Appendix A
Proof of theorem 1

To prove theorem 1, we first establish some necessary lemmas concerning the Laplace transform, initial conditions, and transfer functions. Notation-wise, the Laplace transform is $\mathcal{L}$, the inverse Laplace transform $\mathcal{L}^{-1}$, the dirac delta function is $\delta(t)$, and s is a complex Laplace domain variable. Lowercase letters denote time domain variables and their capitalized counterparts denote their Laplace transform.

Lemma 1:

$$\mathcal{L}^{-1}\{sF(s)\} = f'(t) + f(0)\delta(t) \ .$$

Proof: Since $\mathcal{L}^{-1}\{1\} = \delta(t)$,

$$\mathcal{L}^{-1}\{sF(s)\} = \mathcal{L}^{-1}\{sF(s) - f(0) + f(0)\}$$
$$= \mathcal{L}^{-1}\{sF(s) - f(0)\} + \mathcal{L}^{-1}\{f(0)\} = f'(t) + f(0)\delta(t) \ .$$

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Lemma 2:

$$\mathcal{L}^{-1}\{s^2F(s)\} = f''(t) + f(0)\delta'(t) + f'(0)\delta(t)$$

Proof: Since $\mathcal{L}^{-1}\{s\} = \delta'(t)$ and $\mathcal{L}^{-1}\{1\} = \delta(t)$,

$$\mathcal{L}^{-1}\{s^2F(s)\} = \mathcal{L}^{-1}\{s^2F(s) - sf(0) - f'(0) + sf(0) + f'(0)\}$$
$$= \mathcal{L}^{-1}\{s^2F(s) - sf(0) - f'(0)\} + \mathcal{L}^{-1}\{sf(0)\} + \mathcal{L}^{-1}\{f'(0)\}$$
$$= f''(t) + f(0)\delta'(t) + f'(0)\delta(t) \ .$$

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Lemma 3 (Initial Conditions): Given the assumptions (PI), (FOP), (LSP), and (DVF),

1) $u(0) = k_c q_2$.
2) $v'(0) = k_p k_c q_2$

Proof: (1) follows from plugging (DVF) into (PI). (2) follows from plugging (1) and (LSP) into (FOP). ■

Next, we establish the process, controller, and closed-loop transfer functions of $\mathcal{S}$. Transfer functions express the Laplace domain relationship between two variables of interest. In general, if $x(t)$ and $y(t)$ are two time domain functions, then the transfer function g relating x to y satisfies $X(s) = g(s)Y(s)$. The following lemma derives the transfer function for the process (g_p), controller (g_c), and closed loop (g_{CL}). These functions express the Laplace domain relationships between the control variable and the error, the process and control variable, and the process variable and the set point; respectively.

Lemma 4 (Transfer Functions): Given the system $\mathcal{S}$,

1) The controller transfer function is

$$g_c(s) = \frac{k_c(\tau_I s + 1)}{\tau_I s} \ .$$

2) The process transfer function is

$$g_p(s) = \frac{k_p}{\tau_p s + 1} \ .$$

3) The closed loop transfer function is

$$g_{CL}(s) = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{k_p k_c} s^2 + \frac{\tau_I(1+k_p k_c)}{k_p k_c} s + 1} \ .$$

Proof:

1) g_c must satisfy $U(s) = g_c(s)E(s)$. Taking the Laplace transform of both sides of (PI), we obtain

$$U(s) = k_c E(s) + \frac{k_c}{\tau_I}\frac{E(s)}{s} \ .$$

Manipulating, we arrive at

$$U(s) = \frac{k_c(\tau_I s + 1)}{\tau_I s} E(s) \Rightarrow g_c(s) = \frac{k_c(\tau_I s + 1)}{\tau_I s} \ .$$

2) g_p must satisfy $V(s) = g_p(s)U(s)$. Taking the Laplace transform of both sides of (FOP), we obtain

$$sV(s) - v(0) = -\frac{1}{\tau_p}V(s) + \frac{k_p}{\tau_p}U(s) \ .$$

(DVF) simplifies things to

$$sV(s) = -\frac{1}{\tau_p}V(s) + \frac{k_p}{\tau_p}U(s) \ .$$

Manipulating, we arrive at

$$V(s) = \frac{k_p}{\tau_p s + 1}U(s) \Rightarrow g_p(s) = \frac{k_p}{\tau_p s + 1} \ .$$

3) g_{CL} satisfies $V(s) = g_{CL}(s)R(s)$. We derive g_{CL} by coupling the process and controller transfer functions. First recall that $V(s) = g_p(s)U(s)$. Substituting in the controller relationship $U(s) = g_c(s)E(s)$, we have

$$V(s) = g_p(s)g_c(s)E(s) = g_p(s)g_c(s)[R(s) - V(s)]$$

from the linearity of the Laplace transform. Solving for $V(s)$, we obtain

$$V(s) = \frac{g_p(s)g_c(s)}{1 + g_p(s)g_c(s)}R(s) \ ,$$

and so

$$g_{CL}(s) = \frac{g_p(s)g_c(s)}{1 + g_p(s)g_c(s)} \ .$$

Explicitly,

$$g_{CL}(s) = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{k_p k_c} s^2 + \frac{\tau_I(1+k_p k_c)}{k_p k_c} s + 1}$$

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We can now prove theorem 1:

Proof of Theorem 1:

1) **Proof for process variables:** By definition, g_{CL} satisfies $V(s) = g_{CL}(s)R(s)$. Explicitly,

$$V(s) = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{k_p k_c} s^2 + \frac{\tau_I(1+k_p k_c)}{k_p k_c} s + 1} R(s) \ ,$$

which can be rewritten as

$$\frac{\tau_I \tau_p}{k_p k_c} s^2 V(s) + \frac{\tau_I(1 + k_p k_c)}{k_p k_c} s V(s) + V(s) = \tau_I s R(s) + R(s) \tag{1}$$

Applying the lemmas and taking the Laplace transform of both sides, we have

$$\begin{aligned} &\frac{\tau_I \tau_p}{k_p k_c}[v''(t) + v(0)\delta'(t) + v'(0)\delta(t)] \\ &+ \frac{\tau_I(1 + k_p k_c)}{k_p k_c}[v'(t) + v(0)\delta(t)] + v(t) \\ &= \tau_I[r'(t) + r(0)\delta(t)] + r(t) \Rightarrow \qquad (2) \\ &\frac{\tau_I \tau_p}{k_p k_c} v''(t) + \tau_I q_2 \delta(t) + \frac{\tau_I(1 + k_p k_c)}{k_p k_c} v'(t) + v(t) \\ &= \tau_I r'(t) + \tau_I q_2 \delta(t) + r(t) \Rightarrow \\ &v''(t) + \frac{1 + k_p k_c}{\tau_p} v'(t) + \frac{k_p k_c}{\tau_I \tau_p} v(t) \\ &= \frac{k_p k_c}{\tau_p} q_1 + \frac{k_p k_c}{\tau_I \tau_p} q_2 + \frac{k_p k_c}{\tau_I \tau_p} q_1 t \ . \qquad (3) \end{aligned}$$

Lastly, set γ, k, a and b as specified in the theorem.

2) **Proof for control variables:** Expanding the controller transfer function, we have

$$\begin{aligned} U(s) &= g_c(s)E(s) = g_c(s)(R(s) - V(s)) \\ &= g_c(s)(R(s) - g_p(s)U(s)) \\ &= \frac{k_c(\tau_I s + 1)}{\tau_I s}(R(s) - \frac{k_p}{\tau_p s + 1} U(s)) \qquad (4) \end{aligned}$$

We can manipulate (4) to obtain

$$\begin{aligned} &\tau_I \tau_p s^2 U(s) + \tau_I(k_p k_c + 1) s U(s) + k_p k_c U(s) \\ &= k_c \tau_I \tau_p s^2 R(s) + k_c(\tau_p + \tau_I) s R(s) + k_c R(s) \ . \end{aligned}$$

Define $m_u := \tau_I \tau_p$, $\gamma_u := \tau_I(k_p k_c + 1)$, $k_u := k_p k_c$, $m_r := k_c \tau_I \tau_p$, and $\gamma_r := k_c(\tau_p + \tau_I)$. By the lemmas,

$$\begin{aligned} &m_u[u''(t) + u(0)\delta'(t) + u'(0)\delta(t)] + \gamma_u[u'(t) + u(0)\delta(t)] \\ &+ k_u u(t) = \\ &m_r[r''(t) + r(0)\delta'(t) + r'(0)\delta(t)] + \gamma_r[r'(t) + r(0)\delta(t)] \\ &+ k_c r(t)] \qquad (5) \end{aligned}$$

When we expand (5), a number of terms cancel out, and we obtain

$$\begin{aligned} &m_u u''(t) + \gamma_u u'(t) + k_u u(t) + [m_u u(0) - m_r r(0)]\delta'(t) \\ &+ [m_u u'(0) + \gamma_u u(0) - m_r r'(0) - \gamma_r r(0)]\delta(t) \\ &= m_r r''(t) + \gamma_r r'(t) + k_c r(t) \qquad (6) \end{aligned}$$

Now, the coefficients of $\delta(t)$ and $\delta'(t)$ on the left hand side of (6) are identically zero. For the coefficient of $\delta'(t)$, recall that $u(0) = k_c r(0)$ and then substitute in m_u and m_r. For the coefficient of $\delta(t)$,

$$\begin{aligned} &m_u u'(0) + \gamma_u u(0) - m_r r'(0) - \gamma_r r(0) \\ &= \tau_I \tau_p[k_c r'(0) - \frac{k_p k_c^2}{\tau_p} r(0) + \frac{k_c}{\tau_I} r(0)] + \\ &\tau_I(k_p k_c + 1) k_c r(0) - k_c \tau_I \tau_p r'(0) - k_c(\tau_p + \tau_I) r(0) \qquad (7) \end{aligned}$$

Expanding (7), terms cancel out, and we find that it equals zero. Substitute back in for m_u, γ_u, k_u, m_r, and γ_r, and divide both sides by $\tau_I \tau_p$ to obtain

$$\begin{aligned} &u''(t) + \frac{k_p k_c + 1}{\tau_p} u'(t) + \frac{k_p k_c}{\tau_I \tau_p} u(t) \\ &= k_c r''(t) + \frac{k_c(\tau_p + \tau_I)}{\tau_I \tau_p} r'(t) + \frac{k_c}{\tau_I \tau_p} r(t) \qquad (8) \end{aligned}$$

Now use (LSP) to substitute for $r(t)$ and define γ, k, a, and b as stated in the hypothesis. Rearrange terms to complete the proof.

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APPENDIX B
PROOF OF THEOREM 3

Proof: Let $v(t) = R_v e^{-\gamma t}\cos(\omega t - \phi_v) + c_v t + y_v$ and $u(t) = R_u e^{-\gamma t}\cos(\omega t - \phi_u) + c_u t + y_u$ be the oscillators corresponding to the process and control variables, respectively. (FOP) and (DVF) imply that $k_p = \tau_p v'(0)/u(0)$. Using this result and taking the derivative of (FOP), we have that $v''(0) = -v'(0)/\tau_p + k_p u'(0)/\tau_p = -v'(0)/\tau_p + v'(0)u'(0)/u(0)$, which implies $\tau_p = u(0)v'(0)/(u'(0)v'(0) - v''(0)u(0))$. Substituting back in for k_p, we obtain $k_p = v'(0)^2/(u'(0)v'(0) - v''(0)u(0))$. To recover k_c and τ_I, recall from theorems 1 and 2 that $\gamma = (1 + k_p k_c)/\tau_p$ and $\omega = \sqrt{4k - \gamma^2}/2$, where $k = k_p k_c/(\tau_I \tau_p)$, but also recall that we performed the subtle update $\gamma \leftarrow \gamma/2$ to recover our damped linearly driven oscillator. From this, it follows that k_c and τ_I are as defined in the theorem. q_1 and q_2 also follow from theorems 1 and 2. Now set $u_0 = u(0)$, $u_0' = u'(0)$, $v_0' = v'(0)$, and $v_0'' = v''(0)$, and the proof is complete. ■