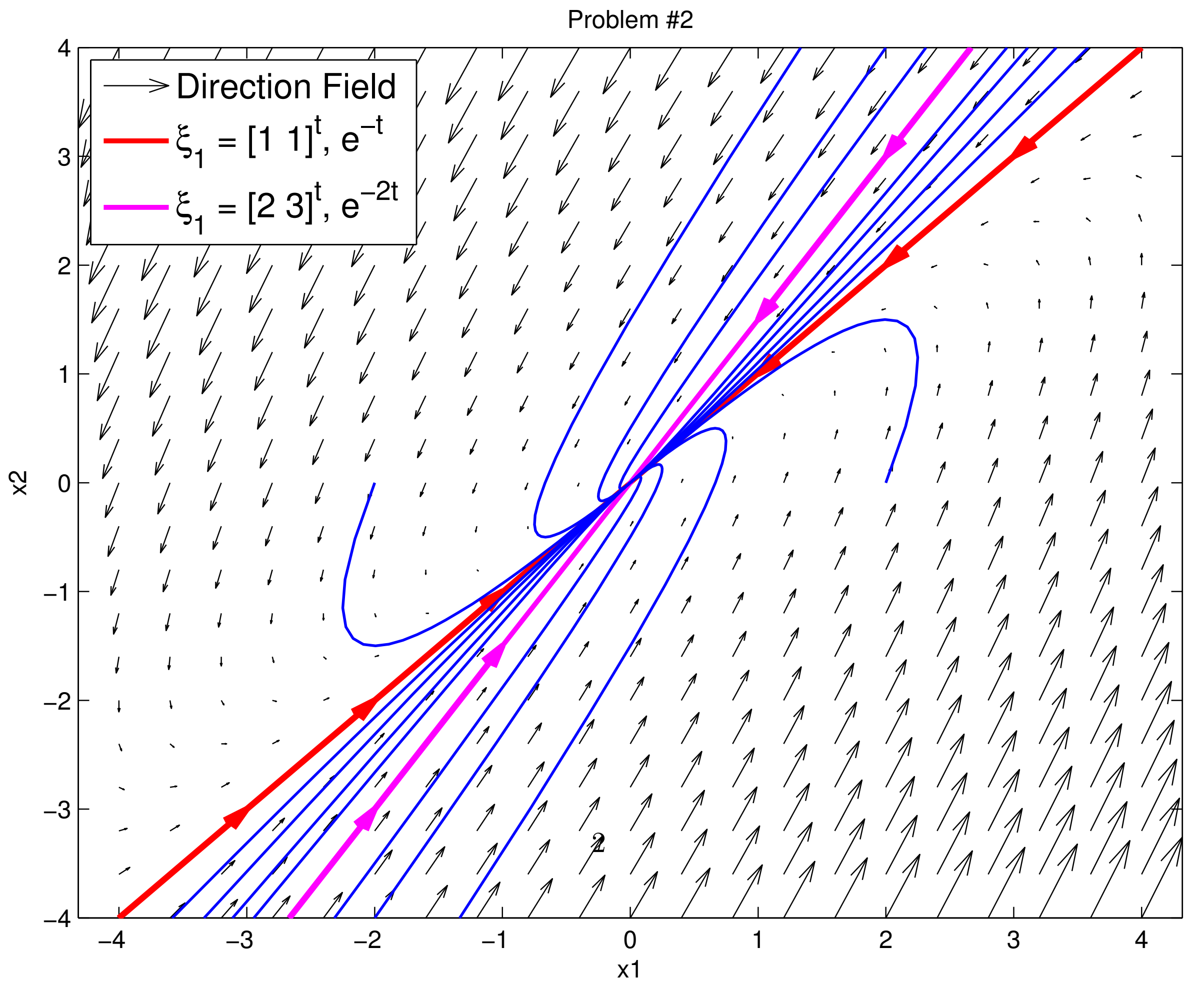
$$\xi^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Thus, the general solution is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}}$$

**(b)** Before using technology to obtain the direction field let us analyze our solution to the system. Because our eigenvalues are both negative, $r_1 = -1, r_2 = -2$, we have a stable node. This means our trajectories will converge to the origin of our plot. To determine where there are possible changes in the direction field we plot a line that has the slope of our eigenvectors.


Problem #2
Direction Field
$\xi_1$ = [1 1]$^t$, $e^{-t}$
$\xi_1$ = [2 3]$^t$, $e^{-2t}$
x1
x2


Problem #2
Direction Field
$\xi_1$ = [1 1]$^t$, $e^{-t}$
$\xi_1$ = [2 3]$^t$, $e^{-2t}$
x1
x2

**Problem 4**
(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.
(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

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**(a)** Solving this system we assume that $\mathbf{x} = \xi e^{rt}$ and substitute into our system to obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

After setting the determinant equal to zero we end up with

$$(r+3)(r-2) = 0 \quad \Rightarrow r_1 = -3, r_2 = 2$$

For $r_1 = -3$ our system becomes

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

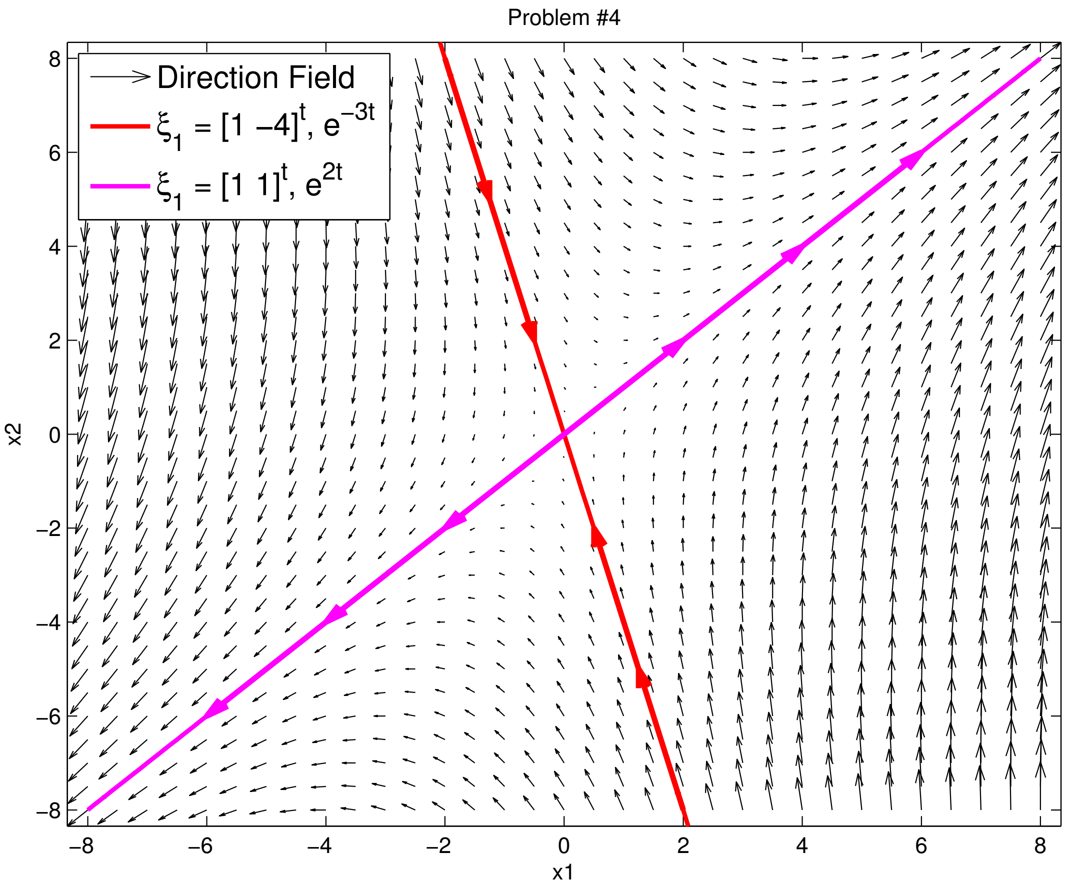
Repeating this process for $r_2 = 2$ our second eigenvector is

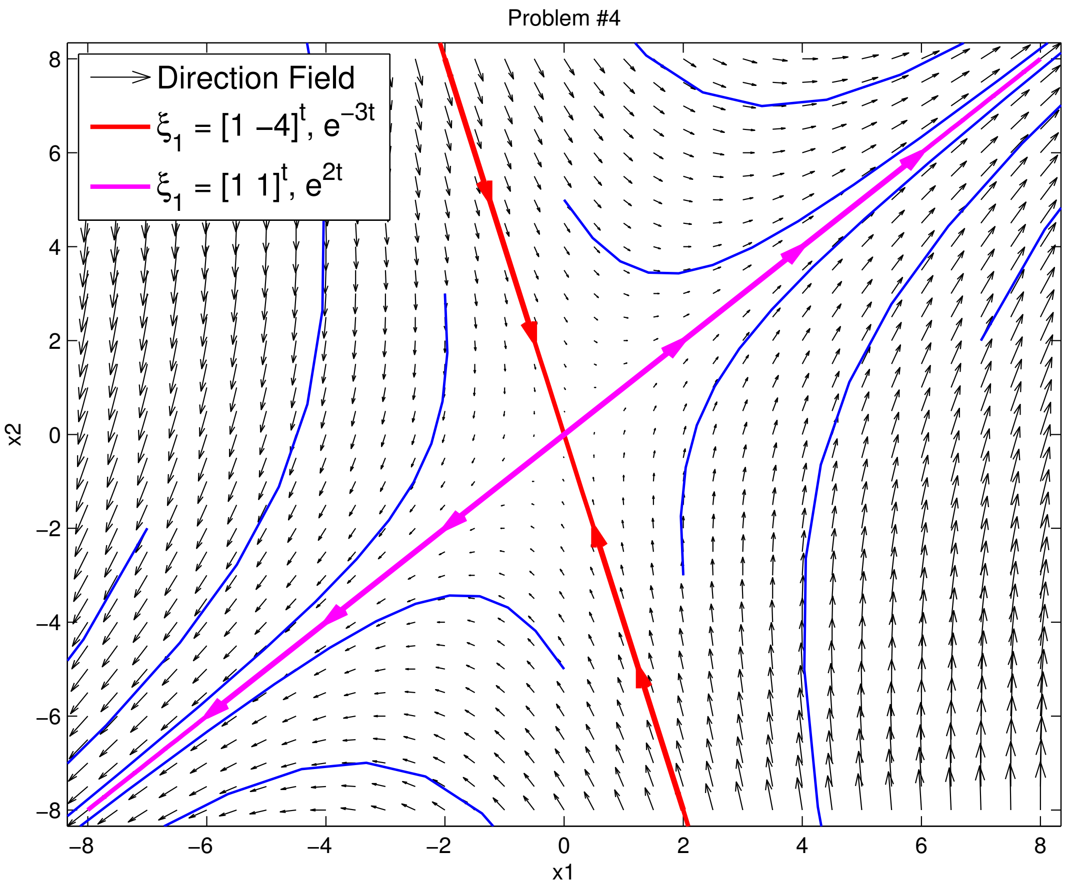
$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

**(b)** Before using technology to obtain the direction field let us analyze our solution to the system. Because our eigenvalues are opposite signs, $r_1 = -3, r_2 = 2$, we have a saddle node. We expect the trajectories to follow along our positive eigenvalue solution, $e^{2t}$, as $t \to \infty$. To determine where there are possible changes in the direction field we plot a line that has the slope of our eigenvectors. Notice how the eigenvector with the increasing exponential goes outward ($e^{2t}$), whereas the eigenvector with the decreasing exponential goes inward ($e^{-3t}$).


Problem #4
Direction Field
ξ1 = [1 −4]t, e−3t
ξ1 = [1 1]t, e2t
x1
x2


Problem #4
Direction Field
ξ1 = [1 −4]t, e−3t
ξ1 = [1 1]t, e2t
x1
x2

**Problem 6**

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$$

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**(a)** Solving this system we assume that $\mathbf{x} = \xi e^{rt}$ and substitute into our system to obtain

$$\begin{pmatrix} \frac{5}{4} - r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

After setting the determinant equal to zero we end up with

$$(2r-1)(r-2) = 0 \qquad \Rightarrow r_1 = \frac{1}{2}, r_2 = 2$$

For $r_1 = \frac{1}{2}$ our system becomes

$$\begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

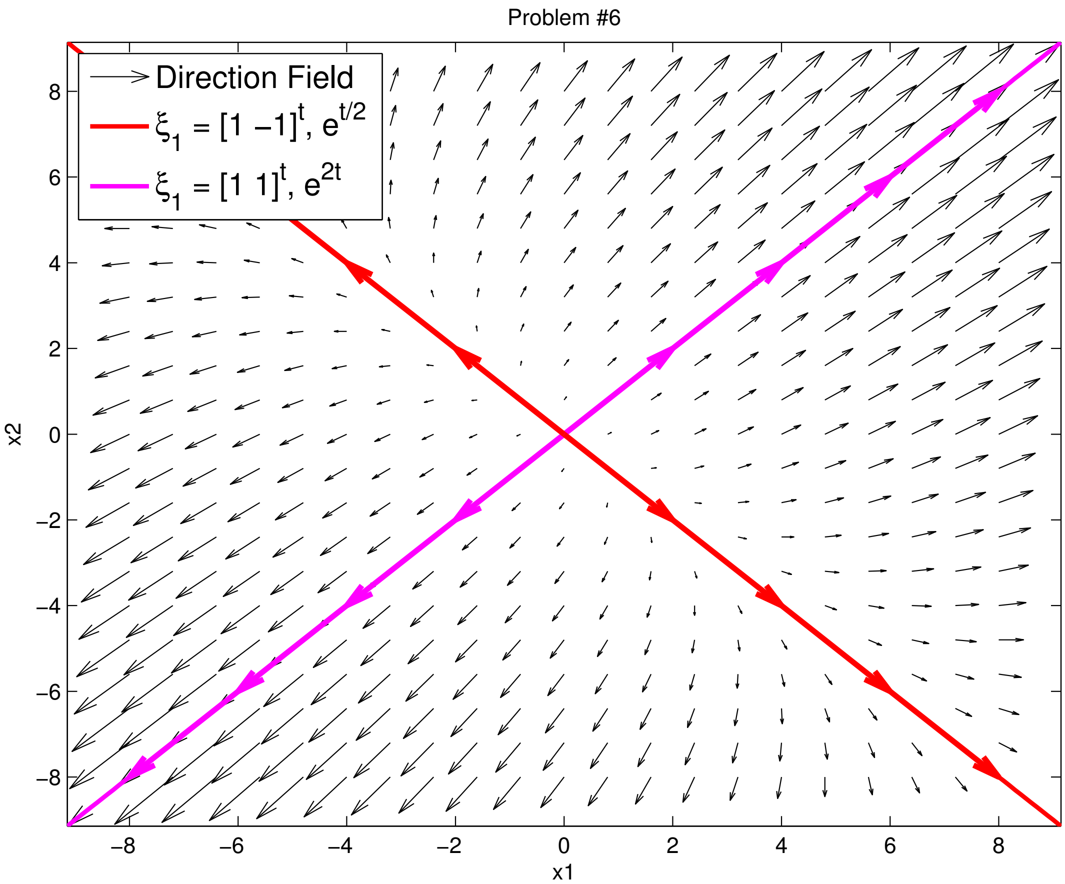
Repeating this process for $r_2 = 2$ our second eigenvector is

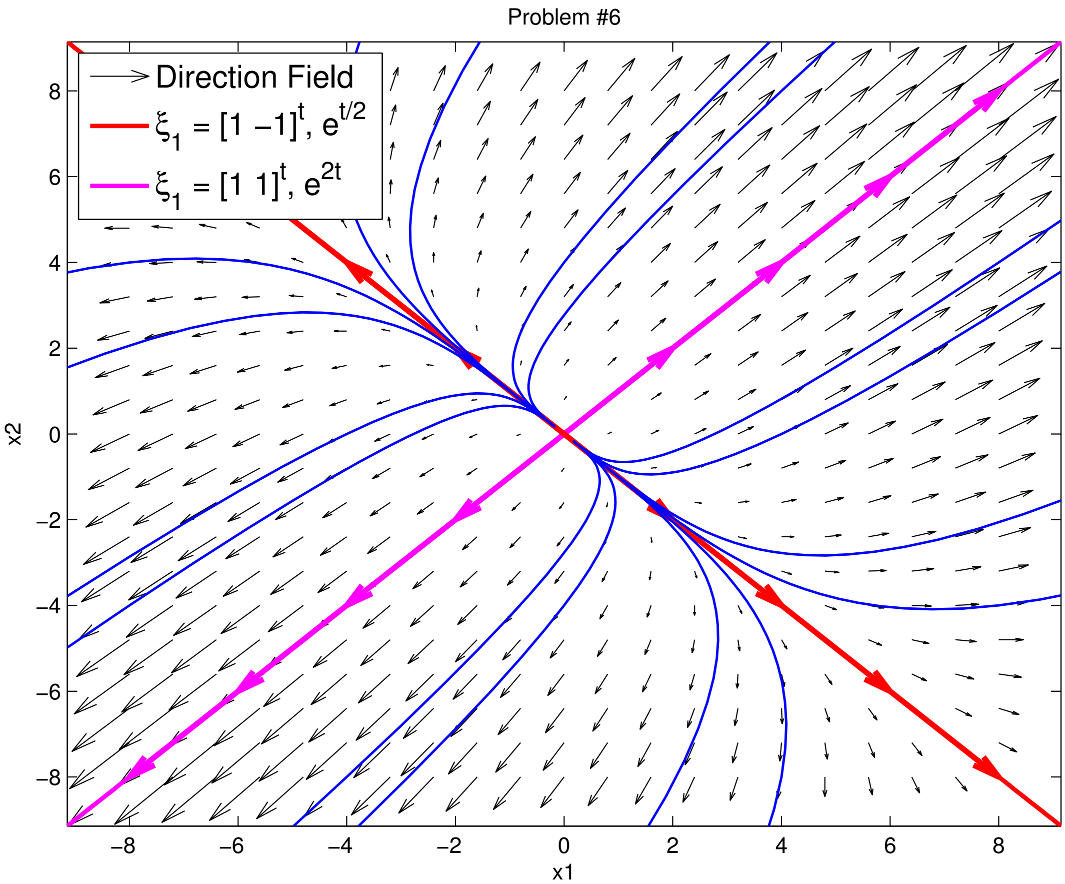
$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, the general solution is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

**(b)** Before using technology to obtain the direction field let us analyze our solution to the system. Because our eigenvalues are same signs, $r_1 = \frac{1}{2}, r_2 = 2$, we have an unstable node. We expect the trajectories to follow along our largest positive eigenvalue solution, $e^{2t}$, as $t \to \infty$. To determine where there are possible changes in the direction field we plot two lines whose direction is determined by our eigenvectors. Notice how both lines go outward. This is because our solution is composed of two exponentially increasing functions.


Problem #6
Direction Field
$\xi_1 = [1\ -1]^t,\ e^{t/2}$
$\xi_1 = [1\ 1]^t,\ e^{2t}$
x1
x2


Problem #6
Direction Field
$\xi_1 = [1\ -1]^t,\ e^{t/2}$
$\xi_1 = [1\ 1]^t,\ e^{2t}$
x1
x2

# Quick Notes

- Determine the exponents of the solution.
  - If the solution has a positive exponent then the direction field will flow outward **near** that line.
  - If the solution has a negative exponent then the direction field will flow inward **near** that line.
- The trajectories will asymptotically follow the larger exponential solution as $t \to \infty$.

$$e^{t/2} < e^{3t} \text{ as } t \to \infty, \text{ thus the trajectories in the end will follow } \quad e^{3t}$$
$$e^{-t/2} < e^{3t} \text{ as } t \to \infty, \text{ thus the trajectories in the end will follow } \quad e^{3t}$$
$$e^{-3t} < e^{-t/2} \text{ as } t \to \infty, \text{ thus the trajectories in the end will follow } \quad e^{-t/2}$$