Calculus 2 Formulas

krista king MATH

Integrals

Midpoint rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\bar{x}_i\right)\Delta x = \Delta x\left[f\left(\bar{x}_1\right) + \ldots + f\left(\bar{x}_n\right)\right]$$

where $\left(x_{i-1}, x_i\right)$ and

$$\Delta x = \frac{b-a}{n}$$ and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$ and

$$x_i = a + i\Delta x$$

Midpoint and trapezoidal error bounds

E_T and E_M are the errors in the trapezoidal and midpoint rules

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$

for $a \leq x \leq b$

Simpson's rule

$$\int_a^b f(x)\ dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and

$$\Delta x = \frac{b-a}{n}$$

Simpson's error bounds

E_S is the error in Simpson's rule

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $\left|f^{(4)}(x)\right| \leq K$

for $a \leq x \leq b$

Symmetric functions

Suppose f is continuous on $[-a, a]$.

If f is **even** $\quad [f(-x) = f(x)]$, then $\quad \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$

If f is **odd** $\quad [f(-x) = -f(x)]$, then $\quad \int_{-a}^{a} f(x)\ dx = 0$

Limit process for area under the curve

$$\int_{a}^{b} f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\quad \Delta x = \frac{b-a}{n}$

and $\quad x_i = a + i\Delta x$

Summation formulas for the limit process

$\sum_{i=1}^{n} k = kn \quad$ where k is any non-zero constant

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30} = \frac{6n^5+15n^4+10n^3+6n^2-n}{30}$$

Net change theorem

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Fundamental theorem of calculus

Suppose f is continuous on $[a,b]$.

Part 1

Given integral	How to solve it
$f(x) = \int_a^x f(t)\ dt$	Plug x in for t.
$f(x) = \int_x^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.

$f(x) = \int_a^{g(x)} f(t)\ dt$

Plug $g(x)$ in for t, then multiply by dg/dx.

$f(x) = \int_{g(x)}^{a} f(t)\ dt$

Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.

$f(x) = \int_{g(x)}^{h(x)} f(t)\ dt$

Split the limits of integration as $\int_{g(x)}^{0} f(t)\ dt + \int_{0}^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^{0} f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Part 2

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

Integration by parts

$$\int u\ dv = uv - \int v\ du$$

Properties of integrals

$$\int_a^b c\ dx = c(b-a)$$

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

$$\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Common indefinite integrals

$$\int k\ dx = kx + C$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$ with $n \neq -1$

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C$$

Integrals of trig functions

$$\int \sin x \ dx = -\cos x + C$$

$$\int \csc x \ dx = \ln \left| \csc x - \cot x \right| + C$$

or $$\int \csc x \ dx = \ln \left(\sin \frac{x}{2} \right) - \ln \left(\cos \frac{x}{2} \right) + C$$

$$\int \cos x \ dx = \sin x + C$$

$$\int \sec x \ dx = \ln \left| \sec x + \tan x \right| + C$$

or

$$\int \sec x \ dx = \ln \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) - \ln \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) + C$$

$$\int \tan x \ dx = -\ln \cos x + C$$

$$\int \cot x \ dx = \ln \sin x + C$$

Other common trig integrals

$$\int \sec^2 x \ dx = \tan x + C$$

$$\int \csc^2 x \ dx = -\cot x + C$$

$$\int \sec x \tan x \ dx = \sec x + C$$

$$\int \csc x \cot x \ dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1} \ dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1 - x^2}} \ dx = \sin^{-1} x + C$$

$$\int \sinh x \ dx = \cosh x + C$$

$$\int \cosh x \ dx = \sinh x + C$$

Rewriting inverse hyperbolic functions

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1}) = \pm \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$\text{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$$

$$\text{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 - x^2}}{|x|}\right)$$

$$\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x + 1}{x - 1}\right)$$

Integrals of inverse hyperbolic trig functions

$$\int \sinh^{-1} x \ dx = x \sinh^{-1} x - \sqrt{x^2 + 1} + C$$

$$\int \cosh^{-1} x \ dx = x \cosh^{-1} x - \sqrt{x - 1}\sqrt{x + 1} + C$$

$$\int \tanh^{-1} x \ dx = \frac{1}{2} \log(1 - x^2) + x \tanh^{-1} x + C$$

$$\int \coth^{-1} x \ dx = \frac{1}{2} \log(1 - x^2) + x \coth^{-1} x + C$$

Integrals resulting in inverse hyperbolic trig functions

$$\int \frac{1}{\sqrt{x^2+1}} \ dx = \sinh^{-1} x$$

$$\int \frac{1}{\sqrt{x-1}\sqrt{x+1}} \ dx = \cosh^{-1} x$$

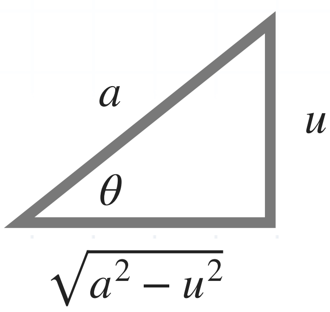
$$\int \frac{1}{1-x^2} \ dx = \tanh^{-1} x$$

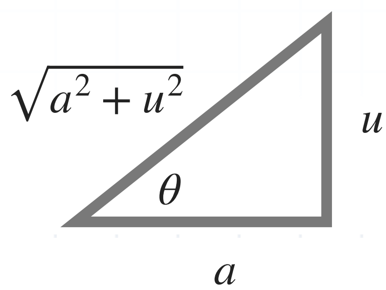
$$\int \frac{1}{1-x^2} \ dx = \coth^{-1} x$$

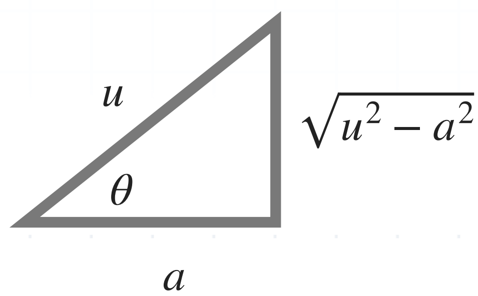
Trig substitution setup

	sin	tan	sec
the integral includes	$\sqrt{a^2-u^2}$	$\sqrt{a^2+u^2}$	$\sqrt{u^2-a^2}$
so substitute	$u=a\sin\theta$	$u=a\tan\theta$	$u=a\sec\theta$
and use the identity	$1-\sin^2\theta=\cos^2\theta$	$1+\tan^2\theta=\sec^2\theta$	$\sec^2\theta-1=\tan^2\theta$
solve for the trig	$\sin\theta=\frac{u}{a}$	$\tan\theta=\frac{u}{a}$	$\sec\theta=\frac{u}{a}$
and for du	$du=a\cos\theta\ d\theta$	$du=a\sec^2\theta\ d\theta$	$du=a\sec\theta\tan\theta\ d\theta$
and for θ	$\theta=\arcsin\frac{u}{a}$	$\theta=\arctan\frac{u}{a}$	$\theta=\operatorname{arcsec}\frac{u}{a}$

reference triangle







Trig substitution simplification

	arcsin x	arccos x	arctan x	arccsc$x$	arcsec$x$	arccot$x$
sin of	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{x^2+1}}$	$\frac{1}{x}$	$\sqrt{1-\frac{1}{x^2}}$	$-\frac{1}{x\sqrt{\frac{1}{x^2}+1}}$
cos of	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{x^2+1}}$	$\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$	$\frac{1}{\sqrt{\frac{1}{x^2}+1}}$
tan of	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$
csc of	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{x^2+1}}{x}$	x	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{\frac{1}{x^2}+1}$
sec of	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{x^2+1}$	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	x	$\sqrt{\frac{1}{x^2}+1}$
cot of	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	x

Applications of Integrals

Average value

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Area between curves

$$A = \int_a^b |f(x) - g(x)|\ dx$$

Arc length

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

for a function $y = f(x)$

on the interval $a \le x \le b$

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

for a function $x = g(y)$

on the interval $c \le y \le d$

Surface area of revolution

The surface area of revolution is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

for a function $y=f(x)$

rotated about the x-axis

on the interval $a\le x\le b$

$$S=\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

for a function $y=f(x)$

rotated about the y-axis

on the interval $a\le x\le b$

$$S=\int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

for a function $x=g(y)$

rotated about the x-axis

on the interval $c\le y\le d$

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

for a function $x=g(y)$

rotated about the y-axis

on the interval $c\le y\le d$

Volume of revolution

Axis	Disks	Washers	Shells
	$\int \text{area width}$	$\int \text{area width}$	$\int \text{circumference height width}$
		Axis of revolution: HORIZONTAL	
x-axis	$\int_a^b \pi \left[f(x)\right]^2\ dx$	$\int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2\ dx$	$\int_c^d 2\pi y \left[f(y) - g(y)\right]\ dy$
$y = -k$		$\int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2\ dx$	$\int_c^d 2\pi (y + k)\left[f(y) - g(y)\right]\ dy$
$y = k$		$\int_a^b \pi \left[k - f(x)\right]^2 - \pi \left[k - g(x)\right]^2\ dx$	$\int_c^d 2\pi (k - y)\left[f(y) - g(y)\right]\ dy$
		Axis of revolution: VERTICAL	
y-axis	$\int_c^d \pi \left[f(y)\right]^2\ dy$	$\int_c^d \pi \left[f(y)\right]^2 - \pi \left[g(y)\right]^2\ dy$	$\int_a^b 2\pi x \left[f(x) - g(x)\right]\ dx$
$x = -k$		$\int_c^d \pi \left[k + f(y)\right]^2 - \pi \left[k + g(y)\right]^2\ dy$	$\int_a^b 2\pi (x + k)\left[f(x) - g(x)\right]\ dx$
$x = k$		$\int_c^d \pi \left[k - f(y)\right]^2 - \pi \left[k - g(y)\right]^2\ dy$	$\int_a^b 2\pi (k - x)\left[f(x) - g(x)\right]\ dx$

Mean value theorem for integrals

If f is continuous on $[a,b]$, then c exists in $[a,b]$ such that

$$f(c)=f_{ave}=\frac{1}{b-a}\int_a^b f(x)\ dx, \quad \text{that is,} \quad \int_a^b f(x)\ dx=f(c)(b-a)$$

Moments of the region

The moment of the region

about the y-axis is $$M_y=\lim_{n\to\infty}\sum_{i=1}^{n}\rho\bar{x}_i f(\bar{x}_i)\Delta x=\rho\int_a^b xf(x)\ dx$$

about the x-axis is $$M_x=\lim_{n\to\infty}\sum_{i=1}^{n}\rho\cdot\frac{1}{2}\left[f(\bar{x}_i)\right]^2\Delta x=\rho\int_a^b\frac{1}{2}\left[f(x)\right]^2\ dx$$

Center of mass of the region

The center of mass is located at $(\bar{x},\bar{y})$ where

$$\bar{x}=\frac{M_y}{m}=\frac{1}{A}\int_a^b xf(x)\ dx \quad \text{and} \quad \bar{y}=\frac{M_x}{m}=\frac{1}{A}\int_a^b\frac{1}{2}\left[f(x)\right]^2\ dx$$

Center of mass of the region bounded by two curves

The center of mass is located at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]\ dx \qquad \text{and} \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{\left[f(x)\right]^2 - \left[g(x)\right]^2\right\}\ dx$$

Polar & Parametric

Parametric

Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ where $$\frac{dx}{dt} \neq 0$$

Area

$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt$$

Surface area of revolution

The surface area of a parametric curve rotated

about the x-axis is $$S_x = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

about the y-axis is $$S_y = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Arc length

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

Volume of revolution

The volume created by rotating a parametric curve

about the x-axis is $$V_x = \int_a^b \pi y^2 \left[x'(t)\right] \ dt$$

about the y-axis is $$V_y = \int_a^b \pi x^2 \left[y'(t)\right] \ dt$$

Polar

Conversion between cartesian and polar coordinates

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $x^2 + y^2 = r^2$

Distance between two points

The distance between two polar coordinate points $\left(r_1, \theta_1\right)$ and $\left(r_2, \theta_2\right)$ is

$$D = \sqrt{\left(r_2 \cos \theta_2 - r_1 \cos \theta_1\right)^2 + \left(r_2 \sin \theta_2 - r_1 \sin \theta_1\right)^2}$$

Area

The area enclosed by a polar curve is

$$A = \int_a^b \frac{1}{2} \left[f(\theta)\right]^2 \ d\theta = \int_a^b \frac{1}{2} r^2 \ d\theta$$

Area between curves

The area between two polar curves is

$$A = \frac{1}{2} \int_\alpha^\beta (r_{\text{outer}})^2 - (r_{\text{inner}})^2 \ d\theta$$

Arc length

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta$$

Surface area of revolution

The surface area of revolution of a polar parametric curve rotated about the x-axis is

$$S_x = \int_{\alpha}^{\beta} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

about the y-axis is

$$S_y = \int_{\alpha}^{\beta} 2\pi x \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Sequences & Series

Limit of a sequence

The limit of a sequence $\{a_n\}$ is L

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms of a_n closer and closer to L as we we make n larger and larger. If

$$\lim_{n\to\infty} a_n$$

exists, the sequence converges (is convergent). Otherwise it diverges (is divergent).

Precise definition of the limit of a sequence

The limit of a sequence $\{a_n\}$ is L

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \epsilon$$

Limit laws for sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}(a_n-b_n)=\lim_{n\to\infty}a_n-\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}ca_n=c\lim_{n\to\infty}a_n$$

$$\lim_{n\to\infty}c=c$$

$$\lim_{n\to\infty}(a_nb_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}$$ if $\lim_{n\to\infty}b_n\neq 0$

$$\lim_{n\to\infty}a_n^p=\left[\lim_{n\to\infty}a_n\right]^p$$ if $p>0$ and $a_n>0$

Squeeze theorem for sequences

If $a_n\le b_n\le c_n$ for $n\ge n_0$ and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}c_n=L$ then $\lim_{n\to\infty}b_n=L$.

Absolute value of a sequence

If $\lim_{n\to\infty}|a_n|=0$, then $\lim_{n\to\infty}a_n=0$.

Convergence of a sequence r^n

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = 0 \quad \text{if} \quad -1 < r < 1$$

$$= 1 \quad \text{if} \quad r = 1$$

Increasing, decreasing, and monotonic sequences

A sequence $\{a_n\}$ is

increasing if $a_n < a_{n+1}$ for all $n \geq 1$, ($a_1 < a_2 < a_3 < \ldots$)

decreasing if $a_n > a_{n+1}$ for all $n \geq 1$, ($a_1 > a_2 > a_3 > \ldots$)

monotonic if it's either increasing or decreasing

Bounded sequences

A sequence $\{a_n\}$ is

bounded above if there's a number M such that $a_n \leq M$ for all $n \geq 1$

bounded below if there's a number m such that $m \leq a_n$ for all $n \geq 1$

a bounded sequence if it's bounded above and below

Monotonic sequence theorem

Every bounded, monotonic sequence is convergent.

Partial sum of the series

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, let s_n denote its nth partial sum.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$$

If the sequence $\{s_n\}$ converges and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ converges and we can say

$$a_1 + a_2 + \dots + a_n + \dots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is the sum of the series. If the sequence $\{s_n\}$ diverges, then the series diverges.

Convergence and sum of a geometric series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

converges if $|r| < 1$, otherwise it diverges (if $|r| \geq 1$). The sum of the convergent series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

Convergence of a_n

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Test for divergence

If $\lim_{n\to\infty} a_n$ doesn't exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Laws of convergent series

If the series a_n and b_n both converge, then so do these (where c is a constant):

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Integral test for convergence

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$, and let $a_n = f(n)$.

If $\int_1^{\infty} f(x)\ dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\int_1^{\infty} f(x)\ dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remainder estimate for the integral test

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ converges. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_n^{\infty} f(x)\ dx$$

p-Series test for convergence

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$

diverges if $p \leq 1$

Comparison test for convergence

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges.

If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.

Limit comparison test for convergence

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$

where c is a finite number and

where $0 < c < \infty$

then either both series converge or both diverge.

Alternating series test for convergence

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \qquad b_n > 0$$

satisfies

$b_{n+1} \le b_n$ for all n

$$\lim_{n \to \infty} b_n = 0$$

then the series converges.

Alternating series estimation theorem

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$b_{n+1} \le b_n$$

$$\lim_{n \to \infty} b_n = 0$$

then $|R_n| = |s - s_n| \le b_{n+1}$.

Absolute convergence

A series $\sum a_n$ is absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

If a series $\sum a_n$ is absolutely convergent, then it's convergent.

Conditional convergence

A series $\sum a_n$ is conditionally convergent if it's convergent but not absolutely convergent.

Ratio test for convergence

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive about the convergence of $\sum_{n=1}^{\infty} a_n$, which means we'll have to use a different convergence test to determine convergence.

Root test for convergence

If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive about the convergence of $\sum_{n=1}^{\infty} a_n$, which means we'll have to use a different convergence test to determine convergence.

Convergence of power series

Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$,

the series converges only when $x = a$ **or**

the series converges for all x **or**

there is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

Differentiation and integration of power series

If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

Note: The radii of convergence of these two power series is R.

Power series representation (expansion)

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and the power series has the form

$$\begin{aligned} f(x) \quad &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

which is the taylor series of the function f at a (or about a or centered at a).

Taylor series

The taylor series of a function f at a (or about a or centered at a) is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Remainder of the taylor series

If $f(x) = T_n(x) + R_n(x)$ where T_n is the nth-degree taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its taylor series on the interval $|x-a| < R$.

Taylor's inequality

If

$$|f^{(n+1)}(x)| \le M \text{ for } |x-a| \le d$$

then the remainder $R_n(x)$ of the taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \qquad |x-a| \le d$$

Maclaurin series

The maclaurin series is a specific instance of the Taylor series where $a = 0$. In other words, it's just the Taylor series of a function f at 0 (or about 0 or centered at 0).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Common maclaurin series and their radii of convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!} x^{2n+1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2{,}835} + \ldots$$

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1{,}152} - \ldots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots \qquad R = 1$$

Exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

Binomial series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$