AUTOMORPHIC REPRESENTATIONS OF $SL_2(\mathbb{R})$

FRANCIS BAHK

Contents

1. Introduction

An *automorphic form* is a function on a reductive algebraic group G that transforms in a very particular way under the action of G. Automorphic forms are prominent in modern number theory, representation theory, algebraic geometry, and harmonic analysis since they are equipped with useful *spectral data* (generalization of eigenvalues of a matrix). Using Borel as my primary reference, I will study the analytic theory of automorphic forms on the group $SL_2(\mathbb{R})$ with respect to discrete subgroup $SL_2(\mathbb{Z})$ with finite covolume. There's plenty of theory for when G is an arbitrary semisimple Lie group, but I will focus on the more concrete theory for $SL_2(\mathbb{R})$. I hope I'm thorough, provide enough motivation, and improve my own understanding.

In addition, I hope to contribute to the Langlands program someday, and so I believe it's good taste to discuss three major achievements of the program. One is the Principle of Functionality, which in simple terms says, given two different groups and an automorphic form attached to each group, when the automorphic forms are related to each other. It is very astonishing that for some groups, one can express the data of an automorphic form of one group in terms of the automorphic form of the other. This notion of course has huge implications on number theory. The second is the Reciprocity Conjecture, which says that for every motive, there exists an automorphic form such that the motive's data and the automorphic form's spectral data match. Since motives are the fundamental building blocks of algebraic varieties, the Reciprocity Conjecture is analogous to the classification of

Date: 3.26.24.

fundamental particles in physics. The third is the general trace formula for any group G, which relates the geometry of G to its spectral data. A breakthrough for the Langlands program was to compare trace formulas for different groups to prove identities between the groups' geometric data.

2. Basic Representation Theory

Suppose G is a matrix Lie group and V a finite-dimensional real or complex vector space. Then a finite-dimensional real or complex *representation* of G is a Lie group homomorphism π, which maps G into the invertible linear transformations of V, denoted $GL(V)$. The notion of a *matrix representation* arises when we choose a basis B for V. In this case, π represents the elements of G as invertible matrices and hence we call it a *matrix representation.*

As for the reducibility of a representation, consider a vector subspace $W \subset V$. We say W is *G-invariant* if for every group element $g \in G$ and vector $w \in W$, the vector $\pi(g)w$ is an element of W (where $\pi(g) : V \to V$ is a bijective linear transformation). This coincides with the notion of a *reducible representation* of G, which is a function $\pi : G \to GL(V)$ which has a nontrivial invariant subspace W. In other words, π has a nonzero vector subspace $W \subset V$ such that $\pi(g)w \in W$ for all $g \in G$ and all $w \in W$. Conversely, an *irreducible representation* (π, V) is nonzero representation that has no proper nontrivial subrepresentation $(\pi|_W, W)$, with $W \subset V$ closed under the action of $\{\rho(g) : g \in G\}$. Thus a representation is *irreducible* if it has only trivial subrepresentations (any representation can form a subrepresentation with the trivial G-invariant subspaces, V and $\{0\}$). Finally, a representation is *unitary* if $\pi(g)$ is a unitary operator for every $g \in G$.

3. Automorphic Forms

Suppose G acts on a topological space X. For each orbit of the group action, choose a representative. Then the set of representatives (typically chosen to create a connected subset of X) is a *fundamental domain.* The *cusps* are intuitively the representatives in the fundamental domain where the congruence subgroup touches the boundary of $\mathbb{H}$.

An *automorphic form* is a function f on G satisfying three conditions. The first is left-invariance, i.e. for any $\gamma \in \Gamma$, $f(\gamma x) = f(x)$ for all $g \in G$. The second condition is that f has "nice analytic properties," enforced by the fact that f must an eigenfunction of certain Casimir operators on G. Finally, in case $\Gamma\backslash G$ is not compact and hence has cusps, the third condition forces f to satisfy moderate growth at cusps. An *automorphic form* is a well-behaved function from a topological group G to the complex numbers (or complex vector space) which is invariant under the action of a discrete subgroup $\Gamma \subset G$. Automorphic forms are a generalization of periodic functions in Euclidean space; they are periodic functions on general topological groups. The Casimir operator acts by a scalar on automorphic functions.

In case others may have the same misunderstanding I had when learning the theory, I must clarify that although an automorphic form is a smooth function on $\Gamma\backslash G$, it need *not* be an element of $L^2(\Gamma\backslash G)$. An explicit example is the holomorphic Eisenstein series. One goal of this paper is to illuminate the connection between representation theory of $SL_2(\mathbb{R})$ and automorphic forms on $\mathbb{H}$. There are two classes of irreducible representations of $SL_2(\mathbb{R})$: the principal series and the discrete series.

Thus there are two types of classical automorphic forms: Maass forms and modular forms. In addition, there's both a geometric way and a more general way to view automorphic forms. The geometric idea is that $\Gamma\backslash\mathbb{H}$ is a Riemann surface, so an automorphic form $f : G \to \mathbb{C}$ is a function on a surface invariant under the action of Γ. But the general perspective is more suitable for the Langlands program, since it views automorphic forms as objects on $\Gamma\backslash G$ which are related to harmonic analysis on it or G.

As we have showed, G acts on $\Gamma\backslash G$ on the right, but aside from invariance under this action, we also want to study symmetries of $\Gamma\backslash G$ under the action. This notion gives rise to representations on various function spaces, such as the Hilbert space $H = L^2(\Gamma\backslash G)$. If a function $f \in H$, then we say f is *square-integrable*. For various reasons, we may want to express or *decompose* a Hilbert space H into the direct sum of closed subspaces, viz. $H = \oplus_i H_i$ (where each $H_i \subset H$ is closed). This allows us to express every function $f \in H$ as a unique sum of functions from the subspaces. An example is the Fourier series decomposition. A big question in Representation Theory is: what irreducible representations of G occur in the decomposition of $L^2(\Gamma\backslash G)$?

4. Operators

4.1. **Casimir Operators.** An operator which commutes with all the elements of a Lie algebra $\mathfrak{g}$ is called a *Casimir operator*. In other words, $[C, X] = 0$ for all $X \in \mathfrak{g}$. For $SL_2(\mathbb{R})$, choose a basis H, E, F for the complexification of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ such that $[H, E] = 2E, [H, F] = -2F, [E, F] = H$. Then the Casimir operator, which generates the center of the universal enveloping algebra, is defined as $C = H^2 + 1 + 2EF + 2FE$.

4.2. **Bi-invariant Differential Operators.** In our setting, an *invariant differential operator* D is a map from functions to functions. The word "differential' means the output depends only on the input $f(x)$ and its derivatives, while "invariant" means there's a group G with a group action on the functions (or other objects in question) such that D preserves the action, viz. $D(g \circ f) = g \circ (Df)$ for $g \in G$, f smooth over G. In this case, the differential operator is left-invariant or invariant under left translation, but it can also be right invariant.

Consider Lie group and Lie algebra G and $\mathfrak{g}$. Then take $Y \in \mathfrak{g}$ so that D_v is the *left-invariant differential operator* defined on a function $f \in C^\infty(G)$ by

$$D_Y f(g) = \frac{d}{dt} f(ge^{tY})\Big|_{t=0}$$

where $g \in G$. In other words, the differential operator D is defined by the Lie derivative. Then $\{Y \mid Y \in \mathfrak{g}\}$ generates the algebra of left-invariant differential operators on G over $\mathbb{C}$ (let us call this algebra $I(G)$.) The Casimir operator C lies in $I(G)$. In addition, $I(G)$ is isomorphic to the *universal enveloping algebra* $U(\mathfrak{g})$, which is the *unital associative algebra* (it has unit or identity element) whose representations correspond to the representations of $\mathfrak{g}$. As for the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, it's exactly the set of bi-invariant differential operators since those commute with all other differential operators.

Recall $\mathfrak{g}$ is the set of 2×2 real matrices with 0 trace. The standard basis of $\mathfrak{g}$ consists of matrices H, E, F defined as

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These basis elements satisfy the commutator relations $[H, E] = HE - EH = 2E$, $[H, F] = -2F$, and $[E, F] = H$. In the standard basis, the Casimir operator is defined as $C = 1/2H^2 + EF + FE = 1/2H^2 + H + 2FE = 1/2H^2 - H + 2EF$. If G acts smoothly on a smooth manifold, then $\mathfrak{g}$ defines differential operators on that manifold.

As we have mentioned, the Casimir operator C is a differential operator of second order on G that is invariant under both left and right translations. However, not every L^2 function is differentiable, which means differential operators are *only defined on a dense subset of a Hilbert space.* So if $\mathfrak{h}$ is a Hilbert space, then an operator on $\mathfrak{h}$ is an ordered pair (T, D_T) where D_T is a dense linear subspace of $\mathfrak{h}$ called the domain of T and $T : D_T \to \mathfrak{h}$ is a linear transformation. Thus the elements of $\mathfrak{g}$ are realized as left-invariant differential operators on G.

5. Decompositions

In this section we will go over the Iwasawa and Bruhat decompositions.

5.1. **Iwasawa Decomposition.**

Now let's study $G = SL_2(\mathbb{R})$ in more depth. Let $A \subset G$ be all of G's diagonal matrices with positive entries, $N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$, and $K = SO(2)$. One can also view N as the group of *unipotent* upper triangular matrices, where unipotent means all eigenvalues are 1. Then $ANK = G$ and the map $f := (a, n, k) \mapsto ank$ is a homeomorphism. Here's a proof of the Iwasawa decomposition.

Proof. We want to show f is continuous, bijective, and has continuous inverse. Well the action on $\mathbb{H}$ is continuous because the map $(g, z) \mapsto gz$ where $z \in \mathbb{H}$ is a continuous map with continuous inverse

For bijectivity, pick $g \in G$ and let $g \cdot i$ equal some complex number $x + iy$. Notice $G/K \to \mathbb{H} := gK \mapsto gi$ is a bijection. Consider

$$a = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 1 & x/y \\ 0 & 1 \end{pmatrix}.$$

We want to show that there's a matrix $k \in K$ such that $g = ank$. Well by computation,

$$an = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & x/y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix},$$

which when applied to i equals

$$\frac{y^{1/2}i + xy^{-1/2}}{y^{-1/2}} = \frac{y^{1/2}i}{y^{-1/2}} + \frac{xy^{-1/2}}{y^{-1/2}} = iy + x = x + iy.$$

Thus the group AN acts transitively on $\mathbb{H}$. Since $gi = ani$ for some $a \in A$, $n \in N$, and now left multiplication by g^{-1} implies $i = g^{-1}ani = 1$. Well since the stabilizer of i is K, this must mean $g^{-1}an \in K$. This must mean there's a rotation matrix k such that $g = ank$.

To determine the inverse $f^{-1} : G \to A \times N \times K$, let $f^{-1}(g) := (a(g), n(g), k(g))$ where we have defined

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that

$$a(g) = \begin{pmatrix} (c^2+d^2)^{-1/2} & 0 \\ 0 & (c^2+d^2)^{1/2} \end{pmatrix},$$

$$n(g) = \begin{pmatrix} 1 & ac+bd \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad k(g) = (c^2+d^2)^{-1/2} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

The fact that $ff^{-1} = f^{-1}f = \mathbf{1}$ proves that f is the bijection we were hoping for. $\square$

5.2. **Bruhat Decomposition.** There's yet another way to decompose $G = SL_2(\mathbb{R})$. Let P be the subgroup of matrices $\begin{pmatrix} a & x \\ 0 & 1/a \end{pmatrix}$, w the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and N the subgroup of matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then the Bruhat decomposition states $G = NwP \sqcup P$.

Proof.

$$NwP = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} xa & x^2 - 1/a \\ a & x \end{pmatrix},$$

whose determinant is $ax^2 - ax^2 + a/a = 1$. According to the Iwasawa decomposition, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c \neq 0$, then it can be written uniquely as a product pwn for $p \in P$, $n \in N$ following from computation. $\square$

Notice if we have $g \in G$ such that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $c = 0$, then we don't have to do anything. On the other hand if $c \neq 0$ then notice

$$\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b - da/c \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b - da/c \\ c & 0 \end{pmatrix}$$

which both have determinant equal to $-(bc - da) = ad - bc$ as required.

Acknowledgments

Thank you to Professor Dan Barbasch for guiding me through the subject in MATH 7390 and MATH 4500, recommending various useful books, and motivating me to continue studying hard.

References

[1] Armand Borel, "Automorphic forms on $SL_2(\mathbb{R})$", 1854.