Chapter 3.5: Undetermined Coefficients

Problem 8
Find the general solution of the given differential equation.

$$y'' + 2y' + y = 2e^{-t}$$

Let's begin with the solution to the homogeneous portion of our differential equation. Since our DE has constant coefficients we will use the characteristic equation.

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ (r+1)^2 &= 0 \\ r &= -1 \end{aligned}$$

Therefore, the homogenous solution is $y_h = C_1e^{-t} + C_2te^{-t}$.

Now to determine the particular solution. We would expect our solution to follow $Y(t) = Ae^{-t}$, but the characteristic equation repeated root matches the right hand side of the nonhomogeneous equation so we must change our expected solution to $Y(t) = At^2e^{-t}$. Taking the first and second derivative of Y we obtain:

$$\begin{aligned} Y'(t) &= A(2te^{-t} - t^2e^{-t}) \\ Y''(t) &= A(2e^{-t} - 4te^{-t} + t^2e^{-t}). \end{aligned}$$

Substituting everything back into the DE we obtain

$$\left[A(2e^{-t} - 4te^{-t} + t^2e^{-t})\right] + 2\left[A(2te^{-t} - t^2e^{-t})\right] + \left[At^2e^{-t}\right] = 2e^{-t}.$$

After simplifying, we arrive at

$$2Ae^{-t} = 2e^{-t}.$$

So, A=1 which makes our particular solution

$$Y(t) = t^2e^{-t}.$$

Thus, the general solution is

$$y = C_1e^{-t} + C_2te^{-t} + t^2e^{-t}.$$

Problem 7
Find the general solution of the given differential equation.

$$y'' + 9y = t^2e^{3t} + 6$$

Let's begin with solution to the homogeneous portion of our differential equation. Since our DE has constant coefficients we will use the characteristic equation.

$$\begin{aligned} r^2 + 9 &= 0 \\ r^2 &= -9 \\ r &= \pm 3i \end{aligned}$$

Therefore, the homogenous solution is $y_h = C_1\cos(3t) + C_2\sin(3t)$.

Now to determine the particular solution. We expect our solution to follow $Y(t) = At^2e^{3t} + Bte^{3t} + Ce^{3t} + D$. Taking the first and second derivative of Y we obtain.

$$\begin{aligned} Y'(t) &= A\left(2te^{3t} + 3t^2e^{3t}\right) + B\left(e^{3t} + 3te^{3t}\right) + C\left(3e^{3t}\right) \\ Y''(t) &= A\left(2e^{3t} + 12te^{3t} + 9t^2e^{3t}\right) + B\left(6e^{3t} + 9te^{3t}\right) + C\left(9e^{3t}\right) \end{aligned}$$

Substituting everything back into our DE and grouping we are left with

$$\left(18Ae^{3t}\right)t^2 + \left(12Ae^{3t} + 18Be^{3t}\right)t + \left(2Ae^{3t} + 6Be^{3t} + 18Ce^{3t}\right) + 9D = t^2e^{3t} + 6$$

So, $A = \frac{1}{18}, B = \frac{1}{27}, C = 1, D = \frac{2}{3}$ which makes our particular solution

$$Y(t) = \frac{1}{18}t^2e^{3t} + \frac{1}{27}te^{3t} + e^{3t} + \frac{2}{3}$$

Thus, the general solution is

$$\begin{aligned} y &= C_1\cos(3t) + C_2\sin(3t) + \frac{1}{18}t^2e^{3t} + \frac{1}{27}te^{3t} + e^{3t} + \frac{2}{3} \\ &= C_1\cos(3t) + C_2\sin(3t) + \frac{1}{162}\left(9t^2 - 6t + 1\right)e^{3t} + \frac{2}{3} \end{aligned}$$

Problem 20
Find the solution of the given initial value problem.

$$y'' + 2y' + 5y = 4e^{-t}\text{cos}(2\text{t}), \qquad y(0) = 1, \quad y'(0) = 0$$

Let's begin with solution the homogeneous portion of our differential equation. Since our DE has constant coefficients we will use the characteristic equation.

$$\begin{aligned} r^2 + 2r + 5 &= 0 \\ (r+1)^2 &= -4 \\ r &= -1 \pm 2i \end{aligned}$$

Therefore, the homogenous solution is $y_h = C_1e^{-t}\text{cos}(2t) + C_2e^{-t}\text{sin}(2t)$.

Now to determine the particular solution. We would expect our initial guess to be $Y(t) = Ae^{-t}\text{cos}(2\text{t}) + Be^{-t}\text{sin}(2\text{t})$, but the characteristic equation root matches the right hand side of the nonhomogeneous equation we must change our guess to $Y(t) = Ate^{-t}\text{cos}(2\text{t}) + Bte^{-t}\text{sin}(2\text{t})$. Taking the first and second derivative of Y we obtain.

$$\begin{aligned} Y'(t) &= A\left(\cos(2t)\,e^{-t} - t\cos(2t)\,e^{-t} - 2t\sin(2t)\,e^{-t}\right) \\ &\quad + B\left(\sin(2t)\,e^{-t} + 2t\cos(2t)\,e^{-t} - t\sin(2t)\,e^{-t}\right) \\ Y''(t) &= A\left(4t\sin(2t)\,e^{-t} - 4\sin(2t)\,e^{-t} - 3t\cos(2t)\,e^{-t} - 2\cos(2t)\,e^{-t}\right) \\ &\quad + B\left(4\cos(2t)\,e^{-t} - 2\sin(2t)\,e^{-t} - 4t\cos(2t)\,e^{-t} - 3t\sin(2t)\,e^{-t}\right) \end{aligned}$$

Substituting everything back into our DE and grouping we are left with

$$4e^{-t}\left(B\cos(2t) - A\sin(2t)\right) = 4e^{-t}\text{cos}(2\text{t})$$

So, $A = 0, B = 1$ which makes our particular solution

$$Y(t) = te^{-t}\text{sin}(2\text{t})$$

Thus, the general solution is

$$y = C_1e^{-t}\text{cos}(2t) + C_2e^{-t}\text{sin}(2t) + te^{-t}\text{sin}(2\text{t})$$

Solving for the exact solution we need use the initial conditions.

$$\begin{aligned} y' &= C_1\left(-\cos(2t)\,e^{-t} - 2\sin(2t)\,e^{-t}\right) + C_2\left(2\cos(2t)\,e^{-t} - \sin(2t)\,e^{-t}\right) \\ &\quad + \left(\sin(2t)\,e^{-t} + 2t\cos(2t)\,e^{-t} - t\sin(2t)\,e^{-t}\right) \end{aligned}$$

Inserting our initial conditions we obtain the system of equations

$y(0) = 1$	$y'(0) = 0$
$C_1 = 1$	$-C_1 + 2C_2 = 0$

So, $C_1 = 1, C_2 = \frac{1}{2}$
Thus, the exact solution is

$$y = e^{-t}\text{cos}(2t) + \frac{1}{2}e^{-t}\text{sin}(2t) + te^{-t}\text{sin}(2\text{t})$$