Topic: Trapezoidal rule error bound

Question: Calculate the area under the curve. Then, use the Trapezoidal Rule, with $n=6$, to approximate the same area. Compare the actual area to the result to determine the error of the Trapezoidal Rule approximation of the area.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

Answer choices:

A Actual area is $\frac{135}{2}$ $TRAP_6$ is $\frac{265}{4}$ Error is $\frac{5}{4}$

B Actual area is $\frac{265}{4}$ $TRAP_6$ is $\frac{135}{2}$ Error is $\frac{5}{4}$

C Actual area is $\frac{10{,}597}{160}$ $TRAP_6$ is $\frac{1{,}353}{20}$ Error is $\frac{227}{160}$

D Actual area is $\frac{1{,}353}{20}$ $TRAP_6$ is $\frac{10{,}597}{160}$ Error is $\frac{227}{160}$

Solution: D

The question asks us to calculate the area under the curve, and then approximate the same area using the trapezoidal Rule, with $n = 6$, and compare the results by identifying the error.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2 \ dx$$

From the integral, the function is

$$g(x) = -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2$$

Let's begin by integrating $g(x)$ using the power rule and evaluating the integral. This will give the actual area under the curve.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2 \ dx$$

$$\left(-\frac{x^6}{15} + \frac{7x^4}{12} + \frac{5x^3}{3} + 2x^2 + 2x\right)\Bigg|_0^3$$

$$-\frac{(3)^6}{15} + \frac{7(3)^4}{12} + \frac{5(3)^3}{3} + 2(3)^2 + 2(3) - \left[-\frac{0^6}{15} + \frac{7(0^4)}{12} + \frac{5(0^3)}{3} + 2(0^2) + 2(0)\right]$$

$$-\frac{243}{5} + \frac{567}{12} + \frac{135}{3} + 18 + 6 = \frac{1{,}353}{20}$$

Now, we'll estimate the area under the curve using the Trapezoidal Rule, with $n = 6$. The table below shows the interval $[0,3]$ divided into 6

subintervals, and the function values at each point. The work is shown below the table.

x	0	0.5	1	1.5	2	2.5	3
$g(x)$	2	$\frac{1,327}{240}$	$\frac{194}{15}$	$\frac{1,927}{80}$	$\frac{538}{15}$	$\frac{1,951}{48}$	$\frac{124}{5}$

For $g(0)$:

$$g(0) = -\frac{2}{5}(0)^5 + \frac{7}{3}(0)^3 + 5(0)^2 + 4(0) + 2 = 2$$

For $g(1/2)$:

$$g\left(\frac{1}{2}\right) = -\frac{2}{5}\left(\frac{1}{2}\right)^5 + \frac{7}{3}\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) + 2$$

$$g\left(\frac{1}{2}\right) = -\frac{2}{5}\left(\frac{1}{32}\right) + \frac{7}{3}\left(\frac{1}{8}\right) + 5\left(\frac{1}{4}\right) + 4\left(\frac{1}{2}\right) + 2$$

$$g\left(\frac{1}{2}\right) = -\frac{1}{80} + \frac{7}{24} + \frac{5}{4} + 2 + 2$$

$$g\left(\frac{1}{2}\right) = \frac{1,327}{240}$$

For $g(1)$:

$$g(1) = -\frac{2}{5}(1)^5 + \frac{7}{3}(1)^3 + 5(1)^2 + 4(1) + 2 = -\frac{2}{5} + \frac{7}{3} + 5 + 4 + 2 = \frac{194}{15}$$

For $g(3/2)$:

$$g\left(\frac{3}{2}\right)=-\frac{2}{5}\left(\frac{3}{2}\right)^5+\frac{7}{3}\left(\frac{3}{2}\right)^3+5\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+2$$

$$g\left(\frac{3}{2}\right)=-\frac{2}{5}\left(\frac{243}{32}\right)+\frac{7}{3}\left(\frac{27}{8}\right)+5\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+2$$

$$g\left(\frac{3}{2}\right)=-\frac{243}{80}+\frac{63}{8}+\frac{45}{4}+6+2$$

$$g\left(\frac{3}{2}\right)=\frac{1,927}{80}$$

For $g(2)$:

$$g(2)=-\frac{2}{5}(2)^5+\frac{7}{3}(2)^3+5(2)^2+4(2)+2$$

$$g(2)=-\frac{2}{5}(32)+\frac{7}{3}(8)+5(4)+4(2)+2$$

$$g(2)=-\frac{64}{5}+\frac{56}{3}+20+8+2$$

$$g(2)=\frac{538}{15}$$

For $g(5/2)$:

$$g\left(\frac{5}{2}\right)=-\frac{2}{5}\left(\frac{5}{2}\right)^5+\frac{7}{3}\left(\frac{5}{2}\right)^3+5\left(\frac{5}{2}\right)^2+4\left(\frac{5}{2}\right)+2$$

$$g\left(\frac{5}{2}\right)=-\frac{2}{5}\left(\frac{3125}{32}\right)+\frac{7}{3}\left(\frac{125}{8}\right)+5\left(\frac{25}{4}\right)+4\left(\frac{5}{2}\right)+2$$

$$g\left(\frac{5}{2}\right) = -\frac{625}{16} + \frac{875}{24} + \frac{125}{4} + 10 + 2$$

$$g\left(\frac{5}{2}\right) = \frac{1,951}{48}$$

For $g(3)$:

$$g(3) = -\frac{2}{5}(3)^5 + \frac{7}{3}(3)^3 + 5(3)^2 + 4(3) + 2$$

$$g(3) = -\frac{2}{5}(243) + \frac{7}{3}(27) + 5(9) + 4(3) + 2$$

$$g(3) = -\frac{486}{5} + 63 + 45 + 12 + 2$$

$$g(3) = \frac{124}{5}$$

The general rule for the Trapezoidal Rule approximation of the area is

$$A = \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + \ \dots \ + 2f(x_{n-1}) + f(x_n)\right]$$

The subinterval widths are all $1/2$, so $\Delta x = 1/2$. To find the Trapezoidal Rule approximation of the area, insert each function value in the table into the general Trapezoidal Rule.

$$A = \frac{1}{4}\left[2 + 2\left(\frac{1,327}{240}\right) + 2\left(\frac{194}{15}\right) + 2\left(\frac{1,927}{80}\right) + 2\left(\frac{538}{15}\right) + 2\left(\frac{1,951}{48}\right) + \frac{124}{5}\right]$$

$$A = \frac{1}{4}\left(2 + \frac{1,327}{120} + \frac{388}{15} + \frac{1,927}{40} + \frac{1,076}{15} + \frac{1,951}{24} + \frac{124}{5}\right)$$

$$A = \frac{1}{4}\left(\frac{10{,}597}{40}\right)$$

$$A = \frac{10{,}597}{160}$$

The error is the actual area minus the estimated area.

$$\frac{1{,}353}{20} - \frac{10{,}597}{160} = \frac{227}{160}$$

Topic: Trapezoidal rule error bound

Question: Calculate the error bound of the Trapezoidal Rule, with $n = 6$.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2 \ dx$$

Answer choices:

A $|E_T| \leq 10.25$

B $|E_T| \leq 10.06$

C $|E_T| \leq 11.13$

D $|E_T| \leq 10.008$

Solution: A

The question asks us to calculate the error bound of the Trapezoidal Rule, with $n = 6$, for the area under the curve.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2\ dx$$

To find the error bound of the Trapezoidal Rule on the interval $[a, b]$, we use this formula.

$$|E_T| \leq k\frac{(b-a)^3}{12n^2}$$

Where $|E_T|$ denotes the maximum error of the Trapezoidal Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

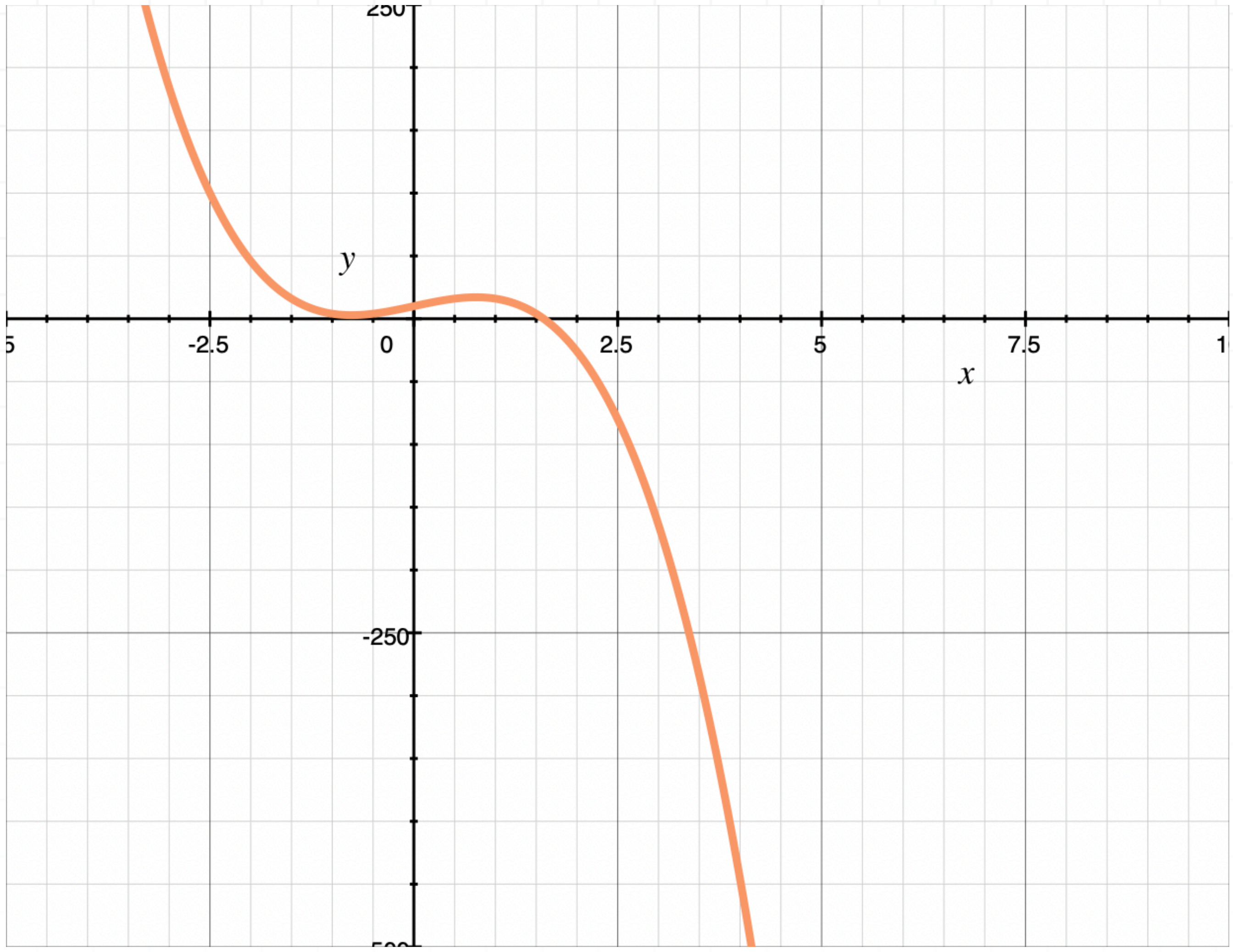
First, let's find k. The value k if often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$g(x) = -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2$$

$$g'(x) = -\frac{2}{5}(5)x^4 + \frac{7}{3}(3)x^2 + 5(2)x^1 + 4 = -2x^4 + 7x^2 + 10x + 4$$

$$g''(x) = -8x^3 + 14x + 10$$

The graph of $g''(x)$ is shown below.



The second derivative, $g''(x)$, will reach its maximum absolute value at the point $(3, -164)$, so the value of $M_{f''}$ is 164.

$$g''(0) = 10, \ \ g''(3) = -164, \ \ k = 164$$

Now in the expression

$$|E_T| \le k\frac{(b-a)^3}{12n^2}$$

$k = 164$, $a = 0$, $b = 3$ and $n = 6$. Evaluate the error bound.

$$k\frac{(b-a)^3}{2412n^2} = (164)\frac{(3-0)^3}{12(6)^2} = \frac{(164)(27)}{(12)(36)} = 10.25$$

Therefore,

$|E_T| \leq 10.25$

Topic: Trapezoidal rule error bound

Question: Find n to get the accuracy of the Trapezoidal Rule to within 0.00001.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

Answer choices:

A $\quad n=6{,}073$

B $\quad n=6{,}072$

C $\quad n=6{,}074$

D $\quad n=6{,}075$

Solution: D

The question asks us to find n to get the accuracy of the Trapezoidal Rule to within 0.00001.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

To find the error bound of the Midpoint Rule on the interval $[a,b]$, we use this formula.

$$|E_T| \le k\frac{(b-a)^3}{12n^2}$$

Where $|E_T|$ denotes the maximum error of the Trapezoidal Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

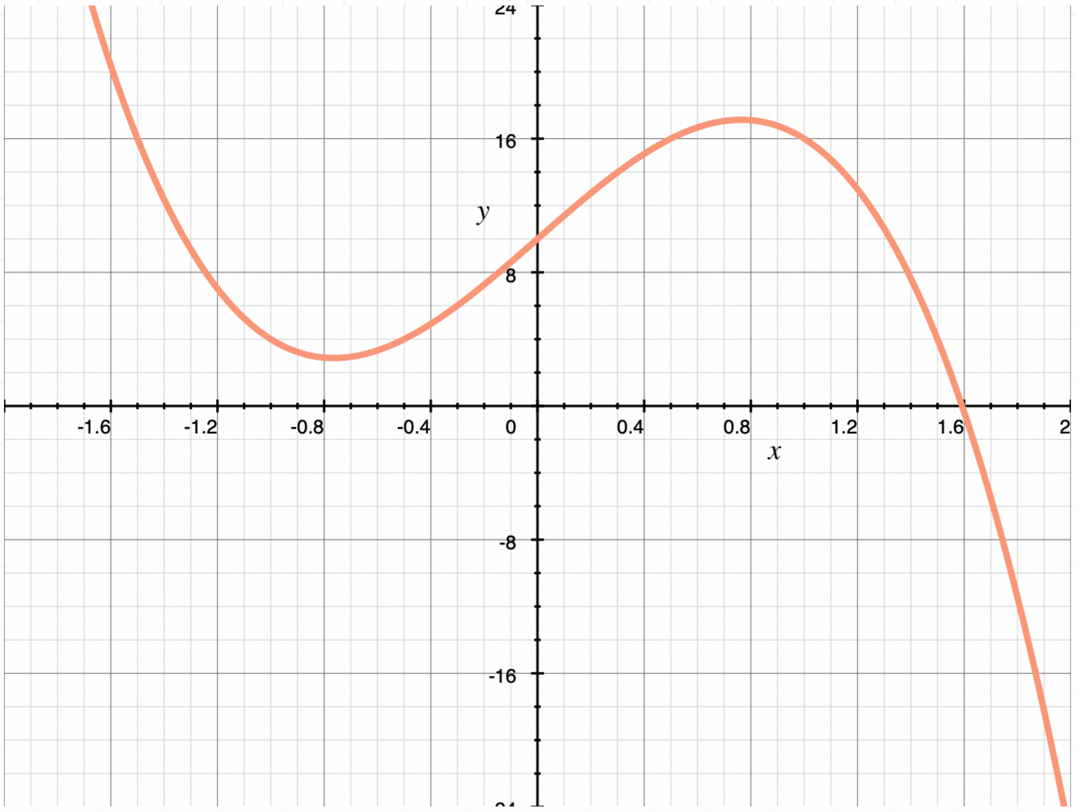
First, let's find k. The value k is often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$g(x)=-\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2$$

$$g'(x)=-\frac{2}{5}(5)x^4+\frac{7}{3}(3)x^2+105+4=-2x^4+7x^2+105+4$$

$$g''(x)=-8x^3+14x+10$$

The graph of $g''(x)$ is shown below.



The second derivative $g''(x)$ will reach its maximum value at $(0.7637,17.1284)$ but its maximum absolute value is at the point $(3,-164)$, so the value of $M_{f''}$ is 164.

$$g''(0)=10, \ \ g''(3)=-164, \ \ k=164$$

Now in the expression

$$|E_T| \leq k\frac{(b-a)^3}{12n^2}$$

$k=164$, $a=0$ and $b=3$. We'll find the value of n. Let's simplify the expression first.

$$|E_T| \leq (164)\frac{(3-0)^3}{12n^2}$$

$$|E_T| \leq \frac{(164)(27)}{12n^2}$$

$$|E_T| \leq \frac{4,428}{12n^2}$$

$$|E_T| \leq \frac{369}{n^2}$$

Since we want the error to be less than 0.00001, we set the maximum error bound expression to be less than 0.00001.

$$\frac{369}{n^2} \leq 0.00001$$

Multiply by n^2 and divide by 0.00001.

$$369 \leq (0.00001)n^2$$

$$\frac{369}{0.00001} \leq n^2$$

Square root both sides of the inequality, ignoring the possibility that n could be negative.

$$\sqrt{\frac{369}{0.00001}} \leq \sqrt{n^2}$$

$$n \geq 6,074.54$$

We found an interval for n. However, since n is the number of subintervals, n has to be a whole number. Thus, to be accurate to within 0.0001, $n = 6,075$.