Calculus 2 Notes

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MATH

Indefinite integrals

An indefinite integral is the opposite of a derivative, which is why indefinite integrals are also called *antiderivatives*.

General rule for indefinite integrals

The general rule for indefinite integration is

$$\int f(x)\ dx = F(x) + C$$

where C is the constant of integration and $F'(x) = f(x)$. In other words, the derivative of $F(x)$ is $f(x)$, which means that the integral of $f(x)$ is $F(x)$, plus the constant of integration C, which we add to account for a constant that might have disappeared when we took the derivative of $F(x)$ to get $f(x)$.

Of course, this means that you can check to see whether or not you took the integral correctly by taking the derivative of your answer. You should get back to the function you integrated originally.

Since indefinite integration happens over an open interval, taking an indefinite integral means we're finding the area under the curve in it's entire domain.

For basic power functions, we can use the integral formula,

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C$$

Let's do an example where we evaluate an integral like this one.

Example

Evaluate the indefinite integral.

$$\int 3x^6 \, dx$$

We need to remember the general rule $\int f(x) \, dx = F(x) + C$ where $F'(x) = f(x)$.

$$\int 3x^6 \, dx = \frac{3x^7}{7} + C$$

Let's check our answer by taking its derivative to see if it gives us our original function, $f(x)$. Remember, the derivative of the constant C is always zero. Using power rule, we get

$$\frac{d}{dx}\left(\frac{3x^7}{7} + C\right) = \frac{21x^6}{7} + 0$$

$$\frac{d}{dx}\left(\frac{3x^7}{7} + C\right) = 3x^6$$

The derivative we just found is equal to our original function, so we know that the indefinite integral we calculated was correct.

Let's do another example, this time with multiple terms.

Example

Find the antiderivative.

$$\int 2x^4 - 6x + 2\ dx$$

Using the formula $\int f(x)\ dx = F(x) + C$ where $F'(x) = f(x)$, we take the integral one term at a time and get

$$\int 2x^4 - 6x + 2\ dx = \frac{2x^5}{5} - \frac{6x^2}{2} + 2x + C$$

$$\int 2x^4 - 6x + 2\ dx = \frac{2}{5}x^5 - 3x^2 + 2x + C$$

Initial value problems

Consider the following situation. You're given the function $f(x) = 2x - 3$ and asked to find its derivative. This function is pretty basic, so unless you're taking calculus out of order, it shouldn't cause you too much stress to figure out that the derivative of $f(x)$ is 2.

Now consider what it would be like to work backwards from our derivative. If you're given the function $f'(x) = 2$ and asked to find its integral, it's impossible for you to get back to the original function, $f(x) = 2x - 3$. As you can see, taking the integral of the derivative we found gives us back the first term of the original function, $2x$, but somewhere along the way we lost the -3. In fact, we always lose the constant (term without a variable attached), when we take the derivative of something. Which means we're never going to get the constant back when we try to integrate our derivative. It's lost forever.

Accounting for that lost constant is why we always add C to the end of our integrals. C is called the "constant of integration" and it acts as a placeholder for our missing constant. In order to get back to our original function, and find our long-lost friend, -3 , we'll need some additional information about this problem, namely, an initial condition, which looks like this:

$$y(0) = -3$$

Problems that provide you with one or more initial conditions are called Initial Value Problems. Initial conditions take what would otherwise be an entire rainbow of possible solutions, and whittles them down to one specific solution.

Remember that the basic idea behind Initial Value Problems is that, once you differentiate a function, you lose some information about that function. More specifically, you lose the constant. By integrating $f'(x)$, you get a family of solutions that only differ by a constant.

$$\int 2\ dx = 2x - 3$$

$$\int 2\ dx = 2x + 7$$

$$\int 2\ dx = 2x - \sqrt{2}$$

Given one point on the function, (the initial condition), you can pick a specific solution out of a much broader solution set.

Example

Given $f'(x) = 2$ and $f(0) = -3$, find $f(x)$.

Integrating $f'(x)$ means we're integrating $2\ dx$, and we'll get $2x + C$, where C is the constant of integration. At this point, C is holding the place of our now familiar friend, -3, but we don't know that yet. We have to use our initial condition to find out.

To use our initial condition, $f(0) = -3$, we plug in the number inside the parentheses for x and the number on the right side of the equation for y. Therefore, in our case, we'll plug in 0 for x and -3 for y.

$$-3 = 2(0) + C$$

$$-3 = C$$

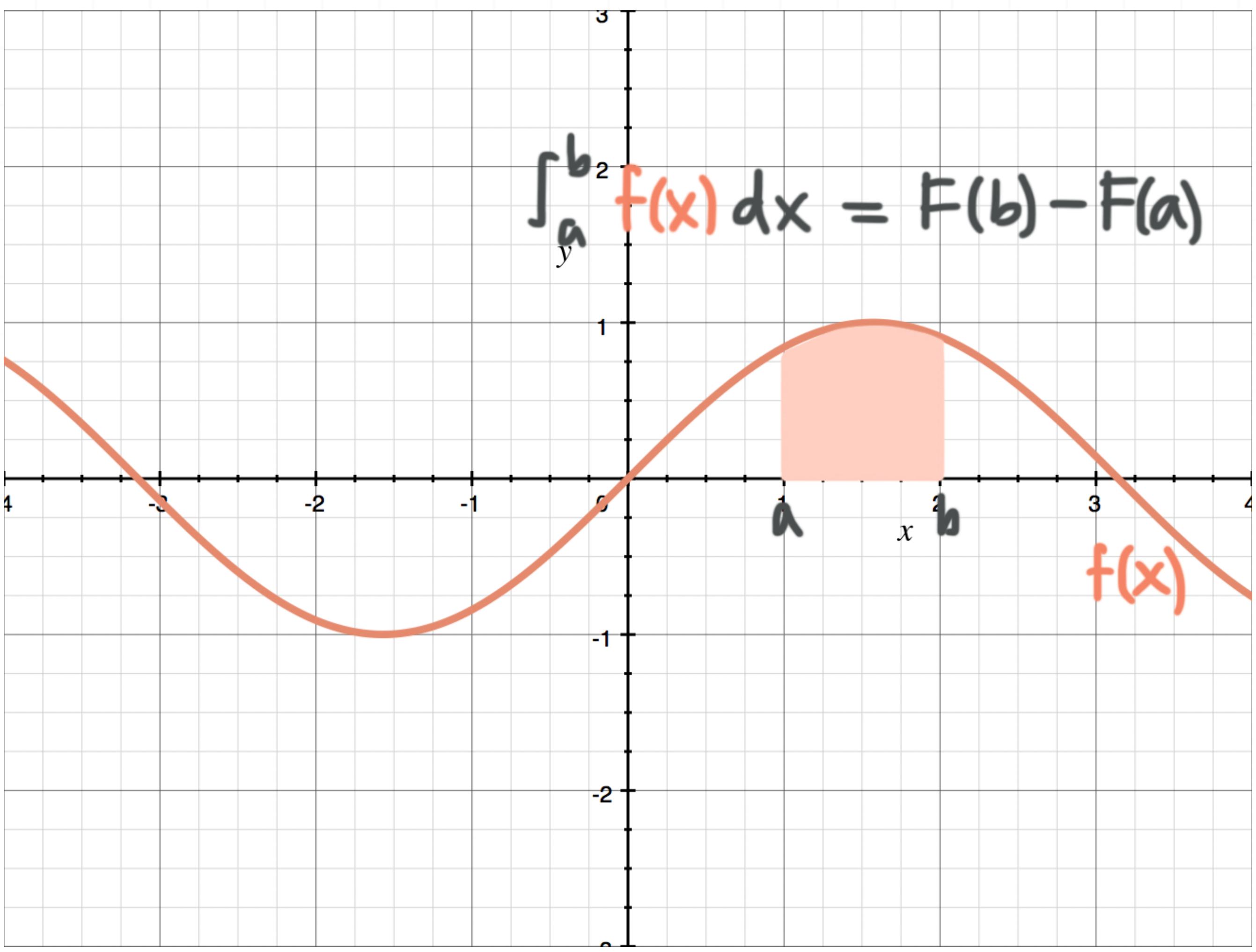
Notice that the solution would have been different had we been given a different initial condition. Now we know exactly what the full solution looks like, and exactly which one of the many possible solutions was originally differentiated. Therefore, the final answer is the function we originally differentiated:

$$f(x) = 2x - 3$$

Definite integrals

Evaluating a definite integral means finding the area enclosed by the graph of the function and the x-axis, over the given interval $[a, b]$.

In the graph below, the shaded area is the integral of $f(x)$ on the interval $[a, b]$. Finding this area means taking the integral of $f(x)$, plugging the upper limit b into the result, and then subtracting from that whatever you get when you plug in the lower limit a.



Let's do an example where we evaluate a definite integral.

Example

Evaluate the integral.

$$\int_0^2 3x^2 - 5x + 2\ dx$$

If we let $f(x) = 3x^2 - 5x + 2$ and then integrate the polynomial, we get

$$F(x) = \left(x^3 - \frac{5}{2}x^2 + 2x + C\right)\Big|_0^2$$

where C is the constant of integration.

Evaluating on the interval $[0,2]$, we get

$$F(x) = \left[(2)^3 - \frac{5}{2}(2)^2 + 2(2) + C\right] - \left[(0)^3 - \frac{5}{2}(0)^2 + 2(0) + C\right]$$

$$F(x) = (8 - 10 + 4 + C) - (0 - 0 + 0 + C)$$

$$F(x) = 8 - 10 + 4 + C - C$$

$$F(x) = 2$$

As you can see, the constant of integration “cancels out” in the end, leaving a definite value as the final answer, not just a function for y defined in terms of x.

Since this will always be the case, you can just leave C out of your answer whenever you’re solving a definite integral.

So, what do we mean when we say $F(x) = 2$? What does this value represent? When we say that $F(x) = 2$, it means that the area

1. below the graph of $f(x)$,

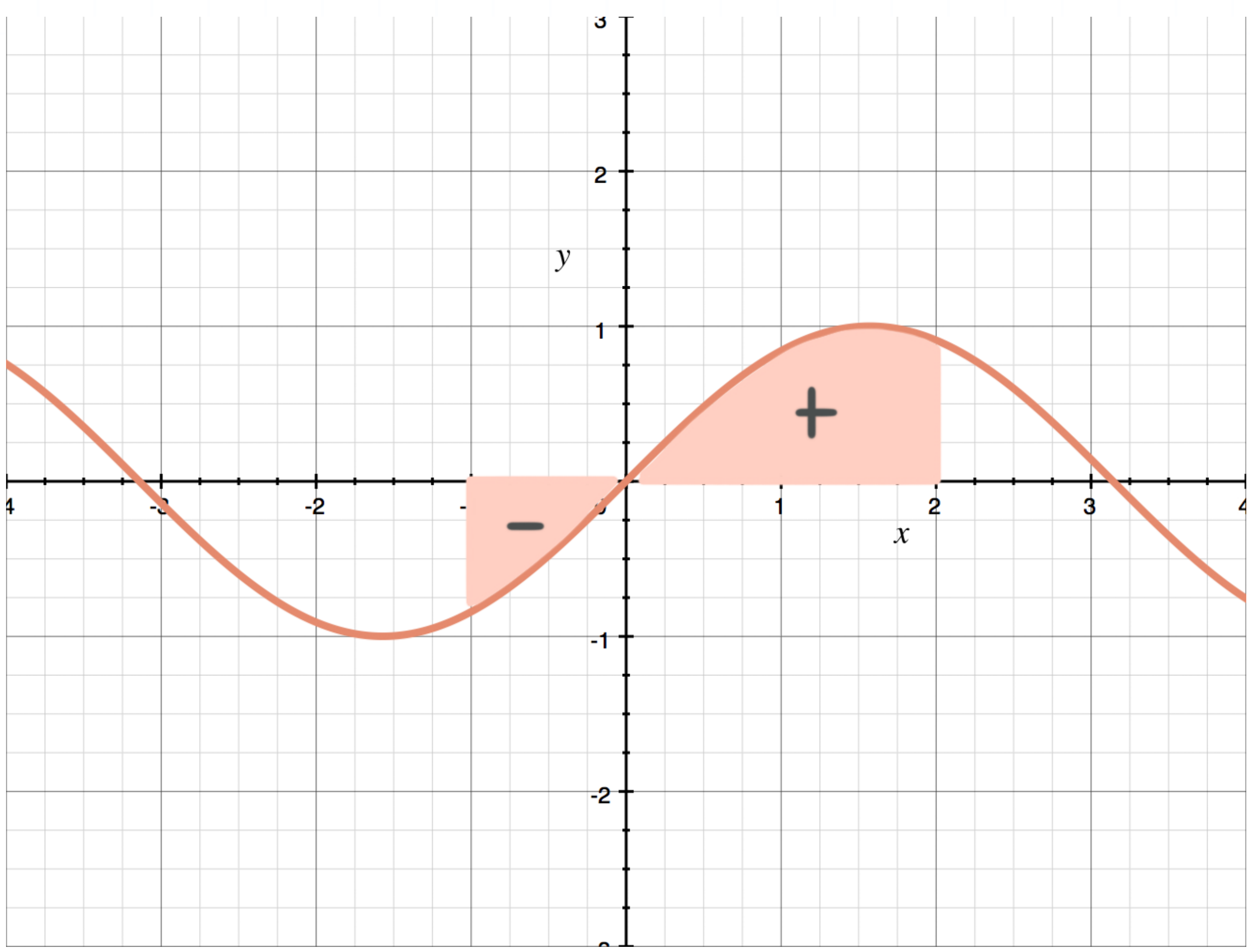
2. above the x-axis, and

3. between the lines $x = 0$ and $x = 2$

is 2 square units.

Keep in mind that we're talking about the area *enclosed* by the graph and the x-axis. If $f(x)$ drops below the x-axis inside $[a, b]$, we treat the area under the x-axis as negative area.

Then finding the value of $F(x)$ means subtracting the area enclosed by the graph under the x-axis from the area enclosed by the graph above the x-axis.



In other words, evaluating the definite integral of $f(x) = \sin x$ on $[-1,2]$ means subtracting the area enclosed by the graph below the x-axis from the area enclosed by the graph above the x-axis.

This means that, if the area enclosed by the graph below the x-axis is larger than the area enclosed by the graph above the x-axis, then the value of $F(x)$ will be negative ($F(x) < 0$).

If the area enclosed by the graph below the x-axis is exactly equal to the area enclosed by the graph above the x-axis, then $F(x) = 0$.

Definite integrals of even and odd functions

Sometimes we can simplify a definite integral if we recognize that the function we're integrating is an even function or an odd function. If the function is neither even nor odd, then we proceed with integration like normal.

To find out whether the function is even or odd, we'll substitute $-x$ into the function for x. If we get back the original function $f(x)$, the function is even. If we get back the original function multiplied by -1, the function is odd. In other words,

- If $f(-x) = f(x)$, the function is even
- If $f(-x) = -f(x)$, the function is odd

If we discover that the function is even or odd, the next step is to check the limits of integration (the interval over which we're integrating). In order to use the special even or odd function rules for definite integrals, our interval must be in the form $[-a, a]$. In other words, the limits of integration have the same number value but opposite signs, like $[-1,1]$ or $[-5,5]$.

If the function is even or odd and the interval is $[-a, a]$, we can apply these rules:

When $f(x)$ is even,

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

When $f(x)$ is odd,

$$\int_{-a}^{a} f(x)\ dx = 0$$

Example

Integrate.

$$\int_{-2}^{2} 3x^2 + 2\ dx$$

First we'll check to see if the function meets the criteria for an even or odd function. To see if it's even, we'll substitute $-x$ for x.

$$f(x) = 3x^2 + 2$$

$$f(-x) = 3(-x)^2 + 2$$

$$f(-x) = 3x^2 + 2$$

After substituting $-x$ for x, we were able to get back to the original function, which means we can say that

$$f(x) = f(-x)$$

and therefore that the function is even.

Looking at the given interval $[-2,2]$, we see that it's in the form $[-a, a]$.

Since we know that our function is even and that our interval is symmetric about the y-axis, we can calculate our answer using the formula

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = 2\int_{0}^{2} 3x^2 + 2\ dx$$

Now, instead of integrating the left-hand side, we can instead integrate the right-hand side, and evaluating over the new interval will be a little easier.

$$\int_{-2}^{2} 3x^2 + 2\ dx = 2\left(\frac{3}{3}x^3 + 2x\right)\Big|_0^2$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left(2x^3 + 4x\right)\Big|_0^2$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = \left[2(2)^3 + 4(2)\right] - \left[2(0)^3 + 4(0)\right]$$

$$\int_{-2}^{2} 3x^2 + 2\ dx = 24$$

Let's try another example.

Example

Integrate.

$$\int_{-7}^{7} 3x^7 + 4\sin x\ dx$$

First we'll check to see if the function meets the criteria for an even or odd function. Let's start by testing it to see if it's an even function by substituting $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(x) \neq f(-x)$$

In order for the function to be even, $f(-x) = f(x)$. Since $f(x) \neq f(-x)$, this function is not even.

So we'll check to see if the function is odd. Remember that an odd function requires $f(-x) = -f(x)$. We can test this by substituting $-x$ for x.

$$f(x) = 3x^7 + 4\sin x$$

$$f(-x) = 3(-x)^7 + 4\sin(-x)$$

$$f(-x) = -3x^7 - 4\sin x$$

$$f(-x) = -\left(3x^7 + 4\sin x\right)$$

$$f(-x) = -f(x)$$

Because $f(-x)$ becomes $-f(x)$, we can say that the function is odd. Looking at the given interval $[-7,7]$, we see that it's in the form $[-a,a]$.

Since we know that our function is odd and that our interval is symmetric about the y-axis, we can calculate the answer using the formula

$$\int_{-a}^{a} f(x)\ dx = 0$$

$$\int_{-7}^{7} 3x^7 + 4\sin x\ dx = 0$$

Riemann sums

Left endpoints, right endpoints, and midpoints

Since we don't know yet how to calculate the exact area under a curve (we'll learn how to do this later with integration), we'll need to use Riemann Sums to find an approximation of the area. The basic idea behind a Riemann Sum approximation is to use rectangles to approximate the area under the curve. The more rectangles we draw, the better our area approximation will be.

To use a Riemann Sum approximation, you'll simply calculate the area of each rectangle under the graph, and then add the areas together to get the total area under that section of the graph. The accuracy of the approximation will depend on whether we use the left-endpoints of the rectangles, the right-endpoints, or their midpoints. All three choices will give us different approximations of the area. Midpoint approximations are usually the most accurate.

Overestimation and underestimation

1. If the function is increasing everywhere in the interval, left endpoints will give an underestimation and right endpoints give an overestimation.

2. If the function is decreasing everywhere in the interval, left endpoints will give an overestimation and right endpoints give an underestimation.

3. Midpoints will usually give us a better approximation of both increasing and decreasing functions.

4. If the function is both increasing and decreasing at different points in the interval, it's going to be difficult for us to determine whether or not we'll get an under or overestimation from using left or right endpoints.

Calculating a Riemann sum

In order to calculate a Riemann Sum, follow these steps:

1. Find Δx if $\Delta x = \dfrac{b-a}{n}$.

2. Divide your interval and separate the x-axis into increments of Δx.

3. Decide whether you'll use the left endpoints, right endpoints, or midpoints of each of the rectangles indicated by the increments you created in Step 2.

4. Evaluate your function at these points, add all of your answers together, and then multiply your final answer by Δx. This is the Riemann sum for the interval.

$$A = \sum_{i=1}^{n} f(x_i)\Delta x$$

A note about integration

There is no limit to the number of rectangles you use to evaluate the area under the graph. The more rectangles you use, the more accurate your approximation will be. If you used an infinite number of rectangles, and therefore the value of Δx approached 0, you'd be integrating and finding the exact value of the area, instead of just an approximation.

Trapezoidal rule

The trapezoidal rule is one method we can use to approximate the area under a function over a given interval. If it's difficult to find area exactly using an integral, we can use trapezoidal rule instead to estimate the integral. It's called trapezoidal rule because we use trapezoids to estimate the area under the curve.

With this method, we divide the given interval into n subintervals, and then find the width of the subintervals. We call the width Δx. The larger the value of n, the smaller the value of Δx, and the more accurate our final answer.

The formula for trapezoidal rule is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

where the limits of integration $[a, b]$ are the endpoints of the interval. Δx is

$$\Delta x = \frac{b-a}{n}$$

where n is the number of trapezoids, and the subintervals are defined by $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$, where

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = x_1 + \Delta x$$

...

$$x_{n-1} = x_{n-2} + \Delta x$$

$$x_n = x_{n-1} + \Delta x$$

Example

Using $n = 4$ and the trapezoidal rule, approximate the value of the integral.

$$\int_2^6 e^{x^2}\ dx$$

First, we need to find the the width of the subintervals using

$$\Delta x = \frac{b-a}{n}$$

where $a = 2$, $b = 6$, and $n = 4$.

$$\Delta x = \frac{6-2}{4}$$

$$\Delta x = 1$$

This means that each sub-interval is 1 unit wide. Now we can solve for our sub-intervals using $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$ where $x_0 = 2$ (we start here because it's our lower limit of integration), $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 6$ (we end here because it's our upper limit of integration).

$$[2,3],\ [3,4],\ [4,5],\ [5,6]$$

Now we're ready to plug these values into our trapezoidal rule formula. Remember, since we include the start and endpoints of our interval, we'll always have $n+1$ terms.

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)\right]$$

$$\int_2^6 e^{x^2}\ dx \approx \frac{1}{2}\left[e^{(2)^2}+2e^{(3)^2}+2e^{(4)^2}+2e^{(5)^2}+e^{(6)^2}\right]$$

$$\int_2^6 e^{x^2}\ dx \approx 2.16\times 10^{15}$$

Using the trapezoidal rule, our approximate area is 2.16×10^{15} units.

Simpson's rule

Simpson's rule is a method we can use to approximate the area under a function over a given interval. If it's difficult to find area exactly using an integral, we can use Simpson's rule instead to estimate the integral.

In order to use Simpson's rule to get an estimate of the area, we need to know the interval over which we're calculating area so that we can divide the area into n subintervals. That'll allow us to calculate the width of each subinterval, Δx. The larger the value of n, the smaller the value of Δx and the more accurate our final answer will be.

In order to use Simpson's rule, n **must** be an even number.

The interval over which we want to find area is $[a, b]$, and the sub-intervals are $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$ where

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = x_1 + \Delta x$$

...

$$x_{n-1} = x_{n-2} + \Delta x$$

$$x_n = x_{n-1} + \Delta x$$

To find Δx, we use

$$\Delta x = \frac{b-a}{n}$$

Putting it all together, we get the formula for the Simpson's rule, which is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

This formula is a little tricky. We have to remember that

- the odd subscripts (like $f(x_1)$) are multiplied by 4 and
- the even subscripts (like $f(x_4)$) are multiplied by 2, except
- the first and last terms ($f(x_0)$ and $f(x_n)$), which have no multiplier.

Example

Using $n = 4$ and Simpson's rule, approximate the value of the integral.

$$\int_0^3 e^{x^2}\ dx$$

First we'll find the width of the subintervals. We'll use $\Delta x = \frac{b-a}{n}$ where $a = 0$, $b = 3$, and $n = 4$.

$$\Delta x = \frac{3-0}{4}$$

$$\Delta x = \frac{3}{4}$$

This means that each sub-interval is $3/4$ units wide. Now we can solve for our subintervals using $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{n-1}, x_n]$, where $x_0 = 0$, $x_1 = 3/4$, $x_2 = 3/2$, $x_3 = 9/4$, and $x_4 = 3$.

$$\left[0,\frac{3}{4}\right],\ \left[\frac{3}{4},\frac{3}{2}\right],\ \left[\frac{3}{2},\frac{9}{4}\right],\ \left[\frac{9}{4},3\right]$$

Now we can plug all of this information into the Simpson's rule formula. Remember that with this method, we'll end up with $n+1$ terms since we include the endpoints of the interval.

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{\frac{3}{4}}{3}\left[e^{(0)^2}+4e^{\left(\frac{3}{4}\right)^2}+2e^{\left(\frac{3}{2}\right)^2}+4e^{\left(\frac{9}{4}\right)^2}+e^{(3)^2}\right]$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{1}{4}\left(e^0+4e^{\frac{9}{16}}+2e^{\frac{9}{4}}+4e^{\frac{81}{16}}+e^9\right)$$

$$\int_0^3 e^{x^2}\ dx \approx \frac{1}{4}\left(1+4e^{\frac{9}{16}}+2e^{\frac{9}{4}}+4e^{\frac{81}{16}}+e^9\right)$$

$$\int_0^3 e^{x^2}\ dx \approx 2{,}191$$

Using Simpson's rule, the approximate area is $2{,}191$ square units.

Error bounds

Remember that midpoint rule, trapezoidal rule, and Simpson's rule are all different ways to come up with an approximation for area under the curve. But how do we know how accurate our approximation is, in comparison to the exact area under the curve? We want to know whether an approximation is very good, and close to actual area, or if it's a very bad approximation of actual area.

That's where the error bound formulas come in. They tell us the maximum possible error in our approximations. So if the error bound is very large, we know that it's possible that our approximation is bad, and far from the actual area. If the error bound is very small, we know that our approximation is pretty good, and close to the actual area.

The error bound formulas are

Midpoint rule error bound	$\left\|E_M\right\| \le \dfrac{K(b-a)^3}{24n^2}$	$\left\|f''(x)\right\| \le K$
Trapezoidal rule error bound	$\left\|E_T\right\| \le \dfrac{K(b-a)^3}{12n^2}$	$\left\|f''(x)\right\| \le K$
Simpson's rule error bound	$\left\|E_S\right\| \le \dfrac{K(b-a)^5}{180n^4}$	$\left\|f^{(4)}(x)\right\| \le K$

where

- E_M, E_T, and E_S are actual error for the midpoint, trapezoidal, and Simpson's rule respectively

- $|E_M|$, $|E_T|$, and $|E_S|$ are the absolute values of the actual errors, which you can also think of as the maximum possible error, or the maximum possible difference between your approximation of area and actual area

- $[a, b]$ is the interval over which you're finding area

- n is the number of subintervals you're using to find area over the interval $[a, b]$

- $f''(x)$ is the second derivative of the given function $f(x)$

- $f^{(4)}(x)$ is the fourth derivative of the given function $f(x)$

You'll want to use information from your problem to plug in for a, b, and n, and you're going to be solving the inequality for $|E_M|$, $|E_T|$, or $|E_S|$. Which means the only value you really need to find is K. Finding K is the only real tricky part when it comes to finding error bound.

Notice that for midpoint and trapezoidal rules, $|f''(x)| \leq K$, and for Simpson's rule $|f^{(4)}(x)| \leq K$. This means that for midpoint and trapezoidal rules, K must always be greater than or equal to the second derivative of the given function, and that for Simpson's rule, K must always be greater than or equal to the fourth derivative of the given function. In other words, what you'll be trying to do is find the maximum possible value of the second derivative (for midpoint and trapezoidal rules) or fourth derivative (for Simpson's rule) and use that value for K.

Example

Find the error bound $|E_S|$ if $n = 4$, and then find the number of subintervals n that will guarantee the area approximation is accurate within 0.00001.

$$\int_0^1 e^{x^2}\ dx$$

We know that the interval we're interested in is $[a, b] = [0,1]$ and that $n = 4$, so plugging these values into the the Simpson's rule error bound formula gives

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

$$|E_S| \le \frac{K(1-0)^5}{180(4)^4}$$

$$|E_S| \le \frac{K}{46{,}080}$$

To find a value for K, we'll need to use the condition that $\left|f^{(4)}(x)\right| \le K$, which means we need to find the fourth derivative of the given function $f(x) = e^{x^2}$.

$$f(x) = e^{x^2}$$

$$f'(x) = 2xe^{x^2}$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}$$

$$f'''(x) = 12xe^{x^2} + 8x^3e^{x^2}$$

$$f^{(4)}(x) = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$$

Remember that the interval we're interested in is $[0,1]$. Therefore, we need to find the maximum value that this fourth derivative can attain over that interval. Since the fourth derivative is a polynomial function and every term is positive, we know it's increasing throughout the interval, which means that the largest value it'll attain is at $x = 1$.

At $x = 1$,

$$f^{(4)}(1) = 12e^{1^2} + 48(1)^2e^{1^2} + 16(1)^4e^{1^2}$$

$$f^{(4)}(1) = 76e$$

Since this is the largest value the fourth derivative will have during the interval, we'll say $K = 76e$.

$$\left|E_S\right| \leq \frac{76e}{46{,}080}$$

$$\left|E_S\right| \leq 0.0045$$

This tells us that the error will be no larger than about 0.0045, so if we used Simpson's rule with $n = 4$ subintervals to approximate the area under the curve, we'd get a pretty accurate estimate of actual area.

To answer the second part of this question, we need to find the number of subintervals n that will guarantee an even more accurate estimation, one in which error is no greater than 0.00001. To do this, we'll take the right side of the error bound formula, plug in for a, b, and K but leave n as-is, and set that less than or equal to 0.00001.

$$\frac{76e(1-0)^5}{180n^4} \le 0.00001$$

$$76e \le 0.0018n^4$$

$$\frac{76e}{0.0018} \le n^4$$

$$\sqrt[4]{\frac{76e}{0.0018}} \le n$$

$$18.41 \le n$$

You can't use 18.41 subintervals, you'd need to use either 18 subintervals, or 19 subintervals. But n must be greater than 18.41 in order to guarantee an approximation of area within 0.00001 of actual area, which means we'll round up and say that $n = 19$ subintervals.

If we were doing this with a midpoint or trapezoidal rule problem, we could stop here. But with Simpson's rule, remember that we always have to use an even number of subintervals. Which means that we actually have to round up to nearest even number, and use $n = 20$ subintervals in order to guarantee an area estimate within 0.00001.

Part 1 of the FTC

Part 1 of the Fundamental Theorem of Calculus (FTC) is the formula that relates the derivative to the integral. In other words, it connects the first big idea of calculus, the derivative, to the second big idea of calculus, the integral. It states that

If $r(x)$ is continuous on $[a, b]$ then

$$f(x) = \int_a^x r(t)\ dt,$$

is continuous on $[a, b]$, it's differentiable on (a, b), and

$$f'(x) = r(x)$$

This means that if you take the integral of the function $r(t)$ over the interval $[a, x]$, the answer you get, $f(x)$, can be differentiated to get back to $r(x)$.

Therefore, the FTC is something you can use to double check your integration for mistakes.

When it comes to solving a problem using Part 1 of the Fundamental Theorem, we can use the chart below to help us figure out how to do it. The chart tells us how to solve for $f'(x)$, depending on the kinds of bounds we find on the integral.

Given integral	**How to solve it**
$f(x) = \int_a^x r(t)\ dt$	Plug x in for t.

$$f(x) = \int_x^a r(t)\ dt$$

Reverse limits of integration and multiply by -1, then plug x in for t.

$$f(x) = \int_a^{g(x)} r(t)\ dt$$

Plug $g(x)$ in for t, then multiply by dg/dx.

$$f(x) = \int_{g(x)}^a r(t)\ dt$$

Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.

$$f(x) = \int_{g(x)}^{h(x)} r(t)\ dt$$

Split the limits of integration as $\int_{g(x)}^0 r(t)\ dt + \int_0^{h(x)} r(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^0 r(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Example

Use Part 1 of the Fundamental Theorem of Calculus to find $f'(x)$.

$$f(x) = \int_0^{x^2} t^2 - 1\ dt$$

Since the given lower bound is a constant and the upper bound is a function in terms of x, the integral we're given follows the pattern of the third integral in the table,

$$f(x) = \int_a^{g(x)} r(t)\ dt$$

So we know we need to plug $g(x)$ (the upper bound) in for t, then multiply by dg/dx (the derivative of the upper bound). Therefore, we can say that $f'(x)$ is

$$f'(x) = \left((x^2)^2 - 1\right)(2x)$$

$$f'(x) = (x^4 - 1)(2x)$$

$$f'(x) = 2x^5 - 2x$$

Part 2 of the FTC

Part 2 of the Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

if $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is the anti-derivative of $f(x)$.

Part 2 of the FTC tells us that we can figure out the exact value of an indefinite integral (area under the curve) when we know the interval over which to evaluate (in this case the interval $[a, b]$).

There are rules to keep in mind. For instance, the function $f(x)$ must be continuous over the interval $[a, b]$ (no holes, breaks, or jumps), and the interval must be closed, which means that both limits of integration must be constants (real numbers only, no infinity allowed).

Example

Use Part 2 of the Fundamental Theorem of Calculus to find the value of the integral.

$$F(x) = \int_1^3 x^3\ dx$$

First, we perform the integration.

$$F(x) = \frac{x^4}{4}\Bigg|_1^3$$

Next, we plug in the upper and lower limits, subtracting the value at the lower bound from the value at the upper bound.

$$F = \frac{3^4}{4} - \frac{1^4}{4}$$

$$F = \frac{81}{4} - \frac{1}{4}$$

$$F = \frac{80}{4}$$

Let's double check that this satisfies Part 2 of the FTC. If we break the equation into parts,

$$F(b) = \int x^3\ dx \text{ where } b = 3 \text{ and } F(a) = \int x^3\ dx \text{ where } a = 1$$

and evaluate the two equations separately, we can double check our answer. First we integrate as an indefinite integral.

$$F(x) = \int x^3\ dx$$

$$F(x) = \frac{x^4}{4} + C$$

Next we plug in $b = 3$ and $a = 1$.

$$F(3) = \frac{3^4}{4} + C$$

$$F(1) = \frac{1^4}{4} + C$$

Finally, we find $F(b) - F(a)$.

$$F(3) - F(1) = \frac{3^4}{4} + C - \left(\frac{1^4}{4} + C\right)$$

$$F(3) - F(1) = \frac{3^4}{4} + C - \frac{1^4}{4} - C$$

$$F(3) - F(1) = \frac{80}{4}$$

As you can see, we've verified that value of F that we found earlier. This answer is what we expected and it confirms Part 2 of the FTC.

Example

Integrate using Part 2 of the FTC.

$$F(t) = \int_2^x t^3 + 2t^4 \ dt$$

When we integrate we get

$$F(t) = \left(\frac{t^4}{4} + \frac{2t^5}{5} + C\right)\Bigg|_2^x$$

Evaluating over the interval, we get

$$F(x) = \frac{x^4}{4} + \frac{2x^5}{5} + C - \left(\frac{2^4}{4} + \frac{2(2)^5}{5} + C\right)$$

$$F(x) = \frac{x^4}{4} + \frac{2x^5}{5} + C - \left(4 + \frac{64}{5} + C\right)$$

$$F(x) = \frac{x^4}{4} + \frac{2x^5}{5} + C - \left(\frac{20}{5} + \frac{64}{5} + C\right)$$

$$F(x) = \frac{x^4}{4} + \frac{2x^5}{5} + C - \frac{84}{5} - C$$

$$F(x) = \frac{x^4}{4} + \frac{2x^5}{5} - \frac{84}{5}$$

For this last example, we could use Part 1 of the FTC to confirm.

$$\frac{dF(x)}{dx} = \frac{d\left(\frac{x^4}{4} + \frac{2x^5}{5} - \frac{84}{5}\right)}{dx}$$

$$\frac{dF(x)}{dx} = \left(\frac{4x^3}{4} + \frac{10x^4}{5} - 0\right)$$

$$\frac{dF(x)}{dx} = x^3 + 2x^4$$

We know that

$$f(t) = t^3 + 2t^4$$

So by substituting x for t we get

$$f(x) = x^3 + 2x^4$$

We can see that $\frac{dF(x)}{dx} = f(x)$.

$$\frac{dF(x)}{dx} = x^3 + 2x^4 = f(x)$$

U-substitution

Finding derivatives of elementary functions was a relatively simple process, because taking the derivative only meant applying the right derivative rules.

This is not the case with integration. Unlike derivatives, it may not be immediately clear which integration rules to use, and every function is like a puzzle.

Most integrals need some work before you can even begin the integration. They have to be transformed or manipulated in order to reduce the function's form into some simpler form. U-substitution is the simplest tool we have to transform integrals.

When you use u-substitution, you'll define u as a differentiable function in terms of the variable in the integral, take the derivative of u to get du, and then substitute these values back into your integrals.

Unfortunately, there are no perfect rules for defining u. If you try a substitution that doesn't work, just try another one. With practice, you'll get faster at identifying the right value for u.

Here are some common substitutions you can try.

For integrals that contain power functions, try using the base of the power function as the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int x\left(x^2+1\right)^4 \ dx$$

Let

$$u = x^2 + 1$$

$$du = 2x \ dx$$

$$dx = \frac{du}{2x}$$

Substituting back into the integral, we get

$$\int x(u)^4 \ \frac{du}{2x}$$

$$\int u^4 \ \frac{du}{2}$$

$$\frac{1}{2}\int u^4 \ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2}\left(\frac{1}{5}u^5\right) + C$$

$$\frac{1}{10}u^5 + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{10}\left(x^2+1\right)^5+C$$

For integrals of rational functions, if the numerator is of equal or greater degree than the denominator, always perform division first. Otherwise, try using the denominator as a possible substitution.

Example

Use u-substitution to evaluate the integral.

$$\int \frac{x}{x^2+1}\ dx$$

Let

$$u=x^2+1$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Substituting back into the integral, we get

$$\int \frac{x}{u}\cdot\frac{du}{2x}$$

$$\int \frac{1}{u} \cdot \frac{du}{2}$$

$$\frac{1}{2} \int \frac{1}{u} \ du$$

This is much simpler than our original integral, and something we can actually integrate.

$$\frac{1}{2} \ln |u| + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{2} \ln \left| x^2 + 1 \right| + C$$

For integrals containing exponential functions, try using the power for the substitution.

Example

Use u-substitution to evaluate the integral.

$$\int e^{\sin x \cos x} \cos 2x \ dx$$

Let $u = \sin x \cos x$, and using the product rule to differentiate,

$$du = \left[\left(\frac{d}{dx} \sin x \right) \cos x + \sin x \left(\frac{d}{dx} \cos x \right) \right] \ dx$$

$$du = [\cos x \cdot \cos x + \sin x \cdot (-\sin x)] \ dx$$

$$du = \cos^2 x - \sin^2 x \ dx$$

$$du = \cos 2x \ dx$$

Substituting back into the integral, we get

$$\int e^u \ du$$

$$e^u + C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$e^{\sin x \cos x} + C$$

Integrals containing trigonometric functions can be more challenging to manipulate. Sometimes, the value of u isn't even part of the original integral. Therefore, the better you know your trigonometric identities, the better off you'll be.

Example

Use u-substitution to evaluate the integral.

$$\int \frac{\tan x}{\cos x} \ dx$$

Since

$$\tan x = \frac{\sin x}{\cos x}$$

we can rewrite the integral as

$$\int \frac{\frac{\sin x}{\cos x}}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\ dx$$

$$\int \frac{\sin x}{\cos^2 x}\ dx$$

Let

$$u = \cos x$$

$$du = -\sin x\ dx$$

$$dx = -\frac{du}{\sin x}$$

Substituting back into the integral, we get

$$\int \frac{\sin x}{u^2} \cdot \left(-\frac{du}{\sin x}\right)$$

$$-\int \frac{1}{u^2}\ du$$

$$-\int u^{-2}\ du$$

$$-\frac{1}{-1}u^{-1}+C$$

$$u^{-1}+C$$

$$\frac{1}{u}+C$$

Now, back-substitute to put the answer back in terms of x instead of u.

$$\frac{1}{\cos x}+C$$

U-substitution in definite integrals

U-substitution in definite integrals is just like substitution in indefinite integrals except that, since the variable is changed, the limits of integration must be changed as well.

Example

Use u-substitution to evaluate the integral.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\ dx$$

Let

$$u = \sin x$$

$$du = \cos x\ dx$$

$$dx = \frac{du}{\cos x}$$

Since we're dealing with a definite integral, we need to use the equation $u = \sin x$ to find limits of integration in terms of u, instead of x.

when $x = 0$, $\qquad u = \sin 0$

$$u = 0$$

when $x = \frac{\pi}{2}$, $\qquad u = \sin \frac{\pi}{2}$

$$u = 1$$

Substituting back into the integral (including for our limits of integration), we get

$$\int_0^1 \frac{\cos x}{1+u^2} \cdot \frac{du}{\cos x}$$

$$\int_0^1 \frac{1}{1+u^2}\ du$$

Using this very common formula,

$$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + C$$

take the integral.

$$\int_0^1 \frac{1}{1+u^2}\ du = \tan^{-1} u \Big|_0^1$$

$$\int_0^1 \frac{1}{1+u^2}\ du = \tan^{-1} 1 - \tan^{-1} 0$$

$$\int_0^1 \frac{1}{1+u^2}\ du = \frac{\pi}{4}$$

Integration by parts

Unlike differentiation, integration is not always straightforward and we can't always express the integral of every function in terms of neat and clean elementary functions.

When your integral is too complicated to solve without a fancy technique and you've ruled out u-substitution, integration by parts should be your next approach for evaluating your integral. If you remember that the product rule was your method for finding derivatives of functions that were multiplied together, you can think about integration by parts as the method often used for *integrating* functions that are multiplied together.

Suppose you want to integrate the following

$$\int xe^{-x}\ dx$$

How can you integrate the above expression quickly and easily? You can't, unless you're a super human genius. But hopefully you can recognize that you have two functions multiplied together inside of this integral, one being x and the other being e^{-x}. If you try u-substitution, you won't find anything to cancel in your integral, and you'll be no better off, which means that your next step should be an attempt at integrating with our new method, integration by parts.

The formula we'll use is derived by integrating the product rule formula, and looks like this:

$$\int u\ dv = uv - \int v\ du$$

In the formula above, everything to the left of the equals sign represents your original function, which means your original function must be composed of u and dv. Your job is to identify which part of your original function will be u, and which will be dv.

My favorite technique for picking u and dv is to assign u to the function in your integral whose derivative is simpler than the original u. Consider again the example from earlier:

$$\int xe^{-x}\ dx$$

I would assign u to x, because the derivative of x is 1, which is much simpler than x. If you have $\ln x$ in your integral, that's usually a good bet for u because the derivative of $\ln x$ is $1/x$; much simpler than $\ln x$. Once you pick which of your functions will be represented by u, the rest is easy because you know that the other function will be represented by dv.

Using this formula can be challenging for a lot of students, but the hardest part is identifying which of your two functions will be u and which will be dv. That's the very first thing you have to tackle with integration by parts, so once you get that over with, you'll be home free.

After completing this first crucial step, you take the derivative of u, called du, and the integral of dv, which will be v. Now that you have u, du, v and dv, you can plug all of your components into the right side of the integration by parts formula. Everything to the right of the equals sign will be part of your answer. If you've correctly assigned u and dv, the integral on the right should now be much easier to integrate.

Example

Evaluate the integral.

$$\int xe^{-x}\ dx$$

Our integral is comprised of two functions, x and e^{-x}. One of them must be u and the other dv. Since the derivative of x is 1, which is much simpler than the derivative of e^{-x}, we'll assign u to x.

$u = x$ → differentiate → $du = 1\ dx$

$dv = e^{-x}\ dx$ → integrate → $v = -e^{-x}$

Plugging all four components into the right side of our formula gives the following transformation of our original function:

$$(x)(-e^{-x}) - \int (-e^{-x})(1\ dx)$$

$$-xe^{-x} + \int e^{-x}\ dx$$

Now that we have something we can work with, we integrate.

$$-xe^{-x} + (-e^{-x}) + C$$

The answer is therefore

$$-xe^{-x} - e^{-x} + C$$

Or factored, we have

$$-e^{-x}(x+1) + C$$

Integration by parts two times

What happens if you apply integration by parts and the integral you're left with still isn't easy to solve?

The first thing you should do in this situation is make sure that you assigned u and dv correctly. Try assigning u and dv to opposite components of your original integral and see if you end up with a better answer.

If you still get an integral you can't evaluate, maybe you need to use u-substitution on the left-over integral after you've already used integration by parts. Or, maybe integration by parts wasn't the right integration technique to use in the first place. Check to see if u-substitution works better on your original integral.

If all else fails, the trick might be to use integration by parts a second or third time. In other words, you might need to apply integration by parts several times in a row to get to an integral you can easily solve.

With practice, you'll start to realize that integrals like these ones often require multiple applications of integration by parts:

Power $\times$ Exponential

$\int x^n e^x\ dx$ Set $u = x^n$, apply IBP n times to reduce x^n to 1

Power $\times$ Trigonometric

$\int x^n \sin x \, dx$ Set $u = x^n$, apply IBP n times to reduce x^n to 1

$\int x^n \cos x \, dx$ Set $u = x^n$, apply IBP n times to reduce x^n to 1

Exponential $\times$ Trigonometric

$\int e^x \sin x \, dx$ Set $u = \sin x$, apply IBP twice to get back to $\sin x$, combine with the left-hand side

$\int e^x \cos x \, dx$ Set $u = \cos x$, apply IBP twice to get back to $\cos x$, combine with the left-hand side

While these aren't the only functions that force you to apply integration by parts multiple times, they are by far the most common, so it's helpful to remember them if you can.

Here's an example of a power function with a trigonometric function.

Example

Use integration by parts to evaluate the integral.

$$\int x^3 \cos x \, dx$$

First, we'll assign u and dv, then differentiate u to get du and integrate dv to get v.

$u = x^3$ differentiate $du = 3x^2\ dx$

$dv = \cos x\ dx$ integrate $v = \sin x$

Plugging all four components into the formula gives

$$(x^3)(\sin x) - \int (\sin x)(3x^2\ dx)$$

$$x^3 \sin x - \int 3x^2 \sin x\ dx$$

What remains inside the integral is not easy to evaluate. Since u-substitution won't get us anywhere, we try integration by parts again, using our most recent integral.

$u = 3x^2$ differentiate $du = 6x\ dx$

$dv = \sin x\ dx$ integrate $v = -\cos x$

Plugging in again to our last integral gives

$$x^3 \sin x - \left[(3x^2)(-\cos x) - \int (-\cos x)(6x\ dx)\right]$$

$$x^3 \sin x - \left(-3x^2 \cos x + \int 6x \cos x\ dx\right)$$

$$x^3 \sin x + 3x^2 \cos x - \int 6x \cos x\ dx$$

We still don't have an easy integral, so we use integration by parts one more time.

$u = 6x$	differentiate	$du = 6\ dx$
$dv = \cos x\ dx$	integrate	$v = \sin x$

Using the integration by parts formula to again transform the integral, we get:

$$x^3 \sin x + 3x^2 \cos x - \left[(6x)(\sin x) - \int (\sin x)(6\ dx)\right]$$

$$x^3 \sin x + 3x^2 \cos x - 6x \sin x + \int 6 \sin x\ dx$$

We finally have something we can easily integrate, so the answer is

$$x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C$$

Sometimes after applying integration by parts twice, you end up with an integral that is the same as your original problem. When that happens, instead of feeling like you're right back where you started, realize that you can add the integral on the right side of your equation to the integral on the left. You're just combining like terms like you did in algebra, except, instead of combining something simple like x^2 and $3x^2$, you're combining equal integrals.

Let's look at an example of an exponential function with a trigonometric function.

Example

Use integration by parts to evaluate the integral.

$$\int e^x \cos x \ dx$$

First, we'll assign u and dv, then differentiate u to get du and integrate dv to get v.

$u = \cos x$ differentiate $du = -\sin x \ dx$

$dv = e^x \ dx$ integrate $v = e^x$

Plugging all four components into the integration by parts formula gives

$$(\cos x)(e^x) - \int (e^x)(-\sin x \ dx)$$

$$e^x \cos x + \int e^x \sin x \ dx$$

We use integration by parts again to simplify the remaining integral.

$u = \sin x$ differentiate $du = \cos x \ dx$

$dv = e^x \ dx$ integrate $v = e^x$

Plugging in again, we get

$$e^x \cos x + \left[(\sin x)(e^x) - \int (e^x)(\cos x \ dx)\right]$$

$$\int e^x \cos x \ dx = e^x \cos x + e^x \sin x - \int e^x \cos x \ dx$$

See how our new integral on the right is the same as the original integral on the left? It seems like we're right back to the beginning of our problem and that all hope is lost. Instead, we can combine like terms and add the integral on the right to the left side of our equation.

$$2\int e^x \cos x \ dx = e^x \cos x + e^x \sin x$$

We'll just simplify the right-hand side and then divide both sides of the equation by 2 to solve for the original integral and get our final answer.

$$2\int e^x \cos x \ dx = e^x(\cos x + \sin x)$$

$$\int e^x \cos x \ dx = \frac{e^x(\cos x + \sin x)}{2}$$

Tabular integration

Tabular integration is an alternative method we can use to deal with problems that would normally be integrated using integration by parts.

Tabular integration is often extremely useful in situations where our integral requires us to use integration by parts multiple times.

To use tabular integration, we create a table with two columns. In the first column, we differentiate u until it goes to 0. In the second column, we integrate dv as many times as we differentiated u to get it to 0.

	1. Differentiate u	2. Integrate dv
a.	$u = 1a$	$dv = 2a$
b.	$u' = 1b$	$v = 2b$
c.	$u'' = 1c$	$\int vdv = 2c$
	...	...
d.	constant	$\dots \int vdv \dots = 2e$
e.	0	$\dots \int vdv \dots = 2f$

Once we've built our table, we can start compiling our answer. Our first term will be the product of the value from the first row of the first column ($1a$), and the from the second row of the second column ($2b$). We continue the same pattern to get the product of $1b$ and $2c$, the product of $1c$ and $2d$, etc. The last term is the one that includes the last non-zero term from the first column (the constant).

It's very important that we alternate the signs of these terms as we add them together. The odd terms (first, third, etc.) are positive; the even terms (second, fourth, etc.) are negative.

Example

Use tabular integration to evaluate the integral. Verify your answer using integration by parts.

$$\int x^4 e^{3x}\ dx$$

First, we need to name our parts. Let

$$u_0 = x^4$$

and

$$dv_0 = e^{3x}\ dx$$

We can then calculate

$$du_0 = 4x^3\ dx$$

and

$$v_0 = \frac{1}{3}e^{3x}$$

Now let's set up our table.

	1. Differentiate u	2. Integrate dv
a.	x^4	e^{3x}
b.	$4x^3$	$\frac{1}{3}e^{3x}$
c.	$12x^2$	$\frac{1}{9}e^{3x}$
d.	$24x$	$\frac{1}{27}e^{3x}$
e.	24	$\frac{1}{81}e^{3x}$
f.	0	$\frac{1}{243}e^{3x}$

Next, we can compile our answer, remembering to alternate signs, starting with $+$. We get,

$$\int x^4e^{3x}\ dx = (x^4)\left(\frac{1}{3}e^{3x}\right) - (4x^3)\left(\frac{1}{9}e^{3x}\right) + (12x^2)\left(\frac{1}{27}e^{3x}\right)$$

$$-(24x)\left(\frac{1}{81}e^{3x}\right) + (24)\left(\frac{1}{243}e^{3x}\right) + C$$

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{9}x^2e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{81}e^{3x} + C$$

We can now verify our answer using integration by parts. Remember, $u_0 = x^4$, $dv_0 = e^{3x}\ dx$, $du_0 = 4x^3\ dx$ and $v_0 = \frac{1}{3}e^{3x}$ and our formula for integration by parts is $\int u\ dv = uv - \int v\ du$.

$$\int x^4e^{3x}\ dx = \left(x^4\right)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)\left(4x^3\ dx\right)$$

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{3}\int x^3e^{3x}\ dx$$

Using integration by parts a second time and letting $u_1 = x^3$, $dv_1 = e^{3x}\ dx$, $du_1 = 3x^2\ dx$, and $v_1 = \frac{1}{3}e^{3x}$, we get

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{3}\left[\left(x^3\right)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)\left(3x^2\ dx\right)\right]$$

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{3}\int x^2e^{3x}\ dx$$

Using integration by parts a third time and letting $u_2 = x^2$, $dv_2 = e^{3x}\ dx$, $du_2 = 2x\ dx$, and $v_2 = \frac{1}{3}e^{3x}$, we get

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{3}\left[\left(x^2\right)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(2x\ dx)\right]$$

$$\int x^4e^{3x}\ dx = \frac{1}{3}x^4e^{3x} - \frac{4}{9}x^3e^{3x} + \frac{4}{9}x^2e^{3x} - \frac{8}{9}\int xe^{3x}\ dx$$

Using integration by parts a fourth time and letting $u_3 = x$, $dv_3 = e^{3x}\ dx$, $du_3 = dx$, and $v_3 = \frac{1}{3}e^{3x}$, we get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\left[(x)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(dx)\right]$$

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{27}\int e^{3x}\ dx$$

We're finally at a point where we can easily integrate the remaining integral, so we integrate and get

$$\int x^4 e^{3x}\ dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}xe^{3x} + \frac{8}{81}e^{3x} + C$$

The answer we just got using integration by parts was the same answer we got when we used tabular integration, which tells us that we did the tabular integration correctly.

In this particular example, we can see how much faster and easier it was to use tabular integration than it was to use integration by parts. Tabular integration will often be the faster method when we have to set u equal to a high-degree power function like x^3, x^7 or x^{12}.

That's because when we set u equal to a high-degree power function and then apply integration by parts, we only reduce the degree of the power function by 1. In other words, x^3 would become x^2, x^7 would become x^6, and x^{12} would become x^{11}. We have to continue applying integration by parts over and over again until the degree is reduced all the way to 0, so that we get x^0, which is just 1, and that part of the function drops away, leaving us with only the dv part.

Now that you know how to use tabular integration, try thinking through your integration by parts problems before you start them to see if tabular integration might be an easier way to solve them.

Partial fractions

The method of partial fractions is an extremely useful tool whenever you need to integrate a fraction with polynomials in both the numerator and denominator; something like this:

$$f(x)=\frac{7x+1}{x^2-1}$$

If you were asked to integrate

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

you shouldn't have too much trouble, because if you don't have a variable in the numerator of your fraction, then your integral is simply the numerator multiplied by the natural log ($\ln$) of the absolute value of the denominator, like this:

$$\int \frac{3}{x+1}+\frac{4}{x-1}\ dx$$

$$3\ln|x+1|+4\ln|x-1|+C$$

where C is the constant of integration. Not *too* hard, right?

Don't forget to use chain rule and divide by the derivative of your denominator. In the case above, the derivatives of both of our denominators are 1, so this step didn't appear. But if your integral is

$$\int \frac{3}{2x+1}\ dx$$

then your answer will be

$$\frac{3}{2}\ln|2x+1|+C$$

because the derivative of our denominator is 2, which means we have to divide by 2, according to chain rule.

So back to the original example. We said at the beginning of this section that

$$f(x)=\frac{7x+1}{x^2-1}$$

would be difficult to integrate, but that we wouldn't have as much trouble with

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

In fact, these two are actually the same function. If we try adding $3/(x+1)$ and $4/(x-1)$ together, you'll see that we get back to the original function.

$$f(x)=\frac{3}{x+1}+\frac{4}{x-1}$$

$$f(x)=\frac{3(x-1)+4(x+1)}{(x+1)(x-1)}$$

$$f(x)=\frac{3x-3+4x+4}{x^2-x+x-1}$$

$$f(x)=\frac{7x+1}{x^2-1}$$

Again, attempting to integrate $f(x) = (7x + 1)/\left(x^2 - 1\right)$ is extremely difficult. But if you can express this function as $f(x) = 3/(x + 1) + 4/(x - 1)$, then integrating is much simpler. This method of converting complicated fractions into simpler fractions that are easier to integrate is called decomposition into “partial fractions”.

Let’s start talking about how to perform a partial fractions decomposition. Before we move forward it’s important to remember that you must perform long division with your polynomials whenever the degree (value of the greatest exponent) of your denominator is not greater than the degree of your numerator, as is the case in the following example.

Example

Evaluate the integral.

$$\int \frac{x^3 - 3x^2 + 2}{x + 3}\, dx$$

Because the degree (the value of the highest exponent in the numerator, 3), is greater than the degree of the denominator, 1, we have to perform long division first.

$$
\begin{array}{r l}
 & x^2-6x+18-\dfrac{52}{x+3} \\ \cline{2-2}
x+3 & \big)\, x^3-3x^2+0x+2 \\
 & -(x^3+3x^2) \\ \cline{2-2}
 & -6x^2 \\
 & -(-6x^2-18x) \\ \cline{2-2}
 & 18x+2 \\
 & -(18x+54) \\ \cline{2-2}
 & -52
\end{array}
$$

After performing long division, our fraction has been decomposed into

$$\left(x^2-6x+18\right)-\frac{52}{x+3}$$

Now the function is easy to integrate.

$$\int x^2-6x+18-\frac{52}{x+3}\ dx$$

$$\frac{1}{3}x^3-3x^2+18x-52\ln|x+3|+C$$

Okay. So now that you've either performed long division or confirmed that the degree of your denominator is greater than the degree of your numerator (such that you don't have to perform long division), it's time for full-blown partial fractions.

The first step is to factor your denominator as much as you can. Your second step will be determining which type of denominator you're dealing with, depending on how it factors. Your denominator will be the product of the following:

1. Distinct linear factors
2. Repeated linear factors
3. Distinct quadratic factors
4. Repeated quadratic factors

Let's take a look at an example of each of these four cases so that you understand the difference between them.

Distinct linear factors

In this first example, we'll look at the first case above, in which the denominator is a product of distinct linear factors.

Example

Evaluate the integral.

$$\int \frac{x^2+2x+1}{x^3-2x^2-x+2}\ dx$$

Since the degree of the denominator is higher than the degree of the numerator, we don't have to perform long division before we start. Instead, we can move straight to factoring the denominator, as follows.

$$\int \frac{x^2+2x+1}{(x-1)(x+1)(x-2)}\ dx$$

We can see that our denominator is a product of distinct linear factors because $(x-1)$, $(x+1)$, and $(x-2)$ are all different first-degree factors.

Once we have it factored, we set our fraction equal to the sum of its component parts, assigning new variables to the numerator of each of our fractions. Since our denominator can be broken down into three different factors, we need three variables A, B and C to go on top of each one of our new fractions, like so:

$$\frac{x^2+2x+1}{(x-1)(x+1)(x-2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}$$

Now that we've separated our original function into its partial fractions, we multiply both sides by the denominator of the left-hand side. The denominator will cancel on the left-hand side, and on the right, each of the three partial fractions will end up multiplied by all the factors other than the one that was previously included in its denominator.

$$x^2+2x+1=A(x+1)(x-2)+B(x-1)(x-2)+C(x-1)(x+1)$$

The next step is to multiply out all of these terms.

$$x^2+2x+1=A\left(x^2-x-2\right)+B\left(x^2-3x+2\right)+C\left(x^2-1\right)$$

$$x^2+2x+1=Ax^2-Ax-2A+Bx^2-3Bx+2B+Cx^2-C$$

Now we collect like terms together, meaning that we re-order them, putting all the x^2 terms next to each other, all the x terms next to each other, and then all the constants next to each other.

$$x^2+2x+1=\left(Ax^2+Bx^2+Cx^2\right)+(-Ax-3Bx)+(-2A+2B-C)$$

Finally, we factor out the x terms.

$$x^2+2x+1=(A+B+C)x^2+(-A-3B)x+(-2A+2B-C)$$

Doing this allows us to equate coefficients on the left and right sides. Do you see how the coefficient on the x^2 term on the left-hand side of the equation is 1? Well, the coefficient on the x^2 term on the right-hand side is $(A+B+C)$, which means those two must be equal. We can do the same for the x term, as well as for the constants. We get the following three equations:

[1] $A+B+C=1$

[2] $-A-3B=2$

[3] $-2A+2B-C=1$

Now that we have these equations, we need to solve for our three constants A, B, and C. This can easily get confusing, but with practice, you should get the hang of it. If you have one equation with only two variables instead of all three, like [2], that's a good place to start. Solving [2] for A gives us

[4] $A=-3B-2$

Now we'll substitute [4] for A into [1] and [3] and then simplify, such that these equations:

$$(-3B - 2) + B + C = 1$$

$$-2(-3B - 2) + 2B - C = 1$$

become these equations:

[5] $-2B + C = 3$

[6] $8B - C = -3$

Now we can add [5] and [6] together to solve for B.

$$-2B + C + 8B - C = 3 - 3$$

$$6B = 0$$

[7] $B = 0$

Plugging [7] back into [4] to find A, we get

$$A = -3(0) - 2$$

[8] $A = -2$

Plugging [7] back into [5] to find B, we get

$$-2(0) + C = 3$$

[9] $C = 3$

Having solved for the values of our three constants in [7], [8] and [9], we're finally ready to plug them back into our partial fractions decomposition. Doing so should produce something that's easier for us to integrate than our original function.

$$\int \frac{x^2+2x+1}{(x-1)(x+1)(x-2)}\ dx = \int \frac{-2}{x-1} + \frac{0}{x+1} + \frac{3}{x-2}\ dx$$

Simplifying the integral on the right side, we get

$$\int \frac{3}{x-2} - \frac{2}{x-1}\ dx$$

Remembering that the integral of $1/x$ is $\ln|x| + C$, we integrate and get

$$3\ln|x-2| - 2\ln|x-1| + C$$

Repeated linear factors

Let's move now to the second of our four case types above, in which the denominator will be a product of linear factors, some of which are repeated.

Example

Evaluate the integral.

$$\int \frac{2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}{x^4 - 2x^2 + 1}\ dx$$

You’ll see that we need to carry out long division before we start factoring, since the degree of the numerator is greater than the degree of the denominator ($5 > 4$).

$$\begin{array}{r|l}
 & 2x - 3 + \dfrac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} \\
\hline
x^4 - 2x^2 + 1 & 2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13 \\
 & -(2x^5 + 0x^4 - 4x^3 + 0x^2 + 2x) \\
\hline
 & -3x^4 + 9x^3 + 3x^2 - 11x + 13 \\
 & -(-3x^4 + 0x^3 + 6x^2 + 0x - 3) \\
\hline
 & 9x^3 - 3x^2 - 11x + 16
\end{array}$$

Now that the degree of the remainder is less than the degree of the original denominator, we can rewrite the problem as

$$\int 2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1}\ dx$$

Integrating the $2x - 3$ will be simple, so for now, let’s focus on the fraction. We’ll factor the denominator.

$$\frac{9x^3 - 3x^2 - 11x + 16}{\left(x^2 - 1\right)\left(x^2 - 1\right)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)(x-1)(x+1)(x-1)}$$

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)^2(x-1)^2}$$

Given the factors involved in our denominator, you might think that the partial fraction decomposition would look like this:

$$\frac{9x^3 - 3x^2 - 11x + 16}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x-1}$$

However, the fact that we're dealing with repeated factors, ($(x+1)$ is a factor twice and $(x-1)$ is a factor twice), the partial fractions decomposition is actually the following:

$$\frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

To see why, let's take a simpler example. The partial fractions decomposition of $x^2/\left[(x+1)^4\right]$ is

$$\frac{x^2}{(x+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$$

Notice how we included $(x+1)^4$, our original factor, as well as each factor of lesser degree? We have to do this every time we have a repeated factor.

Let's continue with our original example.

$$\frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

We'll multiply both sides of our equation by the denominator from the left side, $(x+1)^2(x-1)^2$, which will cancel the denominator on the left and some of the factors on the right.

$$9x^3 - 3x^2 - 11x + 16 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

To simplify, we'll start multiplying all terms on the right side together.

$$9x^3 - 3x^2 - 11x + 16 = A\left(x^3 + x^2 - x - 1\right) + B\left(x^2 + 2x + 1\right)$$

$$+C\left(x^3 - x^2 - x + 1\right) + D\left(x^2 - 2x + 1\right)$$

Now we'll group like terms together.

$$9x^3 - 3x^2 - 11x + 16 = (A + C)x^3 + (A + B - C + D)x^2$$

$$+(-A + 2B - C - 2D)x + (-A + B + C + D)$$

Equating coefficients on both sides of the equation gives us the following equations.

[1] $A + C = 9$

[2] $A + B - C + D = -3$

[3] $-A + 2B - C - 2D = -11$

[4] $-A + B + C + D = 16$

Now we'll start solving for variables. If we subtract A from both sides of [1], we get

[5] $C = 9 - A$

If we plug [5] into [2], [3] and [4], we have

$$A + B - (9 - A) + D = -3$$

$$-A + 2B - (9 - A) - 2D = -11$$

$$-A + B + (9 - A) + D = 16$$

And simplifying, we get the following:

[6] $2A + B + D = 6$

[7] $2B - 2D = -2$

[8] $-2A + B + D = 7$

Let's now solve [7] for B.

$$2B - 2D = -2$$

$$2B = -2 + 2D$$

$$B = -1 + D$$

[9] $B = D - 1$

Plugging [9] into [6] and [8], we get

$$2A + (D - 1) + D = 6$$

$$-2A + (D - 1) + D = 7$$

And simplifying, we get the following:

[10] $2A + 2D = 7$

[11] $-2A + 2D = 8$

We solve [11] for D.

$$-2A + 2D = 8$$

$$2D = 8 + 2A$$

[12] $D = 4 + A$

We plug [12] into [10] to solve for A.

$$2A + 2(4 + A) = 7$$

$$2A + 8 + 2A = 7$$

$$4A = -1$$

[13] $A = -\frac{1}{4}$

At last! We've solved for one variable. Now it's pretty quick to find the other three. With [13], we can use [12] to find D.

$$D = 4 - \frac{1}{4}$$

[14] $D = \frac{15}{4}$

We plug [14] into [9] to find B.

$$B = \frac{15}{4} - 1$$

[15] $B = \frac{11}{4}$

Last but not least, we plug [13] into [5] to solve for C.

$$C = 9 - \left(-\frac{1}{4}\right)$$

$$C = 9 + \frac{1}{4}$$

[16] $C = \frac{37}{4}$

Taking the values of the constants from [13], [14], [15], [16] and bringing back the $2x - 3$ that we put aside following the long division earlier in this example, we'll write out the partial fractions decomposition.

$$\int \frac{2x^5 - 3x^4 + 5x^3 + 3x^2 - 9x + 13}{x^4 - 2x^2 + 1} \ dx$$

$$\int 2x - 3 + \frac{9x^3 - 3x^2 - 11x + 16}{x^4 - 2x^2 + 1} \ dx$$

$$\int 2x - 3 + \frac{-\frac{1}{4}}{x - 1} + \frac{\frac{11}{4}}{(x-1)^2} + \frac{\frac{37}{4}}{x+1} + \frac{\frac{15}{4}}{(x+1)^2} \ dx$$

Now we can integrate. Using the rule from algebra that $1/(x^n) = x^{-n}$, we'll flip the second and fourth fractions so that they are easier to integrate.

$$\int 2x-3\ dx-\frac{1}{4}\int\frac{1}{x-1}\ dx+\frac{11}{4}\int(x-1)^{-2}\ dx+\frac{37}{4}\int\frac{1}{x+1}\ dx+\frac{15}{4}\int(x+1)^{-2}\ dx$$

Now that we've simplified, we'll integrate to get our final answer.

$$x^2-3x-\frac{1}{4}\ln|x-1|-\frac{11}{4(x-1)}+\frac{37}{4}\ln|x+1|-\frac{15}{4(x+1)}+C$$

Distinct quadratic factors

Now let's take a look at an example in which the denominator is a product of distinct quadratic factors.

In order to solve these types of integrals, you'll sometimes need the following formula:

[A] $$\int\frac{m}{x^2+n^2}\ dx=\frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$$

Example

Evaluate the integral.

$$\int\frac{x^2-2x-5}{x^3-x^2+9x-9}\ dx$$

As always, the first thing to notice is that the degree of the denominator is larger than the degree of the numerator, which means that we don't have to perform long division before we can start factoring the denominator. So let's get right to it and factor the denominator.

$$\int \frac{x^2-2x-5}{(x-1)(x^2+9)}\ dx$$

We have one distinct linear factor, $(x-1)$, and one distinct quadratic factor, (x^2+9).

As we already know, linear factors require one constant in the numerator, like this:

$$\frac{A}{x-1}$$

The numerators of quadratic factors require a polynomial, like this:

$$\frac{Ax+B}{x^2+9}$$

Remember though that when we add these fractions together in the partial fractions decomposition, we never want to repeat the same constant, so the partial fractions decomposition is

$$\frac{x^2-2x-5}{(x-1)(x^2+9)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$$

See how we started the second fraction with B instead of A? If we added a second quadratic factor to this example, its numerator would be $Dx+E$.

Multiplying both sides of our decomposition by the denominator on the left gives

$$x^2 - 2x - 5 = A(x^2 + 9) + (Bx + C)(x - 1)$$

$$x^2 - 2x - 5 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

$$x^2 - 2x - 5 = (Ax^2 + Bx^2) + (-Bx + Cx) + (9A - C)$$

$$x^2 - 2x - 5 = (A + B)x^2 + (-B + C)x + (9A - C)$$

Then equating coefficients on the left and right sides gives us the following equations.

[1] $A + B = 1$

[2] $-B + C = -2$

[3] $9A - C = -5$

We solve [1] for A.

[4] $A = 1 - B$

Plugging [4] into [3] leaves us with two equations in terms of B and C.

[2] $-B + C = -2$

[5] $9(1 - B) - C = -5$

Simplifying [5] leaves us with

[2] $-B + C = -2$

[6] $-9B - C = -14$

Solving [2] for C we get

[7] $C = B - 2$

Plugging [7] into [6] gives

$$-9B - (B - 2) = -14$$

$$-10B + 2 = -14$$

$$-10B = -16$$

[8] $B = \frac{8}{5}$

Now that we have a value for B, we'll plug [8] into [7] to solve for C.

$$C = \frac{8}{5} - 2$$

[9] $C = -\frac{2}{5}$

We can also plug [8] into [4] to solve for A.

$$A = 1 - \frac{8}{5}$$

[10] $A = -\frac{3}{5}$

Plugging [8], [9] and [10] into our partial fractions decomposition, we get

$$\int \frac{x^2 - 2x - 5}{(x-1)(x^2+9)}\ dx = \int \frac{-\frac{3}{5}}{x-1} + \frac{\frac{8}{5}x - \frac{2}{5}}{x^2+9}\ dx$$

$$-\frac{3}{5}\int\frac{1}{x-1}\ dx+\frac{8}{5}\int\frac{x}{x^2+9}\ dx-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Integrating the first term only, we get

$$-\frac{3}{5}\ln|x-1|+\frac{8}{5}\int\frac{x}{x^2+9}\ dx-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Using u-substitution to integrate the second integral, letting

$$u=x^2+9$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

we get

$$-\frac{3}{5}\ln|x-1|+\frac{8}{5}\int\frac{x}{u}\cdot\frac{du}{2x}-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\int\frac{1}{u}\ du-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln|u|-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

$$-\frac{3}{5}\ln|x-1|+\frac{4}{5}\ln|x^2+9|-\frac{2}{5}\int\frac{1}{x^2+9}\ dx$$

Using [A] to integrate the third term, we get

[A] $$\int\frac{m}{x^2+n^2}\ dx=\frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right)+C$$

$$m = 1$$

$$n = 3$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln|x^2+9| - \frac{2}{5}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + C$$

$$-\frac{3}{5}\ln|x-1| + \frac{4}{5}\ln|x^2+9| - \frac{2}{15}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\frac{1}{5}\left[4\ln|x^2+9| - 3\ln|x-1| - \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + C$$

Repeated quadratic factors

Last but not least, let's take a look at an example in which the denominator is a product of quadratic factors, at least some of which are repeated.

We'll be using formula [A] like we did in the last example.

Example

Evaluate the integral.

$$\int \frac{-x^3+2x^2-x+1}{x(x^2+1)^2}\,dx$$

Remember, when we're dealing with repeated factors, we have to include every lesser degree of that factor in our partial fractions decomposition, which will be

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Multiplying both sides by the denominator of the left-hand side gives us

$$-x^3+2x^2-x+1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$$

Simplifying the right-hand side, we get

$$-x^3+2x^2-x+1=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+(Dx+E)x$$

$$-x^3+2x^2-x+1=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

Grouping like terms together, we have

$$-x^3+2x^2-x+1=(Ax^4+Bx^4)+(Cx^3)+(2Ax^2+Bx^2+Dx^2)+(Cx+Ex)+(A)$$

And factoring, we get

$$-x^3+2x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+(A)$$

Now we equate coefficients and write down the equations we'll use to solve for each of our constants.

[1] $A+B=0$

[2] $C=-1$

[3] $2A+B+D=2$

[4] $C + E = -1$

[5] $A = 1$

We already have values for A and C. Plugging [5] into [1] to solve for B gives us

$$1 + B = 0$$

[6] $B = -1$

Plugging [2] into [4] to solve for E, we get

$$-1 + E = -1$$

[7] $E = 0$

Plugging [5] and [6] into [3] to solve for D gives us

$$2(1) - 1 + D = 2$$

[8] $D = 1$

Plugging our constants from [2], [5], [6], [7] and [8] back into the decomposition, we get

$$\int \frac{(1)}{x} + \frac{(-1)x + (-1)}{x^2 + 1} + \frac{(1)x + (0)}{(x^2 + 1)^2}\ dx$$

$$\int \frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2}\ dx$$

$$\int \frac{1}{x}\ dx - \int \frac{x + 1}{x^2 + 1}\ dx + \int \frac{x}{(x^2 + 1)^2}\ dx$$

$$\int \frac{1}{x}\ dx - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Evaluating the first integral only, we get

$$\ln|x| - \int \frac{x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Using u-substitution to evaluate the second integral, letting

$$u = x^2 + 1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \int \frac{x}{u} \cdot \frac{du}{2x} - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\int \frac{1}{u}\ du - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|u| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

Using formula **[A]** to evaluate the third integral, we get

[A] $$\int \frac{m}{x^2+n^2}\ dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right) + C$$

$$m = 1$$

$$n = 1$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right) + \int \frac{x}{(x^2+1)^2}\ dx$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \int \frac{x}{(x^2+1)^2}\ dx$$

Using u-substitution to evaluate the fourth integral, letting

$$u = x^2 + 1$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

we get

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \int \frac{x}{u^2} \cdot \frac{du}{2x}$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \frac{1}{2}\int \frac{1}{u^2}\ du$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + \frac{1}{2}\int u^{-2}\ du$$

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2u} + C$$

And plugging back in for u gives us the final answer.

$$\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

In summary, in order to integrate by expressing rational functions (fractions) in terms of their partial fractions decomposition, you should follow these steps:

1. Ensure that the rational function is “proper”, such that the degree (greatest exponent) of the numerator is less than the degree of the denominator. If necessary, use long division to make it proper.

2. Perform the partial fractions decomposition by factoring the denominator, which will always be expressible as the product of either linear or quadratic factors, some of which may be repeated.

    a. If the denominator is a product of distinct linear factors: This is the simplest kind of partial fractions decomposition. Nothing fancy here.

    b. If the denominator is a product of linear factors, some of which are repeated: Remember to include factors of lesser degree than your repeated factors.

    c. If the denominator is a product of distinct quadratic factors: You’ll need the following equation:

[A] $$\int \frac{m}{x^2+n^2}\,dx = \frac{m}{n}\tan^{-1}\left(\frac{x}{n}\right) + C$$

d. If the denominator is a product of quadratic factors, some of which are repeated: Use the two formulas above and remember to include factors of lesser degree than your repeated factors.

Rationalizing substitution

Sometimes we need to use partial fractions to evaluate an integral, but the integral isn't in a form that's ready for partial fractions decomposition. Remember that, in order to use partial fractions, the function has to be a proper rational function, which means that it's the quotient of two polynomials where the degree of the denominator is greater than the degree of the numerator.

If the function is rational (the quotient of two polynomials), but not proper (the degree of the denominator is not greater than the degree of the numerator), then you just need to perform polynomial long division to make it proper.

If, on the other hand, the function isn't rational, you may be able to use what's called a "rationalizing substitution" to make it rational. From there, you can check to make sure it's proper, and then once it's proper, use a partial fractions decomposition to evaluate the integral.

In other words, follow this process:

1. Make sure the function is a **rational** function. If it isn't, try a rationalizing substitution.

2. If the function is rational, make sure it's **proper**. If it isn't, perform polynomial long division to make it proper.

3. If the function is rational and proper, use **partial fractions** to evaluate it.

Example

Evaluate the integral.

$$\int \frac{\sqrt{x+4}}{2x}\ dx$$

We can't evaluate the integral as-is. We'll try making a substitution that will rationalize the function, letting

$$u = \sqrt{x+4}$$

$$du = \frac{1}{2\sqrt{x+4}}\ dx$$

$$dx = 2\sqrt{x+4}\ du$$

We'll plug these values into our integral and get

$$\int \frac{u}{2x} 2\sqrt{x+4}\ du$$

Substituting again, remembering that $u = \sqrt{x+4}$, the integral simplifies to

$$\int \frac{u}{2x} 2u\ du$$

$$\int \frac{u^2}{x}\ du$$

We need to replace the x in the denominator with a function in terms of u. We'll rearrange $u = \sqrt{x+4}$, solving it for x to get $x = u^2 - 4$, and then we'll plug it in for x.

$$\int \frac{u^2}{u^2 - 4}\ du$$

Now that we have a rational function, we can use partial fractions to evaluate the integral, we just need to make sure it's proper a proper rational function before we do.

We'll compare the degree of the numerator to the degree of the denominator. The degrees of the numerator and denominator are both 2. Since the degrees are equal, we'll need to use polynomial long division to make the function proper.

When we divide $u^2 - 4$ into u^2, we get an answer of 1 and a remainder of 4, so the integral simplifies to

$$\int 1 + \frac{4}{u^2 - 4}\ du$$

$$\int 1\ du + \int \frac{4}{u^2 - 4}\ du$$

To evaluate the now proper rational function in the second integral, we'll use partial fractions. We'll start by factoring the denominator.

$$\int 1\ du + \int \frac{4}{(u-2)(u+2)}\ du$$

The partial fractions decomposition gives us

$$\int 1\ du + \int \frac{1}{u-2}\ du + \int \frac{-1}{u+2}\ du$$

$$\int 1\ du + \int \frac{1}{u-2}\ du - \int \frac{1}{u+2}\ du$$

Integrating, we get

$$u + \ln|u-2| - \ln|u+2| + C$$

Plugging back in for u, remembering that $u = \sqrt{x+4}$, the answer becomes

$$\sqrt{x+4} + \ln\left|\sqrt{x+4}-2\right| - \ln\left|\sqrt{x+4}+2\right| + C$$

$$\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

Trigonometric integrals

Trigonometric functions follow standard rules for integration. The integrals of the six standard trigonometric functions are

$$\int \sin(ax)\ dx = \frac{-\cos(ax)}{a} + C$$

$$\int \cos(ax)\ dx = \frac{\sin(ax)}{a} + C$$

$$\int \tan(ax)\ dx = \frac{-\ln\left|\cos(ax)\right|}{a} + C$$

$$\int \cot(ax)\ dx = \frac{\ln\left|\sin(ax)\right|}{a} + C$$

$$\int \sec(ax)\ dx = \frac{\ln\left|\sec(ax) + \tan(ax)\right|}{a} + C$$

$$\int \csc(ax)\ dx = \frac{-\ln\left|\csc(ax) + \cot(ax)\right|}{a} + C$$

We also have a few other standard trigonometric integrals that are based on the standard trigonometric derivatives.

$$\int \sec^2(ax)\ dx = \frac{\tan(ax)}{a} + C$$

$$\int \csc^2(ax)\ dx = \frac{-\cot(ax)}{a} + C$$

$$\int \sec(ax)\tan(ax)\ dx = \frac{\sec(ax)}{a} + C$$

$$\int \csc(ax)\cot(ax)\ dx = \frac{-\csc(ax)}{a} + C$$

Example

Evaluate the integral.

$$\int 6\tan(2x)\ dx$$

We'll simplify the integral by pulling the constant out front.

$$6\int \tan(2x)\ dx$$

Using the formula for the integral of tangent,

$$\int \tan(ax)\ dx = \frac{-\ln\left|\cos(ax)\right|}{a} + C$$

we get

$$6\int \tan(2x)\ dx = 6 \cdot \frac{-\ln\left|\cos(2x)\right|}{2} + C$$

$$6\int \tan(2x)\ dx = -3\ln\left|\cos(2x)\right| + C$$

Let's try a more complex example.

Example

Evaluate the integral.

$$\int e^{4x} - \sec(7x)\tan(7x) + 5\sin x \ dx$$

First we'll separate the terms of the function into different integrals.

$$\int e^{4x} \ dx + \int -\sec(7x)\tan(7x) \ dx + \int 5\sin x \ dx$$

$$\int e^{4x} \ dx - \int \sec(7x)\tan(7x) \ dx + 5\int \sin x \ dx$$

Now we're ready to integrate using the formulas we defined earlier,

$$\int \sin(ax) \ dx = \frac{-\cos(ax)}{a} + C$$

$$\int \sec(ax)\tan(ax) \ dx = \frac{\sec(ax)}{a} + C$$

We get

$$\frac{e^{4x}}{4} - \frac{\sec(7x)}{7} + 5\left(\frac{-\cos x}{1}\right) + C$$

$$\frac{e^{4x}}{4} - \frac{\sec(7x)}{7} - 5\cos x + C$$

Hyperbolic integrals

Hyperbolic functions follow standard rules for integration. The general rules for the six hyperbolic functions are

$$\int \sinh(ax)\ dx = \frac{\cosh(ax)}{a} + C$$

$$\int \cosh(ax)\ dx = \frac{\sinh(ax)}{a} + C$$

$$\int \tanh(ax)\ dx = \frac{\ln\left|\cosh(ax)\right|}{a} + C$$

$$\int \coth(ax)\ dx = \frac{\ln\left|\sinh(ax)\right|}{a} + C$$

$$\int \text{sech}(ax)\ dx = \frac{\arctan[\sinh(ax)]}{a} + C$$

$$\int \text{csch}(ax)\ dx = \frac{\ln\left|\tanh(\frac{ax}{2})\right|}{a} + C$$

We also have a few other standard hyperbolic integrals that are based on the standard hyperbolic derivatives.

$$\int \text{sech}^2(ax)\ dx = \frac{\tanh(ax)}{a} + C$$

$$\int \text{csch}^2(ax)\ dx = \frac{-\coth(ax)}{a} + C$$

$$\int \text{sech}(ax)\tanh(ax)\ dx = \frac{-\text{sech}(ax)}{a} + C$$

$$\int \text{csch}(ax)\coth(ax)\ dx = \frac{-\text{csch}(ax)}{a} + C$$

Example

Evaluate the integral.

$$\int 5\text{sech}(3x)\ dx$$

First, we simplify the integral by factoring out the 5.

$$5\int \text{sech}(3x)\ dx$$

Remembering that $\int \text{sech}(ax)\ dx = \frac{\arctan[\sinh(ax)]}{a} + C$, we integrate and get

$$\int 5\text{sech}(3x)\ dx = \frac{5\arctan[\sinh(3x)]}{3} + C$$

Now let's try a more complex example.

Example

Evaluate the integral.

$$\int 16x^3 - 2\sinh(3x) + 4\text{csch}^2(6x)\ dx$$

First, we will break the integral into parts to simplify it.

$$\int 16x^3\ dx + \int -2\sinh(3x)\ dx + \int 4\text{csch}^2(6x)\ dx$$

$$16\int x^3\ dx - 2\int \sinh(3x)\ dx + 4\int \text{csch}^2(6x)\ dx$$

We'll use our hyperbolic integration formulas to integrate, and we'll get

$$16\left(\frac{1}{4}x^4\right) - 2\left[\frac{\cosh(3x)}{3}\right] + 4\left[\frac{-\coth(6x)}{6}\right] + C$$

$$4x^4 - \frac{2\cosh(3x)}{3} - \frac{2\coth(6x)}{3} + C$$

Inverse hyperbolic integrals

Inverse hyperbolic functions follow standard rules for integration. Remember, an inverse hyperbolic function can be written two ways. For example, inverse hyperbolic sine can be written as

arcsinh or as

$\sinh^{-1}$

Some people argue that the arcsinh form should be used instead of $\sinh^{-1}$ because $\sinh^{-1}$ can be misinterpreted as $1/\sinh$. Whichever form you prefer, you see both, so you should be able to recognize both and understand that they mean the same thing.

The general rules for the six inverse hyperbolic functions are

$$\int \text{arcsinh}(ax)\ dx = x\text{arcsinh}(ax) - \frac{\sqrt{a^2x^2+1}}{a} + C$$

$$\int \text{arccosh}(ax)\ dx = x\text{arccosh}(ax) - \frac{\sqrt{ax+1}\sqrt{ax-1}}{a} + C$$

$$\int \text{arctanh}(ax)\ dx = x\text{arctanh}(ax) + \frac{\ln(1-a^2x^2)}{2a} + C$$

$$\int \text{arccoth}(ax)\ dx = x\text{arccoth}(ax) + \frac{\ln(a^2x^2-1)}{2a} + C$$

$$\int \text{arcsech}(ax)\ dx = x\text{arcsech}(ax) - \frac{2}{a}\arctan\sqrt{\frac{1-ax}{1+ax}} + C$$

$$\int \text{arccsch}(ax)\ dx = x\text{arccsch}(ax) + \frac{1}{a}\text{arccoth}\sqrt{\frac{1}{a^2x^2}+1} + C$$

We also have a few other standard inverse hyperbolic integrals that are based on the standard inverse hyperbolic derivatives. In the following formulas, u represents a function.

$$\int \frac{1}{\sqrt{a^2+u^2}}\ du = \text{arcsinh}\left(\frac{u}{a}\right) + C \qquad \text{where } a > 0$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\ du = \text{arccosh}\left(\frac{u}{a}\right) + C \qquad \text{where } u > a > 0$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\text{arctanh}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 < a^2$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\text{arccoth}\left(\frac{u}{a}\right) + C \qquad \text{if } u^2 > a^2$$

$$\int \frac{1}{u\sqrt{a^2-u^2}}\ du = -\frac{1}{a}\text{arcsech}\left(\frac{u}{a}\right) + C \qquad \text{where } 0 < u < a$$

$$\int \frac{1}{u\sqrt{a^2+u^2}}\ du = -\frac{1}{a}\text{arccsch}\left(\frac{u}{a}\right) + C \qquad \text{where } u \neq 0$$

Example

Evaluate the integral.

$$\int -7\text{arcsech}(5x)\ dx$$

We'll simplify by factoring -7 out of the integral.

$$-7\int \text{arcsech}(5x)\ dx$$

We'll use

$$\int \text{arcsech}(ax)\ dx = x\text{arcsech}(ax) - \frac{2}{a}\arctan\sqrt{\frac{1-ax}{1+ax}} + C$$

to integrate, and get

$$-7x\text{arcsech}(5x) + \frac{14}{5}\arctan\sqrt{\frac{1-5x}{1+5x}} + C$$

Example

Evaluate the integral.

$$\int \frac{1}{\sqrt{9+x^2}} - \text{arccsch}(4x) + 3x^2\ dx$$

First, break the integral into parts.

$$\int \frac{1}{\sqrt{9+x^2}}\ dx + \int -\text{arccsch}(4x)\ dx + \int 3x^2\ dx$$

$$\int \frac{1}{\sqrt{9+x^2}}\ dx - \int \text{arccsch}(4x)\ dx + 3\int x^2\ dx$$

Now we'll integrate using the formulas from this section, and we'll get

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1}{4^2x^2}+1} + \frac{3}{3}x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1}{16x^2}+\frac{16x^2}{16x^2}} + x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\sqrt{\frac{1+16x^2}{16x^2}} + x^3 + C$$

$$\text{arcsinh}\left(\frac{x}{3}\right) - x\text{arccsch}(4x) - \frac{1}{4}\text{arccoth}\left(\frac{\sqrt{16x^2+1}}{4x}\right) + x^3 + C$$

Trigonometric substitution

Trigonometric substitution is another tool we can use to help solve integrals that are too complex for simpler strategies.

You should check to see whether u-substitution, integration by parts, or partial fractions can be used to evaluate the integral before you try trigonometric substitution, because they're often easier and faster. If you want to use trigonometric substitution, the integral must contain one of the values below.

If the integral contains $\sqrt{b^2x^2 - a^2}$ use the substitution $x = \frac{a}{b}\sec\theta$

If the integral contains $\sqrt{a^2 - b^2x^2}$ use the substitution $x = \frac{a}{b}\sin\theta$

If the integral contains $\sqrt{a^2 + b^2x^2}$ use the substitution $x = \frac{a}{b}\tan\theta$

Some integrals can be solved with trigonometric substitution but don't necessarily contain one of the values above. When this is the case, we may have to complete the square or use a simple u-substitution before proceeding with the trigonometric substitution.

Example

Evaluate the integral.

$$\int \frac{x}{\sqrt{2x^2 - 4x - 7}}\ dx$$

Since our function doesn't already contain the format we need for a trigonometric substitution, but the value inside the square root is a quadratic function, we'll complete the square to see if we can get the quadratic function into the right format.

To save some space, let's just work with the value underneath the radical, then we'll plug it back into the integral.

$$2x^2 - 4x - 7$$

$$2\left(x^2 - 2x - \frac{7}{2}\right)$$

$$2\left(x^2 - 2x + 1 - 1 - \frac{7}{2}\right)$$

$$2\left[\left(x^2 - 2x + 1\right) - 1 - \frac{7}{2}\right]$$

$$2\left[(x-1)^2 - \frac{9}{2}\right]$$

$$2(x-1)^2 - 9$$

Plugging this back into the integral, we get

$$\int \frac{x}{\sqrt{2(x-1)^2 - 9}}\ dx$$

Now we have the format we need for the secant substitution,

$$x = \frac{a}{b}\sec\theta$$

If we say that

$$x = x - 1$$

$$a = 3$$

$$b = \sqrt{2}$$

then the substitutions are

$$x - 1 = \frac{3}{\sqrt{2}}\sec\theta$$

$$x = 1 + \frac{3}{\sqrt{2}}\sec\theta$$

$$dx = \frac{3}{\sqrt{2}}\sec\theta\tan\theta\ d\theta$$

$$\sec\theta = \frac{\sqrt{2}(x-1)}{3}$$

Plugging into the integral for x and dx, we get

$$\int \frac{x}{\sqrt{2(x-1)^2 - 9}}\ dx$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(1+\frac{3}{\sqrt{2}}\sec\theta-1\right)^2-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\ d\theta\right)$$

We'll focus on simplifying the denominator.

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(\frac{3}{\sqrt{2}}\sec\theta\right)^2-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{2\left(\frac{9}{2}\sec^2\theta\right)-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{9\sec^2\theta-9}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\sqrt{9\left(\sec^2\theta-1\right)}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

$$\int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{3\sqrt{\sec^2\theta-1}}\left(\frac{3}{\sqrt{2}}\sec\theta\tan\theta\right)\ d\theta$$

Remembering the trigonometric identity $\tan^2\theta = \sec^2\theta - 1$, we get

$$\int \frac{1 + \frac{3}{\sqrt{2}} \sec \theta}{3\sqrt{\tan^2 \theta}} \left(\frac{3}{\sqrt{2}} \sec \theta \tan \theta \right) \ d\theta$$

$$\int \frac{1 + \frac{3}{\sqrt{2}} \sec \theta}{3 \tan \theta} \left(\frac{3}{\sqrt{2}} \sec \theta \tan \theta \right) \ d\theta$$

Now we can cancel $3 \tan \theta$.

$$\int \left(1 + \frac{3}{\sqrt{2}} \sec \theta \right) \left(\frac{1}{\sqrt{2}} \sec \theta \right) \ d\theta$$

We'll find a common denominator so that we can make the whole function one fraction.

$$\int \left(\frac{\sqrt{2}}{\sqrt{2}} + \frac{3 \sec \theta}{\sqrt{2}} \right) \left(\frac{\sec \theta}{\sqrt{2}} \right) \ d\theta$$

$$\int \left(\frac{\sqrt{2} + 3 \sec \theta}{\sqrt{2}} \right) \left(\frac{\sec \theta}{\sqrt{2}} \right) \ d\theta$$

Multiplying these functions together, we get

$$\int \frac{\sqrt{2} \sec \theta + 3 \sec^2 \theta}{2} \ d\theta$$

$$\frac{1}{2} \int \sqrt{2} \sec \theta + 3 \sec^2 \theta \ d\theta$$

We'll break this into two integrals.

$$\frac{1}{2}\int \sqrt{2}\sec\theta\ d\theta + \frac{1}{2}\int 3\sec^2\theta\ d\theta$$

$$\frac{\sqrt{2}}{2}\int \sec\theta\ d\theta + \frac{3}{2}\int \sec^2\theta\ d\theta$$

Now we can integrate.

$$\frac{\sqrt{2}}{2}\ln\left|\sec\theta + \tan\theta\right| + \frac{3}{2}\tan\theta + C$$

To finish the problem, we need to put the answer back in terms of x instead of θ. Remember that we solved for

$$\sec\theta = \frac{\sqrt{2}(x-1)}{3}$$

when we were setting up the substitution. Plugging that into what we have so far, we get

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3} + \tan\theta\right| + \frac{3}{2}\tan\theta + C$$

Now we'll take a break from this function in order to find $\tan\theta$. Once we've found it, we'll come back to this function to plug it in. In order to find $\tan\theta$, we'll remember that

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\cos\theta = \frac{1}{\frac{\sqrt{2}(x-1)}{3}}$$

$$\cos\theta = \frac{3}{\sqrt{2}(x-1)}$$

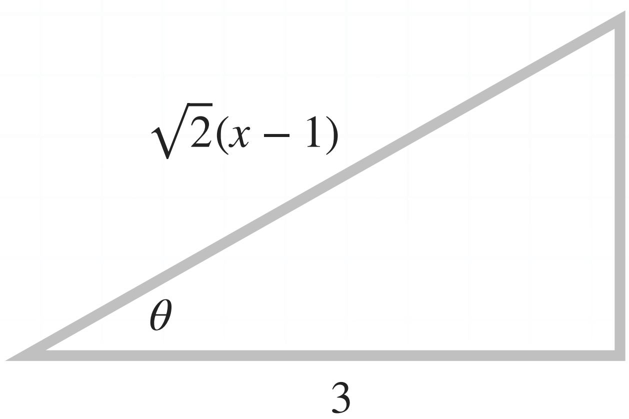
We know that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

so

$$\text{adjacent} = 3$$

$$\text{hypotenuse} = \sqrt{2}(x-1)$$



Using the pythagorean theorem,

$$a^2 + b^2 = c^2$$

we'll plug in the adjacent side and the hypotenuse to solve for the opposite side.

$$(3)^2 + b^2 = \left[\sqrt{2}(x-1)\right]^2$$

$$9 + b^2 = 2(x-1)^2$$

$$9 + b^2 = 2(x-1)(x-1)$$

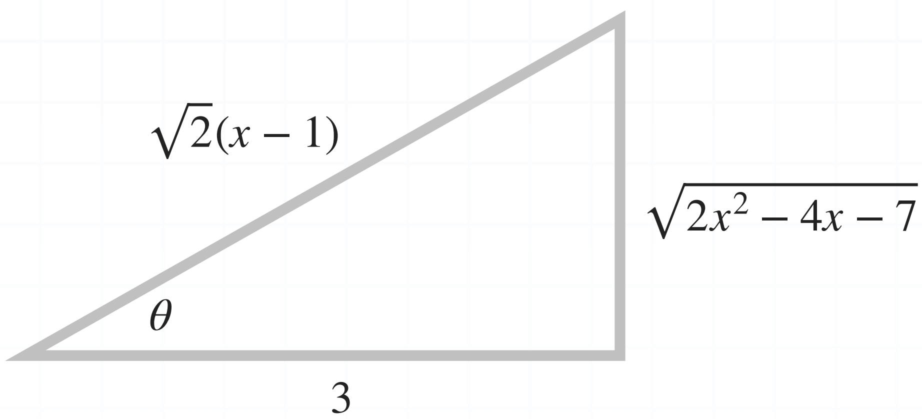
$$9 + b^2 = 2\left(x^2 - 2x + 1\right)$$

$$9 + b^2 = 2x^2 - 4x + 2$$

$$b^2 = 2x^2 - 4x - 7$$

$$b = \sqrt{2x^2 - 4x - 7}$$

Now our triangle is



Knowing that

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

we can say that

$$\tan\theta = \frac{\sqrt{2x^2 - 4x - 7}}{3}$$

Now we can plug this value for $\tan\theta$ into the function we've been working with.

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3} + \tan\theta\right| + \frac{3}{2}\tan\theta + C$$

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)}{3}+\frac{\sqrt{2x^2-4x-7}}{3}\right|+\frac{3}{2}\left(\frac{\sqrt{2x^2-4x-7}}{3}\right)+C$$

$$\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}(x-1)+\sqrt{2x^2-4x-7}}{3}\right|+\frac{\sqrt{2x^2-4x-7}}{2}+C$$

$$\frac{1}{2}\left[\sqrt{2}\ln\left|\frac{\sqrt{2}(x-1)+\sqrt{2x^2-4x-7}}{3}\right|+\sqrt{2x^2-4x-7}\right]+C$$

Quadratic functions

Quadratic functions are functions in the form

$$ax^2 + bx + c = 0$$

Integrating functions that include a quadratic can sometimes be a little difficult. Most often, we'll see an integral problem in the form

$$\int \frac{Ax + B}{ax^2 + bx + c} \ dx$$

There are three methods we'll use to evaluate quadratic integrals:

- Substitution
- Partial fractions
- Trigonometric substitution

You should try using these techniques in the order listed above, because substitution is the easiest and fastest, and trigonometric substitution is the longest and most difficult.

Substitution

Let's look at how to solve a quadratic integral using substitution.

Example

Evaluate the integral.

$$\int \frac{6x}{3x^2 - 1}\ dx$$

We'll try substitution, letting

$$u = 3x^2 - 1$$

$$du = 6x\ dx$$

Plugging these back into the integral, we get

$$\int \frac{1}{u}\ du$$

$$\ln|u| + C$$

$$\ln\left|3x^2 - 1\right| + C$$

Sometimes substitution doesn't work, and we need to use partial fractions instead to evaluate the integral. In order to use partial fractions, we must be able to factor the quadratic.

Partial fractions

Example

Evaluate the integral.

$$\int \frac{9x+8}{3x^2+10x-8}\ dx$$

We notice that the quadratic function in the denominator of the fraction can be factored.

$$\int \frac{9x+8}{(x+4)(3x-2)}\ dx$$

Here we'll use a partial fractions decomposition to split the integral in two. We can always double-check this step by finding a common denominator to bring our separated fractions back together again.

$$\int \frac{2}{x+4}\ dx + \int \frac{3}{3x-2}\ dx$$

$$2\int \frac{1}{x+4}\ dx + 3\int \frac{1}{3x-2}\ dx$$

Now we integrate to get the final answer.

$$2\ln|x+4| + \ln|3x-2| + C$$

If substitution and partial fractions don't work, you might need to use trigonometric substitution.

Trigonometric substitution

Remember that the general formulas for trigonometric substitution are

$$\int \sqrt{b^2x^2 - a^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\sec\theta$$

$$\int \sqrt{a^2 - b^2x^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\sin\theta$$

$$\int \sqrt{a^2 + b^2x^2}\ dx \quad \text{uses the substitution} \quad x = \frac{a}{b}\tan\theta$$

We won't always find our function already in this format, and sometimes we might have to alter it by completing the square before we can use trigonometric substitution.

Example

Evaluate the integral.

$$\int \sqrt{x^2 + 4x + 5}\ dx$$

First, we notice our equation is not already in the form for trigonometric substitution, so let's try completing the square to see if we can get it into the right format.

To complete the square we take

$$x^2 + 4x + 5 = \left(x^2 + 4x + 4\right) + 5 - 4$$

$$x^2 + 4x + 4 + 5 - 4 = (x+2)^2 + 1$$

We can put this back into the integral to get

$$\int \sqrt{(x+2)^2 + 1}\ dx$$

Now we can see that the trigonometric substitution is

$$x + 2 = \tan\theta$$

$$x = \tan\theta - 2$$

$$dx = \sec^2\theta\ d\theta$$

Making the substitution, we get

$$\int \sqrt{(\tan\theta)^2 + 1} \cdot \sec^2\theta\ d\theta$$

$$\int \sqrt{\sec^2\theta} \cdot \sec^2\theta\ d\theta$$

$$\int \sec\theta \cdot \sec^2\theta\ d\theta$$

$$\int \sec^3\theta\ d\theta$$

Integrating, we get

$$\frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) + C$$

To finish this problem, we need to get the answer back in terms of x instead of θ, so we'll back-substitute.

$$\frac{1}{2}\left[\left(\sqrt{x^2+4x+5}\right)(x+2)+\ln\left|\left(\sqrt{x^2+4x+5}\right)+(x+2)\right|\right]+C$$

$$\frac{1}{2}\left[(x+2)\sqrt{x^2+4x+5}+\ln\left|x+2+\sqrt{x^2+4x+5}\right|\right]+C$$

Improper integrals

Improper integrals are just like definite integrals, except that the lower and/or upper limit of integration is infinite.

Remember that a definite integral is an integral that we evaluate over a certain interval. An improper integral is just a definite integral where one end of the interval is $\pm\infty$.

The formulas we use to deal with improper integrals are

1. $\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$ over the interval $[a,\infty)$

2. $\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$ over the interval $(-\infty, b]$

3. $\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx$ over the interval $(-\infty,\infty)$

If the limit in case 1 or 2 exists (if it generates a real number answer), we say that it converges. If the limit in case 1 or 2 does not exist, we say that it diverges. This means the answer isn't a real number (the answer might be $\pm\infty$).

In case 3, the equation will diverge if either of the integrals on the right-hand side diverge. This means that in order for case 3 to converge, both integrals on the right-hand side must have real-number answers.

Example

Evaluate the improper integral.

$$\int_0^{\infty} \frac{1}{x^2+1}\ dx$$

This integral is like case 1, so we use the rule to get

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

$$\int_0^{\infty} \frac{1}{x^2+1}\ dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1} dx$$

Integrating, we get

$$\lim_{b\to\infty} \arctan x \Big|_0^b$$

Evaluating over the interval $[0,b]$, we get

$$\lim_{b\to\infty} \left[\arctan(b) - \arctan(0)\right]$$

$$\arctan(\infty)$$

$$\frac{\pi}{2}$$

In this case, since we get a real number answer, we know that our integral converges.

When infinity occurs as part of the interval, or when one end of the interval approaches infinity, we say that the equation has an infinite discontinuity. In other words, that the equation has an asymptote in the interval or at one end of the interval.

For improper integrals that are infinitely discontinuous somewhere in the interval, we use the following formulas.

4. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then the function has a vertical asymptote at $x = b$ and

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

5. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then the function has a vertical asymptote at $x = a$ and

$$\int_a^b f(x)\ dx = \lim_{c \to a^+} \int_c^b f(x)\ dx$$

6. If f is continuous on the interval $[a, b]$ except for some c in $[a, b]$ where f has an infinite discontinuity, then the function has a vertical asymptote at $x = c$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

If the limit in case 4 or 5 exists (if it generates a real-number answer), we say that it converges. If the limit in case 4 or 5 does not exist, we say that it diverges. This means the answer isn't a real number (the answer might be $\pm\infty$).

In case 6, the equation will diverge if either integral on the right-hand side diverges. This means that in order for case 6 to converge, both integrals on the right-hand side must have real-number answers.

Example

Evaluate the improper integral.

$$\int_0^2 \frac{1}{x^3}\ dx$$

This integral is like case 5, so we use the rule to get

$$\int_a^b f(x)\ dx = \lim_{c\to a^+} \int_c^b f(x)\ dx$$

$$\int_0^2 \frac{1}{x^3}\ dx = \lim_{c\to 0^+} \int_c^2 \frac{1}{x^3}\ dx$$

Integrating, we get

$$\lim_{c\to 0^+} \left(-\frac{1}{2x^2}\right)\Bigg|_c^2$$

Evaluating over the interval $[c,2]$, we get

$$\lim_{c \to 0^+} \left(-\frac{1}{2(2)^2} + \frac{1}{2c^2} \right)$$

$$\lim_{c \to 0^+} \left(-\frac{1}{8} + \frac{1}{2c^2} \right)$$

$$\infty$$

In this case, since we get an infinite answer, we know that our integral diverges.

Comparison theorem

The comparison theorem for improper integrals is very similar to the comparison test for convergence that you'll study as part of Sequences & Series. It allows you to draw a conclusion about the convergence or divergence of an improper integral, without actually evaluating the integral itself.

It states:

If $f(x) \geq g(x) \geq 0$ on $[a, \infty)$, then

If $\int_a^\infty f(x)\ dx$ converges then so does $\int_a^\infty g(x)\ dx$

If $\int_a^\infty g(x)\ dx$ diverges then so does $\int_a^\infty f(x)\ dx$

Notice that it does not state:

If $f(x) \geq g(x) \geq 0$ on $[a, \infty)$, then

If $\int_a^\infty g(x)\ dx$ converges then so does $\int_a^\infty f(x)\ dx$

If $\int_a^\infty f(x)\ dx$ diverges then so does $\int_a^\infty g(x)\ dx$

The comparison theorem will allow you to draw the first two conclusions, but not the others. The reason is that we're assuming $f(x) \geq g(x)$.

Thinking about $f(x)$:

If $f(x)$ is greater than (above) $g(x)$, then if $f(x)$ converges, we know it will force $g(x)$ to also converge. But if $f(x)$ diverges, then we can't draw any conclusion about $g(x)$ because $g(x)$ could diverge or converge below it.

Thinking about $g(x)$:

If $g(x)$ is less than (below) $f(x)$, then if $g(x)$ diverges, we know it will force $f(x)$ to also diverge. But if $g(x)$ converges, then we can't draw any conclusion about $f(x)$ because $f(x)$ could diverge or converge above it.

Given an improper integral and asked to use comparison theorem to say whether it converges or diverges, our goal will be to find a comparison function that we know will either always be greater than the given function, or always be less than the given function.

Since $f(x) \geq g(x)$ in the comparison theorem:

If we find a comparison function that is always greater than the given function, then the given function will be $g(x)$ and the comparison function will be $f(x)$.

In this case, in order to use the comparison theorem to draw a conclusion, we'd have to show that the comparison function $f(x)$ converges. If we can show that the comparison function $f(x)$ converges, then we've proven that the given function $g(x)$ also converges.

If we find a comparison function that is always less than the given function, then the given function will be $f(x)$ and the comparison function will be $g(x)$.

In this case, in order to use the comparison theorem to draw a conclusion, we'd have to show that the comparison function $g(x)$ diverges. If we can show that the comparison function $g(x)$ diverges, then we've proven that the given function $f(x)$ also diverges.

Example

Use the comparison theorem to say whether the integral converges or diverges.

$$\int_1^{\infty} \frac{x-1}{x^4+2x^2}\ dx$$

Often it's really helpful to try to make a guess about whether the given function is converging or diverging so that we know whether to look for a comparison function that is greater than or less than the given function.

If we guess that the given function is converging, we'll look for a comparison function that is greater than the given function, so that we can show that the comparison function is converging, and therefore prove that the given function is converging.

If we guess that the given function is diverging, we'll look for a comparison function that is less than the given function, so that we can show that the comparison function is diverging, and therefore prove that the given function is diverging.

Since the interval is $[1,\infty)$, let's plug in the first few values, $x = 1$, $x = 2$, $x = 3$, etc.

$$x = 1 \qquad \frac{1 - 1}{1^4 + 2(1)^2} \qquad 0$$

$$x = 2 \qquad \frac{2 - 1}{2^4 + 2(2)^2} \qquad \frac{1}{24}$$

$$x = 3 \qquad \frac{3 - 1}{3^4 + 2(3)^2} \qquad \frac{2}{99}$$

$$x = 4 \qquad \frac{4 - 1}{4^4 + 2(4)^2} \qquad \frac{1}{96}$$

These values seem to be approaching 0, so we'll guess that the given function is converging. Which means we want to look for a comparison function that is greater than the given function, and hope that the comparison function will converge, proving that the given function also converges.

The only way to make a fraction bigger is to make the numerator bigger and/or to make the denominator smaller. If we just take away the -1 from the numerator of the given function, that would immediately make the numerator larger, because we wouldn't be subtracting 1 from it. So we know right away that the comparison function

$$\int_1^{\infty} \frac{x}{x^4 + 2x^2}\ dx$$

will be larger than the given function. We can factor out an x and simplify this comparison function to

$$\int_1^\infty \frac{1}{x^3+2x}\,dx$$

We know this comparison function will get even larger if we can make the denominator smaller. Taking away the $2x$ would make the denominator smaller, so our comparison function will simplify to

$$\int_1^\infty \frac{1}{x^3}\,dx$$

We know that

$$\frac{1}{x^3} > \frac{x-1}{x^4+2x^2}$$

and that the comparison function is therefore $f(x)$, that the given function is $g(x)$, and that we're looking to show that the comparison function $f(x)$ converges, which will prove that the given function $g(x)$ also converges.

For functions in the form

$$\frac{1}{x^p}$$

If $p > 1$ then the function converges

If $p < 1$ then the function diverges

Therefore, since $3 > 1$ we can say that the comparison function converges, which proves that the given function also converges.

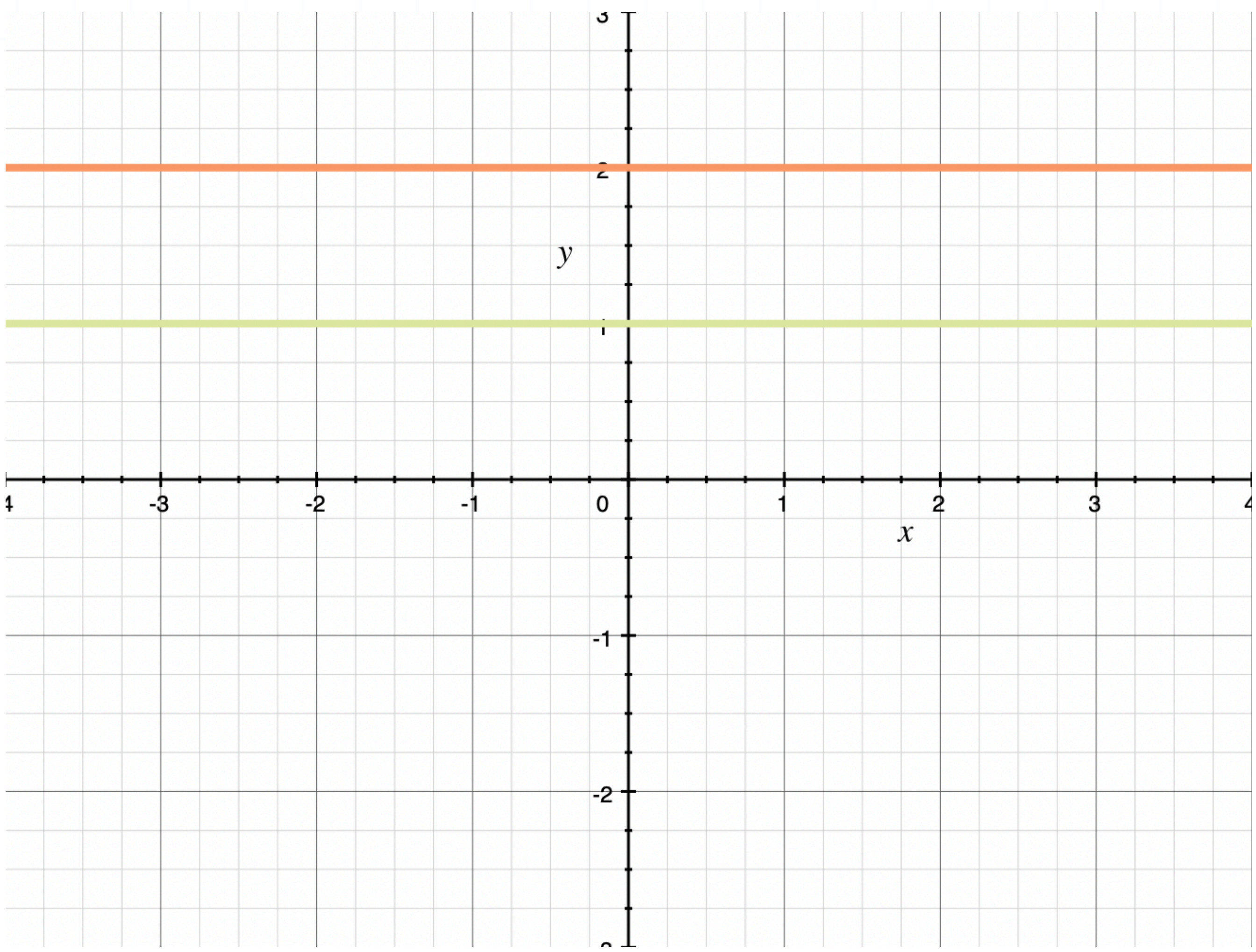
Area between curves

Based on the name of this application, finding area between curves is exactly what you'd expect it to be.

In these kinds of problems, you'll be given the equations of two curves, and asked to find the area between them. Finding the area is always going to require three pieces of information:

1. The orientation of your curves.

    a. Thinking broadly here, do they look more like this, where one curve is higher, and the other is lower?

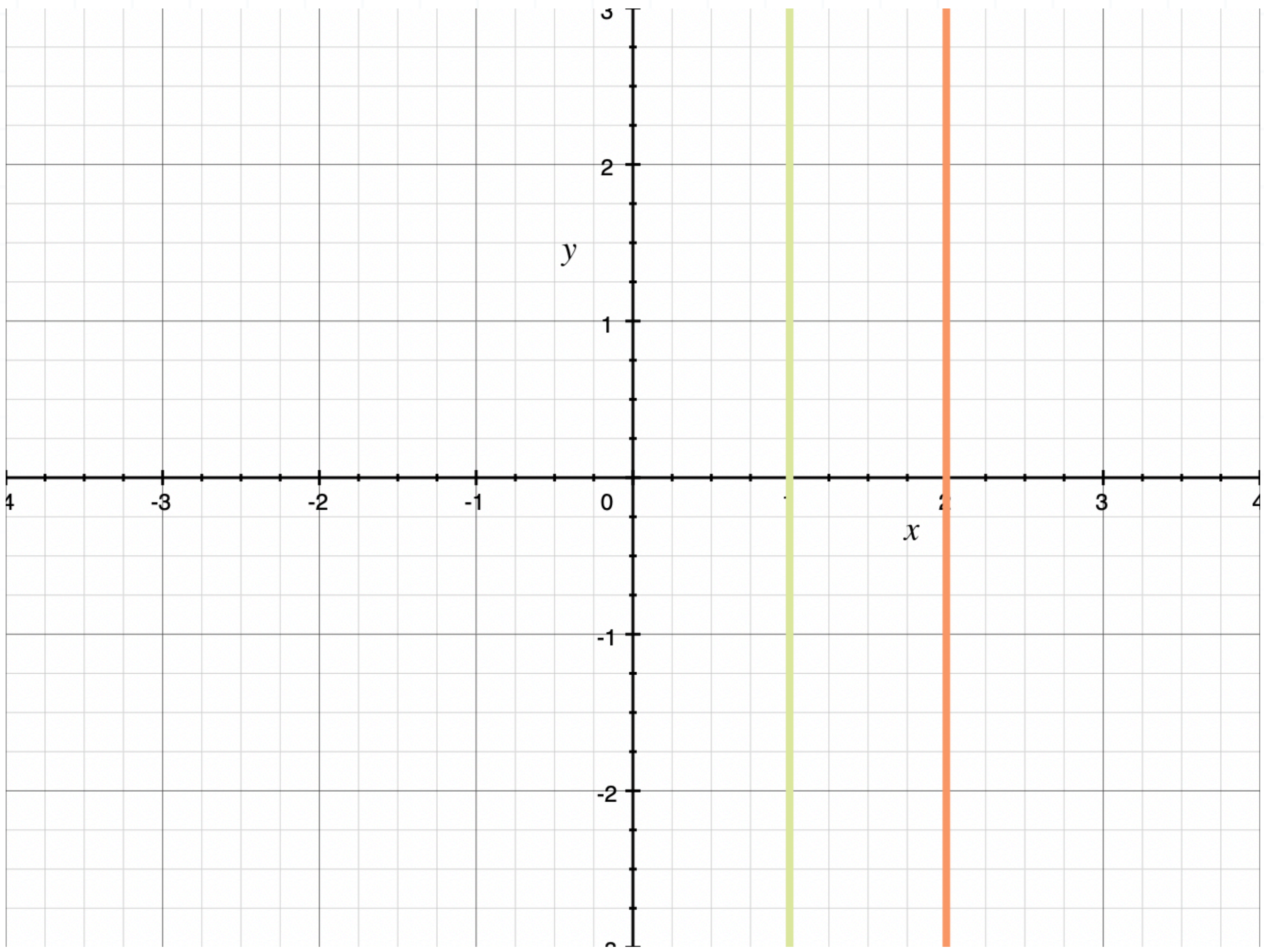


If so, we'll use the the area formula,

$$[A] \quad A = \int_{x=a}^{x=b} f(x) - g(x)\ dx$$

where $[a, b]$ is the interval on which we'll find area, $f(x)$ is the "higher" function, and $g(x)$ is the "lower" function.

b. Or do they look more like this, where one curve is on the left and the other is on the right?



If so, we'll use the area formula,

$$[B] \quad A = \int_{y=a}^{y=b} f(y) - g(y)\ dy$$

where $[a, b]$ is the interval on which we'll find area, $f(y)$ is the "right" function, and $g(y)$ is the "left" function.

2. The points where the curves intersect each other, or a given interval on which to evaluate area.

a. If your problem just says "Find the area between the curves", but doesn't specify an interval, then you need to find the points where the curves intersect each other. Those points of intersection become your interval, $[a, b]$.

b. If your problem says "Find the area between the curves on the interval $[a, b]$", then you need to check to see whether the curves intersect each other inside the given interval (more on this later). Points of intersection outside the given interval can be ignored.

3. Which curve is higher and which is lower, or which is on the left and which is on the right.

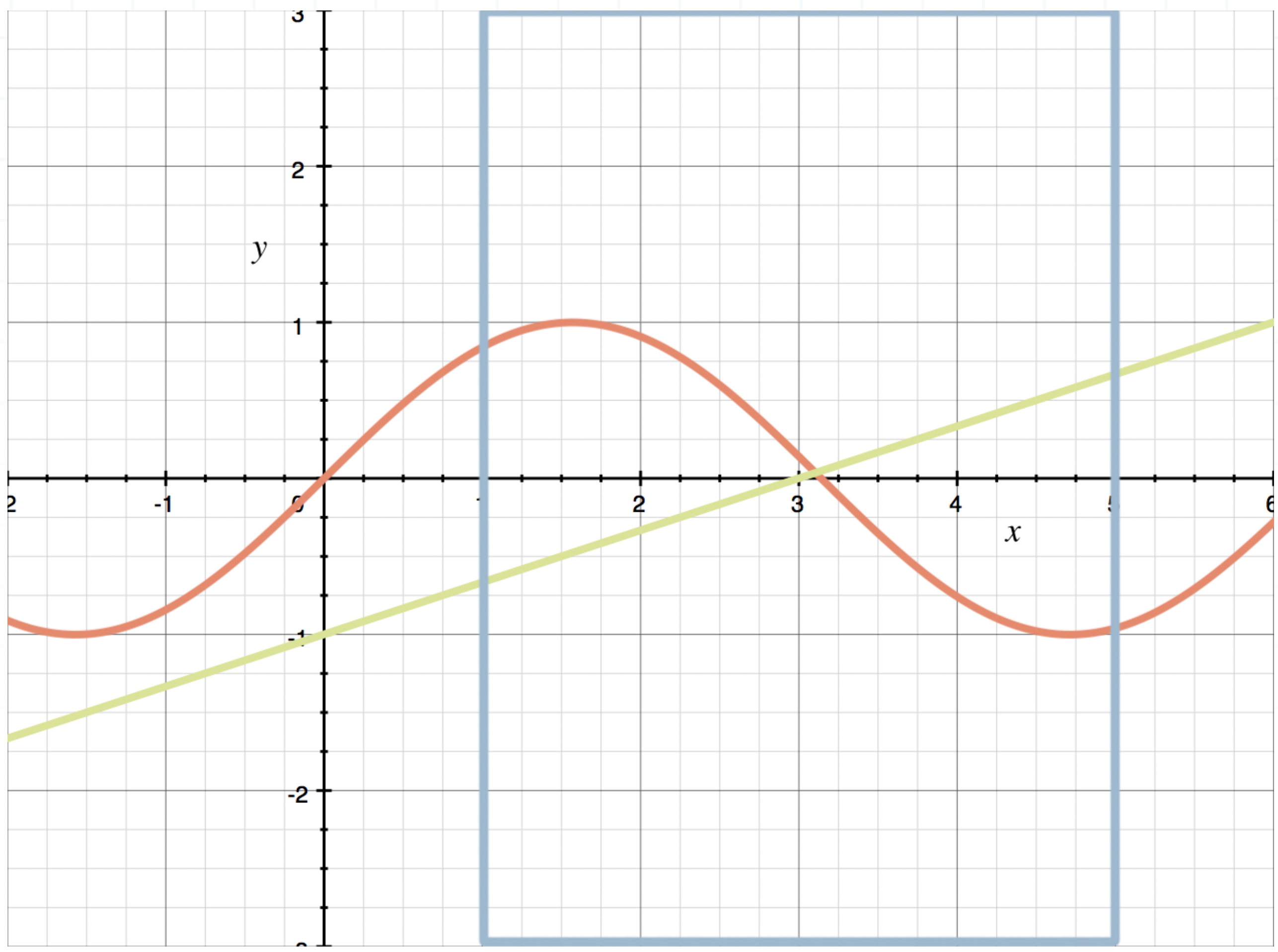
a. If the orientation of your curves is "higher-lower", then you need to figure out which curve is higher and which curve is lower in the given interval.

b. If the orientation of your curves is "left-right", then you need to figure out which curve is on the left and which is on the right in the given interval.

c. Note: As we mentioned in (2b) above, if you have a point of intersection inside the given interval, the curves cross each other. Therefore, which curve is higher/lower or on-the-left/on-the-right will switch at the point of intersection.

1) If you have a higher/lower switch, you'll use the following area formula instead of [A]

[C] $A = \int_{x=a}^{x=c} f(x) - g(x)\ dx + \int_{x=c}^{x=b} g(x) - f(x)\ dx$



On the interval $[1,5]$, the sine function is higher in the first half of the interval than the linear function. Around $x = 3$, the curves cross each other; the line becomes the higher function and the sine curve becomes lower function.

2) If you have a left/right switch, you'll use the following area formula instead of [B]

[D] $A = \int_{y=a}^{y=c} f(y) - g(y)\ dy + \int_{y=c}^{y=b} g(y) - f(y)\ dy$

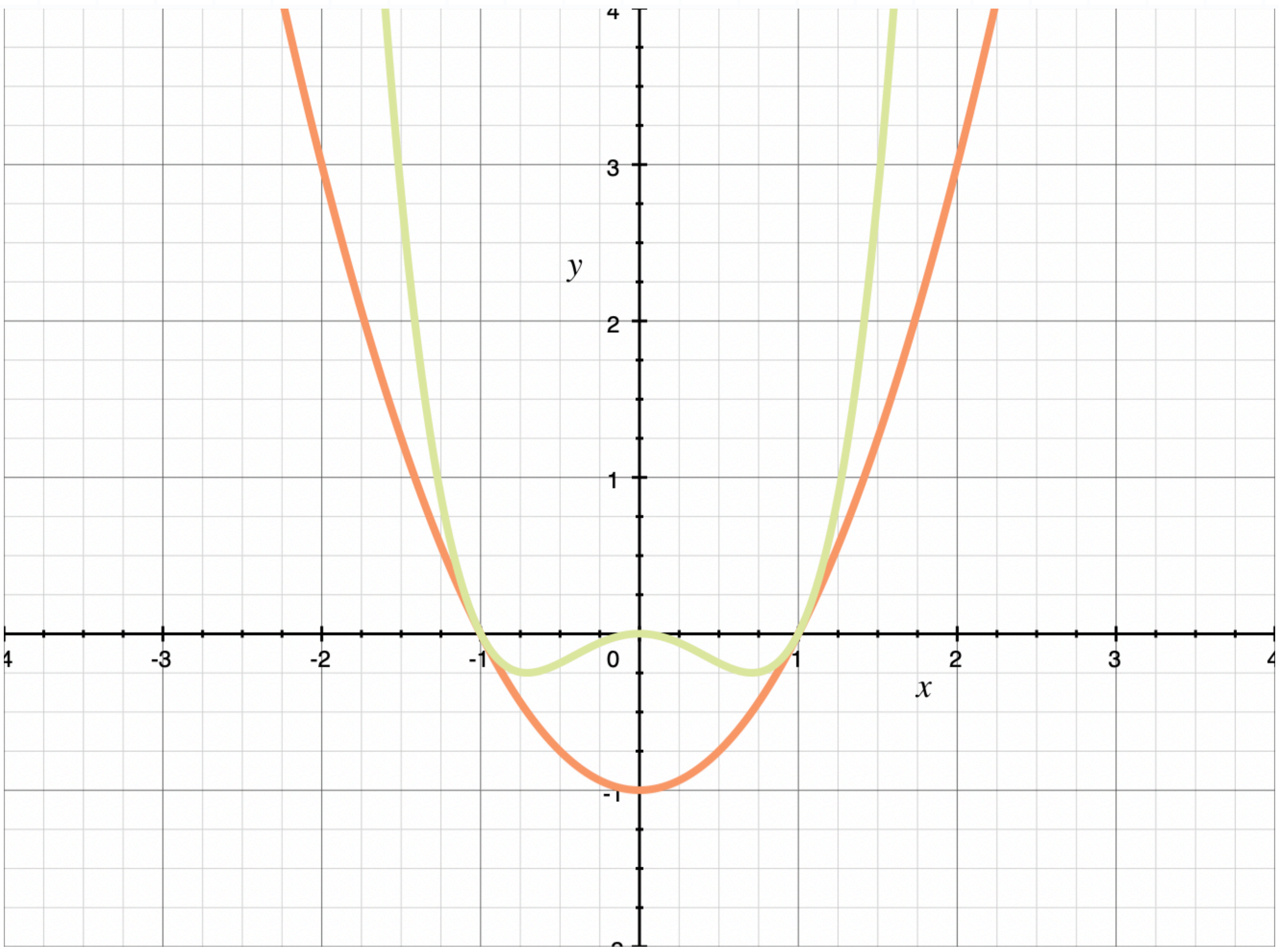
Example

Find the area between the curves.

$$y = x^2 - 1$$

$$y = x^4 - x^2$$

First, let's graph the functions to see their general orientation.



These look like higher-lower curves, with $y = x^4 - x^2$ being the "higher" function and $y = x^2 - 1$ being the "lower" function.

This means we'll be using formula **[A]** to find area.

$$A = \int_{x=a}^{x=b} f(x) - g(x)\ dx$$

Because the question didn't identify an interval, the second step is to find points of intersection. Since both curves are defined for y, we can set them equal to one another and then solve for x.

$$x^2 - 1 = x^4 - x^2$$

$$x^4 - 2x^2 + 1 = 0$$

$$\left(x^2 - 1\right)^2 = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Because the points of intersection are $x = -1$ and $x = 1$, we know we're trying to find the area enclosed by the curves between those points.

The third step is to figure out which curve is higher and which is lower between the points of intersection. Since there are only two points of intersection, $x = \pm 1$, and these are the endpoints of the interval, we know that there is no third point of intersection inside the interval where the curves cross each other. Which means that, across the full interval, one function will always be higher than the other one.

In order to figure out which one is higher, we'll pick an x-value between the points of intersection and plug it into the equations of the original functions. We'll pick $x = 0$ since it lies between the points of intersection, $x = -1$ and $x = 1$.

Plugging $x = 0$ into $y = x^2 - 1$, we get

$$y = (0)^2 - 1$$

$$y = -1$$

Plugging $x = 0$ into $y = x^4 - x^2$, we get

$$y = (0)^4 - (0)^2$$

$$y = 0$$

Because $0 > -1$, we know that $x^4 - x^2 > x^2 - 1$ between the points of intersection. Plugging this information into **[A]**, we get

$$A = \int_{-1}^{1} \left(x^4 - x^2\right) - \left(x^2 - 1\right)\ dx$$

$$A = \int_{-1}^{1} x^4 - 2x^2 + 1\ dx$$

$$A = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x \Big|_{-1}^{1}$$

$$A = \left[\frac{1}{5}(1)^5 - \frac{2}{3}(1)^3 + (1)\right] - \left[\frac{1}{5}(-1)^5 - \frac{2}{3}(-1)^3 + (-1)\right]$$

$$A = \left(\frac{1}{5} - \frac{2}{3} + 1\right) - \left(-\frac{1}{5} + \frac{2}{3} - 1\right)$$

$$A = \frac{1}{5} - \frac{2}{3} + 1 + \frac{1}{5} - \frac{2}{3} + 1$$

$$A = \frac{2}{5} - \frac{4}{3} + 2$$

$$A = \frac{6}{15} - \frac{20}{15} + \frac{30}{15}$$

$A = \dfrac{16}{15}$ square units

Let's look at another example where the orientation of the curves is left-right instead of higher-lower.

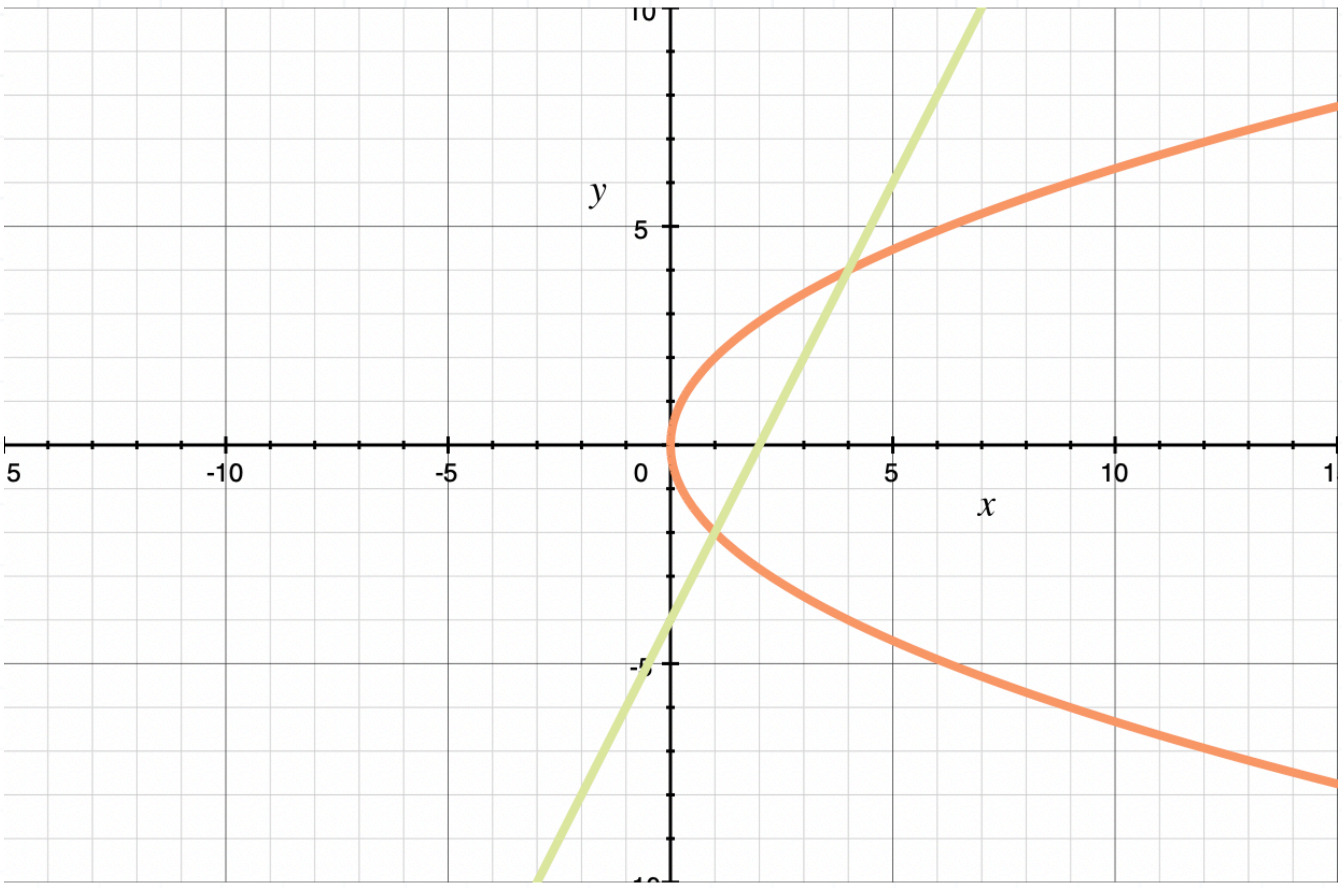
Example

Find the area between the curves.

$$y^2 = 4x$$

$$y = 2x - 4$$

First, let's graph the functions to see their general orientation.



These look like left-right curves, with $y = 2x - 4$ being the "right" curve and $y^2 = 4x$ being the "left" curve.

This means we'll be using formula [B] to find area.

$$A = \int_{y=a}^{y=b} f(y) - g(y)\ dy$$

Because the question didn't identify an interval, the second step is to find points of intersection. We'll solve both equations for x so that we can set them equal to one another and then solve for y to get points of intersection.

The first equation becomes

$$y^2 = 4x$$

[1] $x = \dfrac{y^2}{4}$

The second equation becomes

$$y = 2x - 4$$

[2] $x = \dfrac{y+4}{2}$

Setting them equal to each other gives

$$\frac{y^2}{4} = \frac{y+4}{2}$$

$$2y^2 = 4(y+4)$$

$$2y^2 = 4y + 16$$

$$y^2 = 2y + 8$$

$$y^2 - 2y - 8 = 0$$

$$(y+2)(y-4) = 0$$

$y = -2$ and $y = 4$

Because the points of intersection are $y = -2$ and $y = 4$, we know we're trying to find the area enclosed by the curves between those points.

The third step is to figure out which curve is on the right and which is on the left between the points of intersection. Since there are only two points of intersection, $y = -2$ and $y = 4$, and these are the endpoints of the interval, we know that there is no third point of intersection inside the interval where the curves cross each other. Which means that, across the full interval, one function will always be on the right and the other will always be on the left.

In order to figure out which one is on the right, we'll pick a y-value between the points of intersection and plug it into [1] and [2]. We'll pick $y = 0$ since it lies between the points of intersection, $y = -2$ and $y = 4$.

Plugging $y = 0$ into [1], we get

$$x = \frac{(0)^2}{4}$$

$$x = 0$$

Plugging $y = 0$ into [2], we get

$$x = \frac{(0) + 4}{2}$$

$$x = 2$$

Because $2 > 0$, we know that [2]>[1] between the points of intersection. Plugging this information into [B], we get

$$A = \int_{-2}^{4} \frac{y+4}{2} - \frac{y^2}{4}\ dy$$

$$A = \int_{-2}^{4} \frac{2(y+4)}{4} - \frac{y^2}{4}\ dy$$

$$A = \frac{1}{4}\int_{-2}^{4} 2(y+4) - y^2\ dy$$

$$A=\frac{1}{4}\int_{-2}^{4}2y+8-y^2\ dy$$

$$A=\frac{1}{4}\left(y^2+8y-\frac{1}{3}y^3\right)\Big|_{-2}^{4}$$

$$A=\frac{1}{4}\left[(4)^2+8(4)-\frac{1}{3}(4)^3\right]-\frac{1}{4}\left[(-2)^2+8(-2)-\frac{1}{3}(-2)^3\right]$$

$$A=\frac{1}{4}\left[\left(16+32-\frac{64}{3}\right)-\left(4-16+\frac{8}{3}\right)\right]$$

$$A=\frac{1}{4}\left(16+32-\frac{64}{3}-4+16-\frac{8}{3}\right)$$

$$A=\frac{1}{4}\left(\frac{48}{3}+\frac{96}{3}-\frac{64}{3}-\frac{12}{3}+\frac{48}{3}-\frac{8}{3}\right)$$

$$A=\frac{1}{4}\left(\frac{108}{3}\right)$$

$$A=\frac{108}{12}$$

$A=9$ square units

Arc length

We can use integration to calculate the arc length of a function, which is the length the function would be if we took the line of its graph and stretched it out straight and measured it.

Sometimes we'll need to find arc length of a function in the form $y = f(x)$, but other times the function will be in the form $x = g(y)$.

1. If the equation is in the form $y = f(x)$, the interval will be $a \le x \le b$ and the equation for arc length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. If the equation is in the form $x = g(y)$, the interval will be $c \le y \le d$ and the equation for arc length is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Example

Calculate the arc length of the curve over the interval.

$$y = \ln(\sec x)$$

on $0 \le x \le \frac{\pi}{3}$

Since the function we're given is in the form $y = f(x)$, we have to use the formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

First, we'll calculate dy/dx and then plug it back into the arc length formula.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x}$$

$$\frac{dy}{dx} = \tan x$$

Plugging the derivative into the arc length formula, we get

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\tan x)^2}\ dx$$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\ dx$$

Remembering that $\sec^2(x) = 1 + \tan^2(x)$, we get

$$L = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\ dx$$

$$L = \int_0^{\frac{\pi}{3}} \sec x\ dx$$

Integrate.

$$L = \ln \left| \sec(x) + \tan(x) \right| \Big|_0^{\frac{\pi}{3}}$$

Now we evaluate over the interval and get

$$L = \ln \left| \sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) \right| - \ln \left| \sec 0 + \tan 0 \right|$$

$$L = 1.32$$

The arc length of $y = \ln(\sec x)$ over the interval $0 \le x \le \frac{\pi}{3}$ is $L = 1.32$.

Now let's try an example where the curve is defined for x in terms of y.

Example

Calculate the arc length of the curve over the interval.

$$x = \frac{2}{3}(y-1)^{\frac{3}{2}}$$

on $2 \le y \le 5$

Since the function we're given is in the form $x = g(y)$, we have to use the formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

First, we'll calculate dx/dy and then plug it back into the arc length formula.

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}}$$

Plugging the derivative into the arc length formula, we get

$$L = \int_2^5 \sqrt{1 + \left[(y-1)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_2^5 \sqrt{y}\ dy$$

Integrate.

$$L = \frac{2}{3}y^{\frac{3}{2}}\Big|_2^5$$

Now we evaluate over the interval and get

$$L = \frac{2}{3}(5)^{\frac{3}{2}} - \frac{2}{3}(2)^{\frac{3}{2}}$$

$$L = 5.6$$

The arc length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ over the interval $2 \le y \le 5$ is $L = 5.6$.

Average value of a function

In the same way that we can find the average of set of numbers, we can also find the average value of a function over a specific interval.

The formula we use to find the average value of a function $f(x)$ over the interval $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Think about the average value of a function as the average height the function attains above the x-axis. If the function were $y = 3$, then the height of the function is always 3 everywhere, so the average height of the function would also be 3. When the function gets more complicated, we can use the average value formula to find its average height on $[a, b]$.

Example

Calculate the average value of the function over the interval.

$$f(x) = x^3 - 2x^2 + e^{2x}$$

on $[3,7]$

We'll use the formula for average value

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

and get

$$f_{avg} = \frac{1}{7-3}\int_3^7 x^3 - 2x^2 + e^{2x}\ dx$$

$$f_{avg} = \frac{1}{4}\int_3^7 x^3 - 2x^2 + e^{2x}\ dx$$

Next we can break the integral apart by term.

$$f_{avg} = \frac{1}{4}\int_3^7 x^3\ dx + \frac{1}{4}\int_3^7 -2x^2\ dx + \frac{1}{4}\int_3^7 e^{2x}\ dx$$

$$f_{avg} = \frac{1}{4}\int_3^7 x^3\ dx - \frac{2}{4}\int_3^7 x^2\ dx + \frac{1}{4}\int_3^7 e^{2x}\ dx$$

Integrate.

$$f_{avg} = \frac{1}{4}\left(\frac{x^4}{4}\right)\Bigg|_3^7 - \frac{2}{4}\left(\frac{x^3}{3}\right)\Bigg|_3^7 + \frac{1}{4}\left(\frac{e^{2x}}{2}\right)\Bigg|_3^7$$

$$f_{avg} = \frac{x^4}{16} - \frac{x^3}{6} + \frac{e^{2x}}{8}\Bigg|_3^7$$

Now we can evaluate on the interval.

$$f_{avg} = \left[\frac{(7)^4}{16} - \frac{(7)^3}{6} + \frac{e^{2(7)}}{8}\right] - \left[\frac{(3)^4}{16} - \frac{(3)^3}{6} + \frac{e^{2(3)}}{8}\right]$$

$$f_{avg} = 150{,}367$$

The average value of the function $f(x) = x^3 - 2x^2 + e^{2x}$ over the interval $[3,7]$ is $150{,}367$.

Mean value theorem for integrals

The mean value theorem for integrals tells us that, for a continuous function $f(x)$, there's at least one point c inside the interval $[a, b]$ at which the value of the function will be equal to the average value of the function over that interval.

This means we can equate the average value of the function over the interval to the value of the function at the single point.

In other words,

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c)$$

The equation above sets the average value of the function over the interval $[a, b]$ (on the left), equal to the value of the function at the point c (on the right). If we multiply both sides by $(b-a)$, we get the mean value theorem for integrals:

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

Example

Find the point c that satisfies the mean value theorem for integrals on the interval $[1,4]$.

$$f(x) = 3x^2 - 2x$$

Looking at the equation we can see that it is a polynomial and is therefore continuous. This means that we can go ahead and use the mean value theorem for integrals.

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

$$\int_1^4 3x^2 - 2x\ dx = \left(3c^2 - 2c\right)(4-1)$$

$$\int_1^4 3x^2 - 2x\ dx = 9c^2 - 6c$$

Now we can break up the integral to make it easier to solve.

$$\int_1^4 3x^2\ dx + \int_1^4 -2x\ dx = 9c^2 - 6c$$

$$3\int_1^4 x^2\ dx - 2\int_1^4 x\ dx = 9c^2 - 6c$$

Integrate.

$$\left[3\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right)\right]\Bigg|_1^4 = 9c^2 - 6c$$

$$\left(x^3 - x^2\right)\Big|_1^4 = 9c^2 - 6c$$

Now we can evaluate over the interval.

$$\left[(4)^3 - (4)^2\right] - \left[(1)^3 - (1)^2\right] = 9c^2 - 6c$$

$$48 = 9c^2 - 6c$$

$$0 = 9c^2 - 6c - 48$$

Now we need to solve for c.

$$0 = 3\left(3c^2 - 2c - 16\right)$$

$$0 = 3c^2 - 2c - 16$$

$$0 = (3c - 8)(c + 2)$$

Setting each of the factors equal to 0 individually to solve for c, we get

$$3c - 8 = 0$$

$$c = \frac{8}{3}$$

and

$$c + 2 = 0$$

$$c = -2$$

Only one of these values, $c = 8/3$, falls in the interval $[1,4]$, which means it's the only solution.

It's possible to find more than one valid answer for c, but in the last example, there's only one point, $c = 8/3$, at which the value of the function is equal to the average value of the function over the interval.

Surface area of revolution

We can use integrals to find the surface area of the three-dimensional figure that's created when we take a function and rotate it around an axis and over a certain interval.

The formulas we use to find surface area of revolution are different depending on the form of the original function and the axis of rotation.

1. When the function is in the form $y = f(x)$ and you're rotating around the y-axis, the interval is $a \le x \le b$ and the formula is

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. When the function is in the form $y = f(x)$ and you're rotating around the x-axis, the interval is $a \le x \le b$ and the formula is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

3. When the function is in the form $x = g(y)$ and you're rotating around the y-axis, the interval is $c \le y \le d$ and the formula is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

4. When the function is in the form $x = g(y)$ and you're rotating around the x-axis, the interval is $c \le y \le d$ and the formula is

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Example

Find the area of the surface generated by rotating the function about the given axis over the given interval.

$$y = x^3$$

about the x-axis

$$0 \le x \le 3$$

Since the equation is in the form $y = f(x)$, and we're rotating around the x-axis, we'll use the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

We'll calculate dy/dx and then substitute it back into the equation.

$$\frac{dy}{dx} = 3x^2$$

$$S = \int_0^3 2\pi x^3 \sqrt{1 + \left(3x^2\right)^2}\ dx$$

$$S = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4}\ dx$$

Using u-substitution and setting $u = 1 + 9x^4$ and $du = 36x^3\ dx$, we calculate

$$x = \left(\frac{u-1}{9}\right)^{\frac{1}{4}}$$

$$dx = \frac{1}{36x^3}\ du$$

$$dx = \frac{1}{36\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3}\ du$$

Plugging these values back into the integral, we get

$$S = \int_0^3 2\pi\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3\sqrt{u}\frac{1}{36\left[\left(\frac{u-1}{9}\right)^{\frac{1}{4}}\right]^3}\ du$$

$$S = \int_0^3 2\pi\left(\frac{u-1}{9}\right)^{\frac{3}{4}}\sqrt{u}\frac{1}{36\left(\frac{u-1}{9}\right)^{\frac{3}{4}}}\ du$$

$$S = \frac{\pi}{18}\int_0^3 \left(\frac{u-1}{9}\right)^{\frac{3}{4}}\sqrt{u}\frac{1}{\left(\frac{u-1}{9}\right)^{\frac{3}{4}}}\ du$$

$$S = \frac{\pi}{18}\int_0^3 \sqrt{u}\ du$$

Integrate.

$$S=\left(\frac{\pi}{18}\right)\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_0^3$$

$$S=\frac{\pi}{27}u^{\frac{3}{2}}\Bigg|_0^3$$

We'll plug back in for u, remembering that $u=1+9x^4$, and then evaluate over the interval.

$$S=\frac{\pi}{27}\left(1+9x^4\right)^{\frac{3}{2}}\Bigg|_0^3$$

$$S=\frac{\pi}{27}\left[1+9(3)^4\right]^{\frac{3}{2}}-\left[\frac{\pi}{27}\left(1+9(0)^4\right)^{\frac{3}{2}}\right]$$

$S=2{,}294.8$ square units

The surface area obtained by rotating $y=x^3$ around the x-axis over the interval $0\le x\le 3$ is $S=2{,}294.8$.

Volume of revolution, disk method

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the x-axis (or some other horizontal axis with the equation $y = b$) or around the y-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. To use the disk method, the volume generated by rotating the function has to be a solid volume with no holes in the middle. It'll be bounded by the function we're rotating and the axis of rotation.

The disk method formulas we use to find volume of rotation are different depending on the form of the function and the axis of rotation.

1. If the function is in the form $y = f(x)$ and we're rotating around the x-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b \pi(f(x))^2 dx$$

2. If the function is in the form $x = f(y)$ and we're rotating around the y-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V = \int_c^d \pi(f(y))^2 dy$$

To use the formula table to find the right volume of revolution formula, first determine the axis of rotation or revolution. The problem will usually tell us the line of rotation.

The best way to figure out whether we need to use disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

Axis	**Disks**
	$\int$ area width
x-axis	$\int_a^b \pi(f(x))^2 dx$
y-axis	$\int_c^d \pi(f(y))^2 dy$

Let's do an example where we find the volume of revolution when we rotate around the x-axis.

Example

Find the volume of the solid created by rotating the function about the x-axis over the interval $[1,3]$.

$$y = 2x^3$$

Since the function is in the form $y = f(x)$, we'll use the formula

$$V = \int_a^b \pi(f(x))^2 \ dx$$

and plug in what we've been given.

$$V = \int_1^3 \pi(2x^3)^2 dx$$

$$V = \int_1^3 4\pi x^6 \ dx$$

$$V = 4\pi \int_1^3 x^6 \ dx$$

We'll integrate,

$$V = 4\pi \left(\frac{x^7}{7}\right)\Bigg|_1^3$$

$$V = \frac{4\pi x^7}{7}\Bigg|_1^3$$

and then evaluate over the interval.

$$V = \frac{4\pi(3)^7}{7} - \frac{4\pi(1)^7}{7}$$

$$V = \frac{8{,}744\pi}{7}$$

$$V \approx 3{,}924.30$$

The volume of the solid object created by rotating $y = 2x^3$ about the x-axis over the interval $[1,3]$ is $V \approx 3{,}924.30$.

Let's do another example, this time where we rotate around the y-axis.

Example

Find the volume of the solid created by rotating the function about the y-axis over the interval $y = 3$ to $y = 5$.

$$x = 3y^{-2}$$

Since the function is in the form $x = f(y)$, we'll use the formula

$$V = \int_c^d \pi (f(y))^2 \ dy$$

and plug in what we've been given.

$$V = \int_3^5 \pi (3y^{-2})^2 \ dy$$

$$V = \int_3^5 9\pi y^{-4} \ dy$$

$$V = 9\pi \int_3^5 y^{-4} \ dy$$

We'll integrate,

$$V = 9\pi \left(\frac{y^{-3}}{-3} \right) \Bigg|_3^5$$

$$V = -3\pi y^{-3} \Big|_3^5$$

and then evaluate over the interval.

$$V = -3\pi(5)^{-3} - (-3\pi(3)^{-3})$$

$$V = -\frac{3\pi}{5^3} + \frac{3\pi}{3^3}$$

$$V = \frac{\pi}{3^2} - \frac{3\pi}{5^3}$$

$$V = \frac{\pi}{9} - \frac{3\pi}{125}$$

$$V = \frac{125\pi}{1,125} - \frac{27\pi}{1,125}$$

$$V = \frac{98\pi}{1,125}$$

$$V \approx 0.27$$

The volume of the solid object created by rotating $x = 3y^{-2}$ about the y-axis over the interval $[3,5]$ is $V \approx 0.27$.

Volume of revolution, washer method

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the x-axis (or some other horizontal axis with the equation $y = b$) or around the y-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. To use the washer method, the volume generated by rotating the function has to be a ring (like a washer, or a donut) with a hole in the middle.

In order to generate a volume like this, the region has to be bounded by two functions, either $y = f(x)$ and $y = g(x)$ or $x = f(y)$ and $x = g(y)$.

The washer method formulas we use to find volume of rotation are different depending on the form of the functions and the axis of rotation.

1. If the region is bounded by $y = f(x)$ and $y = g(x)$, and if $f(x) > g(x)$, and if we're rotating around the x-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2 \ dx$$

2. If the region is bounded by $x = f(y)$ and $x = g(y)$, and if $f(y) > g(y)$, and if we're rotating around the y-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V=\int_c^d \pi[f(y)]^2-\pi[g(y)]^2\ dy$$

The table below will help guide you through how to solve a volume problem when you're using disks or washers to find the volume. Start in the first row of the table, and determine the line of rotation or revolution. The problem will usually tell you the line of rotation. If you're asked to rotate about the x-axis or some line defined for y in terms of x, then stay in the first column of the table. If you're asked to rotate about the y-axis or some line defined for x in terms of y, then stay in the second column of the table.

The best way to figure out whether you need to use disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

Axis	Disks	Washers	Shells
	$\int$ area width	$\int$ area width	$\int$ circumference height width
	Axis of revolution: HORIZONTAL		
x-axis	$\int_a^b \pi \left[f(x)\right]^2 \ dx$	$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$	$\int_c^d 2\pi y \left[f(y) - g(y)\right] \ dy$
$y = -k$		$\int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2 \ dx$	$\int_c^d 2\pi(y + k)\left[f(y) - g(y)\right] \ dy$
$y = k$		$\int_a^b \pi \left[k - g(x)\right]^2 - \pi \left[k - f(x)\right]^2 \ dx$	$\int_c^d 2\pi(k - y)\left[f(y) - g(y)\right] \ dy$
	Axis of revolution: VERTICAL		
y-axis	$\int_c^d \pi \left[f(y)\right]^2 \ dy$	$\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2 \ dy$	$\int_a^b 2\pi x \left[f(x) - g(x)\right] \ dx$
$x = -k$		$\int_c^d \pi \left[k + f(y)\right]^2 - \pi \left[k + g(y)\right]^2 \ dy$	$\int_a^b 2\pi(x + k)\left[f(x) - g(x)\right] \ dx$
$x = k$		$\int_c^d \pi \left[k - g(y)\right]^2 - \pi \left[k - f(y)\right]^2 \ dy$	$\int_a^b 2\pi(k - x)\left[f(x) - g(x)\right] \ dx$

Example

Find the volume of the solid created by rotating the region bounded by the curves about the x-axis.

$$y = x^2 + 2$$

$$y = x + 4$$

Looking at the question, we can see that the functions that bound our region are in the form $y = f(x)$ and $y = g(x)$, so we'll use the formula

$$V = \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2 \ dx$$

The problem doesn't give us an interval over which to integrate, but when we're using washer method the interval is defined by the points of intersection of the two curves. To find the points of intersection and generate the interval, we'll set the functions equal to each other and solve for x.

$$x^2 + 2 = x + 4$$

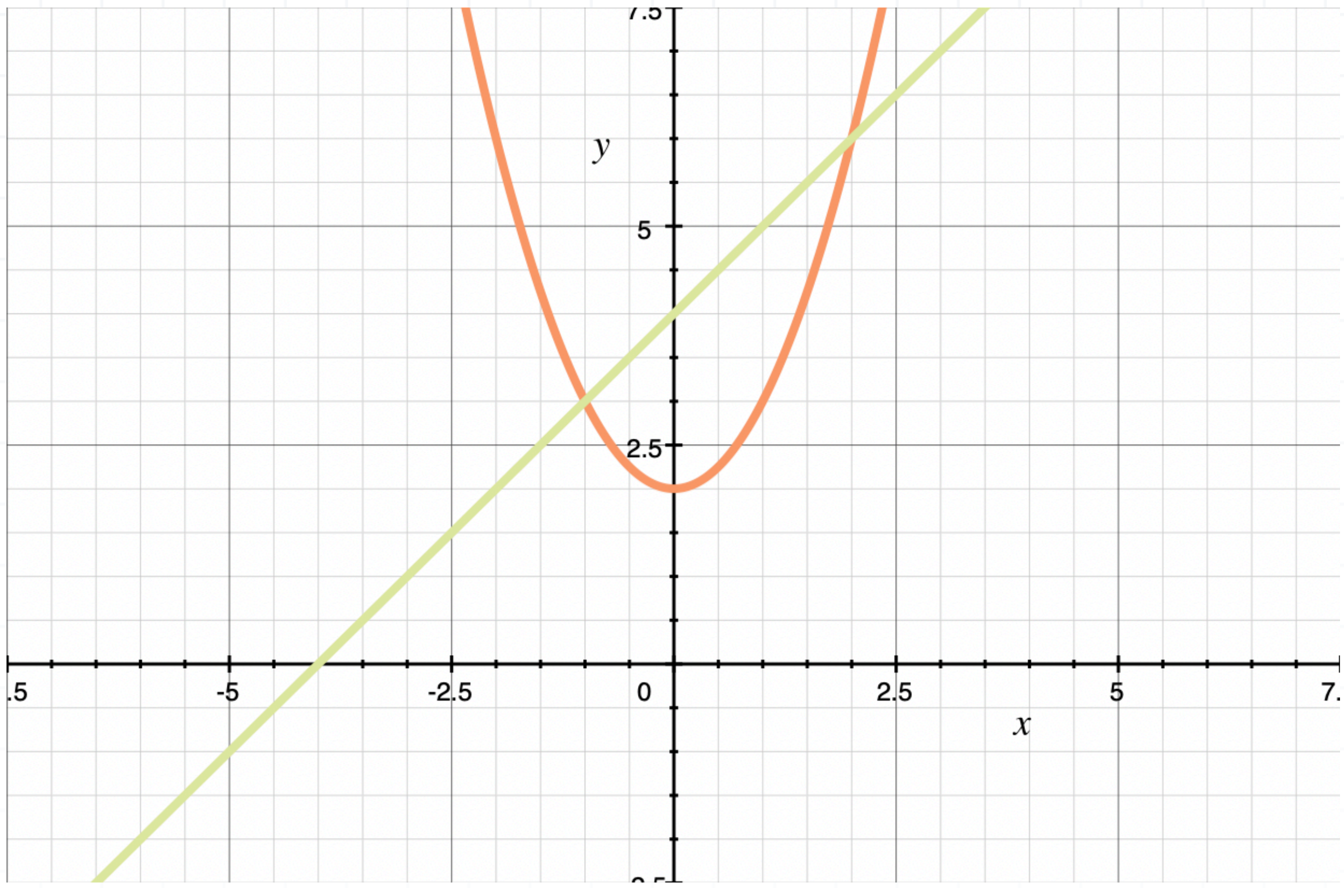
$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$x = -1$ and $x = 2$

Based on the values we found for x, the interval is $[-1,2]$.

The next step is deciding which of the functions will be $f(x)$ and which will be $g(x)$, remembering that $f(x) \geq g(x)$. The simplest way to see this is to graph the two equations. Remember, we are only interested in what is happening in the interval $[-1,2]$.



Based on the graph, $f(x) = x + 4$ and $g(x) = x^2 + 2$, because $y = x + 4$ is greater than $y = x^2 + 2$ over the interval $[-1,2]$.

If you can't graph the functions to see which one is greater than the other over the interval, you can always pick a value in the interval and plug it into both functions. For example, since 0 is in the interval $[-1,2]$, we can plug it into both of the functions and we get

$$y = x + 4$$

$$y = 0 + 4 = 4$$

and

$$y = x^2 + 2$$

$$y = (0)^2 + 2 = 2$$

Because $y = x + 4$ gives us a greater value back than $y = x^2 + 2$ $(4 > 2)$, we know that $f(x) = x + 4$ and $g(x) = x^2 + 2$.

Remember also that, if you're dealing with a problem in which the functions are defined as $x = f(y)$ and $x = g(y)$, then you're plugging in a y-value to see which of the functions returns a greater x-value. The right-most function (the one with the larger x-value), is greater than the left-most function (the one with the smaller x-value).

Now we can solve for volume.

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) \ dx$$

$$V = \int_{-1}^2 \pi \left([x+4]^2 - [x^2+2]^2 \right) \ dx$$

$$V = \int_{-1}^2 \pi \left(x^2 + 8x + 16 - [x^4 + 4x^2 + 4] \right) \ dx$$

$$V = \int_{-1}^2 \pi \left(-x^4 - 3x^2 + 8x + 12 \right) \ dx$$

$$V = \pi \int_{-1}^2 -x^4 - 3x^2 + 8x + 12 \ dx$$

Now we can break the integral into smaller parts.

$$V = \pi \int_{-1}^2 -x^4 \ dx + \pi \int_{-1}^2 -3x^2 \ dx + \pi \int_{-1}^2 8x \ dx + \pi \int_{-1}^2 12 \ dx$$

$$V = -\pi \int_{-1}^{2} x^4 \ dx - 3\pi \int_{-1}^{2} x^2 \ dx + 8\pi \int_{-1}^{2} x \ dx + 12\pi \int_{-1}^{2} 1 \ dx$$

Integrating, we get

$$V = -\pi \left(\frac{x^5}{5} \right) - 3\pi \left(\frac{x^3}{3} \right) + 8\pi \left(\frac{x^2}{2} \right) + 12\pi(x) \Big|_{-1}^{2}$$

$$V = \frac{-\pi x^5}{5} - \pi x^3 + 4\pi x^2 + 12\pi x \Big|_{-1}^{2}$$

Evaluating over the interval, we get

$$V = \frac{-\pi(2)^5}{5} - \pi(2)^3 + 4\pi(2)^2 + 12\pi(2) - \left[\frac{-\pi(-1)^5}{5} - \pi(-1)^3 + 4\pi(-1)^2 + 12\pi(-1) \right]$$

$$V = \frac{162}{5}\pi$$

This is the volume of the solid object created by rotating the region bounded by $y = x^2 + 2$ and $y = x + 4$ about the x-axis.

Volume of revolution, cylindrical shells

We can use integrals to find the volume of the three-dimensional object created by rotating a function around either the x-axis (or some other horizontal axis with the equation $y = b$) or around the y-axis (or some other vertical axis with the equation $x = a$).

We can do this using the disk method, the washer method, or using cylindrical shells. The cylindrical shell method rotates the function in a perpendicular fashion. This means that $y = f(x)$ is rotated around the y-axis and $x = g(y)$ is rotated around the x-axis.

The cylindrical shells method formulas we use to find volume of rotation are different depending on the form of the function and the axis of rotation.

1. If the function is in the form $y = f(x)$ and we're rotating around the y-axis over the interval $[a, b]$, the formula for the volume of the solid is

$$V = \int_a^b 2\pi x f(x)\ dx$$

2. If the function is in the form $x = g(y)$ and we're rotating around the x-axis over the interval $[c, d]$, the formula for the volume of the solid is

$$V = \int_c^d 2\pi y f(y)\ dy$$

The table below will help guide you through how to solve a volume problem when you're using cylindrical shells to find the volume. Start in the first row of the table, and determine the line of rotation or revolution. The problem will usually tell you the line of rotation. If you're asked to rotate about the y-axis or some line defined for x in terms of y, then stay in the first column of the table. If you're asked to rotate about the x-axis or some line defined for y in terms of x, then stay in the second column of the table.

The best way to figure out whether you need to use cylindrical shells instead of either disks or washers is to graph the functions and the axis of rotation and draw a picture of the rotated volume.

Axis	Disks	Washers	Shells
	$\int$ area width	$\int$ area width	$\int$ circumference height width
Axis of revolution: HORIZONTAL			
x-axis	$\int_a^b \pi \left[f(x)\right]^2 \ dx$	$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$	$\int_c^d 2\pi y \left[f(y) - g(y)\right] \ dy$
$y = -k$		$\int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2 \ dx$	$\int_c^d 2\pi(y + k)\left[f(y) - g(y)\right] \ dy$
$y = k$		$\int_a^b \pi \left[k - g(x)\right]^2 - \pi \left[k - f(x)\right]^2 \ dx$	$\int_c^d 2\pi(k - y)\left[f(y) - g(y)\right] \ dy$
Axis of revolution: VERTICAL			
y-axis	$\int_c^d \pi \left[f(y)\right]^2 \ dy$	$\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2 \ dy$	$\int_a^b 2\pi x \left[f(x) - g(x)\right] \ dx$
$x = -k$		$\int_c^d \pi \left[k + f(y)\right]^2 - \pi \left[k + g(y)\right]^2 \ dy$	$\int_a^b 2\pi(x + k)\left[f(x) - g(x)\right] \ dx$
$x = k$		$\int_c^d \pi \left[k - g(y)\right]^2 - \pi \left[k - f(y)\right]^2 \ dy$	$\int_a^b 2\pi(k - x)\left[f(x) - g(x)\right] \ dx$

Example

Find the volume of the solid created by rotating the curve about the y-axis over the interval $[1,2]$.

$$y = x^3$$

Looking at the function, we can see that it's in the form $y = f(x)$ and revolved around the y-axis, which means the volume is given by

$$V = \int_a^b 2\pi x f(x)\ dx$$

$$V = \int_1^2 2\pi x \left(x^3\right)\ dx$$

$$V = \int_1^2 2\pi x^4\ dx$$

$$V = 2\pi \int_1^2 x^4\ dx$$

Integrating, we get

$$V = 2\pi \left(\frac{x^5}{5}\right)\Bigg|_1^2$$

$$V = \frac{2\pi x^5}{5}\Bigg|_1^2$$

Now we can evaluate over the interval.

$$V = \frac{2\pi(2)^5}{5} - \left[\frac{2\pi(1)^5}{5}\right]$$

$$V = \frac{62}{5}\pi$$

This is the volume of the solid object created by rotating $y = x^3$ about the y-axis over the interval $[1,2]$.

Let's try another example where the curve is defined for x in terms of y.

Example

Find the volume of the solid created by rotating the curve about the x-axis over the interval $y = 2$ to $y = 4$.

$$x = 3y^4$$

Looking at the function, we can see that it's in the form $x = g(y)$ and revolved around the x-axis, which means the volume is given by

$$V = \int_a^b 2\pi y f(y)\ dy$$

$$V = \int_2^4 2\pi y\left(3y^4\right)\ dy$$

$$V = \int_2^4 6\pi y^5\ dy$$

$$V = 6\pi \int_2^4 y^5\ dy$$

Integrating, we get

$$V = 6\pi\left(\frac{y^6}{6}\right)\Bigg|_2^4$$

$$V = \pi y^6\Big|_2^4$$

Now we can evaluate over the interval.

$$V = \pi(4)^6 - \pi(2)^6$$

$$V = 4{,}032\pi$$

This is the volume of rotation for the solid object created by rotating $x = 3y^4$ about the x-axis over the interval $y = 2$ to $y = 4$.

Work done to lift a mass or weight

To calculate the work done when we lift a weight or mass vertically some distance, we'll use the formula

$$W = \int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the force equation, and $[a, b]$ is the starting and ending height of the weight or mass.

Oftentimes problems like these will have us use a rope or cable to lift an object up some vertical height. In a problem like this, we'll need to determine the combined force required to lift the rope and the object. The formula for force is

$$F = mg$$

where F is force, m is the mass of the object, and g is the gravitational constant 9.8 m/s^2.

If we're given a weight instead of a mass, we can say that the force required to lift the weight is the same as the weight itself, because gravity has already been factored in when we have the "weight" of the object. Otherwise, if we have a mass, we have to multiply by the gravitational constant in order to get the force required to lift it.

Example

Movers are trying to bring a piano into a third floor apartment. Since the front door is too small, they're going to hoist the piano up the side of the building and in through a window. The steel cable they're using has a mass of 3 kg/m and the piano has a mass of 550 kg. Find the work done to lift the piano to the third floor window, 7 m above the ground.

In order to find the work required to lift the piano, we'll find the work required to lift just the piano, then we'll find the work required to lift the cable, and then we'll add the two together to find the total work required.

Remember that work is just the integral of force over the interval $[a, b]$,

$$W = \int_a^b F(x)\ dx$$

which means that we need to find the force for the piano and the force for the cable. If we start with the piano, we know the mass of the piano is 550 kg, so the force required to lift it is

$$F_p = m_p \cdot g$$

$$F_p = 550 \cdot 9.8$$

$$F_p = 5{,}390$$

Now we can find the work required to lift the piano from 0 m at ground level to 7 m at third floor level.

$$W = \int_{a}^{b} F(x)\ dx$$

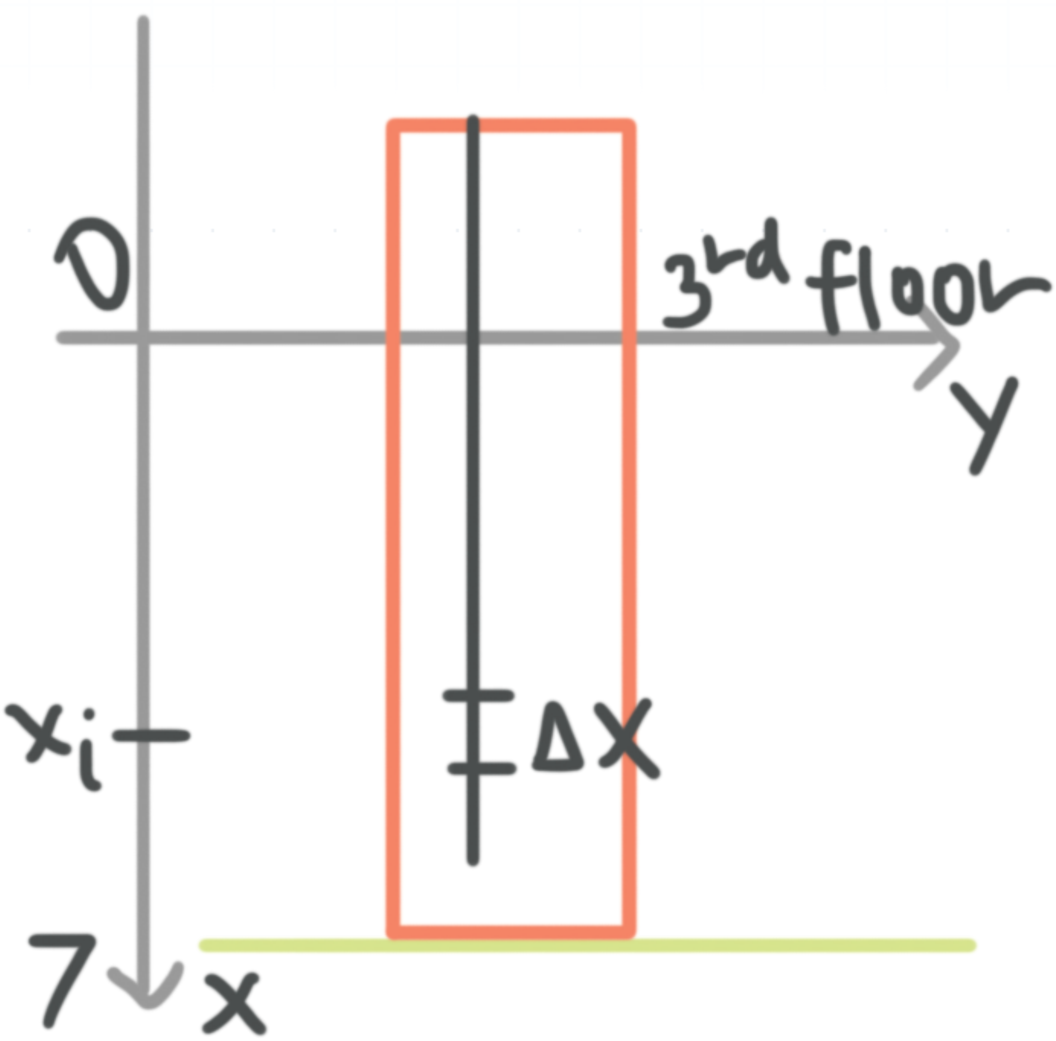
$$W_p = \int_{0}^{7} 5{,}390\ dx$$

$$W_p = 5{,}390x \Big|_{0}^{7}$$

$$W_p = 5{,}390(7) - 5{,}390(0)$$

$$W_p = 37{,}730$$

Next, we'll find the force for the cable F_c. If we put the apartment building into a coordinate system, with the third floor of the building at the origin and the x-axis extending out toward the base of the building, then we get



We divide the cable into small parts, each with a length of Δx. If we try to model the force for just a single section of Δx, we can say that we have to move that section a distance of x_i to get it to the third floor window, 7 m above the ground. Which means the force exerted on one slice of the cable is

$$F_i = m_i g$$

$$F_i = 3 \cdot 9.8$$

$$F_i = 29.4$$

Now we can find the work required to lift just the same slice of cable.

$$W_i = F_i d$$

$$W_i = 29.4 \cdot x_i$$

$$W_i = 29.4x_i$$

If we want to find the work required to lift the entire cable, and not just a single slice, we'll use an infinite number of slices n and sum the work required to lift every slice.

$$W_c = \lim_{n\to\infty} \sum_{i=1}^{n} 29.4x_i \Delta x$$

Taking the infinite sum is the same as taking the integral, so we'll convert this to integral notation, changing x_i to x and Δx to dx when we do.

$$W_c = \int 29.4x \ dx$$

Since the section of cable at the third floor has to be pulled up 0 m to get to the third floor, and the section of cable at the ground has to be pulled up 7 m to get to the third floor, the integral becomes

$$W_c = \int_0^7 29.4x \ dx$$

$$W_c = \frac{29.4}{2}x^2 \Big|_0^7$$

$$W_c = 14.7x^2 \Big|_0^7$$

$$W_c = 14.7(7)^2 - 14.7(0)^2$$

$$W_c = 14.7(49)$$

$$W_c = 720.3$$

To find the total work required, we'll add together the work required to lift both the piano and the cable.

$$W = W_p + W_c$$

$$W = 37{,}730 + 720.3$$

$$W = 38{,}450.3 \text{ J}$$

Work done on elastic springs

To calculate the work done when we stretch or compress an elastic spring, we'll use the formula

$$W = \int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the force equation, and $[a, b]$ is the distance over which the spring is stretched or compressed.

Every spring has its own spring constant k. This spring constant is part of Hooke's Law, which states that

$$F(x) = kx$$

where $F(x)$ is the force required to stretch or compress the spring, k is the spring constant, and x is the difference between the natural length and the stretched or compressed length. Since k is unique to each spring, we'll need to calculate it prior to determining work, unless it's given in the problem.

Keep in mind that we'll want to find work in terms of Joules J, which is the same as Newton-meters N-m.

Example

A spring has a natural length of 30 cm. A 50 N force is required to stretch and hold the spring at a length of 40 cm.

1. How much work is done to stretch the spring from 42 cm to 48 cm?

2. How much work is done to compress the spring from 30 cm to 25 cm?

We'll use Hooke's Law to find $F(x)$, but first we need to find k.

Since we know that a 50 N force is required to stretch and hold the spring at a length of 40 cm, from its natural length of 30 cm, we'll set $F(x) = 50$ and $x = 0.10$ m, which is the difference between the natural length and the stretched length, converted from cm to m. Remember that we'll be finding work in terms of Newtons and meters, which is why we converted 10 cm to 0.10 m.

$$50 = 0.10k$$

$$k = 500$$

With k, we can develop a generic equation for our spring using Hooke's Law.

$$F(x) = kx$$

$$F(x) = 500x$$

Work done to stretch the spring

To calculate the work required to stretch the spring from 42 cm to 48 cm, we pretend that the spring at its natural length of 30 cm ends at the origin, which means that stretching it to 42 cm means we've stretched it to 12, because $42-30=12$. Stretching it to 48 cm means we've stretched it from the origin to 18, because $48-30=18$.

Therefore, the work equation would be

$$W=\int_a^b F(x)\ dx$$

$$W=\int_{12}^{18} 500x\ dx$$

But we need to convert the units from cm to m, so the interval becomes 0.12 m to 0.18 m.

$$W=\int_{0.12}^{0.18} 500x\ dx$$

$$W=250x^2\Big|_{0.12}^{0.18}$$

$$W=250(0.18)^2-250(0.12)^2$$

$$W=4.5$$

The work done to stretch a spring with natural length 30 cm and spring constant $k=500$ from 42 cm to 48 cm is 4.5 J.

Work done to compress the spring

To calculate the work required to compress the spring from 30 cm to 25 cm, we pretend that the spring ends at the origin, which means that compressing it to 25 cm means we've compressed it to -5, because $25-30=-5$.

Therefore, the work equation would be

$$W=\int_a^b F(x)\ dx$$

$$W=\int_0^{-5} 500x\ dx$$

But we need to convert the units from cm to m, so the interval becomes 0 m to -0.05 m.

$$W=\int_0^{-0.05} 500x\ dx$$

$$W=250x^2\Big|_0^{-0.05}$$

$$W=250(-0.05)^2-250(0)^2$$

$$W=0.625$$

The work done to compress a spring with natural length 30 cm and spring constant $k=500$ from 30 cm to 25 cm is 0.625 J.

Work done to empty a tank

Finding the work required to empty a tank of the substance it contains is a common calculus application.

We always solve these problems the same way.

1. We divide the tank into an infinite number of slices n.

2. We calculate the work required to remove that single slice of substance from the tank.

3. We develop an equation to solve for the work needed to empty the entire tank, based on the work that was required to remove the single slice.

While this process seems fairly simple, it'll take a little effort to calculate the work required to remove a single slice of substance.

We know that work is equal to force multiplied by distance ($W = Fd$), so we'll find force and distance, and then use them to find work. To get force, we use the equation $F = mg$, where m is mass and g is the gravitational constant, $g = 9.8$ m/s^2.

To solve for mass, we'll remember that mass is equal to density multiplied by volume $m = \delta V$.

Density of the substance in the tank is usually provided in the problem, or the substance in the tank is water, which has a known density of $1{,}000$ kg/m^3. Volume of the single slice will depend on the shape of the tank.

As you can see, we'll start by calculating the most basic pieces of information, and then work our way up to finding work. In other words, follow these steps:

1. Find density and volume, then use them to calculate mass.

2. Multiply mass by the gravitational constant to find force.

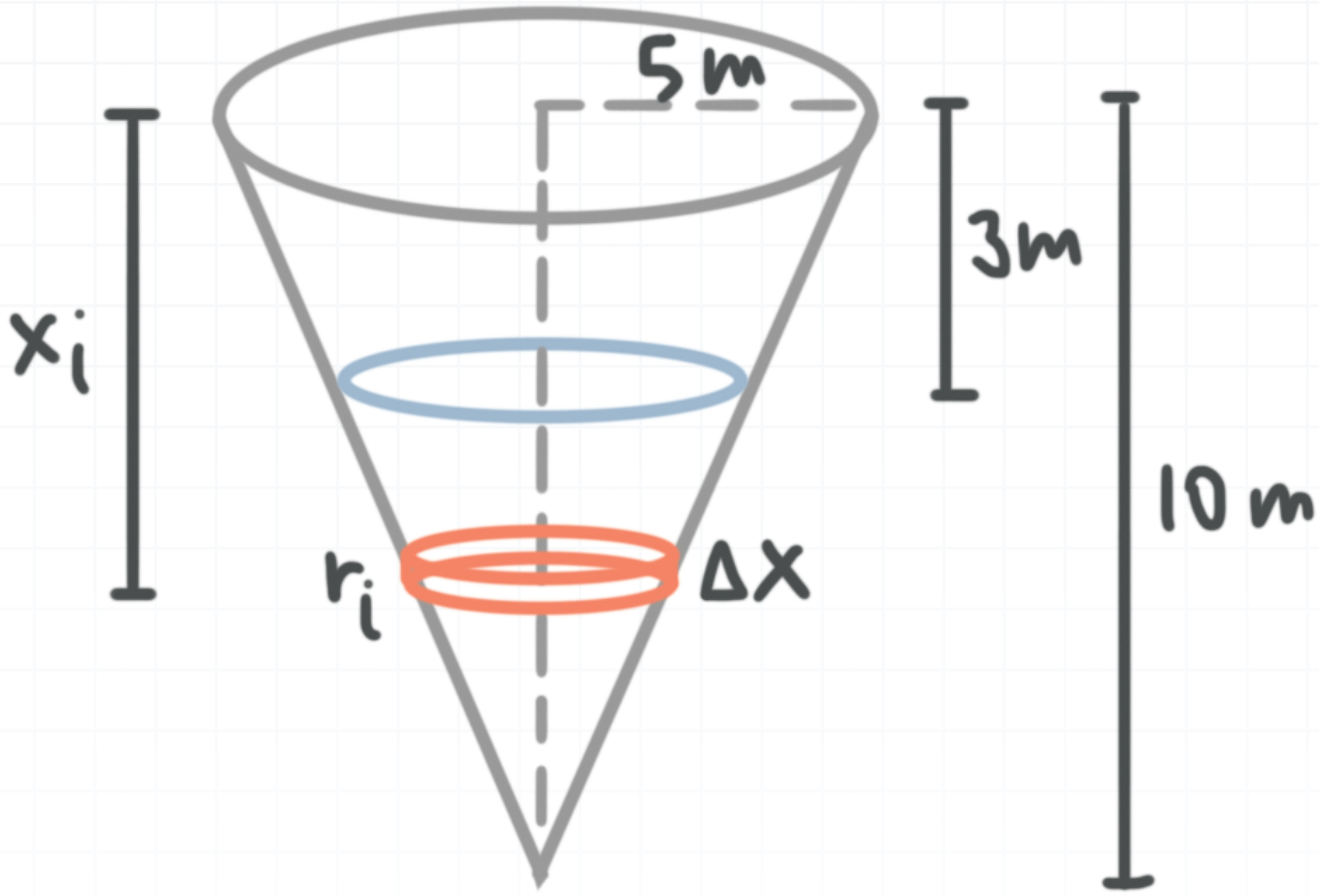
3. Multiply force by distance to get work.

Remember that it's extremely useful to draw a diagram of the tank and the substance inside, labeling everything we know, before we start making calculations.

Example

A tank in the shape of an inverted circular cone has a height of 10 m and base radius of 5 m. It's filled with water to a height of 7 m. Find the work required to empty the tank by pumping the water out through the top. Remember, the density of water is $1{,}000$ kg/m^3.

We'll always start by drawing a diagram first.



If the height of the water is 7 m, then the top 3 m of the tank is empty. We want to divide the water into n slices, figure out the work required to lift one slice, and then add the work for all of the slices together.

One slice has a height of Δx. We can call the radius of this slice r_i. We can also say that the distance from the slice to the top of the tank is x_i.

Following the three steps given above, we need to find density and volume, then use them to calculate mass. We know that the density of water is $\delta = 1{,}000$ kg/m^3. Volume of the circular slice is given by the formula for the volume of a cylinder.

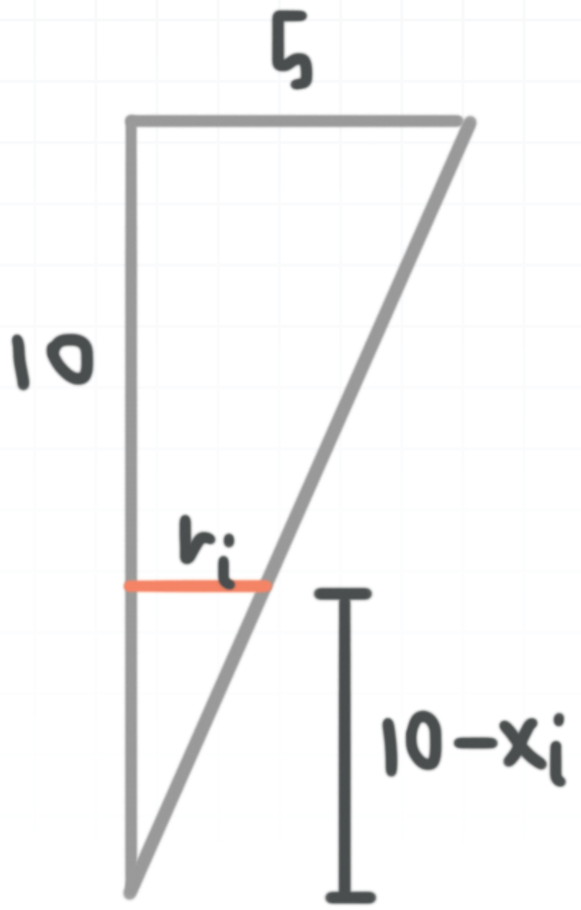
$$V_i = (\text{circular area})(\text{height})$$

$$V_i = \left(\pi r_i^2\right)(\Delta x)$$

$$V_i = \pi r_i^2 \Delta x$$

In order to get a real value for volume, we need to solve for the radius r_i. To solve for the radius we can use the property of similar triangles.

The diagram below shows how a triangular section of the entire tank relates to the the triangular section underneath the slice.



If we know that the height of the tank is 10 m, and if we know (from the last diagram) that the distance between the slice and the top of the tank is x_i, then that means the distance between the slice and the bottom of the tank is $10 - x_i$.

Let's set up the equation for r_i using the property of similar triangles, which tells us that the following ratios are equal:

$$\frac{r_i}{10 - x_i} = \frac{5}{10}$$

$$r_i = \frac{1}{2}(10 - x_i)$$

Next we'll solve for volume using $V_i = \pi r_i^2 \Delta x$ and the value we just found for r_i.

$$V_i = \pi r_i^2 \Delta x$$

$$V_i = \pi \left[\frac{1}{2}(10 - x_i)\right]^2 \Delta x$$

$$V_i = \frac{\pi}{4}(10 - x_i)^2 \Delta x$$

Now that we finally have the volume, we can multiply it by the density to get an equation for mass.

$$m = \delta V$$

$$m_i = (1{,}000)\left[\frac{\pi}{4}(10 - x_i)^2 \Delta x\right]$$

$$m_i = 250\pi(10 - x_i)^2 \Delta x$$

With a value for mass, we can calculate force using $F_i = m_i g$ and $g = 9.8$ m/s^2.

$$F_i = m_i g$$

$$F_i = \left[250\pi(10 - x_i)^2 \Delta x\right](9.8)$$

$$F_i = 2{,}450\pi(10 - x_i)^2 \Delta x$$

With a value for force, we can calculate work using $W_i = F_i d$. Remember that the distance from the slice to the top of the tank is x_i.

$$W_i = F_i d$$

$$W_i = \left[2{,}450\pi(10 - x_i)^2\Delta x\right](x_i)$$

$$W_i = 2{,}450\pi x_i(10 - x_i)^2\Delta x$$

At this point we have an equation for the work required to lift a single slice of the water to the top of the tank to remove it. Our next step is to modify the work equation we just found so that it models the work required to remove *all* the water from the tank (all n slices), not just a single slice.

In order to get the most accurate result, we want to use as many slices as possible, which means we'll take $n \to \infty$. In other words, we'll take the limit as $n \to \infty$ of the work equation that sums up the work required for all the slices, and we get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2{,}450\pi x_i(10 - x_i)^2\Delta x$$

Of course, this is the same thing as taking the integral over the given interval. Since we're now looking at all values of x, we can let $x_i = x$. Since we're changing to integral notation, we also let $\Delta x = dx$. The interval of integration is $[3,10]$ because the top-most slice of water has to be lifted 3 m to be removed from the tank, and the bottom-most slice of water has to be lifted 10 m to be removed from the tank.

$$W = \int_3^{10} 2{,}450\pi x(10 - x)^2\ dx$$

$$W = 2{,}450\pi \int_3^{10} x(10 - x)^2 \ dx$$

$$W = 2{,}450\pi \int_3^{10} 100x - 20x^2 + x^3 \ dx$$

$$W = 2{,}450\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right] \Bigg|_3^{10}$$

$$W = 2{,}450\pi \left[50(10)^2 - \frac{20(10)^3}{3} + \frac{(10)^4}{4} \right] - 2{,}450\pi \left[50(3)^2 - \frac{20(3)^3}{3} + \frac{(3)^4}{4} \right]$$

$$W = 4{,}179{,}802.6$$

$$W = 4.18 \times 10^6$$

The work required to empty the tank is 4.18×10^6 J.

Work done by a variable force

To calculate the work done when a variable force is applied to lift an object of some mass or weight, we'll use the formula

$$W = \int_a^b F(x)\ dx$$

where W is the work done, $F(x)$ is the equation of the variable force, and $[a, b]$ is the starting and ending height of the object.

If W is positive, it means that the force is doing work in the given interval. If W is negative, then work needs to be done on the interval. The answer to these types of work problems is usually given in Joules J.

Example

Find the work done to lift a 20 kg box from the floor to a height of 3 m when the variable force $F(x)$ is given in Newtons.

$$F(x) = 4x^2 - 2x + 3$$

Using the formula from this section, and defining the interval $[a, b]$ as $[0,3]$, we get

$$W = \int_0^3 4x^2 - 2x + 3\ dx$$

$$W=\int_0^3 4x^2\ dx+\int_0^3 -2x\ dx+\int_0^3 3\ dx$$

$$W=4\int_0^3 x^2\ dx-2\int_0^3 x\ dx+3\int_0^3 1\ dx$$

Integrating, we get

$$W=\left[4\left(\frac{x^3}{3}\right)-2\left(\frac{x^2}{2}\right)+3(x)\right]\Bigg|_0^3$$

$$W=\frac{4x^3}{3}-x^2+3x\Bigg|_0^3$$

Now we'll evaluate over the interval.

$$W=\left[\frac{4(3)^3}{3}-(3)^2+3(3)\right]-\left[\frac{4(0)^3}{3}-(0)^2+3(0)\right]$$

$$W=36$$

36 J of force are required to lift a 20 kg box from the floor to a height of 3 m when the variable force applied is defined by $F(x)=4x^2-2x+3$.

Moments and center of mass of the system

The center of mass of a region is the single point where the system is balanced. In other words, if you could take the region into physical space and set it on a pencil point, there's one point in the region where it would balance on that point. Setting it on the pencil at any other point will make the system fall to one side or the another.

With this idea in mind, realize that a perfectly symmetrical region, like a square or a circle, has it's center or mass right in the center. Any region which is not symmetrical (asymmetrical), will have a center of mass closer to the larger part of the region.

When we're looking for the center of mass of a region, we'll first calculate the area bounded by the two curves that define the region, and by the given interval. Remember, if the problem doesn't specify an interval, you'll need to use the points of intersection of the curves by setting them equal to one another and solving for x.

The equation for this area is

[A] $$A = \int_a^b f(x) - g(x)\ dx$$

Next we'll find the moments of the system, which are values that tell us how easily the function can be rotated around the x- and y-axes. We'll calculate the moment of the system in the x direction, and the moment in the y direction using

[B] $M_x=\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2-\left[g(x)\right]^2\right)\ dx$

[C] $M_y=\int_a^b x\left[f(x)-g(x)\right]\ dx$

Once we have area and the moments of the system, we need to use both to calculate the coordinates of the center of mass using

[D] $\bar{x}=\frac{M_y}{A}=\frac{1}{A}\int_a^b x\left[f(x)-g(x)\right]\ dx$

[E] $\bar{y}=\frac{M_x}{A}=\frac{1}{A}\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2-\left[g(x)\right]^2\right)\ dx$

Example

Find the center of mass of the region bounded by the curves over the interval $[0,2]$.

$$y=x^4$$

$$y=0$$

We'll solve for the area of the region using formula [A].

$$A=\int_0^2 x^4-0\ dx$$

$$A = \int_0^2 x^4 \, dx$$

$$A = \frac{1}{5}x^5 \Big|_0^2$$

$$A = \frac{1}{5}(2)^5 - \frac{1}{5}(0)^5$$

$$A = \frac{32}{5}$$

Now we'll find moments of the system. Using formula [B] to find M_x, we get

$$M_x = \int_0^2 \frac{1}{2}\left[\left(x^4\right)^2 - (0)^2\right] \, dx$$

$$M_x = \int_0^2 \frac{1}{2}x^8 \, dx$$

$$M_x = \frac{1}{2}\int_0^2 x^8 \, dx$$

$$M_x = \frac{x^9}{18}\Big|_0^2$$

$$M_x = \frac{2^9}{18} - \frac{0^9}{18}$$

$$M_x = \frac{512}{18}$$

$$M_x = \frac{256}{9}$$

Using formula [C] to find M_y, we get

$$M_y = \int_0^2 x\left(x^4 - 0\right)\ dx$$

$$M_y = \int_0^2 x^5\ dx$$

$$M_y = \frac{x^6}{6}\Bigg|_0^2$$

$$M_y = \frac{(2)^6}{6} - \frac{(0)^6}{6}$$

$$M_y = \frac{64}{6}$$

$$M_y = \frac{32}{3}$$

With area and moments of the system, we can find the coordinates of the center or mass, $(\bar{x}, \bar{y})$.

Using formula [D] to find $\bar{x}$, we get

$$\bar{x} = \frac{M_y}{A}$$

$$\bar{x} = \frac{\frac{32}{3}}{\frac{32}{5}}$$

$$\bar{x} = \left(\frac{32}{3}\right)\left(\frac{5}{32}\right)$$

$$\bar{x} = \frac{5}{3}$$

Using formula [E] to find $\bar{y}$, we get

$$\bar{y} = \frac{M_x}{A}$$

$$\bar{y} = \frac{\frac{256}{9}}{\frac{32}{5}}$$

$$\bar{y} = \left(\frac{256}{9}\right)\left(\frac{5}{32}\right)$$

$$\bar{y} = \frac{40}{9}$$

The center of mass of the region bounded by $y = x^4$ and $y = 0$ on the interval $[0,2]$ is

$$(\bar{x}, \bar{y}) = \left(\frac{5}{3}, \frac{40}{9}\right)$$

Hydrostatic pressure and force

The word "hydrostatic" refers to a liquid at rest. So "hydrostatic force" refers to the force exerted on a solid object by a liquid at rest, and "hydrostatic pressure" refers to the pressure exerted on a solid object by a liquid at rest.

Calculus problems involving hydrostatic pressure and force usually involve calculating the hydrostatic pressure and force that a liquid exerts on the container it's being held in. In these types of problems, your first step will be to solve for hydrostatic pressure using the formula

$$P = \rho g d$$

where P is hydrostatic pressure, ρ is density of the liquid, g is the gravitational constant 9.8m/s^2, and d is the depth of the liquid (not the depth of the container). The units for pressure are Pascals Pa, or equivalently, kg/ms^2.

With a value for pressure, you'll then solve for hydrostatic force exerted by the liquid on the bottom of the container, using the formula

$$F = PA$$

where F is hydrostatic force, P is hydrostatic pressure you found earlier, and A is the square area of the bottom of the container. The units for force are Newtons N, or equivalently, kg m/s^2.

If you need to solve for hydrostatic force on the *end* of your container, instead of on the *bottom*, or on an upright plate inserted into your container, you'll use the modified force equation

$$F = WAd$$

where F is hydrostatic force, W is weight (density $\times$ gravity), A is the area of the vertical surface, and d is the depth of the liquid (not the depth of the container).

Example

A tank is 6 m wide, 12 m long and 4 m deep. It's filled with water of density $1{,}000$ kg/m^3 to a depth of 3 m.

a) Find the hydrostatic pressure at the bottom of the tank.

b) Find the hydrostatic force on the bottom of the tank.

c) Find the hydrostatic force on one end of the tank.

Hydrostatic pressure

To find hydrostatic pressure at the bottom of the tank, we'll use $P = \rho g d$. We're told that $\rho = 1{,}000$ kg/m^3 and that $d = 3$ m. Remember that depth refers to the depth of the liquid, not the depth of the tank (although the two are equal when the tank is completely full). Finally, we know that $g = 9.8$ m/s^2.

$$P = \left(\frac{1{,}000\text{kg}}{\text{m}^3}\right)\left(\frac{9.8\text{m}}{\text{s}^2}\right)(3\text{m})$$

$$P = \frac{29{,}400\text{kg}}{\text{ms}^2}$$

$$P = 2.94 \times 10^4 \text{ kg/ms}^2$$

or

$$P = 2.94 \times 10^4 \text{ Pa}$$

The hydrostatic pressure at the bottom of the tank is $P = 2.94 \times 10^4$ Pa.

Hydrostatic force on the bottom

To find hydrostatic force on the bottom of the tank, we'll use $F = PA$. From the previous part, we know that $P = 29{,}400$ Pa. To calculate area, we'll use the fact that the tank is 6 m wide and 12 m long. Which means the area of the bottom of the tank is

$$A = (6 \text{ m})(12 \text{ m})$$

$$A = 72 \text{ m}^2$$

Next, we can solve for the hydrostatic force at the bottom of the pool.

$$F = \left(\frac{29{,}400 \text{ kg}}{\text{ms}^2}\right)\left(72 \text{ m}^2\right)$$

$$F = \frac{2{,}116{,}800 \text{ kg} \cdot \text{m}}{\text{s}^2}$$

$$F = 2.12 \times 10^6 \text{ kg} \cdot \text{m/s}^2$$

or

$$F = 2.12 \times 10^6 \text{ N}$$

The hydrostatic force at the bottom of the tank is $F = 2.12 \times 10^6$ N.

Hydrostatic force on the end

To find hydrostatic force on one end of the tank, we'll use the modified force equation $F = WAd$.

Since weight is density $\times$ gravity, weight is

$$W = \left(\frac{1{,}000 \text{ kg}}{\text{m}^3}\right)\left(\frac{9.8 \text{ m}}{\text{s}^2}\right)$$

$$W = \left(\frac{9{,}800 \text{ kg}}{\text{m}^2\text{s}^2}\right)$$

Since we're looking for force against a vertical surface and force at deeper depths is greater than force at shallower depths, we can't use the area of the entire surface in our force equation. Instead, we have to divide the surface into small horizontal strips so that we can assume that the force against each strip is roughly the same throughout the strip.

If we divide the end of the tank into tiny slices of equal depth, then each strip is 6 m wide and Δx tall, and sitting at a depth of x_i. The area of one strip is $A_i = 6 \cdot \Delta x$. The force against one strip is

$$F = WAd$$

$$F_i = (9{,}800)(6\Delta x)(x_i)$$

In order to solve for the force against the end of the tank, instead of against a small strip of it, we need to sum together the force against all of the slices, and take the limit as the number of slices approaches infinity, $n \to \infty$. Let's put this all together and see how it looks.

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} (9{,}800)(6\Delta x)(x_i)$$

We need to remember that taking the limit as $n \to \infty$ of the sum of the force against all of the slices is the same as taking the integral of our force equation over the interval of the depth, $[0,3]$. Remember, when we move this into an integral, x_i becomes x, and Δx becomes dx. Let's put this all together and see how it looks.

$$F = \int_0^3 (9{,}800)(6\ dx)x$$

$$F = 58{,}800 \int_0^3 x\ dx$$

$$F = 58{,}800 \left(\frac{x^2}{2} \right) \Bigg|_0^3$$

$$F = 29{,}400x^2 \Big|_0^3$$

$$F = 29{,}400(3)^2 - 29{,}400(0)^2$$

$$F = 264{,}600$$

$$F = 2.65 \times 10^5$$

The hydrostatic force on the end is $F = 2.65 \times 10^5$ N.

Vertical motion

Vertical motion problems are just what they sound like: problems that deal with an object dropped straight down from some height, or thrown straight up in the air so that they go up and then come back down. The motion of the object is vertical.

Problems like these require you to know the relationship between position $x(t)$, velocity $v(t)$, and acceleration $a(t)$. The important thing to know is that the derivative of position is velocity, and the derivative of velocity is acceleration.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

We can also describe the above relationship using integrals instead of derivatives, and we see that the integral of acceleration is velocity, and the integral of velocity is position.

$$a(t)$$

$$\int a(t)\ dt = v(t)$$

$$\int\int a(t)\ dt = \int v(t) = x(t)$$

Example

A pumpkin is thrown straight up into the air from the ground with an initial velocity $v(t_0) = 24$ m/s. The acceleration is due to gravity only, so $a(t) = -9.8$ m/s^2. The velocity of the pumpkin is modeled by $v(t) = at + v_0$.

What is the maximum height the pumpkin reaches, and how much time passes before the pumpkin hits the ground again?

Before we start, we need to figure out what's happening to the pumpkin. Here's what we know: The path of the pumpkin has three important points. The first point occurs when it's initially thrown into the air, where

$t_0 = 0$

$x(t_0) = 0$ or $x(0) = 0$

$v(t_0) = 24$ or $v(0) = 24$

At the start, time and position are both 0 because it's the beginning of the pumpkin's flight and it hasn't moved yet. Initial velocity is given in the question as $v(t_0) = 24$.

The second important point occurs when the pumpkin reaches its maximum height, where we know its velocity is $v(t_1) = 0$, because this is the transition moment when the pumpkin stops traveling up and starts traveling down. If the pumpkin is traveling neither up nor down, then it has no direction, and therefore velocity is 0. We don't know the time t_1 or the position $x(t_1)$. In other words,

$t_1 = ?$

$$x(t_1) = ?$$

$$v(t_1) = 0$$

The first part of the question asks us to solve for $x(t_1)$.

The third important point occurs when the pumpkin hits the ground, where the position will be 0 again, and $s(t_2) = 0$. We don't know the time t_2 or the velocity $v(t_2)$. So

$$t_2 = ?$$

$$s(t_2) = 0$$

$$v(t_2) = ?$$

The second part of the question asks us to solve for t_2.

Now that we've identified what we know, we need to solve for the position equation. The problem tells us that $v(t) = at + v_0$, $a(t) = -9.8$ and $v(t_0) = 24$. Plugging these values into $v(t)$, we get

$$v(t) = -9.8t + 24$$

Now we can integrate $v(t)$ to get $x(t)$.

$$x(t) = \int -9.8t + 24 \ dt$$

$$x(t) = \int -9.8t \ dt + \int 24 \ dt$$

$$x(t) = -9.8\int t\ dt + 24\int 1\ dt$$

$$x(t) = -9.8\left(\frac{t^2}{2}\right) + 24t + C$$

$$x(t) = -4.9t^2 + 24t + C$$

Since we know $x(0) = 0$ we can plug this into the position function we just found in order to solve for C.

$$0 = -4.9(0)^2 + 24(0) + C$$

$$C = 0$$

Therefore, the position function is

$$x(t) = -4.9t^2 + 24t$$

With the position function in hand, we can start working on solving for the maximum height of the pumpkin. We'll start by finding t_1. Since we know that $v(t_1) = 0$ and $v(t) = -9.8t + 24$, we get

$$0 = -9.8t_1 + 24$$

$$-24 = -9.8t_1$$

$$t_1 \approx 2.45$$

Now that we know that $t_1 \approx 2.45$ when the pumpkin reaches maximum height, we can use $x(t) = -4.9t^2 + 24t$ to solve for $x(t_1)$.

$$x(2.45) = -4.9(2.45)^2 + 24(2.45)$$

$$x(2.45) \approx 29.39$$

The maximum height the pumpkin reaches is 29.39 m, and that answers the first part of the question.

To find out how much time passes before the pumpkin hits the ground, we have to remember that $x(t_2) = 0$ and $x(t) = -4.9t^2 + 24t$. We can plug in this information and solve for t_2.

$$0 = -4.9t_2^2 + 24t_2$$

$$0 = -4.9t(t_2 - 4.9)$$

$$t_2 = 0 \text{ or } t_2 = 4.9$$

We know $t_0 = 0$ corresponds to the pumpkin's initial position, which means $t_2 = 4.9$ has to correspond to the pumpkin's final position.

The amount of time that passes before the pumpkin hits the ground again is 4.9 s, which answers the second part of the question.

Rectilinear motion

Rectilinear motion problems deal with an object that moves laterally, or horizontally. The object can be moving along the ground or at any other height, as long as it's moving horizontally. We call this type of motion "rectilinear" motion.

Problems like these require you to know the relationship between position $x(t)$, velocity $v(t)$, and acceleration $a(t)$. The important thing to know is that the derivative of position is velocity, and the derivative of velocity is acceleration.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

We can also describe the above relationship using integrals instead of derivatives, and we see that the integral of acceleration is velocity, and the integral of velocity is position.

$$a(t)$$

$$\int a(t)\ dt = v(t)$$

$$\int \int a(t)\ dt = \int v(t) = x(t)$$

Example

An object is moving along the ground. Its acceleration is $a(t) = 3t + 5$, its velocity at time $t = 4$ is $v(4) = 6$, and its position at $t = 5$ is $x(5) = 25$. Find the equation for position that describes this object's motion.

We can integrate the acceleration function to get a velocity function, $v(t)$.

$$v(t) = \int a(t)\ dt$$

$$v(t) = \int 3t + 5\ dt$$

$$v(t) = \int 3t\ dt + \int 5\ dt$$

$$v(t) = 3\int t\ dt + 5\int 1\ dt$$

$$v(t) = \frac{3t^2}{2} + 5t + C$$

Now we need to solve for C using $v(4) = 6$.

$$6 = \frac{3(4)^2}{2} + 5(4) + C$$

$$C = -38$$

So the equation for velocity, $v(t)$, is

$$v(t) = \frac{3t^2}{2} + 5t - 38$$

Now we can integrate the velocity function to get the position function.

$$x(t) = \int v(t)\ dt$$

$$x(t) = \int \frac{3t^2}{2} + 5t - 38\ dt$$

$$x(t) = \int \frac{3t^2}{2}\ dt + \int 5t\ dt + \int -38\ dt$$

$$x(t) = \frac{3}{2}\int t^2\ dt + 5\int t\ dt - 38\int 1\ dt$$

$$x(t) = \frac{3}{2}\left(\frac{t^3}{3}\right) + 5\left(\frac{t^2}{2}\right) - 38t + C$$

$$x(t) = \frac{t^3}{2} + \frac{5t^2}{2} - 38t + C$$

Now we need to solve for C using $x(5) = 25$.

$$25 = \frac{(5)^3}{2} + \frac{5(5)^2}{2} - 38(5) + C$$

$$C = 90$$

So the equation for position, $x(t)$, is

$$x(t) = \frac{t^3}{2} + \frac{5t^2}{2} - 38t + 90$$

Centroids of plane regions

The centroid of a plane region is the region's exact center point. If we imagine the plane region as a flat sheet of paper, and we attached a string to its centroid, the paper would hang perfectly flat from the string. In other words, the centroid of a plane region is like the region's balancing point.

To find the centroid of a region over the interval $[a, b]$, we have to start by calculating the area of the region. If the region is defined above by $f(x)$ and below by $g(x)$, over the interval $[a, b]$, then the area of the region is given by

$$A = \int_a^b f(x) - g(x)\ dx$$

Keep in mind that, if only one curve is given, then it's likely implied that $g(x) = 0$. Once we've found the area of the plane region, we can find the coordinates coordinates of the centroid $(\bar{x}, \bar{y})$ as

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\ dx$$

Let's work through an example where we find the centroid of a rectangular region.

Example

Find the centroid of the region bounded by the curves.

$x = 1$ and $x = 6$

$y = 0$ and $y = 4$

We know $[a, b] = [1,6]$, and because $y = 4$ is above $y = 0$, we'll say $f(x) = 4$ and $g(x) = 0$. Then the area of the plane region will be

$$A = \int_1^6 4 - 0\ dx$$

$$A = 4\int_1^6 dx$$

$$A = 4x\Big|_1^6$$

$$A = 4(6) - 4(1)$$

$$A = 20$$

Then the coordinates of the centroid will be

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{x} = \frac{1}{20}\int_1^6 x(4 - 0)\ dx$$

$$\bar{x} = \frac{1}{5}\int_1^6 x\ dx$$

$$\bar{x}=\frac{1}{5}\left(\frac{x^2}{2}\right)\Big|_1^6$$

$$\bar{x}=\frac{x^2}{10}\Big|_1^6$$

$$\bar{x}=\frac{6^2}{10}-\frac{1^2}{10}$$

$$\bar{x}=\frac{35}{10}$$

$$\bar{x}=\frac{7}{2}$$

and

$$\bar{y}=\frac{1}{A}\int_a^b\frac{1}{2}[(f(x))^2-(g(x))^2]\ dx$$

$$\bar{y}=\frac{1}{20}\int_1^6\frac{1}{2}(4^2-0^2)\ dx$$

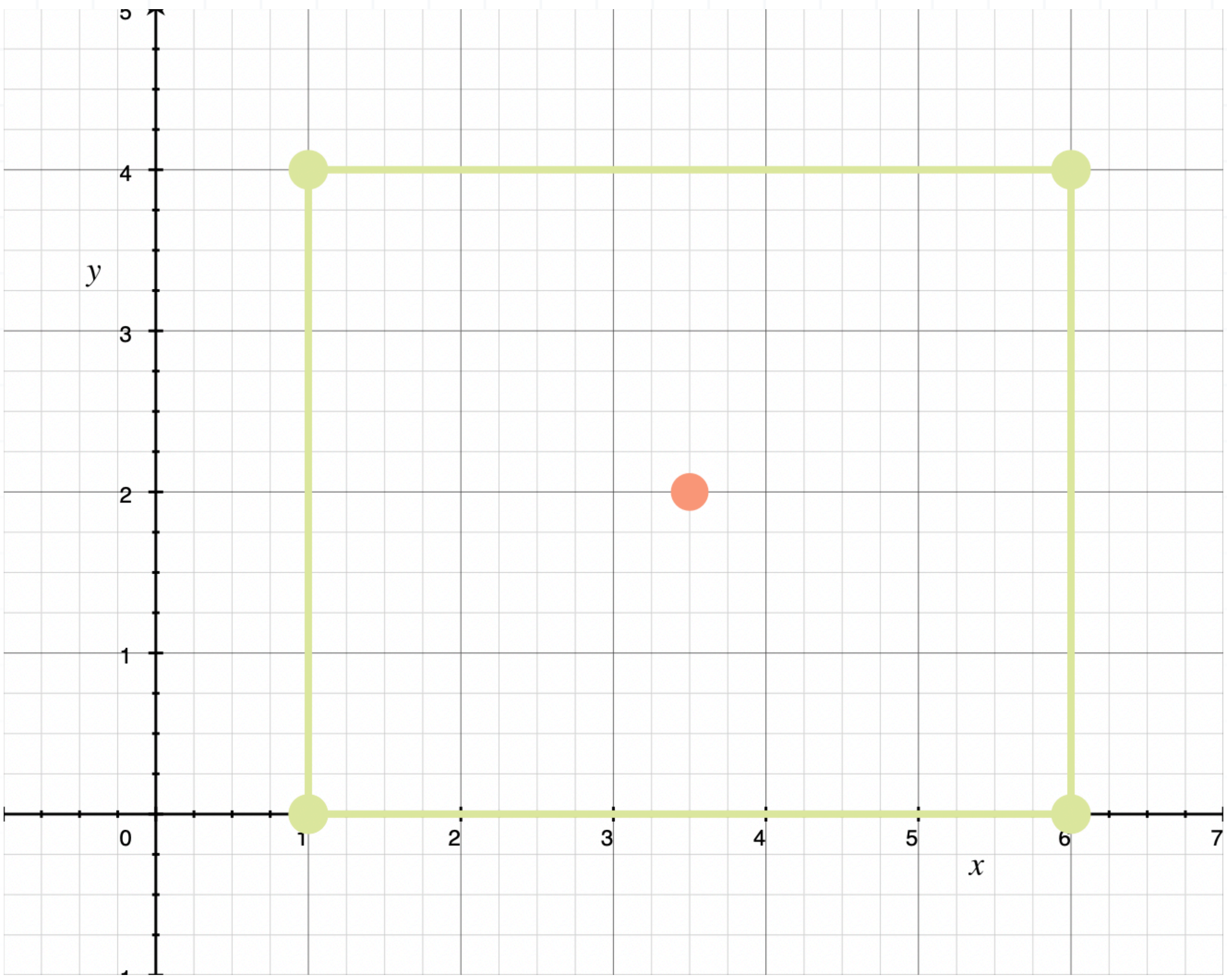
$$\bar{y}=\frac{2}{5}\int_1^6 dx$$

$$\bar{y}=\frac{2x}{5}\Big|_1^6$$

$$\bar{y}=\frac{2(6)}{5}-\frac{2(1)}{5}$$

$$\bar{y}=2$$

So the centroid of the region is at $(7/2,2)$, which we can confirm visually by graphing the region and the centroid that we found.



Economics

When we study present and future value in calculus, usually we're trying to calculate the amount a sum of money will be worth in the *future* after it's had time to grow and earn interest, or we're trying to calculate how much money we had in the *past* given the sum of money in the account today.

The present and future value formulas we use will vary depending on the rate at which interest is compounded, and whether we're calculating the value of a single deposit, or a continuous income stream. Use the table below to determine which formula to use.

Compounding	**Future value**	**Present value**
n times annually, single deposit	$FV=PV\left(1+\frac{r}{n}\right)^{nt}$	$PV=\frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$

FV is the future value

PV is the present value

r is the yearly rate

n is the number of times compounded annually

t is the number of years

Continuously, single deposit	$FV=PVe^{rt}$	$PV=\frac{FV}{e^{rt}}$

FV is the future value

PV is the present value

r is the yearly rate

t is the number of years

Continuously, continuous stream $FV = \int_0^T S(t)e^{r(T-t)}\ dt$ $\quad PV = \int_0^T S(t)e^{-rt}\ dt$

FV is the future value

$S(t)$ is the continuous income stream

r is the yearly rate in decimal form

T is the number of years elapsed

t is the variable we'll solve for

Let's do an example in which interest is compounded n times annually for a single deposit.

Example

Find the value of a $\$3{,}000$ investment after 3 years, if the interest rate is 3% and interest is compounded every 3 months (4 times per year).

Here's what we know.

$$PV = 3{,}000$$

$$r = 0.03$$

$$n = 4$$

$$t = 3$$

Plugging these into the future value formula for interest compounded n times per year for a single deposit, we get

$$FV = 3{,}000\left(1 + \frac{0.03}{4}\right)^{(4)(3)}$$

$$FV = 3{,}281.42$$

The value of the account after 3 years is $\$3{,}281.42$.

Let's try an example in which interest is compounded continuously for a single deposit.

Example

Find the value after 5 years of an investment that's worth $\$1{,}500$ right now, if the interest rate is $6\,\%$ compounded continuously.

Here's what we know.

$$PV = 1{,}500$$

$$r = 0.06$$

$$t = 5$$

Plugging these into the future value equation for interest compounded continuously for a single deposit, we get

$$FV = 1{,}500e^{(0.06)(5)}$$

$$FV = 2{,}024.79$$

The value of the account after 5 years is $\$2{,}024.79$.

Now let's do an example where interest is compounded continuously for a continuous income stream.

Example

Find the future value after 3 years of an account that has $\$2{,}000$ added to it annually, if the interest rate is $10\,\%$ compounded continuously. Assume that no money is withdrawn from the account during these 3 years, and that no money is added to the account other than the $\$2{,}000$ annual deposit.

Here's what we know.

$$S(t) = 2{,}000$$

$$r = 0.10$$

$$T = 3$$

Plugging these into the future value equation for interest compounded continuously for a continuous income stream, we get

$$FV = \int_0^3 2{,}000e^{0.10(3-t)}\ dt$$

$$FV = \int_0^3 2{,}000e^{0.30-0.10t}\ dt$$

$$FV = \int_0^3 2{,}000e^{0.30}e^{-0.10t}\ dt$$

$$FV = 2{,}000e^{0.30}\int_0^3 e^{-0.10t}\ dt$$

$$FV = \left(2{,}000e^{0.30}\right)\left(\frac{e^{-0.10t}}{-0.10}\right)\Bigg|_0^3$$

$$FV = \left(-20{,}000e^{0.30}\right)\left(e^{-0.10t}\right)\Big|_0^3$$

$$FV = \left(-20{,}000e^{0.30}\right)\left[e^{-0.10(3)} - e^{-0.10(0)}\right]$$

$$FV = 7{,}020.00$$

The value of the account after 3 years is $\$7{,}020.00$.

We'll do one last example for compounding interest n times annually for a continuous income stream.

Example

You deposit $\$10{,}000$ every year for 5 years into a new bank account. Interest on the account is compounded continuously at $8\,\%$. What is the present value?

Here's what we know.

$$S(t) = 10{,}000$$

$$r = 0.08$$

$$T = 5$$

Plugging these into the present value equation for interest compounded n times annually for a continuous income stream, we get

$$PV = \int_0^5 10{,}000e^{-0.08t}\ dt$$

$$PV = 10{,}000\int_0^5 e^{-0.08t}\ dt$$

$$PV = 10{,}000\left(\frac{e^{-0.08t}}{-0.08}\right)\Bigg|_0^5$$

$$PV = -125{,}000\left(e^{-0.08t}\right)\Big|_0^5$$

$$PV = -125{,}000\left[e^{-0.08(5)} - e^{-0.08(0)}\right]$$

$$PV = 41{,}209.99$$

The value of the account today, assuming you make the $\$10{,}000$ annual deposits for 5 years and get the interest rate you've been promised, is $\$41{,}209.99$.

Consumer and producer surplus

Consumer and producer surplus are values that a company can calculate to see when they have excess demand or production. If a company can better balance demand and production, they can be more profitable.

Consumer surplus is calculated using

$$CS = \int_0^{q_e} D(q)\ dq - p_e q_e$$

where CS is consumer surplus, $D(q)$ is the demand curve, p_e is the equilibrium price and q_e is the equilibrium quantity.

Producer surplus is calculated using

$$PS = p_e q_e - \int_0^{q_e} S(q)\ dq$$

where PS is producer surplus, $S(q)$ is the supply curve, p_e is the equilibrium price and q_e is the equilibrium quantity.

Example

Find equilibrium quantity and price, and then consumer and producer surplus.

$$D(q) = -0.25q + 13$$

$$S(q) = 0.05q^2 - 2$$

In order to find the equilibrium quantity, we need to remember that our system will achieve equilibrium when supply equals demand. In other words, if we set the supply curve equal to the demand curve, the resulting q value will be the equilibrium quantity q_e.

$$-0.25q + 13 = 0.05q^2 - 2$$

$$0 = 0.05q^2 + 0.25q - 15$$

$$0 = 5q^2 + 25q - 1500$$

$$0 = (5q + 100)(q - 15)$$

Setting each factor equal to 0 separately, we get

$$5q + 100 = 0$$

$$q = -20$$

or

$$q - 15 = 0$$

$$q = 15$$

Since the equilibrium quantity must be positive, $q_e = 15$ is the equilibrium quantity for the given demand and supply curves $D(q) = -0.25q + 13$ and $S(q) = 0.05q^2 - 2$.

Now we can solve for the equilibrium price p_e. We can find the equilibrium price by plugging equilibrium quantity into either the demand or supply curve (they will both give us the same answer). Let's use the supply curve $S(q) = 0.05q^2 - 2$.

$$S(15) = 0.05(15)^2 - 2$$

$$S(15) = 9.25$$

The equilibrium price p_e for the demand curve $D(q) = -0.25q + 13$ and the supply curve $S(q) = 0.05q^2 - 2$ is $p_e = 9.25$, and that's the answer to the first part of the question.

To solve for consumer surplus, we'll plug the demand curve, plus the equilibrium price and quantity into the consumer surplus formula, and get

$$CS = \int_0^{15} -0.25q + 13\ dq - (9.25)(15)$$

$$CS = \int_0^{15} -0.25q + 13\ dq - 138.75$$

$$CS = \left(\frac{-0.25q^2}{2} + 13q\right)\Big|_0^{15} - 138.75$$

$$CS = \left(-0.125q^2 + 13q\right)\Big|_0^{15} - 138.75$$

$$CS = -0.125(15)^2 + 13(15) - \left[-0.125(0)^2 + 13(0)\right] - 138.75$$

$$CS = 28.125$$

The consumer surplus is 28.125.

Now we can solve for the producer surplus by plugging the supply curve and the equilibrium price and quantity into the producer surplus equation.

$$PS=(9.25)(15)-\int_0^{15}0.05q^2-2\ dq$$

$$PS=138.75-\int_0^{15}0.05q^2-2\ dq$$

$$PS=138.75-\left(\frac{0.05q^3}{3}-2q\right)\Big|_0^{15}$$

$$PS=138.75-\left(0.017q^3-2q\right)\Big|_0^{15}$$

$$PS=138.75-\left[0.017(15)^3-2(15)-\left[0.017(0)^3-2(0)\right]\right]$$

$$PS=111.375$$

The producer surplus is 111.375.

Probability density functions

Probability density refers to the probability that a continuous random variable X will exist within a set of conditions. It follows that using the probability density equations will tell us the likelihood of an X existing in the interval $[a, b]$.

A probability density function $f(x)$ must meet these conditions:

1. $f(x) \geq 0$ for all values of x

2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$

The equation for probability density is

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\ dx$$

where $P(a \leq X \leq b)$ is the probability that X exists in $[a, b]$.

Example

Show that $f(x)$ is a probability density function and find $P(1 \leq X \leq 4)$.

$$f(x) = \left(\frac{x^3}{5{,}000}\right)(10 - x)$$

for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x

The first thing we need to do is show that $f(x)$ is a probability density function. We can see that the interval $0 \le x \le 10$ is positive. For all other possibilities we know that $f(x) = 0$. This means we've satisfied the first criteria for a probability density equation. Now we need to verify that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

We can set the interval to $[0,10]$ since it's only in this interval that the equation doesn't equal 0.

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \left(\frac{x^3}{5{,}000}\right)(10 - x)\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \frac{x^3}{500} - \frac{x^4}{5{,}000}\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{0}^{10} \frac{x^3}{500}\ dx + \int_{0}^{10} -\frac{x^4}{5{,}000}\ dx$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \frac{x^4}{2{,}000} - \frac{x^5}{25{,}000}\Bigg|_{0}^{10}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \left[\frac{(10)^4}{2{,}000} - \frac{(10)^5}{25{,}000}\right] - \left[\frac{(0)^4}{2{,}000} - \frac{(0)^5}{25{,}000}\right]$$

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

The equation has met both of the criteria, so we've verified that it's a probability density function.

In order to solve for $P(1 \leq X \leq 4)$, we'll identify the interval $[1,4]$ and plug it into the probability density equation.

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

$$P(1 \leq X \leq 4) = \int_1^4 \left(\frac{x^3}{5{,}000}\right)(10 - x)\ dx$$

$$P(1 \leq X \leq 4) = \int_1^4 \frac{x^3}{500} - \frac{x^4}{5{,}000}\ dx$$

$$P(1 \leq X \leq 4) = \int_1^4 \frac{x^3}{500}\ dx + \int_0^{10} -\frac{x^4}{5{,}000}\ dx$$

$$P(1 \leq X \leq 4) = \frac{x^4}{2{,}000} - \frac{x^5}{25{,}000}\Bigg|_1^4$$

$$P(1 \leq X \leq 4) = \left[\frac{(4)^4}{2{,}000} - \frac{(4)^5}{25{,}000}\right] - \left[\frac{(1)^4}{2{,}000} - \frac{(1)^5}{25{,}000}\right]$$

$$P(1 \leq X \leq 4) = 0.0866$$

The answer tell us that the probability of X existing between 1 and 4 is about $8.66\,\%$.

Theorem of Pappus

The Theorem of Pappus tells us that the volume of a three-dimensional solid object that's created by rotating a two-dimensional shape around an axis is given by

$$V = Ad$$

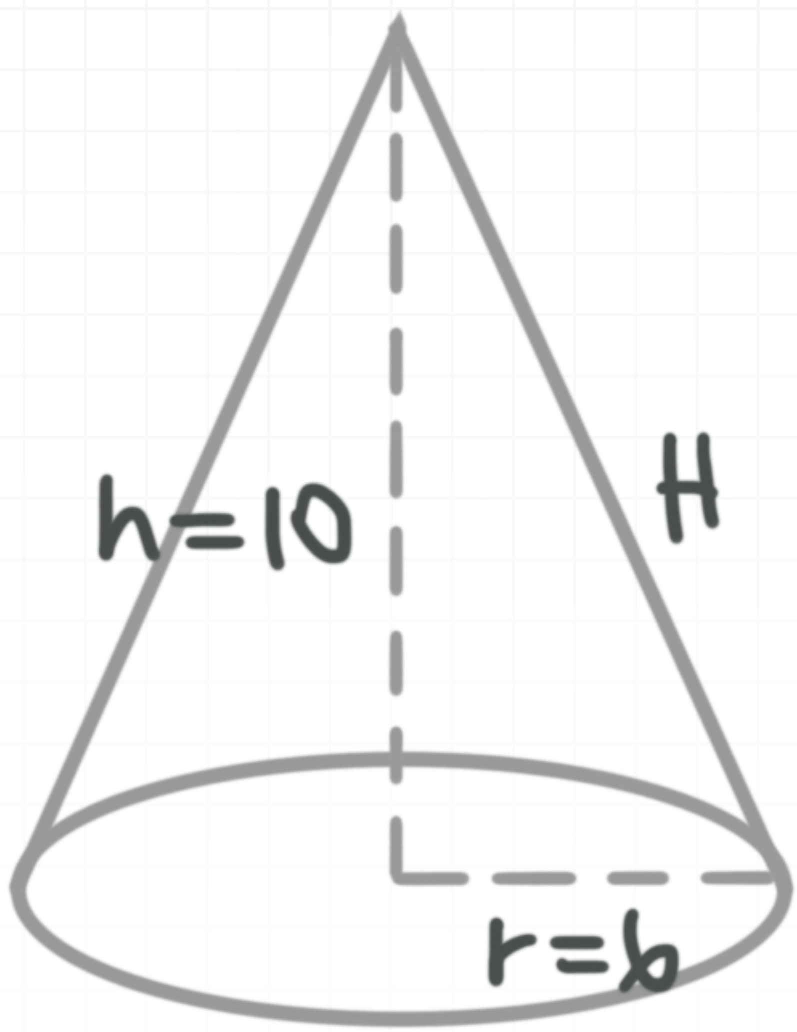
where V is the volume of the three-dimensional object, A is the area of the two-dimensional figure being revolved, and d is the distance traveled by the centroid of the two-dimensional figure.

Example

Use the Theorem of Pappus to find the volume of a right circular cone with radius $r = 6$ and height $h = 10$.

The Theorem of Pappus defines volume as $V = Ad$. Before we can solve for volume we need to find the area of the triangle we're revolving. Our shape, the right circular cone, can be described as a triangle rotated around an axis. The formula for area of a triangle is

$$A = \frac{1}{2}bh$$



The base of the triangle will be the radius $r = b = 6$, and the height of the triangle will be $h = 10$.

$$A = \frac{1}{2}(6)(10)$$

$$A = 30$$

Next, we need to solve for distance, d. Distance will involve the relationship of the triangle's centroid and the rotation it experiences. In other words, $d = 2\pi\bar{x}$ where $\bar{x}$ is the x-coordinate of the centroid and 2π refers to the fact that the object is being rotated around an axis. The equation for $\bar{x}$ is

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

Looking at this equation we realize we're still missing $f(x)$, which is the third side of the triangle, H. If we position the center of the base of the cone at the origin $(0,0)$, then the right edge of the base of the cone sits at $(6,0)$, and the point at the top of the cone sits at $(0,10)$.

Therefore, the equation that models the hypotenuse H is the equation of the line passing through $(0,10)$ and $(6,0)$. The slope of that line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 0}{0 - 6} = \frac{10}{-6} = -\frac{5}{3}$$

Then the equation of the line modeling the hypotenuse is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{5}{3}(x - 6)$$

$$y = -\frac{5}{3}x + 10$$

Now we can solve for $\bar{x}$.

$$\bar{x} = \frac{1}{30}\int_0^6 x\left(-\frac{5}{3}x + 10\right)\ dx$$

$$\bar{x} = -\frac{1}{30}\int_0^6 \frac{5}{3}x^2 - 10x\ dx$$

$$\bar{x} = -\frac{1}{30}\left(\frac{5}{9}x^3 - 5x^2\right)\Big|_0^6$$

$$\bar{x} = \frac{1}{6}x^2 - \frac{1}{54}x^3 \Big|_0^6$$

$$\bar{x} = \frac{1}{6}(6)^2 - \frac{1}{54}(6)^3 - \left(\frac{1}{6}(0)^2 - \frac{1}{54}(0)^3\right)$$

$$\bar{x} = 6 - 4$$

$$\bar{x} = 2$$

Now we can solve for distance $d = 2\pi\bar{x}$.

$$d = 2\pi(2)$$

$$d = 4\pi$$

Finally, we can solve for volume using $V = Ad$.

$$V = 30(4\pi)$$

$$V = 120\pi$$

Eliminating the parameter

Given a parametric curve where our function is defined by two equations, one for x and one for y, and both of them in terms of a parameter t,

$$x = f(t)$$

$$y = g(t)$$

we can eliminate the parameter value in a few different ways. We can

1. Solve each equation for the parameter t, then set the equations equal to one another, or

2. Solve one equation for the parameter t, then plug that value into the second equation, or

3. Solve each equation for part of an identity, then plug both values into the identity.

Let's try an example using the second method, where we eliminate the parameter by solving for t in one of our functions, and then plugging the value we find into the other function.

Example

Eliminate the parameter.

$$x = 2t^2 + 6$$

$$y = 5t$$

We'll solve $y = 5t$ for t, since this will be easier than solving $x = 2t^2 + 6$ for t.

$$y = 5t$$

$$t = \frac{y}{5}$$

Plugging this into the equation for x, we get

$$x = 2\left(\frac{y}{5}\right)^2 + 6$$

$$x = \frac{2y^2}{25} + 6$$

Removing the fraction, we get

$$25x = 2y^2 + 150$$

$$25x - 2y^2 = 150$$

Now let's try an example using the third method, where we solve each equation for part of an identity, and then plug both values into the identity.

Example

Eliminate the parameter.

$$x = e^t$$

$$y = e^{4t}$$

We know that $y = e^{ab}$ is the same as $y = (e^a)^b$. If we use this property, we can take $y = e^{4t}$ and rewrite it as $y = (e^t)^4$. Since $x = e^t$, we can substitute x into $y = (e^t)^4$ for e^t.

$$y = x^4$$

And because we have e^t in the original parametric equations, and $e^t > 0$ for all t, that requires that $x > 0$, and we have to transfer this condition to our final answer.

$$y = x^4, \text{ where } x > 0$$

Let's try another example using the third method.

Example

Eliminate the parameter.

$$x = 2\cos\theta$$

$$y = 3\sin\theta$$

$$0 \leq \theta \leq 2\pi$$

Rearranging $x = 2\cos\theta$ and $y = 3\sin\theta$ to isolate the trigonometric functions, we get

$$x = 2\cos\theta$$

$$\cos\theta = \frac{x}{2}$$

and

$$y = 3\sin\theta$$

$$\sin\theta = \frac{y}{3}$$

Since we know that $\sin^2\theta + \cos^2\theta = 1$, we can substitute the values we just found for $\cos\theta$ and $\sin\theta$.

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$

$$\frac{y^2}{9} + \frac{x^2}{4} = 1$$

$$y^2 + \frac{9x^2}{4} = 9$$

$$4y^2 + 9x^2 = 36$$

$$9x^2 + 4y^2 = 36$$

Derivative of a parametric curve

Given a parametric curve where our function is defined by two equations, one for x and one for y, and both of them in terms of a parameter t,

$$x = f(t)$$

$$y = g(t)$$

we calculate the derivative of the parametric curve using the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

where dy/dx is the first derivative of the parametric curve, dx/dt is the derivative of $x = f(t)$ and dy/dt is the derivative of $y = g(t)$.

Example

Find the derivative of the parametric curve.

$$x = 3t^4 - 6$$

$$y = 2e^{4t}$$

We'll start by finding dy/dt and dx/dt.

$$y = 2e^{4t}$$

$$\frac{dy}{dt} = 8e^{4t}$$

and

$$x = 3t^4 - 6$$

$$\frac{dx}{dt} = 12t^3$$

Plugging these into the derivative formula for dy/dx, we get

$$\frac{dy}{dx} = \frac{8e^{4t}}{12t^3}$$

$$\frac{dy}{dx} = \frac{2e^{4t}}{3t^3}$$

Second derivative of a parametric curve

To find the second derivative of a parametric curve, we need to find its first derivative dy/dx, using the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

and then plug it into this formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

where d^2y/dx^2 is the second derivative of the parametric curve, dy/dx is its first derivative and dx/dt is the first derivative of the equation for x. The d/dt is notation that tells us to take the derivative of dy/dx with respect to t.

Example

Find the second derivative of the parametric curve.

$$x = 5t^3 + 6t$$

$$y = t^4 - 3$$

To find the first derivative, we'll solve for dx/dt and dy/dt.

This means we will have to solve for dy/dt, dx/dt and dy/dx first. Let's start with dy/dt.

$$x = 5t^3 + 6t$$

$$\frac{dx}{dt} = 15t^2 + 6$$

and

$$y = t^4 - 3$$

$$\frac{dy}{dt} = 4t^3$$

Plugging these two derivatives into the formula for the first derivative,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

we get

$$\frac{dy}{dx} = \frac{4t^3}{15t^2 + 6}$$

Now plugging the first derivative dy/dx and the value we found earlier for dx/dt into the formula for the second derivative,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{4t^3}{15t^2+6}\right)}{15t^2+6}$$

We'll use quotient rule to take the derivative of dy/dx with respect to t.

$$\frac{d^2y}{dx^2} = \frac{\frac{(12t^2)(15t^2+6)-(4t^3)(30t)}{(15t^2+6)^2}}{15t^2+6}$$

$$\frac{d^2y}{dx^2} = \frac{(12t^2)(15t^2+6)-(4t^3)(30t)}{(15t^2+6)^2} \cdot \frac{1}{15t^2+6}$$

$$\frac{d^2y}{dx^2} = \frac{180t^4+72t^2-120t^4}{(15t^2+6)^3}$$

$$\frac{d^2y}{dx^2} = \frac{60t^4+72t^2}{(15t^2+6)^3}$$

$$\frac{d^2y}{dx^2} = \frac{12t^2(5t^2+6)}{(15t^2+6)^3}$$

Sketching parametric curves by plotting points

To sketch a parametric curve, we'll follow these steps:

1. Create a table where we find x- and y-values based on specific parameter values of t.

2. Eliminate the parameter to find a cartesian equation in terms of just x and y.

3. Sketch the parametric curve.

Let's walk through an example, so we can see these steps in action

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Example

Sketch the parametric curve.

$$y = \sin t$$

$$x = \cos t$$

$$0 \leq t \leq 2\pi$$

Let's create a table of x- and y-values based on parameter values of t inside the given interval. Since the interval is given as $0 \leq t \leq 2\pi$, we'll choose well-known parameter values inside this interval so that they're easy to plug into our equations for x and y.

	x	y
$t_1 = 0$	1	0
$t_2 = \frac{\pi}{2}$	0	1
$t_3 = \pi$	-1	0
$t_4 = \frac{3\pi}{2}$	0	-1
$t_5 = 2\pi$	1	0

Now we'll eliminate the parameter to find a cartesian equation that represents our parametric equation. If we remember that $\sin^2 t + \cos^2 t = 1$, we can just square our parametric equations,

$$y = \sin t$$

$$y^2 = \sin^2 t$$

and

$$x = \cos t$$

$$x^2 = \cos^2 t$$

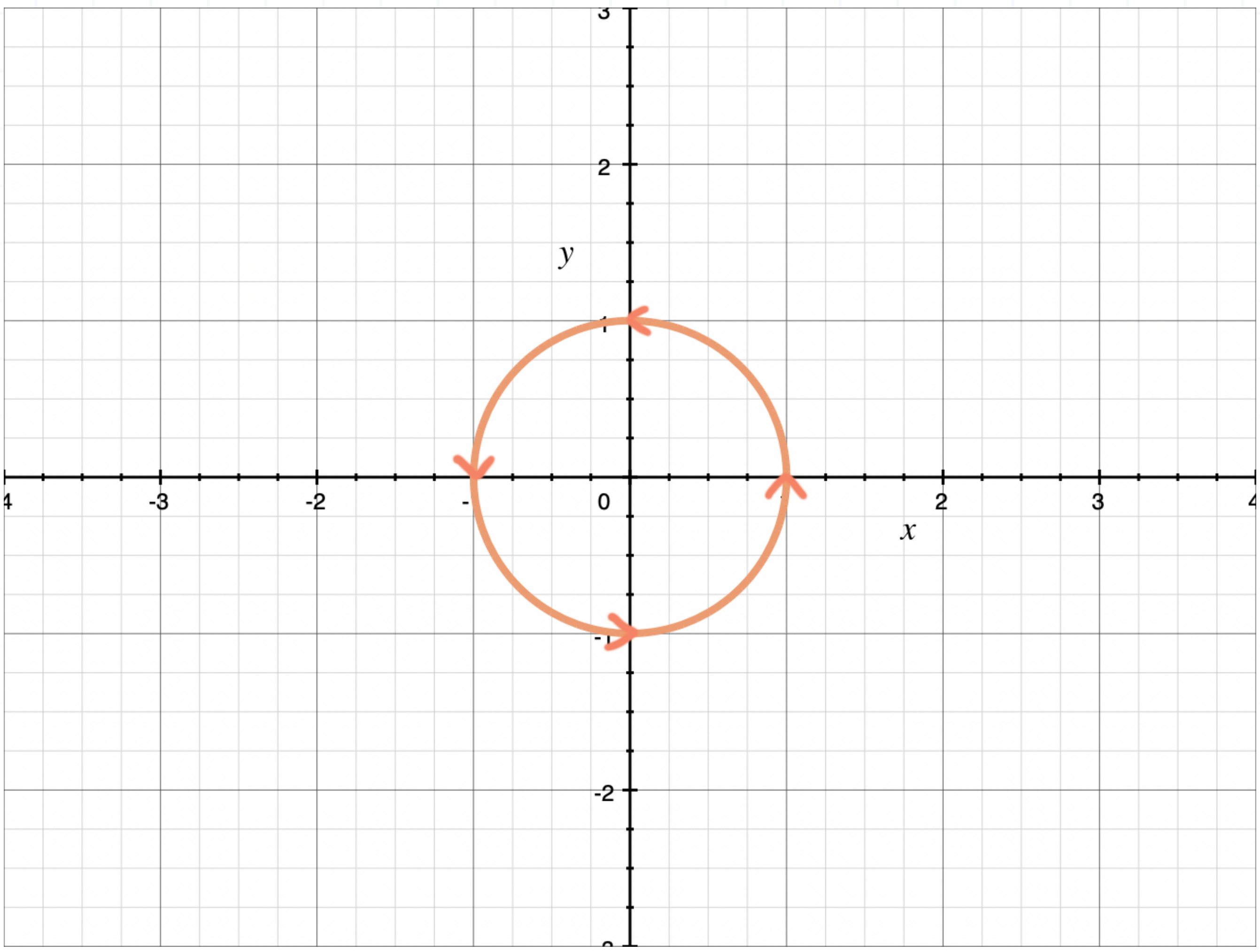
and then plug them into the identity.

$$\sin^2 t + \cos^2 t = 1$$

$$y^2 + x^2 = 1$$

$$x^2 + y^2 = 1$$

Since we have a list of points to plot and we know our cartesian equation, we can sketch the parametric curve. When we plot the points following the direction of the parameter t, we'll see that the parameter is moving counter-clockwise around the circle.



Tangent line to the parametric curve

We'll use the same point-slope formula to define the equation of the tangent line to the parametric curve that we used to define the tangent line to a cartesian curve, which is

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is the point where the tangent line intersects the curve.

To find the slope m, we'll use the formula for the derivative of a parametric curve.

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Once we find the derivative of the parametric curve using this formula, we'll plug the given point into the derivative to find the slope at that particular point.

Then we'll plug the slope and the given point into the point-slope formula for the equation of a line and simplify to get our tangent line equation.

Example

Find the tangent line(s) to the parametric curve.

$$x = t^5 - 4t^3$$

$$y = t^2$$

at $(0,4)$

To find the derivative of the parametric curve, we'll first need to calculate dy/dt and dx/dt.

$$x = t^5 - 4t^3$$

$$\frac{dx}{dt} = 5t^4 - 12t^2$$

and

$$y = t^2$$

$$\frac{dy}{dt} = 2t$$

Plugging these into the derivative formula, we get

$$\frac{dy}{dx} = \frac{2t}{5t^4 - 12t^2}$$

We need to plug the given point into the derivative we just found, but the given point is a cartesian point, and we only have t in our derivative equation. Therefore, in order to plug the given point into the derivative, we need to convert it from a cartesian point into a parameter value for t. To do this, we'll plug the given point into both of the original parametric equations, and look for matching solutions.

$$x = t^5 - 4t^3$$

$$0 = t^5 - 4t^3$$

$$0 = t^3\left(t^2 - 4\right)$$

$$t = 0, \ \ \pm 2$$

and

$$y = t^2$$

$$4 = t^2$$

$$t = \pm 2$$

In order for the parameter value to be valid, it has to be a solution in both equations, which means the parameter value we're interested in are $t = \pm 2$. Since we got two solutions, we're going to have two tangent lines.

To solve for the slope of each tangent line, we'll plug $t = \pm 2$ into the derivative equation we found above.

For $t = 2$:

$$f(2) = \frac{2(2)}{5(2)^4 - 12(2)^2}$$

$$f(2) = \frac{4}{80 - 48}$$

$$f(2) = \frac{1}{8}$$

For $t = -2$:

$$f(-2) = \frac{2(-2)}{5(-2)^4 - 12(-2)^2}$$

$$f(-2) = \frac{-4}{80 - 48}$$

$$f(-2) = -\frac{1}{8}$$

Finally, plugging the slopes we found and the given point $(0,4)$ into the point-slope formula for the equation of a line, we get the following two tangent lines.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{8}(x - 0)$$

$$y = \frac{1}{8}x + 4$$

and

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{8}(x - 0)$$

$$y = -\frac{1}{8}x + 4$$

The equations can also be manipulated into this form:

$$8y = x + 32$$

$$-x + 8y = 32$$

and

$$8y = -x + 32$$

$$x + 8y = 32$$

Area under a parametric curve

Given a parametric curve where our function is defined by two equations, one for x and one for y, and both of them in terms of a parameter t,

$$x = f(t)$$

$$y = g(t)$$

we'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

where A is the area under the curve, $y(t)$ is $y = g(t)$, and $x'(t)$ is the derivative of $x = f(t)$.

Keep in mind as you're working these kinds of problems that this area formula won't give us a real-number answer. Instead, it'll give us a function that represents the area under any part of the parametric curve. In order to find a number value for the area, we'll have to use a definite integral by defining an interval for the area.

Example

Find the function that defines the area under the parametric curve.

$$x = 2\theta - \cos\theta$$

$$y = 2 + \sin\theta$$

Don't be confused by the fact that the parameter is θ instead of t. It's still a parameter value, because x and y are both defined in terms of θ.

We've already been given $y(\theta)$, but we need to find $x'(\theta)$ before we can plug into the area formula.

$$x = 2\theta - \cos\theta$$

$$x'(\theta) = 2 + \sin\theta$$

Plugging $y(\theta)$ and $x'(\theta)$ into the area formula, we get

$$A = \int (2 + \sin\theta)(2 + \sin\theta)\ d\theta$$

$$A = \int 4 + 4\sin\theta + \sin^2\theta\ d\theta$$

Using the formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

we'll make a substitution for $\sin^2\theta$.

$$A = \int 4 + 4\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\ d\theta$$

$$A = \int 4 + 4\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = \int \frac{9}{2} + 4\sin\theta - \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = \int \frac{9}{2}\ d\theta + \int 4\sin\theta\ d\theta - \int \frac{1}{2}\cos 2\theta\ d\theta$$

$$A = \frac{9}{2}\theta - 4\cos\theta - \frac{1}{4}\sin 2\theta$$

Area under one arc or loop of a parametric curve

Sometimes we need to find the area under just one arc or loop of a parametric curve. In order to do it, we'll use the area formula

$$A = \int_a^b y(t)x'(t)\ dt$$

where $[a, b]$ is the interval that contains the loop, and $x'(t)$ is the derivative of $x(t)$. So when we're given equations for x and y, we simply plug y and the derivative of x into the formula for area.

In order to find the bounds $[a, b]$, we want to look at the values of x and y as the parameter traces out its arc. When the parameter is equal to 0, that will correspond to a particular coordinate point (x, y). Once the arc arrives back at the same starting point (x, y), you've closed one loop of the curve, and therefore you'll use the limits of integration that correspond to those parameter values.

Example

Find the area under the parametric curve over the given interval.

$$x = 3 + \sin\theta$$

$$y = 3 + \cos\theta$$

$$0 \leq \theta \leq 2\pi$$

The first thing we need to do is find the limits of integration. We'll set up a chart for θ starting at 0, and then fill in the corresponding values of x and y.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	3	4	3	2	3
y	4	3	2	3	4

Because we started at the point $(3,4)$ and didn't return to the same point until the parameter value reached 2π, the bounds for the integral will be $[0,2\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta) = \cos\theta$$

Plugging everything into the area formula, we get

$$A = \int_0^{2\pi} (3 + \cos\theta)(\cos\theta) \ d\theta$$

$$A = \int_0^{2\pi} 3\cos\theta + \cos^2\theta \ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2\theta$ using the formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

We'll make the substitution.

$$A=\int_0^{2\pi}3\cos\theta+\frac{1}{2}(1+\cos 2\theta)\ d\theta$$

$$A=\int_0^{2\pi}3\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\ d\theta$$

$$A=3\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\Big|_0^{2\pi}$$

$$A=3\sin(2\pi)+\frac{1}{2}(2\pi)+\frac{1}{4}\sin(2(2\pi))-\left[3\sin(0)+\frac{1}{2}(0)+\frac{1}{4}\sin(2(0))\right]$$

$$A=3(0)+\pi+\frac{1}{4}(0)-\left[3(0)+\frac{1}{2}(0)+\frac{1}{4}(0)\right]$$

$$A=\pi$$

Parametric arc length

The arc length of a parametric curve over the interval $\alpha \leq t \leq \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

where α and β are the limits of the interval

where dx/dt is the derivative of $x(t)$

where dy/dt is the derivative of $y(t)$

Example

Find the length of the parametric curve.

$$x = 5\sin t$$

$$y = 5\cos t$$

for $0 \leq t \leq 2\pi$

We need to find the derivatives of the parametric equations.

$$x = 5\sin t$$

$$\frac{dx}{dt} = 5\cos t$$

and

$$y = 5\cos t$$

$$\frac{dy}{dt} = -5\sin t$$

Since we were given the limits of integration in the problem, we're ready to plug everything into the arc length formula.

$$L = \int_0^{2\pi} \sqrt{(5\cos t)^2 + (-5\sin t)^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25\cos^2 t + 25\sin^2 t}\ dt$$

$$L = \int_0^{2\pi} \sqrt{25\left(\cos^2 t + \sin^2 t\right)}\ dt$$

Since $\cos^2 t + \sin^2 t = 1$, we get

$$L = \int_0^{2\pi} \sqrt{25(1)}\ dt$$

$$L = \int_0^{2\pi} 5\ dt$$

$$L = 5t\Big|_0^{2\pi}$$

$$L = 5(2\pi) - 5(0)$$

$$L = 10\pi$$

Surface area of revolution of a parametric curve, horizontal axis

The surface area of the solid created by revolving a parametric curve around the x-axis is given by

$$S_x = \int_a^b 2\pi y\sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\ dt$$

where the curve is defined over the interval $[a, b]$,

where $f'(t)$ is the derivative of the curve $f(t)$

where $g'(t)$ is the derivative of the curve $g(t)$

Example

Find the surface area of revolution of the solid created when the parametric curve is rotated around the given axis over the given interval.

$$x = \cos^3 t$$

$$y = \sin^3 t$$

for $0 \le t \le \dfrac{\pi}{2}$, rotated around the x-axis

We'll call the parametric equations

$$f(t) = \cos^3 t$$

$$g(t) = \sin^3 t$$

The limits of integration are defined in the problem, but we need to find both derivatives before we can plug into the formula.

$$f'(t) = -3\cos^2 t \sin t$$

$$g'(t) = 3\sin^2 t \cos t$$

Now we'll plug into the formula for the surface area of revolution.

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi\left(\sin^3 t\right)\sqrt{\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t \cos t\right)^2}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t\left(\cos^2 t + \sin^2 t\right)}\ dt$$

Since $\sin^2 t + \cos^2 t = 1$, we get

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t\,(1)}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{9\sin^2 t \cos^2 t}\ dt$$

$$S_x = \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\left(3\sin t \cos t\right)\ dt$$

$$S_x = 6\pi\int_0^{\frac{\pi}{2}} \sin^4 t \cos t\ dt$$

We'll use u-substitution, letting

$$u = \sin t$$

$$du = \cos t \ dt$$

We'll make the substitution.

$$S_x = 6\pi \int_{x=0}^{x=\frac{\pi}{2}} u^4 \ du$$

$$S_x = \frac{6\pi}{5} u^5 \Big|_{x=0}^{x=\frac{\pi}{2}}$$

Back-substituting for u, we get

$$S_x = \frac{6\pi}{5} \sin^5 t \Big|_0^{\frac{\pi}{2}}$$

$$S_x = \left(\frac{6\pi}{5} \sin^5 \frac{\pi}{2} \right) - \left(\frac{6\pi}{5} \sin^5 0 \right)$$

$$S_x = \frac{6\pi}{5}(1)^5 - \frac{6\pi}{5}(0)^5$$

$$S_x = \frac{6\pi}{5}$$

Surface area of revolution of a parametric curve, vertical axis

The surface area of the solid created by revolving a parametric curve around the y-axis is given by

$$S_y = \int_a^b 2\pi x\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$$

where the curve is defined over the interval $[a, b]$, $f'(t)$ is the derivative of the curve $x = f(t)$, and $g'(t)$ is the derivative of the curve $y = g(t)$.

Let's do an example where we calculate the surface area of revolution around the y-axis over a specific interval.

Example

Find the surface area of revolution of the solid created when the parametric curve is rotated around the y-axis over $0 \le t \le 3$.

$$x = 2t^2$$

$$y = 2t^3$$

We'll call the parametric equations

$$f(t) = 2t^2$$

$$g(t) = 2t^3$$

The limits of integration are defined in the problem, but we need to find both derivatives before we can plug into the formula.

$$f'(t) = 4t$$

$$g'(t) = 6t^2$$

Now we'll plug into the formula for the surface area of revolution.

$$S_y = \int_0^3 2\pi(2t^2)\sqrt{(4t)^2 + (6t^2)^2}\ dt$$

$$S_y = \int_0^3 4\pi t^2\sqrt{16t^2 + 36t^4}\ dt$$

$$S_y = \int_0^3 4\pi t^2\sqrt{4t^2(4 + 9t^2)}\ dt$$

$$S_y = \int_0^3 8\pi t^3\sqrt{4 + 9t^2}\ dt$$

$$S_y = 8\pi\int_0^3 t^3\sqrt{4 + 9t^2}\ dt$$

We'll use a substitution, letting $u = 4 + 9t^2$, $t^2 = (u - 4)/9$, and $dt = du/18t$.

$$S_y = 8\pi\int_{t=0}^{t=3} t^3\sqrt{u}\ \frac{du}{18t}$$

$$S_y = \frac{8\pi}{18}\int_{t=0}^{t=3} t^2\sqrt{u}\ du$$

$$S_y = \frac{8\pi}{18}\int_{t=0}^{t=3} \frac{u-4}{9}\sqrt{u}\ du$$

$$S_y = \frac{8\pi}{18}\int_{t=0}^{t=3} \left(\frac{u}{9} - \frac{4}{9}\right)u^{\frac{1}{2}}\ du$$

$$S_y = \frac{8\pi}{18}\int_{t=0}^{t=3} \frac{u^{\frac{3}{2}}}{9} - \frac{4u^{\frac{1}{2}}}{9}\ du$$

$$S_y = \frac{8\pi}{162}\int_{t=0}^{t=3} u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\ du$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=3}$$

Back-substituting for u, we get

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}(4+9t^2)^{\frac{5}{2}} - \frac{8}{3}(4+9t^2)^{\frac{3}{2}}\right)\Bigg|_0^3$$

Evaluate over the interval.

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}(4+9(3)^2)^{\frac{5}{2}} - \frac{8}{3}(4+9(3)^2)^{\frac{3}{2}}\right) - \frac{4\pi}{81}\left(\frac{2}{5}(4+9(0)^2)^{\frac{5}{2}} - \frac{8}{3}(4+9(0)^2)^{\frac{3}{2}}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}(4+81)^{\frac{5}{2}} - \frac{8}{3}(4+81)^{\frac{3}{2}}\right) - \frac{4\pi}{81}\left(\frac{2}{5}(4+0)^{\frac{5}{2}} - \frac{8}{3}(4+0)^{\frac{3}{2}}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}(85)^{\frac{5}{2}} - \frac{8}{3}(85)^{\frac{3}{2}} - \frac{2}{5}(4)^{\frac{5}{2}} + \frac{8}{3}(4)^{\frac{3}{2}}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[(85)^5]^{\frac{1}{2}} - \frac{8}{3}[(85)^3]^{\frac{1}{2}} - \frac{2}{5}[(4)^{\frac{1}{2}}]^5 + \frac{8}{3}[(4)^{\frac{1}{2}}]^3\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[85(85)^4]^{\frac{1}{2}} - \frac{8}{3}[85(85)^2]^{\frac{1}{2}} - \frac{2}{5}(2)^5 + \frac{8}{3}(2)^3\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2}{5}[(85)^2\sqrt{85}] - \frac{8}{3}[85\sqrt{85}] - \frac{2}{5}(32) + \frac{8}{3}(8)\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2(85)^2\sqrt{85}}{5} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{2 \cdot 5 \cdot 5 \cdot 17 \cdot 17 \cdot \sqrt{85}}{5} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(2{,}890\sqrt{85} - \frac{680\sqrt{85}}{3} - \frac{64}{5} + \frac{64}{3}\right)$$

$$S_y = \frac{4\pi}{81}\left(2{,}890\sqrt{85} - \frac{64}{5} + \frac{64 - 680\sqrt{85}}{3}\right)$$

Find a common denominator.

$$S_y = \frac{4\pi}{81}\left(\frac{43{,}350\sqrt{85}}{15} - \frac{192}{15} + \frac{320 - 3{,}400\sqrt{85}}{15}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{43{,}350\sqrt{85} - 3{,}400\sqrt{85} - 192 + 320}{15}\right)$$

$$S_y = \frac{4\pi}{81}\left(\frac{39{,}950\sqrt{85} + 128}{15}\right)$$

Volume of revolution of a parametric curve

In the same way that we could find the volume of a three-dimensional object generated by rotating a two-dimensional area around in axis when we studied applications of integrals, we can find the volume of revolution generated by revolving the area enclosed by two parametric curves.

The formulas we use to find the volume of revolution for a parametric curve are

rotation around the x-axis $\quad V_x = \int_{\alpha}^{\beta} \pi y^2 \frac{dx}{dt}\ dt$

rotation around the y-axis $\quad V_y = \int_{\alpha}^{\beta} \pi x^2 \frac{dy}{dt}\ dt$

Example

Find the volume of revolution of the parametric curve.

$x = 3t^2 + 4$

$y = t^4$

for $0 \leq t \leq 2$, rotated around the y-axis

Since we're rotating around the y-axis, we'll use the formula

$$V_y = \int_{\alpha}^{\beta} \pi x^2 \frac{dy}{dt}\ dt$$

The problem gave the interval $0 \le t \le 2$, so $\alpha = 0$ and $\beta = 2$. Now we need to find dy/dt so that we can plug it into the volume formula.

$$y = t^4$$

$$\frac{dy}{dt} = 4t^3$$

Plugging everything into the volume formula, we get

$$V_y = \int_0^2 \pi \left(3t^2 + 4\right)^2 \left(4t^3\right)\ dt$$

$$V_y = 4\pi \int_0^2 t^3 \left(3t^2 + 4\right)^2\ dt$$

$$V_y = 4\pi \int_0^2 t^3 \left(9t^4 + 24t^2 + 16\right)\ dt$$

$$V_y = 4\pi \int_0^2 9t^7 + 24t^5 + 16t^3\ dt$$

$$V_y = 4\pi \left(\frac{9t^8}{8} + \frac{24t^6}{6} + \frac{16t^4}{4}\right)\Bigg|_0^2$$

$$V_y = 4\pi \left(\frac{9t^8}{8} + 4t^6 + 4t^4\right)\Bigg|_0^2$$

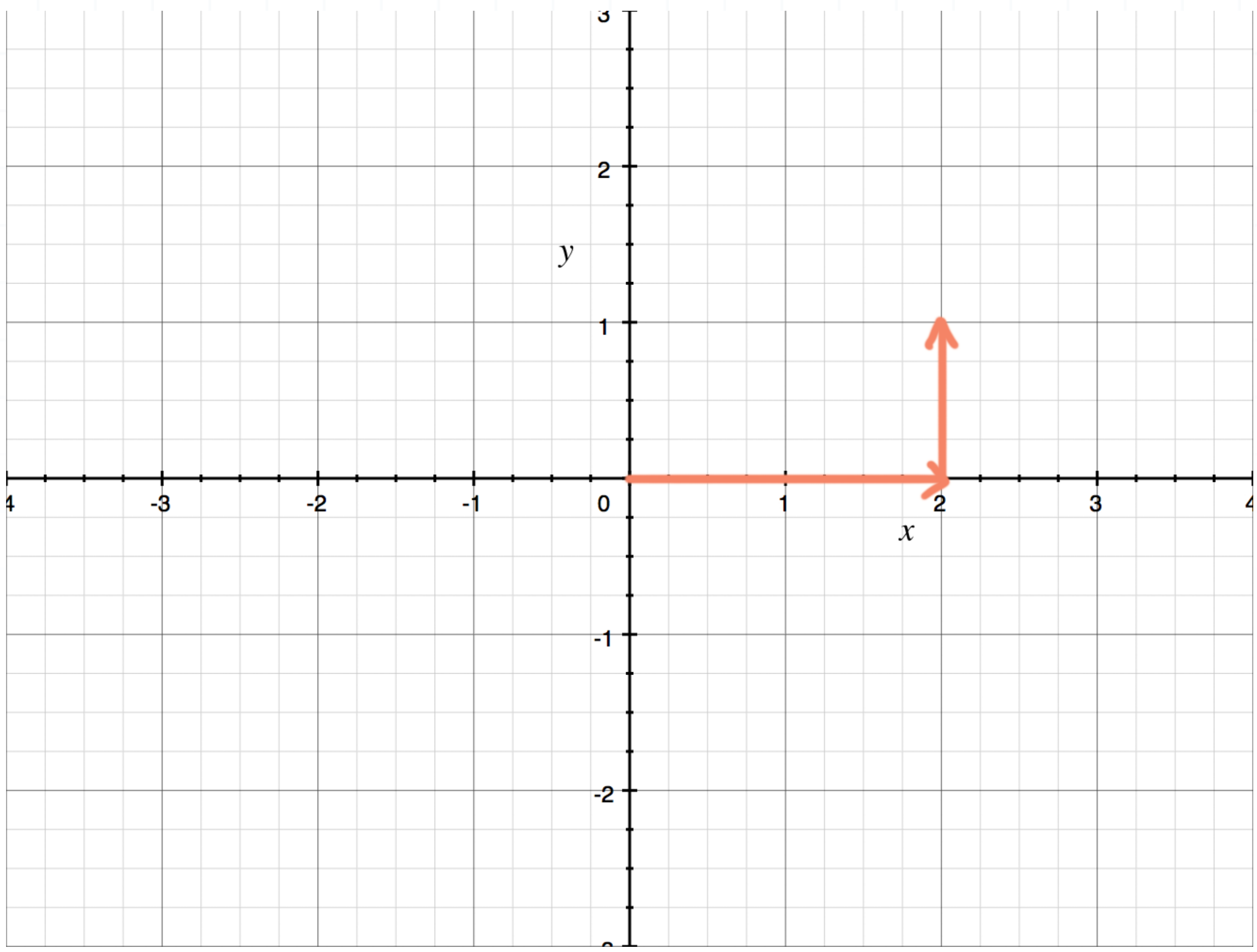
$$V_y = 4\pi\left[\frac{9(2)^8}{8} + 4(2)^6 + 4(2)^4\right] - 4\pi\left[\frac{9(0)^8}{8} + 4(0)^6 + 4(0)^4\right]$$

$$V_y = 4\pi\left[9(2)^5 + 256 + 64\right]$$

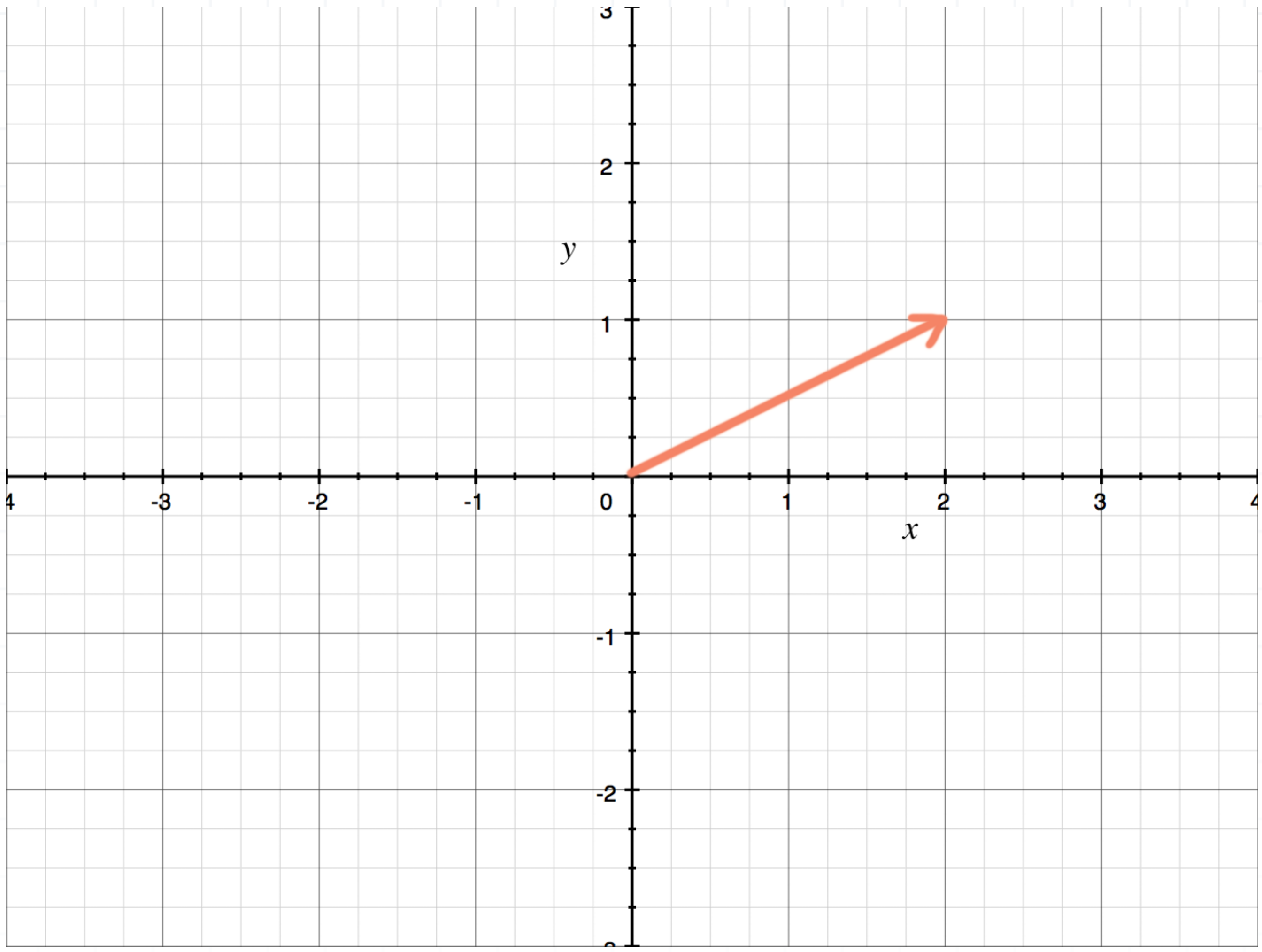
$$V_y = 2{,}432\pi$$

Converting to polar coordinates

Rectangular coordinates, or cartesian coordinates, come in the form (x, y). It's easy to remember that they're called rectangular coordinates, because if you start at the origin and move first to the x-coordinate, and then to the y-coordinate, your path is a horizontal line, followed by a vertical line, which form two sides of a rectangle.



Polar coordinates, on the other hand, come in the form (r, θ). Instead of moving out from the origin using horizontal and vertical lines, we instead pick the angle θ, which is the direction, and then move out from the origin a certain distance r.



Rectangular to polar

To convert rectangular coordinates to polar coordinates, we'll use the conversion formulas

$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

We'll start by plugging the x and y values from the rectangular point into the left side of $x^2 + y^2 = r^2$, and we'll get a value for r.

Then we'll use

r and the x-value and plug them into $x = r\cos\theta$

r and the y-value and plug them into $y = r\sin\theta$.

The value of θ that turns out to be a solution to both equations is the value of θ we should use in our converted polar point.

Polar to rectangular

To convert polar coordinates to rectangular coordinates, we'll use the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

All we have to do is take the values of r and θ from the polar point, plug them into the right sides of these conversion formulas, and solve for x and y, the values we need for the equivalent rectangular coordinate point.

Example

Convert the rectangular point to a polar point.

$$(1, -1)$$

We'll plug the x- and y-values from our rectangular point into $x^2 + y^2 = r^2$ to find r.

$$x^2+y^2=r^2$$

$$(1)^2+(-1)^2=r^2$$

$$1+1=r^2$$

$$2=r^2$$

$$\sqrt{2}=r$$

Note: There are multiple ways to indicate the same polar point. Even though $r=\pm\sqrt{2}$, we're only using the positive solution because the negative solution will actually return the same polar point. This will always be true, so you can always get away with only using the positive solution for r.

Plugging $r=\sqrt{2}$ and $x=1$ into $x=r\cos\theta$ to find θ, we get

$$x=r\cos\theta$$

$$1=\sqrt{2}\cos\theta$$

$$\frac{1}{\sqrt{2}}=\cos\theta$$

$$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\cos\theta$$

$$\frac{\sqrt{2}}{2}=\cos\theta$$

$$\theta=\frac{\pi}{4},\frac{7\pi}{4}$$

and

$$y = r\sin\theta$$

$$-1 = \sqrt{2}\sin\theta$$

$$-\frac{1}{\sqrt{2}} = \sin\theta$$

$$-\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \sin\theta$$

$$-\frac{\sqrt{2}}{2} = \sin\theta$$

$$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$$

Since $\theta = 7\pi/4$ is a solution to both equations, this is the one we'll use.

The equivalent polar coordinate point is

$$\left(\sqrt{2}, \frac{7\pi}{4}\right)$$

Let's look at another example where we convert from polar coordinates to rectangular coordinates.

Example

Convert the polar point to a rectangular point.

$$\left(1, \frac{\pi}{2}\right)$$

We'll use the conversion formulas, plugging 1 in for r and $\pi/2$ in for θ.

$$x = r\cos\theta$$

$$x = 1\cos\frac{\pi}{2}$$

$$x = 0$$

and

$$y = r\sin\theta$$

$$y = 1\sin\frac{\pi}{2}$$

$$y = 1$$

The equivalent rectangular coordinate point is $(0,1)$.

Converting rectangular equations

To convert rectangular equations to polar equations, we'll use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Example

Convert the rectangular equation to a polar equation.

$$x^2 - 3x + y^2 + 2y = 0$$

Use $r^2 = x^2 + y^2$.

$$x^2 + y^2 - 3x + 2y = 0$$

$$r^2 - 3x + 2y = 0$$

Use $x = r\cos\theta$ and $y = r\sin\theta$.

$$r^2 - 3r\cos\theta + 2r\sin\theta = 0$$

$$r^2 = 3r\cos\theta - 2r\sin\theta$$

$$r = 3\cos\theta - 2\sin\theta$$

Once we've eliminated all x and y variables, and replaced them with r and θ variables, we're done with the conversion.

Converting polar equations

To convert polar equations to rectangular equations, we'll use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Example

Convert the polar equation to a rectangular equation.

$$r = 3\cos\theta$$

Modify the conversion formula $x = r\cos\theta$.

$$x = r\cos\theta$$

$$\cos\theta = \frac{x}{r}$$

Plug this into the given polar equation.

$$r = 3\left(\frac{x}{r}\right)$$

$$r^2 = 3x$$

Use $r^2 = x^2 + y^2$.

$$x^2 + y^2 = 3x$$

$$x^2 - 3x + y^2 = 0$$

Once we've eliminated all r and θ variables, and replaced them with x and y variables, we're done with the conversion.

Distance between polar points

To find the distance between two polar coordinates, we have two options. We can

1. Convert our polar coordinates points into cartesian coordinate points using the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and then use the distance formula from the cartesian coordinate system.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) are the converted rectangular points. It doesn't matter which point we use for (x_1, y_1) and (x_2, y_2).

2. Use the distance formula from the polar coordinate system

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

where (r_1, θ_1) and (r_2, θ_2) are the given polar points. It doesn't matter which point we use for (r_1, θ_1) and (r_2, θ_2), but it's easier to make θ_1 the larger of the two θ values, since we subtract θ_2 from θ_1.

Let's try an example to test out the first method.

Example

Find the distance between the polar points.

$$\left(2,\frac{\pi}{2}\right) \text{ and } \left(3,\frac{\pi}{4}\right)$$

We'll convert the polar points into cartesian points.

Using the conversion formulas to change $\left(2,\frac{\pi}{2}\right)$ into polar, we get

$$x_1 = 2\cos\frac{\pi}{2}$$

$$x_1 = 0$$

and

$$y_1 = 2\sin\frac{\pi}{2}$$

$$y_1 = 2$$

The new point is $(0,2)$.

Using the conversion formulas to change $\left(3,\frac{\pi}{4}\right)$ into polar, we get

$$x_2 = 3\cos\frac{\pi}{4}$$

$$x_2 = \frac{3\sqrt{2}}{2}$$

and

$$y_2 = 3\sin\frac{\pi}{4}$$

$$y_2 = \frac{3\sqrt{2}}{2}$$

The new point is $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

Setting

$$(x_1, y_1) = (0,2)$$

$$(x_2, y_2) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

and plugging these points into the distance formula from the cartesian coordinate system, we get

$$D = \sqrt{\left(\frac{3\sqrt{2}}{2} - 0\right)^2 + \left(\frac{3\sqrt{2}}{2} - 2\right)^2}$$

$$D = \sqrt{\frac{18}{4} + \frac{18}{4} - \frac{12\sqrt{2}}{2} + 4}$$

$$D = \sqrt{13 - 6\sqrt{2}}$$

The distance between the polar points is $D = \sqrt{13 - 6\sqrt{2}}$.

Let's look at a different example where we use the second method.

Example

Find the distance between the polar points.

$(1,2\pi)$ and $(2,\pi)$

Setting

$$\left(r_1, \theta_1\right) = (1,2\pi)$$

$$\left(r_2, \theta_2\right) = (2,\pi)$$

and plugging these points into the distance formula from the polar coordinate system, we get

$$D = \sqrt{(1)^2 + (2)^2 - 2(1)(2)\cos(2\pi - \pi)}$$

$$D = \sqrt{5 - 4\cos\pi}$$

$$D = \sqrt{5 - 4(-1)}$$

$$D = \sqrt{5 + 4}$$

$$D = \sqrt{9}$$

$$D = 3$$

The distance between the polar points is $D = 3$.

Sketching polar curves

We'll sketch polar curves by plotting values for r at known values of θ. We can also use the table below to quickly graph polar curves given in these standard forms.

1. Lines

$\theta = \beta$	The line from $(0,0)$ set at the angle β
$r\cos\theta = a$	The vertical line through $x = a$
$r\sin\theta = b$	The horizontal line through $y = b$

2. Circles

$r = a$	The circle centered at $(0,0)$ with radius a
$r = 2a\cos\theta$	The circle centered at $(a,0)$ with radius $\lvert a \rvert$
$r = 2b\sin\theta$	The circle centered at $(0,b)$ with radius $\lvert b \rvert$
$r = 2a\cos\theta + 2b\sin\theta$	The circle centered at (a, b) with radius $\sqrt{a^2 + b^2}$

3. Cardioids, limaçons and others

$r = a \pm a\cos\theta$	The cardioid through the origin
$r = a \pm a\sin\theta$	

$r = a \pm b\cos\theta$, $a < b$	The limaçon with an inner loop
$r = a \pm b\sin\theta$, $a < b$	
$r = a \pm b\cos\theta$, $a > b$	The limaçon without an inner loop
$r = a \pm b\sin\theta$, $a > b$	

If we can't use the table above to find a standard form for the polar curve we're given, then we can always generate a table of coordinate points (r, θ). In order to do that, we'll take the value inside the trigonometric function that includes θ, set it equal to $\pi/2$, then solve for θ. For example, given the polar curve $r = 6\sin 3\theta$,

$$3\theta = \frac{\pi}{2}$$

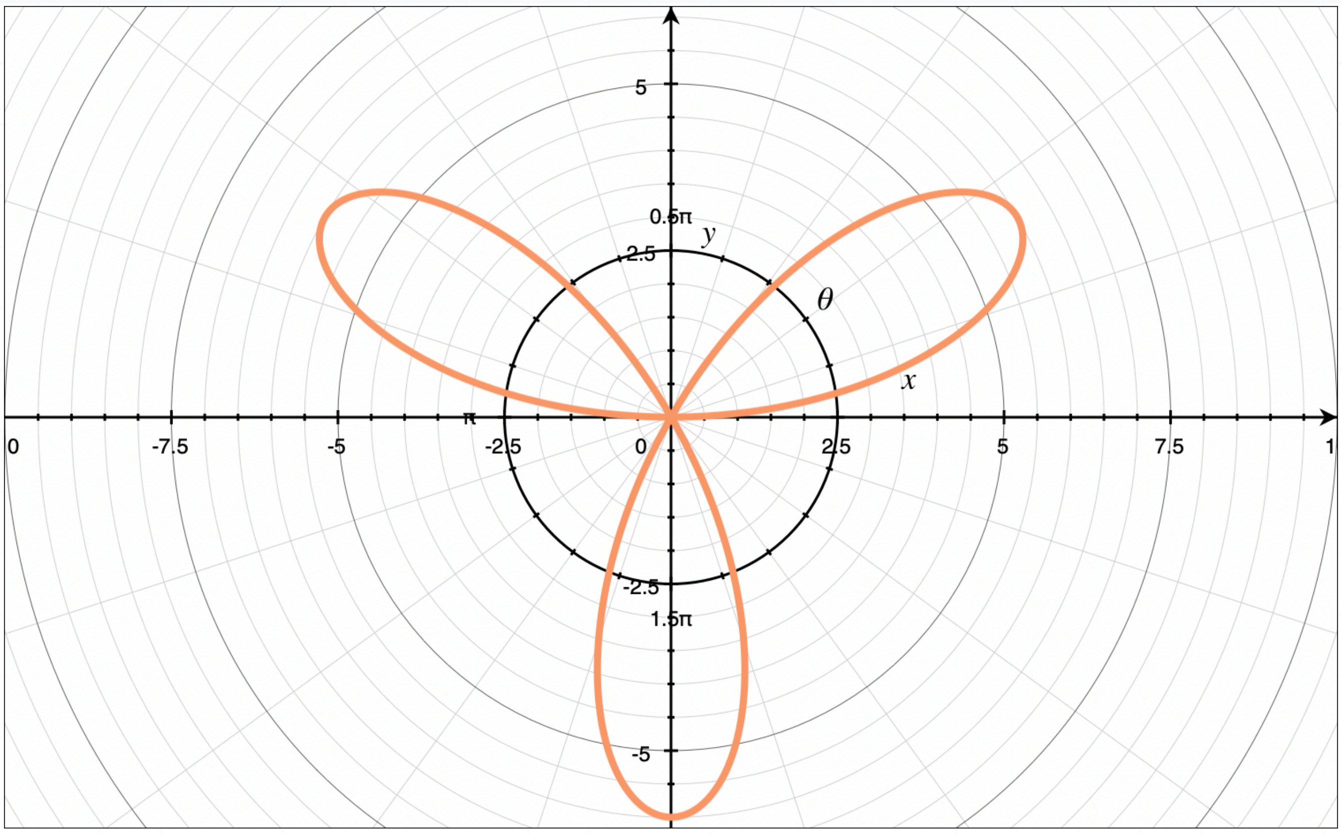
$$\theta = \frac{\pi}{6}$$

Then we'll find r for the increments of $\pi/6$ on the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 6\sin 3\theta$	0	6	0	-6
θ	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
$r = 6\sin 3\theta$	0	6	0	-6
θ	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$

$r = 6\sin 3\theta$	0	6	0	-6

Plotting these points on polar axes, we get



Let's try some examples with lines defined in terms of polar coordinates.

Example

Graph the polar curves on the same axes.

$$\theta = \frac{\pi}{3}$$

$$r\cos\theta = 3$$

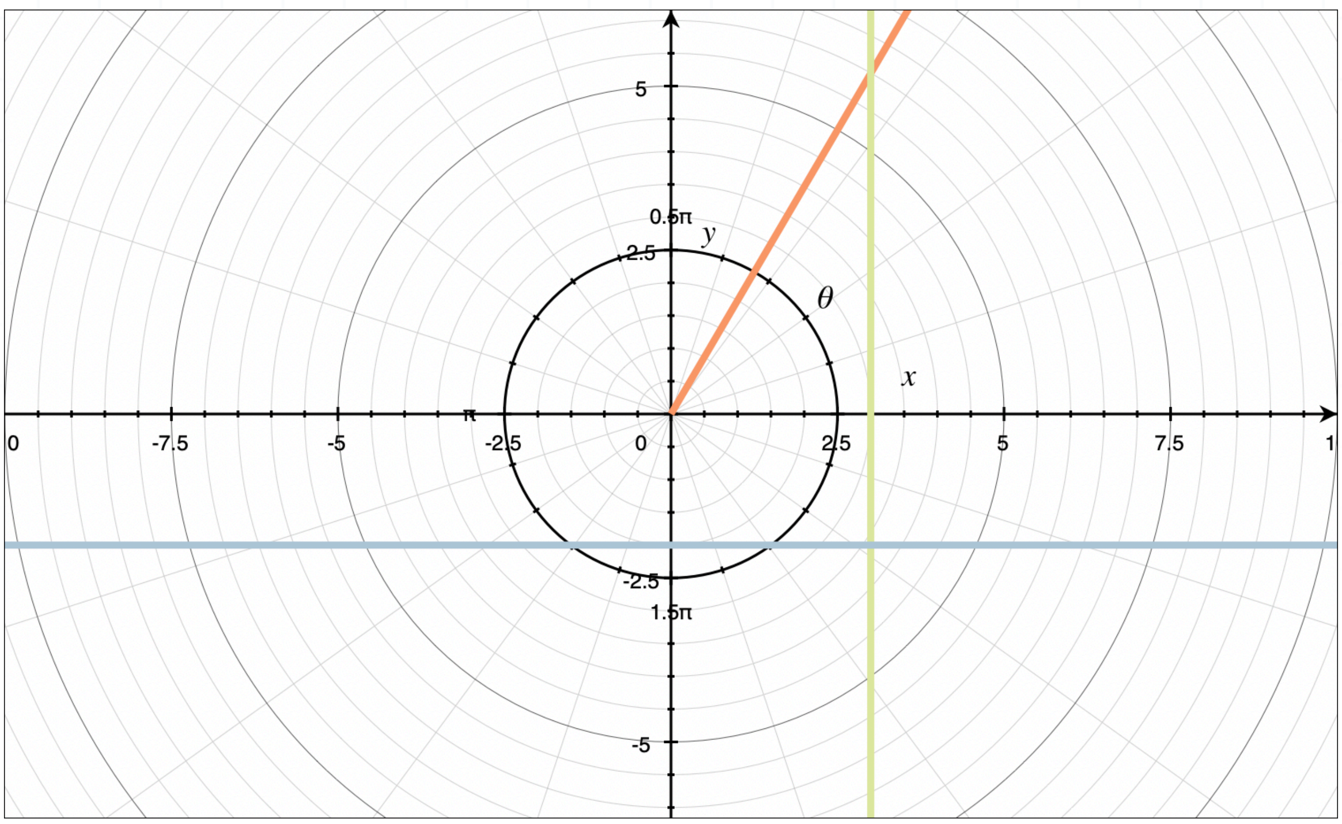
$$r\sin\theta = -2$$

Using the table of standard curves, we can plot all of these on the same axes.

1. $\theta = \pi/3$ is like $\theta = \beta$, so it's a straight line through the origin at the angle $\pi/3$.

2. $r\cos\theta = 3$ is like $r\cos\theta = a$, so it's a vertical line through $x = 3$.

3. $r\sin\theta = -2$ is like $r\sin\theta = b$, so it's a horizontal line through $y = -2$.



Let's try some examples with circles defined in terms of polar coordinates.

Example

Graph the polar curves on the same axes.

$$r = 4$$

$$r = 6\cos\theta$$

$$r = -4\sin\theta$$

$$r = 2\cos\theta + 6\sin\theta$$

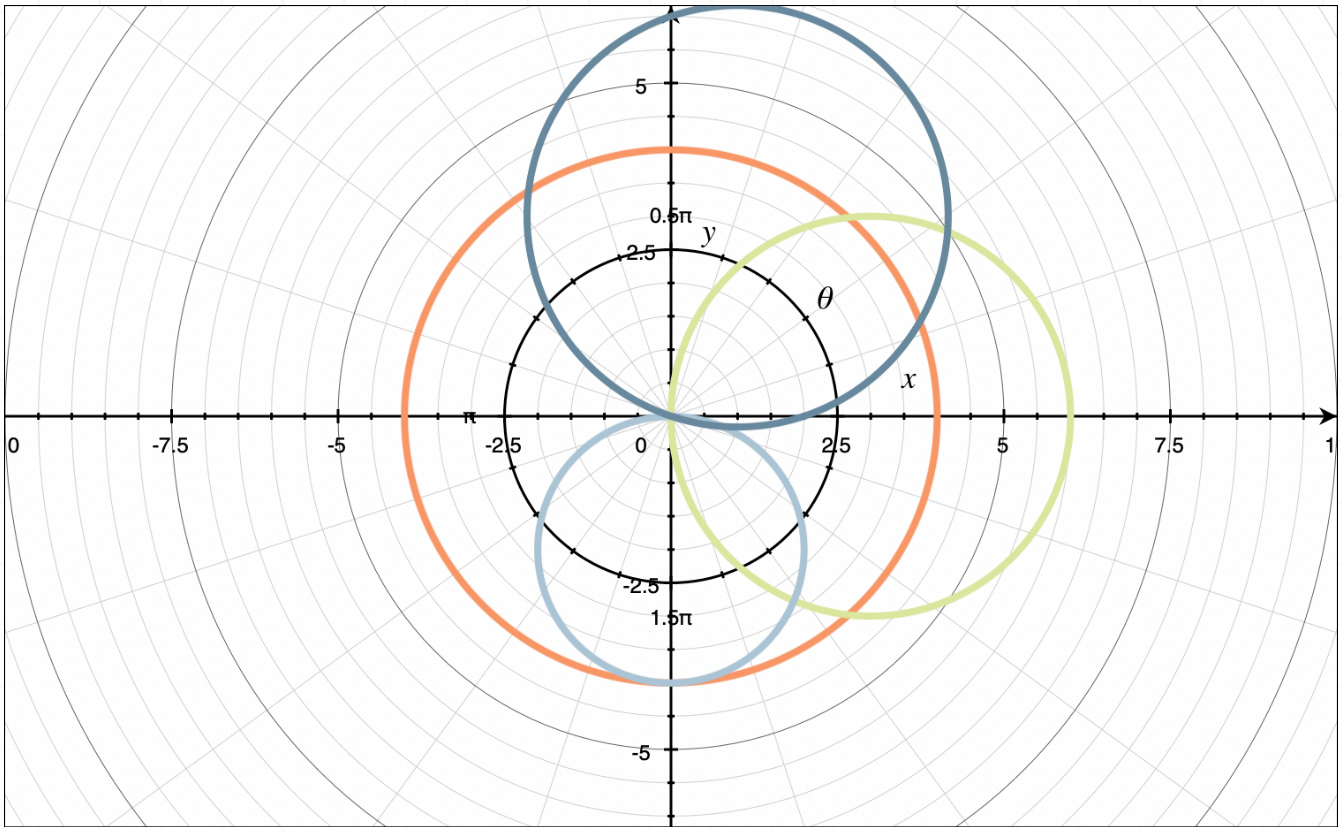
Using the table of standard curves, we can plot all of these on the same axes.

1. $r = 4$ is like $r = a$, so it's a circle centered at $(0,0)$ with radius 4.

2. $r = 6\cos\theta$ is like $r = 2a\cos\theta$, so it's a circle centered at $(3,0)$ with radius $|3|$.

3. $r = -4\sin\theta$ is like $r = 2b\sin\theta$, so it's a circle centered at $(0,-2)$ with radius $|-2|$.

4. $r = 2\cos\theta + 6\sin\theta$ is like $r = 2a\cos\theta + 2b\sin\theta$, so it's a circle centered at $(1,3)$ with radius $\sqrt{a^2 + b^2} = \sqrt{10}$.



Let's try some examples with more complex curves defined in terms of polar coordinates.

Example

Graph the polar curves.

$$r = 3 + 3\sin\theta$$

$$r = 2 + 4\cos\theta$$

$$r = 7 + 6\cos\theta$$

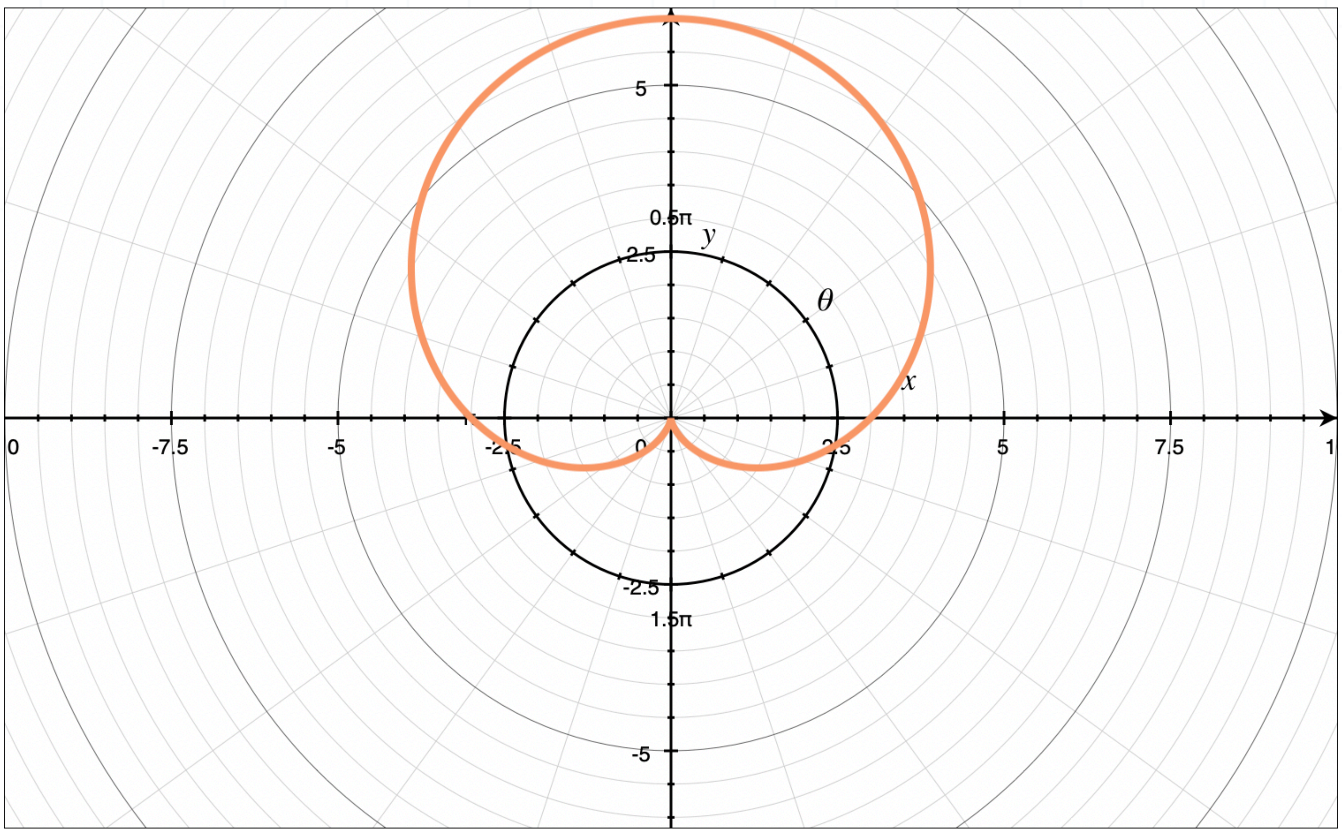
$$r = 6\sin 2\theta$$

For $r = 3 + 3\sin\theta$:

$r = 3 + 3\sin\theta$ is like $r = a \pm a\sin\theta$, so it's a cardioid through the origin. We'll generate a table of values over the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$r = 3 + 3\sin\theta$	3	6	3	0	3

With these points and knowing the shape of our polar curve, we can sketch the graph.

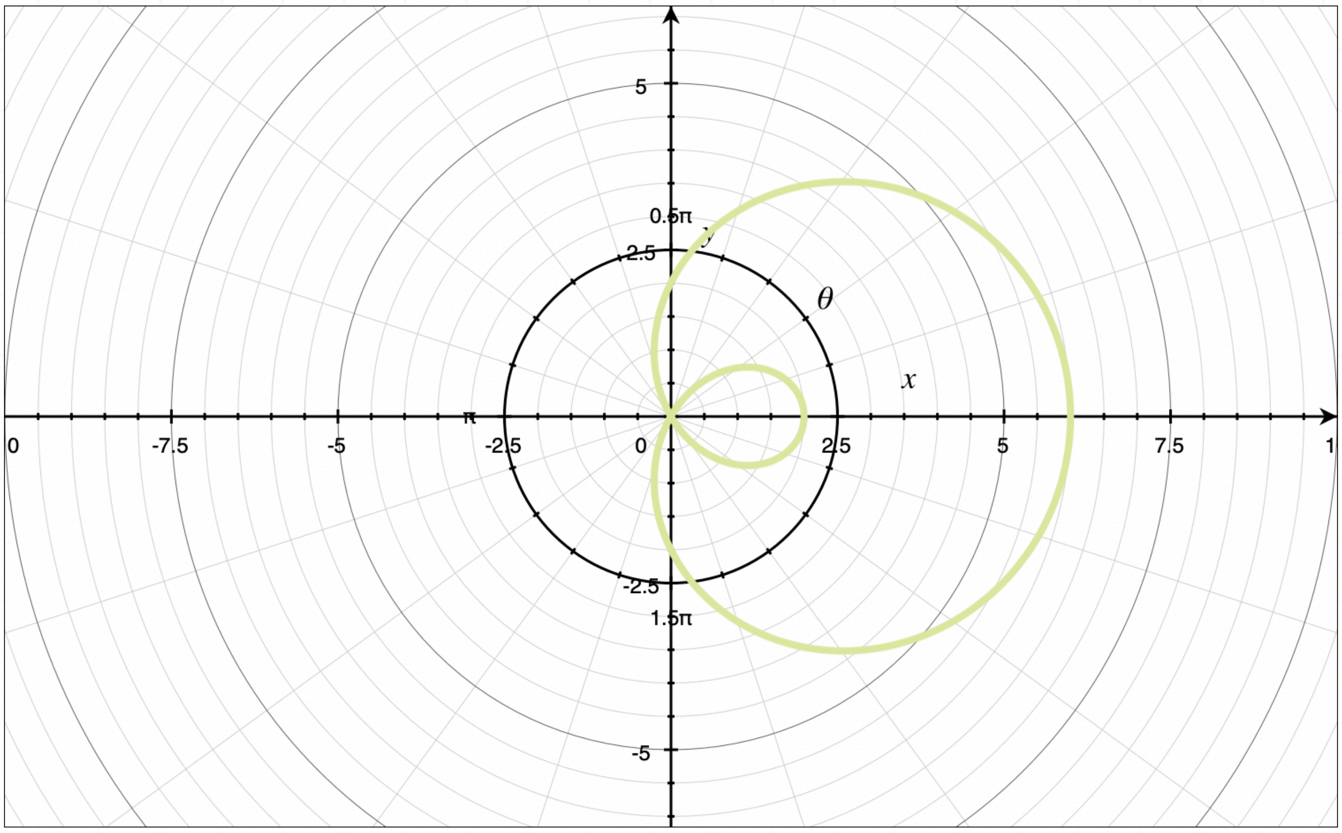


For $r = 2 + 4\cos\theta$:

$r = 2 + 4\cos\theta$ is like $r = a \pm b\cos\theta$ with $a < b$, so it's a limaçon with an inner loop. We'll generate a table of values over the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$r = 2 + 4\cos\theta$	6	2	-2	2	6

With these points and knowing the shape of our polar curve, we can sketch the graph.

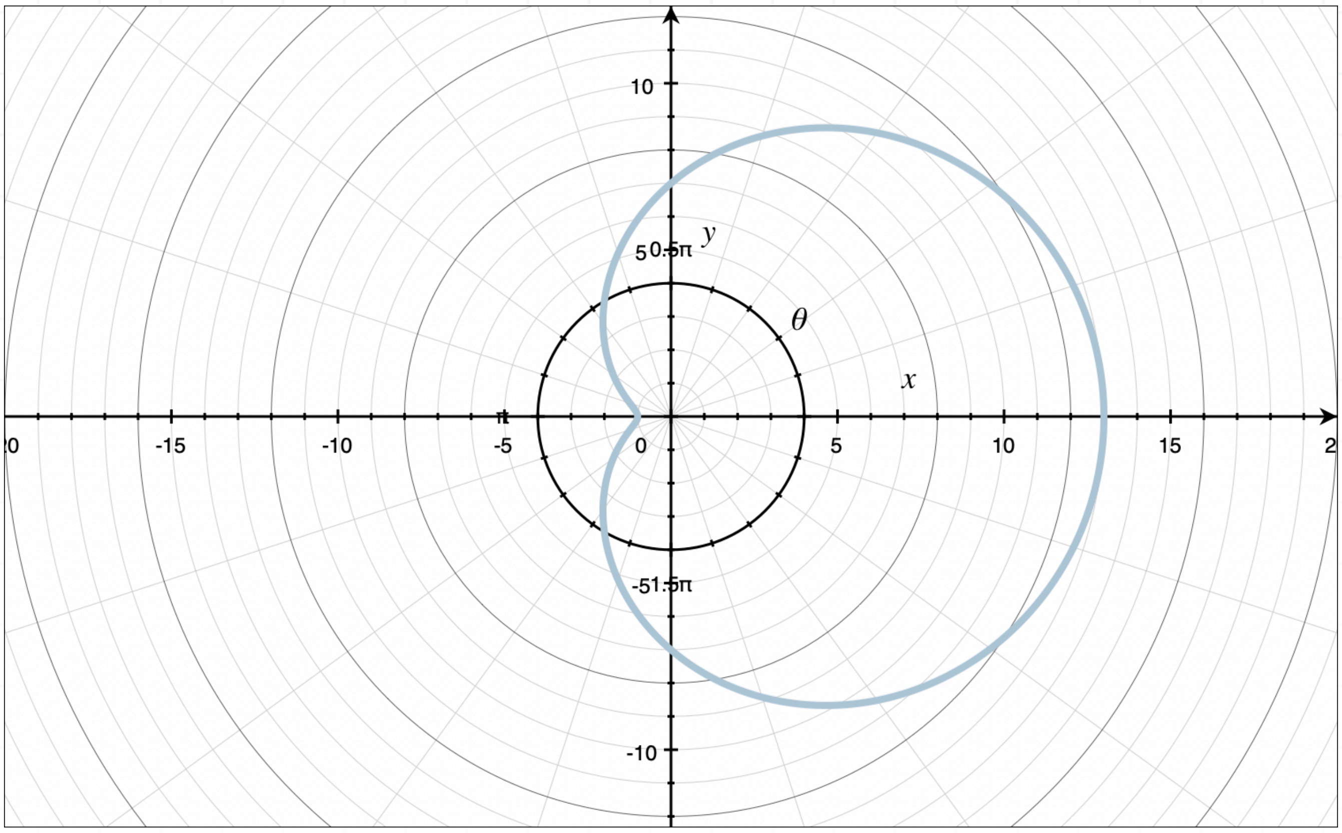


For $r = 7 + 6\cos\theta$:

$r = 7 + 6\cos\theta$ is like $r = a \pm b\cos\theta$ with $a > b$, so it's a limaçon without an inner loop. We'll generate a table of values over the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$r = 7 + 6\cos\theta$	13	7	1	7	13

With these points and knowing the shape of our polar curve we can sketch the graph.



For $r = 6\sin 2\theta$:

$r = 6\sin 2\theta$ doesn't match any of the standard forms in our table. In this case, we'll set the value inside our trigonometric function equal to $\pi/2$ and then solve for θ.

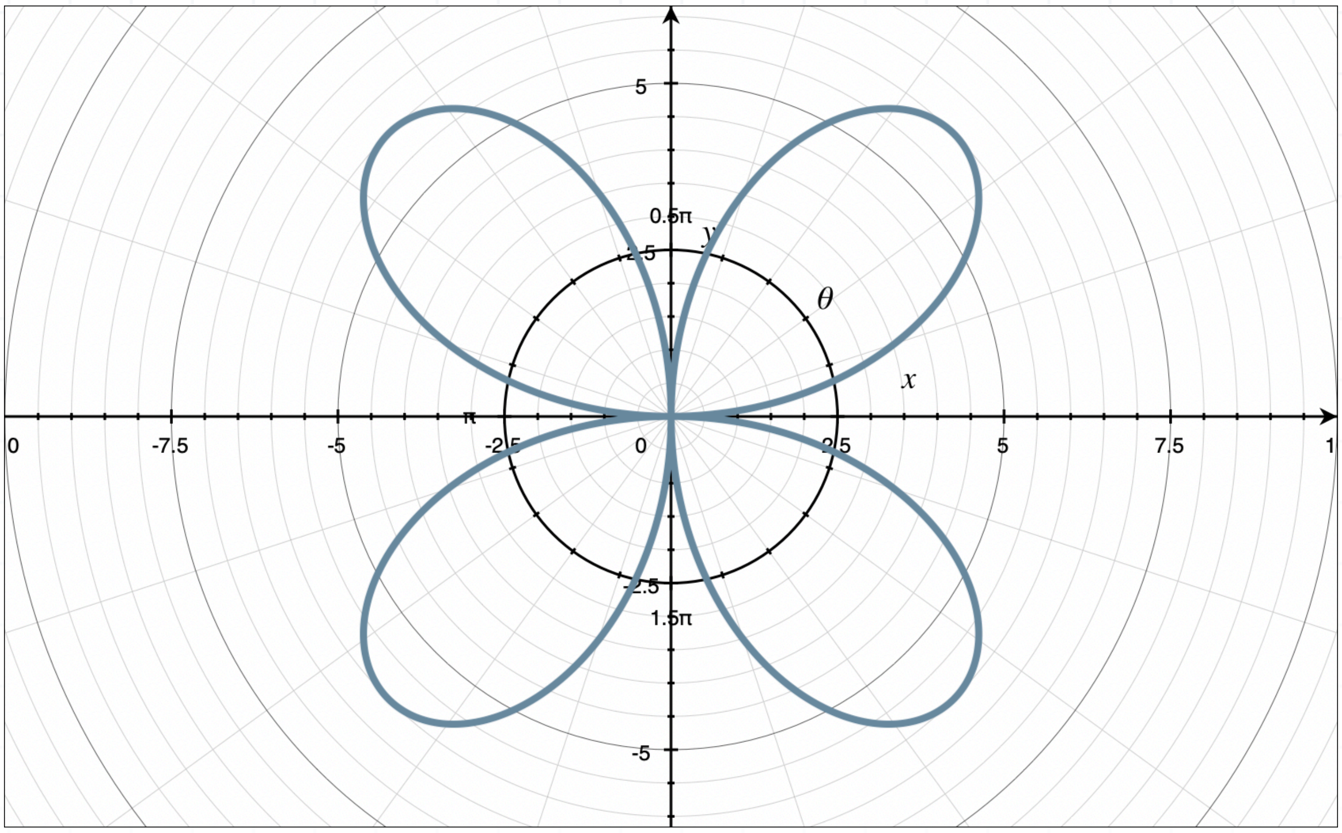
$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

Then we'll find r for the increments of $\pi/4$ on the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$r = 6\sin 2\theta$	0	6	0	-6	0	6	0	-6	0

Plotting these points on polar axes, we get


5
0.5π
2.5
θ
y
x
0
-7.5
-5
π
-2.5
0
2.5
5
7.5
-2.5
1.5π
-5

Tangent line to the polar curve

We'll find the equation of the tangent line to a polar curve in much the same way that we find the tangent line to a cartesian curve. We'll follow these steps:

1. Find the **slope** of the tangent line m, using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

   remembering to plug the value of θ at the tangent point into dy/dx to get a real-number value for the slope m.

2. **Find** x_1 **and** y_1 by plugging the value of θ at the tangent point into the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

3. Plug the slope m and the point (x_1, y_1) into the **point-slope formula** for the equation of a line

$$y - y_1 = m(x - x_1)$$

Example

Find the tangent line to the polar curve at the given point.

$$r = 1 + 2\cos\theta$$

at $\theta = \frac{\pi}{4}$

We'll start by calculating $dr/d\theta$, the derivative of the given polar equation, so that we can plug it into the formula for the slope of the tangent line.

$$r = 1 + 2\cos\theta$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

Plugging $dr/d\theta$ and the given polar equation $r = 1 + 2\cos\theta$ into the formula for dy/dx, we get

$$m = \frac{dy}{dx} = \frac{(-2\sin\theta)\sin\theta + (1 + 2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (1 + 2\cos\theta)\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta - 2\sin\theta\cos\theta}$$

$$m = \frac{dy}{dx} = \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-4\sin\theta\cos\theta - \sin\theta}$$

Plugging the value of $\theta = \pi/4$ into the slope equation, we'll get a real-number value for the slope m.

$$m = \frac{dy}{dx} = \frac{-2\sin^2\frac{\pi}{4} + \cos\frac{\pi}{4} + 2\cos^2\frac{\pi}{4}}{-4\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}}$$

$$m=\frac{dy}{dx}=\frac{-2\left(\frac{\sqrt{2}}{2}\right)^2+\frac{\sqrt{2}}{2}+2\left(\frac{\sqrt{2}}{2}\right)^2}{-4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{-2\left(\frac{2}{4}\right)+\frac{\sqrt{2}}{2}+2\left(\frac{2}{4}\right)}{-4\cdot\frac{2}{4}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{-1+\frac{\sqrt{2}}{2}+1}{-2-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\frac{\sqrt{2}}{2}}{-\frac{4}{2}-\frac{\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\frac{\sqrt{2}}{2}}{\frac{-4-\sqrt{2}}{2}}$$

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{2}\left(\frac{2}{-4-\sqrt{2}}\right)$$

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{-4-\sqrt{2}}$$

If we want to get rid of the square root in the denominator, we can multiply by the conjugate.

$$m=\frac{dy}{dx}=\frac{\sqrt{2}}{-4-\sqrt{2}}\left(\frac{-4+\sqrt{2}}{-4+\sqrt{2}}\right)$$

$$m=\frac{dy}{dx}=\frac{-4\sqrt{2}+2}{16-2}$$

$$m=\frac{dy}{dx}=\frac{-4\sqrt{2}+2}{14}$$

$$m=\frac{dy}{dx}=\frac{-2\sqrt{2}+1}{7}$$

$$m=\frac{dy}{dx}=\frac{1-2\sqrt{2}}{7}$$

Now we want to find x_1 and y_1 by plugging the value of θ at the tangent point and the given polar equation $r=1+2\cos\theta$ into the conversion formulas

$$x=r\cos\theta$$

$$x_1=\left(1+2\cos\frac{\pi}{4}\right)\cos\frac{\pi}{4}$$

$$x_1=\left[1+2\left(\frac{\sqrt{2}}{2}\right)\right]\frac{\sqrt{2}}{2}$$

$$x_1=\left(1+\sqrt{2}\right)\frac{\sqrt{2}}{2}$$

$$x_1=\frac{\sqrt{2}+2}{2}$$

$$x_1 = \frac{2+\sqrt{2}}{2}$$

and

$$y = r\sin\theta$$

$$y_1 = \left(1 + 2\cos\frac{\pi}{4}\right)\sin\frac{\pi}{4}$$

$$y_1 = \left[1 + 2\left(\frac{\sqrt{2}}{2}\right)\right]\frac{\sqrt{2}}{2}$$

$$y_1 = \left(1+\sqrt{2}\right)\frac{\sqrt{2}}{2}$$

$$y_1 = \frac{\sqrt{2}+2}{2}$$

$$y_1 = \frac{2+\sqrt{2}}{2}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2+\sqrt{2}}{2} = \frac{1-2\sqrt{2}}{7}\left(x - \frac{2+\sqrt{2}}{2}\right)$$

$$y - \frac{2+\sqrt{2}}{2} = \frac{1-2\sqrt{2}}{7}x - \frac{2+\sqrt{2}-4\sqrt{2}-4}{14}$$

$$y - \frac{2+\sqrt{2}}{2} = \frac{1-2\sqrt{2}}{7}x - \frac{-3\sqrt{2}-2}{14}$$

$$y - \frac{2+\sqrt{2}}{2} = \frac{1-2\sqrt{2}}{7}x + \frac{3\sqrt{2}+2}{14}$$

$$2y - \left(2+\sqrt{2}\right) = \frac{2-4\sqrt{2}}{7}x + \frac{3\sqrt{2}+2}{7}$$

Eliminate the fractions by multiplying through by 7.

$$14y - 7\left(2+\sqrt{2}\right) = \left(2-4\sqrt{2}\right)x + 3\sqrt{2} + 2$$

$$14y = \left(2-4\sqrt{2}\right)x + 3\sqrt{2} + 2 + 7\left(2+\sqrt{2}\right)$$

$$14y = \left(2-4\sqrt{2}\right)x + 3\sqrt{2} + 2 + 14 + 7\sqrt{2}$$

$$14y = \left(2-4\sqrt{2}\right)x + 16 + 10\sqrt{2}$$

$$14y - \left(2-4\sqrt{2}\right)x = 16 + 10\sqrt{2}$$

$$7y - \left(1-2\sqrt{2}\right)x = 8 + 5\sqrt{2}$$

The equation of the tangent line is $7y - \left(1-2\sqrt{2}\right)x = 8 + 5\sqrt{2}$.

Vertical and horizontal tangent lines to the polar curve

We'll find equations of the vertical and horizontal tangent lines to a polar curve by following these steps:

1. Convert the polar equation into rectangular equations using the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

2. Find the slope of the tangent line m using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3. Find horizontal tangent lines

   a. Set $m = 0$ and solve for θ

   b. Plug these values of θ into the original polar equation to find associated values of r

   c. Pair up values of r and θ to find the coordinate points where the polar equation has horizontal tangent lines

4. Find **vertical tangent lines**

   a. Find the values of θ where m is undefined

b. Plug these values of θ into the original polar equation to find associated values of r

c. Pair up values of r and θ to find the coordinate points where the polar equation has vertical tangent lines

Example

Find the points on the polar curve where the graph of the tangent line is vertical or horizontal.

$$r = 2\sin\theta$$

We'll convert the polar equation to a rectangular equation using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Plugging $r = 2\sin\theta$ into these conversion formulas, we get equations for x and y.

$$x = r\cos\theta$$

$$x = 2\sin\theta\cos\theta$$

and

$$y = r\sin\theta$$

$$y = 2\sin\theta\sin\theta$$

$$y = 2\sin^2\theta$$

We'll find the derivatives $dy/d\theta$ and $dx/d\theta$.

$$\frac{dy}{d\theta} = 4\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 2(2\sin\theta\cos\theta)$$

Because $2\sin\theta\cos\theta = \sin(2\theta)$,

$$\frac{dy}{d\theta} = 2\sin(2\theta)$$

and

$$\frac{dx}{d\theta} = 2\cos\theta\cos\theta - 2\sin\theta\sin\theta$$

$$\frac{dx}{d\theta} = 2\left(\cos^2\theta - \sin^2\theta\right)$$

Because $\cos^2\theta - \sin^2\theta = \cos(2\theta)$,

$$\frac{dx}{d\theta} = 2\cos(2\theta)$$

Plugging both derivatives into the formula for dy/dx, we get

$$\frac{dy}{dx} = \frac{2\sin(2\theta)}{2\cos(2\theta)}$$

$$\frac{dy}{dx} = \frac{\sin(2\theta)}{\cos(2\theta)}$$

With an equation for dy/dx in hand, we're ready to find vertical and horizontal tangent lines.

Horizontal tangent lines exist where $dy/dx = 0$. In order for dy/dx to be 0, the numerator has to be 0.

$$\sin(2\theta) = 0$$

So

$$2\theta = 0$$

$$\theta = 0$$

or

$$2\theta = \pi$$

$$\theta = \frac{\pi}{2}$$

To find the r-values associated with these θ values, we'll plug them back into the original polar equation.

$$r = 2\sin\theta$$

$$r = 2\sin(0)$$

$$r = 2(0)$$

$$r = 0$$

and

$$r = 2\sin\theta$$

$$r = 2\sin\frac{\pi}{2}$$

$$r = 2(1)$$

$$r = 2$$

Putting our values together, we can say that $r = 2\sin\theta$ has horizontal tangent lines at $(0,0)$ and $(2,\pi/2)$.

Vertical tangent lines exist where dy/dx is undefined. In order for dy/dx to be undefined, the denominator has to be 0.

$$\cos(2\theta) = 0$$

So

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

or

$$2\theta = \frac{3\pi}{2}$$

$$\theta = \frac{3\pi}{4}$$

To find the r-values associated with these θ values, we'll plug them back into the original polar equation.

$$r = 2\sin\theta$$

$$r = 2\sin\frac{\pi}{4}$$

$$r = 2\cdot\frac{\sqrt{2}}{2}$$

$$r = \sqrt{2}$$

and

$$r = 2\sin\theta$$

$$r = 2\sin\frac{3\pi}{4}$$

$$r = 2\cdot\left(-\frac{\sqrt{2}}{2}\right)$$

$$r = -\sqrt{2}$$

Putting our values together, we can say that $r = 2\sin\theta$ has vertical tangent lines at $\left(\sqrt{2},\pi/4\right)$ and $\left(-\sqrt{2},3\pi/4\right)$.

We'll summarize our findings.

Horizontal tangent lines at $(0,0)$ and $(2,\pi/2)$

Vertical tangent lines at $\left(\sqrt{2},\pi/4\right)$ and $\left(-\sqrt{2},3\pi/4\right)$

Intersection of polar curves

To find the points of intersection of two polar curves,

solve both curves for r,

set the two curves equal to each other

solve for θ

Using these steps, we might get more intersection points than actually exist, or fewer intersection points than actually exist. To verify that we've found all of the intersection points, and only real intersection points, we graph our curves and visually confirm the intersection points.

We can also convert our polar equations to rectangular equations, solve for the points of intersection of the rectangular curves, and then convert the rectangular points of intersection back into polar coordinates. Even though it's extra work to convert everything from polar to rectangular, using this method guarantees that we'll find all of the points of intersection, and only the real points of intersection.

Let's try an example where we keep everything in polar coordinates.

Example

Find the points of intersection of the polar curves.

$$r = \sin\theta$$

$$r = 1 - \sin\theta$$

To find the points of intersection of these polar curves, we'll set them equal to each other and solve for θ.

$$\sin\theta = 1 - \sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

To find the values of r that are associated with these values of θ, we'll plug the θ values back into either of the original polar curves; we'll choose $r = \sin\theta$.

For $\theta = \dfrac{\pi}{6}$:

$$r = \sin\frac{\pi}{6}$$

$$r = \frac{1}{2}$$

For $\theta = \dfrac{5\pi}{6}$:

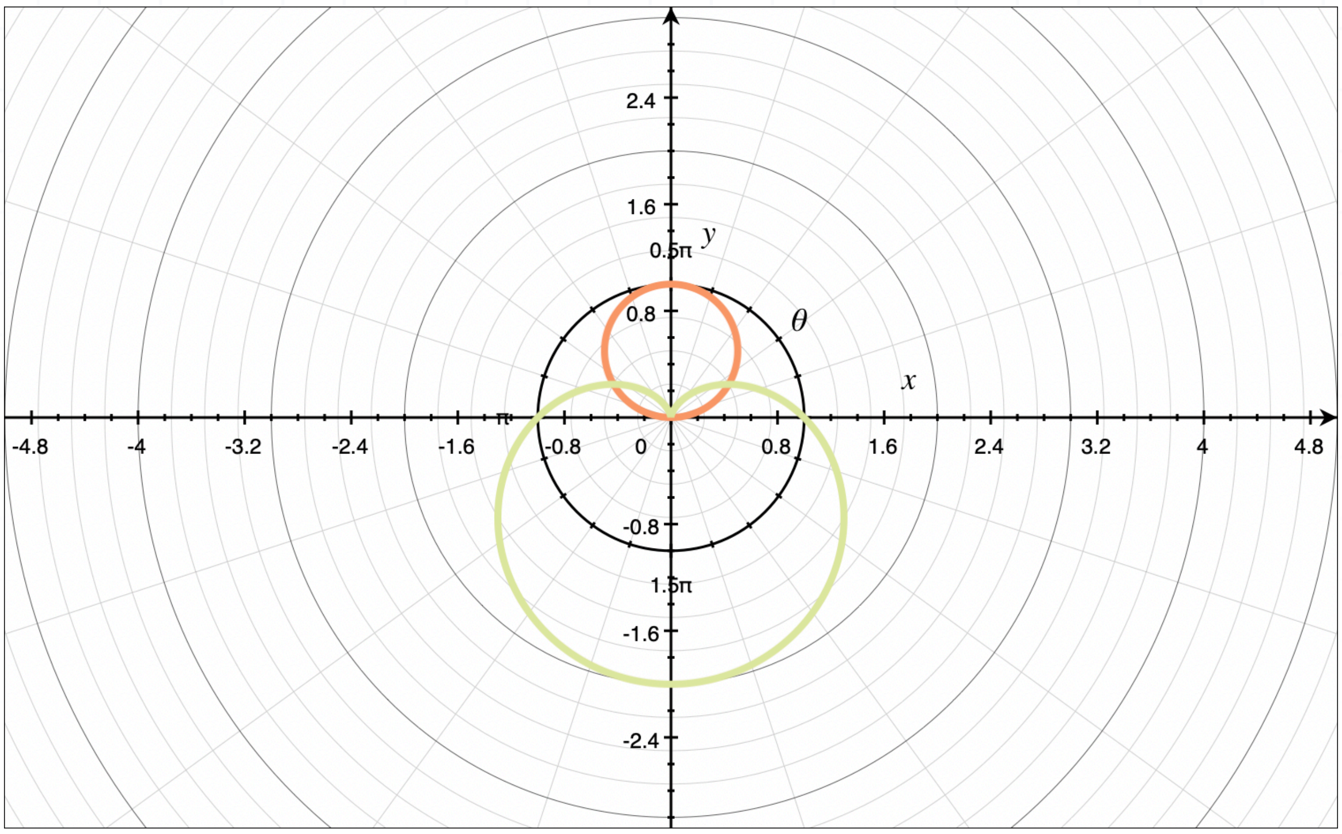
$$r = \sin\frac{5\pi}{6}$$

$$r = \frac{1}{2}$$

Putting these values together, the points of intersection are

$$\left(\frac{1}{2},\frac{\pi}{6}\right) \text{ and } \left(\frac{1}{2},\frac{5\pi}{6}\right)$$

To confirm that these are the points of intersection, we can graph both curves.



Looking at the graph, we see that $(0,0)$ is also a point of intersection, so in total, the graphs intersect each other at

$$\left(\frac{1}{2},\frac{\pi}{6}\right) \text{ and } \left(\frac{1}{2},\frac{5\pi}{6}\right) \text{ and } (0,0)$$

In the previous example, we had to graph the polar curves in order to find all of the points of intersection. That's because we left everything in polar form.

Let's try another example where we convert our polar curves into rectangular coordinates.

Example

Find the points of intersection of the polar curves.

$$r = \cos\theta$$

$$r = 2 - \cos\theta$$

We'll convert the polar curves to rectangular coordinates using the conversion formula

$$x = r\cos\theta$$

$$\cos\theta = \frac{x}{r}$$

Plugging x/r into the given polar equations for $\cos\theta$, we get

$$r = \cos\theta$$

$$r = \frac{x}{r}$$

$$x = r^2$$

and

$$r = 2 - \cos\theta$$

$$r = 2 - \frac{x}{r}$$

$$x = 2r - r^2$$

We've gotten rid of θ, but now we need to get rid of r, which we'll do using the conversion formula

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

Plugging $x^2 + y^2$ into the given polar equations for r^2, and $\sqrt{x^2 + y^2}$ in for r, we get

$$x = r^2$$

$$x = x^2 + y^2$$

$$x^2 + y^2 - x = 0$$

and

$$x = 2r - r^2$$

$$x = 2\sqrt{x^2 + y^2} - \left(x^2 + y^2\right)$$

$$x = 2\sqrt{x^2 + y^2} - x^2 - y^2$$

$$x^2 + y^2 - 2\sqrt{x^2 + y^2} + x = 0$$

Since both of our rectangular equations are equal to 0, we can set them equal to each other.

$$x^2 + y^2 - x = x^2 + y^2 - 2\sqrt{x^2 + y^2} + x$$

$$-x = -2\sqrt{x^2 + y^2} + x$$

$$-2\sqrt{x^2 + y^2} = -2x$$

$$\sqrt{x^2 + y^2} = x$$

Since we found that $x^2 + y^2 = x$ when we were converting $r = \cos\theta$ to rectangular coordinates, we can say

$$\sqrt{x} = x$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0, 1$$

To find the y-values associated with these x-values, we'll plug them into $x^2 + y^2 - x = 0$.

For $x = 0$:

$$(0)^2 + y^2 - (0) = 0$$

$$y = 0$$

For $x = 1$:

$$(1)^2 + y^2 - (1) = 0$$

$$y = 0$$

Putting our values together, we know that the points of intersection are $(0,0)$ and $(1,0)$.

We need to convert these rectangular coordinate points back into polar coordinates, which we'll do using the conversion formulas

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Plugging the rectangular coordinate points into these formulas, we get

For $(0,0)$:

$$r = \sqrt{(0)^2 + (0)^2} \qquad \theta = \tan^{-1}\left(\frac{0}{0}\right)$$

$$r = 0$$

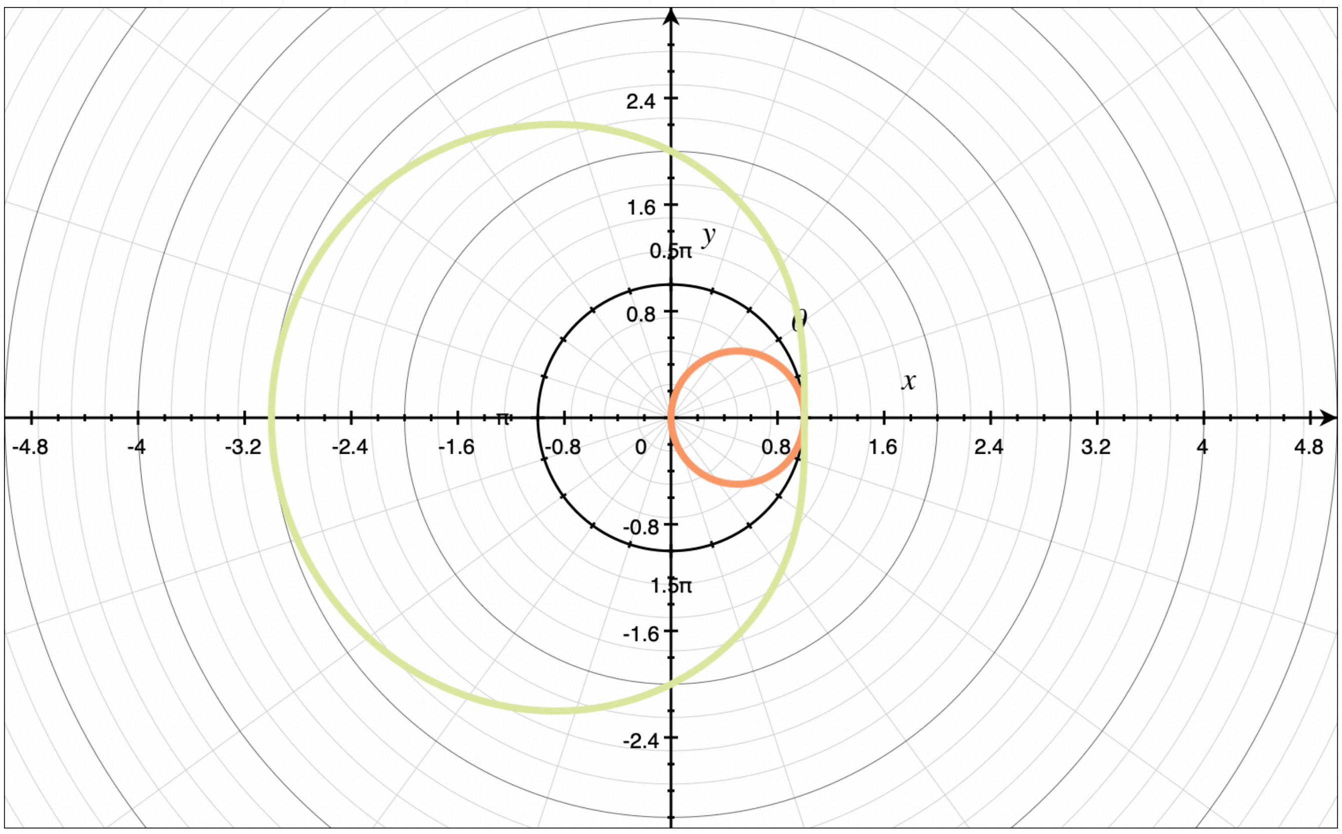
Since the equation for θ is undefined, the rectangular point $(0,0)$ can't be defined in polar coordinates and therefore isn't a polar point of intersection.

For $(1,0)$:

$$r = \sqrt{(1)^2 + (0)^2} \qquad \theta = \tan^{-1}\left(\frac{0}{1}\right)$$

$$r = 1 \qquad \theta = 0$$

The only point of intersection of the given polar curves is the polar point $(1,0)$. If we want to double-check ourselves, we can sketch the polar curves and confirm this point of intersection.



Area inside a polar curve

The area inside a polar curve is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \ d\theta$$

where $[\alpha, \beta]$ is the interval

where r is the equation of the polar curve

Example

Find the area inside the polar curve.

$$r = 4\sin\theta$$

We need to find the interval, which we'll do by setting $r = 0$ and solving for any values of θ.

$$r = 4\sin\theta$$

$$0 = \sin\theta$$

$\theta = \pi$ and $\theta = 2\pi$

$\alpha = \pi$ and $\beta = 2\pi$

Now we can plug the interval we found and the given polar equation into the formula for the area inside a polar curve.

$$A = \int_{\pi}^{2\pi} \frac{1}{2}\left(4\sin\theta\right)^2 \ d\theta$$

$$A = \int_{\pi}^{2\pi} \frac{1}{2} \cdot 16\sin^2\theta \ d\theta$$

$$A = 8\int_{\pi}^{2\pi} \sin^2\theta \ d\theta$$

Since $\sin^2\theta = \frac{1}{2}\left[1 - \cos(2\theta)\right]$, we get

$$A = 8\int_{\pi}^{2\pi} \frac{1}{2}\left[1 - \cos(2\theta)\right] \ d\theta$$

$$A = 4\int_{\pi}^{2\pi} 1 - \cos(2\theta) \ d\theta$$

$$A = 4\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\pi}^{2\pi}$$

$$A = 4\left[2\pi - \frac{\sin\left(2(2\pi)\right)}{2} - \left(\pi - \frac{\sin(2\pi)}{2}\right)\right]$$

$$A = 4\left[2\pi - \frac{\sin(4\pi)}{2} - \pi + \frac{\sin(2\pi)}{2}\right]$$

$$A = 4\left(2\pi - \frac{0}{2} - \pi + \frac{0}{2}\right)$$

$$A = 4\left(2\pi - \pi\right)$$

$$A = 4\pi$$

Area bounded by one loop of a polar curve

When we need to find the area bounded by a single loop of the polar curve, we'll use the same formula we used to find area inside the polar curve in general.

$$A = \int_{\alpha}^{\beta} \frac{1}{2}r^2 \ d\theta$$

where $[\alpha, \beta]$ is the interval

where r is the equation of the polar curve

The best way to find the interval that defines one loop of the curve is to graph the curve.

Example

Find the area bounded by one loop of the the polar curve.

$$r = 3\sin(2\theta)$$

We'll start by finding points that we can use to graph the curve. In order to do so, we'll take the value inside the trigonometric function, set it equal to $\pi/2$, and solve for θ.

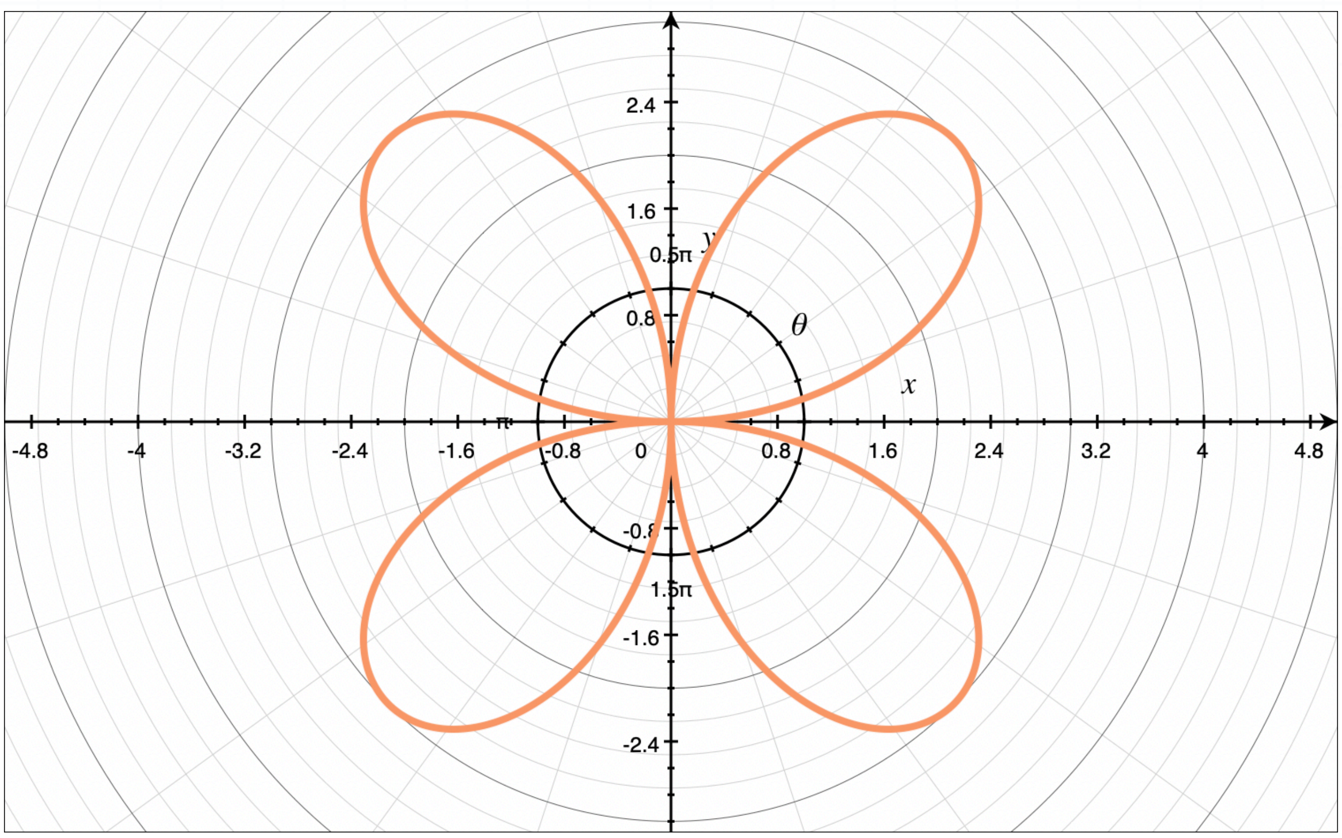
$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

We need to find coordinate points for multiples of $\pi/4$ in the interval $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$r = 3\sin(2\theta)$	0	3	0	-3	0	3	0	-3	0

Plotting these points on polar axes, we get



From the graph, we can see that the curve starts at $(0,0)$, goes out to 3 at an angle $\pi/4$, then curves back to the origin at the angle $\pi/2$. Plugging this into the area formula, we get

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[3\sin(2\theta)\right]^2\ d\theta$$

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[9 \sin^2(2\theta) \right] \ d\theta$$

$$A = \frac{9}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \ d\theta$$

We'll use u-substitution, letting

$$u = 2\theta$$

$$du = 2 \ d\theta$$

$$d\theta = \frac{du}{2}$$

We'll substitute into the integral.

$$A = \frac{9}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2 u \ \frac{du}{2}$$

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2 u \ du$$

Since $\sin^2 u = \frac{1}{2} \left[1 - \cos(2u) \right]$, we get

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2} \left[1 - \cos(2u) \right] \ du$$

$$A = \frac{9}{4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2} \cos(2u) \ du$$

$$A=\frac{9}{4}\left[\frac{1}{2}u-\frac{1}{4}\sin(2u)\right]\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}}$$

Back-substituting for u, we get

$$A=\frac{9}{4}\left[\frac{1}{2}(2\theta)-\frac{1}{4}\sin(2(2\theta))\right]\Bigg|_{0}^{\frac{\pi}{2}}$$

$$A=\frac{9}{4}\left[\theta-\frac{1}{4}\sin(4\theta)\right]\Bigg|_{0}^{\frac{\pi}{2}}$$

$$A=\frac{9}{4}\left[\frac{\pi}{2}-\frac{1}{4}\sin\left(4\cdot\frac{\pi}{2}\right)-\left(0-\frac{1}{4}\sin(4\cdot 0)\right)\right]$$

$$A=\frac{9}{4}\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi+\frac{1}{4}\sin 0\right)$$

$$A=\frac{9}{4}\left(\frac{\pi}{2}-\frac{1}{4}(0)+\frac{1}{4}(0)\right)$$

$$A=\frac{9}{4}\left(\frac{\pi}{2}\right)$$

$$A=\frac{9\pi}{8}$$

Area between polar curves

In order to calculate the area between two polar curves, we'll

1. Find the points of intersection if the interval isn't given

2. Graph the curves to confirm the points of intersection

3. For each enclosed region, use the points of intersection to find upper and lower limits of integration $[\alpha, \beta]$

4. For each enclosed region, determine which curve is the outer curve and which is the inner

5. Plug this into the formula for area between curves,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_O^2 - r_I^2)\ d\theta$$

where $[\alpha, \beta]$ is the interval that defines the area, r_O is the outer curve, and r_I is the inner curve

Example

Find the area between the polar curves $r = 2$ and $r = 3 + 2\sin\theta$.

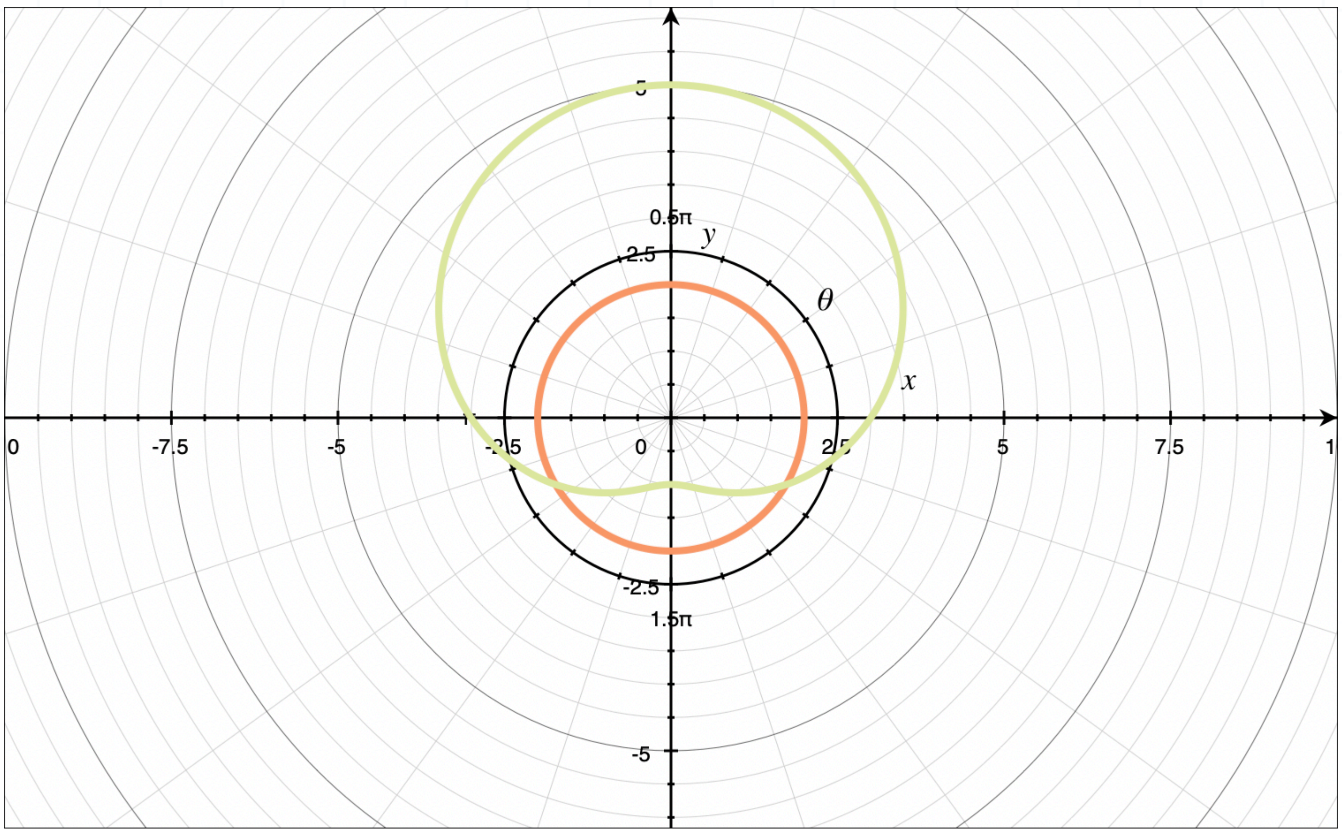
Since the problem doesn't give us an interval over which to evaluate the area, we'll need to find the points of intersection of the curves. We'll set the polar curves equal to each other and solve for θ.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

We'll graph the curves to confirm the points of intersection.



Based on the graph above, the area between curves is given by

$$A_T = A_1 + A_2$$

where A_T is total area, A_1 is the larger section, and A_2 is the smaller section. We always want to work in a counterclockwise direction, which means that, in order to find A_1 and A_2, we'll use the intervals

$A_1 \qquad \left[\frac{11\pi}{6}, \frac{7\pi}{6}\right]$

$A_2 \qquad \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

However, we always need $\alpha < \beta$ in our interval, so we'll change the interval for A_1 into its equivalent $-\theta$, and we'll get

$A_1 \qquad \left[-\frac{\pi}{6}, \frac{7\pi}{6}\right]$

$A_2 \qquad \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

We'll also need to use the graph to indicate which curve is the outer curve and which is the inner curve. We'll say

	Interval	**Outer**	**Inner**
A_1	$\left[-\frac{\pi}{6}, \frac{7\pi}{6}\right]$	$r_O = 3 + 2\sin\theta$	$r_I = 2$
A_2	$\left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$	$r_O = 2$	$r_I = 3 + 2\sin\theta$

Now we can plug everything we've found into the area formula.

$$A_T = A_1 + A_2$$

$$A_T = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}((3 + 2\sin\theta)^2 - (2)^2)\ d\theta + \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}((2)^2 - (3 + 2\sin\theta)^2)\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (3 + 2\sin\theta)(3 + 2\sin\theta) - 4\ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - (3 + 2\sin\theta)(3 + 2\sin\theta)\ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 9 + 12\sin\theta + 4\sin^2\theta - 4 \ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - (9 + 12\sin\theta + 4\sin^2\theta) \ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5 \ d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4 - 9 - 12\sin\theta - 4\sin^2\theta \ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5 \ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 4\sin^2\theta + 12\sin\theta + 5 \ d\theta$$

Since $2\sin^2\theta = 1 - \cos(2\theta)$,

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2(1 - \cos(2\theta)) + 12\sin\theta + 5 \ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2(1 - \cos(2\theta)) + 12\sin\theta + 5 \ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2 - 2\cos(2\theta) + 12\sin\theta + 5 \ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 2 - 2\cos(2\theta) + 12\sin\theta + 5 \ d\theta$$

$$A_T = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 12\sin\theta - 2\cos(2\theta) + 7 \ d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} 12\sin\theta - 2\cos(2\theta) + 7 \ d\theta$$

$$A_T = \frac{1}{2}(-12\cos\theta - \sin(2\theta) + 7\theta)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} - \frac{1}{2}(-12\cos\theta - \sin(2\theta) + 7\theta)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

Evaluate over the interval.

$$A_T = \frac{1}{2}\left[-12\cos\frac{7\pi}{6} - \sin\frac{14\pi}{6} + 7\left(\frac{7\pi}{6}\right) - \left(-12\cos\left(-\frac{\pi}{6}\right) - \sin\left(-\frac{2\pi}{6}\right) + 7\left(-\frac{\pi}{6}\right)\right)\right]$$

$$-\frac{1}{2}\left[-12\cos\frac{11\pi}{6}-\sin\frac{22\pi}{6}+7\left(\frac{11\pi}{6}\right)-\left(-12\cos\frac{7\pi}{6}-\sin\frac{14\pi}{6}+7\left(\frac{7\pi}{6}\right)\right)\right]$$

$$A_T=\frac{1}{2}\left[-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}-\left(-12\cos\frac{11\pi}{6}-\sin\frac{5\pi}{3}-\frac{7\pi}{6}\right)\right]$$

$$-\frac{1}{2}\left[-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+\frac{77\pi}{6}-\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}\right)\right]$$

$$A_T=\frac{1}{2}\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+\frac{49\pi}{6}+12\cos\frac{11\pi}{6}+\sin\frac{5\pi}{3}+\frac{7\pi}{6}\right)$$

$$-\frac{1}{2}\left(-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+\frac{77\pi}{6}+12\cos\frac{7\pi}{6}+\sin\frac{7\pi}{3}-\frac{49\pi}{6}\right)$$

$$A_T=\frac{1}{2}\left(-12\cos\frac{7\pi}{6}-\sin\frac{7\pi}{3}+12\cos\frac{11\pi}{6}+\sin\frac{5\pi}{3}+\frac{28\pi}{3}\right)$$

$$-\frac{1}{2}\left(-12\cos\frac{11\pi}{6}-\sin\frac{11\pi}{3}+12\cos\frac{7\pi}{6}+\sin\frac{7\pi}{3}+\frac{14\pi}{3}\right)$$

Simplify the trigonometric functions.

$$A_T=\frac{1}{2}\left[-12\left(-\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}+12\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}+\frac{28\pi}{3}\right]$$

$$-\frac{1}{2}\left[-12\left(\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)+12\left(-\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}+\frac{14\pi}{3}\right]$$

$$A_T=\frac{1}{2}\left(\frac{12\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\frac{12\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\frac{28\pi}{3}\right)$$

$$-\frac{1}{2}\left(-\frac{12\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\frac{12\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{14\pi}{3}\right)$$

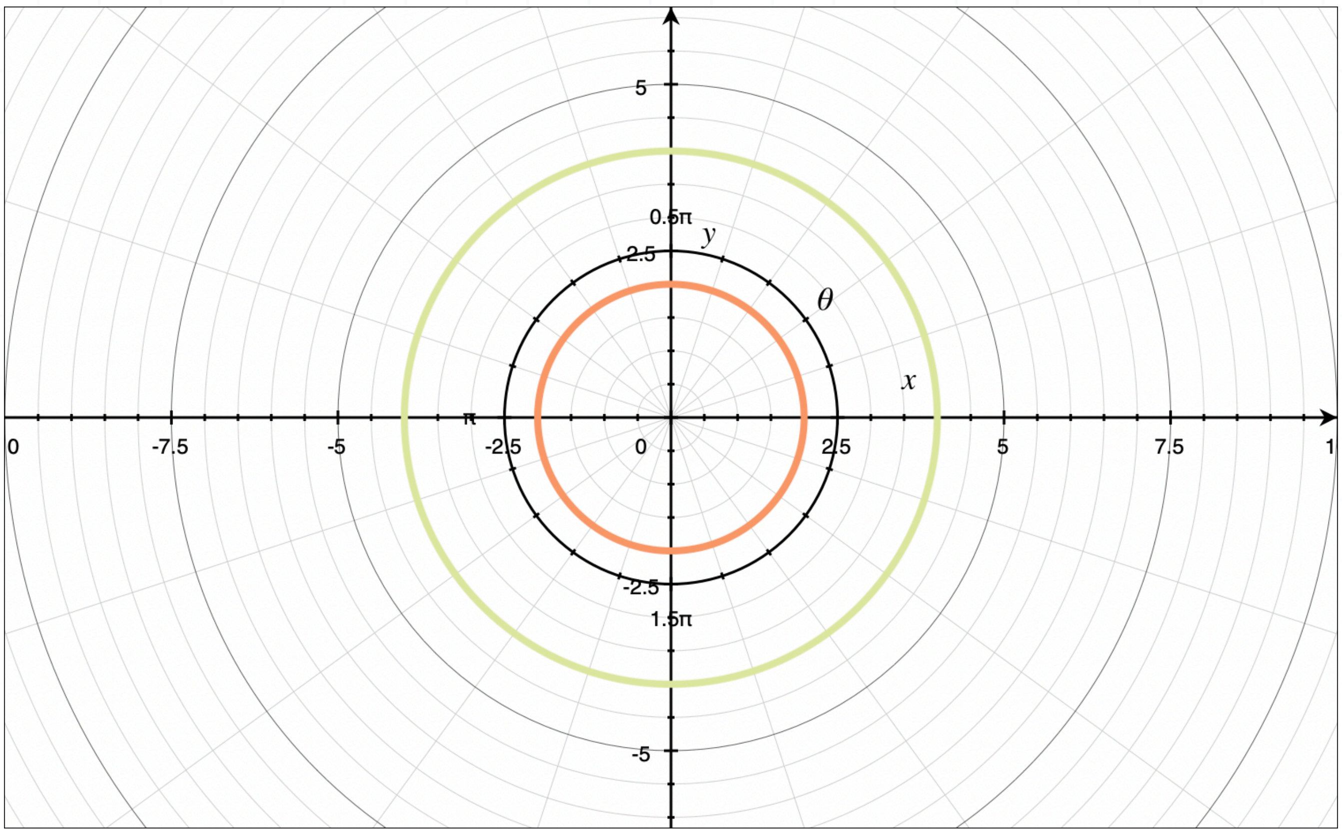
$$A_T=\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{28\pi}{6}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}+\frac{12\sqrt{3}}{4}-\frac{\sqrt{3}}{4}-\frac{14\pi}{6}$$

$$A_T=\frac{44\sqrt{3}}{4}+\frac{14\pi}{6}$$

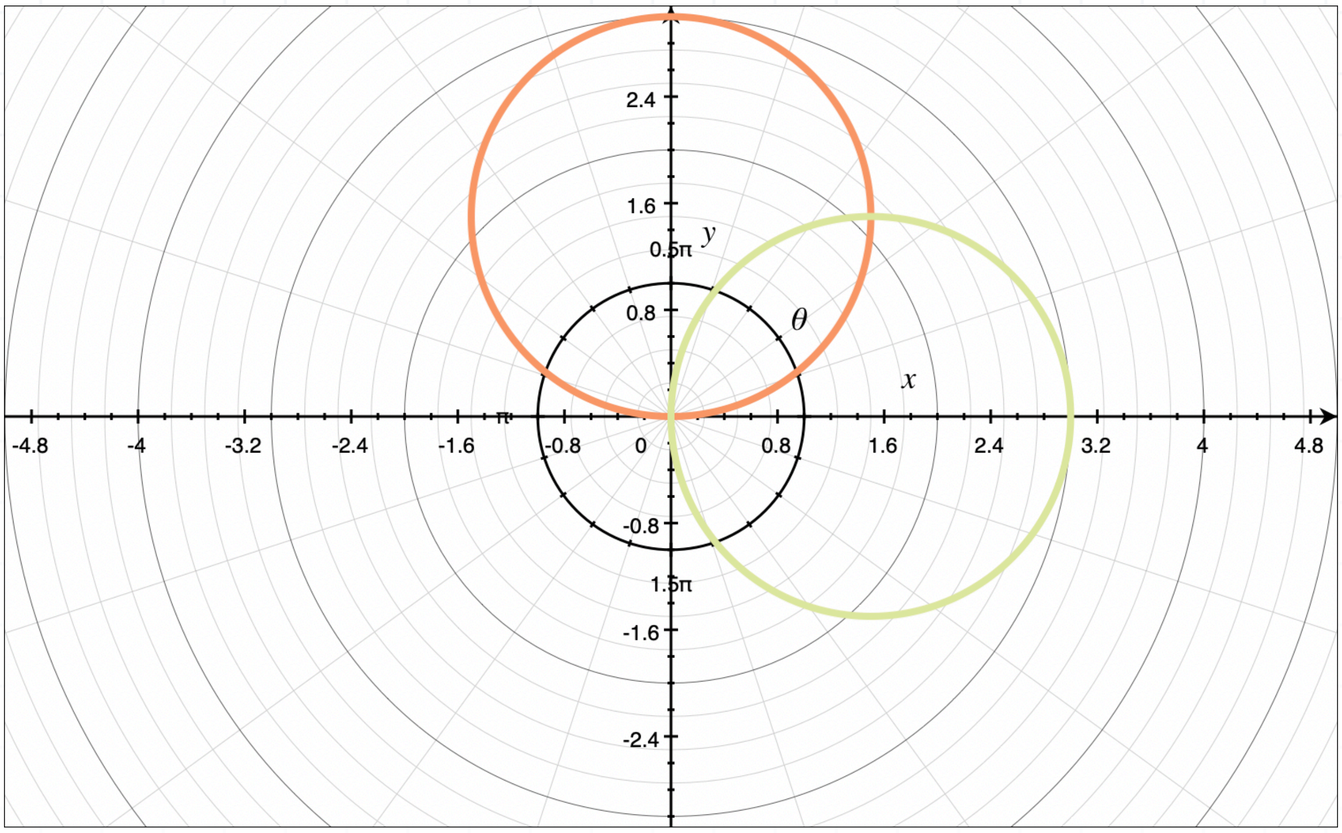
$$A_T=11\sqrt{3}+\frac{7\pi}{3}$$

Area inside both polar curves

The area inside both curves is the area which is enclosed by both curves. For example, given the polar curves 1. $r=2$ and 2. $r=4$, only the area inside $r=2$ is inside both curves. Some of the area inside $r=4$ is outside of $r=2$, so that area isn't inside both curves.



Unfortunately, not all of these kinds of problems will be this easy. There will always be multiple ways to go about finding the area inside both polar curves. For example, given the curves 1. $r=3\sin\theta$ and 2. $r=3\cos\theta$ whose graphs are



we could

find the area inside $r = 3\sin\theta$ and then subtract the area inside $r = 3\sin\theta$ but outside $r = 3\cos\theta$

find the area inside $r = 3\cos\theta$ and then subtract the area inside $r = 3\cos\theta$ but outside $r = 3\sin\theta$

Some sections of area will be easier to solve for than others. Therefore, the best way to solve for the area inside both curves is to graph them, then based on the graphs, look for the easiest areas to calculate and use those to go about finding the area inside both curves.

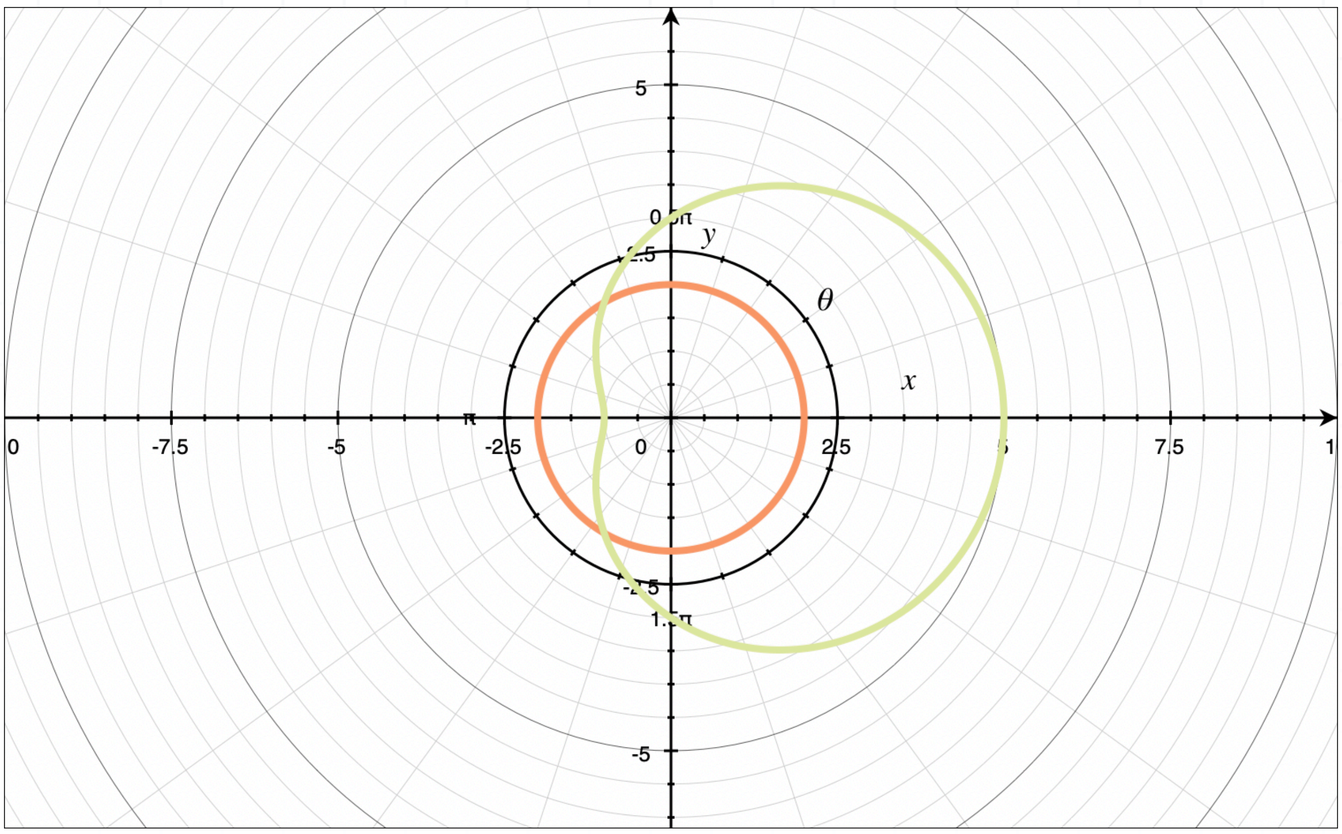
Example

Find the area of the region enclosed by both polar curves.

$$r = 2$$

$$r = 3 + 2\cos\theta$$

We'll graph the given curves to see what we're dealing with.



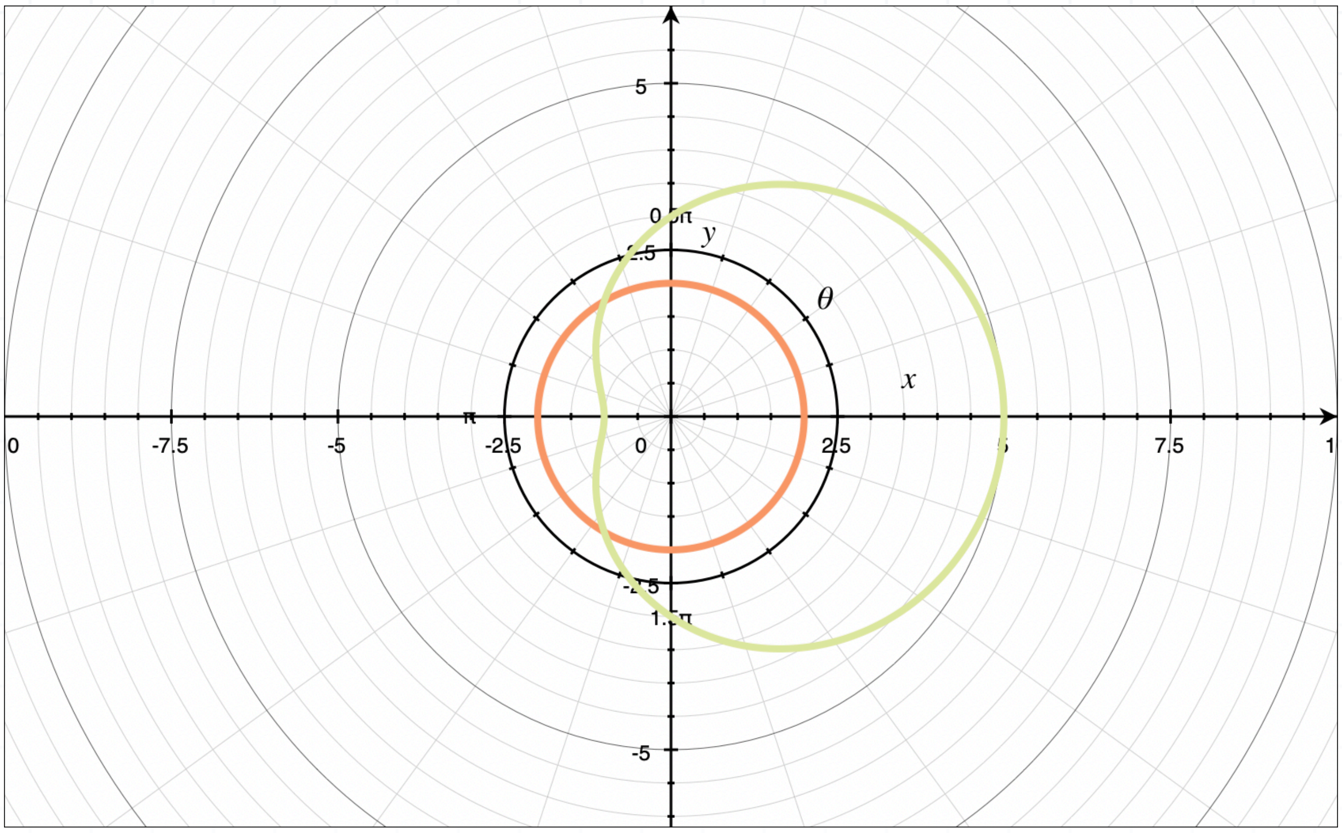
From the graph, we can see that most of 1. $r = 2$ is also enclosed by 2. $r = 3 + 2\cos\theta$. To find the area inside both curves, we could

use $A_T = A_1 - A_2$, where

A_T is the area inside both curves

A_1 is all of the area inside $r = 3 + 2\cos\theta$

A_2 is the area inside $r = 3 + 2\cos\theta$ but outside $r = 2$

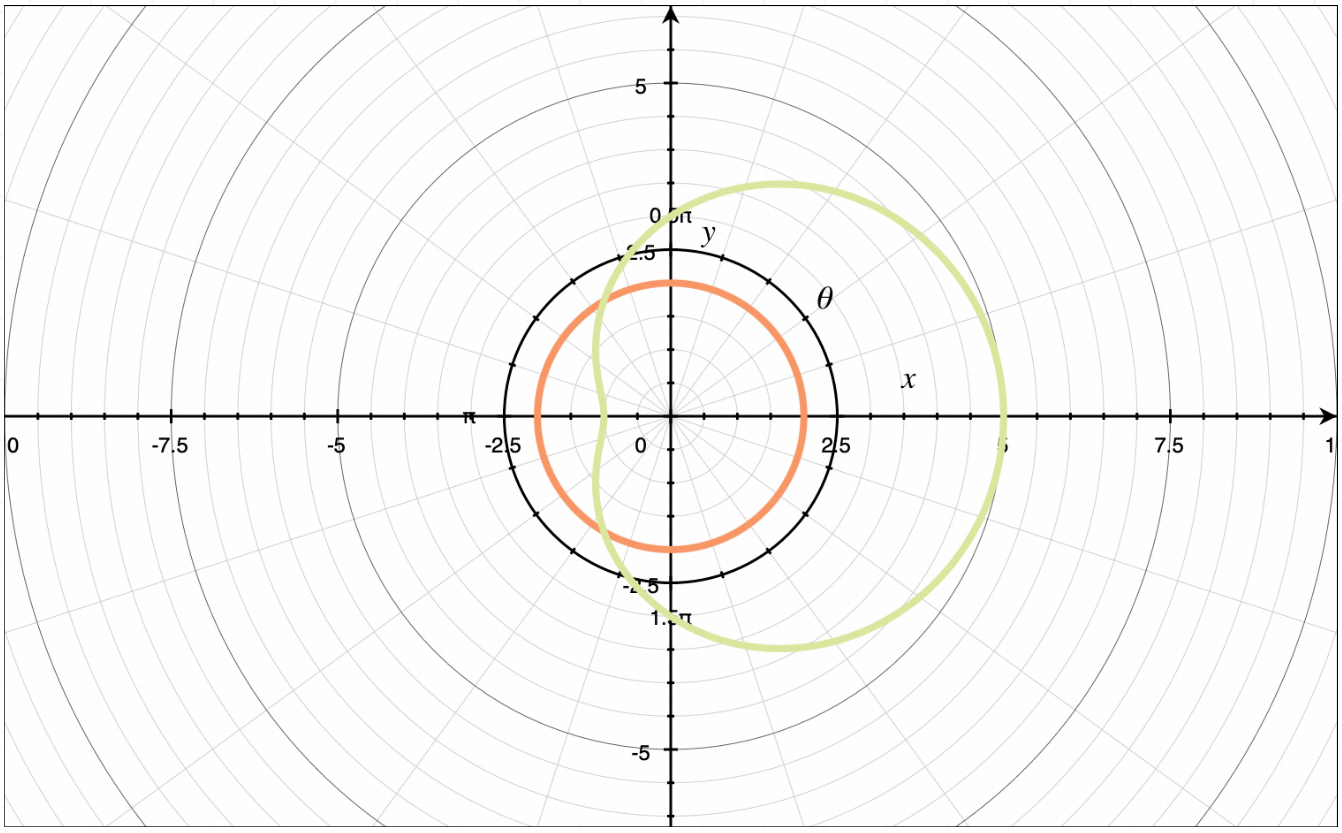


use $A_T = A_1 - A_2$, where

A_T is the area inside both curves

A_1 is all of the area inside $r = 2$

A_2 is the area inside $r = 2$ but outside $r = 3 + 2\cos\theta$



Since finding the area inside $r = 2$ is a little easier than finding the area inside $r = 3 + 2\cos\theta$, we'll use the second option.

To find the area inside $r = 2$, we'll use the area formula over the interval $[0,2\pi]$.

$$A_1 = \int_{\alpha}^{\beta} \frac{1}{2}r^2\ d\theta$$

$$A_1 = \int_{0}^{2\pi} \frac{1}{2}(2)^2\ d\theta$$

$$A_1 = \int_{0}^{2\pi} 2\ d\theta$$

$$A_1 = 2\theta\Big|_0^{2\pi}$$

$$A_1 = 2(2\pi) - 2(0)$$

$$A_1 = 4\pi$$

To find the area inside $r = 2$ but outside $r = 3 + 2\cos\theta$, we'll find the intersection points and then use the formula

$$A_2 = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_O^2 - r_I^2\right)\ d\theta, \text{ where}$$

$$r_O = 2$$

$$r_I = 3 + 2\cos\theta$$

We'll find the intersection points by setting the polar equations equal to each other and solving for θ.

$$2 = 3 + 2\cos\theta$$

$$-\frac{1}{2} = \cos\theta$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

The interval for this section of area is

$$[\alpha, \beta] = \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$$

If instead we wanted to find the area of outside $r = 2$ but inside $r = 3 + 2\cos\theta$, we'd have to change the interval to $[\alpha, \beta] = \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$.

Plugging everything we know about that A_2 into the area formula, we get

$$A_2 = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left[(2)^2 - (3 + 2\cos\theta)^2\right]\ d\theta$$

$$A_2 = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4 - \left(9 + 12\cos\theta + 4\cos^2\theta\right)\ d\theta$$

$$A_2 = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4 - 9 - 12\cos\theta - 4\cos^2\theta\ d\theta$$

$$A_2 = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} -5 - 12\cos\theta - 4\cos^2\theta\ d\theta$$

$$A_2 = -\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4\cos^2\theta + 12\cos\theta + 5\ d\theta$$

Since $\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$,

$$A_2 = -\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4\left[\frac{1}{2}\left[1 + \cos(2\theta)\right]\right] + 12\cos\theta + 5\ d\theta$$

$$A_2 = -\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2\left[1 + \cos(2\theta)\right] + 12\cos\theta + 5\ d\theta$$

$$A_2 = -\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 2\cos(2\theta) + 12\cos\theta + 5\ d\theta$$

$$A_2 = -\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2\cos(2\theta) + 12\cos\theta + 7\ d\theta$$

$$A_2 = -\frac{1}{2}\left[\sin(2\theta) + 12\sin\theta + 7\theta\right]\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

Evaluate over the interval.

$$A_2 = -\frac{1}{2}\left[\sin\left(2\cdot\frac{4\pi}{3}\right) + 12\sin\frac{4\pi}{3} + 7\cdot\frac{4\pi}{3}\right] + \frac{1}{2}\left[\sin\left(2\cdot\frac{2\pi}{3}\right) + 12\sin\frac{2\pi}{3} + 7\cdot\frac{2\pi}{3}\right]$$

$$A_2 = -\frac{1}{2}\left(\sin\frac{8\pi}{3} + 12\sin\frac{4\pi}{3} + \frac{28\pi}{3}\right) + \frac{1}{2}\left(\sin\frac{4\pi}{3} + 12\sin\frac{2\pi}{3} + \frac{14\pi}{3}\right)$$

Simplify the trigonometric functions.

$$A_2 = -\frac{1}{2}\left[\frac{\sqrt{3}}{2} + 12\left(-\frac{\sqrt{3}}{2}\right) + \frac{28\pi}{3}\right] + \frac{1}{2}\left[\left(-\frac{\sqrt{3}}{2}\right) + 12\left(\frac{\sqrt{3}}{2}\right) + \frac{14\pi}{3}\right]$$

$$A_2 = -\frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{12\sqrt{3}}{2} + \frac{28\pi}{3}\right) + \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + \frac{12\sqrt{3}}{2} + \frac{14\pi}{3}\right)$$

$$A_2 = -\frac{1}{2}\left(-\frac{11\sqrt{3}}{2} + \frac{28\pi}{3}\right) + \frac{1}{2}\left(\frac{11\sqrt{3}}{2} + \frac{14\pi}{3}\right)$$

$$A_2 = \frac{11\sqrt{3}}{4} - \frac{28\pi}{6} + \frac{11\sqrt{3}}{4} + \frac{14\pi}{6}$$

$$A_2 = \frac{22\sqrt{3}}{4} - \frac{14\pi}{6}$$

$$A_2 = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}$$

Find a common denominator.

$$A_2 = \frac{33\sqrt{3}}{6} - \frac{14\pi}{6}$$

$$A_2 = \frac{33\sqrt{3} - 14\pi}{6}$$

Our last step is to solve for A_T using $A_T = A_1 - A_2$.

$$A_T = 4\pi - \frac{33\sqrt{3} - 14\pi}{6}$$

$$A_T = \frac{24\pi - 33\sqrt{3} + 14\pi}{6}$$

$$A_T = \frac{38\pi - 33\sqrt{3}}{6}$$

Arc length of a polar curve

The arc length of a polar curve is simply the length of a section of a polar parametric curve between two points a and b. We use the formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

where L is the arc length

r is the equation of the polar curve

$\frac{dr}{d\theta}$ is the derivative of the polar curve

a and b are the endpoints of the section

Example

Find the arc length of the polar curve over the given interval.

$$r = \cos^2 \frac{\theta}{2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

Before we can plug into the arc length formula, we need to find $dr/d\theta$.

$$\frac{dr}{d\theta} = 2\cos\frac{\theta}{2}\left[-\sin\frac{\theta}{2}\right]\left(\frac{1}{2}\right)$$

$$\frac{dr}{d\theta} = -\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

Now we can go ahead and solve for the arc length

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^2\frac{\theta}{2}\right)^2 + \left(-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\cos^4\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}\ d\theta$$

Since $\cos^2 x + \sin^2 x = 1$, we get

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^2\frac{\theta}{2}\right)(1)}\ d\theta$$

$$L = \int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2}\ d\theta$$

$$L = 2\sin\frac{\theta}{2}\Big|_0^{\frac{\pi}{2}}$$

$$L = 2\sin\left(\frac{\frac{\pi}{2}}{2}\right) - 2\sin\left(\frac{0}{2}\right)$$

$$L = 2\sin\frac{\pi}{4} - 2\sin 0$$

$$L = 2 \cdot \frac{\sqrt{2}}{2} - 2(0)$$

$$L = \sqrt{2}$$

Let's do another example.

Example

Find the arc length of the polar curve over the given interval.

$$r = e^{2\theta}$$

$$0 \le \theta \le \pi$$

Before we can plug into the arc length formula, we need to find $dr/d\theta$.

$$\frac{dr}{d\theta} = 2e^{2\theta}$$

Plugging everything into the formula, we get

$$s = \int_0^{\pi} \sqrt{\left(e^{2\theta}\right)^2 + \left(2e^{2\theta}\right)^2} \ d\theta$$

$$s = \int_0^{\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} \ d\theta$$

$$s = \int_0^{\pi} \sqrt{5e^{4\theta}} \ d\theta$$

$$s = \sqrt{5}\int_0^{\pi} e^{2\theta}\ d\theta$$

$$s = \frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^{\pi}$$

$$s = \frac{\sqrt{5}}{2}e^{2(\pi)} - \frac{\sqrt{5}}{2}e^{2(0)}$$

$$s = \frac{\sqrt{5}}{2}e^{2\pi} - \frac{\sqrt{5}}{2}$$

$$s = \frac{\sqrt{5}\left(e^{2\pi} - 1\right)}{2}$$

Surface area of revolution of a polar curve

We can find the surface area of the object created when we rotate a polar curve around either the x-axis or the y-axis using the formulas

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \qquad \text{around the } x\text{-axis}$$

$$S_y = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \qquad \text{around the } y\text{-axis}$$

where r is the equation of the polar curve

$\frac{dr}{d\theta}$ is the derivative of the polar curve

$[\alpha, \beta]$ is the interval

We can solve for x and y as needed using the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example

Find the surface area of revolution of the polar curve over the given interval.

$$r = 5\cos\theta$$

$0 \le \theta \le \pi$

around the y-axis

Before we can plug into the formula, we need to find x and $dr/d\theta$.

Since $x = r\cos\theta$, we get

$$x = 5\cos\theta\cos\theta$$

$$x = 5\cos^2\theta$$

To find $dr/d\theta$, we'll take the derivative of the given polar equation.

$$r = 5\cos\theta$$

$$\frac{dr}{d\theta} = -5\sin\theta$$

We'll plug everything into the formula for the surface area of revolution about the y-axis.

$$S_y = \int_\alpha^\beta 2\pi x\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S_y = \int_0^\pi 2\pi\left(5\cos^2\theta\right)\sqrt{\left(5\cos\theta\right)^2 + \left(-5\sin\theta\right)^2}\ d\theta$$

$$S_y = 10\pi\int_0^\pi \cos^2\theta\sqrt{25\cos^2\theta + 25\sin^2\theta}\ d\theta$$

$$S_y = 10\pi \int_0^{\pi} \cos^2\theta \sqrt{25\left(\cos^2\theta + \sin^2\theta\right)}\ d\theta$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$S_y = 50\pi \int_0^{\pi} \cos^2\theta \sqrt{1}\ d\theta$$

$$S_y = 50\pi \int_0^{\pi} \cos^2\theta\ d\theta$$

Since $\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$,

$$S_y = 50\pi \int_0^{\pi} \frac{1}{2}\left[1 + \cos(2\theta)\right]\ d\theta$$

$$S_y = 50\pi \int_0^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\ d\theta$$

$$S_y = 25\pi \int_0^{\pi} 1 + \cos(2\theta)\ d\theta$$

$$S_y = 25\pi \left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{\pi}$$

$$S_y = 25\pi\theta + \frac{25\pi}{2}\sin(2\theta)\Bigg|_0^{\pi}$$

$$S_y = 25\pi(\pi) + \frac{25\pi}{2}\sin(2\pi) - \left[25\pi(0) + \frac{25\pi}{2}\sin(2(0))\right]$$

$$S_y = 25\pi^2 + \frac{25\pi}{2}(0) - 25\pi(0) - \frac{25\pi}{2}(0)$$

$$S_y = 25\pi^2$$

Sequences vs. series

Sequences and series are almost always studied together, because they're so closely related.

A **sequence** is just a list of terms in a specific order, and is denoted by

$$a_n$$

A **series** is the sum of a sequence, and is denoted by

$$\sum_{n=b}^{c} a_n$$

where b is the beginning of the interval you're calculating, typically $b = 1$, where c is the end of the interval being calculated, and where a_n is the sequence we're taking the sum of.

Both sequences and series can be defined over a closed or infinite interval.

Example

Calculate the series.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

This question is asking us to calculate an infinite series. The first step will be to calculate the first few terms. Let's calculate $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

When $n = 1$, $\quad a_1 = \frac{1}{2^1} \quad$ so $\quad a_1 = \frac{1}{2}$

When $n = 2$, $\quad a_2 = \frac{1}{2^2} \quad$ so $\quad a_2 = \frac{1}{4}$

When $n = 3$, $\quad a_3 = \frac{1}{2^3} \quad$ so $\quad a_3 = \frac{1}{8}$

When $n = 4$, $\quad a_4 = \frac{1}{2^4} \quad$ so $\quad a_4 = \frac{1}{16}$

Now that we have the first four terms we can start our summation. Let's add the first four terms to see what we get

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$\frac{8}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16}$$

$$\frac{15}{16}$$

From our answer, it looks like our series is approaching 1 as $n \to \infty$. Let's calculate the fifth term to see if our hypothesis holds.

When $n = 5$ $\quad a_5 = \frac{1}{2^5} \quad$ so $\quad a_5 = \frac{1}{32}$

Now let's add the fifth term to the sum of the first four terms

$$\frac{15}{16}+\frac{1}{32}$$

$$\frac{30}{32}+\frac{1}{32}$$

$$\frac{31}{32}$$

This number is even closer to 1 therefore our hypothesis is correct. We can write our answer as

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^{\infty}}$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=1$$

The series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is equal to 1.

Formula for the general term

The general term of a sequence a_n is a term that can represent every other term in the sequence. It relates each term is the sequence to its place in the sequence. For example, given the sequence

$$\{-1, -2, -3, -4, -5, \ \dots\}$$

we need to realize that

the first term of the sequence is -1

the second term of the sequence is -2

the third term of the sequence is -3

the fourth term of the sequence is -4

the fifth term of the sequence is -5

In other words

when $n = 1$, the value of the sequence is -1

when $n = 2$, the value of the sequence is -2

when $n = 3$, the value of the sequence is -3

when $n = 4$, the value of the sequence is -4

when $n = 5$, the value of the sequence is -5

Based on this information, the value of the sequence is always $-n$, so a formula for the general term of the sequence is

$$a_n = -n$$

This was an easy example, but we'll always follow this same process to find the general term of any sequence. We'll

Match the terms of the sequence to their place in the sequence n

Find the pattern that relates each term to its corresponding value of n

We always have to pay special attention to the signs of the terms in the sequence.

If all of the terms in the sequence are positive, a_n will be positive.

If all of the terms in the sequence are negative, a_n will be negative ($-a_n$).

If the signs of the terms are alternating, and

if the odd terms ($n = 1, 3, 5, \ldots$) are negative, a_n will include $(-1)^n$

if the even terms ($n = 2, 4, 6, \ldots$) are negative, a_n will include $(-1)^{n-1}$ or $(-1)^{n+1}$

Example

Find a formula for the general term a_n of the sequence.

$$\left\{-\frac{1}{4}, \frac{8}{5}, -\frac{27}{6}, \frac{64}{7}\right\}$$

Our first step is to match the terms of the sequence to their place in the sequence n. We have

$a_1 = -\frac{1}{4}$ when $n = 1$

$a_2 = \frac{8}{5}$ when $n = 2$

$a_3 = -\frac{27}{6}$ when $n = 3$

$a_4 = \frac{64}{7}$ when $n = 4$

Now we can start examining the sequence. First we notice that the signs of the terms are alternating, and the odd terms are negative, which means that our formula for a_n will include $(-1)^n$.

Next, we'll look at just the numerator of each term in our sequence. Taking the numerator only, we see that

$a_1 = 1$ when $n = 1$

$a_2 = 8$ when $n = 2$

$a_3 = 27$ when $n = 3$

$a_4 = 64$ when $n = 4$

We should recognize that these are perfect cubes. In other words

$a_1 = (1)^3$ when $n = 1$

$a_2 = (2)^3$ when $n = 2$

$a_3 = (3)^3$ when $n = 3$

$a_4 = (4)^3$ when $n = 4$

This tells us that the numerator of every term in our sequence can be represented by n^3, so the numerator in the general term a_n will be n^3.

Now let's look at the denominator of each term in our sequence. Taking the denominator only, we see that

$a_1 = 4$ when $n = 1$

$a_2 = 5$ when $n = 2$

$a_3 = 6$ when $n = 3$

$a_4 = 7$ when $n = 4$

The denominator of each term is always 3 higher than the corresponding value of n, which means that the denominator of every term in our sequence can be represented by $n + 3$, so the denominator of the general term a_n will be $n + 3$.

We've looked at every part of each term (the sign, the numerator, and the denominator), so it's time to start putting it all together.

We said that we had to include $(-1)^n$ to address the negative odd terms, that the numerator was n^3, and that the denominator was $n+3$. So a formula for the general term a_n of the sequence

$$\left\{-\frac{1}{4},\frac{8}{5},-\frac{27}{6},\frac{64}{7}\right\}$$

is

$$a_n=(-1)^n\frac{n^3}{n+3}$$

Convergence of a sequence

If we say that a sequence converges, it means that the limit of the sequence exists as $n \to \infty$. If the limit of the sequence as $n \to \infty$ does not exist, we say that the sequence diverges. A sequence always either converges or diverges, there is no other option. This doesn't mean we'll always be able to tell whether the sequence converges or diverges, sometimes it can be very difficult for us to determine convergence or divergence.

There are many ways to test a sequence to see whether or not it converges.

Sometimes all we have to do is evaluate the limit of the sequence at $n \to \infty$. If the limit exists then the sequence converges, and the answer we found is the value of the limit.

Sometimes it's convenient to use the squeeze theorem to determine convergence because it'll show whether or not the sequence has a limit, and therefore whether or not it converges. Then we'll take the limit of our sequence to get the real value of the limit.

Example

Say whether or not the sequence converges and find the limit of the sequence if it does converge.

$$a_n = \frac{\sin^2(n)}{3^n}$$

Remember, when a sequence converges, its limit exists at $n \to \infty$.

Let's evaluate the sequence using the squeeze theorem. We'll start by evaluating the numerator of a_n, $\sin^2(n)$. We know that the sine function exists between -1 and 1, so we can say that

$$-1 \le \sin(n) \le 1$$

We also know that when the sine function is squared, it only exists between 0 and 1, so we can modify the inequality to say that

$$0 \le \sin^2(n) \le 1$$

Finally, we can multiply the above inequality by $1/3^n$ to make it match our original sequence.

$$\left(0 \le \sin^2(n) \le 1\right)\frac{1}{3^n}$$

$$\frac{0}{3^n} \le \frac{\sin^2(n)}{3^n} \le \frac{1}{3^n}$$

$$0 \le \frac{\sin^2(n)}{3^n} \le \frac{1}{3^n}$$

Now, we have our original sequence bounded by two values. When we take the limit as $n \to \infty$, $1/3^n$ on the right side of the inequality will approach 0.

$$0 \le \lim_{n\to\infty} \frac{\sin^2(n)}{3^n} \le \lim_{n\to\infty} \frac{1}{3^n}$$

$$0 \le \lim_{n \to \infty} \frac{\sin^2(n)}{3^n} \le 0$$

Since the limit of the sequence is bounded by two real numbers, this means that our limit exists and our sequence converges. Finally, we can take the limit of our sequence as it approaches infinity.

$$\lim_{n \to \infty} \frac{\sin^2(n)}{3^n} = \frac{k}{\infty}$$

where k represents the constant number from 0 to 1 that we derived from the inequality $0 \le \sin^2(n) \le 1$. We get ∞ in the denominator because as $n \to \infty$, 3^n will approach ∞. Since we have a constant in the numerator and an infinity large value in the denominator, we know that

$$\lim_{n \to \infty} \frac{\sin^2(n)}{3^n} = 0$$

We can conclude that the sequence

$$a_n = \frac{\sin^2(n)}{3^n}$$

converges and that its limit as $n \to \infty$ is 0.

$$\lim_{n \to \infty} \frac{\sin^2(n)}{3^n} = \lim_{n \to \infty} a_n = 0$$

Limit of a convergent sequence

Remember that a sequence is convergent if its limit exists as $n \to \infty$. So it makes sense that once we know that a sequence is convergent, we should be able to evaluate the limit as $n \to \infty$ and get a real-number answer.

The way that we simplify and evaluate the limit will depend on the kind of functions we have in our sequence (trigonometric, exponential, etc.), but we know that the limit as $n \to \infty$ exists.

Example

Find the limit of the convergent sequence.

$$a_n = \ln\left(4n^3 + 3\right) - \ln\left(3n^3 - 5\right)$$

We've been told the sequence converges, so we already know that the limit will exist as $n \to \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(4n^3 + 3\right) - \ln\left(3n^3 - 5\right)$$

If we remember our laws of logarithms, we know that

$$\ln a - \ln b = \ln \frac{a}{b}$$

so we can simplify the limit to

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \lim_{n\to\infty} \ln\left(\frac{4n^3+3}{3n^3-5}\right)$$

We'll divide each term in our rational function by the variable of the highest degree, n^3.

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \lim_{n\to\infty} \ln\left(\frac{\frac{4n^3}{n^3}+\frac{3}{n^3}}{\frac{3n^3}{n^3}-\frac{5}{n^3}}\right)$$

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \lim_{n\to\infty} \ln\left(\frac{4+\frac{3}{n^3}}{3-\frac{5}{n^3}}\right)$$

Now we will evaluate the limit.

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \ln\left(\frac{4+\frac{3}{\infty}}{3-\frac{5}{\infty}}\right)$$

We know that any fraction that has a constant in the numerator and an infinitely large denominator will approach 0, so

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \ln\left(\frac{4+0}{3-0}\right)$$

$$\lim_{n\to\infty} \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right) = \ln\frac{4}{3}$$

The limit of the convergent sequence $a_n = \ln\left(4n^3+3\right) - \ln\left(3n^3-5\right)$ is $\ln\frac{4}{3}$.

Increasing, decreasing, and not monotonic

Sequences are always either monotonic or not monotonic. If a sequence is monotonic, it means that it's always increasing or always decreasing. If a sequence is sometimes increasing and sometimes decreasing and therefore doesn't have a consistent direction, it means that the sequence is not monotonic. In other words, a non-monotonic sequence is increasing for parts of the sequence and decreasing for others.

The fastest way to make a **guess** about the behavior of a sequence is to calculate the first few terms of the sequence and visually determine if it's increasing, decreasing or not monotonic.

If we want to get more technical and **prove** the behavior of the sequence, we can use the following inequalities.

A sequence is increasing if $a_n \le a_{n+1}$

A sequence is decreasing if $a_n \ge a_{n+1}$

A sequence is not monotonic if $a_n \le a_{n+1} \ge a_{n+2}$ or $a_n \ge a_{n+1} \le a_{n+2}$.

Example

Is the sequence increasing, decreasing or not monotonic?

$$a_n = n^3 + 9$$

We can start by determining the first few values of the sequence. Let's calculate $n=1$, $n=2$, $n=3$ and $n=4$.

When $n=1$ $\quad a_1=(1)^3+9$ $\quad$ so $\quad a_1=10$

When $n=2$ $\quad a_2=(2)^3+9$ $\quad$ so $\quad a_2=17$

When $n=3$ $\quad a_3=(3)^3+9$ $\quad$ so $\quad a_3=36$

When $n=4$ $\quad a_4=(4)^3+9$ $\quad$ so $\quad a_4=73$

The first four terms of our sequence are $\{10,\ 17,\ 36,\ 73\}$. Looking at these first few terms, we can see that the sequence is increasing. If we want to be more strict, we can prove it by showing that $a_n \leq a_{n+1}$.

$$n^3+9 \leq (n+1)^3+9$$

$$n^3+9 \leq n^3+3n^2+3n+10$$

Looking at our inequality we can see that $a_n \leq a_{n+1}$ is true for our sequence. After all, n^3+10 on its own has to be greater than or equal to n^3+9, and adding $3n^2+3n$ will definitely make it greater, so we know for sure that

$$n^3+9 \leq \left(n^3+10\right)+3n^2+3n$$

If we're still unsure, we can always plug in a few values of n to confirm our conclusion.

The sequence $a_n=n^3+9$ is increasing, which means it's also monotonic.

Bounded sequences

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Example

Say whether or not the sequence is bounded, and if it is, find its bounds.

$$a_n = \frac{n^2 + 6}{3n^2 - 1}$$

In order for a sequence to be bounded the sequence needs to be monotonic (either increasing or decreasing). Let's assess our sequence to see if it's monotonic. We can do this by calculating the first few terms of the sequence. Let's calculate $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

When $n = 1$ $\quad a_1 = \frac{(1)^2 + 6}{3(1)^2 - 1}$ so $\quad a_1 = \frac{7}{2}$

When $n = 2$ $\quad a_2 = \frac{(2)^2 + 6}{3(2)^2 - 1}$ so $\quad a_2 = \frac{10}{11}$

When $n = 3$ $\quad a_3 = \frac{(3)^2 + 6}{3(3)^2 - 1}$ so $\quad a_3 = \frac{15}{26}$

When $n = 4$ $\quad a_4 = \frac{(4)^2 + 6}{3(4)^2 - 1}$ so $\quad a_4 = \frac{22}{47}$

The first four terms of the sequence are

$$\left\{ \frac{7}{2}, \frac{10}{11}, \frac{15}{26}, \frac{22}{47} \right\}$$

Looking at the first four terms we can see that the sequence is decreasing, which means it's also monotonic. Since the sequence is decreasing and monotonic, it means it'll also be bounded above.

Now we need to find the bounds of our sequence. In the case of a decreasing sequence, the first term of the sequence $n = 1$ will be the largest term of the sequence. In this case, $a_1 = 7/2$ is our first term. We can say that our sequence is bounded above at $a_n \leq 7/2$.

Now, we can check to see if our sequence is also bounded below. To do this we'll need to take the limit of our sequence as $n \to \infty$.

$$\lim_{n\to\infty} \frac{n^2+6}{3n^2-1}$$

We can divide each term in the numerator and denominator by the highest-degree variable, n^2.

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{6}{n^2}}{\frac{3n^2}{n^2}-\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{1+\frac{6}{n^2}}{3-\frac{1}{n^2}}$$

Now we can evaluate the limit.

$$\frac{1+\frac{6}{\infty}}{3-\frac{1}{\infty}}$$

$$\frac{1+0}{3-0}$$

$$\frac{1}{3}$$

Since we got a real answer for our limit, we know our sequence is also bounded below. In this case our sequence is bounded below at $a_n > 1/3$. Remember, we calculated this answer by taking the limit of our sequence as it approaches infinity, so our sequence will be greater then the bounded limit but not equal to it. Therefore, the sequence

$$a_n = \frac{n^2+6}{3n^2-1}$$

is bounded above at $a_n \leq 7/2$ and bounded below at $a_n > 1/3$.

Sum of the sequence of partial sums

Remember, a normal series is given by

$$\sum_{n=1}^{\infty} a_n$$

where a_n is a sequence whose n values increase by increments of 1. For example, this series could be

$$\sum_{n=1}^{\infty} a_n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \dots\ a_n$$

On the other hand, a partial sums sequence is called s_n, and its n values increase by additive increments. This means that the first term in a partial sums sequence is the $n = 1$ term, the second term is the $n = 1$ term plus the $n = 2$ term, the third term is $(n = 1) + (n = 2) + (n = 3)$, etc.

A normal series is related to its corresponding partial sums sequence by

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

This equation is critical, because it allows us to work backwards from the partial sums sequence to the original series, a_n.

Example

Find the sum of the sequence of the partial sums.

$$s_n = 1 - 2(0.4)^n$$

This question is asking us to find the sum of the series a_n, given its corresponding sequence of partial sums, so we can use

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} 1 - 2(0.4)^n$$

Now we can evaluate the limit.

$$\sum_{n=1}^{\infty} a_n = 1 - 2(0.4)^\infty$$

When 0.4 is raised to the power of ∞, it'll become smaller and smaller and eventually approach 0.

$$\sum_{n=1}^{\infty} a_n = 1 - 2(0)$$

$$\sum_{n=1}^{\infty} a_n = 1$$

The sum of the series a_n given the sequence of the partial sums $s_n = 1 - 2(0.4)^n$ is 1.

Geometric series test

Before we can learn how to determine the convergence or divergence of a geometric series, we have to define a geometric series.

The general form of a geometric series is ar^{n-1} when the index of n begins at $n=1$. Therefore, the sum of a convergent geometric series is given by

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Sometimes you'll come across a geometric series with an index shift, where n starts at $n=0$ instead of $n=1$. In that case, the standard form of the geometric series is ar^n, and if it's convergent, its sum is given by

$$\sum_{n=0}^{\infty} ar^n$$

Both of these are valid geometric series. The important thing is that the exponent on r matches the index. So, if the index starts at $n=1$, we want to make sure we have r^{n-1}. If the index begins at $n=0$, we want to have r^n.

If we look at the expanded forms of both of these series by calculating the first few terms ($n=1$, $n=2$, $n=3$ and $n=4$, ...), we'll see that they're identical.

$$\sum_{n=1}^{\infty} ar^{n-1} = \qquad \sum_{n=0}^{\infty} ar^n =$$

$$\left\{ar^{1-1} + ar^{2-1} + ar^{3-1} + ar^{4-1} + \dots\right\}$$

$$\left\{ar^0 + ar^1 + ar^2 + ar^3 + \dots\right\} \qquad \left\{ar^0 + ar^1 + ar^2 + ar^3 + \dots\right\}$$

$$a\left\{r^0 + r^1 + r^2 + r^3 + \dots\right\} \qquad a\left\{r^0 + r^1 + r^2 + r^3 + \dots\right\}$$

$$a\left\{1 + r + r^2 + r^3 + \dots\right\} \qquad a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

Which means that, regardless of the kind of geometric series we start with, ar^{n-1} with $n = 1$ or ar^n with $n = 0$, we can find the values of a and r in the same way: by expanding the series through its first few terms and then factoring out the a. Then a will be the coefficient we factored out of the series, and r will be the second term in the series, the term immediately following the 1.

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

$$\sum_{n=0}^{\infty} ar^{n} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

Sometimes we won't even need to expand the series. If we can just make the form of the series match one of the standard forms of a geometric series given above, then we'll be able to prove that the series is geometric and identify a and r.

It's important to be able to find the values of a and r because we'll use r to say whether or not the geometric series is convergent or divergent. If we find that it's convergent, then we'll use a and r to find the sum of the series.

Convergence of a geometric series

We can use the value of r in the geometric series test for convergence to determine whether or not the geometric series converges.

The geometric series test says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Let's do an example where we use the geometric series test.

Example

Show that the series is a geometric series, then use the geometric series test to say whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

Since the index starts at $n = 0$, we need to get the series into the form ar^n, which we can do using simple exponent rules.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{2^n 2^{-1}}{3^n}$$

$$\sum_{n=0}^{\infty} 2^{-1}\left(\frac{2^n}{3^n}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$

Now that we have the series in the right form, we can say

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n \text{ where}$$

$$a = \frac{1}{2}$$

$$r = \frac{2}{3}$$

The fact that we've been able to put the series in this form and identify values of a and r proves that it's a geometric series. Now we just need to say whether or not the series converges.

Remember that the geometric series test for converges tells us that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Since

$$\left|\frac{2}{3}\right| = \frac{2}{3} < 1$$

we can say that $|r| < 1$ and therefore that the series converges.

Sum of the geometric series

We already know from the last section that the standard form of a geometric series is

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or

$$\sum_{n=0}^{\infty} ar^{n}$$

Given either of these forms, the geometric series test for convergence says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

When a geometric series converges, we can find its sum.

Sum of a geometric series

We can use the values of a and r and the formula for the sum of a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

or

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

to find the sum of the geometric series.

Example

Calculate the sum of the geometric series.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

We showed in the last section that this series was geometric by rewriting it as

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{2^n 2^{-1}}{3^n}$$

$$\sum_{n=0}^{\infty} 2^{-1}\left(\frac{2^n}{3^n}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$

Now that we have the series in the right form, we can say

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n$$ where

$$a = \frac{1}{2}$$

$$r = \frac{2}{3}$$

Since the sum of a geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

we can say that the sum is

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{2}}{1-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{3}{3}-\frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{1}{3}}$$

$$\frac{1}{2}\cdot\frac{3}{1}$$

$$\frac{3}{2}$$

We could have also found the sum by expanding the series through its first few terms and identifying values for a and r.

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{1}{2}\left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots\right]$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{1}{2}\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots\right)$$

So

$$a = \frac{1}{2}$$

$$r = \frac{2}{3}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{2}}{1 - \frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{3}{3} - \frac{2}{3}}$$

$$\frac{\frac{1}{2}}{\frac{1}{3}}$$

$$\frac{1}{2} \cdot \frac{3}{1}$$

$$\frac{3}{2}$$

Geometric series for repeating decimals

We can use the formula for the sum of a geometric series to quickly and accurately convert a repeating decimal into a ratio of integers, in other words, into a fraction with whole numbers in the numerator and denominator.

We'll follow these steps:

1. Separate the non-repeating part from the repeating part of the decimal.

2. In a table, match each repeated part with its last decimal place.

3. Create a sum of each part, dividing the repeated parts by their ending decimal places.

4. Identify the geometric series within the sum and set values for a and r from the formula for the sum of a geometric series, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

5. Use the formula to convert the series into one fraction.

6. Add this geometric series fraction to the fraction for the non-repeated part.

Example

Express the repeating decimal as a ratio of integers.

$$1.6\overline{73}$$

The bar over the $.073$ indicates that this is the portion of the decimal that repeats. This tells us that the decimal looks like

$$1.67373737373737373...$$

We've been asked to convert this decimal value into a fraction with a real-number numerator and denominator.

Our first step is to separate the non-repeating part from the repeating part of the decimal.

$$1.6 + .07373737373737373...$$

We add a 0 in the tenths place of our repeating part because it's holding the place of the $.6$ we pulled out into the non-repeating part. The repeating sequence starts with the first 7 in the hundredths place, and we need to keep it there when we separate the decimals, so it's critical to put in the 0.

Next, we'll separate each part of the repeated sequence into its own row of the table below, replacing the decimal places before it with 0s. Once we've built out the left column, we'll put the corresponding place in the second column.

1.6		
73	$.073$	ends at the $1{,}000$s place
73	$.00073$	ends at the $100{,}000$s place

73	$.0000073$	ends at the $10{,}000{,}000$s place

We'll create a sum of the non-repeated part and each of the repeated parts, dividing each repeated part by its ending decimal place.

$$1.6\overline{73} = 1.6 + \frac{73}{1{,}000} + \frac{73}{100{,}000} + \frac{73}{10{,}000{,}000} + \dots$$

The sum we just created is a geometric series, so we can use the formula for the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

to turn the sum into just one fraction. We'll factor out the first fraction from the repeated part.

$$1.6\overline{73} = 1.6 + \frac{73}{1{,}000}\left(1 + \frac{1}{100} + \frac{1}{10{,}000} + \dots\right)$$

With our series in this form, we can identify a and r from the formula for the sum of a geometric series. a is always the value we factored that's sitting right in front of the parentheses. r is the always the second term inside the parentheses, the value immediately following the 1.

$$a = \frac{73}{1{,}000}$$

$$r = \frac{1}{100}$$

Plugging these into the formula for the sum of a geometric series, remembering to keep the non-repeated part of our decimal, 1.6, we get

$$1.6+\sum_{n=0}^{\infty}\frac{73}{1{,}000}\left(\frac{1}{100}\right)^n=1.6+\frac{\frac{73}{1{,}000}}{1-\frac{1}{100}}$$

$$1.6+\frac{\frac{73}{1{,}000}}{\frac{100}{100}-\frac{1}{100}}$$

$$1.6+\frac{\frac{73}{1{,}000}}{\frac{99}{100}}$$

$$1.6+\frac{73}{1{,}000}\cdot\frac{100}{99}$$

$$1.6+\frac{73}{10}\cdot\frac{1}{99}$$

$$1.6+\frac{73}{990}$$

Now we just need to change the non-repeated part of our original decimal into a fraction and then combine these two fractions.

$$\left(1+\frac{6}{10}\right)+\frac{73}{990}$$

$$\left(\frac{10}{10}+\frac{6}{10}\right)+\frac{73}{990}$$

$$\frac{16}{10}+\frac{73}{990}$$

$$\frac{99}{99}\left(\frac{16}{10}\right)+\frac{73}{990}$$

$$\frac{1{,}584}{990} + \frac{73}{990}$$

$$\frac{1{,}657}{990}$$

If you have access to a calculator, you can always double-check yourself. In this case, just use your calculator to divide $1{,}657$ by 990. If you did this correctly, you should get the original repeating decimal, $1.6\overline{73}$.

Convergence of a telescoping series

Telescoping series are series in which all but the first and last terms cancel out. If you think about the way that a long telescope collapses on itself, you can better understand how the middle of a telescoping series cancels itself.

To determine whether a series is telescoping, we'll need to calculate at least the first few terms to see whether the middle terms start canceling with each other.

Convergence of the telescoping series

To see whether or not a telescoping series converges or diverges, we'll need to look at its series of partial sums s_n, which is just the sum of the series through the first n terms.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

Looking at this equation for s_n, we can imagine that the sum of the series through the first four terms would be the partial sum s_4, or

$$s_4 = \sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

What we want to figure out is whether or not we'll get a real-number answer when we take the sum of the entire series, because if we take the sum of the entire series and we get a real-number answer, this means that the series converges. Otherwise, if the sum of the entire series turns out to be infinite, that means the series diverges. In other words, we want to get a real-number answer s, when we use an infinite number of terms n in the series of partial sums s_n. s is the sum of the series, where

$$s = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n$$

So if we calculate the limit as $n \to \infty$ of s_n and we get a real-number answer s, then we can say that the series of partial sums s_n converges, and this lets us also conclude that the series a_n converges. If we cannot find a real-number answer for s, then s_n diverges, and therefore a_n also diverges.

To find s_n, we'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums s_n.

Example

Show that the series is a telescoping series, then say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{1}{1} - \frac{1}{1+1} \qquad 1 - \frac{1}{2}$$

$$n = 2 \qquad \frac{1}{2} - \frac{1}{2+1} \qquad \frac{1}{2} - \frac{1}{3}$$

$$n = 3 \qquad \frac{1}{3} - \frac{1}{3+1} \qquad \frac{1}{3} - \frac{1}{4}$$

$$n = 4 \qquad \frac{1}{4} - \frac{1}{4+1} \qquad \frac{1}{4} - \frac{1}{5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$s = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \right]$$

The series is telescoping if we can cancel all of the terms in the middle (every term but the first and last). When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} 1 - \frac{1}{n+1}$$

If this series of partial sums s_n converges as $n \to \infty$ (if we get a real-number value for s), then we can say that the series of partial sums converges, which allows us to conclude that the telescoping series a_n also converges.

$$s = \lim_{n \to \infty} 1 - \frac{1}{n+1}$$

$$s = \lim_{n \to \infty} 1 - \frac{\frac{1}{n}}{\frac{n}{n} + \frac{1}{n}}$$

$$s = \lim_{n \to \infty} 1 - \frac{\frac{1}{n}}{1 + \frac{1}{n}}$$

$$s = 1 - \frac{0}{1+0}$$

$$s = 1 - 0$$

$$s = 1$$

Since s exists as a real number, the sum of the series is $s = 1$, and we can conclude that the series of partial sums s_n converges, and therefore that the series a_n also converges.

Sum of a telescoping series

Telescoping series are series in which all but the first and last terms cancel out. If you think about the way that a long telescope collapses on itself, you can better understand how the middle of a telescoping series cancels itself.

To determine whether a series is telescoping, we'll need to calculate at least the first few terms to see whether the middle terms start canceling with each other.

Sum of the telescoping series

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

Example

Show that the series is a telescoping series, then find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{1}{1} - \frac{1}{1+1} \qquad 1 - \frac{1}{2}$$

$$n = 2 \qquad \frac{1}{2} - \frac{1}{2+1} \qquad \frac{1}{2} - \frac{1}{3}$$

$$n = 3 \qquad \frac{1}{3} - \frac{1}{3+1} \qquad \frac{1}{3} - \frac{1}{4}$$

$$n = 4 \qquad \frac{1}{4} - \frac{1}{4+1} \qquad \frac{1}{4} - \frac{1}{5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

The series is telescoping if we can cancel all of the terms in the middle (every term but the first and last). When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = 1 - \frac{1}{n+1}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \lim_{n \to \infty} 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{\infty + 1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1$$

The sum of the series is 1.

Limit vs. sum of the series

Sometimes it's easy to forget that there's a difference between the *limit* of an infinite series and the *sum* of an infinite series.

The **limit** of a series is the value the series' terms are approaching as $n \to \infty$.

The **sum** of a series is the value of all the series' terms added together.

They're two very different things, and we use a different calculation to find each one. Let's find both the limit and the sum of the same series so that we can see the difference.

Example

Find the limit and the sum of the series.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}}$$

To find the limit of the series, we'll identify the series as a_n, and then take the limit of a_n as $n \to \infty$.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2^{3n}}{64^{\frac{n}{2}}}$$

$$\lim_{n\to\infty} \frac{\left(2^3\right)^n}{\left(64^{\frac{1}{2}}\right)^n}$$

$$\lim_{n\to\infty} \frac{8^n}{\left(\sqrt{64}\right)^n}$$

$$\lim_{n\to\infty} \frac{8^n}{8^n}$$

$$\lim_{n\to\infty} 1$$

$$1$$

The limit of the series is 1.

To find the sum of the series, we'll expand the series.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{2^{3(1)}}{64^{\frac{1}{2}}} + \frac{2^{3(2)}}{64^{\frac{2}{2}}} + \frac{2^{3(3)}}{64^{\frac{3}{2}}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{2^3}{\sqrt{64}} + \frac{2^6}{\left(\sqrt{64}\right)^2} + \frac{2^9}{\left(\sqrt{64}\right)^3} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{8}{8} + \frac{64}{8^2} + \frac{512}{8^3} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \frac{8}{8} + \frac{64}{64} + \frac{512}{512} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = 1 + 1 + 1 + \dots$$

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{64^{\frac{n}{2}}} = \infty$$

Every term in our series will be equal to 1. Since we have an infinite number of terms in our series, we can say that the sum is infinite.

We can see that the limit of the series is 1, but the sum of the same series is ∞.

Integral test

The integral test for convergence is only valid for series that are

Positive: all of the terms in the series are positive

Decreasing: every term is less than the one before it, $a_{n-1} > a_n$

Continuous: the series is defined everywhere in its domain

If the given series meets these three criteria, then we can use the integral test for convergence to integrate the series and say whether the series is converging or diverging.

Given the series

$$\sum_{n=1}^{\infty} a_n$$

we set $f(x) = a_n$ and evaluate the integral

$$\int_1^{\infty} f(x)\ dx$$

According to the integral test, the series and the integral always have the same result, meaning that they either both converge or they both diverge. This means that if the value of the of the integral

converges to a **real number**, then the series also **converges**

diverges to **infinity**, then the series also **diverges**

Example

Use the integral test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3}{n^2}$$

Before we apply the integral test, we need to confirm that the series is positive, decreasing, and continuous.

We'll find the first few terms of the series using $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{3}{(1)^2} \qquad 3$$

$$n = 2 \qquad \frac{3}{(2)^2} \qquad \frac{3}{4}$$

$$n = 3 \qquad \frac{3}{(3)^2} \qquad \frac{1}{3}$$

$$n = 4 \qquad \frac{3}{(4)^2} \qquad \frac{3}{16}$$

Looking at the first four terms, we can already see that our terms will always be positive. There's no positive value of n that will make its term negative, so we know that our series is positive.

We can also tell by looking at the first four terms that our series is decreasing. The larger the value of n, the larger the denominator becomes, and the smaller the terms become. Therefore, every term in the series will be smaller than the one before it, $a_{n-1} > a_n$. We can also prove it this way:

$$\frac{3}{(n-1)^2} > \frac{3}{n^2}$$

We can flip both fractions if we flip the inequality.

$$\frac{(n-1)^2}{3} < \frac{n^2}{3}$$

$$(n-1)^2 < n^2$$

$$(n-1)^2 - n^2 < 0$$

$$n^2 - 2n + 1 - n^2 < 0$$

$$-2n + 1 < 0$$

$$-2n < -1$$

$$n > \frac{1}{2}$$

For all values $n > 1/2$, the series is decreasing. Since this series starts at $n = 1$, that means the series is decreasing everywhere in its domain.

Finally, we need to confirm that our series is continuous. The series is defined from 1 to ∞, so in order for the series to be discontinuous, the denominator would have to be equal to 0. Since $n^2 \neq 0$ for $1 \leq n \leq \infty$, we know that the series is continuous.

Now that we know the series is positive, decreasing, and continuous, we can use the integral test to say whether the series converges or diverges.

Plugging the given series into the integral, we get

$$\int_1^\infty \frac{3}{x^2}\ dx$$

$$\int_1^\infty 3x^{-2}\ dx$$

$$\frac{3x^{-1}}{-1}\Bigg|_1^\infty$$

$$-3x^{-1}\Big|_1^\infty$$

$$-\frac{3}{x}\Bigg|_1^\infty$$

$$-\frac{3}{\infty}-\left(-\frac{3}{1}\right)$$

$$0+3$$

$$3$$

Since the integral converges to a real number, we know that series also converges.

Note: The value of the integral is not necessarily the value to which the series converges. Don't confuse the value you find for the integral with the limit or the sum of the corresponding series. They're not necessarily the same.

p-series test

If we have a series a_n in the form

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

then we can use the p-series test for convergence to say whether or not a_n will converge. The p-series test says that

a_n will converge when $p > 1$

a_n will diverge when $p \leq 1$

The key is to make sure that the given series matches the format above for a p-series, and then to look at the value of p to determine convergence.

Example

Use the p-series test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

In order to use the p-series test, we need to make sure the format of the given series matches the format above for a p-series, so we'll rewrite the given series as

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

In this format, we can see that $p = 1/2$. The p-series test tells us that a_n diverges when $p \le 1$, so we can say that this series diverges.

Let's try a second example.

Example

Use the p-series test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$$

In order to use the p-series test, we need to make sure the format of the given series matches the format above for a p-series, so we'll rewrite the given series as

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}} = \sum_{n=1}^{\infty} \frac{1}{\left(n^4\right)^{\frac{1}{3}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$$

In this format, we can see that $p = 4/3$. The p-series test tells us that a_n converges when $p > 1$, so we can say that this series converges.

nth term test

When the terms of a series decrease toward 0, we say that the series is converging. Otherwise, the series is diverging.

The nth term test is inspired by this idea, and we can use it to show that a series is diverging. Ironically, even though the nth term test is one of the *convergence* tests that we learn when we study sequences and series, it can only test for *divergence*, it can never confirm convergence.

The nth term test says that

if $\lim_{n \to \infty} a_n \neq 0$

then $\sum a_n$ diverges

In other words,

If we take the limit as $n \to \infty$ and the result is **non-zero**, then the series **diverges**

If we take the limit as $n \to \infty$ and the result is **zero**, then the test is **inconclusive**

Notice that the only conclusion we can draw is that the series diverges. It's possible that the series we're testing converges, but we can't use the nth term test to show convergence. It can only be used to show divergence, and if it doesn't prove divergence, then the test is inconclusive.

Example

Use the nth term test to show whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{4n^3 - 4}{3n^3 + 2}$$

To use the nth term test we'll take the limit of the series as it approaches ∞.

If the result is non-zero, then the series diverges

If the result is zero, then the test is inconclusive

Taking the limit, we get

$$\lim_{n \to \infty} \frac{4n^3 - 4}{3n^3 + 2}$$

We'll simplify the limit by dividing each term in the fraction by the variable of the highest degree, n^3.

$$\lim_{n \to \infty} \frac{4n^3 - 4}{3n^3 + 2} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$\lim_{n \to \infty} \frac{\frac{4n^3}{n^3} - \frac{4}{n^3}}{\frac{3n^3}{n^3} + \frac{2}{n^3}}$$

$$\lim_{n \to \infty} \frac{4 - \frac{4}{n^3}}{3 + \frac{2}{n^3}}$$

Evaluating the limit at ∞, we get

$$\frac{4-\frac{4}{\infty^3}}{3+\frac{2}{\infty^3}}$$

When we have a fraction in which the numerator is constant and the denominator is infinite, the whole fraction approaches 0.

$$\frac{4-0}{3+0}$$

$$\frac{4}{3}$$

Since our answer is non-zero, the nth term test proves that the series diverges.

Comparison test

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Example

Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}+n}$$

We need to find a series that's similar to the original series, but simpler. The original series is

$$a_n = \frac{n}{\sqrt{n^5}+n}$$

For the comparison series, we'll use the same numerator as the original series, since it's already pretty simple. Looking at the denominator, we can see that the first term $\sqrt{n^5}$ carries more weight and will affect our series more than the second term n, so we'll just use the first term from the original denominator for the denominator of our comparison series, and the comparison series is

$$b_n = \frac{n}{\sqrt{n^5}}$$

$$b_n = \frac{n}{n^{\frac{5}{2}}}$$

$$b_n = n^{1-\frac{5}{2}}$$

$$b_n = n^{-\frac{3}{2}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

We can see that this simplified version of b_n is just a p-series, where $p = 3/2$. We'll use the p-series test for convergence to say whether or not b_n converges. Remember, the p-series test says that the series will

converge when $p > 1$

diverge when $p \leq 1$

Since $p = 3/2$ in b_n, we know that b_n converges.

That means we need to show that $0 \leq a_n \leq b_n$ to prove that the original series a_n is also converging. If we can't show that $0 \leq a_n \leq b_n$, then the test is inconclusive with this particular comparison series.

Let's try to verify that $0 \leq a_n \leq b_n$ by checking a few points for both a_n and b_n, like $n = 1$, $n = 4$ and $n = 9$.

	a_n		b_n	
$n = 1$	$\frac{1}{\sqrt{(1)^5 + (1)}}$	$\frac{1}{2}$	$\frac{1}{(1)^{\frac{3}{2}}}$	1
$n = 4$	$\frac{4}{\sqrt{(4)^5 + (4)}}$	$\frac{1}{9}$	$\frac{1}{(4)^{\frac{3}{2}}}$	$\frac{1}{8}$
$n = 9$	$\frac{9}{\sqrt{(9)^5 + (9)}}$	$\frac{1}{28}$	$\frac{1}{(9)^{\frac{3}{2}}}$	$\frac{1}{27}$

Looking at these three terms, we can see that $0 \leq a_n \leq b_n$, since a_n is always positive and always smaller than b_n.

Therefore, we can say that the original series a_n converges.

Limit comparison test

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n \geq 0$ and $b_n > 0$,

$L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < L < \infty$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n \geq 0$ and $b_n > 0$,

$L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and $0 < L < \infty$, and

the comparison series b_n is converging

Example

Use the limit comparison test to say whether or not the series is converging.

$$\sum_{n=1}^{\infty} \frac{6n}{2n^3 + 3}$$

We need to find a series that's similar to the original series, but simpler. The original series is

$$a_n = \frac{6n}{2n^3 + 3}$$

For the comparison series, we'll use the same numerator as the original series, since it's already pretty simple, but we'll drop the 6 since it has little effect on the series as $n \to \infty$. Looking at the denominator, we can see that the first term $2n^3$ carries more weight and will affect our series more than the second term 3, so we'll just use the first term from the original denominator for the denominator of our comparison series, but drop the 2, and the comparison series is

$$b_n = \frac{n}{n^3}$$

$$b_n = \frac{1}{n^2}$$

We can see that this simplified version of b_n is just a p-series, where $p = 2$. We'll use the p-series test for convergence to say whether or not b_n converges. Remember, the p-series test says that the series will

converge when $p > 1$

diverge when $p \le 1$

Since $p = 2$ in b_n, we know that b_n converges.

That means we need to show that $a_n > 0$ and $b_n > 0$ and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also converging.

Let's try to verify that $a_n > 0$ and $b_n > 0$ by checking a few points for both a_n and b_n, like $n = 1$, $n = 2$ and $n = 3$.

		a_n		b_n
$n = 1$	$\frac{6(1)}{2(1)^3+3}$	$\frac{6}{5}$	$\frac{1}{(1)^2}$	1
$n = 2$	$\frac{6(2)}{2(2)^3+3}$	$\frac{12}{19}$	$\frac{1}{(2)^2}$	$\frac{1}{4}$
$n = 3$	$\frac{6(3)}{2(3)^3+3}$	$\frac{18}{57}$	$\frac{1}{(3)^2}$	$\frac{1}{9}$

Looking at these three terms, we can see that $a_n > 0$ and $b_n > 0$. There's no positive value of n that will make a term in a_n or b_n negative.

The last thing we need to verify is

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

Plugging a_n and b_n into the limit formula gives

$$L = \lim_{n \to \infty} \frac{\frac{6n}{2n^3+3}}{\frac{1}{n^2}}$$

$$L = \lim_{n \to \infty} \frac{6n}{2n^3+3}\left(\frac{n^2}{1}\right)$$

$$L = \lim_{n \to \infty} \frac{6n^3}{2n^3+3}$$

$$L = \lim_{n \to \infty} \frac{6n^3}{2n^3+3}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)$$

$$L = \lim_{n \to \infty} \frac{\frac{6n^3}{n^3}}{\frac{2n^3}{n^3}+\frac{3}{n^3}}$$

$$L = \lim_{n \to \infty} \frac{6}{2+\frac{3}{n^3}}$$

$$L = \frac{6}{2+\frac{3}{\infty}}$$

$$L = \frac{6}{2+0}$$

$$L = 3$$

Since

$L = 3 > 0,$

$a_n > 0$ and $b_n > 0$, and

the comparison series b_n is converging,

we can say the the original series a_n is also converging.

Error or remainder of a series

Imagine that you need to find the sum of a series, but you don't have a formula that you can use to do it. Instead, you have to manually add all of the series' terms together, one at a time. Of course you could never do this, because the series has an infinite number of terms, and you'd be adding forever.

But what if you knew that the sum of just the first five terms of the series was only $.00001$ less than the sum of the entire series? If that were the case, maybe you could just use the first five terms, and say that it was a *good enough estimate* of the total sum, since it's only $.00001$ different and it saves you from manually adding infinitely more terms to the sum.

If you use the estimate, then you want to be able to report next to your answer that the value you found is only $.00001$ off of the total sum. This $.00001$ value is called the remainder, or error, of the series, and it tells you how close your estimate is to the real sum.

To find the remainder of the series, we'll need to

1. Estimate the total sum by calculating a partial sum for the series.

2. Use the comparison test to say whether the series converges or diverges.

3. Use the integral test to solve for the remainder.

Example

Use the first six terms to estimate the remainder of the series.

$$\sum_{n=1}^{\infty} \frac{n}{2n^4+3}$$

The first thing we need to do is to find the sum of the first six terms s_6 of our original series a_n.

$$n=1 \qquad a_1=\frac{(1)}{2(1)^4+3} \qquad a_1=\frac{1}{5}$$

$$n=2 \qquad a_2=\frac{(2)}{2(2)^4+3} \qquad a_2=\frac{2}{35}$$

$$n=3 \qquad a_3=\frac{(3)}{2(3)^4+3} \qquad a_3=\frac{1}{54}$$

$$n=4 \qquad a_4=\frac{(4)}{2(4)^4+3} \qquad a_4=\frac{4}{515}$$

$$n=5 \qquad a_5=\frac{(5)}{2(5)^4+3} \qquad a_5=\frac{5}{1{,}253}$$

$$n=6 \qquad a_6=\frac{(6)}{2(6)^4+3} \qquad a_6=\frac{6}{2{,}595}$$

The sum of the first six terms of the series a_n is

$$s_6=\frac{1}{5}+\frac{2}{35}+\frac{1}{54}+\frac{4}{515}+\frac{5}{1{,}253}+\frac{6}{2{,}595}$$

$$s_6=0.2000+0.0571+0.0185+0.0078+0.0040+0.0023$$

$$s_6 = 0.2897$$

Since we've rounded our decimals, we'll say

$$s_6 \approx 0.2897$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already pretty simple. For the denominator, we can use n^4, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{n}{n^4}$$

$$b_n = \frac{1}{n^3}$$

The comparison series b_n is a p-series where $p = 3$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \leq 1$

Since $p = 3$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \leq a_n \leq b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$n = 1 \qquad b_1 = \frac{1}{(1)^3} \qquad b_1 = 1$

$n = 2 \qquad b_2 = \frac{1}{(2)^3} \qquad b_2 = \frac{1}{8}$

$n = 3 \qquad b_3 = \frac{1}{(3)^3} \qquad b_3 = \frac{1}{27}$

Looking at these three terms and their corresponding terms from a_n, we can see that $0 \le a_n \le b_n$, which means that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first six terms, R_6. We'll call the remainder of the comparison series b_n after the first six terms, T_6. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_6 \le T_6$, which will be less than the total area under b_n.

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty f(x) \ dx$$

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty \frac{1}{x^3} \ dx$$

$$R_6 \le T_6 \le \int_6^\infty b_n \ dx = \int_6^\infty x^{-3} \ dx$$

$$R_6 \le \frac{x^{-2}}{-2}\Bigg|_6^\infty$$

$$R_6 \le \lim_{a \to \infty} \frac{x^{-2}}{-2}\Bigg|_6^a$$

$$R_6 \le \lim_{a\to\infty} -\frac{1}{2x^2}\Big|_6^a$$

$$R_6 \le \lim_{a\to\infty} -\frac{1}{2a^2} - \left(-\frac{1}{2(6)^2}\right)$$

$$R_6 \le \lim_{a\to\infty} \frac{1}{2(6)^2} - \frac{1}{2a^2}$$

$$R_6 \le \lim_{a\to\infty} \frac{1}{72} - \frac{1}{2a^2}$$

$$R_6 \le \frac{1}{72} - \frac{1}{2\infty^2}$$

$$R_6 \le \frac{1}{72} - 0$$

$$R_6 \le \frac{1}{72}$$

$$R_6 \le 0.0139$$

The sixth partial sum of the series a_n is $s_6 \approx 0.2897$, with error $R_6 \le 0.0139$.

Ratio test

The ratio test for convergence lets us determine the convergence or divergence of a series a_n using the limit

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Once we find a value for L, we can say that

the series converges absolutely if $L < 1$.

the series diverges if $L > 1$ or if L is infinite.

the test is inconclusive if $L = 1$.

The ratio test is used most often when our series includes a factorial or something raised to the nth power.

Example

Use the ratio test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3}{4^n}$$

To use the ratio test, we need to solve for the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then evaluate the value of L.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}} \right|$$

We can drop the absolute value bars since all of our terms will be positive.

$$L = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{4^{n+1}} \left(\frac{4^n}{n^3} \right)$$

Grouping like bases together, we get

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(\frac{4^n}{4^{n+1}} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(4^{n-(n+1)} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(4^{-1} \right)$$

$$L = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \left(\frac{1}{4} \right)$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{(n+1)^3}{n^3}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{\frac{n^3}{n^3} + \frac{3n^2}{n^3} + \frac{3n}{n^3} + \frac{1}{n^3}}{\frac{n^3}{n^3}}$$

$$L = \frac{1}{4} \lim_{n \to \infty} \frac{1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{1}$$

$$L = \left(\frac{1}{4} \right) \frac{1 + \frac{3}{\infty} + \frac{3}{\infty} + \frac{1}{\infty}}{1}$$

$$L = \left(\frac{1}{4} \right) \frac{1 + 0 + 0 + 0}{1}$$

$$L = \frac{1}{4}$$

Since $L < 1$, we can say that the original series a_n converges absolutely.

Root test

The root test for convergence lets us determine the convergence or divergence of a series a_n using the limit

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

The convergence or divergence of the series depends on the value of L.

- the series converges absolutely if $L < 1$.
- the series diverges if $L > 1$ or if L is infinite.
- the test is inconclusive if $L = 1$.

The root test is used most often when our series includes something raised to the nth power.

Example

Use the root test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{6^n}{(n+2)^n}$$

To use the root test, we need to solve for the limit

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

and then evaluate the value of L.

$$L = \lim_{n\to\infty} \sqrt[n]{\left| \frac{6^n}{(n+2)^n} \right|}$$

We can drop the absolute value bars since all of our terms will be positive.

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{6^n}{(n+2)^n}}$$

$$L = \lim_{n\to\infty} \left[\frac{6^n}{(n+2)^n} \right]^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \left[\left(\frac{6}{n+2} \right)^n \right]^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \left(\frac{6}{n+2} \right)^{\frac{n}{n}}$$

$$L = \lim_{n\to\infty} \frac{6}{n+2}$$

$$L = \frac{6}{\infty + 2}$$

$$L = \frac{6}{\infty}$$

$$L = 0$$

Since $L < 1$, we can say that the original series a_n converges absolutely.

Absolute and conditional convergence

We can calculate the convergence of a series using any of the many convergence tests, depending on the format of the series we've been given. Sometimes we're asked to say whether the series converges conditionally or converges absolutely.

A series converges conditionally if

a_n converges but $\left|a_n\right|$ diverges

In other words, $a_n \neq \left|a_n\right|$ for all possible values of n.

A series converges absolutely if

a_n and $\left|a_n\right|$ both converge

In other words, $a_n = |a_n|$ for all possible values of n.

The ratio and root tests both use absolute value bars. If we can drop the absolute value bars by the end of the problem because the expression inside the absolute value bars will always be positive anyway, then we know that the series converges absolutely.

On the other hand, if we can't drop the absolute value bars because it's possible that the expression inside them could be negative, then we know that the series converges conditionally.

Let's try an example where we use the root test to determine absolute or conditional convergence.

Example

Determine whether the series converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \left(\frac{n^3 - 1}{6n^3 + 4} \right)^n$$

We know we can use the ratio test or the root test to determine absolute convergence, and this series looks like a great candidate for the root test, since the whole thing is raised to the nth power.

To use the root test, we need to solve for the limit

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

and then evaluate the value of L.

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{n^3 - 1}{6n^3 + 4} \right)^n \right|}$$

$$L = \lim_{n \to \infty} \left[\left| \left(\frac{n^3 - 1}{6n^3 + 4} \right)^n \right| \right]^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \frac{n^3 - 1}{6n^3 + 4} \right|^{\frac{n}{n}}$$

$$L = \lim_{n\to\infty}\left|\frac{n^3-1}{6n^3+4}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{n^3-1}{6n^3+4}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)\right|$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{n^3}{n^3}-\frac{1}{n^3}}{\frac{6n^3}{n^3}+\frac{4}{n^3}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{1-\frac{1}{n^3}}{6+\frac{4}{n^3}}\right|$$

$$L = \left|\frac{1-\frac{1}{\infty}}{6+\frac{4}{\infty}}\right|$$

$$L = \left|\frac{1-0}{6+0}\right|$$

$$L = \left|\frac{1}{6}\right|$$

Since $L < 1$, and since we can drop the absolute value bars and it wouldn't change the value of L, we can say that a_n and $\left|a_n\right|$ both converge, and therefore that a_n converges absolutely.

Alternating series test

The alternating series test for convergence lets us say whether an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

where $a_n > 0$

is converging or diverging.

The alternating series test for convergence tells us that

an alternating series converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

Example

Use the alternating series test to say whether the series converges or diverges

$$\sum_{n=5}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

First, we separate the series from the part that makes it alternating.

$$\sum_{n=5}^{\infty} (-1)^{n-3} \frac{\sqrt{n}}{n+4}$$

Matching this up to the standard form of an alternating series given above, we can say that the series is

$$a_n = \frac{\sqrt{n}}{n+4}$$

Now we need to show that

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n\to\infty} a_n = 0$$

for the series a_n, in order to say that a_n converges. Remembering that this series starts at $n = 5$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 5$	$\frac{\sqrt{5}}{5+4}$	$\frac{\sqrt{5}}{9}$	$\frac{\sqrt{5+1}}{5+1+4}$	$\frac{\sqrt{6}}{10}$
$n = 6$	$\frac{\sqrt{6}}{6+4}$	$\frac{\sqrt{6}}{10}$	$\frac{\sqrt{6+1}}{6+1+4}$	$\frac{\sqrt{7}}{11}$
$n = 7$	$\frac{\sqrt{7}}{7+4}$	$\frac{\sqrt{7}}{11}$	$\frac{\sqrt{7+1}}{7+1+4}$	$\frac{\sqrt{8}}{12}$

$n=8$	$\frac{\sqrt{8}}{8+4}$	$\frac{\sqrt{8}}{12}$	$\frac{\sqrt{8+1}}{8+1+4}$	$\frac{\sqrt{9}}{13}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 5$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convince by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{x+4}\right)$$

Using the quotient rule, we get

$$\frac{\frac{1}{2}(x)^{-\frac{1}{2}}(x+4)-(x)^{\frac{1}{2}}(1)}{(x+4)^2}$$

$$\frac{\frac{1}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}-x^{\frac{1}{2}}}{(x+4)^2}$$

$$\frac{-\frac{1}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}}{(x+4)^2}$$

$$\frac{-\frac{x^{\frac{1}{2}}}{2}+\frac{2}{x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{-\frac{x}{2x^{\frac{1}{2}}}+\frac{4}{2x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{\frac{4-x}{2x^{\frac{1}{2}}}}{(x+4)^2}$$

$$\frac{4-x}{2x^{\frac{1}{2}}}\cdot\frac{1}{(x+4)^2}$$

$$\frac{4-x}{2x^{\frac{1}{2}}(x+4)^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n=5$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing, and therefore that it converges.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+4}$$

$$\lim_{n\to\infty} a_n = \frac{\sqrt{\infty}}{\infty+4}$$

$$\lim_{n\to\infty} a_n = \frac{\sqrt{\infty}}{\infty}$$

Since the numerator will be significantly smaller than the denominator, especially as n gets really big, we can say that

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Alternating series estimation theorem

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series with a remainder or error that we can calculate. To use this theorem, our series must follow two rules:

1. The series must be decreasing, $b_n \geq b_{n+1}$

2. The limit of the series must be zero, $\lim_{n \to \infty} b_n = 0$

Once we confirm that our alternating series meets these two conditions, we can calculate the error using

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example

Approximate the sum of the series to three decimal places.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{10^n}$$

We'll calculate the first few terms of the series until we have a stable answer to three decimal places.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)}{10^1} \qquad a_1 = 0.1$$

$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)}{10^2} \qquad a_2 = -0.02$

$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)}{10^3} \qquad a_3 = 0.003$

$n = 4 \qquad a_4 = \frac{(-1)^{4-1}(4)}{10^4} \qquad a_4 = -0.0004$

$n = 5 \qquad a_5 = \frac{(-1)^{5-1}(5)}{10^5} \qquad a_5 = 0.00005$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.02)$$

$$a_1 + a_2 = 0.1 - 0.02$$

$$a_1 + a_2 = 0.08$$

$$s_2 = 0.08$$

Since we're not to three decimal places, we'll add another term to the sum

$$a_1 + a_2 + a_3 = 0.1 + (-0.02) + 0.003$$

$$a_1 + a_2 + a_3 = 0.1 - 0.02 + 0.003$$

$$a_1 + a_2 + a_3 = 0.083$$

$$s_3 = 0.083$$

We've made it to three decimal places, but we need to make sure that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.02) + 0.003 + (-0.0004)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.02 + 0.003 - 0.0004$$

$$a_1 + a_2 + a_3 + a_4 = 0.0826$$

$$s_4 = 0.0826$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_3 \approx 0.083$$

In order to use the alternating series estimation theorem, we need to show that the series is decreasing, $b_n \geq b_{n+1}$. Pulling out b_n from the given series, we get

$$b_n = \frac{n}{10^n}$$

Which means that

$$b_{n+1} = \frac{n+1}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

	b_n	b_{n+1}

$n=1$	$\frac{1}{10^1}$	$\frac{1}{10}$	$\frac{1+1}{10^{1+1}}$	$\frac{1}{50}$
$n=2$	$\frac{2}{10^2}$	$\frac{1}{50}$	$\frac{2+1}{10^{2+1}}$	$\frac{3}{1{,}000}$
$n=3$	$\frac{3}{10^3}$	$\frac{3}{1{,}000}$	$\frac{3+1}{10^{3+1}}$	$\frac{1}{2{,}500}$

Looking at these results, we can see that $b_n \geq b_{n+1}$, so b_n is decreasing.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much faster than the numerator. This means that the fraction will approach 0.

$$\lim_{n\to\infty} \frac{n}{10^n} = 0$$

Now that we've shown that our series meets the two criteria, we can use the alternating series estimation theorem. We'll use the inequality

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Plugging in the values we have, we get

$$|R_3| = |s - s_3| \leq b_{3+1}$$

$$|R_3| \leq b_4$$

$$|R_3| \le \frac{4}{10^4}$$

$$|R_3| \le \frac{1}{2{,}500}$$

$$|R_3| \le 0.0004$$

The approximate sum of the series to three decimal places is 0.083 with an error of $|R_3| \le 0.0004$.

Power series representation

We can convert functions into a power series using the standard form of a power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$$

The interval of convergence of the series

$$a<x<b$$

will be the set of values for which the series is converging. Remember, even if we can find an interval of convergence for a series, it doesn't mean that the entire series is converging, only that the series is converging in the specific interval.

The radius of convergence of the series

$$R=\frac{b-a}{2}$$

will always be half of the interval of convergence. You can remember this if you think about the interval of convergence as the diameter of a circle, and the radius of convergence as the half of the diameter.

Example

Find the power series representation, then find the radius and interval of convergence.

$$f(x) = \frac{x}{8 + x^2}$$

Power series representation

In order to find a power series representation, we need to manipulate the function into the form

$$\frac{1}{1-x}$$

First, we'll factor the x out of the numerator.

$$\frac{1}{1-x} = (x)\frac{1}{8 + x^2}$$

Next we'll remove the 8 from the denominator.

$$\frac{1}{1-x} = (x)\frac{1}{8\left(1 + \frac{x^2}{8}\right)}$$

$$\frac{1}{1-x} = \left(\frac{x}{8}\right)\frac{1}{1 + \frac{x^2}{8}}$$

Now we'll make the sign in between the terms in the denominator negative.

$$\frac{1}{1-x} = \left(\frac{x}{8}\right)\frac{1}{1 - \left(-\frac{x^2}{8}\right)}$$

We can see that $-x^2/8$ is the value of x from the standard form of the power series, so we'll plug that into the power series formula.

$$\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

We can't forget that our power series is also multiplied by the $x/8$ that we factored out, and we'll need to multiply our sum by this term.

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

Now we can simplify.

$$\frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^2}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x}{8}\left(-\frac{x^2}{8}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{8^1}\left[\frac{(-1)x^2}{8}\right]^n$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{8^1}\left(\frac{x^{2n}}{8^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

This is the power series representation of the function.

Radius of convergence

To find the radius of convergence, we need to identify a_n from the power series representation we just found.

$$a_n = \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

Using a_n, we need to generate a_{n+1}.

$$a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{8^{(n+1)+1}}$$

$$a_{n+1} = \frac{(-1)^{n+1} x^{2n+2+1}}{8^{n+1+1}}$$

$$a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{8^{n+2}}$$

We can use the ratio test to say whether or not the series converges. The ratio test tells us that

If $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$, then

the series converges absolutely if $L < 1$.

the series diverges if $L > 1$ or if L is infinite.

the test is inconclusive if $L=1$.

Plugging a_n and a_{n+1} into the formula for L from the ratio test, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+3}}{8^{n+2}}}{\frac{(-1)^n x^{2n+1}}{8^{n+1}}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{8^{n+2}}\left(\frac{8^{n+1}}{(-1)^n x^{2n+1}}\right)\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{8^{n+1}}{8^{n+2}}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)^{n+1-n}\cdot 8^{n+1-(n+2)}\cdot x^{2n+3-(2n+1)}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)^1\cdot 8^{n+1-n-2}\cdot x^{2n+3-2n-1}\right|$$

$$L=\lim_{n\to\infty}\left|(-1)\cdot 8^{-1}\cdot x^2\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)x^2}{8}\right|$$

Since it's inside absolute brackets, we can drop the -1.

$$L=\lim_{n\to\infty}\left|\frac{x^2}{8}\right|$$

We no longer have n in this function, which means the limit as $n \to \infty$ will have no effect, so we can remove it.

$$L = \left| \frac{x^2}{8} \right|$$

The ratio test tells us that L will converge when $L < 1$, so we generate the following inequality.

$$\left| \frac{x^2}{8} \right| < 1$$

$$\left| x^2 \right| < 8$$

$$-\sqrt{8} < x < \sqrt{8}$$

Based on this inequality, the radius of convergence is $R = \sqrt{8}$.

Interval of convergence

The interval of convergence is given by the inequality $-\sqrt{8} < x < \sqrt{8}$, but we still need to test the endpoints of the interval to say whether the series converges at one or both of them.

We'll plug the endpoints back into the original series and then test each of them for convergence.

Let's start by testing $x = -\sqrt{8}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(-\sqrt{8}\right)^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left[-(8)^{\frac{1}{2}}\right]^{2n} \left[-(8)^{\frac{1}{2}}\right]^{1}}{8^n 8^1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (8)^n \left[-(8)^{\frac{1}{2}}\right]}{8^n 8^1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(-\sqrt{8}\right)}{8}$$

$$\sum_{n=0}^{\infty} -\frac{\sqrt{8}}{8}(-1)^n$$

$$-\frac{\sqrt{8}}{8}\sum_{n=0}^{\infty}(-1)^n$$

$$-\frac{\sqrt{8}}{8}\left[(-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + \dots\right]$$

$$-\frac{\sqrt{8}}{8}(1 - 1 + 1 - 1 + 1 - \dots)$$

This is a geometric series with $r = -1$. The geometric series test tells us that the series will converge if $|r| < 1$.

$$|-1| < 1$$

$$1 < 1$$

Since 1 is not less than 1, the series diverges, which means it's divergent at the endpoint $x = -\sqrt{8}$.

Now we'll test $x = \sqrt{8}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{8}\right)^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left[(8)^{\frac{1}{2}}\right]^{2n+1}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 8^{n+\frac{1}{2}}}{8^{n+1}}$$

$$\sum_{n=0}^{\infty} (-1)^n 8^{n+\frac{1}{2}-(n+1)}$$

$$\sum_{n=0}^{\infty} (-1)^n 8^{n+\frac{1}{2}-n-1}$$

$$\sum_{n=0}^{\infty}(-1)^n 8^{-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{8}}(-1)^n$$

$$\frac{1}{\sqrt{8}}\sum_{n=0}^{\infty}(-1)^n$$

We already know from testing the other endpoint that this is a divergent geometric series. Since the series is divergent at both endpoints, the interval of convergence is $-\sqrt{8} < x < \sqrt{8}$.

Power series multiplication

Previously we learned how to create a power series representation for a function by modifying a similar, known series to match the function.

Sometimes the function we're given is the product of two other functions, like

$$f(x) = \frac{\cos x}{1 - x}$$

This function is the product of $g(x) = \cos x$ and $h(x) = 1/(1 - x)$.

$$f(x) = \frac{1}{1 - x} \cdot \cos x$$

If we already know the power series representations of $g(x) = \cos x$ and $h(x) = 1/(1 - x)$, we can multiply the expanded power series together to find a power series representation of $f(x)$, since $f(x)$ is the product of $g(x)$ and $h(x)$.

In other words, since we already know from a table to standard Maclaurin series that

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots$$

$$\frac{1}{1 - x} \approx 1 + x + x^2 + x^3 + \dots$$

we can say that

$$f(x) \approx \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots\right)\left(1 + x + x^2 + x^3 + \dots\right)$$

Keep in mind also that multiplying power series together is just like multiplying two simple polynomials together. We use the distributive property from algebra, and we multiply the first term in the first series by all of the terms in the second series, then we multiply the second term in the first series by all of the terms in the second series, and so on.

$$f(x) \approx 1\left(1 + x + x^2 + x^3 + \dots\right) - \frac{1}{2}x^2\left(1 + x + x^2 + x^3 + \dots\right)$$

$$+\frac{1}{24}x^4\left(1 + x + x^2 + x^3 + \dots\right) - \frac{1}{720}x^6\left(1 + x + x^2 + x^3 + \dots\right) + \dots$$

$$f(x) \approx 1 + x + x^2 + x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^4 - \frac{1}{2}x^5$$

$$+\frac{1}{24}x^4 + \frac{1}{24}x^5 + \frac{1}{24}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^6 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9$$

We'll collect like-terms, and get

$$f(x) \approx 1 + x + x^2 - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^3 - \frac{1}{2}x^4 + \frac{1}{24}x^4 - \frac{1}{2}x^5 + \frac{1}{24}x^5$$

$$+\frac{1}{24}x^6 - \frac{1}{720}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9$$

$$f(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4 - \frac{11}{24}x^5$$

$$+\frac{1}{24}x^6 - \frac{1}{720}x^6 + \frac{1}{24}x^7 - \frac{1}{720}x^7 - \frac{1}{720}x^8 - \frac{1}{720}x^9$$

Remember to list the terms in the series in ascending order of degree, with any constant first, followed by x, followed by x^2, x^3, x^4, etc.

For most power series multiplication problems, we'll be asked to find a specific number of non-zero terms in the expanded power series representation of $f(x)$. With this in mind, we can actually stop multiplying once we have the number of non-zero terms we've been asked for. In the above example, if we were asked for the first five non-zero terms, we could have stopped multiplying once we had all of our x^4 terms. We would have collected like terms for all fourth- or lesser-degree terms, and given an answer of

$$f(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4$$

Example

Use power series multiplication to find the first four non-zero terms of the Maclaurin series of the given function.

$$y = \sin(2x)e^{3x}$$

We know that the expanded versions of the Maclaurin series for $\sin x$ and e^x are

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5{,}040}x^7 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Since we have $\sin(2x)$ instead of $\sin x$ and since we have e^{3x} instead of e^x, we'll need to modify both series.

We'll start with the $\sin x$ series, letting $x = 2x$, and the expanded series will be

$$\sin(2x) = 2x - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 - \frac{1}{5{,}040}(2x)^7 + \dots$$

$$\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots$$

Now we'll modify the e^x series, letting $x = 3x$, and the expanded series will be

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$$

We'll use the distributive property from algebra to multiply the series together, as if we're multiplying two simple polynomials.

$$\sin(2x)e^{3x} = \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots\right)\left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots\right)$$

$$\sin(2x)e^{3x} = 2x + 6x^2 + 9x^3 + 9x^4$$

$$-\frac{4}{3}x^3 - 4x^4 - 6x^5 - 6x^6$$

$$+\frac{4}{15}x^5+\frac{4}{5}x^6+\frac{6}{5}x^7+\frac{6}{5}x^8$$

$$-\frac{8}{315}x^7-\frac{24}{315}x^8-\frac{72}{630}x^9-\frac{72}{630}x^{10}+\dots$$

Cancel everything that isn't an x, x^2, x^3, or x^4 term.

$$\sin(2x)e^{3x}=2x+6x^2+9x^3+9x^4-\frac{4}{3}x^3-4x^4$$

$$\sin(2x)e^{3x}=2x+6x^2+\frac{27}{3}x^3-\frac{4}{3}x^3+5x^4$$

$$\sin(2x)e^{3x}=2x+6x^2+\frac{23}{3}x^3+5x^4$$

Since this is an approximation, the answer will be

$$\sin(2x)e^{3x}=2x+6x^2+\frac{23}{3}x^3+5x^4$$

Power series division

Sometimes we'll want to use polynomial long division to simplify a fraction, but either the numerator and/or denominator isn't a polynomial. In this case, we may be able to replace the non-polynomial with its power series expansion, which will be a polynomial.

The simplest way to do this for the non-polynomial is to find a similar, known power series expansion and then modify it to match the non-polynomial function. Once we have polynomial expressions for both the numerator and denominator, we'll do polynomial long division until we have the number of non-zero terms we've been asked for.

Example

Use power series division to find the first three non-zero terms of the Maclaurin series of the given function.

$$y = \frac{x}{e^{3x}}$$

In order to use long division, we need polynomials in the numerator and denominator of our function. The numerator is already a polynomial, but we need to find a power series expansion for e^{3x} so that we can change it into a polynomial.

We know that the expanded version of the Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Since we have e^{3x} instead of e^x, we'll need to modify the series, letting $x = 3x$, such that the expanded series will be

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$$

Now that both the numerator and denominator are represented as polynomials, we'll do the long division.

$$\begin{array}{r|l}
 & x - 3x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4 \\
\hline
1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots & x \\
 & -\left(x + 3x^2 + \frac{9}{2}x^3 + \frac{9}{2}x^4 + \dots\right) \\
\cline{2-2}
 & -3x^2 - \frac{9}{2}x^3 - \frac{9}{2}x^4 + \dots \\
 & -\left(-3x^2 - 9x^3 - \frac{27}{2}x^4 - \frac{27}{2}x^5 + \dots\right) \\
\cline{2-2}
 & \frac{9}{2}x^3 + 9x^4 + \frac{27}{2}x^5 + \dots \\
 & -\left(\frac{9}{2}x^3 + \frac{27}{2}x^4 + \frac{81}{4}x^5 + \frac{81}{4}x^6 + \dots\right) \\
\cline{2-2}
 & -\frac{9}{2}x^4 - \frac{27}{4}x^5 - \frac{81}{4}x^6 + \dots
\end{array}$$

$$-\frac{9}{2}x^4-\frac{27}{2}x^5-\frac{81}{4}x^6-\frac{81}{4}x^7+\ldots$$

Remember, we only need to find the first three non-zero terms. We'll take the first three terms from our quotient and say that the first three non-zero terms are

$$y=\frac{x}{e^{3x}}\approx x-3x^2+\frac{9}{2}x^3$$

Power series differentiation

Sometimes we can generate the power series representation of a function by manipulating a standard power series like

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

and then differentiating the result. The goal is to use differentiation to get the left side of this equation to match exactly the function we've been given. When we differentiate, we have to remember to differentiate all three parts of the equation.

We'll try to simplify the sum on the right as much as possible, and the result will be the power series representation of our function. If we need to, we can then use the power series representation to find the radius and interval of convergence.

Example

Differentiate to find a power series representation for the function, then find the radius of convergence.

$$f(x) = \frac{1}{(1+x)^2}$$

The function we've been given is similar to the known power series

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty}x^n$$

We need to manipulate this known power series so that it matches the function in our problem.

The first thing we'll do is change the denominator of the known power series from $1-x$ to $1+(-x)$ by replacing x with $-x$.

$$\frac{1}{1-(-x)}=1+(-x)+(-x)^2+(-x)^3+\ldots=\sum_{n=0}^{\infty}(-x)^n$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots=\sum_{n=0}^{\infty}(-1)^n x^n$$

If we differentiate this equation, we'll have to use quotient rule to find the derivative of $1/(1+x)$, and the new denominator will be $(1+x)^2$, which is the denominator from our original function. Differentiating every part of the equation with respect to x, we get

$$\frac{(0)(1+x)-(1)(1)}{(1+x)^2}=0-1+2x-3x^2+\ldots=\sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

$$\frac{-1}{(1+x)^2}=-1+2x-3x^2+\ldots=\sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

The left-hand side of the equation is now almost identical to the given function. To make them truly identical, we just have to multiply by -1.

$$\frac{1}{(1+x)^2}=1-2x+3x^2-\ldots=-1^1\sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

$$\frac{1}{(1+x)^2}=1-2x+3x^2-\ldots=\sum_{n=0}^{\infty}(-1)^{n+1}nx^{n-1}$$

Now that the left side of this equation is identical to the given function, we want to use the right side as a power series representation. If at all possible, we want the exponent of x to be just n. To make that happen, we can substitute $n+1$ for n.

$$\sum_{n=0}^{\infty}(-1)^{n+1}nx^{n-1}$$

$$\sum_{n=0}^{\infty}(-1)^{(n+1)+1}(n+1)x^{(n+1)-1}$$

$$\sum_{n=0}^{\infty}(-1)^{n+2}(n+1)x^{n}$$

$$\sum_{n=0}^{\infty}(-1)^{2}(-1)^{n}(n+1)x^{n}$$

$$\sum_{n=0}^{\infty}(-1)^{n}(n+1)x^{n}$$

This is the power series representation of $f(x)$.

With the power series representation in hand, we can find the radius of convergence using the ratio test. We'll need to identify that a_n is the power series representation, and a_{n+1} is whatever we get when we substitute $n+1$ into the power series representation for n.

$$a_n=(-1)^n(n+1)x^n$$

$$a_{n+1} = (-1)^{n+1}(n+2)x^{n+1}$$

Plugging these into the limit formula from the ratio test, we get

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{n+2}{n+1} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$L = \lim_{n\to\infty} \left| (-1)^{n+1-n} \cdot \frac{n+2}{n+1} \cdot x^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| -1 \cdot \frac{n+2}{n+1} \cdot x \right|$$

$$L = \lim_{n\to\infty} \left| \frac{n+2}{n+1} \cdot x \right|$$

Since the limit only effects n, we can pull the x out in front.

$$L = |x| \lim_{n\to\infty} \left| \frac{n+2}{n+1} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{(n+2)\left(\frac{1}{n}\right)}{(n+1)\left(\frac{1}{n}\right)} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \right|$$

$$L = |x| \left| \frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right|$$

$$L = |x| \left| \frac{1+0}{1+0} \right|$$

$$L = |x|$$

The ratio test tells us that that series converges when $L < 1$. Since we know $L = |x|$, we'll say that the series converges when

$$|x| < 1$$

This inequality is already in the form $|x - a| < R$ as

$$|x - 0| < 1$$

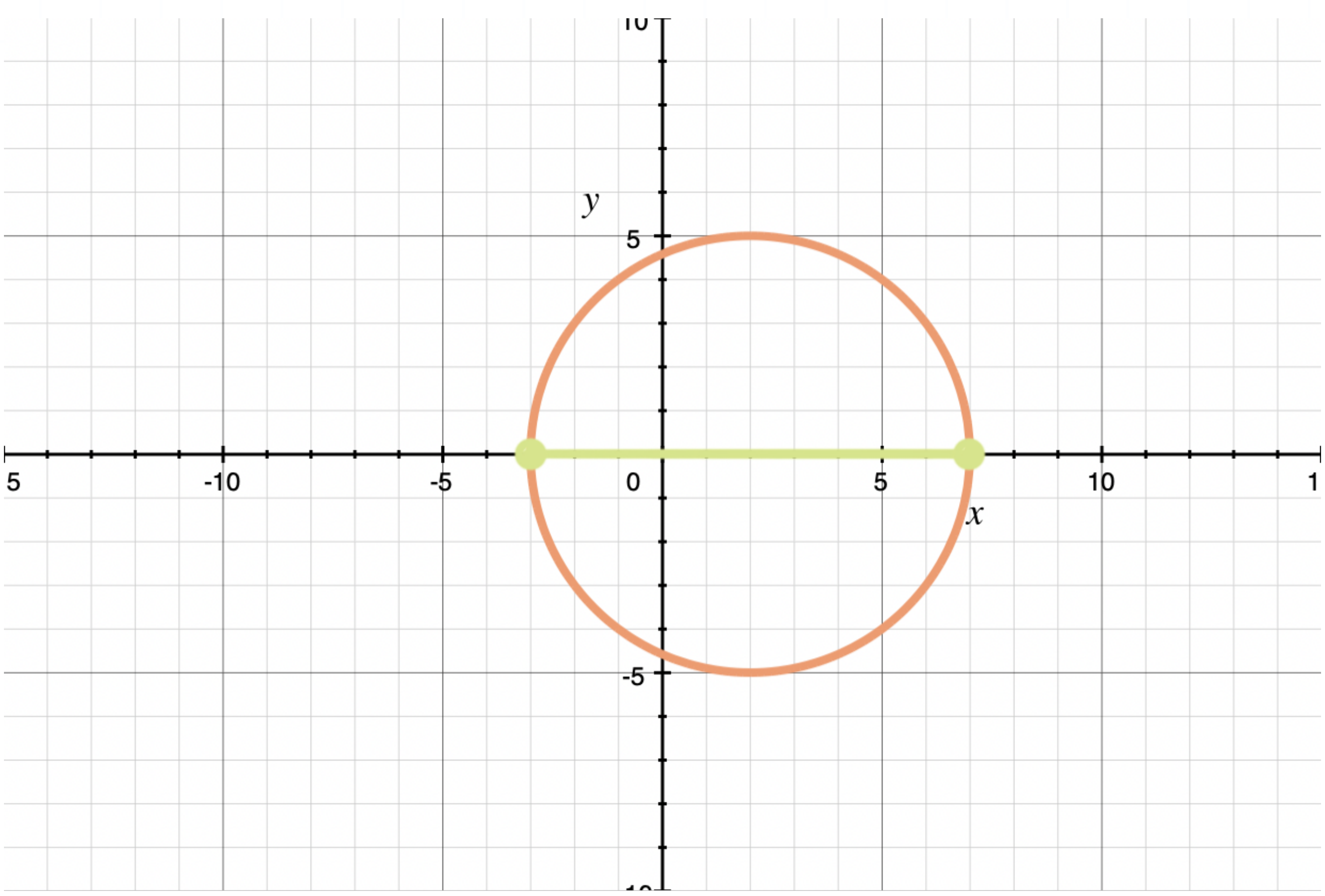
so we can say that the radius of convergence is $R = 1$.

Radius and interval of convergence

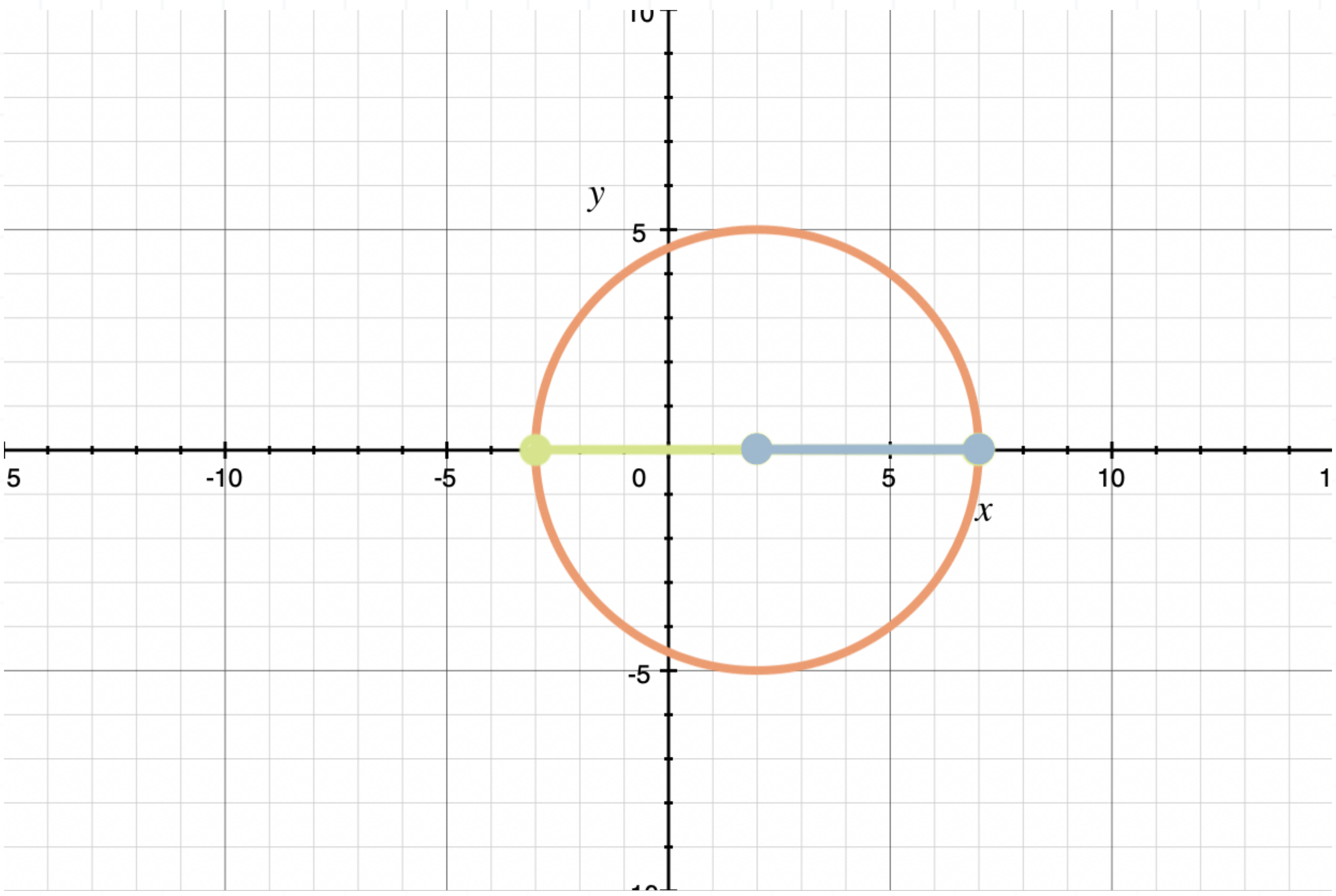
The **interval of convergence** of a series is the set of values for which the series is converging. Remember, even if we can find an interval of convergence for a series, it doesn't mean that the entire series is converging, only that the series is converging in the specific interval.

The **radius of convergence** of a series is always half of the interval of convergence. You can remember this if you think about the interval of convergence as the diameter of a circle.

For example, imagine that the interval of convergence of a series is $-3 < x < 7$. If we graph the interval of convergence along the x-axis and then draw a circle where the endpoints of the interval lie along the circle's perimeter, we get the following picture.



If the interval of convergence is represented by the diameter, then the radius of convergence will be half of the diameter.



With this in mind, we can state the universal fact that, given an interval of convergence

$$a < x < b$$

the radius of convergence is

$$R = \frac{b-a}{2}$$

To find the radius and interval of convergence of a given series, we'll use the ratio test, which tell us that

If $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, then the series converges absolutely if $L < 1$.

the series diverges if $L > 1$ or if L is infinite.

the test is inconclusive if $L = 1$.

Since we know that the series converges when $L < 1$, we can find L, set it $L < 1$, and then find the values for which the series converges.

The **radius** of convergence R of the series will be given by $|x - a| < R$.

The **interval** of convergence will be given by $a - R < x < R + a$.

Once we have the interval of convergence, we'll need to check the convergence of the endpoints of the interval by plugging the endpoints into the original series and using any convergence test that we can to say whether or not the series converges at the endpoint.

If the series diverges at both endpoints, the interval of convergence is $a - R < x < R + a$.

If the series diverges at the left endpoint and converges at the right endpoint, the interval of convergence is $a - R < x \le R + a$.

If the series diverges at the right endpoint and converges at the left endpoint, the interval of convergence is $a - R \le x < R + a$.

Example

Find the radius and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(x-4)^n}{3^n}$$

The series is

$$a_n = \frac{(-1)^n n(x-4)^n}{3^n}$$

To generate a_{n+1}, we'll replace n with $n+1$, and get

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)(x-4)^{n+1}}{3^{n+1}}$$

Now we can plug a_n and a_{n+1} into

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and use the ratio test to find the radius of convergence.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)(x-4)^{n+1}}{3^{n+1}}}{\frac{(-1)^n n(x-4)^n}{3^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n(x-4)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)}{n} \cdot \frac{(x-4)^{n+1}}{(x-4)^n} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| (-1)^{n+1-n} \cdot \frac{n+1}{n} \cdot (x-4)^{n+1-n} \cdot \frac{1}{3^{n+1-n}} \right|$$

$$L = \lim_{n \to \infty} \left| -\frac{n+1}{n} \cdot x - 4 \cdot \frac{1}{3} \right|$$

$$L = \lim_{n \to \infty} \left| -\frac{(n+1)(x-4)}{3n} \right|$$

Since we're dealing with absolute value brackets, the -1 can be dropped.

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(x-4)}{3n} \right|$$

Since it contains no n terms and therefore won't be effected by the limit, we can pull the $x-4$ out in front, as long as we keep in inside absolute value brackets since there are some values of x for which $x-4$ would be negative.

$$L = |x-4| \lim_{n \to \infty} \left| \frac{n+1}{3n} \right|$$

Because evaluating the limit at this point would result in the indeterminate form ∞/∞, we'll need to manipulate our fraction.

$$L = |x-4| \lim_{n\to\infty} \left| \frac{n+1}{3n} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |x-4| \lim_{n\to\infty} \left| \frac{\frac{n}{n} + \frac{1}{n}}{\frac{3n}{n}} \right|$$

$$L = |x-4| \lim_{n\to\infty} \left| \frac{1 + \frac{1}{n}}{3} \right|$$

Evaluate the limit.

$$L = |x-4| \frac{1 + \frac{1}{\infty}}{3}$$

$$L = |x-4| \frac{1+0}{3}$$

$$L = \frac{1}{3}|x-4|$$

The ratio test tells us that our series converges when $L < 1$, so we'll set $L < 1$ and manipulate the inequality into the form $|x-a| < R$, where R is the radius of convergence.

$$\frac{1}{3}|x-4| < 1$$

$$|x-4| < 3$$

With the inequality in this form, we can say that the radius of convergence of our series is $R = 3$.

To find the interval of convergence, we simply solve $|x-4| < 3$ for x. To do this, we just take away the absolute value brackets and add $-R$ to the left side of the inequality, like this:

$$-3 < x - 4 < 3$$

$$-3 + 4 < x - 4 + 4 < 3 + 4$$

$$1 < x < 7$$

Before we can say that this is the interval of convergence, we have to check the endpoints of the interval to see if the series converges at either or both of the endpoints. We can do this by plugging the endpoints back into the original series and then testing for convergence.

For $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(1-4)^n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(-3)^n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(-1)^n(3)^n}{3^n}$$

$$\sum_{n=1}^{\infty} (-1)^{2n} n$$

We know that any value of n will result in an even exponent on the -1 term, since the exponent is $2n$. So $(-1)^{2n} = 1$ for all values of n. Therefore

$$\sum_{n=1}^{\infty} (1)n$$

$$\sum_{n=1}^{\infty} n$$

We can test this remaining series for convergence using the nth term test (also called the zero test, or divergence test). We'll say that the series is $a_n = n$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \neq 0$$

then the series diverges at $x = 1$ by the divergence test.

For $x = 7$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(7-4)^n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(3)^n}{3^n}$$

$$\sum_{n=1}^{\infty} (-1)^n n$$

We can test this remaining series for convergence using the nth term test (also called the zero test, or divergence test). We'll say that the series is $a_n = n$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n n \neq 0$$

then the series diverges at $x = 7$ by the divergence test.

Since the series does not converge at either endpoint, the interval of convergence is $1 < x < 7$.

Estimating definite integrals

We can use power series to estimate definite integrals in the same way we used them to estimate indefinite integrals. The only difference is that we'll evaluate over the given interval once we find a power series that represents the original integral.

To evaluate over the interval, we'll expand the power series through its first few terms, and then evaluate each term separately over the interval.

Oftentimes we'll be asked to use a power series to approximate the definite integral to a certain number of decimal places. If this is the case, we need to make sure we keep more decimals than we're asked for when we evaluate over the interval. That way, we'll be able to give an accurate answer to the requested number of decimal places when we sum all of our decimal values together.

Example

Use power series to estimate the definite integral to five decimal places.

$$\int_0^{0.2} 4x \arctan(2x)\ dx$$

Since this integral includes an $\arctan$ function, we'll use the standard power series

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

and manipulate it to get the given definite integral.

We can start by replacing x with $2x$ inside the $\arctan$ function in order to match the given function.

$$\arctan(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

Next, we'll multiply both sides by $4x$ in order to make the power series match the function.

$$4x\arctan(2x) = 4x \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

$$4x\arctan(2x) = 4x^1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$$

$$4x\arctan(2x) = \sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+2}}{2n+1}$$

Since the problem asks us to find the integral of the left-hand side of this equation, we'll integrate both sides.

$$\int 4x\arctan(2x)\ dx = \int \sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+2}}{2n+1}\ dx$$

Because we want to solve for what we have on the left-hand side, we only need to integrate the right side. We're integrating with respect to x, so we can treat the n's as constants.

$$\int 4x\arctan(2x)\ dx = \sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+3}}{(2n+1)(2n+3)}$$

Adding in the interval from the original question, we get

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \left[\sum_{n=0}^{\infty} \frac{4(-1)^n 2^{2n+1} x^{2n+3}}{(2n+1)(2n+3)}\right]\Bigg|_0^{0.2}$$

In order to evaluate over the interval, we'll expand the power series through the first few terms. Remember, we need to approximate the final answer to five decimal places, which means we'll have to calculate results beyond five decimals until we get to a point where the first five decimal places aren't changing.

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \frac{8x^3}{3} - \frac{32x^5}{15} + \frac{128x^7}{35} - \frac{512x^9}{63} + \frac{2{,}048x^{11}}{99} + \dots\Bigg|_0^{0.2}$$

Evaluating each term separately over the interval, we get

$$\int_0^{0.2} 4x\arctan(2x)\ dx = \left(\frac{8(0.2)^3}{3} - \frac{8(0)^3}{3}\right) - \left(\frac{32(0.2)^5}{15} - \frac{32(0)^5}{15}\right)$$

$$+\left(\frac{128(0.2)^7}{35} - \frac{128(0)^7}{35}\right) - \left(\frac{512(0.2)^9}{63} - \frac{512(0)^9}{63}\right)$$

$$+\left(\frac{2{,}048(0.2)^{11}}{99}-\frac{2{,}048(0)^{11}}{99}\right)+\ldots$$

$$\int_0^{0.2} 4x\arctan(2x)\ dx \approx 0.0213333-0.0006827+0.0000468-0.0000041+0.0000004+\ldots$$

Let's start adding the terms together.

$$n_0+n_1=0.0206506$$

$$n_0+n_1+n_2=0.0206974$$

$$n_0+n_1+n_2+n_3=0.0206933$$

$$n_0+n_1+n_2+n_3+n_4=0.0206937$$

Remember, we only need the first five decimal places. When we analyze our results, we can see that $n_0+n_1+n_2+n_3=0.0206933$ is as far as we need to add in order to get five stable decimal points. Therefore, rounding the answer to five decimal places gives

$$n_0+n_1+n_2+n_3\approx 0.02069$$

Estimating indefinite integrals

Sometimes we'll be given an indefinite integral to evaluate, and we can't easily evaluate it with our normal integration techniques, like u-substitution, integration by parts, and partial fractions. In this case, we might be able to replace the function in the integral with its power series representation, whose expanded form is just a polynomial that's much easier to integrate.

As usual, we'll try to find a well-known power series that's similar to the given function, and then manipulate the power series until it matches the function. Most often, we'll use the power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Once we've modified this equation so that the left-hand side matches our function, we'll expand the power series through the first few terms. Then we'll integrate the individual terms from the expanded form to find a power series representation for the indefinite integral.

If we need to, we can go back to the power series representation that we found originally and use it to find the radius and interval of convergence.

Example

Evaluate the indefinite integral as a power series, then find the radius of convergence.

$$\int \frac{r}{1-r^6}\ dr$$

We need to manipulate the function we've been given in this integral until it matches the power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

We'll start by factoring r out of the numerator.

$$\frac{r}{1-r^6} = (r)\frac{1}{1-r^6}$$

Comparing the remaining fraction to the standard form of the power series, we can see that $x = r^6$. Therefore, we'll have to substitute r^6 for x, and multiply by the r that we factored out.

$$r\sum_{n=0}^{\infty} \left(r^6\right)^n$$

$$r^1\sum_{n=0}^{\infty} r^{6n}$$

$$\sum_{n=0}^{\infty} r^{6n+1}$$

This is the power series representation of the given function. If we expand the series, we get

$$\sum_{n=0}^{\infty} r^{6n+1} = r^1 + r^7 + r^{13} + r^{19} + \ldots + r^{6n+1}$$

Now we can replace the original function with the expanded power series, and the integral becomes

$$\int \frac{r}{1-r^6}\, dr = \int r^1 + r^7 + r^{13} + r^{19} + \ldots + r^{6n+1}\, dr$$

$$\int \frac{r}{1-r^6}\, dr = \frac{r^2}{2} + \frac{r^8}{8} + \frac{r^{14}}{14} + \frac{r^{20}}{20} + \ldots + \frac{r^{6n+2}}{6n+2} + C$$

That means the indefinite integral

$$\int \frac{r}{1-r^6}\, dr$$

is equal to the sum of the series

$$\sum_{n=0}^{\infty} \frac{r^{6n+2}}{6n+2} + C$$

To find the radius of convergence, we'll use the power series representation we found earlier,

$$\sum_{n=0}^{\infty} r^{6n+1}$$

We'll identify a_n and a_{n+1} as

$$a_n = r^{6n+1}$$

$$a_{n+1} = r^{6n+7}$$

We'll plug both of these into the limit formula from the ratio test, and get

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{r^{6n+7}}{r^{6n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| r^{(6n+7)-(6n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| r^{6n+7-6n-1} \right|$$

$$L = \lim_{n\to\infty} \left| r^6 \right|$$

The limit only effects n, not r, and we only have r left over in the limit. Therefore, we can remove the limit.

$$L = \left| r^6 \right|$$

The ratio test tells us that that series converges when $L < 1$. Since we know $L = \left| r^6 \right|$, we'll say that the series converges when

$$\left| r^6 \right| < 1$$

$$-1 < r^6 < 1$$

$$0 < r < 1$$

This tells us that the radius of convergence is $R = 1$.

Binomial series

Just as we did with Maclaurin series, we can use the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

to find the power series representation of functions that are similar to $(1+x)^k$. We'll just try to manipulate the binomial series until it matches the function we've been given.

Once we have the new, manipulated binomial series, we'll expand it through its first few terms and then use the pattern that we see in those terms to find a power series representation for the original function.

With the power series representation in hand, we'll be able to find the radius and interval of convergence of the series.

Example

Use the binomial series to expand the function as a power series, and then find the radius and interval of convergence.

$$f(x) = (1-x)^{-2}$$

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left-hand side match our function, we'll replace x with $-x$ and k with -2.

$$\left[1+(-x)\right]^{-2}=\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n=1-2(-x)+\frac{-2(-2-1)}{2!}(-x)^2+\frac{-2(-2-1)(-2-2)}{3!}(-x)^3+\dots$$

$$(1-x)^{-2}=\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n=1+2x+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1+2x+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$1x^0+2x^1+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$\frac{1}{0!}x^0+\frac{2}{1!}x^1+\frac{6}{2!}x^2+\frac{24}{3!}x^3+\dots$$

$$\frac{1}{1}x^0+\frac{2\cdot 1}{1}x^1+\frac{3\cdot 2\cdot 1}{2\cdot 1}x^2+\frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}x^3+\dots$$

$$1x^0+2x^1+3x^2+4x^3+\dots$$

When we match up these terms with their corresponding n-values, we get

$n=0$	$1x^0$
$n=1$	$2x^1$

$n=2$	$3x^2$
$n=3$	$4x^3$

We can see that the coefficients can all be represented by $n+1$, and that the exponents can all be represented by n. Therefore, the power series representation of the function is

$$(1-x)^{-2}=\sum_{n=0}^{\infty}(n+1)x^n$$

If we want to find the radius of convergence of this power series, we first identify that

$$a_n=(n+1)x^n$$

Then we generate a_{n+1}.

$$a_{n+1}=(n+1+1)x^{n+1}$$

$$a_{n+1}=(n+2)x^{n+1}$$

We plug both a_n and a_{n+1} into

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and then we'll set $L<1$, since the ratio test tells us that the series converges absolutely if $L<1$.

$$L=\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot x^{n+1-n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot x\right|$$

The limit only effects n, which means we can pull x out of the limit, as long as we keep it inside absolute value bars.

$$L = |x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

Since in our series n starts at 0, it's impossible for either the numerator or the denominator of the fraction to be negative, which means we can drop its absolute value bars.

$$L = |x|\lim_{n\to\infty}\frac{n+2}{n+1}$$

$$L = |x|\lim_{n\to\infty}\frac{n+2}{n+1}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$L = |x|\lim_{n\to\infty}\frac{\frac{n}{n}+\frac{2}{n}}{\frac{n}{n}+\frac{1}{n}}$$

$$L = |x| \lim_{n\to\infty} \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}}$$

$$L = |x| \left(\frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right)$$

$$L = |x| \left(\frac{1 + 0}{1 + 0} \right)$$

$$L = |x|(1)$$

$$L = |x|$$

Now we can set $L < 1$.

$$|x| < 1$$

The result is already in the form $|x - a| < R$, as

$$|x - 0| < 1$$

We can see from this equation that the radius of convergence is $R = 1$. That means that the interval of convergence is

$$|x| < 1$$

$$-1 < x < 1$$

Remember though that we always have to test the endpoints of the interval of convergence to say whether or not the series converges there.

We'll do this by plugging the endpoints back into the power series representation.

For $x = -1$:

$$\sum_{n=0}^{\infty} (n+1)(-1)^n$$

$$\sum_{n=0}^{\infty} (-1)^n(n+1)$$

By the divergence test (nth-term test), this series diverges.

For $x = 1$:

$$\sum_{n=0}^{\infty} (n+1)(1)^n$$

$$\sum_{n=0}^{\infty} n+1$$

By the divergence test (nth-term test), this series diverges.

Therefore, we can confirm that the interval of convergence is still

$$-1 < x < 1$$

If we summarize our results, we can say that the power series representation is $(1-x)^{-2}=\sum_{n=0}^{\infty}(n+1)x^n$

the radius of convergence is $R=1$

the interval of convergence is $-1<x<1$

Taylor series

We already know how to use the power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

to find a polynomial representation of a function. But sometimes we'll want to represent a function as a series, and we won't be able to easily relate the function to $1/(1-x)$.

Taylor series let us find a series representation for any function, whether or not we can relate it to the power series $1/(1-x)$.

In order to create a Taylor series representation for a function, we'll need

a - the value about which the function is defined

n - the degree to which we want to evaluate the function

Both of these are usually given in the problem. With a value for a and n, we can build the chart below.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$f(x)$	$f(a)$	$f(a)$
1	1	$f'(x)$	$f'(a)$	$f'(a)$
2	2	$f''(x)$	$f''(a)$	$\frac{f''(a)}{2}$

3	6	$f'''(x)$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	$f''''(x)$	$f''''(a)$	$\frac{f''''(a)}{24}$
		...		
n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$

When we're done with the chart, the value in the far right column becomes the coefficient on each term in the Taylor polynomial, in the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

The sum of all these terms is the Taylor series for the function.

Example

Find the fourth-degree Taylor polynomial about $a=1$.

$$f(x)=x^4+5x^3-x^2+3x-1$$

Since we're looking for the fourth-degree polynomial, we can say that $n=4$. As always, we'll start n at 0, so the values of n we'll include in our chart are $n=0,1,2,3,4$. We can also include all of the corresponding values of $n!$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$f(x)$	$f(a)$	$f(a)$
1	1	$f'(x)$	$f'(a)$	$f'(a)$
2	2	$f''(x)$	$f''(a)$	$\frac{f''(a)}{2}$
3	6	$f'''(x)$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	$f''''(x)$	$f''''(a)$	$\frac{f''''(a)}{24}$

To find the values for the third column, we'll put the original function in the first row, followed by its derivatives in the following rows.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4 + 5x^3 - x^2 + 3x - 1$	$f(a)$	$f(a)$
1	1	$4x^3 + 15x^2 - 2x + 3$	$f'(a)$	$f'(a)$
2	2	$12x^2 + 30x - 2$	$f''(a)$	$\frac{f''(a)}{2}$
3	6	$24x + 30$	$f'''(a)$	$\frac{f'''(a)}{6}$
4	24	24	$f''''(a)$	$\frac{f''''(a)}{24}$

To find the values for the fourth column, $f^{(n)}(a)$, we'll evaluate the values in the third column at $a = 1$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4+5x^3-x^2+3x-1$	$(1)^4+5(1)^3-(1)^2+3(1)-1=7$	$f(a)$
1	1	$4x^3+15x^2-2x+3$	$4(1)^3+15(1)^2-2(1)+3=20$	$f'(a)$
2	2	$12x^2+30x-2$	$12(1)^2+30(1)-2=40$	$\frac{f''(a)}{2}$
3	6	$24x+30$	$24(1)+30=54$	$\frac{f'''(a)}{6}$
4	24	24	24	$\frac{f''''(a)}{24}$

To get the values for the last column, we'll divide the result of the fourth column by $n!$ from the second column.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$x^4+5x^3-x^2+3x-1$	$(1)^4+5(1)^3-(1)^2+3(1)-1=7$	7
1	1	$4x^3+15x^2-2x+3$	$4(1)^3+15(1)^2-2(1)+3=20$	20
2	2	$12x^2+30x-2$	$12(1)^2+30(1)-2=40$	$\frac{40}{2}=20$
3	6	$24x+30$	$24(1)+30=54$	$\frac{54}{6}=9$
4	24	24	24	$\frac{24}{24}=1$

With the whole chart filled in, we can build each term of the Taylor polynomial.

$n=0$	$\frac{f^{(n)}(a)}{n!}(x-a)^n=7(x-1)^0$	7
$n=1$	$\frac{f^{(n)}(a)}{n!}(x-a)^n=20(x-1)^1$	$20(x-1)$
$n=2$	$\frac{f^{(n)}(a)}{n!}(x-a)^n=20(x-1)^2$	$20(x-1)^2$
$n=3$	$\frac{f^{(n)}(a)}{n!}(x-a)^n=9(x-1)^3$	$9(x-1)^3$
$n=4$	$\frac{f^{(n)}(a)}{n!}(x-a)^n=1(x-1)^4$	$(x-1)^4$

Putting all of the terms together, we get the fourth-degree Taylor polynomial.

$$7+20(x-1)+20(x-1)^2+9(x-1)^3+(x-1)^4$$

Radius and interval of convergence of a Taylor series

Sometimes we'll be asked for the radius and interval of convergence of a Taylor series. In order to find these things, we'll first have to find a power series representation for the Taylor series.

Once we have the Taylor series represented as a power series, we'll identify a_n and a_{n+1} and plug them into the limit formula from the ratio test in order to say where the series is convergent.

Example

Using the chart below, find the third-degree Taylor series about $a = 3$ for $f(x) = \ln(2x)$. Then find the power series representation of the Taylor series, and the radius and interval of convergence.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	$\ln(2x)$	$\ln 6$	$\ln 6$
1	1	$\frac{1}{x}$	$\frac{1}{3}$	$\frac{1}{3}$
2	2	$-\frac{1}{x^2}$	$-\frac{1}{9}$	$-\frac{1}{18}$
3	6	$\frac{2}{x^3}$	$\frac{2}{27}$	$\frac{1}{81}$

Taylor series

Since we already have the chart done, the value in the far right column becomes the coefficient on each term in the Taylor polynomial, in the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

With the whole chart filled in, we can build each term of the Taylor polynomial.

$$n=0 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \ln(6)(x-3)^0 \qquad \ln 6$$

$$n=1 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{1}{3}(x-3)^1 \qquad \frac{1}{3}(x-3)$$

$$n=2 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = -\frac{1}{18}(x-3)^2 \qquad -\frac{1}{18}(x-3)^2$$

$$n=3 \qquad \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{1}{81}(x-3)^3 \qquad \frac{1}{81}(x-3)^3$$

Putting all of the terms together, we get the third-degree Taylor polynomial.

$$\ln 6 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3$$

Power series representation

We want to find a power series representation for the Taylor series above. The first thing we can see is that the exponent of each $(x-3)$ is equal to the n value of that term, which means that

$$(x-3)^n$$

will be part of the power series representation. The fractional coefficient in front of the $(x-3)$ terms can be represented by

$$\frac{1}{n3^n}$$

Finally, we need to deal with the negative sign in front of the $n=2$ term. If we multiply our terms by

$$(-1)^{n+1}$$

the $n=2$ term will be negative and the $n=1$ and $n=3$ terms will be positive. Remember, none of these generalizations apply to our $n=0$ term, so we'll leave this term outside of the power series representation.

$$\ln 6 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$$

Notice the sum starts at $n=1$, since the $n=0$ term is not included in the sum.

Radius and interval of convergence

To find the radius of convergence, we'll identify a_n and a_{n+1} using the power series representation we just found.

$$a_n = \frac{(-1)^{n+1}(x-3)^n}{n3^n}$$

$$a_{n+1} = \frac{(-1)^{n+2}(x-3)^{n+1}}{3^{n+1}(n+1)}$$

We can plug a_n and a_{n+1} into the limit formula from the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)3^{n+1}}}{\frac{(-1)^{n+1}(x-3)^n}{n3^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(-1)^{n+1}(x-3)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| (-1)^{n+2-(n+1)} \cdot (x-3)^{n+1-n} \cdot \frac{n}{n+1} \cdot 3^{n-(n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| (-1)^{n+2-n-1} \cdot (x-3)^{n+1-n} \cdot 3^{n-n-1} \cdot \frac{n}{n+1} \right|$$

$$L=\lim_{n\to\infty}\left|(-1)^1\cdot(x-3)^1\cdot 3^{-1}\cdot\frac{n}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|-\frac{1}{3}(x-3)\frac{n}{n+1}\right|$$

Since we're dealing with absolute value, the -1 can be removed.

$$L=\lim_{n\to\infty}\left|\frac{n(x-3)}{3(n+1)}\right|$$

The limit only effects n, so we can remove the $(x-3)$.

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3(n+1)}\right|$$

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3n+3}\right|$$

Since we'll get the indeterminate form ∞/∞ if we try to evaluate the limit, we'll divide the numerator and denominator by the highest-degree variable in order to reduce the fraction.

$$L=|x-3|\lim_{n\to\infty}\left|\frac{n}{3n+3}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right|$$

$$L=|x-3|\lim_{n\to\infty}\left|\frac{\frac{n}{n}}{\frac{3n}{n}+\frac{3}{n}}\right|$$

$$L = |x-3| \lim_{n \to \infty} \left| \frac{1}{3+\frac{3}{n}} \right|$$

$$L = |x-3| \left| \frac{1}{3+\frac{3}{\infty}} \right|$$

$$L = |x-3| \left| \frac{1}{3+0} \right|$$

$$L = |x-3| \left| \frac{1}{3} \right|$$

$$L = \frac{1}{3}|x-3|$$

Since the ratio test tells us that the series will converge when $L < 1$, so we'll set up the inequality.

$$\frac{1}{3}|x-3| < 1$$

$$|x-3| < 3$$

Since the inequality is in the form $|x-a| < R$, we can say that the radius of convergence is $R = 3$.

To find the interval of convergence, we'll take the inequality we used to find the radius of convergence, and solve it for x.

$$|x-3| < 3$$

$$-3 < x - 3 < 3$$

$$-3 + 3 < x - 3 + 3 < 3 + 3$$

$$0 < x < 6$$

We need to test the endpoints of the inequality by plugging them into the power series representation. We'll start with $x = 0$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n(3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$$

The exponent on the -1 will always be odd, so the sum is going to simplify to

$$\sum_{n=1}^{\infty} -\frac{1}{n}$$

This is a divergent p-series, so the series diverges at the endpoint $x = 0$. Now we'll test $x = 6$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(6-3)^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{n3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

The alternating series test for convergence says that a series converges if $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} \frac{1}{n}$$

$$\frac{1}{\infty}$$

$$0$$

The series converges at the endpoint $x = 6$.

We've shown that the series diverges at $x = 0$ and converges at $x = 6$, which means the interval of convergence is

$$0 < x \le 6$$

We'll summarize our findings.

3rd-degree Taylor polynomial $\ln 6 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3$

Power series representation	$\ln 6 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$
Radius of convergence	$R = 3$
Interval of convergence	$0 < x \leq 6$

Taylor's inequality

Taylor's inequality states that, for a function $f(x)$,

$$\text{if} \quad \left| f^{n+1}(x) \right| \leq M \qquad \text{for} \quad |x-a| \leq d$$

$$\text{then} \quad \left| R_n(x) \right| \leq \frac{M}{(n+1)!} \left| x-a \right|^{n+1} \quad \text{for} \quad |x-a| \leq d$$

This theorem looks elaborate, but it's nothing more than a tool to find the remainder of a series.

For example, oftentimes we're asked to find the nth-degree Taylor polynomial that represents a function $f(x)$. The sum of the terms after the n th term that aren't included in the Taylor polynomial is the remainder. We can use Taylor's inequality to find that remainder and say whether or not the nth-degree polynomial is a good approximation of the function's actual value.

Sometimes we can use Taylor's inequality to show that the remainder of a power series is $R_n(x)=0$. If the remainder is 0, then we know that the series representation of the function is equal to the exact value of the original function.

If we want to use the theorem to show that the power series representation of the function is equal to the the function itself, then we'll need to show that both parts of Taylor's inequality are true and that the remainder is 0.

Example

Use Taylor's inequality to show that the the Maclaurin series representation of the function is equal to the original function.

$$f(x) = \sin x$$

Using a table of common Maclaurin series, we know that the power series representation of the Maclaurin series for $f(x) = \sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

In order to show that this equation is true, that the sum of the Maclaurin series is in fact equal to the original function, we'll need to use Taylor's inequality to show that the remainder of the power series is 0.

Since we're dealing with a Maclaurin series, $a = 0$, and we can adjust the inequalities from the theorem from

$$\text{if} \quad \left|f^{n+1}(x)\right| \le M \quad \text{for} \quad |x-a| \le d$$

$$\text{then} \quad \left|R_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d$$

to

$$\text{if} \quad \left|f^{n+1}(x)\right| \le M \quad \text{for} \quad |x-0| \le d$$

then $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x-0\right|^{n+1}$ for $\left|x-0\right| \leq d$

or

if $\left|f^{n+1}(x)\right| \leq M$ for $|x| \leq d$

then $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$

Next, we'll need to take the first few derivatives of $f(x)=\sin x$ in order to plug in a value for $f^{n+1}(x)$.

$n=0$ $\quad f^{n+1}(x)=f^{0+1}(x)=f^1(x)=f'(x)=\cos x$

$n=1$ $\quad f^{n+1}(x)=f^{1+1}(x)=f^2(x)=f''(x)=-\sin x$

$n=2$ $\quad f^{n+1}(x)=f^{2+1}(x)=f^3(x)=f'''(x)=-\cos x$

$n=3$ $\quad f^{n+1}(x)=f^{3+1}(x)=f^4(x)=f''''(x)=\sin x$

Since we'll be taking the absolute value of $f^{n+1}(x)$, we can say that

$$\left|f^{n+1}(x)\right| = \cos x$$

or

$$\left|f^{n+1}(x)\right| = \sin x$$

We know that $\cos x$ and $\sin x$ only exist between -1 and 1, so we could say

$$-1 \leq \left|f^{n+1}(x)\right| \leq 1$$

However, since we're dealing with absolute value, $\left|f^{n+1}(x)\right|$ can't be negative, so

$$0 \leq \left|f^{n+1}(x)\right| \leq 1$$

This inequality tells us that the value of $|f^{n+1}(x)|$ is somewhere on the interval $[0,1]$. Let's pick a few values in the interval and plug them into the first inequality from Taylor's inequality.

$\left\|f^{n+1}(x)\right\|$	$\left\|f^{n+1}(x)\right\| \leq M$
0	$0 \leq M$
$1/3$	$1/3 \leq M$
$3/4$	$3/4 \leq M$
1	$1 \leq M$

What we can see is that, if we pick any value $|f^{n+1}(x)| < 1$, then we won't be including the whole interval $[0,1]$. But if we pick $|f^{n+1}(x)| = 1$, then we know that M will always be greater than or equal to any value in the interval. Therefore, as a rule, we'll always pick the right-hand side of the interval. In this case, that's $M = 1$, which we'll plug into our already simplified version of Taylor's inequality.

if $\quad \left|f^{n+1}(x)\right| \leq 1 \qquad$ for $\quad |x| \leq d$

then $\quad \left|R_n(x)\right| \leq \dfrac{1}{(n+1)!}|x|^{n+1} \qquad$ for $\quad |x| \leq d$

With Taylor's inequality simplified after plugging in $a = 0$ and $M = 1$, we'll use squeeze theorem to evaluate the remainder inequality and try to show that the remainder is 0.

$$\left|R_n(x)\right| \leq \frac{1}{(n+1)!}\left|x\right|^{n+1}$$

$$\lim_{n\to\infty}\left|R_n(x)\right| \leq \lim_{n\to\infty}\frac{1}{(n+1)!}\left|x\right|^{n+1}$$

$$\lim_{n\to\infty}\left|R_n(x)\right| \leq \frac{1}{(\infty+1)!}\left|x\right|^{\infty+1}$$

$$\lim_{n\to\infty}\left|R_n(x)\right| \leq 0\cdot\left|x\right|^{\infty}$$

$$\lim_{n\to\infty}\left|R_n(x)\right| \leq 0$$

By definition, it's impossible for a remainder to ever be negative, so it must be true that

$$\lim_{n\to\infty}\left|R_n(x)\right| = 0$$

Since the remainder is 0, we know that the power series representation of the Maclaurin series of the function is exactly equal to the original function $f(x) = \sin x$.

Maclaurin series

A Maclaurin series is the specific instance of the Taylor series when $a = 0$. Remember that we can choose any value of a in order to find a Taylor polynomial. Maclaurin series eliminate that choice and force us to choose $a = 0$.

Remember that we would always use the formula

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

to build each term in the Taylor series. Since $a = 0$ in every Maclaurin series, this formula simplifies to

$$\frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$\frac{f^{(n)}(0)}{n!}x^n$$

Everything else about the Maclaurin series is the same.

Example

Find the seventh-degree Maclaurin series of the function.

$$f(x) = \sin(3x)$$

We'll start by creating the chart we've always made for Taylor polynomials. Since we're finding the series to the seventh-degree, we'll use n from 0 to 7. Since it's a Maclaurin series, we'll use $a=0$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	1	$\sin(3x)$	$\sin(3\cdot 0)=0$	$\frac{0}{1}=0$
1	1	$3\cos(3x)$	$3\cos(3\cdot 0)=3$	$\frac{3}{1}=3$
2	2	$-9\sin(3x)$	$-9\sin(3\cdot 0)=0$	$\frac{0}{2}=0$
3	6	$-27\cos(3x)$	$-27\cos(3\cdot 0)=-27$	$\frac{-27}{6}=-\frac{27}{6}$
4	24	$81\sin(3x)$	$81\sin(3\cdot 0)=0$	$\frac{0}{24}=0$
5	120	$243\cos(3x)$	$243\cos(3\cdot 0)=243$	$\frac{243}{120}=\frac{81}{40}$
6	720	$-729\sin(3x)$	$-729\sin(3\cdot 0)=0$	$\frac{0}{720}=0$
7	5,040	$-2{,}187\cos(3x)$	$-2{,}187\cos(3\cdot 0)=-2{,}187$	$\frac{-2{,}187}{5{,}040}=-\frac{243}{560}$

With the whole chart filled in, we can build each term of the Maclaurin series.

$n = 0$	$\frac{f^{(n)}(0)}{n!}x^n = 0x^0$	0
$n = 1$	$\frac{f^{(n)}(0)}{n!}x^n = 3x^1$	$3x$
$n = 2$	$\frac{f^{(n)}(0)}{n!}x^n = 0x^2$	0
$n = 3$	$\frac{f^{(n)}(0)}{n!}x^n = -\frac{27}{6}x^3$	$-\frac{27}{6}x^3$
$n = 4$	$\frac{f^{(n)}(0)}{n!}x^n = 0x^4$	0
$n = 5$	$\frac{f^{(n)}(0)}{n!}x^n = \frac{81}{40}x^5$	$\frac{81}{40}x^5$
$n = 6$	$\frac{f^{(n)}(0)}{n!}x^n = 0x^6$	0
$n = 7$	$\frac{f^{(n)}(0)}{n!}x^n = -\frac{243}{560}x^7$	$-\frac{243}{560}x^7$

Putting all of the terms together, we get the seventh-degree Maclaurin series.

$$0 + 3x + 0 - \frac{27}{6}x^3 + 0 + \frac{81}{40}x^5 + 0 - \frac{243}{560}x^7$$

$$3x - \frac{27}{6}x^3 + \frac{81}{40}x^5 - \frac{243}{560}x^7$$

Sum of the Maclaurin series

To find the sum of a Maclaurin series, we'll try to use a common Maclaurin series for which we already know the sum, manipulating the given series until it matches the standard series.

Example

Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n \pi^{2n}}{(2n)!}$$

From a table of standard Maclaurin series, we already know that the sum of the Maclaurin series of $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Since this series is really similar to the series we're given in this problem, we want to try to manipulate our series until it matches the form of this standard series.

We'll start by changing the 4-based term so that its exponent becomes $2n$, like the exponent in the standard series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(2^2\right)^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}$$

Since they have the same exponent, we can combine the 2-based term with the π-based term.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

With the changes we've made, the given series now matches the standard series for $\cos x$, except that $x = 2\pi$. Knowing that $x = 2\pi$, we can make the substitution on the left-hand side of the formula for the sum of the Maclaurin series of $\cos x$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

We know that $\cos(2\pi) = 1$, so

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$$

Since the right side of this equation is equal to the sum of the given series, we can say that the sum of the given series is 1.

Radius and interval of convergence of a Maclaurin series

Sometimes we'll be asked for the radius and interval of convergence of a Maclaurin series. In order to find these things, we'll first have to find a power series representation for the Maclaurin series, which we can do by hand, or using a table of common Maclaurin series.

Once we have the Maclaurin series represented as a power series, we'll identify a_n and a_{n+1} and plug them into the limit formula from the ratio test in order to say where the series is convergent and give the radius of convergence.

Then we'll use the radius of convergence to find the interval of convergence, making sure to test the endpoints of the interval to verify whether or not the series converges at one or both endpoints.

Example

Find the radius and interval of convergence of the Maclaurin series of the function.

$$f(x) = \ln(1 + 2x)$$

Using a table of common Maclaurin series, we know that the power series representation of the Maclaurin series for $f(x) = \ln(1 + x)$ is

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Since our series has a $2x$ in place of x, we'll make that substitution on both sides of the equation and get a power series representation for the given function.

$$\ln(1+2x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2x)^n$$

Now that we have a power series representation for the given function, we're able to find the radius of convergence using the ratio test.

Since the ratio test tells us that a series converges if $L<1$ when

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

we just need to find a_n and a_{n+1}, and then plug them into the equation for L. Using the power series we generated for our function, we'll say that

$$a_n=\frac{(-1)^{n+1}}{n}(2x)^n$$

$$a_{n+1}=\frac{(-1)^{n+2}}{n+1}(2x)^{n+1}$$

Plugging these into the equation for L from the ratio test, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(2x)^{n+1}}{n+1}}{\frac{(-1)^{n+1}(2x)^n}{n}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(2x)^{n+1}}{n+1}\left[\frac{n}{(-1)^{n+1}(2x)^n}\right]\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}}\cdot\frac{(2x)^{n+1}}{(2x)^n}\cdot\frac{n}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+2-(n+1)}(2x)^{n+1-n}n}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+2-n-1}(2x)^1n}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^1 2xn}{n+1}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{-2xn}{n+1}\right|$$

The absolute value brackets cancel the -1, so we get

$$L=\lim_{n\to\infty}\left|\frac{2xn}{n+1}\right|$$

The limit only deals with n, not x, so we can pull $2x$ out of the limit, as long as we keep it in absolute value bars.

$$L=\left|2x\right|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{n}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |2x| \lim_{n\to\infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = |2x| \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$L = |2x| \left| \frac{1}{1+0} \right|$$

$$L = |2x| \, |1|$$

$$L = |2x|$$

Since by the ratio test we know that the series will converge when $L < 1$, we'll set

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

With the inequality in the form $|x-a|<R$, we can say that the radius of convergence of the Maclaurin series is

$$R=\frac{1}{2}$$

To find the interval of convergence of the Maclaurin series, we'll remove the absolute value bars from the radius of convergence.

$$|x|<\frac{1}{2}$$

$$-\frac{1}{2}<x<\frac{1}{2}$$

But before we can call this the interval of convergence, we have to verify whether or not the series converges at one or both endpoints, $x=-1/2$ and $x=1/2$. To do this, we'll plug each endpoint into the original series.

For $x=-1/2$,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(2x)^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[2\left(-\frac{1}{2}\right)\right]^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

which means we can use the alternating series test to say whether or not it converges. Remember, the alternating series test tells us that a series converges if $\lim_{n \to \infty} a_n = 0$.

$$\lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} \frac{1}{n}$$

$$\frac{1}{\infty}$$

$$0$$

Because the limit is 0, the series converges by the alternating series test, which means the Maclaurin series converges at the left endpoint of the interval, $x = -1/2$.

Now we'll test $x = 1/2$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (2x)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[2\left(\frac{1}{2}\right)\right]^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n$$

Since $1^n = 1$ for all values of n, we can cancel it out.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$$

This is an alternating series where

$$a_n = \frac{1}{n}$$

This is the same series we used to find the convergence of the left endpoint of the interval, and we already know that it converges by the alternating series test. Therefore, we can say that the series also converges at the right endpoint of the interval, $x = 1/2$.

Since the series converges at both endpoints of the interval, the interval of convergence of the Maclaurin series of $f(x) = \ln(1 + 2x)$ is

$$-\frac{1}{2} \le x \le \frac{1}{2}$$
