Calculus 2 Quizzes

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Topic: Indefinite integrals

Question: Evaluate the indefinite integral.

$$\int (2+x)\left(x^2-4\right)\ dx$$

Answer choices:

A $\frac{1}{6}x^5+\frac{2}{3}x^4-2x^3-8x^2+C$

B $2x+C$

C $\frac{1}{4}x^4+\frac{2}{3}x^3-2x^2-8x+C$

D $3x^2+4x-4+C$

Solution: C

In order to integrate, we must first rewrite the function by multiplying the two binomial terms together.

$$\int x^3 + 2x^2 - 4x - 8 \ dx$$

$$\frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{4}{2}x^2 - 8x + C$$

$$\frac{1}{4}x^4 + \frac{2}{3}x^3 - 2x^2 - 8x + C$$

Topic: Indefinite integrals

Question: Evaluate the indefinite integral.

$$\int \frac{2x^3 - x^2 + 4}{x^2}\ dx$$

Answer choices:

A $\qquad \dfrac{x^3 - x^2 - 4}{x} + C$

B $\qquad \dfrac{2x^3 - 8}{x^3} + C$

C $\qquad \dfrac{\frac{1}{4}x^4 - \frac{1}{3}x^3 + 4x}{\frac{1}{3}x^3} + C$

D $\qquad 3x - 1 + C$

Solution: A

Before we can integrate, we must rewrite by dividing each term in the numerator by the denominator.

$$\int \frac{2x^3 - x^2 + 4}{x^2} \ dx$$

$$\int 2x - 1 + 4x^{-2} \ dx$$

$$\frac{2}{2}x^2 - x + \frac{4}{-1}x^{-1} + C$$

$$x^2 - x - \frac{4}{x} + C$$

$$\frac{x^3}{x} - \frac{x^2}{x} - \frac{4}{x} + C$$

$$\frac{x^3 - x^2 - 4}{x} + C$$

Topic: Indefinite integrals

Question: Evaluate the indefinite integral.

$$\int x^3\sqrt{x}\ dx$$

Answer choices:

A $\frac{1}{6}x^{\frac{11}{2}}+C$

B $\frac{7}{2}x^{\frac{5}{2}}+C$

C $\frac{3}{2}x^{\frac{3}{2}}+C$

D $\frac{2}{9}x^{\frac{9}{2}}+C$

Solution: D

In order to integrate, we must first rewrite by multiplying the two factors.

$$\int x^3\sqrt{x}\ dx$$

$$\int x^3x^{\frac{1}{2}}\ dx$$

$$\int x^{3+\frac{1}{2}}\ dx$$

$$\int x^{\frac{6}{2}+\frac{1}{2}}\ dx$$

$$\int x^{\frac{7}{2}}\ dx$$

$$\frac{1}{\frac{9}{2}}x^{\frac{7}{2}+\frac{2}{2}}+C$$

$$\frac{2}{9}x^{\frac{9}{2}}+C$$

Topic: Properties of integrals

Question: Use properties of integrals to simplify the integral as much as possible.

$$\int_0^2 6x^2 - 5x + 3 \ dx$$

Answer choices:

A $\quad 6\int_0^2 x^2 \ dx - 5\int_0^2 x \ dx + 3\int_0^2 dx$

B $\quad 6\int_0^2 x^2 \ dx - 5\int_2^4 x \ dx + 3\int_4^6 dx$

C $\quad \int_0^2 6x^2 \ dx + \int_0^2 5x \ dx + \int_0^2 3 \ dx$

D $\quad \int_0^2 6x^2 \ dx + \int_2^4 5x \ dx + \int_4^6 3 \ dx$

Solution: A

When our function is the sum or difference of two terms, we can separate those terms into different integrals.

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Separating the terms in our integral based on these rules, we get

$$\int_0^2 6x^2 - 5x + 3\ dx = \int_0^2 6x^2\ dx - \int_0^2 5x\ dx + \int_0^2 3\ dx$$

We also know that the a constant coefficient which is multiplied by the entire function inside the integral can be pulled out in front of the integral.

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

Pulling these coefficients out of our integral, we get

$$\int_0^2 6x^2 - 5x + 3\ dx = 6\int_0^2 x^2\ dx - 5\int_0^2 x\ dx + 3\int_0^2 dx$$

Topic: Properties of integrals

Question: Use properties of integrals to simplify the integral as much as possible.

$$\int_{-1}^{5} 9x^3 - 4x^2 - 7x + 18\ dx$$

Answer choices:

A $\quad 9\int_{-1}^{5} x^3\ dx + 4\int_{-1}^{5} x^2\ dx + 7\int_{-1}^{5} x\ dx + 18\int_{-1}^{5}\ dx$

B $\quad 9\int_{-1}^{5} x^3\ dx + 4\int_{1}^{5} x^2\ dx - 7\int_{1}^{5} x\ dx + 18\int_{-1}^{5}\ dx$

C $\quad 9\int_{-1}^{5} x^3\ dx - 4\int_{-1}^{5} x^2\ dx - 7\int_{-1}^{5} x\ dx + 18\int_{-1}^{5}\ dx$

D $\quad 9\int_{-1}^{5} x^3\ dx - 4\int_{1}^{5} x^2\ dx - 7\int_{1}^{5} x\ dx + 18\int_{-1}^{5}\ dx$

Solution: C

When our function is the sum or difference of two terms, we can separate those terms into different integrals.

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Separating the terms in our integral based on these rules, we get

$$\int_{-1}^5 9x^3 - 4x^2 - 7x + 18\ dx = \int_{-1}^5 9x^3\ dx - \int_{-1}^5 4x^2\ dx - \int_{-1}^5 7x\ dx + \int_{-1}^5 18\ dx$$

We also know that the a constant coefficient which is multiplied by the entire function inside the integral can be pulled out in front of the integral.

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

Pulling these coefficients out of our integral, we get

$$\int_{-1}^5 9x^3 - 4x^2 - 7x + 18\ dx = 9\int_{-1}^5 x^3\ dx - 4\int_{-1}^5 x^2\ dx - 7\int_{-1}^5 x\ dx + 18\int_{-1}^5 dx$$

Topic: Properties of integrals

Question: Use properties of integrals to simplify the integral as much as possible.

$$\int_0^{\pi} 8x \ln x - 3x^3 + 4\sin(2x)\ dx$$

Answer choices:

A $\quad 8\int_0^{\pi} x \ln x\ dx + 3\int_0^{\pi} x^3\ dx + 8\int_0^{\pi} \sin(x)\ dx$

B $\quad 8\int_0^{\pi} x\ dx + \int_0^{\pi} \ln x\ dx + 3\int_0^{\pi} x^3\ dx + 4\int_0^{\pi} \sin(2x)\ dx$

C $\quad 8\int_0^{\pi} x\ dx + \int_0^{\pi} \ln x\ dx - 3\int_0^{\pi} x^3\ dx + 4\int_0^{\pi} \sin(2x)\ dx$

D $\quad 8\int_0^{\pi} x \ln x\ dx - 3\int_0^{\pi} x^3\ dx + 4\int_0^{\pi} \sin(2x)\ dx$

Solution: D

When our function is the sum or difference of two terms, we can separate those terms into different integrals.

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

$$\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Separating the terms in our integral based on these rules, we get

$$\int_0^\pi 8x \ln x - 3x^3 + 4\sin(2x)\ dx = \int_0^\pi 8x \ln x\ dx - \int_0^\pi 3x^3\ dx + \int_0^\pi 4\sin(2x)\ dx$$

We also know that the a constant coefficient which is multiplied by the entire function inside the integral can be pulled out in front of the integral.

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

Pulling these coefficients out of our integral, we get

$$\int_0^\pi 8x \ln x - 3x^3 + 4\sin(2x)\ dx = 8\int_0^\pi x \ln x\ dx - 3\int_0^\pi x^3\ dx + 4\int_0^\pi \sin(2x)\ dx$$

Topic: Find f given f''

Question: Find $f(x)$ if $f''(x) = 6x - 4$.

Answer choices:

A $\quad f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + x^2 + x$

B $\quad f(x) = 3x^2 - 4x + C$

C $\quad f(x) = x^3 - 2x^2 + Cx + D$

D $\quad f(x) = \dfrac{x^3}{4} - \dfrac{2x^2}{3} + \dfrac{Cx}{D} + \dfrac{D}{C}$

Solution: C

The question asks us to find the function $f(x)$ if the second derivative of the function is $f''(x) = 6x - 4$.

Note that the question does not provide initial values of $f(x)$ or $f'(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The second derivative is a polynomial function. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is.

First we'll write the second derivative showing all exponents.

$$f''(x) = 6x - 4 = 6x^1 - 4x^0$$

$$f'(x) = \int 6x - 4 \ dx$$

$$f'(x) = \frac{6x^{1+1}}{2} - \frac{4x^{0+1}}{1} + C$$

Simplify each term to finish finding the first derivative.

$$f'(x) = 3x^2 - 4x + C$$

Now, find the function by repeating the process.

$$f'(x) = 3x^2 - 4x + C = 3x^2 - 4x^1 + Cx^0$$

Once again, we'll add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f(x) = \int 3x^2 - 4x^1 + Cx^0 \ dx$$

$$f(x) = \frac{3x^{2+1}}{3} - \frac{4x^{1+1}}{2} + \frac{Cx^{0+1}}{1} + D$$

After we simplify each term, the function is

$$f(x) = x^3 - 2x^2 + Cx + D$$

Topic: Find f given f''

Question: Find $f(x)$ if $f''(x) = 42x^5 - 40x^3 + 12x^2 - 12x + 2$.

Answer choices:

A $\quad f(x) = \frac{x^8}{7} - \frac{x^6}{3} + \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + \frac{x^2}{2} + x + C$

B $\quad f(x) = x^7 - 2x^5 + x^4 - 2x^3 + x^2 + 2x + 1$

C $\quad f(x) = 7x^6 - 10x^4 + 4x^3 - 6x^2 + 2x + C$

D $\quad f(x) = x^7 - 2x^5 + x^4 - 2x^3 + x^2 + Cx + D$

Solution: D

The question asks us to find the function $f(x)$ if the second derivative of the function is

$$f''(x) = 42x^5 - 40x^3 + 12x^2 - 12x + 2$$

Note that the question does not provide initial values of $f(x)$ or $f'(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The second derivative is a polynomial function. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is.

We will, first, write the second derivative showing all exponents.

$$f''(x) = 42x^5 - 40x^3 + 12x^2 - 12x + 2 = 42x^5 - 40x^3 + 12x^2 - 12x^1 + 2x^0$$

$$f'(x) = \int 42x^5 - 40x^3 + 12x^2 - 12x^1 + 2x^0 \ dx$$

$$f'(x) = \frac{42x^{5+1}}{6} - \frac{40x^{3+1}}{4} + \frac{12x^{2+1}}{3} - \frac{12x^{1+1}}{2} + \frac{2x^{0+1}}{1} + C$$

After we simplify each term, the first derivative function is

$$f'(x) = 7x^6 - 10x^4 + 4x^3 - 6x^2 + 2x + C$$

Now, find the function by repeating the process.

$$f'(x) = 7x^6 - 10x^4 + 4x^3 - 6x^2 + 2x + C$$

$$f'(x) = 7x^6 - 10x^4 + 4x^3 - 6x^2 + 2x^1 + Cx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f(x) = \int 7x^6 - 10x^4 + 4x^3 - 6x^2 + 2x^1 + Cx^0 \ dx$$

$$f(x) = \frac{7x^{6+1}}{7} - \frac{10x^{4+1}}{5} + \frac{4x^{3+1}}{4} - \frac{6x^{2+1}}{3} + \frac{2x^{1+1}}{2} + \frac{Cx^{0+1}}{1} + D$$

Now, simplify each term and the function is

$$f(x) = x^7 - 2x^5 + x^4 - 2x^3 + x^2 + Cx + D$$

Topic: Find f given f’’

Question: Find $f(x)$.

$$f''(x) = \frac{45\sqrt{x}}{16} - \frac{1}{2\sqrt{x}} - \frac{1}{20}x^{-\frac{3}{2}}$$

Answer choices:

A $\quad \frac{45}{16}x^{\frac{5}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{10}\sqrt{x}$

B $\quad \frac{3}{4}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{\sqrt{x}}{5} + Cx + D$

C $\quad \frac{3}{4}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{\sqrt{x}}{5} + x + 1$

D $\quad \frac{3}{4}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{\sqrt{x}}{5}$

Solution: B

The question asks us to find the function $f(x)$ if the second derivative of the function is

$$f''(x) = \frac{45\sqrt{x}}{16} - \frac{1}{2\sqrt{x}} - \frac{1}{20}x^{-\frac{3}{2}}$$

Note that the question does not provide initial values of $f(x)$ or $f'(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The second derivative is a function with radicals and rational exponents. In order to use the exponent rule when finding the anti-derivative, we will convert all terms to rational exponent terms. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions “could” contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled “c” to add the possibility of a constant term in the function, although we do not know what that constant is.

First we’ll write the second derivative, showing all exponents.

$$f''(x) = \frac{45\sqrt{x}}{16} - \frac{1}{2\sqrt{x}} - \frac{1}{20}x^{-\frac{3}{2}}$$

$$f''(x) = \frac{45}{16}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{20}x^{-\frac{3}{2}}$$

$$f'(x) = \int \frac{45}{16}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{20}x^{-\frac{3}{2}}\ dx$$

$$f'(x) = \frac{\frac{45}{16}x^{\frac{1}{2}+1}}{\frac{3}{2}} - \frac{\frac{1}{2}x^{-\frac{1}{2}+1}}{\frac{1}{2}} - \frac{\frac{1}{20}x^{-\frac{3}{2}+1}}{-\frac{1}{2}} + C$$

Since we are dividing fractions by fractions, let's change the expressions to multiplying by the reciprocal of the fraction in the denominator. We will also simplify the exponents of each term.

$$f'(x) = \frac{45}{16} \times \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2} \times \frac{2}{1}x^{\frac{1}{2}} - \frac{1}{20} \times -\frac{2}{1}x^{-\frac{1}{2}} + C$$

Simplify each term to finish finding the first derivative.

$$f'(x) = \frac{15}{8}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{1}{10}x^{-\frac{1}{2}} + C$$

Now, find the function by repeating the process.

$$f'(x) = \frac{15}{8}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{1}{10}x^{-\frac{1}{2}} + Cx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f(x) = \int \frac{15}{8}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{1}{10}x^{-\frac{1}{2}} + Cx^0 \ dx$$

$$f(x) = \frac{\frac{15}{8}x^{\frac{3}{2}+1}}{\frac{5}{2}} - \frac{x^{\frac{1}{2}+1}}{\frac{3}{2}} + \frac{\frac{1}{10}x^{-\frac{1}{2}+1}}{\frac{1}{2}} + \frac{Cx^{0+1}}{1} + D$$

Since we are dividing fractions by fractions again, let's change the expressions to multiplying by the reciprocal of the fraction in the denominator, as we did before. We will also simplify the exponents of each term.

$$f(x) = \frac{15}{8} \times \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{10} \times \frac{2}{1}x^{\frac{1}{2}} + Cx + D$$

After we simplify each term, the function is

$$f(x) = \frac{3}{4}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{\sqrt{x}}{5} + Cx + D$$

Topic: Find f given f’’’

Question: Find $f(x)$ if $f'''(x) = 24x$.

Answer choices:

A $\quad f(x) = x^4 + Cx^2 + Dx + E$

B $\quad f(x) = x^4$

C $\quad f(x) = x^3 - 2x^2 + Cx + D$

D $\quad f(x) = \frac{x^4}{4} - \frac{x^2}{2} + x + C$

Solution: A

The question asks us to find the function $f(x)$ if the third derivative of the function is $f'''(x) = 24x$.

Note that the question does not provide initial values of $f(x)$, $f'(x)$ or $f''(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same third derivative.

We are given the third derivative of the function. To find the second derivative of the function, find the anti-derivative of the third derivative. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The third derivative is a polynomial monomial function. To find the anti-derivative, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is.

First we'll write the third derivative showing the exponent.

$$f'''(x) = 24x = 24x^1$$

Then, we integrate to find the anti-derivative.

$$f''(x) = \int 24x^1 \ dx$$

$$f''(x) = \frac{24x^{1+1}}{2} + C$$

Simplify to finish finding the second derivative.

$$f''(x) = 12x^2 + C$$

Next, find the first derivative by repeating the process.

$$f''(x) = 12x^2 + Cx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f'(x) = \int 12x^2 + Cx^0 \ dx$$

$$f'(x) = \frac{12}{3}x^{2+1} + \frac{Cx^{0+1}}{1} + D$$

$$f'(x) = 4x^3 + Cx + D$$

Now, find the function by repeating the process again.

$$f'(x) = 4x^3 + Cx + D$$

$$f'(x) = 4x^3 + Cx^1 + Dx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constants so we will call it "E".

$$f(x) = \int 4x^3 + cx^1 + dx^0 \ dx$$

$$f(x) = \frac{4x^{3+1}}{4} + \frac{Cx^{1+1}}{2} + \frac{Dx^{0+1}}{1} + E$$

Since the letter "C" is an arbitrary constant, we can ignore its division by 2. After we simplify each term, the function is

$$f(x) = x^4 + Cx^2 + Dx + E$$

Topic: Find f given f'''

Question: Find $f(x)$ if $f'''(x) = 336x^5 - 120x^3 + 24x$.

Answer choices:

A $\quad f(x) = \frac{x^8}{7} - \frac{x^6}{3} + \frac{x^4}{2} + Cx^2 + Dx + E$

B $\quad f(x) = x^8 - x^6 + x^4$

C $\quad f(x) = x^8 - x^6 + x^4 + C$

D $\quad f(x) = x^8 - x^6 + x^4 + Cx^2 + Dx + E$

Solution: D

The question asks us to find the function $f(x)$ if the third derivative of the function is $f'''(x) = 336x^5 - 120x^3 + 24x$.

Note that the question does not provide initial values of $f(x)$, $f'(x)$, or $f''(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same third derivative.

We are given the third derivative of the function. To find the second derivative of the function, find the anti-derivative of the third derivative. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The third derivative is a polynomial function. To find the anti-derivative, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is.

We will, first, write the third derivative showing the exponent.

$$f'''(x) = 336x^5 - 120x^3 + 24x^1$$

Next, we integrate to find the anti-derivative.

$$f''(x) = \int 336x^5 - 120x^3 + 24x^1 \ dx$$

$$f''(x) = \frac{336x^{5+1}}{6} - \frac{120x^{3+1}}{4} + \frac{24x^{1+1}}{2} + C$$

Simplify to finish finding the second derivative.

$$f''(x) = 56x^6 - 30x^4 + 12x^2 + C$$

Next, find the first derivative by repeating the process.

$$f''(x) = 56x^6 - 30x^4 + 12x^2 + Cx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f'(x) = \int 56x^6 - 30x^4 + 12x^2 + Cx^0 \ dx$$

$$f'(x) = \frac{56x^{6+1}}{7} - \frac{30x^{4+1}}{5} + \frac{12x^{2+1}}{3} + \frac{Cx^{0+1}}{1} + D$$

$$f'(x) = 8x^7 - 6x^5 + 4x^3 + Cx + D$$

Now, find the function by repeating the process.

$$f'(x) = 8x^7 - 6x^5 + 4x^3 + Cx + D$$

$$f'(x) = 8x^7 - 6x^5 + 4x^3 + Cx^1 + Dx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constants so we will call it “E”.

$$f(x) = \int 8x^7 - 6x^5 + 4x^3 + Cx^1 + Dx^0 \ dx$$

$$f(x) = \frac{8x^{7+1}}{8} - \frac{6x^{5+1}}{6} + \frac{4x^{3+1}}{4} + \frac{Cx^{1+1}}{2} + \frac{Dx^{0+1}}{1} + E$$

Since the letter “C” is an arbitrary constant, we can ignore its division by 2. After we simplify each term, the function is

$$f(x) = x^8 - x^6 + x^4 + Cx^2 + Dx + E$$

Topic: Find f given f'''

Question: Find $f(x)$.

$$f'''(x) = -\frac{2}{9}x^{-\frac{4}{3}} - \frac{5}{16}x^{-\frac{9}{4}}$$

Answer choices:

A $\frac{3}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{-\frac{3}{4}}$

B $x^{\frac{5}{3}} - x^{-\frac{3}{4}} + Cx^2 + Dx + E$

C $\frac{3}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{\frac{3}{4}} + Cx^2 + Dx + E$

D $\frac{3}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{-\frac{3}{4}} + C$

Solution: C

The question asks us to find the function $f(x)$ if the third derivative of the function is

$$f'''(x) = -\frac{2}{9}x^{-\frac{4}{3}} - \frac{5}{16}x^{-\frac{9}{4}}$$

Note that the question does not provide initial values of $f(x)$, $f'(x)$, or $f''(x)$ so our answer will be a family of possible $f(x)$ functions that could have the same third derivative.

We are given the third derivative of the function. To find the second derivative of the function, find the anti-derivative of the third derivative. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative.

The third derivative is a function with rational exponents. To find the anti-derivative, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is.

First, we integrate to find the anti-derivative.

$$f''(x) = \int -\frac{2}{9}x^{-\frac{4}{3}} - \frac{5}{16}x^{-\frac{9}{4}}\ dx$$

$$f''(x) = -\left(\frac{2}{9}\right)\frac{x^{-\frac{4}{3}+1}}{-\frac{1}{3}} - \left(\frac{5}{16}\right)\frac{x^{-\frac{9}{4}+1}}{-\frac{5}{4}} + C$$

Since we are dividing fractions by fractions, let's multiply the fraction in the numerator by the reciprocal of the fraction in the denominator and simplify the exponents.

$$f''(x) = -\left(\frac{2}{9}\right)\left(-\frac{3}{1}\right)x^{-\frac{1}{3}} - \left(\frac{5}{16}\right)\left(-\frac{4}{5}\right)x^{-\frac{5}{4}} + C$$

Simplify to finish finding the second derivative.

$$f''(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{4}x^{-\frac{5}{4}} + C$$

Next, find the first derivative by repeating the process.

$$f''(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{4}x^{-\frac{5}{4}} + Cx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D".

$$f'(x) = \int \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{4}x^{-\frac{5}{4}} + Cx^0\ dx$$

$$f'(x) = \left(\frac{2}{3}\right)\frac{x^{-\frac{1}{3}+1}}{\frac{2}{3}} + \left(\frac{1}{4}\right)\frac{x^{-\frac{5}{4}+1}}{-\frac{1}{4}} + \frac{Cx^{0+1}}{1} + D$$

Once again, we are dividing fractions by fractions, so let's multiply the fraction in the numerator by the reciprocal of the fraction in the denominator and simplify the exponents.

$$f'(x) = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{\frac{2}{3}} + \left(\frac{1}{4}\right)\left(-\frac{4}{1}\right)x^{-\frac{1}{4}} + Cx + D$$

$$f'(x) = x^{\frac{2}{3}} - x^{-\frac{1}{4}} + Cx + D$$

Now, find the function by repeating the process again.

$$f'(x) = x^{\frac{2}{3}} - x^{-\frac{1}{4}} + Cx + D$$

$$f'(x) = x^{\frac{2}{3}} - x^{-\frac{1}{4}} + Cx^1 + Dx^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constants so we will call it "E".

$$f(x) = \int x^{\frac{2}{3}} - x^{-\frac{1}{4}} + Cx^1 + Dx^0 \ dx$$

$$f(x) = \frac{x^{\frac{2}{3}+1}}{\frac{5}{3}} - \frac{x^{-\frac{1}{4}+1}}{\frac{3}{4}} + \frac{Cx^{1+1}}{2} + \frac{Dx^{0+1}}{1} + E$$

We are, once again, dividing fractions by fractions, so let's multiply by the reciprocal of the denominator again and simplify the exponents.

$$f(x) = \frac{3}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{\frac{3}{4}} + Cx^2 + Dx + E$$

Since the letter "C" is an arbitrary constant, we can ignore its division by 2. After we simplify each term, the function is

$$f(x) = \frac{3}{5}x^{\frac{5}{3}} - \frac{4}{3}x^{\frac{3}{4}} + Cx^2 + Dx + E$$

Topic: Initial value problems

Question: Solve the initial value problem.

$$\frac{dy}{dx} = 2x + 3$$

$y = 5$ when $x = 0$

Answer choices:

A $y = x^2 + 3x + 5$

B $y = 5$

C $y = 4x^2 + 3x + 5$

D $y = x^2 + 3x - 40$

Solution: A

In order to find y, we multiply both sides of the equation by dx and then integrate both sides.

$$dy = (2x + 3)\ dx$$

$$\int dy = \int 2x + 3\ dx$$

$$y = x^2 + 3x + C$$

Now, in order to find the specific equation that passes through $y = 5$ when $x = 0$, we substitute these values into the general equation we found and solve for C.

$$5 = 0^2 + 3(0) + C$$

$$5 = C$$

Therefore, the specific equation we are looking for it

$$y = x^2 + 3x + 5$$

Topic: Initial value problems

Question: Solve the initial value problem.

$$f''(x) = \cos x$$

$f'(0) = 1$ and $f(0) = 3$

Answer choices:

A $\quad f(x) = \sin x + 1$

B $\quad f(x) = -\cos x + x + 4$

C $\quad f(x) = -\sin x + 1$

D $\quad f(x) = \cos x + x + 2$

Solution: B

Before we can find the equation for $f(x)$, we must first find the equation for $f'(x)$, which we do by integrating $f''(x)$.

$$f'(x) = \int \cos x \ dx$$

$$f'(x) = \sin x + C$$

Now we find the specific equation for $f'(x)$ by solving for C with the initial condition given.

$$f'(0) = \sin 0 + C = 1$$

$$C = 1$$

$$f'(x) = \sin x + 1$$

In order to find $f(x)$, we integrate $f'(x)$ and find C by using the initial condition for $f(x)$.

$$f(x) = \int (\sin x + 1) \ dx$$

$$f(x) = - \cos x + x + C$$

$$f(0) = - \cos 0 + 0 + C = 3$$

$$-1 + C = 3$$

$$C = 4$$

Therefore,

$$f(x) = -\cos x + x + 4$$

Topic: Initial value problems

Question: Solve the initial value problem.

$$\frac{dy}{dx} = 11x^2 - 5x + 6$$

$$y(0) = 7$$

Answer choices:

A $\quad y = \frac{11}{3}x^3 - \frac{5}{2}x^2 + 6x$

B $\quad y = \frac{11}{3}x^3 - \frac{5}{2}x^2 + 6x + C$

C $\quad y = x^3 - x^2 + 6x + 7$

D $\quad y = \frac{11}{3}x^3 - \frac{5}{2}x^2 + 6x + 7$

Solution: D

The question asks us to solve the initial value problem.

$$\frac{dy}{dx} = 11x^2 - 5x + 6$$

$$y(0) = 7$$

In an initial value problem, you're given two things; a differential equation, and a function value at a specific input value. We know that the given equation is a differential equation because it begins with dy/dx, which is the notation for the first derivative of a function with respect to x.

To solve a differential equation, we separate the variables and integrate. The result of the integration gives us a general function because the function "could" contain a constant term, which becomes zero when we differentiate the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is. When we use the initial condition, we will find the specific value of "C". The initial value enables us to find the value of "C".

First, we'll rewrite the differential equation, separating the variables, and then integrate.

$$\frac{dy}{dx} = 11x^2 - 5x + 6$$

$$dy = 11x^2 - 5x + 6\ dx$$

$$\int dy = \int 11x^2 - 5x + 6\ dx$$

Since the integrand is a polynomial, we can change its terms so each term has an exponent. Then we'll perform the integration using the exponent rule.

$$\int y^0\ dy = \int 11x^2 - 5x^1 + 6x^0\ dx$$

$$y = \frac{11}{3}x^3 - \frac{5}{2}x^2 + 6x + C$$

Now we use the initial value $y(0) = 7$ to find "C".

$$7 = \frac{11}{3}(0)^3 - \frac{5}{2}(0)^2 + 6(0) + C$$

We can see that $C = 7$. Replace the "C" with 7. The answer to the initial value problem is

$$y = \frac{11}{3}x^3 - \frac{5}{2}x^2 + 6x + 7$$

Topic: Find f given f'' and initial conditions

Question: Find $f(x)$.

$$f''(x) = 18x + 8$$

$$f'(2) = 46 \text{ and } f(0) = 8$$

Answer choices:

A $f(x) = x^3 + x^2 - x + 8$

B $f(x) = 3x^3 + 4x^2 - 6x + 8$

C $f(x) = 3x^3 + 4x^2 - 6x - 8$

D $f(x) = \frac{x^3}{3} + \frac{x^2}{4} - \frac{x}{6} + 8$

Solution: B

The question asks us to find the function $f(x)$ if the second derivative of the function is $f''(x) = 18x + 8$, $f'(2) = 46$, and $f(0) = 8$.

Note that the question provides initial values of $f(x)$ and $f'(x)$ so our answer will be a specific $f(x)$ function with the given second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative. Once we find the general first derivative and the general function, we will use the initial conditions to find the specific function.

The second derivative is a polynomial function. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is. When we use the initial condition, we will find the specific value of "C".

We will, first, write the second derivative showing all exponents.

$$f''(x) = 18x + 8 = 18x^1 + 8x^0$$

$$f'(x) = \int 18x^1 + 8x^0 \ dx$$

$$f'(x) = \frac{18x^{1+1}}{2} + \frac{8x^{0+1}}{1} + C$$

Simplify each term to finish finding the general first derivative.

$$f'(x) = 9x^2 + 8x + C$$

The question states that $f'(2) = 46$, so to find "C" let's make the derivative equal to 46 when $x = 2$.

$$46 = 9(2)^2 + 8(2) + C$$

$$46 = 36 + 16 + C$$

$$46 = 52 + C$$

$$-6 = C$$

Therefore,

$$f'(x) = 9x^2 + 8x - 6$$

Now, find the function by repeating the process.

$$f'(x) = 9x^2 + 8x - 6 = 9x^2 + 8x^1 - 6x^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D", but we will find its value using the initial condition.

$$f(x) = \int 9x^2 + 8x^1 - 6x^0 \ dx$$

$$f(x) = \frac{9x^{2+1}}{3} + \frac{8x^{1+1}}{2} - \frac{6x^{0+1}}{1} + D$$

After we simplify each term, the general function is

$$f(x) = 3x^3 + 4x^2 - 6x + D$$

The question further states that $f(0) = 8$, so to find "D" let's make the function equal to 8 when $x = 0$.

$$8 = 3(0)^3 + 4(0)^2 - 6(0) + D$$

$$D = 8$$

The specific function in this problem is

$$f(x) = 3x^3 + 4x^2 - 6x + 8$$

Topic: Find f given f'' and initial conditions

Question: Find $f(x)$.

$$f''(x) = 60x^2 - 36x + 6$$

$$f'(1) = 0 \text{ and } f(1) = 8$$

Answer choices:

A $\quad f(x) = 5x^4 - 6x^3 + 3x^2 + Cx + D$

B $\quad f(x) = \frac{x^4}{5} - \frac{x^3}{6} + \frac{x^2}{3} - \frac{x}{8} + 14$

C $\quad f(x) = x^4 - x^3 + x^2 - x + 8$

D $\quad f(x) = 5x^4 - 6x^3 + 3x^2 - 8x + 14$

Solution: D

The question asks us to find the function $f(x)$ if the second derivative of the function is $f''(x) = 60x^2 - 36x + 6$, and if $f'(1) = 0$ and $f(1) = 8$.

Note that the question provides initial values of $f(x)$ and $f'(x)$ so our answer will be a specific $f(x)$ function with the given second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative. Once we find the general first derivative and the general function, we will use the initial conditions to find the specific function.

The second derivative is a polynomial function. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes zero when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is. When we use the initial condition, we will find the specific value of "C".

We will, first, write the second derivative showing all exponents.

$$f''(x) = 60x^2 - 36x + 6 = 60x^2 - 36x^1 + 6x^0$$

$$f'(x) = \int 60x^2 - 36x^1 + 6x^0 \ dx$$

$$f'(x) = \frac{60x^{2+1}}{3} - \frac{36x^{1+1}}{2} + \frac{6x^{0+1}}{1} + C$$

Simplify each term to finish finding the general first derivative.

$$f'(x) = 20x^3 - 18x^2 + 6x + C$$

The question states that $f'(1) = 0$, so to find "C" let's make the derivative equal to 0 when $x = 1$.

$$0 = 20(1)^3 - 18(1)^2 + 6(1) + C$$

$$0 = 20 - 18 + 6 + C$$

$$0 = 8 + C$$

$$C = -8$$

So

$$f'(x) = 20x^3 - 18x^2 + 6x - 8$$

Now find the function by repeating the process.

$$f'(x) = 20x^3 - 18x^2 + 6x - 8$$

$$f'(x) = 20x^3 - 18x^2 + 6x^1 - 8x^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became zero when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D", but we will find its value using the initial condition.

$$f(x) = \int 20x^3 - 18x^2 + 6x^1 - 8x^0\ dx$$

$$f(x) = \frac{20x^{3+1}}{4} - \frac{18x^{2+1}}{3} + \frac{6x^{1+1}}{2} - \frac{8x^{0+1}}{1} + D$$

After we simplify each term, the general function is

$$f(x) = 5x^4 - 6x^3 + 3x^2 - 8x + D$$

The question further states that $f(1) = 8$, so to find "D" let's make the function equal to 8 when $x = 1$.

$$8 = 5(1)^4 - 6(1)^3 + 3(1)^2 - 8\,(1) + D$$

$$8 = 5 - 6 + 3 - 8 + D$$

$$8 = -6 + D$$

$$D = 14$$

The specific function in this problem is

$$f(x) = 5x^4 - 6x^3 + 3x^2 - 8x + 14$$

Topic: Find f given f'' and initial conditions

Question: Find $f(x)$.

$$f''(x) = \frac{35}{4}x^{\frac{3}{2}} + \frac{15}{2}x^{\frac{1}{2}} - \frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$f'(1) = 12 \text{ and } f(4) = 189$$

Answer choices:

A $\quad f(x) = x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 4x - 15$

B $\quad f(x) = x^{\frac{7}{2}} + x^{\frac{5}{2}} - x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$

C $\quad f(x) = x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

D $\quad f(x) = x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + Cx + D$

Solution: A

The question asks us to find the function $f(x)$ if the second derivative of the function is

$$f''(x)=\frac{35}{4}x^{\frac{3}{2}}+\frac{15}{2}x^{\frac{1}{2}}-\frac{3}{4}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$

and if $f'(1)=12$ and $f(4)=189$.

Note that the question provides initial values of $f(x)$ and $f'(x)$, so our answer will be a specific $f(x)$ function with the given second derivative.

We are given the second derivative of the function. To find the first derivative of the function, find the anti-derivative of the second derivative. Then, to find the function, we repeat the process by finding the anti-derivative of the first derivative. Once we find the general first derivative and the general function, we will use the initial conditions to find the specific function.

The second derivative is a polynomial function. To find the anti-derivative, in each term, add 1 to the exponent and divide the term by the new exponent.

Additionally, all functions "could" contain a constant term, which becomes 0 when we take the derivative of the function. Thus, when we find the anti-derivative, we add a constant labeled "C" to add the possibility of a constant term in the function, although we do not know what that constant is. When we use the initial condition, we will find the specific value of "C".

First we'll integrate the second derivative to find the first derivative.

$$f'(x) = \int \frac{35}{4}x^{\frac{3}{2}} + \frac{15}{2}x^{\frac{1}{2}} - \frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\ dx$$

$$f'(x) = \frac{\frac{35}{4}x^{\frac{3}{2}+1}}{\frac{5}{2}} + \frac{\frac{15}{2}x^{\frac{1}{2}+1}}{\frac{3}{2}} - \frac{\frac{3}{4}x^{-\frac{1}{2}+1}}{\frac{1}{2}} - \frac{\frac{1}{2}x^{-\frac{3}{2}+1}}{-\frac{1}{2}} + C$$

Since we're dividing each term by a fraction, change each term to multiplying by the reciprocal of the fraction in the denominator.

$$f'(x) = \left(\frac{35}{4}\right)\left(\frac{2}{5}\right)x^{\frac{5}{2}} + \left(\frac{15}{2}\right)\left(\frac{2}{3}\right)x^{\frac{3}{2}} - \left(\frac{3}{4}\right)\left(\frac{2}{1}\right)x^{\frac{1}{2}} - \left(\frac{1}{2}\right)\left(-\frac{2}{1}\right)x^{-\frac{1}{2}} + C$$

Simplify each term to finish finding the general first derivative.

$$f'(x) = \frac{7}{2}x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + C$$

The question states that $f'(1) = 12$, so to find "C" let's make the derivative equal to 12 when $x = 1$.

$$12 = \frac{7}{2}(1)^{\frac{5}{2}} + 5(1)^{\frac{3}{2}} - \frac{3}{2}(1)^{\frac{1}{2}} + (1)^{-\frac{1}{2}} + C$$

$$12 = \frac{7}{2} + 5 - \frac{3}{2} + (1) + C$$

$$12 = 2 + 5 + 1 + C$$

$$12 = 8 + C$$

$$C = 4$$

So

$$f'(x) = \frac{7}{2}x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 4$$

Now find the function by repeating the process.

$$f'(x) = \frac{7}{2}x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 4x^0$$

Once again, we will add a constant to cover the likely event that the original function had a constant term that became 0 when the first derivative was taken. We do not know that the new constant is the same as the old constant so we will call it "D", but we will find its value using the initial condition.

$$f(x) = \int \frac{7}{2}x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 4x^0 \ dx$$

$$f(x) = \frac{\frac{7}{2}x^{\frac{5}{2}+1}}{\frac{7}{2}} + \frac{5x^{\frac{3}{2}+1}}{\frac{5}{2}} - \frac{\frac{3}{2}x^{\frac{1}{2}+1}}{\frac{3}{2}} + \frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}} + \frac{4x^{0+1}}{1} + D$$

Again, since we are dividing each term by a fraction, change each term to multiplying by the reciprocal of the fraction in the denominator.

$$f(x) = \left(\frac{7}{2}\right)\left(\frac{2}{7}\right)x^{\frac{7}{2}} + (5)\left(\frac{2}{5}\right)x^{\frac{5}{2}} - \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{\frac{3}{2}} + (1)\left(\frac{2}{1}\right)x^{\frac{1}{2}} + 4x + D$$

$$f(x) = x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 4x + D$$

The question further states that $f(4) = 189$, so to find "D" let's make the function equal to 189 when $x = 4$.

$$189 = (4)^{\frac{7}{2}} + 2(4)^{\frac{5}{2}} - (4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}} + 4(4) + D$$

$$189 = 128 + 64 - 8 + 4 + 16 + D$$

$$189 = 204 + D$$

$$D = -15$$

The specific function in this problem is

$$f(x) = x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 4x - 15$$

Topic: Definite integrals

Question: Evaluate the definite integral.

$$\int_{2}^{4} \sqrt{\frac{3}{x}}\, dx$$

Answer choices:

A $\quad 2.03$

B $\quad 0.20$

C $\quad 0.79$

D $\quad 3.51$

Solution: A

First, we rewrite the integrand, and then we integrate.

$$\int_2^4 \sqrt{\frac{3}{x}}\ dx$$

$$\int_2^4 \sqrt{3}x^{-\frac{1}{2}}\ dx$$

$$\frac{\sqrt{3}x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_2^4$$

$$2\sqrt{3}\sqrt{4} - 2\sqrt{3}\sqrt{2} \approx 2.03$$

Topic: Definite integrals

Question: Evaluate the definite integral.

Given $\int_0^5 f(x)\ dx = 2$ and $\int_5^8 f(x)\ dx = -4$, find $\int_0^8 f(x)\ dx$

Answer choices:

A 6

B -6

C 2

D -2

Solution: D

$$\int_0^8 f(x)\ dx = \int_0^5 f(x)\ dx + \int_5^8 f(x)\ dx$$

$$\int_0^8 f(x)\ dx = 2 + (-4)$$

$$\int_0^8 f(x)\ dx = -2$$

Topic: Definite integrals

Question: Evaluate the definite integral.

$$\int_1^3 x^2 - 7x + 5\ dx$$

Answer choices:

A $-\frac{26}{3}$

B $-\frac{28}{3}$

C $-\frac{4}{7}$

D $-\frac{2}{7}$

Solution: B

Integrate one term at a time.

$$\frac{1}{2+1}x^{2+1}-\frac{7}{1+1}x^{1+1}+5x\Big|_1^3$$

$$\frac{1}{3}x^3-\frac{7}{2}x^2+5x\Big|_1^3$$

Evaluate the antiderivative at the upper limit, then subtract the value of the antiderivative at the lower limit.

$$\frac{1}{3}(3)^3-\frac{7}{2}(3)^2+5(3)-\left(\frac{1}{3}(1)^3-\frac{7}{2}(1)^2+5(1)\right)$$

$$9-\frac{63}{2}+15-\frac{1}{3}+\frac{7}{2}-5$$

$$19-\frac{56}{2}-\frac{1}{3}$$

$$19-28-\frac{1}{3}$$

$$-9-\frac{1}{3}$$

$$-\frac{27}{3}-\frac{1}{3}$$

$$-\frac{28}{3}$$

Topic: Area under or enclosed by the curve

Question: Find the area under the graph of the function over the given interval.

$$\int_{-2}^{2} x^2 + 3\ dx$$

Answer choices:

A $-\frac{26}{3}$

B $-\frac{52}{3}$

C $\frac{26}{3}$

D $\frac{52}{3}$

Solution: D

Sometimes in calculus we need to distinguish between the area *under* a graph between the area *enclosed* by a graph.

When we talk about the area under the graph, we're talking about net area, which means we treat the area above the x-axis as positive and the area below the x-axis as negative. When we add the two together, the negative area will reduce the positive area. If our result is positive, we know we have more area above the x-axis than below it. On the other hand, if our result if negative, we know we have more area below the x-axis than above it.

In contrast, when we talk about the area enclosed by the graph, we're talking about gross area, which means we treat the area above and below the x-axis as positive. Because of this, any area below the x-axis just increases the total area, and our result will always be positive.

In this particular problem we're asked to find the area under the graph, which just means that we can take the integral of the function over the interval.

$$\int_{-2}^{2} x^2 + 3 \ dx$$

$$\left(\frac{x^3}{3} + 3x\right)\Bigg|_{-2}^{2}$$

$$\left[\frac{(2)^3}{3} + 3(2)\right] - \left[\frac{(-2)^3}{3} + 3(-2)\right]$$

$$\frac{26}{3} - \left(\frac{-26}{3}\right)$$

$$\frac{52}{3}$$

Topic: Area under or enclosed by the curve

Question: Find the area enclosed by the graph of the function over the given interval.

$$\int_{-1}^{1} x^3 \, dx$$

Answer choices:

A $\frac{1}{2}$

B $\frac{1}{4}$

C $\frac{3}{4}$

D 0

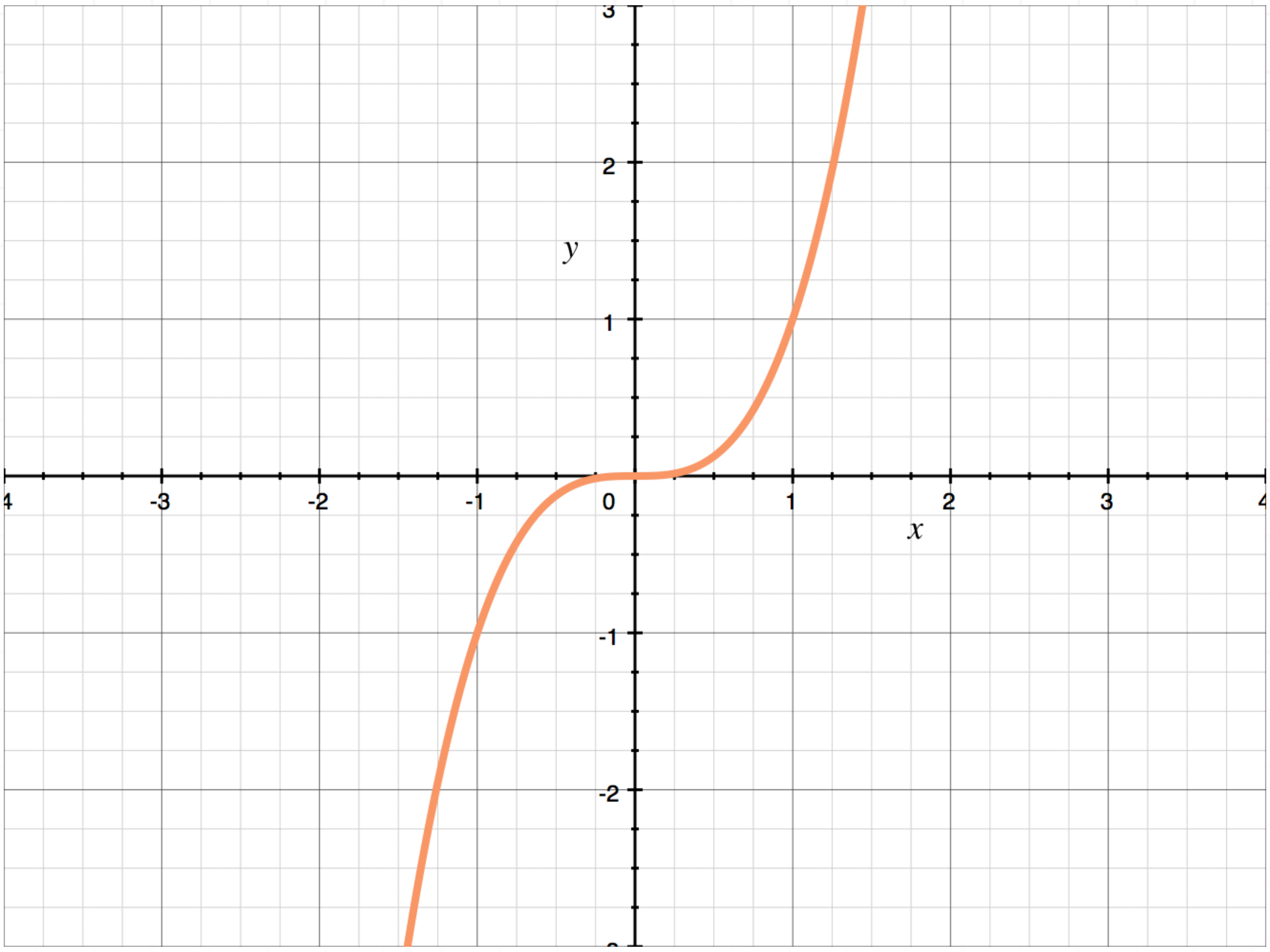
Solution: A

Sometimes in calculus we need to distinguish between the area *under* a graph between the area *enclosed* by a graph.

When we talk about the area under the graph, we're talking about net area, which means we treat the area above the x-axis as positive and the area below the x-axis as negative. When we add the two together, the negative area will reduce the positive area. If our result is positive, we know we have more area above the x-axis than below it. On the other hand, if our result if negative, we know we have more area below the x-axis than above it.

In contrast, when we talk about the area enclosed by the graph, we're talking about gross area, which means we treat the area above and below the x-axis as positive. Because of this, any area below the x-axis just increases the total area, and our result will always be positive.

If we graph the given function,



we can see that from -1 to 0 the function is below the x-axis and from 0 to 1 the function is above the x-axis. We'll break the integral into these two sections.

The area of the first section is

$$\int_{-1}^{0} x^3\ dx = \frac{x^4}{4}\Bigg|_{-1}^{0}$$

$$\int_{-1}^{0} x^3\ dx = \frac{(0)^4}{4} - \frac{(-1)^4}{4}$$

$$\int_{-1}^{0} x^3\ dx = -\frac{1}{4}$$

The area of the second section is

$$\int_0^1 x^3 \, dx = \frac{x^4}{4} \Big|_0^1$$

$$\int_0^1 x^3 \, dx = \frac{(1)^4}{4} - \frac{(0)^4}{4}$$

$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

Because we're looking for area enclosed by the graph (gross area), we take the absolute value of each area, and then sum the areas together.

$$A = \left| -\frac{1}{4} \right| + \left| \frac{1}{4} \right|$$

$$A = \frac{2}{4}$$

$$A = \frac{1}{2}$$

Topic: Area under or enclosed by the curve

Question: Find the area enclosed by the graph of the function over the given interval.

$$\int_{-\pi}^{2\pi} \sin x \ dx$$

Answer choices:

A $\quad -6$

B $\quad 4$

C $\quad 6$

D $\quad -4$

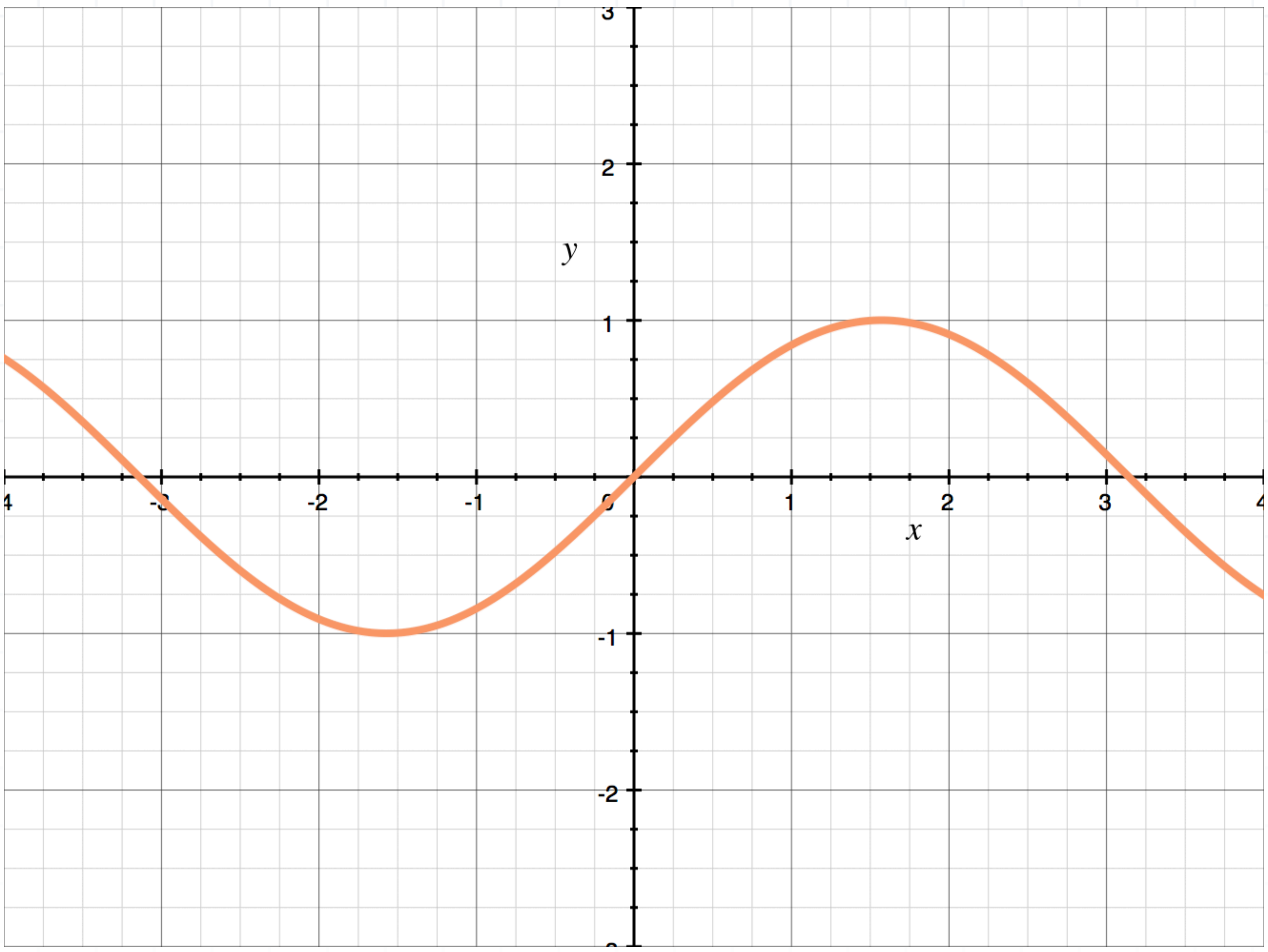
Solution: C

Sometimes in calculus we need to distinguish between the area *under* a graph between the area *enclosed* by a graph.

When we talk about the area under the graph, we're talking about net area, which means we treat the area above the x-axis as positive and the area below the x-axis as negative. When we add the two together, the negative area will reduce the positive area. If our result is positive, we know we have more area above the x-axis than below it. On the other hand, if our result if negative, we know we have more area below the x-axis than above it.

In contrast, when we talk about the area enclosed by the graph, we're talking about gross area, which means we treat the area above and below the x-axis as positive. Because of this, any area below the x-axis just increases the total area, and our result will always be positive.

If we graph the given function,



we can see that from $-\pi$ to 0 the function is below the x-axis and from 0 to π the function is above the x-axis. From π to 2π, the function is below the x-axis again. We'll break the integral into these three sections.

The area of the first section is

$$\int_{-\pi}^{0} \sin x \ dx$$

$$-\cos x \Big|_{-\pi}^{0}$$

$$-\cos 0 - \left[-\cos(-\pi)\right]$$

$$-2$$

The area of the second section is

$$\int_0^{\pi} \sin x \ dx$$

$$-\cos x \Big|_0^{\pi}$$

$$-\cos \pi - (-\cos 0)$$

$$2$$

The area of the third section is

$$\int_{\pi}^{2\pi} \sin x \ dx$$

$$-\cos x \Big|_{\pi}^{2\pi}$$

$$-\cos(2\pi) - (-\cos \pi)$$

$$-2$$

Because we're looking for area enclosed by the graph (gross area), we take the absolute value of each area, and then sum the areas together.

$$A = |-2| + |2| + |-2|$$

$$A = 6$$

Topic: Definite integrals of even and odd functions

Question: If this is the integral of an even function, rewrite the integral.

$$\int_{-4}^{4} x^4 - 2x^2 \, dx$$

Answer choices:

A The function isn't even or can't be rewritten.

B The function is even and can be rewritten as $\int_{0}^{4} x^4 - 2x^2 \, dx$

C The function is even and can be rewritten as $2\int_{0}^{4} x^4 - 2x^2 \, dx$

D The function is even and can be rewritten as $2\int_{-2}^{2} x^4 - 2x^2 \, dx$

Solution: C

In order for us to be able to rewrite the integral, we need to know that the area under the function to the left of the y-axis is equal to the area under the function to the right of the y-axis. We can say that these two areas are equal if we can show two things:

1. That the function is even, which means it's symmetrical about the y-axis.

2. That the limits of integration are symmetrical about the y-axis.

We can use simple algebra to determine whether or not the function is even. The way we do this is by substituting $-x$ for x in our original function. If we simplify and the result is equal to our original function, then we know that the function is even.

$$f(x) = x^4 - 2x^2$$

$$f(-x) = (-x)^4 - 2(-x)^2$$

$$f(-x) = x^4 - 2x^2$$

$$f(x) = f(-x)$$

Since we've shown that $f(x) = f(-x)$, we know that the function is even. We can also easily see that the limits of integration are symmetrical about the y-axis, because the interval is $[-4,4]$, which is in the form $[-a, a]$.

With these two requirements satisfied, we can rewrite the integral, changing the limits of integration from $[-a, a]$ to $[0,a]$ and multiply the integral by 2. So we get

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

$$\int_{-4}^{4} x^4 - 2x^2\ dx = 2\int_{0}^{4} x^4 - 2x^2\ dx$$

Topic: Definite integrals of even and odd functions

Question: If this is the integral of an even function, rewrite the integral.

$$\int_0^3 x^2 + 18\ dx$$

Answer choices:

A The function is even and can be rewritten as $\int_0^3 x^2 + 18\ dx$

B The function isn't even or can't be rewritten.

C The function is even and can be rewritten as $2\int_0^{\frac{1}{2}} x^2 + 18\ dx$

D The function is even and can be rewritten as $2\int_{-3}^{0} x^2 + 18\ dx$

Solution: B

In order for us to be able to rewrite the integral, we need to know that the area under the function to the left of the y-axis is equal to the area under the function to the right of the y-axis. We can say that these two areas are equal if we can show two things:

1. That the function is even, which means it's symmetrical about the y-axis.

2. That the limits of integration are symmetrical about the y-axis.

We can use simple algebra to determine whether or not the function is even. The way we do this is by substituting $-x$ for x in our original function. If we simplify and the result is equal to our original function, then we know that the function is even.

$$f(x) = x^2 + 18$$

$$f(-x) = (-x)^2 + 18$$

$$f(-x) = x^2 + 18$$

$$f(x) = f(-x)$$

Since we've shown that $f(x) = f(-x)$, we know that the function is even. However, the limits of integration are $[0,3]$. Since that doesn't match the form $[-a, a]$, we know that the limits of integration are not symmetrical about the y-axis.

So even though the function is even, we can't rewrite the integral.

Topic: Definite integrals of even and odd functions

Question: Definite integrals of odd functions evaluated on the interval $[-a, a]$...

Answer choices:

A ... will have different values depending on the function.

B ... will always equal 0.

C ... will never exist.

D ... will always equal ∞.

Solution: B

Odd functions are symmetric about the origin. If a function is symmetric about the origin, it means that any area above the x-axis in the first quadrant will be reflected below the x-axis in the third quadrant. Or that any area above the x-axis in the second quadrant will be reflected below the x-axis in the fourth quadrant.

Therefore, if we take the integral of an odd function on the interval $[-a, a]$, it means that the area above the x-axis will be equal to the area below the x-axis, and therefore that the value of the integral will always be 0.

If the interval is anything other than $[-a, a]$, we know that value of the integral will be non-zero.

Topic: Summation notation, finding the sum

Question: Calculate the exact sum.

$$\sum_{n=1}^{5} \frac{n^2}{2^n}$$

Answer choices:

A $\quad 4.4$

B $\quad \dfrac{141}{32}$

C $\quad 27.5$

D $\quad \dfrac{57}{32}$

Solution: B

The question asks you to find

$$\sum_{n=1}^{5} \frac{n^2}{2^n}$$

To find the sum, write each term with the value of n for that term. Since the summation shows that n begins with a value of 1 and ends with a value of 5, there are five terms.

$$n = 1 \qquad \frac{1^2}{2^1} = \frac{1}{2}$$

$$n = 2 \qquad \frac{2^2}{2^2} = \frac{4}{4} = 1$$

$$n = 3 \qquad \frac{3^2}{2^3} = \frac{9}{8}$$

$$n = 4 \qquad \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$n = 5 \qquad \frac{5^2}{2^5} = \frac{25}{32}$$

Add all the terms together.

$$\frac{1}{2} + 1 + \frac{9}{8} + 1 + \frac{25}{32}$$

Rewrite the terms with a least common denominator of 32 and combine the five terms.

$$\frac{16}{32}+\frac{32}{32}+\frac{36}{32}+\frac{32}{32}+\frac{25}{32}=\frac{141}{32}$$

Topic: Summation notation, finding the sum

Question: Calculate the exact sum.

$$\sum_{n=1}^{4} \frac{3n}{2n+1}$$

Answer choices:

A $\frac{1,788}{910}$

B $\frac{2,081}{910}$

C $\frac{482}{105}$

D $\frac{506}{105}$

Solution: D

The question asks you to find

$$\sum_{n=1}^{4} \frac{3n}{2n+1}$$

To find the sum, write each term with the value of n for that term. Since the summation shows that n begins with a value of 1 and ends with 4, there are four terms.

$$n = 1 \qquad \frac{3 \times 1}{2 \times 1 + 1} = \frac{3}{3} = 1$$

$$n = 2 \qquad \frac{3 \times 2}{2 \times 2 + 1} = \frac{6}{5}$$

$$n = 3 \qquad \frac{3 \times 3}{2 \times 3 + 1} = \frac{9}{7}$$

$$n = 4 \qquad \frac{3 \times 4}{2 \times 4 + 1} = \frac{12}{9} = \frac{4}{3}$$

Add the terms.

$$1 + \frac{6}{5} + \frac{9}{7} + \frac{4}{3}$$

Rewrite the terms with a least common denominator of 105 and combine the four terms.

$$\frac{105}{105} + \frac{126}{105} + \frac{135}{105} + \frac{140}{105} = \frac{506}{105}$$

Topic: Summation notation, finding the sum

Question: Calculate the exact sum.

$$\sum_{n=0}^{6} (-1)^n \frac{5}{n+2}$$

Answer choices:

A $\frac{481}{56}$

B $-\frac{481}{56}$

C $\frac{307}{168}$

D $-\frac{307}{168}$

Solution: C

The question asks you to find

$$\sum_{n=0}^{6} (-1)^n \frac{5}{n+2}$$

To find the sum, write each term with the value of n for that term. Since the summation shows that n begins with a value of 0, and ends with 6, there are seven terms.

$$n = 0 \qquad (-1)^0 \times \frac{5}{0+2} = 1 \times \frac{5}{2} = \frac{5}{2}$$

$$n = 1 \qquad (-1)^1 \times \frac{5}{1+2} = (-1) \times \frac{5}{3} = -\frac{5}{3}$$

$$n = 2 \qquad (-1)^2 \times \frac{5}{2+2} = 1 \times \frac{5}{4} = \frac{5}{4}$$

$$n = 3 \qquad (-1)^3 \times \frac{5}{3+2} = (-1) \times \frac{5}{5} = -\frac{5}{5} = -1$$

$$n = 4 \qquad (-1)^4 \times \frac{5}{4+2} = 1 \times \frac{5}{6} = \frac{5}{6}$$

$$n = 5 \qquad (-1)^5 \times \frac{5}{5+2} = (-1) \times \frac{5}{7} = -\frac{5}{7}$$

$$n = 6 \qquad (-1)^6 \times \frac{5}{6+2} = 1 \times \frac{5}{8} = \frac{5}{8}$$

Add the terms.

$$\frac{5}{2}-\frac{5}{3}+\frac{5}{4}-1+\frac{5}{6}-\frac{5}{7}+\frac{5}{8}$$

Rewrite the terms with a least common denominator of 168 and combine the seven terms.

$$\frac{420}{168}-\frac{280}{168}+\frac{210}{168}-\frac{168}{168}+\frac{140}{168}-\frac{120}{168}+\frac{105}{168}=\frac{307}{168}$$

Topic: Summation notation, expanding

Question: Expand the summation.

$$\sum_{n=0}^{4} \frac{4n+1}{3^n}$$

Answer choices:

A $\quad \frac{5}{3} + 1 + \frac{13}{27} + \frac{17}{81}$

B $\quad 1 + \frac{5}{3} + 1 + \frac{13}{27} + \frac{17}{81} + \frac{7}{81}$

C $\quad 1 + \frac{5}{3} + 1 + \frac{13}{27} + \frac{17}{81}$

D $\quad \frac{353}{81}$

Solution: C

To expand the summation

$$\sum_{n=0}^{4} \frac{4n+1}{3^n}$$

write each term with the value of n for that term. Since the summation shows that n begins with a value of 0 and ends with 4, there are five terms.

$$n=0 \qquad \frac{4(0)+1}{3^0}=\frac{1}{1}=1$$

$$n=1 \qquad \frac{4(1)+1}{3^1}=\frac{4+1}{3}=\frac{5}{3}$$

$$n=2 \qquad \frac{4(2)+1}{3^2}=\frac{8+1}{9}=\frac{9}{9}=1$$

$$n=3 \qquad \frac{4(3)+1}{3^3}=\frac{12+1}{27}=\frac{13}{27}$$

$$n=4 \qquad \frac{4(4)+1}{3^4}=\frac{16+1}{81}=\frac{17}{81}$$

Write the terms as a sum to expand the summation.

$$1+\frac{5}{3}+1+\frac{13}{27}+\frac{17}{81}$$

Topic: Summation notation, expanding

Question: Expand the summation.

$$\sum_{x=1}^{7} \left(x^2 + 4x + 3\right)$$

Answer choices:

A $\quad 273$

B $\quad 8 + 15 + 24 + 35 + 48 + 63 + 80$

C $\quad 3 + 8 + 15 + 24 + 35 + 48 + 63 + 80$

D $\quad 3 + 8 + 15 + 24 + 35 + 48 + 63$

Solution: B

To expand the summation

$$\sum_{x=1}^{7}\left(x^2+4x+3\right)$$

write each term with the value of x for that term. Since the summation shows that x begins with a value of 1, and ends with 7, there are seven terms.

$$x=1 \qquad (1)^2+4(1)+3=1+4+3=8$$

$$x=2 \qquad (2)^2+4(2)+3=4+8+3=15$$

$$x=3 \qquad (3)^2+4(3)+3=9+12+3=24$$

$$x=4 \qquad (4)^2+4(4)+3=16+16+3=35$$

$$x=5 \qquad (5)^2+4(5)+3=25+20+3=48$$

$$x=6 \qquad (6)^2+4(6)+3=36+24+3=63$$

$$x=7 \qquad (7)^2+4(7)+3=49+28+3=80$$

Write the terms as a sum to expand the summation.

$$8+15+24+35+48+63+80$$

Topic: Summation notation, expanding

Question: Expand the summation.

$$\sum_{x=0}^{4}\left(\frac{1}{2}x^3-\frac{1}{3}x^2+\frac{1}{4}x-\frac{1}{5}\right)$$

Answer choices:

A $\frac{13}{60}+\frac{89}{30}+\frac{221}{20}+\frac{412}{15}+\frac{3{,}313}{60}$

B $\frac{83}{2}$

C $\frac{1{,}163}{12}$

D $-\frac{1}{5}+\frac{13}{60}+\frac{89}{30}+\frac{221}{20}+\frac{412}{15}$

Solution: D

To expand the summation

$$\sum_{x=0}^{4}\left(\frac{1}{2}x^3-\frac{1}{3}x^2+\frac{1}{4}x-\frac{1}{5}\right)$$

write each term with the value of x for that term. Since the summation shows that x begins with a value of 0 and ends with 4, there are five terms.

$x=0$ $$\frac{1}{2}(0)^3-\frac{1}{3}(0)^2+\frac{1}{4}(0)-\frac{1}{5}$$

$$=-\frac{1}{5}$$

$x=1$ $$\frac{1}{2}(1)^3-\frac{1}{3}(1)^2+\frac{1}{4}(1)-\frac{1}{5}$$

$$=\frac{30}{60}-\frac{20}{60}+\frac{15}{60}-\frac{12}{60}=\frac{13}{60}$$

$x=2$ $$\frac{1}{2}(2)^3-\frac{1}{3}(2)^2+\frac{1}{4}(2)-\frac{1}{5}$$

$$=\frac{8}{2}-\frac{4}{3}+\frac{2}{4}-\frac{1}{5}=4-\frac{4}{3}+\frac{1}{2}-\frac{1}{5}$$

$$=\frac{120}{30}-\frac{40}{30}+\frac{15}{30}-\frac{6}{30}=\frac{89}{30}$$

$x=3$ $$\frac{1}{2}(3)^3-\frac{1}{3}(3)^2+\frac{1}{4}(3)-\frac{1}{5}=\frac{27}{2}-\frac{9}{3}+\frac{3}{4}-\frac{1}{5}$$

$$=\frac{27}{2}-3+\frac{3}{4}-\frac{1}{5}=\frac{270}{20}-\frac{60}{20}+\frac{15}{20}-\frac{4}{20}=\frac{221}{20}$$

$x = 4$ $$\frac{1}{2}(4)^3 - \frac{1}{3}(4)^2 + \frac{1}{4}(4) - \frac{1}{5} = \frac{64}{2} - \frac{16}{3} + \frac{4}{4} - \frac{1}{5}$$

$$= 32 - \frac{16}{3} + 1 - \frac{1}{5} = \frac{480}{15} - \frac{80}{15} + \frac{15}{15} - \frac{3}{15} = \frac{412}{15}$$

Write the terms as a sum to expand the summation.

$$-\frac{1}{5} + \frac{13}{60} + \frac{89}{30} + \frac{221}{20} + \frac{412}{15}$$

Topic: Summation notation, collapsing

Question: Use summation notation to rewrite the sum.

$$\frac{(x-1)}{1-2}+\frac{(x-1)^2}{2-4}+\frac{(x-1)^3}{3-8}+\frac{(x-1)^4}{4-16}+\frac{(x-1)^5}{5-32}$$

Answer choices:

A $\displaystyle\sum_{n=1}^{5}\frac{(x-1)^n}{2^n-n}$

B $\displaystyle\sum_{n=0}^{4}\frac{(x-1)^n}{n-2^n}$

C $\displaystyle\sum_{n=1}^{5}\frac{(x-1)^n}{n-2^n}$

D $\displaystyle\sum_{n=1}^{5}\frac{(x-1)^{n+1}}{n-2^{n+1}}$

Solution: C

To rewrite the summation

$$\frac{(x-1)}{1-2}+\frac{(x-1)^2}{2-4}+\frac{(x-1)^3}{3-8}+\frac{(x-1)^4}{4-16}+\frac{(x-1)^5}{5-32}$$

we notice that in the terms of the sum there are three subsequences, one in the numerator and two in the denominator. We can find the summation rule for each of these subsequences separately and then put them together in one summation at the end.

In the numerator, we have

n (term #)	1	2	3	4	5
expression	$(x-1)^1$	$(x-1)^2$	$(x-1)^3$	$(x-1)^4$	$(x-1)^5$

We can see that the numerator is the expression $x-1$ raised to the power of n.

We see a changing difference in the denominator. Associating these differences with the term number we have

n (term #)	1	2	3	4	5
expression	$1-2$	$2-4$	$3-8$	$4-16$	$5-32$

We can see that the first number in each difference is the term number, n. Now let's look at the second number. That number appears to be a power of 2.

n (term #)	1	2	3	4	5

expression	$1-2$	$2-4$	$3-8$	$4-16$	$5-32$
2nd number	2	4	8	16	32
As a power of 2		2^1	2^2	2^3	2^4 2^5

Again, after careful observation, we can see that the denominator contains the difference of n and 2 raised to the power of n.

The sum contains five terms, starting with $n = 1$, and ending with $n = 5$. Therefore, we can rewrite the sum as a summation.

$$\sum_{n=1}^{5} \frac{(x-1)^n}{n-2^n} = \frac{(x-1)}{1-2} + \frac{(x-1)^2}{2-4} + \frac{(x-1)^3}{3-8} + \frac{(x-1)^4}{4-16} + \frac{(x-1)^5}{5-32}$$

Topic: Summation notation, collapsing

Question: Use summation notation to rewrite the sum.

$$3x + \frac{9x^2}{4} + 3x^3 + \frac{81x^4}{16} + \frac{243x^5}{25}$$

Answer choices:

A $\displaystyle\sum_{n=1}^{5} \frac{3^n x^n}{2n+2}$

B $\displaystyle\sum_{n=0}^{4} \frac{3^n x^n}{n^2}$

C $\displaystyle\sum_{n=1}^{5} \frac{3^{n+1} x^{n+1}}{n^2}$

D $\displaystyle\sum_{n=1}^{5} \frac{3^n x^n}{n^2}$

Solution: D

To rewrite the summation

$$3x + \frac{9x^2}{4} + 3x^3 + \frac{81x^4}{16} + \frac{243x^5}{25}$$

it may be helpful to make every term appear to be a fraction to find a pattern of the denominators.

We can see that the denominators are all squared numbers

$$1^2, 2^2, 1^2, 4^2, 5^2$$

The third denominator is either out of place or should be 9. To form a pattern of denominators, we will change the denominator of that term, as well as the numerator of that term so the value of the term remains the same as before. The new sum is

$$\frac{3x}{1} + \frac{9x^2}{4} + \frac{27x^3}{9} + \frac{81x^4}{16} + \frac{243x^5}{25}$$

Now we can see three patterns in the sum; the coefficients of the numerator, the exponents in the numerator, and the denominator.

The coefficient in the numerator is a power of 3.

n (term #)	1	2	3	4	5
coefficient	3	9	27	81	243
as a power of 3	3^1	3^2	3^3	3^4	3^5

We can see that the coefficient in the numerator is the number 3 raised to the power of n, the term number. Now let's look at the power of x in the numerator.

n (term #)	1	2	3	4	5
x term	x^1	x^2	x^3	x^4	x^5

We can see that the exponent of x in the numerator is the same as the value of n, the term number. Next, let's look at the denominator of each term in the sum.

n (term #)	1	2	3	4	5
denominator	1	4	9	16	25
square of n	$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$

We can see that each denominator is the square of the term number. Now we can rewrite the sum as a summation.

$$\sum_{n=1}^{5} \frac{3^n x^n}{n^2} = \frac{3x}{1} + \frac{9x^2}{4} + \frac{27x^3}{9} + \frac{81x^4}{16} + \frac{243x^5}{25}$$

Topic: Summation notation, collapsing

Question: Use summation notation to rewrite the sum.

$$\frac{1}{3}-\frac{x}{4}+\frac{x^2}{5}-\frac{x^3}{6}+\frac{x^4}{7}-\frac{x^5}{8}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{5}\frac{(-1)^n x^n}{n+3}$

B $\displaystyle\sum_{n=1}^{6}\frac{(-1)^n x^n}{n+2}$

C $\displaystyle\sum_{n=0}^{5}\frac{x^n}{n+3}$

D $\displaystyle\sum_{n=0}^{5}\frac{-x^n}{n+3}$

Solution: A

To rewrite the summation

$$\frac{1}{3}-\frac{x}{4}+\frac{x^2}{5}-\frac{x^3}{6}+\frac{x^4}{7}-\frac{x^5}{8}$$

we need to notice that there's a pattern in the denominators, the sign of the terms alternate between positive and negative, and there is a pattern of the exponents, if we change the first term. We will begin by changing the first term so it contains a power of x, and we will place an exponent of 1 in the second term.

$$\frac{x^0}{3}-\frac{x^1}{4}+\frac{x^2}{5}-\frac{x^3}{6}+\frac{x^4}{7}-\frac{x^5}{8}$$

Now we see three definite patterns. The changing signs, the exponent of x, and the denominator. Let's look at each one. Recall that multiplying by (-1) changes the sign of terms. So the summation must contain this multiplication in each term. If we consider the value of n to be the term number, the signs of each term will be incorrect, unless we begin n with 0.

n	0	1	2	3	4	5
Sign	$+$	$-$	$+$	$-$	$+$	$-$
Power of -1	$(-1)^0$	$(-1)^1$	$(-1)^2$	$(-1)^3$	$(-1)^4$	$(-1)^5$
Sign	$+$	$-$	$+$	$-$	$+$	$-$

We can see that if n begins with a value of 0, and we raise -1 to the power of n, we have the correct sign of each term. Next let's look at the power of x in the numerators.

n	0	1	2	3	4	5
x term	x^0	x^1	x^2	x^3	x^4	x^5
Power of x	0	1	2	3	4	5

We can see that the power of x is the same as the value of n. Now, let's look at the denominator of each term.

n	0	1	2	3	4	5
Denominator	3	4	5	6	7	8
$n+3$	$0+3$	$1+3$	$2+3$	$3+3$	$4+3$	$5+3$
Answer	3	4	5	6	7	8

After careful study, we can see that the denominator of each term in the sum is three more than the value of n.

Now we have the pattern of all of the parts of the sum. Let's write the summation.

$$\sum_{n=0}^{5} \frac{(-1)^n x^n}{n+3} = \frac{1}{3} - \frac{x}{4} + \frac{x^2}{5} - \frac{x^3}{6} + \frac{x^4}{7} - \frac{x^5}{8}$$

Topic: Riemann sums, left endpoints

Question: Use a lower sum (inscribed rectangles) to find the area under the curve.

$$y = \sqrt{x} + 1$$

on the interval $[0,3]$

with $n = 6$

Answer choices:

A 3.75

B 6.48

C 6.83

D 5.96

Solution: D

The Riemann sum is a tool we can use to approximate the area under a function over a set interval $a \leq x \leq b$.

We'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Riemann sums we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Riemann sum formula is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is the number of rectangles we're using to approximate area. If we expand the Riemann sum, we get the formula

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right]$$

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use an endpoint of each segment as the values of

x_n. When we're using a Riemann sum to approximate area, we can choose the left endpoints, right endpoints, or midpoints of our rectangles.

Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{3-0}{6}$$

$$\Delta x = \frac{1}{2}$$

Next, we need to figure out whether to use left endpoints, right endpoints, or a combination in order to get the lower sum, so we take the derivative of the function to see where it's increasing and decreasing.

$$y' = \frac{1}{2}x^{-\frac{1}{2}}$$

Since this derivative function is always positive, the original function is increasing on the interval $[0,3]$, and so we use the left endpoints to find the lower sum.

$$c_i = a + (i-1)\Delta x$$

$$c_1 = 0 \qquad c_2 = \frac{1}{2} \qquad c_3 = 1 \qquad c_4 = \frac{3}{2} \qquad c_5 = 2 \qquad c_6 = \frac{5}{2}$$

Finally, we plug everything into the lower sum formula.

$$s(n) = \sum_{i=1}^{n} f(c_i)\Delta x$$

$$s(6) = \sum_{i=1}^{6} f(c_i)\Delta x$$

$$s(6) = \left(\sqrt{0} + 1\right)\left(\frac{1}{2}\right) + \left(\sqrt{\frac{1}{2}} + 1\right)\left(\frac{1}{2}\right) + \left(\sqrt{1} + 1\right)\left(\frac{1}{2}\right)$$

$$+\left(\sqrt{\frac{3}{2}} + 1\right)\left(\frac{1}{2}\right) + \left(\sqrt{2} + 1\right)\left(\frac{1}{2}\right) + \left(\sqrt{\frac{5}{2}} + 1\right)\left(\frac{1}{2}\right)$$

$$s(6) \approx 5.96$$

Topic: Riemann sums, left endpoints

Question: Approximate the area under the curve using a left rectangular approximation method, and five equal subintervals.

$$f(x) = \frac{x^2 + 6x + 5}{x + 4}$$

on the interval $[0,5]$

Answer choices:

A $\frac{20,171}{840}$

B $\frac{4,857}{280}$

C $\frac{4,507}{280}$

D $\frac{19,121}{840}$

Solution: B

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a left rectangular approximation method, we will find the height of the rectangle by calculating the function value at the left endpoint of each subinterval.

The five equal subintervals in the interval $[0,5]$ are $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$. Each subinterval is 1 unit wide. We will calculate the function values at 0, 1, 2, 3, and 4.

$$f(0)=\frac{(0)^2+6(0)+5}{0+4}=\frac{5}{4}$$

$$f(1)=\frac{(1)^2+6(1)+5}{1+4}=\frac{1+6+5}{5}=\frac{12}{5}$$

$$f(2)=\frac{(2)^2+6(2)+5}{2+4}=\frac{4+12+5}{6}=\frac{21}{6}=\frac{7}{2}$$

$$f(3)=\frac{(3)^2+6(3)+5}{3+4}=\frac{9+18+5}{7}=\frac{32}{7}$$

$$f(4)=\frac{(4)^2+6(4)+5}{4+4}=\frac{16+24+5}{8}=\frac{45}{8}$$

Now that we know the height of each rectangle at the left endpoints of the subintervals, we will add the areas together to get the final approximation.

Since the widths of the subintervals are all one unit, we do not have to multiply the heights by the width in this question.

$$\frac{5}{4}+\frac{12}{5}+\frac{7}{2}+\frac{32}{7}+\frac{45}{8}$$

$$\frac{350}{280}+\frac{672}{280}+\frac{980}{280}+\frac{1,280}{280}+\frac{1,575}{280}$$

$$\frac{4,857}{280}$$

Topic: Riemann sums, left endpoints

Question: Approximate the area under the curve using a left rectangular approximation method, and three equal subintervals.

$$g(x) = -\frac{1}{2}x^3 + 5x^2 - 3x - 8$$

on the interval $[2,8]$

Answer choices:

A $\frac{349}{2}$

B 76

C 106

D 152

Solution: D

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a left rectangular approximation method, we will find the height of the rectangle by calculating the function value at the left endpoint of each subinterval.

The three equal subintervals in the interval $[2,8]$ are $[2,4]$, $[4,6]$, and $[6,8]$. Each subinterval is 2 units wide. We will calculate the function values at 2, 4, and 6, and then multiply each value by 2 to find the area of the rectangles.

$$g(2) = -\frac{1}{2}(2)^3 + 5(2)^2 - 3(2) - 8 = -4 + 20 - 6 - 8 = 2$$

Area: $2 \times 2 = 4$

$$g(4) = -\frac{1}{2}(4)^3 + 5(4)^2 - 3(4) - 8 = -32 + 80 - 12 - 8 = 28$$

Area: $2 \times 28 = 56$

$$g(6) = -\frac{1}{2}(6)^3 + 5(6)^2 - 3(6) - 8 = -108 + 180 - 18 - 8 = 46$$

Area: $2 \times 46 = 92$

Now we know the area of each rectangle at the left endpoints of the subintervals, we will add the areas together to get the final approximation.

$$4 + 56 + 92 = 152$$

Topic: Riemann sums, right endpoints

Question: Use Riemann Sums and right endpoints to approximate the integral.

$$\int_0^2 x^2\, dx$$

when $n = 3$

Answer choices:

A $-\frac{56}{9}$

B $\frac{56}{9}$

C $-\frac{112}{27}$

D $\frac{112}{27}$

Solution: D

The Riemann sum is a tool we can use to approximate the area under a function over a set interval $a \leq x \leq b$.

We'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Riemann sums we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Riemann sum formula is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is the number of rectangles we're using to approximate area. If we expand the Riemann sum, we get the formula

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n)\right]$$

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use an endpoint of each segment as the values of

x_n. When we're using a Riemann sum to approximate area, we can choose the left endpoints, right endpoints, or midpoints of our rectangles.

Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-0}{3}$$

$$\Delta x = \frac{2}{3}$$

Since the interval is $[0,2]$, we know that $x_0 = 0$ and that $x_n = 2$. Using $\Delta x = 2/3$ to find the subintervals, we get

$$x_0 = 0$$

$$x_1 = 0 + \frac{2}{3} \qquad x_1 = \frac{2}{3}$$

$$x_2 = \frac{2}{3} + \frac{2}{3} \qquad x_2 = \frac{4}{3}$$

$$x_3 = \frac{4}{3} + \frac{2}{3} \qquad x_3 = \frac{6}{3} \qquad x_3 = 2$$

Since we're using right endpoints, we'll use all but $x_0 = 0$, since this is a left endpoint. Plugging all of this into our Riemann sum formula, remembering that $f(x) = x^2$, we get

$$R_3 = \frac{2}{3}\left[f(x_1) + f(x_2) + f(x_3)\right]$$

$$R_3 = \frac{2}{3}\left[f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right) + f(2)\right]$$

$$R_3 = \frac{2}{3}\left[\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + (2)^2\right]$$

$$R_3 = \frac{2}{3}\left(\frac{4}{9} + \frac{16}{9} + 4\right)$$

$$R_3 = \frac{112}{27}$$

Topic: Riemann sums, right endpoints

Question: Use Riemann Sums and right endpoints to approximate the integral.

$$\int_{-2}^{2} x^2 - 2\ dx$$

when $n = 4$

Answer choices:

A -2

B -1

C 2

D 0

Solution: A

The Riemann sum is a tool we can use to approximate the area under a function over a set interval $a \leq x \leq b$.

We'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Riemann sums we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Riemann sum formula is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is the number of rectangles we're using to approximate area. If we expand the Riemann sum, we get the formula

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \dots + f(x_n)\right]$$

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use an endpoint of each segment as the values of

x_n. When we're using a Riemann sum to approximate area, we can choose the left endpoints, right endpoints, or midpoints of our rectangles.

Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-(-2)}{4}$$

$$\Delta x = 1$$

Since the interval is $[-2,2]$, we know that $x_0 = -2$ and that $x_n = 2$. Using $\Delta x = 1$ to find the subintervals, we get

$$x_0 = -2$$

$$x_1 = -2 + 1 \qquad x_1 = -1$$

$$x_2 = -1 + 1 \qquad x_2 = 0$$

$$x_3 = 0 + 1 \qquad x_3 = 1$$

$$x_4 = 1 + 1 \qquad x_4 = 2$$

Since we're using right endpoints, we'll use all but $x_0 = -2$, since this is a left endpoint. Plugging all of this into our Riemann sum formula, remembering that $f(x) = x^2 - 2$, we get

$$R_4 = 1\left[f(x_1) + f(x_2) + f(x_3) + f(x_4)\right]$$

$$R_4 = 1\left[f(-1) + f(0) + f(1) + f(2)\right]$$

$$R_4 = 1\left[\left((-1)^2 - 2\right) + \left((0)^2 - 2\right) + \left((1)^2 - 2\right) + \left((2)^2 - 2\right)\right]$$

$$R_4 = 1 - 2 + 0 - 2 + 1 - 2 + 4 - 2$$

$$R_4 = -2$$

Topic: Riemann sums, right endpoints

Question: Approximate the area under the curve using a right rectangular approximation method, and four equal subintervals.

$$f(x) = \frac{x^2 + 7x - 3}{x + 1}$$

on the interval $[2,14]$

Answer choices:

A $\quad 131.25$

B $\quad 58.15$

C $\quad 174.45$

D $\quad 152.85$

Solution: C

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a right rectangular approximation method, we will find the height of the rectangle by calculating the function value at the right endpoint of each subinterval.

The four equal subintervals in the interval $[2,14]$ are $[2,5]$, $[5,8]$, $[8,11]$, and $[11,14]$. Each subinterval is 3 units wide. We will calculate the function values at 5, 8, 11 and 14. Then, we will multiply each of these values by 3, the width of each subinterval to find the area of that rectangle.

$$f(5) = \frac{(5)^2 + 7(5) - 3}{5+1} = \frac{25 + 35 - 3}{6} = \frac{19}{2} = 9.5$$

Area: $9.5 \times 3 = 28.5$

$$f(8) = \frac{(8)^2 + 7(8) - 3}{8+1} = \frac{64 + 56 - 3}{9} = \frac{117}{9} = 13$$

Area: $13 \times 3 = 39$

$$f(11) = \frac{(11)^2 + 7(11) - 3}{11+1} = \frac{121 + 77 - 3}{12} = \frac{195}{12} = \frac{65}{4} = 16.25$$

Area: $16.25 \times 3 = 48.75$

$$f(14) = \frac{(14)^2 + 7(14) - 3}{14+1} = \frac{196 + 98 - 3}{15} = \frac{291}{15} = \frac{97}{5} = 19.4$$

Area: $19.4 \times 3 = 58.2$

Now we know the area of each rectangle with the function value at the right endpoints of the subintervals. Add the areas together.

$$28.5 + 39 + 48.75 + 58.2 = 174.45$$

Topic: Riemann sums, midpoints

Question: Use Riemann Sums and midpoints to approximate area.

$$f(x) = -x^2 - x$$

on the interval $-1 \leq x \leq 2$

when $n = 5$

Answer choices:

A $-\frac{69}{50}$

B $\frac{441}{100}$

C $-\frac{441}{100}$

D $\frac{69}{50}$

Solution: C

The Riemann sum is a tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

We'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Riemann sums we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Riemann sum formula is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is the number of rectangles we're using to approximate area. If we expand the Riemann sum, we get the formula

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right]$$

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use an endpoint of each segment as the values of

x_n. When we're using a Riemann sum to approximate area, we can choose the left endpoints, right endpoints, or midpoints of our rectangles.

Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-(-1)}{5}$$

$$\Delta x = \frac{3}{5}$$

Since the interval is $[-1,2]$, we know that $x_0 = -1$ and that $x_n = 2$. Using $\Delta x = 3/5$ to find the subintervals, we get

$$x_0 = -1$$

$$x_1 = -1 + \frac{3}{5} \qquad x_1 = -\frac{2}{5}$$

$$x_2 = -\frac{2}{5} + \frac{3}{5} \qquad x_2 = \frac{1}{5}$$

$$x_3 = \frac{1}{5} + \frac{3}{5} \qquad x_3 = \frac{4}{5}$$

$$x_4 = \frac{4}{5} + \frac{3}{5} \qquad x_4 = \frac{7}{5}$$

$$x_5 = \frac{7}{5} + \frac{3}{5} \qquad x_5 = \frac{10}{5} \qquad x_5 = 2$$

Since we're using midpoints, we need to find the value of x that's halfway between each of the x_n values above.

The first interval is $[-1,-2/5]$, so

$$M_1 = \frac{-1 + \left(-\frac{2}{5}\right)}{2}$$

$$M_1 = -\frac{7}{10}$$

The second interval is $[-2/5,1/5]$, so

$$M_2 = \frac{-\frac{2}{5} + \frac{1}{5}}{2}$$

$$M_2 = -\frac{1}{10}$$

The third interval is $[1/5,4/5]$, so

$$M_3 = \frac{\frac{1}{5} + \frac{4}{5}}{2}$$

$$M_3 = \frac{1}{2}$$

The fourth interval is $[4/5,7/5]$, so

$$M_4 = \frac{\frac{4}{5} + \frac{7}{5}}{2}$$

$$M_4 = \frac{11}{10}$$

The fifth interval is $[7/5,2]$, so

$$M_5=\frac{\frac{7}{5}+2}{2}$$

$$M_5=\frac{17}{10}$$

Plugging all of this into our Riemann sum formula, remembering that $f(x)=-x^2-x$, we get

$$R_5=\frac{3}{5}\left[f\left(-\frac{7}{10}\right)+f\left(-\frac{1}{10}\right)+f\left(\frac{1}{2}\right)+f\left(\frac{11}{10}\right)+f\left(\frac{17}{10}\right)\right]$$

$$R_5=\frac{3}{5}\left\{\left[-\left(-\frac{7}{10}\right)^2-\left(-\frac{7}{10}\right)\right]+\left[-\left(-\frac{1}{10}\right)^2-\left(-\frac{1}{10}\right)\right]+\left[-\left(\frac{1}{2}\right)^2-\frac{1}{2}\right]\right.$$

$$\left.+\left[-\left(\frac{11}{10}\right)^2-\frac{11}{10}\right]+\left[-\left(\frac{17}{10}\right)^2-\frac{17}{10}\right]\right\}$$

$$R_5=\frac{3}{5}\left(-\frac{49}{100}+\frac{7}{10}-\frac{1}{100}+\frac{1}{10}-\frac{1}{4}-\frac{1}{2}-\frac{121}{100}-\frac{11}{10}-\frac{289}{100}-\frac{17}{10}\right)$$

$$R_5=\frac{3}{5}\left(-\frac{460}{100}-\frac{20}{10}-\frac{1}{4}-\frac{1}{2}\right)$$

$$R_5=\frac{3}{5}\left(-\frac{46}{10}-\frac{20}{10}-\frac{1}{4}-\frac{1}{2}\right)$$

$$R_5=\frac{3}{5}\left(-\frac{33}{5}-\frac{1}{4}-\frac{1}{2}\right)$$

$$R_5 = \frac{3}{5}\left(-\frac{132}{20} - \frac{5}{20} - \frac{10}{20}\right)$$

$$R_5 = -\frac{441}{100}$$

Topic: Riemann sums, midpoints

Question: Approximate the area under the curve using a midpoint rectangular approximation method, and five equal subintervals.

$$f(x) = \frac{1}{20}x^2 + x - 7$$

on the interval $[6,26]$

Answer choices:

A 576

B 472

C 468

D 368

Solution: C

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a midpoint rectangular approximation method, we will find the height of the rectangle by calculating the function value at the midpoint of each subinterval.

The five equal subintervals in the interval $[6,26]$ are $[6,10]$, $[10,14]$, $[14,18]$, $[18,22]$, and $[22,26]$. We will next find the midpoint of each subinterval by adding the two endpoints and dividing that sum by 2. We will calculate the function values at 8, 12, 16, 20, and 24. Each subdivision is 4 units wide, so we will find the area of each triangle by multiplying the function values by 4.

$$f(8) = \frac{1}{20}(8)^2 + 8 - 7 = \frac{1}{20}(64) + 1 = 3.2 + 1 = 4.2$$ Area: $4.2 \times 4 = 16.8$

$$f(12) = \frac{1}{20}(12)^2 + 12 - 7 = \frac{1}{20}(144) + 5 = 7.2 + 5 = 12.2$$ Area: $12.2 \times 4 = 48.8$

$$f(16) = \frac{1}{20}(16)^2 + 16 - 7 = \frac{1}{20}(256) + 9 = 12.8 + 9 = 21.8$$ Area: $21.8 \times 4 = 87.2$

$$f(20) = \frac{1}{20}(20)^2 + 20 - 7 = \frac{1}{20}(400) + 13 = 20 + 13 = 33$$ Area: $33 \times 4 = 132$

$f(24) = \frac{1}{20}(24)^2 + 24 - 7 = \frac{1}{20}(576) + 17 = 28.8 + 17 = 45.8$ Area: $45.8 \times 4 = 183.2$

Now that we know the area of each rectangle calculated at the midpoints of the subintervals, we will add the areas together to get the final approximation.

$$16.8 + 48.8 + 87.2 + 132 + 183.2 = 468$$

Topic: Riemann sums, midpoints

Question: Approximate the area under the curve using a midpoint rectangular approximation method, and four equal subintervals.

$$g(x) = -\frac{1}{2}x^3 + 6x^2 - 4x + 2$$

on the interval $[1,9]$

Answer choices:

A 568

B 496

C 484

D 400

Solution: B

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

Because we are using a midpoint rectangular approximation method, we will find the height of the rectangle by calculating the function value at the midpoint of each subinterval.

The four equal subintervals in the interval $[1,9]$ are $[1,3]$, $[3,5]$, $[7,7]$ and $[7,9]$. Each subinterval is 2 units wide. We will calculate the midpoint of each subinterval by finding the average of the endpoints.

We will calculate the function values at 2, 4, 6, and 8, and then multiply each value by 2 to find the area of the rectangles.

$$g(2) = -\frac{1}{2}(2)^3 + 6(2)^2 - 4(2) + 2 = 14$$

Area: $14 \times 2 = 28$

$$g(4) = -\frac{1}{2}(4)^3 + 6(4)^2 - 4(4) + 2 = 50$$

Area: $50 \times 2 = 100$

$$g(6) = -\frac{1}{2}(6)^3 + 6(6)^2 - 4(6) + 2 = 86$$

Area: $86 \times 2 = 172$

$$g(8) = -\frac{1}{2}(8)^3 + 6(8)^2 - 4(8) + 2 = 98$$

Area: $98 \times 2 = 196$

Now that we know the area of each rectangle calculated at the midpoints of the subintervals, we will add the areas together to get the final approximation.

$$28 + 100 + 172 + 196 = 496$$

Topic: Moving from summation notation to the integral

Question: How do you convert the Riemann sum to a definite integral?

$$\sum_{i=1}^{n} \left(7x_i^3 + 3x_i^2 - 2x_i + 1\right) \ \Delta x$$

on the interval $[-1,5]$

Answer choices:

A $\int_{-1}^{5} \left(7x^3 + 3x^2 - 2x + 1\right) \ \Delta x$

B $\int_{-1}^{5} \left(7x_i^3 + 3x_i^2 - 2x_i + 1\right) \ \Delta x$

C $\int_{-1}^{5} \left(7x^3 + 3x^2 - 2x + 1\right) \ dx$

D $\lim_{n \to 5} \int_{-1}^{n} \left(7x^2 + 3x - 2x + 1\right) \ dx$

Solution: C

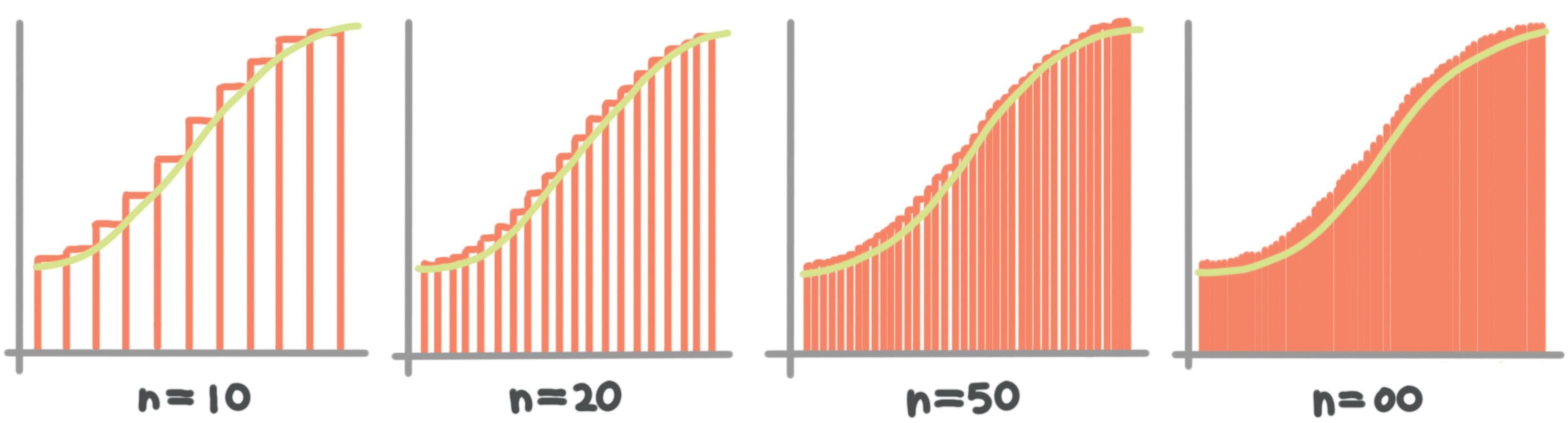
The question asks us to convert a Riemann sum to a definite integral.

Recall that a Riemann sum is used to estimate the area under a curve on an interval $[a, b]$, with a finite number of subintervals, and is written as a sum

$$\sum_{i=1}^{n} f(x_i)\ \Delta x$$

$i = 1$ represents the starting rectangle. The summation adds up all of the rectangles, and the n indicates to stop adding when you get to the last rectangle. The function $f(x_i)$ represents the height of each rectangle and Δx, which is $(b - a)/n$, represents the width of each rectangle.

An integral calculates the area under the curve in the interval exactly. All we need to do is increase the number of subintervals n in the interval $[a, b]$. In fact, increase the number indefinitely. The figure below shows how the exact area under a curve is calculated, as the number of subintervals increases.



To convert a Riemann sum to a definite integral, we need an interval, $[a,b]$. Then, we rewrite the Riemann sum as a limit as the number of subintervals n approaches infinity. The Riemann sum changes to

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$$

From there,

$\lim_{n\to\infty}\sum_{i=1}^{n}$	changes to	$\int_a^b$
$f(x_i)$	changes to	$f(x)$
Δx	changes to	dx

Therefore, in the interval $[-1,5]$, the Riemann sum changes to the integral this way:

$$\sum_{i=1}^{n}\left(7x_i^3+3x_i^2-2x_i+1\right)\ \Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(7x_i^3+3x_i^2-2x_i+1\right)\ \Delta x$$

$$\int_{-1}^{5}\left(7x^3+3x^2-2x+1\right)\ dx$$

Topic: Moving from summation notation to the integral

Question: How do you convert the Riemann sum to a definite integral?

$$\sum_{i=1}^{n}\left(5x_i^2\sqrt{x_i}-1\right)\ \Delta x$$

on the interval $[2,9]$

Answer choices:

A $\lim_{i\to 9}\sum_{i=2}^{n}\left(5x_i^2\sqrt{x_i}-1\right)\ \Delta x$

B $\int_2^9\left(5x^2\sqrt{x}-1\right)\ dx$

C $\int_2^9\left(5x_i^2\sqrt{x_i}-1\right)\ \Delta x$

D $\int_9^2\left(5x^2\sqrt{x}-1\right)\ dx$

Solution: B

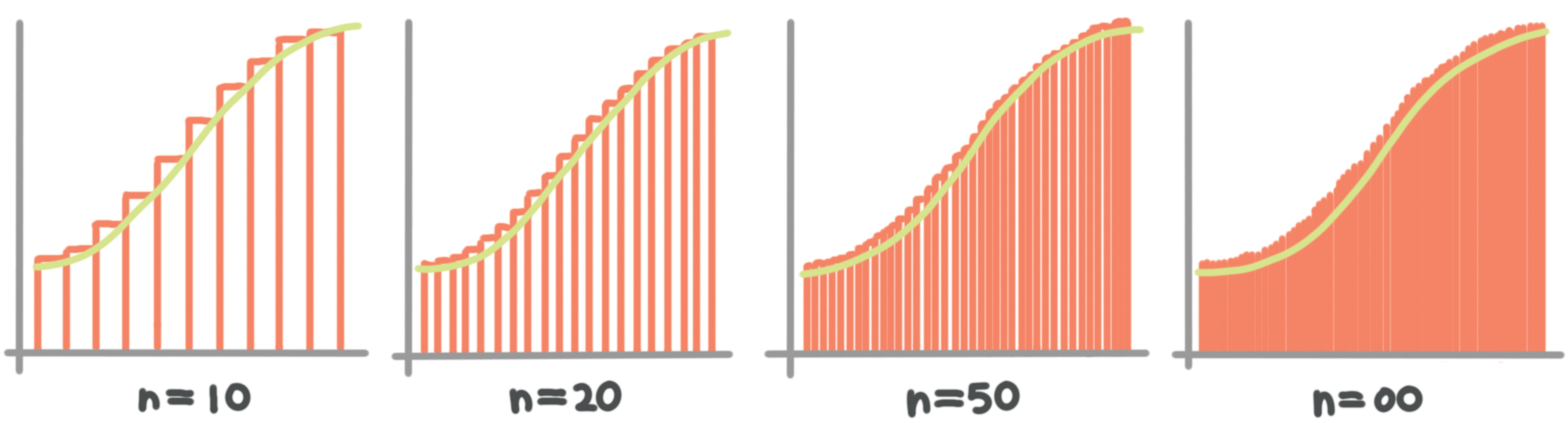
The question asks us to convert a Riemann sum to a definite integral.

Recall that a Riemann sum is used to estimate the area under a curve on an interval $[a, b]$, with a finite number of subintervals, and is written as a sum

$$\sum_{i=1}^{n} f(x_i)\ \Delta x$$

$i = 1$ represents the starting rectangle. The summation adds up all of the rectangles, and the n indicates to stop adding when you get to the last rectangle. The function $f(x_i)$ represents the height of each rectangle and Δx, which is $(b - a)/n$, represents the width of each rectangle.

An integral calculates the area under the curve in the interval exactly. All we need to do is increase the number of subintervals n in the interval $[a, b]$. In fact, increase the number indefinitely. The figure below shows how the exact area under a curve is calculated, as the number of subintervals increases.



To convert a Riemann sum to a definite integral, we need an interval, $[a, b]$. Then, we rewrite the Riemann sum as a limit as the number of subintervals n approaches infinity. The Riemann sum changes to

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

From there,

$\lim_{n\to\infty} \sum_{i=1}^{n}$	changes to	$\int_a^b$
$f(x_i)$	changes to	$f(x)$
Δx	changes to	dx

Therefore, in the interval $[2,9]$, the Riemann sum changes to the integral this way:

$$\sum_{i=1}^{n} \left(5x_i^2\sqrt{x}_i - 1\right)\ \Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5x_i^2\sqrt{x_i} - 1\right)\ \Delta x$$

$$\int_2^9 \left(5x^2\sqrt{x} - 1\right)\ dx$$

Topic: Moving from summation notation to the integral

Question: How do you convert the Riemann sum to a definite integral?

$$\sum_{i=1}^{n} 4\sec^2\left(\frac{\pi x_i}{4}\right)\ \Delta x$$

on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{4}\right]$

Answer choices:

A $\quad \lim_{n\to i} \sum_{i=-\frac{\pi}{3}}^{\frac{\pi}{4}} 4\sec^2\left(\frac{\pi x}{4}\right)\ dx$

B $\quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} 4\sec^2\left(\frac{\pi x}{4}\right)\ dx$

C $\quad \lim_{n\to \frac{\pi}{4}} \int_{-\frac{\pi}{3}}^{n} 4\sec^2\left(\frac{\pi x}{4}\right)\ dx$

D $\quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} 4\sec^2\left(\frac{\pi x_i}{4}\right)\ dx_i$

Solution: B

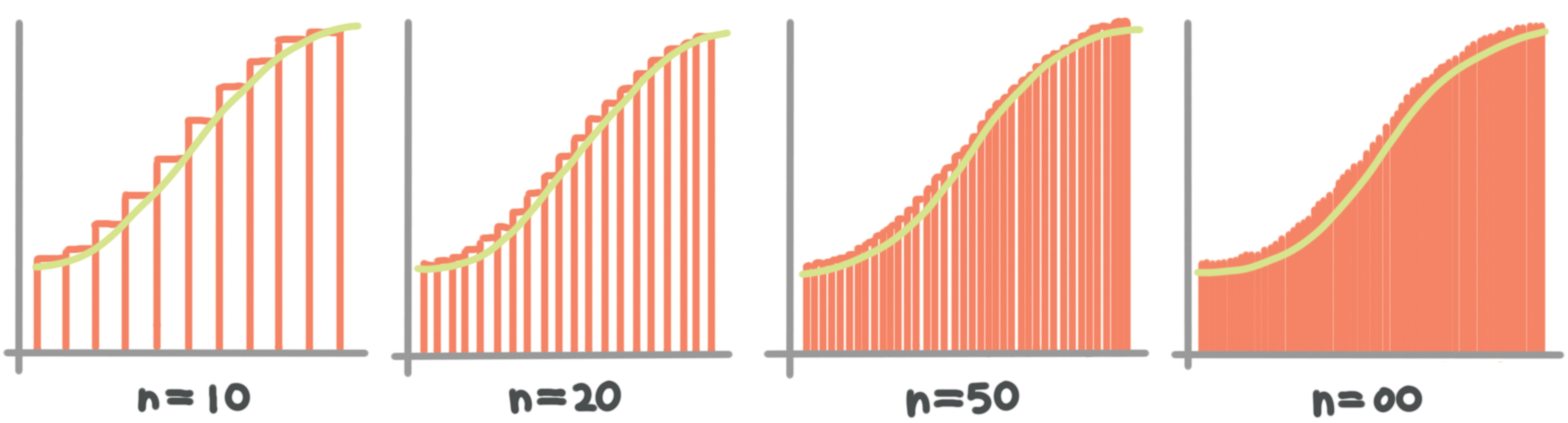
The question asks us to convert a Riemann sum to a definite integral.

Recall that a Riemann sum is used to estimate the area under a curve on an interval $[a, b]$, with a finite number of subintervals, and is written as a sum

$$\sum_{i=1}^{n} f(x_i)\ \Delta x$$

$i = 1$ represents the starting rectangle. The summation adds up all of the rectangles, and the n indicates to stop adding when you get to the last rectangle. The function $f(x_i)$ represents the height of each rectangle and Δx, which is $(b - a)/n$, represents the width of each rectangle.

An integral calculates the area under the curve in the interval exactly. All we need to do is increase the number of subintervals n in the interval $[a, b]$. In fact, increase the number indefinitely. The figure below shows how the exact area under a curve is calculated, as the number of subintervals increases.



To convert a Riemann sum to a definite integral, we need an interval, $[a,b]$. Then, we rewrite the Riemann sum as a limit as the number of subintervals n approaches infinity. The Riemann sum changes to

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$$

From there,

$\lim_{n\to\infty}\sum_{i=1}^{n}$	changes to	$\int_a^b$
$f(x_i)$	changes to	$f(x)$
Δx	changes to	dx

Therefore, in the interval $\left[-\pi/3,\pi/4\right]$, the Riemann sum changes to the integral this way:

$$\sum_{i=1}^{n} 4\sec^2\left(\frac{\pi x_i}{4}\right)\ \Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} 4\sec^2\left(\frac{\pi x_i}{4}\right)\ \Delta x$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} 4\sec^2\left(\frac{\pi x}{4}\right)\ dx$$

Topic: Over and underestimation

Question: Use a Riemann sum to estimate the maximum area and minimum area under this curve on $[1,6]$. Use rectangular approximation methods with 5 equal subintervals.

$$f(x) = \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{2}$$

Answer choices:

A Minimum of $\frac{48}{8}$ Maximum of $\frac{73}{8}$

B Minimum of $\frac{23}{4}$ Maximum of $\frac{37}{4}$

C Minimum of $\frac{45}{8}$ Maximum of $\frac{35}{4}$

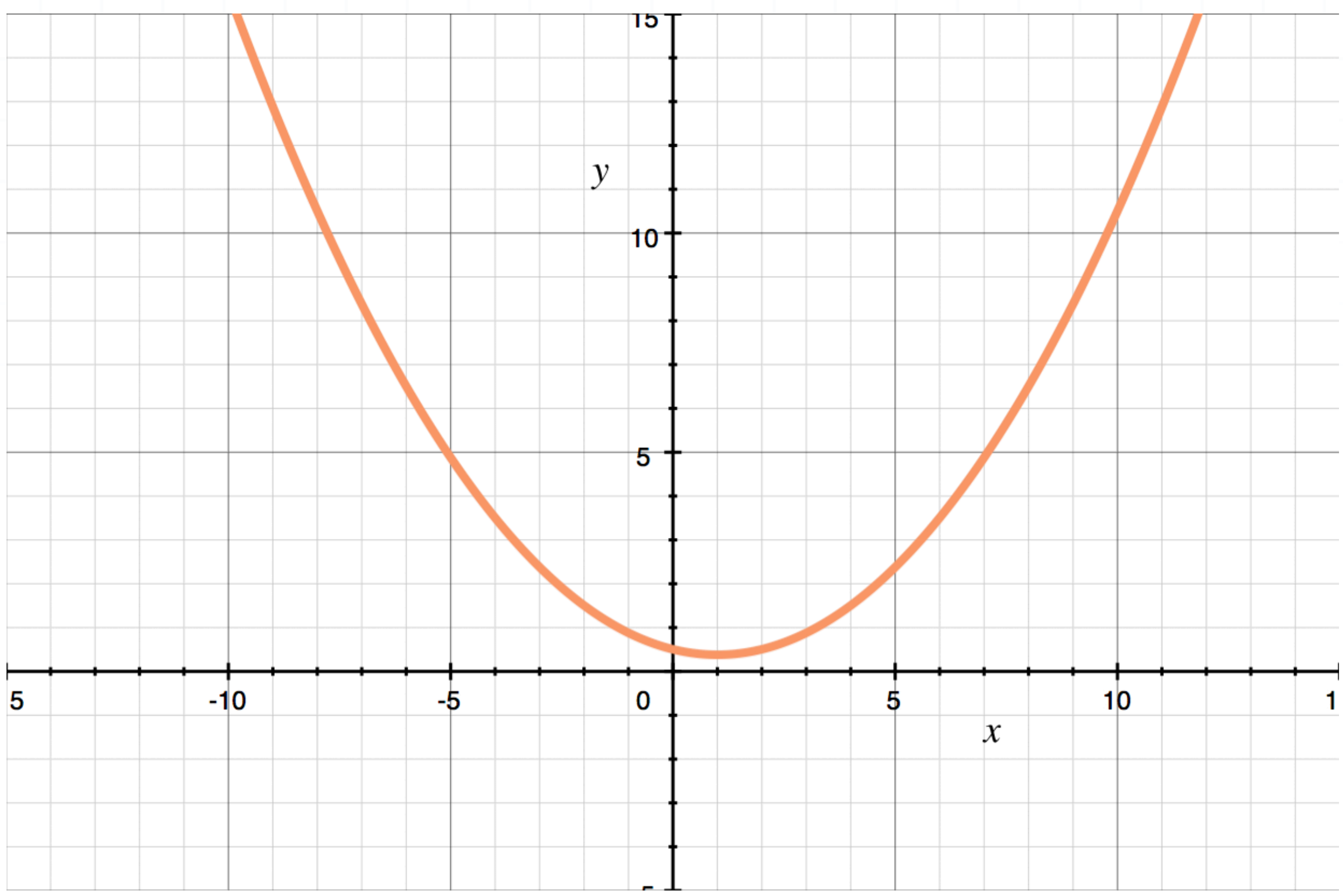
D Minimum of $\frac{35}{4}$ Maximum of $\frac{45}{8}$

Solution: C

The question asks us to estimate the minimum area and maximum area under this curve on the interval $[1,6]$, using rectangular approximation methods with 5 equal subintervals. We will use the left rectangular approximation method (LRAM) and the right rectangular approximation method (RRAM) to accomplish this task.

$$f(x) = \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{2}$$

A graph of $f(x)$ is shown below.



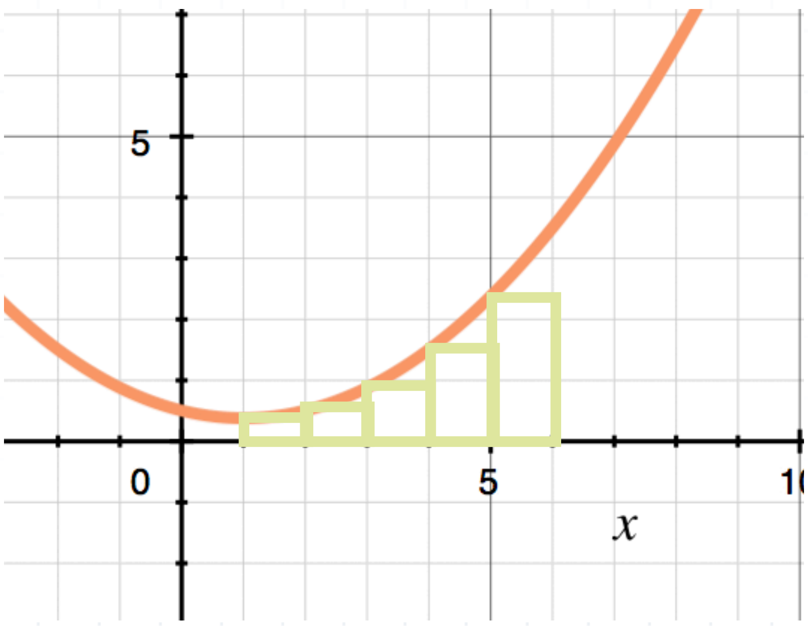
A quick observation of the graph shows that $f(x)$ is an increasing function everywhere on the interval $[1,6]$.

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each

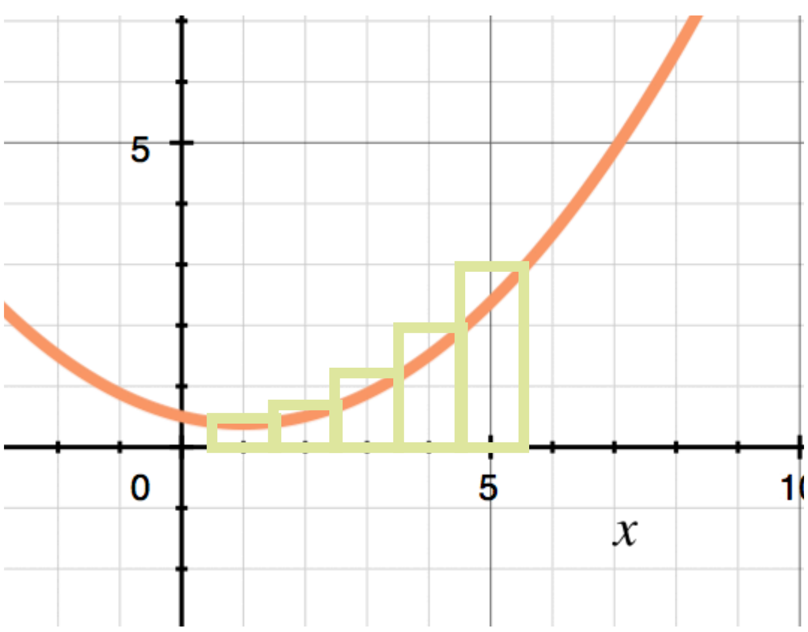
rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

In the LRAM, the height of each rectangle is the function value at the x-value at the left end of each subinterval. In the RRAM, the height of each rectangle is the function value at the x-value at the right end of each subinterval. Since the function is consistently increasing, the function values will be lower in the LRAM than in the RRAM.

Therefore, the LRAM will underestimate the area under the curve,



and the RRAM will overestimate the area under the curve.



Now let's calculate the heights of the endpoints of each subinterval. We begin with the interval $[1,6]$ and make 5 equal subintervals. Thus, we will calculate the value of $f(x)$ at $x = 1,2,3,4,5,$ and 6. The values are in the table below, followed by the work.

x	1	2	3	4	5	6
$f(x)$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{2}$	$\frac{19}{8}$	$\frac{7}{2}$

$$f(1) = \frac{1}{8}(1)^2 - \frac{1}{4}(1) + \frac{1}{2} = \frac{1}{8} - \frac{1}{4} + \frac{1}{2} = \frac{3}{8}$$

$$f(2) = \frac{1}{8}(2)^2 - \frac{1}{4}(2) + \frac{1}{2} = \frac{4}{8} - \frac{2}{4} + \frac{1}{2} = \frac{1}{2}$$

$$f(3) = \frac{1}{8}(3)^2 - \frac{1}{4}(3) + \frac{1}{2} = \frac{9}{8} - \frac{3}{4} + \frac{1}{2} = \frac{7}{8}$$

$$f(4) = \frac{1}{8}(4)^2 - \frac{1}{4}(4) + \frac{1}{2} = \frac{16}{8} - \frac{4}{4} + \frac{1}{2} = \frac{3}{2}$$

$$f(5) = \frac{1}{8}(5)^2 - \frac{1}{4}(5) + \frac{1}{2} = \frac{25}{8} - \frac{5}{4} + \frac{1}{2} = \frac{19}{8}$$

$$f(6) = \frac{1}{8}(6)^2 - \frac{1}{4}(6) + \frac{1}{2} = \frac{36}{8} - \frac{6}{4} + \frac{1}{2} = \frac{7}{2}$$

The LRAM uses the function values at the left endpoints of each subinterval, at $x = 1, 2, 3, 4,$ and 5. The width of each subinterval is 1, so the area of each rectangle is the function value. The area under the curve is the sum of the areas of the five rectangles.

$$LRAM = \frac{3}{8} + \frac{1}{2} + \frac{7}{8} + \frac{3}{2} + \frac{19}{8} = \frac{45}{8}$$

The RRAM uses the function values at the right endpoints of each subinterval, at $x = 2, 3, 4, 5,$ and 6. The width of each subinterval is still 1, so the area of each rectangle is the function value. The area under the curve is the sum of the areas of the five rectangles.

$$RRAM = \frac{1}{2} + \frac{7}{8} + \frac{3}{2} + \frac{19}{8} + \frac{7}{2} = \frac{35}{4}$$

Now that we know the minimum area under the curve is $45/8$ and the maximum area under the curve is $35/4$.

Topic: Over and underestimation

Question: Use a Riemann sum to estimate the maximum area and minimum area under the curve on $[0,10]$, using 5 equal subintervals.

$$g(x) = 3(0.85)^x$$

Answer choices:

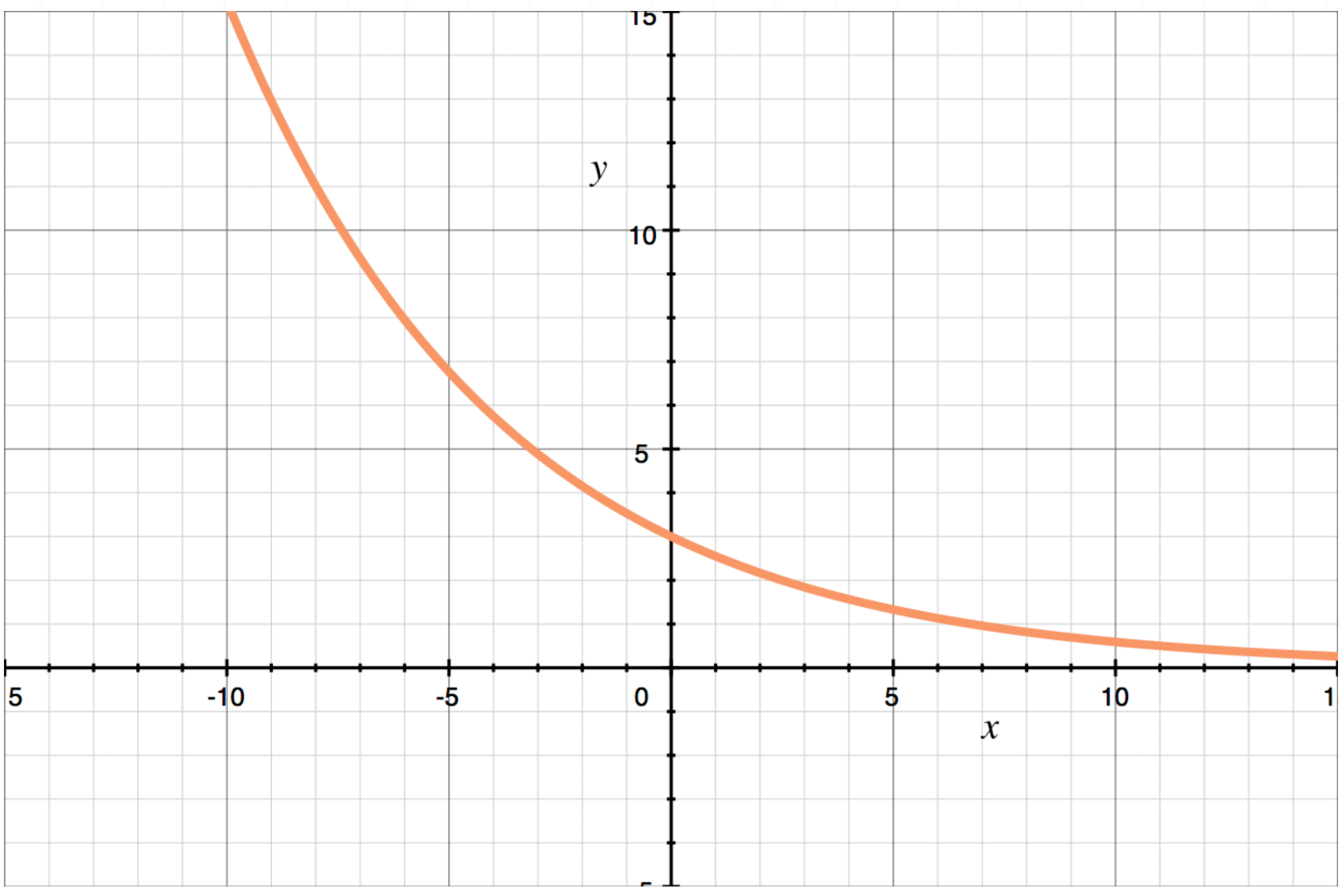
A	Minimum of 17.365	Maximum of 18.546
B	Minimum of 12.546	Maximum of 17.365
C	Minimum of 8.683	Maximum of 9.273
D	Minimum of 6.273	Maximum of 12.273

Solution: B

The question asks us to estimate the minimum area and maximum area under this curve on the interval $[0,10]$, using rectangular approximation methods with 5 equal subintervals. We will use the left rectangular approximation method (LRAM) and the right rectangular approximation method (RRAM) to accomplish this task.

$$g(x) = 3(0.85)^x$$

A graph of $g(x)$ is shown below.



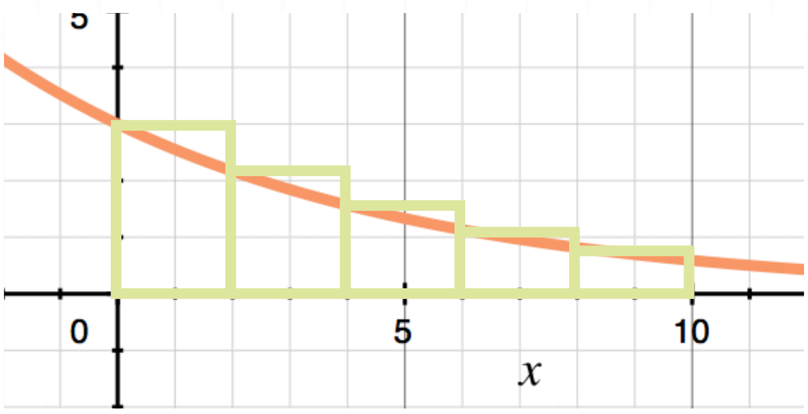
A quick observation of the graph shows that $g(x)$ is a decreasing function everywhere on the interval $[0,10]$.

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each

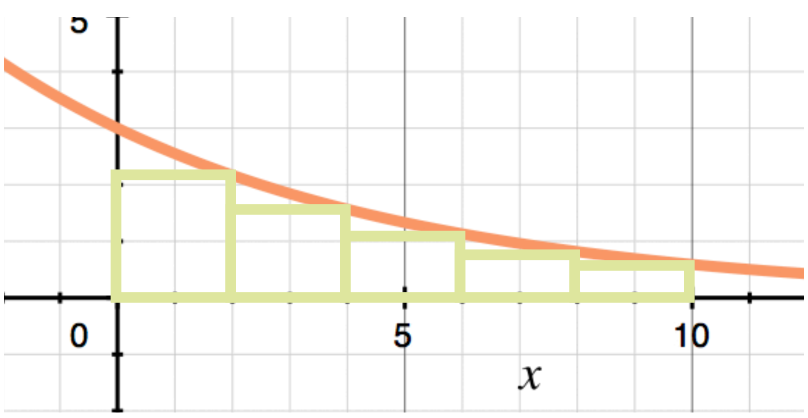
rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

In the LRAM, the height of each rectangle is the function value at the x-value at the left end of each subinterval. In the RRAM, the height of each rectangle is the function value at the x-value at the right end of each subinterval. Since the function is consistently decreasing, the function values will be higher in the LRAM than in the RRAM.

Therefore, the LRAM will overestimate the area under the curve,



and the RRAM will underestimate the area under the curve.



Now we'll calculate the height of the function at the endpoints of each subinterval. When we divide $[0,10]$ into 5 equal subintervals, the endpoints of the subintervals are at $x=0$, 2, 4, 6, 8, and 10, and the height of the function at each endpoint is

$$g(0) = 3(0.85)^0 = 3(1) = 3$$

$$g(2) = 3(0.85)^2 = 3(0.7225) = 2.1675$$

$$g(4) = 3(0.85)^4 = 3(0.5220) = 1.5660$$

$$g(6) = 3(0.85)^6 = 3(0.37715) = 1.1314$$

$$g(8) = 3(0.85)^8 = 3(0.27249) = 0.8175$$

$$g(10) = 3(0.85)^{10} = 3(0.19687) = 0.5906$$

The LRAM uses the function values at the left endpoints of each subinterval, at $x = 0, 2, 4, 6,$ and 8. The width of each subinterval is 2, so the area of each rectangle is two times of the function's value. The area under the curve is the sum of the areas of the five rectangles.

$$LRAM = 2(3) + 2(2.1675) + 2(1.5660) + 2(1.1314) + 2(0.8175) = 17.365$$

The RRAM uses the function values at the right endpoints of each subinterval, at $x = 2, 4, 6, 8,$ and 10. The width of each subinterval is still 2, so the area of each rectangle is two times of the function's value. The area under the curve is the sum of the areas of the five rectangles.

$$RRAM = 2(2.1675) + 2(1.5660) + 2(1.1314) + 2(0.8175) + 2(0.5906) = 12.546$$

Now that we know the minimum area under the curve is 12.546 and the maximum area under the curve is 17.365.

Topic: Over and underestimation

Question: Use a Riemann sum to estimate the maximum area and minimum area under this curve on $[0,64]$. Use rectangular approximation methods with 8 unequal subintervals, beginning and ending at x-values that are perfect square numbers.

$$h(x) = 2\sqrt{x}$$

Answer choices:

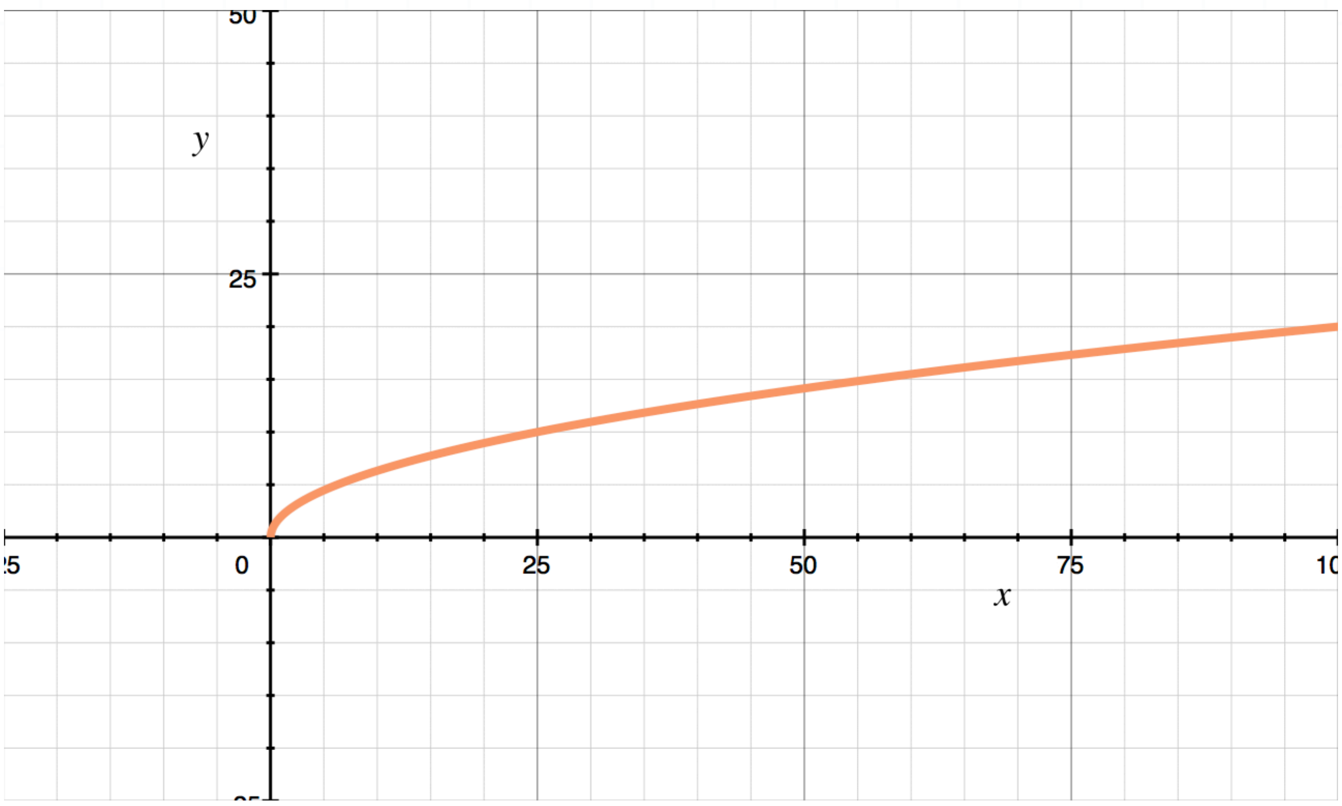
A	Minimum of 299	Maximum of 373
B	Minimum of 305	Maximum of 373
C	Minimum of 599	Maximum of 745
D	Minimum of 616	Maximum of 744

Solution: D

The question asks us to estimate the minimum area and maximum area under this curve on the interval $[0,64]$, using rectangular approximation methods with 8 unequal subintervals. The intervals are to begin and end with perfect square numbers. We will use the Left Rectangular Approximation Method (LRAM) and the Right Rectangular Approximation Method (RRAM) to accomplish this task.

$$h(x) = 2\sqrt{x}$$

A graph of $h(x)$ is shown below.



A quick observation of the graph shows that $h(x)$ is an increasing function everywhere on the interval $[0,64]$.

The term rectangular approximation method means we will approximate the area under the curve using rectangles. We calculate the area of each

rectangle by multiplying the height of the rectangle (the function value) times the width of the rectangle (the length of the subinterval).

In the LRAM, the height of each rectangle is the function value at the x-value at the left end of each subinterval. In the RRAM, the height of each rectangle is the function value at the x-value at the right end of each subinterval. Since the function is consistently increasing, the function values will be lower in the LRAM than in the RRAM.

Therefore, the LRAM will underestimate the area under the curve, and the RRAM will overestimate the area under the curve.

Now let's calculate the heights of the endpoints of each subinterval. We begin with the interval $[0,64]$ and make 8 unequal subintervals that begin and end with perfect square numbers. Thus, we will calculate the value of $h(x)$ at $x = 0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49$ and 64. The values are in the table below.

x	0	1	4	9	16	25	36	49	64	
$\sqrt{x}$	0	1	2	3	4	5	6	7	8	
$h(x)$	0	2	4	6	8		10	12	14	16

Next, we'll calculate the area of the rectangle in each subdivision, using the left endpoint of each subdivision (LRAM). Each subinterval is identified as $[a, b]$. The height of the rectangle is $h(a)$ at the appropriate endpoint of the subinterval. The width of the subinterval is $b - a$, the upper endpoint minus the lower endpoint. The area of the rectangle is the product of the height and the width, $h(a)(b - a)$, in each subinterval. The figures are shown in the table below.

$[a,b]$	$[0,1]$	$[1,4]$	$[4,9]$	$[9,16]$	$[16,25]$	$[25,36]$	$[36,49]$	$[49,64]$
$h(a)$	0	2	4	6	8	10	12	14
$b-a$	1	3	5	7	9	11	13	15
$h(a)(b-a)$	0	6	20	42	72	110	156	210

Since $h(x)$ is strictly increasing on $[0,64]$, the minimum area under the curve is the sum of the areas of the 8 rectangles.

$$LRAM = 0 + 6 + 20 + 42 + 72 + 110 + 156 + 210 = 616$$

Next, we'll calculate the area of the rectangle in each subdivision, using the right endpoint of each subdivision (RRAM). Each subinterval is identified as $[a,b]$. The height of the rectangle is $h(b)$ at the appropriate endpoint of the subinterval. The width of the subinterval is $b-a$, the upper endpoint minus the lower endpoint. The area of the rectangle is the product of the height and the width, $h(b)(b-a)$, in each subinterval. The figures are shown in the table below.

$[a,b]$	$[0,1]$	$[1,4]$	$[4,9]$	$[9,16]$	$[16,25]$	$[25,36]$	$[36,49]$	$[49,64]$
$h(b)$	2	4	6	8	10	12	14	16
$b-a$	1	3	5	7	9	11	13	15
$h(b)(b-a)$	2	12	30	56	90	132	182	240

Since $h(x)$ is strictly increasing on $[0,64]$, the maximum area under the curve is the sum of the areas of the 8 rectangles.

$$LRAM = 2 + 12 + 30 + 56 + 90 + 132 + 182 + 240 = 744$$

Now we that know the minimum area under the curve is 616 and the maximum area under the curve is 744.

Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the area of the region between the function and the x-axis over the given interval.

$$f(x) = 4 - x^2$$

on the interval $[1,2]$

Answer choices:

A $\frac{5}{3}$

B $\frac{3}{5}$

C $\frac{32}{3}$

D $-\frac{3}{5}$

Solution: A

The question asks us to find the area between the function $f(x) = 4 - x^2$ and the x-axis over the interval $[1,2]$. A quick look at the graph of $f(x)$ reveals that the function is above the x-axis over the entire interval. Thus, we can clearly expect a positive answer for the area.

We know that the limit process to find an area in an interval is

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a, b]$, divided into n subdivisions,

$$\Delta x = \frac{b - a}{n}$$

The interval is $[1,2]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x = \frac{2 - 1}{n} = \frac{1}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n}$$

Next, substitute the $f(x)=4-x^2$ into the summation, recalling that the x -value is $1+(i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{i}{n}\right)^2\right]\frac{1}{n}$$

Next, square the expression in the summation, and since the summation is in terms of i, remove $1/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[4-\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\right]$$

Distribute the negative inside the summation.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(4-1-\frac{2i}{n}-\frac{i^2}{n^2}\right)$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty} \frac{1}{n}\left[4n - 1n - \frac{2}{n} \times \frac{n(n+1)}{2} - \frac{1}{n^2} \times \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{1}{n}\left[3n - (n+1) - \frac{(n+1)(2n+1)}{6n}\right]$$

Distribute and multiply to remove the parentheses.

$$\lim_{n\to\infty} \frac{1}{n}\left[3n - n - 1 - \frac{2n^2 + 3n + 1}{6n}\right]$$

Distribute the negative and separate the fraction into individual terms.

$$\lim_{n\to\infty} \frac{1}{n}\left[2n - 1 - \frac{2n^2}{6n} - \frac{3n}{6n} - \frac{1}{6n}\right]$$

$$\lim_{n\to\infty} \left[\frac{2n}{n} - \frac{1}{n} - \frac{2n^2}{6n^2} - \frac{3n}{6n^2} - \frac{1}{6n^2}\right]$$

$$\lim_{n\to\infty} \left[2 - \frac{1}{n} - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right]$$

Take the limit.

$$2 - 0 - \frac{1}{3} - 0 - 0 = \frac{5}{3}$$

The area between $f(x) = 4 - x^2$ and the x-axis in the interval $[1,2]$ is $5/3$.

Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the net area of the region between the function and the x-axis over the given interval.

$$f(x) = 4 - x^2$$

on the interval $[1,3]$

Answer choices:

A $\frac{2}{3}$

B 4

C $-\frac{2}{3}$

D $\frac{3}{2}$

Solution: C

The question asks us to find the area between the function $f(x) = 4 - x^2$ and the x-axis over the interval $[1,3]$. A quick look at the graph of $f(x)$ reveals that the function is above the x-axis on the interval $[1,2]$, but below the x-axis on the interval $[2,3]$. Thus, we can expect a net area as an answer. If the area below the x-axis is larger than the area above the x-axis, then the net area will be a negative value.

We know that the limit process to find an area in an interval is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a, b]$, divided into n subdivisions,

$$\Delta x = \frac{b-a}{n}$$

The interval is $[1,3]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}$$

Next, substitute the $f(x)=4-x^2$ into the summation, recalling that the x -value is $1+(2i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[4-\left(1+\frac{2i}{n}\right)^2\right]\frac{2}{n}$$

Next, square the expression in the summation, and since the summation is in terms of i, remove $2/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[4-\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\right]$$

Distribute the negative inside the summation.

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(4-1-\frac{4i}{n}-\frac{4i^2}{n^2}\right)$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\displaystyle\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty} \frac{2}{n}\left[4n - 1n - \frac{4}{n} \times \frac{n(n+1)}{2} - \frac{4}{n^2} \times \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{2}{n}\left[3n - 2(n+1) - \frac{2}{3n}(n+1)(2n+1)\right]$$

Distribute and multiply to remove the parentheses.

$$\lim_{n\to\infty} \frac{2}{n}\left[3n - 2n - 2 - \frac{2}{3n}\left(2n^2 + 3n + 1\right)\right]$$

Distribute the negative and separate the fraction into individual terms.

$$\lim_{n\to\infty} \frac{2}{n}\left[n - 2 - \frac{4n^2}{3n} - \frac{6n}{3n} - \frac{2}{3n}\right]$$

$$\lim_{n\to\infty} \frac{2}{n}\left[n - 2 - \frac{4n}{3} - 2 - \frac{2}{3n}\right]$$

$$\lim_{n\to\infty} \left[\frac{2n}{n} - \frac{4}{n} - \frac{8n}{3n} - \frac{4}{n} - \frac{4}{3n^2}\right]$$

$$\lim_{n \to \infty} \left[2 - \frac{4}{n} - \frac{8}{3} - \frac{4}{n} - \frac{4}{3n^2} \right]$$

Take the limit.

$$2 - 0 - \frac{8}{3} - 0 - 0 = -\frac{2}{3}$$

The area between $f(x) = 4 - x^2$ and the x-axis in the interval $[1,3]$ is $-2/3$. This means the area below the x-axis was larger than the area above the x-axis.

Topic: Limit process to find area on [a,b]

Question: Use the limit process to find the net area of the region between the function and the x-axis over the given interval.

$$g(x) = x^2 - 5x + 7$$

on the interval $[1,4]$

Answer choices:

A $-\frac{8}{3}$

B $\frac{8}{3}$

C $-\frac{9}{2}$

D $\frac{9}{2}$

Solution: D

The question asks us to find the area between the function $g(x) = x^2 - 5x + 7$ and the x-axis over the interval $[1,4]$. A quick look at the graph of $g(x)$ reveals that the function is above the x-axis in the entire interval $[1,4]$. Thus, we can clearly expect a positive answer for the area.

We know that the limit process to find an area in an interval is

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\ \Delta x$$

We must find Δx and x_i.

Step 1: Recall that in the interval $[a, b]$, divided into n subdivisions,

$$\Delta x = \frac{b-a}{n}$$

The interval is $[1,4]$. Divide the region in the interval into n rectangles to find Δx.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

Step 2: Find x_i by adding the left bound to $i\Delta x$. The left bound is 1.

$$x_i = 1 + i\Delta x = 1 + \frac{3i}{n}$$

Step 3: Write the limit of the sum to find the area. The formula is

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\ \Delta x$$

Using the information from Step 1 and Step 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} g\left(1+\frac{3i}{n}\right)\frac{3}{n}$$

Next, substitute the $g(x)=x^2-5x+7$ into the summation, recalling that the x-value is $1+(3i/n)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2-5\left(1+\frac{3i}{n}\right)+7\right]\frac{3}{n}$$

Next, square the first expression in the summation, distribute the middle term, and since the summation is in terms of i, remove $3/n$ and place it in front of the summation.

$$\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1+\frac{6i}{n}+\frac{9i^2}{n^2}-5-\frac{15i}{n}+7\right]$$

Now, recall that in calculating the summation involving i, we have the following identities.

For a constant term, $\sum_{i=1}^{n} a = an$

For a term containing i, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For a term containing i^2, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Substitute these expressions into the limit of the summation, taking the sum.

$$\lim_{n\to\infty}\frac{3}{n}\left[n+\frac{6}{n}\times\frac{n(n+1)}{2}+\frac{9}{n^2}\times\frac{n(n+1)(2n+1)}{6}-5n-\frac{15}{n}\times\frac{n(n+1)}{2}+7n\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left[n+3(n+1)+\frac{3}{2n}(n+1)(2n+1)-5n-\frac{15}{2}(n+1)+7n\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left[n+3n+3+\frac{3}{2n}(2n^2+3n+1)-5n-\frac{15n}{2}-\frac{15}{2}+7n\right]$$

$$\lim_{n\to\infty}\frac{3}{n}\left(n+3n+3+\frac{6n^2}{2n}+\frac{9n}{2n}+\frac{3}{2n}-5n-\frac{15n}{2}-\frac{15}{2}+7n\right)$$

$$\lim_{n\to\infty}\frac{3}{n}\left(n+3n+3+3n+\frac{9}{2}+\frac{3}{2n}-5n-\frac{15n}{2}-\frac{15}{2}+7n\right)$$

Consolidate the n terms, then the constants, and then distribute the $3/n$ across the terms in the parentheses.

$$\lim_{n\to\infty}\frac{3}{n}\left(\frac{3n}{2}+\frac{3}{2n}+3+\frac{9}{2}-\frac{15}{2}\right)$$

$$\lim_{n\to\infty}\frac{3}{n}\left(\frac{3n}{2}+\frac{3}{2n}\right)$$

$$\lim_{n\to\infty}\frac{9n}{2n}+\frac{9}{2n^2}$$

$$\lim_{n\to\infty}\frac{9}{2}+\frac{9}{2n^2}$$

Take the limit.

$$\frac{9}{2}+0$$

$$\frac{9}{2}$$

The area between $g(x) = x^2 - 5x + 7$ and the x-axis on the interval $[1,4]$ is $9/2$.

Topic: Limit process to find area on [-a,a]

Question: Use the limit process to find the area of the region between $f(x) = 9 - x^2$ and the x-axis on the interval $[-3,3]$.

Answer choices:

A 92

B 18

C 54

D 36

Solution: D

The function is even because $f(x)=f(-x)$. So instead of calculating area under the curve over the interval $[-3,3]$, we'll calculate area over $[0,3]$ and then double the result.

$$\Delta x=\frac{b-a}{n}=\frac{3-0}{n}=\frac{3}{n}$$

Find x_i by adding $i\Delta x$ to the left edge of the interval, 0.

$$x_i=0+i\Delta x=0+\frac{3i}{n}=\frac{3i}{n}$$

Set up the limit expression,

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[9-\left(\frac{3i}{n}\right)^2\right]\frac{3}{n}$$

then simplify.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{27}{n}-\frac{3}{n}\left(\frac{3i}{n}\right)^2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{27}{n}-\frac{3}{n}\left(\frac{9i^2}{n^2}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{27}{n}-\frac{27i^2}{n^3}$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{27}{n}-\sum_{i=1}^{n}\frac{27i^2}{n^3}\right]$$

$$\lim_{n\to\infty}\left[\frac{27}{n}\sum_{i=1}^{n}1-\frac{27}{n^3}\sum_{i=1}^{n}i^2\right]$$

For values of i, we know

$$\sum_{i=1}^{n}a=an$$

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$$

Use these equations to make substitutions.

$$\lim_{n\to\infty}\left[\frac{27}{n}(n)-\frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$\lim_{n\to\infty}27-\frac{9(n+1)(2n+1)}{2n^2}$$

$$\lim_{n\to\infty}27-\frac{9(2n^2+3n+1)}{2n^2}$$

$$\lim_{n\to\infty} 27 - \frac{18n^2 + 27n + 9}{2n^2}$$

Split up the fraction,

$$\lim_{n\to\infty} 27 - \frac{18n^2}{2n^2} - \frac{27n}{2n^2} - \frac{9}{2n^2}$$

$$\lim_{n\to\infty} 27 - 9 - \frac{27}{2n} - \frac{9}{2n^2}$$

$$\lim_{n\to\infty} 18 - \frac{27}{2n} - \frac{9}{2n^2}$$

then evaluate the limit.

$$18 - 0 - 0$$

$$18$$

This is the area over the interval $[0,4]$, so the area over the interval $[-4,4]$ is double this value, which is 36.

Topic: Limit process to find area on [-a,a]

Question: Use the limit process to find the area of the region between $f(x) = x^2 + 2$ and the x-axis on the interval $[-1,1]$.

Answer choices:

A $\frac{14}{3}$

B $\frac{2}{3}$

C $\frac{28}{3}$

D $\frac{8}{3}$

Solution: A

The function is even because $f(x) = f(-x)$. So instead of calculating area under the curve over the interval $[-1,1]$, we'll calculate area over $[0,1]$ and then double the result.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Find x_i by adding $i\Delta x$ to the left edge of the interval, 0.

$$x_i = 0 + i\Delta x = 0 + \frac{1i}{n} = \frac{i}{n}$$

Set up the limit expression,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + 2\right]\frac{1}{n}$$

then simplify.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\frac{1}{n} + \frac{2}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} + \frac{2}{n}$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{i^2}{n^3}+\sum_{i=1}^{n}\frac{2}{n}\right]$$

$$\lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n}i^2+\frac{2}{n}\sum_{i=1}^{n}1\right]$$

For values of i, we know

$$\sum_{i=1}^{n}a=an$$

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$$

Use these equations to make substitutions.

$$\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{2}{n}(n)$$

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}+2$$

$$\lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}+2$$

Split up the fraction,

$$\lim_{n\to\infty}\frac{2n^2}{6n^2}+\frac{3n}{6n^2}+\frac{1}{6n^2}+2$$

$$\lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} + 2$$

$$\lim_{n\to\infty} \frac{7}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

then evaluate the limit.

$$\frac{7}{3} + 0 + 0$$

$$\frac{7}{3}$$

This is the area over the interval $[0,1]$, so the area over the interval $[-1,1]$ is double this value, which is $14/3$.

Topic: Limit process to find area on [-a,a]

Question: Use the limit process to find the area of the region between $f(x) = x^3 + 4$ and the x-axis on the interval $[-2,2]$.

Answer choices:

A 12

B 16

C 20

D 24

Solution: B

First, find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{n} = \frac{4}{n}$$

Find x_i by adding $i\Delta x$ to the left edge of the interval, 0.

$$x_i = -2 + i\Delta x = -2 + \frac{4i}{n}$$

Set up the limit expression,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{4i}{n}\right)\frac{4}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-2+\frac{4i}{n}\right)^3 + 4\right)\frac{4}{n}$$

then simplify.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(-2+\frac{4i}{n}\right)^3 \frac{4}{n} + \frac{16}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(4-\frac{16i}{n}+\frac{16i^2}{n^2}\right)\left(-2+\frac{4i}{n}\right)\frac{4}{n} + \frac{16}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(-8+\frac{16i}{n}+\frac{32i}{n}-\frac{64i^2}{n^2}-\frac{32i^2}{n^2}+\frac{64i^3}{n^3}\right)\frac{4}{n}+\frac{16}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}-\frac{32}{n}+\frac{64i}{n^2}+\frac{128i}{n^2}-\frac{256i^2}{n^3}-\frac{128i^2}{n^3}+\frac{256i^3}{n^4}+\frac{16}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}-\frac{16}{n}+\frac{192i}{n^2}-\frac{384i^2}{n^3}+\frac{256i^3}{n^4}$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}-\frac{16}{n}+\sum_{i=1}^{n}\frac{192i}{n^2}-\sum_{i=1}^{n}\frac{384i^2}{n^3}+\sum_{i=1}^{n}\frac{256i^3}{n^4}\right]$$

$$\lim_{n\to\infty}\left[-\frac{16}{n}\sum_{i=1}^{n}1+\frac{192}{n^2}\sum_{i=1}^{n}i-\frac{384}{n^3}\sum_{i=1}^{n}i^2+\frac{256}{n^4}\sum_{i=1}^{n}i^3\right]$$

For values of i, we know

$$\sum_{i=1}^{n}a=an$$

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n}i^3=\frac{n^2(n+1)^2}{4}$$

Use these equations to make substitutions.

$$\lim_{n\to\infty}\left[-\frac{16}{n}(n)+\frac{192}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{384}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right.$$

$$\left.+\frac{256}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$\lim_{n\to\infty}-16+\frac{96(n+1)}{n}-\frac{64(n+1)(2n+1)}{n^2}+\frac{64(n+1)^2}{n^2}$$

$$\lim_{n\to\infty}-16+\frac{96n+96}{n}-\frac{64(2n^2+3n+1)}{n^2}+\frac{64(n^2+2n+1)}{n^2}$$

$$\lim_{n\to\infty}-16+\frac{96n+96}{n}-\frac{128n^2+192n+64}{n^2}+\frac{64n^2+128n+64}{n^2}$$

Split up the fraction,

$$\lim_{n\to\infty}-16+\frac{96n}{n}+\frac{96}{n}-\frac{128n^2}{n^2}-\frac{192n}{n^2}-\frac{64}{n^2}+\frac{64n^2}{n^2}+\frac{128n}{n^2}+\frac{64}{n^2}$$

$$\lim_{n\to\infty}-16+96+\frac{96}{n}-128-\frac{192}{n}-\frac{64}{n^2}+64+\frac{128}{n}+\frac{64}{n^2}$$

then evaluate the limit.

$$-16+96+0-128-0-0+64+0+0$$

$$-16+96-128+64$$

$$16$$

Topic: Trapezoidal rule

Question: Use trapezoidal rule to approximate the integral.

$$\int_0^1 \sqrt{2 - x^2}\ dx$$

when $n = 4$

Answer choices:

A 7.66

B 1.28

C 4.20

D 1.67

Solution: B

Trapezoidal rule is another tool we can use to approximate the area under a function over a set interval $a \leq x \leq b$.

Instead of dividing the area into rectangles, as we did with Riemann sums, we'll divide the area into trapezoids and then sum the areas of all of the trapezoids in order to get an approximation of area. The greater the number of trapezoids, the more accurate the approximation will be. Of course, if we use an infinite number of trapezoids, taking the limit as $n \to \infty$ of the sum of the area of each trapezoid, then we'd be taking the integral and calculating exact area.

When we approximate area with Trapezoidal rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Trapezoidal rule formula is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and Δx is the width of each trapezoid, and where n is the number of trapezoids we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Trapezoidal rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. All the other terms between them are multiplied by 2. That's because the formula for the area of a

trapezoid uses the length of both bases of the trapezoid. Since the right base of every trapezoid in our approximation is the same line as the left base of the trapezoid next to it, every base except the first and last ones get used twice in our approximation.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{1-0}{4}$$

$$\Delta x = \frac{1}{4}$$

Since the interval is $[0,1]$, we know that $x_0 = 0$ and that $x_n = 1$. Using $\Delta x = 1/4$ to find the subintervals, we get

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{4} \qquad x_1 = \frac{1}{4}$$

$$x_2 = \frac{1}{4} + \frac{1}{4} \qquad x_2 = \frac{1}{2}$$

$$x_3 = \frac{1}{2} + \frac{1}{4} \qquad x_3 = \frac{3}{4}$$

$$x_4 = \frac{3}{4} + \frac{1}{4} \qquad x_4 = \frac{4}{4} \qquad x_4 = 1$$

Plugging all of this into the Trapezoidal rule formula, remembering that $f(x) = \sqrt{2 - x^2}$, we get

$$\frac{\frac{1}{4}}{2}\left[\sqrt{2 - 0^2} + 2\sqrt{2 - \left(\frac{1}{4}\right)^2} + 2\sqrt{2 - \left(\frac{1}{2}\right)^2} + 2\sqrt{2 - \left(\frac{3}{4}\right)^2} + \sqrt{2 - 1^2}\right]$$

$$\frac{1}{8}\left(\sqrt{2} + 2\sqrt{2 - \frac{1}{16}} + 2\sqrt{2 - \frac{1}{4}} + 2\sqrt{2 - \frac{9}{16}} + \sqrt{2 - 1}\right)$$

$$\frac{1}{8}\left(\sqrt{2} + 2\sqrt{\frac{32}{16} - \frac{1}{16}} + 2\sqrt{\frac{8}{4} - \frac{1}{4}} + 2\sqrt{\frac{32}{16} - \frac{9}{16}} + \sqrt{1}\right)$$

$$\frac{1}{8}\left(\sqrt{2} + 2\sqrt{\frac{31}{16}} + 2\sqrt{\frac{7}{4}} + 2\sqrt{\frac{23}{16}} + 1\right)$$

$$\frac{1}{8}\left(\sqrt{2} + \frac{2\sqrt{31}}{4} + \frac{2\sqrt{7}}{2} + \frac{2\sqrt{23}}{4} + 1\right)$$

$$\frac{1}{8}\left(\frac{2\sqrt{2}}{2} + \frac{\sqrt{31}}{2} + \frac{2\sqrt{7}}{2} + \frac{\sqrt{23}}{2} + \frac{2}{2}\right)$$

$$\frac{1}{8}\left(\frac{2\sqrt{2} + \sqrt{31} + 2\sqrt{7} + \sqrt{23} + 2}{2}\right)$$

$$\frac{2\sqrt{2}+\sqrt{31}+2\sqrt{7}+\sqrt{23}+2}{16}$$

1.28

Topic: Trapezoidal rule

Question: Use trapezoidal rule to approximate the integral.

$$\int_0^2 x^2 \, dx$$

when $n = 4$

Answer choices:

A $\frac{11}{4}$

B $\frac{8}{3}$

C $-\frac{11}{4}$

D $-\frac{8}{3}$

Solution: A

Trapezoidal rule is another tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

Instead of dividing the area into rectangles, as we did with Riemann sums, we'll divide the area into trapezoids and then sum the areas of all of the trapezoids in order to get an approximation of area. The greater the number of trapezoids, the more accurate the approximation will be. Of course, if we use an infinite number of trapezoids, taking the limit as $n \to \infty$ of the sum of the area of each trapezoid, then we'd be taking the integral and calculating exact area.

When we approximate area with Trapezoidal rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Trapezoidal rule formula is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and Δx is the width of each trapezoid, and where n is the number of trapezoids we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Trapezoidal rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. All the other terms between them are multiplied by 2. That's because the formula for the area of a

trapezoid uses the length of both bases of the trapezoid. Since the right base of every trapezoid in our approximation is the same line as the left base of the trapezoid next to it, every base except the first and last ones get used twice in our approximation.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-0}{4}$$

$$\Delta x = \frac{1}{2}$$

Since the interval is $[0,2]$, we know that $x_0 = 0$ and that $x_n = 2$. Using $\Delta x = 1/2$ to find the subintervals, we get

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{2} \qquad x_1 = \frac{1}{2}$$

$$x_2 = \frac{1}{2} + \frac{1}{2} \qquad x_2 = 1$$

$$x_3 = 1 + \frac{1}{2} \qquad x_3 = \frac{3}{2}$$

$$x_4 = \frac{3}{2} + \frac{1}{2} \qquad x_4 = 2$$

Plugging all of this into the Trapezoidal rule formula, remembering that $f(x) = x^2$, we get

$$\frac{\frac{1}{2}}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$\frac{\frac{1}{2}}{2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + f(2)\right]$$

$$\frac{\frac{1}{2}}{2}\left[(0)^2 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right]$$

$$\frac{1}{4}\left[2\left(\frac{1}{4}\right) + 2 + 2\left(\frac{9}{4}\right) + 4\right]$$

$$\frac{1}{4}\left(\frac{1}{2} + 2 + \frac{9}{2} + 4\right)$$

$$\frac{1}{4}\left(6 + \frac{10}{2}\right)$$

$$\frac{6}{4} + \frac{10}{8}$$

$$\frac{6}{4} + \frac{5}{4}$$

$$\frac{11}{4}$$

Topic: Trapezoidal rule

Question: Use trapezoidal rule to approximate the integral.

$$\int_1^4 x^3 - 4 \ dx$$

when $n = 2$

Answer choices:

A $-\frac{963}{16}$

B 16

C -16

D $\frac{963}{16}$

Solution: D

Trapezoidal rule is another tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

Instead of dividing the area into rectangles, as we did with Riemann sums, we'll divide the area into trapezoids and then sum the areas of all of the trapezoids in order to get an approximation of area. The greater the number of trapezoids, the more accurate the approximation will be. Of course, if we use an infinite number of trapezoids, taking the limit as $n \to \infty$ of the sum of the area of each trapezoid, then we'd be taking the integral and calculating exact area.

When we approximate area with Trapezoidal rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Trapezoidal rule formula is

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and Δx is the width of each trapezoid, and where n is the number of trapezoids we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Trapezoidal rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. All the other terms between them are multiplied by 2. That's because the formula for the area of a

trapezoid uses the length of both bases of the trapezoid. Since the right base of every trapezoid in our approximation is the same line as the left base of the trapezoid next to it, every base except the first and last ones get used twice in our approximation.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{4-1}{2}$$

$$\Delta x = \frac{3}{2}$$

Since the interval is $[1,4]$, we know that $x_0 = 1$ and that $x_n = 4$. Using $\Delta x = 3/2$ to find the subintervals, we get

$$x_0 = 1$$

$$x_1 = 1 + \frac{3}{2} \qquad x_1 = \frac{5}{2}$$

$$x_2 = \frac{5}{2} + \frac{3}{2} \qquad x_2 = \frac{8}{2} \qquad x_2 = 4$$

Plugging all of this into the Trapezoidal rule formula, remembering that $f(x) = x^3 - 4$, we get

$$\frac{3}{4}\left[f(x_0)+2f(x_1)+f(x_2)\right]$$

$$\frac{3}{4}\left[f(1)+2f\left(\frac{5}{2}\right)+f(4)\right]$$

$$\frac{3}{4}\left[(1)^3-4+2\left[\left(\frac{5}{2}\right)^3-4\right]+(4)^3-4\right]$$

$$\frac{3}{4}\left[1-4+2\left(\frac{125}{8}-4\right)+64-4\right]$$

$$\frac{3}{4}\left(1-4+\frac{125}{4}-8+64-4\right)$$

$$\frac{3}{4}\left(49+\frac{125}{4}\right)$$

$$\frac{147}{4}+\frac{375}{16}$$

$$\frac{588}{16}+\frac{375}{16}$$

$$\frac{963}{16}$$

Topic: Simpson's rule

Question: Use Simpson's rule to approximate the area under the curve.

$$\int_1^2 x^3\ dx$$

when $n = 2$

Answer choices:

A -3.75

B 3.57

C 3.75

D -3.57

Solution: C

Simpson's rule is another tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

Just as we did with Riemann sums, we'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. When we use Simpson's rule, the number of rectangles n must be an even number. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Simpson's rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Simpson's rule formula is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b-a)/n$ and Δx is the width of each rectangle, and where n is even and the number of rectangles we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Simpson's rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. The second and second-to-last

terms are multiplied by 4, and the third and third-to-last terms are multiplied by 2. All of the terms in between from $f(x_3)$ to $f(x_{n-3})$ alternate coefficients starting with 4, so

$$\dots + 4f(x_3) + 2f(x_4) + 4f(x_5) + \dots + 4f(x_{n-5}) + 2f(x_{n-4}) + 4f(x_{n-3}) + \dots$$

The number of rectangles n must be even in order for the pattern to work out correctly in this way.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-1}{2}$$

$$\Delta x = \frac{1}{2}$$

Since the interval is $[1,2]$, we know that $x_0 = 1$ and that $x_n = 2$. Using $\Delta x = 1/2$ to find the subintervals, we get

$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{2} \qquad x_1 = \frac{3}{2}$$

$$x_2 = \frac{3}{2} + \frac{1}{2} \qquad x_2 = 2$$

Plugging all of this into the Simpson's rule formula, remembering that $f(x) = x^3$, we get

$$S_n = \frac{\frac{1}{2}}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$S_n = \frac{\frac{1}{2}}{3}\left[(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right]$$

$$S_n = \frac{1}{6}\left(1 + \frac{27}{2} + 8\right)$$

$$S_n = \frac{1}{6}\left(\frac{45}{2}\right)$$

$$S_n = \frac{45}{12}$$

$$S_n = \frac{15}{4}$$

$$S_n = 3.75$$

Topic: Simpson's rule

Question: Use Simpson's rule to approximate the area under the curve.

$$\int_2^6 4\sqrt{x} - 1 \ dx$$

when $n = 4$

Answer choices:

A -27.684

B 27.684

C -27.648

D 27.648

Solution: D

Simpson's rule is another tool we can use to approximate the area under a function over a set interval $a \leq x \leq b$.

Just as we did with Riemann sums, we'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. When we use Simpson's rule, the number of rectangles n must be an even number. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Simpson's rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Simpson's rule formula is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is even and the number of rectangles we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Simpson's rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. The second and second-to-last

terms are multiplied by 4, and the third and third-to-last terms are multiplied by 2. All of the terms in between from $f(x_3)$ to $f(x_{n-3})$ alternate coefficients starting with 4, so

$$... + 4f(x_3) + 2f(x_4) + 4f(x_5) + ... + 4f(x_{n-5}) + 2f(x_{n-4}) + 4f(x_{n-3}) + ...$$

The number of rectangles n must be even in order for the pattern to work out correctly in this way.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{6-2}{4}$$

$$\Delta x = 1$$

Since the interval is $[2,6]$, we know that $x_0 = 2$ and that $x_n = 6$. Using $\Delta x = 1$ to find the subintervals, we get

$$x_0 = 2$$

$$x_1 = 2 + 1 \qquad x_1 = 3$$

$$x_2 = 3 + 1 \qquad x_2 = 4$$

$$x_3 = 4 + 1 \qquad x_3 = 5$$

$x_4 = 5 + 1 \qquad\qquad x_4 = 6$

Plugging all of this into the Simpson's rule formula, remembering that $f(x) = 4\sqrt{x} - 1$, we get

$$S_n = \frac{1}{3}\left[\left(4\sqrt{2} - 1\right) + 4\left(4\sqrt{3} - 1\right) + 2\left(4\sqrt{4} - 1\right) + 4\left(4\sqrt{5} - 1\right) + \left(4\sqrt{6} - 1\right)\right]$$

$$S_n = \frac{1}{3}\left[4\sqrt{2} - 1 + 16\sqrt{3} - 4 + 8\sqrt{4} - 2 + 16\sqrt{5} - 4 + 4\sqrt{6} - 1\right]$$

$$S_n = \frac{1}{3}\left[4\sqrt{2} + 16\sqrt{3} + 16\sqrt{5} + 4\sqrt{6} + 4\right]$$

$$S_n = 27.648$$

Topic: Simpson's rule

Question: Use Simpson's rule to approximate the area under the curve.

$$\int_1^4 -x^2+3\ dx$$

when $n=4$

Answer choices:

A -12

B 13

C 12

D -13

Solution: A

Simpson's rule is another tool we can use to approximate the area under a function over a set interval $a \le x \le b$.

Just as we did with Riemann sums, we'll divide the area into rectangles and then sum the areas of all of the rectangles in order to get an approximation of area. When we use Simpson's rule, the number of rectangles n must be an even number. The greater the number of rectangles, the more accurate the approximation will be. Of course, if we use an infinite number of rectangles, taking the limit as $n \to \infty$ of the sum of the area of each rectangle, then we'd be taking the integral and calculating exact area.

When we approximate area with Simpson's rule we consider the area above the x-axis to be positive, and the area below the x-axis to be negative. If our final result is positive, it tells us that there's more area above the x-axis than below it. On the other hand, if our final result is negative, it means that there's more area below the x-axis than above it.

The Simpson's rule formula is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and Δx is the width of each rectangle, and where n is even and the number of rectangles we're using to approximate area, and where $[a, b]$ is the given interval.

Notice in the Simpson's rule formula that the first and last terms $f(x_0)$ and $f(x_n)$ are not multiplied by any coefficient. The second and second-to-last

terms are multiplied by 4, and the third and third-to-last terms are multiplied by 2. All of the terms in between from $f(x_3)$ to $f(x_{n-3})$ alternate coefficients starting with 4, so

$$... + 4f(x_3) + 2f(x_4) + 4f(x_5) + ... + 4f(x_{n-5}) + 2f(x_{n-4}) + 4f(x_{n-3}) + ...$$

The number of rectangles n must be even in order for the pattern to work out correctly in this way.

Our plan is to solve for Δx, divide the interval into even segments that are each Δx wide, and then use the right endpoint of each segment as the values of x_n. Plugging the interval and the value of n we've been given into the formula for Δx, we get

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{4-1}{4}$$

$$\Delta x = \frac{3}{4}$$

Since the interval is $[1,4]$, we know that $x_0 = 1$ and that $x_n = 4$. Using $\Delta x = 3/4$ to find the subintervals, we get

$$x_0 = 1$$

$$x_1 = 1 + \frac{3}{4} \qquad x_1 = \frac{7}{4}$$

$$x_2 = \frac{7}{4} + \frac{3}{4} \qquad x_2 = \frac{10}{4} \qquad x_2 = \frac{5}{2}$$

$$x_3 = \frac{10}{4} + \frac{3}{4} \qquad x_3 = \frac{13}{4}$$

$$x_4 = \frac{13}{4} + \frac{3}{4} \qquad x_4 = \frac{16}{4} \qquad x_4 = 4$$

Plugging all of this into the Simpson's rule formula, remembering that $f(x) = -x^2 + 3$, we get

$$S_n = \frac{\frac{3}{4}}{3}\left\{\left[-(1)^2+3\right] + 4\left[-\left(\frac{7}{4}\right)^2 + 3\right] + 2\left[-\left(\frac{10}{4}\right)^2 + 3\right] + 4\left[-\left(\frac{13}{4}\right)^2 + 3\right] + \left[-(4)^2+3\right]\right\}$$

$$S_n = \frac{1}{4}\left(2 - \frac{1}{4} - \frac{13}{2} - \frac{121}{4} - 13\right)$$

$$S_n = \frac{1}{4}(-48)$$

$$S_n = -12$$

Topic: Midpoint rule error bound

Question: Calculate the area under the curve. Then, use the Midpoint Rule with $n = 4$ to approximate the same area. Compare the actual area to the result to determine the error of the Midpoint Rule approximation of the area.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

Answer choices:

A Actual area is $\frac{44}{3}$ $MRAM_4$ is $\frac{115}{8}$ Error is $\frac{7}{24}$

B Actual area is $\frac{115}{8}$ $MRAM_4$ is $\frac{44}{3}$ Error is $\frac{7}{24}$

C Actual area is $\frac{32}{3}$ $MRAM_4$ is $\frac{21}{2}$ Error is $\frac{1}{6}$

D Actual area is $\frac{1,315}{128}$ $MRAM_4$ is $\frac{32}{3}$ Error is $\frac{151}{384}$

Solution: C

The question asks us to calculate the area under the curve, and then approximate the same area using the Midpoint Rule, with $n = 4$, and compare the results by identifying the error.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

From the integral, we know that the function is

$$f(x) = x^3 + x^2 + x + 1$$

Let's begin by integrating $f(x)$ using the power rule and evaluating the integral. This will give the actual area under the curve.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

$$\left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x\right)\Bigg|_0^2$$

$$\left(\frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + 2\right) - \left(\frac{0^4}{4} + \frac{0^3}{3} + \frac{0^2}{2} + 0\right)$$

$$\frac{16}{4} + \frac{8}{3} + \frac{4}{2} + 2 = \frac{32}{3}$$

Now, we'll estimate the area under the curve using the Midpoint Rule, with $n = 4$. The table below shows the interval divided into 4 subintervals, the midpoint of each interval, and the function values at each midpoint. The work is shown below the table.

$[a,b]$	$[0,0.5]$	$[0.5,1]$	$[1,1.5]$	$[1.5,2]$
$a+\dfrac{b-a}{2}$	0.25	0.75	1.25	1.75
$f\left(a+\dfrac{b-a}{2}\right)$	$\dfrac{85}{64}$	$\dfrac{175}{64}$	$\dfrac{369}{64}$	$\dfrac{715}{64}$

$$f\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^2+\frac{1}{4}+1=\frac{85}{64}$$

$$f\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^2+\frac{3}{4}+1=\frac{175}{64}$$

$$f\left(\frac{5}{4}\right)=\left(\frac{5}{4}\right)^3+\left(\frac{5}{4}\right)^2+\frac{5}{4}+1=\frac{369}{64}$$

$$f\left(\frac{7}{4}\right)=\left(\frac{7}{4}\right)^3+\left(\frac{7}{4}\right)^2+\frac{7}{4}+1=\frac{715}{64}$$

The subinterval widths are all $1/2$, so to find the Midpoint Rule approximation of the area, multiply each function value by $1/2$ and add the areas.

$$\left(\frac{85}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{175}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{369}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{715}{64}\right)\left(\frac{1}{2}\right)=\frac{1{,}344}{128}=\frac{21}{2}$$

The error is the actual area minus the estimated area.

$$\frac{32}{3} - \frac{21}{2} = \frac{1}{6}$$

Topic: Midpoint rule error bound

Question: Calculate the error bound of the Midpoint Rule, with $n = 4$.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

Answer choices:

A $\quad |E_M| \leq \dfrac{5}{23}$

B $\quad |E_M| \leq \dfrac{4}{21}$

C $\quad |E_M| \leq \dfrac{7}{24}$

D $\quad |E_M| \leq \dfrac{5}{24}$

Solution: C

The question asks us to calculate the error bound of the Midpoint Rule, with $n = 4$, for the area under the curve.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

To find the error bound of the Midpoint Rule on the interval $[a, b]$, we use this formula.

$$|E_M| \leq k\frac{(b-a)^3}{24n^2}$$

Where $|E_M|$ denotes the maximum error of the Midpoint Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

First, let's find k. The value k is often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$f(x) = x^3 + x^2 + x + 1$$

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

The second derivative, $f''(x)$, will never equal 0 on $[0,2]$, so to find k, find the maximum value on the interval. Evaluate $f''(x)$ at the endpoints.

$$f''(0) = 2, \ \ f''(2) = 14, \ \ k = 14$$

Now in the expression

$$|E_M| \leq k\frac{(b-a)^3}{24n^2}$$

$k = 14$, $a = 0$, $b = 2$, and $n = 4$.

Evaluate the error bound.

$$k\frac{(b-a)^3}{24n^2} = (14)\frac{(2-0)^3}{(24)(4)^2} = \frac{(14)(8)}{(24)(16)} = \frac{7}{24}$$

Therefore,

$$|E_M| \leq \frac{7}{24}$$

Topic: Midpoint rule error bound

Question: Find n to get the accuracy of the Midpoint Rule of the approximation of the area under the curve to be within 0.00001.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

Answer choices:

A $\quad n \geq 683$

B $\quad n \geq 684$

C $\quad n \geq 632$

D $\quad n \geq 633$

Solution: B

The question asks us to find n to get the accuracy of the Midpoint Rule to within 0.00001.

$$\int_0^2 x^3 + x^2 + x + 1 \ dx$$

To find the error bound of the Midpoint Rule on the interval $[a, b]$, we use this formula.

$$|E_M| \leq k\frac{(b-a)^3}{24n^2}$$

Where $|E_M|$ denotes the maximum error of the Midpoint Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

First, let's find k. The value k if often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$f(x) = x^3 + x^2 + x + 1$$

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

The second derivative, $f''(x)$, will never equal 0 on $[0,2]$, so to find k, find the maximum value on the interval. Evaluate $f''(x)$ at the endpoints.

$$f''(0) = 2, \ \ f''(2) = 14, \ \ k = 14$$

Now in the expression

$$|E_M| \le k\frac{(b-a)^3}{24n^2}$$

$k = 14$, $a = 0$, and $b = 2$.

We will find the value of n. Let's simplify the expression first.

$$|E_M| \le (14)\frac{(2-0)^3}{24n^2}$$

$$|E_M| \le \frac{14(8)}{24n^2}$$

$$|E_M| \le \frac{14}{3n^2}$$

Since we want the error to be less than 0.00001, we set the maximum error bound expression to be less than 0.00001.

$$\frac{14}{3n^2} \le 0.00001$$

Multiply by $3n^2$ and divide by 0.00003.

$$\frac{14}{3n^2} \le 0.00001$$

$$14 \le (0.00003)n^2$$

$$\frac{14}{0.00003} \le n^2$$

Square root both sides of the inequality, ignoring the possibility that n could be negative.

$$\sqrt{\frac{14}{0.00003}} \le \sqrt{n^2}$$

$$n \ge 683.1300511$$

We found an interval for n. However, since n is the number of subintervals, n has to be a whole number. Thus, to be accurate to within 0.0001, $n \ge 684$.

Topic: Trapezoidal rule error bound

Question: Calculate the area under the curve. Then, use the Trapezoidal Rule, with $n = 6$, to approximate the same area. Compare the actual area to the result to determine the error of the Trapezoidal Rule approximation of the area.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2\ dx$$

Answer choices:

A Actual area is $\frac{135}{2}$ $TRAP_6$ is $\frac{265}{4}$ Error is $\frac{5}{4}$

B Actual area is $\frac{265}{4}$ $TRAP_6$ is $\frac{135}{2}$ Error is $\frac{5}{4}$

C Actual area is $\frac{10,597}{160}$ $TRAP_6$ is $\frac{1,353}{20}$ Error is $\frac{227}{160}$

D Actual area is $\frac{1,353}{20}$ $TRAP_6$ is $\frac{10,597}{160}$ Error is $\frac{227}{160}$

Solution: D

The question asks us to calculate the area under the curve, and then approximate the same area using the trapezoidal Rule, with $n=6$, and compare the results by identifying the error.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

From the integral, the function is

$$g(x)=-\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2$$

Let's begin by integrating $g(x)$ using the power rule and evaluating the integral. This will give the actual area under the curve.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

$$\left(-\frac{x^6}{15}+\frac{7x^4}{12}+\frac{5x^3}{3}+2x^2+2x\right)\Bigg|_0^3$$

$$-\frac{(3)^6}{15}+\frac{7(3)^4}{12}+\frac{5(3)^3}{3}+2(3)^2+2(3)-\left[-\frac{0^6}{15}+\frac{7(0^4)}{12}+\frac{5(0^3)}{3}+2(0^2)+2(0)\right]$$

$$-\frac{243}{5}+\frac{567}{12}+\frac{135}{3}+18+6=\frac{1{,}353}{20}$$

Now, we'll estimate the area under the curve using the Trapezoidal Rule, with $n=6$. The table below shows the interval $[0,3]$ divided into 6

subintervals, and the function values at each point. The work is shown below the table.

x	0	0.5	1	1.5	2	2.5	3
$g(x)$	2	$\frac{1,327}{240}$	$\frac{194}{15}$	$\frac{1,927}{80}$	$\frac{538}{15}$	$\frac{1,951}{48}$	$\frac{124}{5}$

For $g(0)$:

$$g(0)=-\frac{2}{5}(0)^5+\frac{7}{3}(0)^3+5(0)^2+4(0)+2=2$$

For $g(1/2)$:

$$g\left(\frac{1}{2}\right)=-\frac{2}{5}\left(\frac{1}{2}\right)^5+\frac{7}{3}\left(\frac{1}{2}\right)^3+5\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)+2$$

$$g\left(\frac{1}{2}\right)=-\frac{2}{5}\left(\frac{1}{32}\right)+\frac{7}{3}\left(\frac{1}{8}\right)+5\left(\frac{1}{4}\right)+4\left(\frac{1}{2}\right)+2$$

$$g\left(\frac{1}{2}\right)=-\frac{1}{80}+\frac{7}{24}+\frac{5}{4}+2+2$$

$$g\left(\frac{1}{2}\right)=\frac{1,327}{240}$$

For $g(1)$:

$$g(1)=-\frac{2}{5}(1)^5+\frac{7}{3}(1)^3+5(1)^2+4(1)+2=-\frac{2}{5}+\frac{7}{3}+5+4+2=\frac{194}{15}$$

For $g(3/2)$:

$$g\left(\frac{3}{2}\right)=-\frac{2}{5}\left(\frac{3}{2}\right)^5+\frac{7}{3}\left(\frac{3}{2}\right)^3+5\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+2$$

$$g\left(\frac{3}{2}\right)=-\frac{2}{5}\left(\frac{243}{32}\right)+\frac{7}{3}\left(\frac{27}{8}\right)+5\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+2$$

$$g\left(\frac{3}{2}\right)=-\frac{243}{80}+\frac{63}{8}+\frac{45}{4}+6+2$$

$$g\left(\frac{3}{2}\right)=\frac{1{,}927}{80}$$

For $g(2)$:

$$g(2)=-\frac{2}{5}(2)^5+\frac{7}{3}(2)^3+5(2)^2+4(2)+2$$

$$g(2)=-\frac{2}{5}(32)+\frac{7}{3}(8)+5(4)+4(2)+2$$

$$g(2)=-\frac{64}{5}+\frac{56}{3}+20+8+2$$

$$g(2)=\frac{538}{15}$$

For $g(5/2)$:

$$g\left(\frac{5}{2}\right)=-\frac{2}{5}\left(\frac{5}{2}\right)^5+\frac{7}{3}\left(\frac{5}{2}\right)^3+5\left(\frac{5}{2}\right)^2+4\left(\frac{5}{2}\right)+2$$

$$g\left(\frac{5}{2}\right)=-\frac{2}{5}\left(\frac{3125}{32}\right)+\frac{7}{3}\left(\frac{125}{8}\right)+5\left(\frac{25}{4}\right)+4\left(\frac{5}{2}\right)+2$$

$$g\left(\frac{5}{2}\right) = -\frac{625}{16} + \frac{875}{24} + \frac{125}{4} + 10 + 2$$

$$g\left(\frac{5}{2}\right) = \frac{1{,}951}{48}$$

For $g(3)$:

$$g(3) = -\frac{2}{5}(3)^5 + \frac{7}{3}(3)^3 + 5(3)^2 + 4(3) + 2$$

$$g(3) = -\frac{2}{5}(243) + \frac{7}{3}(27) + 5(9) + 4(3) + 2$$

$$g(3) = -\frac{486}{5} + 63 + 45 + 12 + 2$$

$$g(3) = \frac{124}{5}$$

The general rule for the Trapezoidal Rule approximation of the area is

$$A = \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + \ \dots \ + 2f(x_{n-1}) + f(x_n)\right]$$

The subinterval widths are all $1/2$, so $\Delta x = 1/2$. To find the Trapezoidal Rule approximation of the area, insert each function value in the table into the general Trapezoidal Rule.

$$A = \frac{1}{4}\left[2 + 2\left(\frac{1{,}327}{240}\right) + 2\left(\frac{194}{15}\right) + 2\left(\frac{1{,}927}{80}\right) + 2\left(\frac{538}{15}\right) + 2\left(\frac{1{,}951}{48}\right) + \frac{124}{5}\right]$$

$$A = \frac{1}{4}\left(2 + \frac{1{,}327}{120} + \frac{388}{15} + \frac{1{,}927}{40} + \frac{1{,}076}{15} + \frac{1{,}951}{24} + \frac{124}{5}\right)$$

$$A = \frac{1}{4}\left(\frac{10{,}597}{40}\right)$$

$$A = \frac{10{,}597}{160}$$

The error is the actual area minus the estimated area.

$$\frac{1{,}353}{20} - \frac{10{,}597}{160} = \frac{227}{160}$$

Topic: Trapezoidal rule error bound

Question: Calculate the error bound of the Trapezoidal Rule, with $n=6$.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

Answer choices:

A $\quad |E_T| \leq 10.25$

B $\quad |E_T| \leq 10.06$

C $\quad |E_T| \leq 11.13$

D $\quad |E_T| \leq 10.008$

Solution: A

The question asks us to calculate the error bound of the Trapezoidal Rule, with $n = 6$, for the area under the curve.

$$\int_0^3 -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2\ dx$$

To find the error bound of the Trapezoidal Rule on the interval $[a, b]$, we use this formula.

$$|E_T| \le k\frac{(b-a)^3}{12n^2}$$

Where $|E_T|$ denotes the maximum error of the Trapezoidal Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

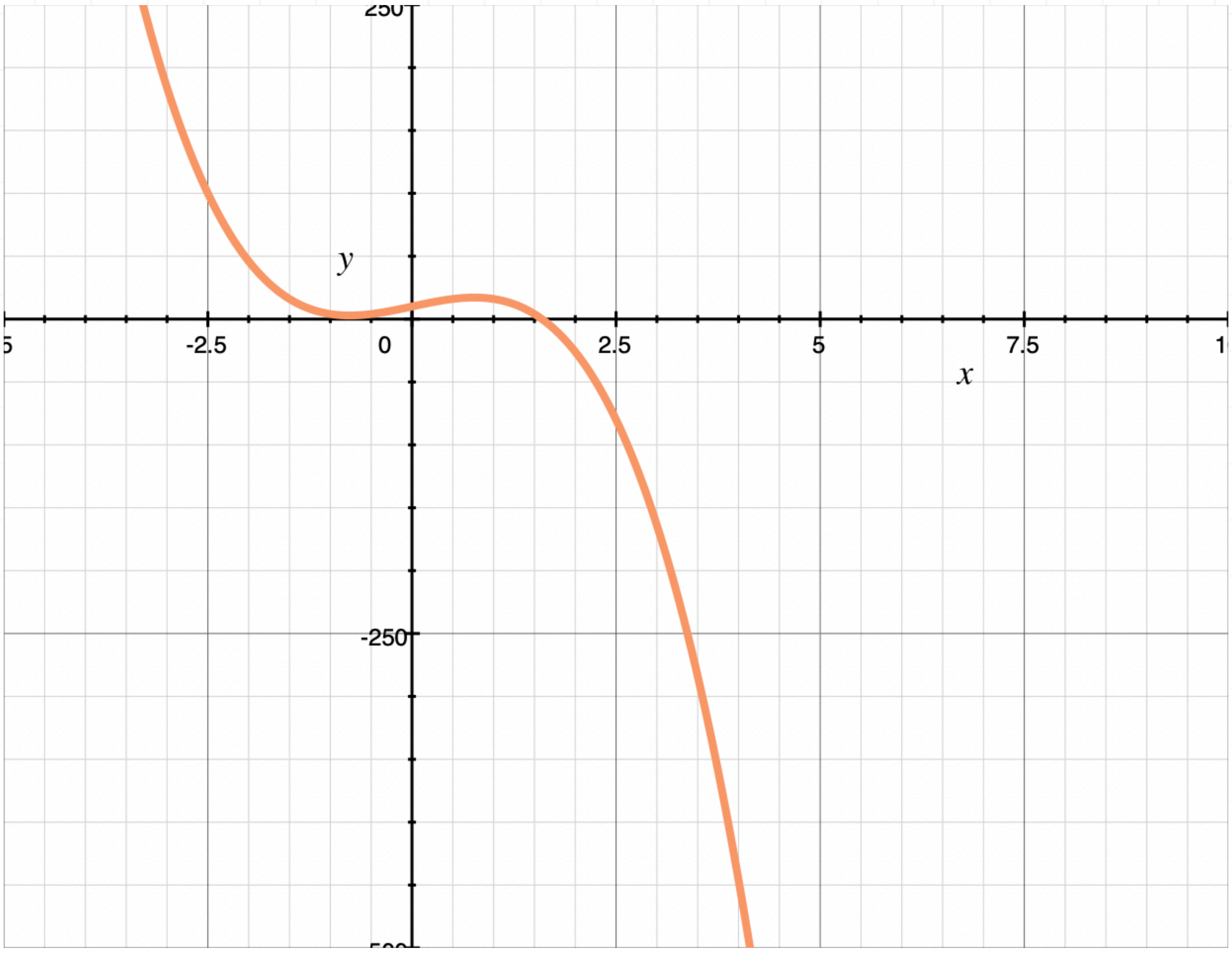
First, let's find k. The value k if often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$g(x) = -\frac{2}{5}x^5 + \frac{7}{3}x^3 + 5x^2 + 4x + 2$$

$$g'(x) = -\frac{2}{5}(5)x^4 + \frac{7}{3}(3)x^2 + 5(2)x^1 + 4 = -2x^4 + 7x^2 + 10x + 4$$

$$g''(x) = -8x^3 + 14x + 10$$

The graph of $g''(x)$ is shown below.



The second derivative, $g''(x)$, will reach its maximum absolute value at the point $(3, -164)$, so the value of $M_{f''}$ is 164.

$$g''(0) = 10, \ \ g''(3) = -164, \ \ k = 164$$

Now in the expression

$$|E_T| \le k\frac{(b-a)^3}{12n^2}$$

$k = 164$, $a = 0$, $b = 3$ and $n = 6$. Evaluate the error bound.

$$k\frac{(b-a)^3}{2412n^2} = (164)\frac{(3-0)^3}{12(6)^2} = \frac{(164)(27)}{(12)(36)} = 10.25$$

Therefore,

$$|E_T| \leq 10.25$$

Topic: Trapezoidal rule error bound

Question: Find n to get the accuracy of the Trapezoidal Rule to within 0.00001.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

Answer choices:

A $\quad n=6{,}073$

B $\quad n=6{,}072$

C $\quad n=6{,}074$

D $\quad n=6{,}075$

Solution: D

The question asks us to find n to get the accuracy of the Trapezoidal Rule to within 0.00001.

$$\int_0^3 -\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2\ dx$$

To find the error bound of the Midpoint Rule on the interval $[a,b]$, we use this formula.

$$|E_T| \le k\frac{(b-a)^3}{12n^2}$$

Where $|E_T|$ denotes the maximum error of the Trapezoidal Rule, k is a constant based on the function, which we will find, a is the lower limit of the interval, b is the upper limit of the interval, and n is the number of subintervals.

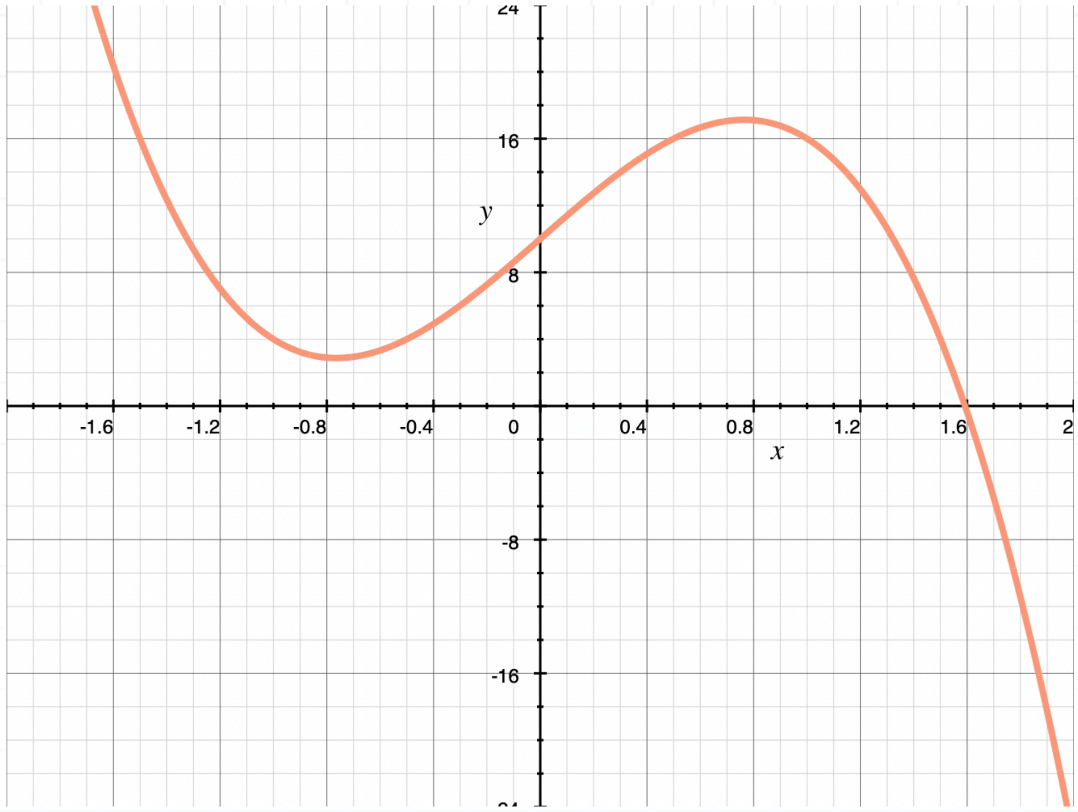
First, let's find k. The value k is often denoted by the notation $M_{f''}$ which means the maximum absolute value of the function's second derivative in the interval. Let's find k for the function and interval in this problem.

$$g(x)=-\frac{2}{5}x^5+\frac{7}{3}x^3+5x^2+4x+2$$

$$g'(x)=-\frac{2}{5}(5)x^4+\frac{7}{3}(3)x^2+105+4=-2x^4+7x^2+105+4$$

$$g''(x)=-8x^3+14x+10$$

The graph of $g''(x)$ is shown below.



The second derivative $g''(x)$ will reach its maximum value at $(0.7637,17.1284)$ but its maximum absolute value is at the point $(3, -164)$, so the value of $M_{f''}$ is 164.

$$g''(0) = 10, \ \ g''(3) = -164, \ \ k = 164$$

Now in the expression

$$|E_T| \leq k\frac{(b-a)^3}{12n^2}$$

$k = 164$, $a = 0$ and $b = 3$. We'll find the value of n. Let's simplify the expression first.

$$|E_T| \leq (164)\frac{(3-0)^3}{12n^2}$$

$$|E_T| \leq \frac{(164)(27)}{12n^2}$$

$$|E_T| \leq \frac{4,428}{12n^2}$$

$$|E_T| \leq \frac{369}{n^2}$$

Since we want the error to be less than 0.00001, we set the maximum error bound expression to be less than 0.00001.

$$\frac{369}{n^2} \leq 0.00001$$

Multiply by n^2 and divide by 0.00001.

$$369 \leq (0.00001)n^2$$

$$\frac{369}{0.00001} \leq n^2$$

Square root both sides of the inequality, ignoring the possibility that n could be negative.

$$\sqrt{\frac{369}{0.00001}} \leq \sqrt{n^2}$$

$$n \geq 6,074.54$$

We found an interval for n. However, since n is the number of subintervals, n has to be a whole number. Thus, to be accurate to within 0.0001, $n = 6,075$.

Topic: Simpson's rule error bound

Question: Calculate area under the curve on the interval $[0,4]$. Then use Simpson's Rule with $n=4$ to approximate area on the same interval and determine the error of the Simpson's Rule approximation.

$$f(x)=\frac{1}{2}(e^x+e^{-x})$$

Answer choices:

A	$A \approx 27.2899$	$\vert E_S \vert \approx 0.1354$
B	$A \approx 27.2899$	$\vert E_S \vert \approx 0.0192$
C	$A \approx 27.3091$	$\vert E_S \vert \approx 0.1354$
D	$A \approx 27.3091$	$\vert E_S \vert \approx 0.0192$

Solution: A

Find actual area under the curve on the interval $[0,4]$.

$$A=\int_0^4 \frac{1}{2}(e^x+e^{-x})\ dx$$

$$A=\frac{1}{2}\int_0^4 e^x+e^{-x}\ dx$$

$$A=\frac{1}{2}(e^x-e^{-x})\Big|_0^4$$

$$A=\frac{1}{2}\left[(e^4-e^{-4})-(e^0-e^{-0})\right]$$

$$A=\frac{1}{2}(e^4-e^{-4}-(1-1))$$

$$A=\frac{1}{2}\left(e^4-\frac{1}{e^4}\right)$$

$$A\approx 27.2899$$

With $n=4$,

$$\Delta x=\frac{b-a}{n}=\frac{4-0}{4}=\frac{4}{4}=1$$

The subinterval widths are all 1, so the boundaries of the subintervals are

$$x_0=0 \qquad x_1=1 \qquad x_2=2 \qquad x_3=3 \qquad x_4=4$$

Evaluate the function at each of these boundaries.

$$f(0)=\frac{1}{2}\left(e^0+\frac{1}{e^0}\right)=1$$

$$f(1)=\frac{1}{2}\left(e+\frac{1}{e}\right)\approx 1.5431$$

$$f(2)=\frac{1}{2}\left(e^2+\frac{1}{e^2}\right)\approx 3.7622$$

$$f(3)=\frac{1}{2}\left(e^3+\frac{1}{e^3}\right)\approx 10.0677$$

$$f(4)=\frac{1}{2}\left(e^4+\frac{1}{e^4}\right)\approx 27.3082$$

Then Simpson's Rule approximates the area as

$$S_n=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\ \cdots\ +2f(x_{x-2})+4f(x_{x-1})+f(x_n)\right]$$

$$S_4=\frac{\Delta x}{3}[f(0)+4f(1)+2f(2)+4f(3)+f(4)]$$

$$S_n\approx\frac{1}{3}[1+4(1.5431)+2(3.7622)+4(10.0677)+27.3082]$$

$$S_n\approx\frac{1}{3}(1+6.1724+7.5244+40.2708+27.3082)$$

$$S_n\approx\frac{1}{3}(82.2758)$$

$$S_n\approx 27.4253$$

The error is the actual area minus the estimated area, as an absolute value.

$$|E_S| \approx |27.2899 - 27.4253|$$

$$|E_S| \approx |-0.1354|$$

$$|E_S| \approx 0.1354$$

Topic: Simpson's rule error bound

Question: Calculate the error bound of Simpson's Rule for the function $f(x)$ with $n = 4$ on the interval $[0,4]$.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

Answer choices:

A $|E_S| \leq 0.3793$

B $|E_S| \leq 0.0038$

C $|E_S| \leq 1.0379$

D $|E_S| \leq 0.6068$

Solution: D

To find the error bound of Simpson's Rule on the interval $[a,b]$, we use

$$|E_S| \le k\frac{(b-a)^5}{180n^4}$$

The value of k is the maximum absolute value of the function's fourth derivative in the interval, so to find k, we'll start by finding the function's fourth derivative.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f^{(4)}(x) = \frac{1}{2}(e^x + e^{-x})$$

To determine where the fourth derivative reaches its maximum absolute value, we'll find the derivative of the fourth derivative function,

$$f^{(5)}(x) = \frac{1}{2}(e^x - e^{-x})$$

and set it equal to 0 to find critical numbers.

$$\frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x}$$

$$x = 0$$

The only critical number is at the left edge of the interval $[0,4]$, so we only need to determine whether the fourth derivative function is increasing or decreasing in the interval, which we can do by evaluating the fifth derivative at a point in the interval. We'll choose $x = 1$.

$$f^{(5)}(1) = \frac{1}{2}(e^1 - e^{-1})$$

$$f^{(5)}(1) = \frac{1}{2}(e - e^{-1})$$

$$f^{(5)}(1) \approx 1.1752$$

Because we get a positive value, the fourth derivative function is increasing everywhere on $[0,4]$, which means the fourth derivative function will reach its maximum absolute value at the far right edge of the interval, at $x = 4$.

$$k = f^{(4)}(4) = \frac{1}{2}(e^4 + e^{-4})$$

$$k = f^{(4)}(4) \approx 27.3082$$

Then the error bound formula gives

$$|E_S| \le k\frac{(b-a)^5}{180n^4}$$

$$|E_S| \le 27.3082\left(\frac{(4-0)^5}{180(4)^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4^5}{180(4)^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4}{180}\right)$$

$$|E_S| \le 27.3082\left(\frac{1}{45}\right)$$

$$|E_S| \le 0.6068$$

Topic: Simpson's rule error bound

Question: Find the smallest value of n that keeps the Simpson's rule error bound within 0.00001 for the function $f(x)$ on the interval $[0,4]$.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

Answer choices:

A $\quad n = 62$

B $\quad n = 63$

C $\quad n = 64$

D $\quad n = 62.7813$

Solution: C

To find the error bound of Simpson's Rule on the interval $[a, b]$, we use

$$|E_S| \leq k\frac{(b-a)^5}{180n^4}$$

The value of k is the maximum absolute value of the function's fourth derivative in the interval, so to find k, we'll start by finding the function's fourth derivative.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f^{(4)}(x) = \frac{1}{2}(e^x + e^{-x})$$

To determine where the fourth derivative reaches its maximum absolute value, we'll find the derivative of the fourth derivative function,

$$f^{(5)}(x) = \frac{1}{2}(e^x - e^{-x})$$

and set it equal to 0 to find critical numbers.

$$\frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x}$$

$$x = 0$$

The only critical number is at the left edge of the interval $[0,4]$, so we only need to determine whether the fourth derivative function is increasing or decreasing in the interval, which we can do by evaluating the fifth derivative at a point in the interval. We'll choose $x = 1$.

$$f^{(5)}(1) = \frac{1}{2}(e^1 - e^{-1})$$

$$f^{(5)}(1) = \frac{1}{2}(e - e^{-1})$$

$$f^{(5)}(1) \approx 1.1752$$

Because we get a positive value, the fourth derivative function is increasing everywhere on $[0,4]$, which means the fourth derivative function will reach its maximum absolute value at the far right edge of the interval, at $x = 4$.

$$k = f^{(4)}(4) = \frac{1}{2}(e^4 + e^{-4})$$

$$k = f^{(4)}(4) \approx 27.3082$$

Then the error bound formula gives

$$|E_S| \le k\frac{(b-a)^5}{180n^4}$$

$$|E_S| \le 27.3082\left(\frac{(4-0)^5}{180n^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4^5}{180n^4}\right)$$

$$|E_S| \le 27.3082\left(\frac{4^4}{45n^4}\right)$$

Since we want to bound the error at 0.00001,

$$27.3082\left(\frac{4^4}{45n^4}\right) \le 0.00001$$

$$27.3082\left(\frac{4^4}{45}\right) \le 0.00001n^4$$

$$\frac{27.3082}{0.00001}\left(\frac{4^4}{45}\right) \le n^4$$

$$\sqrt[4]{\frac{27.3082}{0.00001}\left(\frac{4^4}{45}\right)} \le n$$

$$62.7813 \le n$$

Because n is the number of subintervals, n has to be a whole number, and because we're using Simpson's rule n has to be an *even* whole number. Therefore, $n = 64$.

Topic: Part 1 of the FTC

Question: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative.

$$f(x) = \int_3^x 5t^3 + 1 \ dt$$

Answer choices:

A $\quad f'(x) = \frac{5}{4}x^4 - x$

B $\quad f'(x) = 5x^3 - 1$

C $\quad f'(x) = \frac{5}{4}x^4 + x$

D $\quad f'(x) = 5x^3 + 1$

Solution: D

We've been given an integral with a constant lower limit of integration, and the variable x as the upper limit of integration.

Given integral	**How to solve it**
$f(x) = \int_a^x f(t)\ dt$	Plug x in for t.
$f(x) = \int_x^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.
$f(x) = \int_a^{g(x)} f(t)\ dt$	Plug $g(x)$ in for t, then multiply by dg/dx.
$f(x) = \int_{g(x)}^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.
$f(x) = \int_{g(x)}^{h(x)} f(t)\ dt$	Split the limits of integration as $\int_{g(x)}^0 f(t)\ dt + \int_0^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^0 f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Looking at the chart, we can see that this is the situation described by the first row, which means the steps for solving this using FTC Part 1 are:

1. Plug x in for t.

$$f(x) = \int_3^x 5t^3 + 1 \ dt$$

$$f'(x) = 5x^3 + 1$$

Topic: Part 1 of the FTC

Question: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative.

$$f(x) = \int_{2x^2}^{6} 6t^3 - 6t \ dt$$

Answer choices:

A $\quad f'(x) = -\frac{3}{2}x^4 + 3x^2$

B $\quad f'(x) = 48x^3 - 192x^7$

C $\quad f'(x) = 192x^7 - 48x^3$

D $\quad f'(x) = \frac{3}{2}x^4 - 3x^2$

Solution: B

We've been given an integral with a function as the lower limit of integration, and a constant as the upper limit of integration.

Given integral	**How to solve it**
$f(x) = \int_{a}^{x} f(t)\ dt$	Plug x in for t.
$f(x) = \int_{x}^{a} f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.
$f(x) = \int_{a}^{g(x)} f(t)\ dt$	Plug $g(x)$ in for t, then multiply by dg/dx.
$f(x) = \int_{g(x)}^{a} f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.
$f(x) = \int_{g(x)}^{h(x)} f(t)\ dt$	Split the limits of integration as $\int_{g(x)}^{0} f(t)\ dt + \int_{0}^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^{0} f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Looking at the chart, we can see that this is the situation described by the fourth row, which means the steps for solving this using FTC Part 1 are:

1. Reverse the limits of integration and multiply by -1.

$$f(x) = -\int_{6}^{2x^2} 6t^3 - 6t\ dt$$

2. Plug $g(x)$ in for t and multiply by dg/dx.

$$f'(x) = -\left[6\left(2x^2\right)^3 - 6\left(2x^2\right)\right](4x)$$

$$f'(x) = -\left(48x^6 - 12x^2\right)(4x)$$

$$f'(x) = -192x^7 + 48x^3$$

$$f'(x) = 48x^3 - 192x^7$$

Topic: Part 1 of the FTC

Question: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative.

$$f(x) = \int_{6x-1}^{x^3} \cos(2t) - 4t \ dt$$

Answer choices:

A $f'(x) = -12x^5 + 3x^2 \cos\left(2x^3\right) + 144x - 6\cos(12x - 2) - 24$

B $f'(x) = 3x^2 \cos\left(2x^3\right) + 6\cos(12x - 2) - 12x^5 - 144x + 24$

C $f'(x) = x^2 \cos\left(2x^3\right) - 2\cos(12x - 2) - 4x^5 + 48x - 8$

D $f'(x) = x^2 \cos\left(2x^3\right) + 2\cos(12x - 2) - 4x^5 - 48x + 8$

Solution: A

We've been given an integral with functions as the upper and lower limits of integration.

Given integral	**How to solve it**
$f(x)=\int_a^x f(t)\ dt$	Plug x in for t.
$f(x)=\int_x^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug x in for t.
$f(x)=\int_a^{g(x)} f(t)\ dt$	Plug $g(x)$ in for t, then multiply by dg/dx.
$f(x)=\int_{g(x)}^a f(t)\ dt$	Reverse limits of integration and multiply by -1, then plug $g(x)$ in for t and multiply by dg/dx.
$f(x)=\int_{g(x)}^{h(x)} f(t)\ dt$	Split the limits of integration as $\int_{g(x)}^0 f(t)\ dt+\int_0^{h(x)} f(t)\ dt$. Reverse limits of integration on $\int_{g(x)}^0 f(t)\ dt$ and multiply by -1, then plug $g(x)$ and $h(x)$ in for t, multiplying by dg/dx and dh/dx respectively.

Looking at the chart, we can see that this is the situation described by the fifth row, which means the steps for solving this using FTC Part 1 are:

1. Split the limits of integration at 0.

$$f(x)=\int_{6x-1}^{0}\cos(2t)-4t\ dt+\int_{0}^{x^3}\cos(2t)-4t\ dt$$

2. For the first integral, reverse the limits of integration and multiply by -1.

$$f(x)=-\int_{0}^{6x-1}\cos(2t)-4t\ dt+\int_{0}^{x^3}\cos(2t)-4t\ dt$$

3. Plug in $g(x)$ and $h(x)$ and multiply by dg/dx and dh/dx.

$$f'(x)=-\left\{\cos\left[2(6x-1)\right]-4(6x-1)\right\}(6)+\left\{\cos\left[2\left(x^3\right)\right]-4\left(x^3\right)\right\}\left(3x^2\right)$$

$$f'(x)=-6\left[\cos(12x-2)-24x+4\right]+3x^2\left[\cos\left(2x^3\right)-4x^3\right]$$

$$f'(x)=-6\cos(12x-2)+144x-24+3x^2\cos\left(2x^3\right)-12x^5$$

$$f'(x)=-12x^5+3x^2\cos\left(2x^3\right)+144x-6\cos(12x-2)-24$$

Topic: Part 2 of the FTC

Question: Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_1^4 x^2 - 3x + 2\ dx$$

Answer choices:

A $\frac{9}{2}$

B 6

C $\frac{11}{2}$

D 4

Solution: A

The Fundamental Theorem of Calculus says that

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is the antiderivative function of f. Since

$$F(x) = \int x^2 - 3x + 2\ dx$$

$$F(x) = \int x^2\ dx - \int 3x\ dx + \int 2\ dx$$

$$F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$$

we have

$$\int_1^4 x^2 - 3x + 2\ dx = \left[\frac{1}{3}(4)^3 - \frac{3}{2}(4)^2 + 2(4)\right] - \left[\frac{1}{3}(1)^3 - \frac{3}{2}(1)^2 + 2(1)\right]$$

$$\frac{64}{3} - \frac{48}{2} + 8 - \frac{1}{3} + \frac{3}{2} - 2$$

$$\frac{128}{6} - \frac{144}{6} + \frac{48}{6} - \frac{2}{6} + \frac{9}{6} - \frac{12}{6}$$

$$\frac{27}{6}$$

$$\frac{9}{2}$$

Topic: Part 2 of the FTC

Question: Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_2^4 6x^2\,dx$$

Answer choices:

A 16

B 112

C 128

D 24

Solution: B

The Fundamental Theorem of Calculus says that

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is the antiderivative function of f. Since

$$F(x) = \int 6x^2\ dx$$

$$F(x) = 6\int x^2\ dx$$

$$F(x) = \frac{6}{3}x^3$$

$$F(x) = 2x^3$$

we have

$$\int_2^4 6x^2\ dx = 2(4)^3 - 2(2)^3$$

$$112$$

Topic: Part 2 of the FTC

Question: Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_{-1}^{3} 4x^2 - 5x \ dx$$

Answer choices:

A $\quad 32$

B $\quad -\dfrac{23}{6}$

C $\quad \dfrac{27}{2}$

D $\quad \dfrac{52}{3}$

Solution: D

The Fundamental Theorem of Calculus says that

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is the antiderivative function of f. Since

$$F(x) = \int 4x^2 - 5x\ dx$$

$$F(x) = 4\int x^2\ dx - 5\int x\ dx$$

$$F(x) = \frac{4}{3}x^3 - \frac{5}{2}x^2$$

we have

$$\int_{-1}^{3} 4x^2 - 5x\ dx = \left[\frac{4}{3}(3)^3 - \frac{5}{2}(3)^2\right] - \left[\frac{4}{3}(-1)^3 - \frac{5}{2}(-1)^2\right]$$

$$\frac{108}{3} - \frac{45}{2} + \frac{4}{3} + \frac{5}{2}$$

$$\frac{112}{3} - \frac{40}{2}$$

$$\frac{112}{3} - \frac{60}{3}$$

$$\frac{52}{3}$$

Topic: Net change theorem

Question: Water is being pumped into a tank at a rate (in gallons per minute) given by $w(t) = 60 - 10\sqrt{t}$, with $0 \le t \le 120$ where t is the time in minutes since the pumping began. The tank had $1{,}200$ gallons of water in it when pumping began. Use the Net Change Theorem to determine how much water will be in the tank after 49 minutes of pumping.

Answer choices:

A $653\frac{1}{3}$ gallons

B $1{,}853\frac{1}{3}$ gallons

C $6{,}426\frac{1}{3}$ gallons

D $4{,}140$ gallons

Solution: B

The question asks us to determine the amount of water in a tank that has an initial volume of $1{,}200$ gallons, with water being pumped into the tank at a rate (in gallons per minute) of $w(t) = 60 - 10\sqrt{t}$ for 49 minutes. We'll use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the tank initially has $1{,}200$ gallons in it, we'll add $1{,}200$ to the integral. Also, since the question asks us to find the volume of water in the tank after 49 minutes of pumping, the integration of the pumping rate will be performed on the interval $[0,49]$.

$$1{,}200 + \int_0^{49} 60 - 10\sqrt{t}\ dt$$

To integrate, we'll find the anti-derivative of the integrand, so to make the integration a little easier, we will write the integrand using exponents.

$$1{,}200 + \int_0^{49} 60 - 10t^{\frac{1}{2}}\ dt$$

Now, integrate using the power rule for integration; adding 1 to the exponent and dividing by the new exponent in each term.

$$1{,}200 + \int_0^{49} 60 - 10t^{\frac{1}{2}}\ dt = 1{,}200 + \left[60t - 10\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]\Bigg|_0^{49}$$

We can change the division by a fraction to multiplication by the inverse of the fraction. Then we'll evaluate the anti-derivative at the upper and lower limits.

$$1{,}200 + \left[60t - 10\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]\Bigg|_0^{49}$$

$$1{,}200 + \left[60(49) - \frac{20}{3}(49)^{\frac{3}{2}}\right] - \left[60(0) - \frac{20}{3}(0)^{\frac{3}{2}}\right]$$

$$1{,}200 + \left[60(49) - \frac{20}{3}(49)^{\frac{3}{2}}\right]$$

$$1{,}200 + 2{,}940 - \frac{20}{3}(343)$$

$$\frac{3{,}600}{3} + \frac{8{,}820}{3} - \frac{6{,}860}{3}$$

$$\frac{5{,}560}{3}$$

$1,853.33$

After 49 minutes of pumping, this is the number of gallons in the tank.

Topic: Net change theorem

Question: Beginning at $15{,}000$ feet, a commercial jetliner begins climbing at a rate (in feet per minute) given by $f(t) = e^{0.4t} + 8$, with $0 \le t \le 20$, where t is the time in minutes since the aircraft began climbing. Use the Net Change Theorem to determine the aircraft's elevation after climbing for 15 minutes. Round the answer to the nearest foot.

Answer choices:

A $\quad$ $1{,}126$ feet

B $\quad$ $15{,}526$ feet

C $\quad$ $16{,}126$ feet

D $\quad$ $13{,}874$ feet

Solution: C

The question asks us to determine the elevation of an aircraft that, from an initial elevation of $15{,}000$ feet, climbs at a rate (in feet per minute) of $f(t) = e^{0.4t} + 8$ for 15 minutes. We will use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the aircraft is initially at an altitude of $15{,}000$ feet, we'll add $15{,}000$ to the integral. Also, since the question asks us to find the aircraft's elevation after 15 minutes of climbing, the integration of the climbing rate will be performed on the interval $[0,15]$.

$$15{,}000 + \int_0^{15} e^{0.4t} + 8\ dt$$

To integrate, we'll find the anti-derivative of the integrand, so we'll use the chain rule to integrate $e^{0.4t}$. The integration rule of an expression e^u is e^u/du. In this case, $u = 0.4t$ so $du = 0.4$.

Now, integrate using the chain rule and the power rule for integration; adding 1 to the exponent and dividing by the new exponent.

$$15{,}000 + \int_0^{15} e^{0.4t} + 8\ dt = 15{,}000 + \left(\frac{e^{0.4t}}{0.4} + 8t\right)\Bigg|_0^{15}$$

Now, we will evaluate the result of the integration at the upper and lower limits.

$$15{,}000 + \left(\frac{e^{0.4t}}{0.4} + 8t\right)_0^{15} = 15{,}000 + \left[\frac{e^{0.4(15)}}{0.4} + 8(15)\right] - \left[\frac{e^{0.4(0)}}{0.4} + 8(0)\right]$$

$$15{,}000 + \left[\frac{e^{0.4(15)}}{0.4} + 8(15)\right] - \left[\frac{e^{0.4(0)}}{0.4} + 8(0)\right]$$

$$15{,}000 + 1008.571984 + 120 - 2.5$$

$$16{,}126.071984$$

Thus, after climbing for 15 minutes from an elevation of $15{,}000$ feet, the aircraft is flying at an elevation of $16{,}126$ feet.

Topic: Net change theorem

Question: A tank with $1{,}000$ gallons of solvent is leaking at a rate (in gallons per hour) given by the the function $l(t)$, where t is the time in hours since the tank began leaking. Use the Net Change Theorem to determine the amount of solvent in the tank after 2 days. Round the answer to the nearest gallon.

$$l(t) = \frac{1}{4}\sqrt{t} + 3$$

with $0 \le t \le 240$

Answer choices:

A 801 gallons

B $1{,}199$ gallons

C 199 gallons

D 994 gallons

Solution: A

The question asks us to determine the amount of solvent in a tank that initially contains $1,000$ gallons, and leaks at a rate (in gallons per hour) of

$$l(t) = \frac{1}{4}\sqrt{t} + 3$$

for 2 days. We'll use the Net Change Theorem to answer this question.

The Net Change Theorem states that if we integrate a rate of change expression of a function, we will find the net amount of the change of the function during the period of integration. Thus, if $f'(x)$ is the derivative of $f(x)$, or in other words, if $f'(x)$ is the rate of change of $f(x)$, then

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

This means that the integral of the rate of change is the net change. This theorem uses the concepts of the Fundamental Theorem of Calculus for integration.

Let's begin by writing the integral for this problem. Since the tank initially has $1,000$ gallons of solvent in it, and it is leaking, we will subtract the integral from $1,000$ to find the amount of solvent remaining in the tank after two days. Furthermore, the question states the rate of the leak is in gallons per hour, and we are asked to find the amount in the tank after two days of leaking, we know that the tank leaked for 48 hours. Thus, our integration will be performed on the interval $[0,48]$.

$$1{,}000-\int_0^{48}\frac{1}{4}\sqrt{t}+3\ dt$$

To integrate, we will find the anti-derivative of the integrand, so we will use the exponent rule to integrate the term $(1/4)\sqrt{x}$. The integration rule of an expression $\sqrt{x}$ is

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

To integrate, we'll find the anti-derivative of the integrand, so to make the integration a little easier, we'll write the integrand using exponents.

$$1{,}000-\int_0^{48}\frac{1}{4}t^{\frac{1}{2}}+3\ dt$$

Now, integrate using the power rule for integration; adding 1 to the exponent and dividing by the new exponent.

$$1{,}000-\left[\left(\frac{1}{4}\right)\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+3t\right]\Bigg|_0^{48}$$

Since we're dividing by a fraction, we can change to multiplying by the reciprocal of the denominator. Then we will evaluate the anti-derivative at the upper and lower limits.

$$1{,}000-\left[\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)t^{\frac{3}{2}}+3t\right]\Bigg|_0^{48}$$

$$1{,}000 - \left(\frac{1}{6}t^{\frac{3}{2}} + 3t\right)\Bigg|_0^{48}$$

$$1{,}000 - \frac{1}{6}(48)^{\frac{3}{2}} - 3(48)$$

$$1{,}000 - 55.425358 - 144$$

$$800.574642$$

The question asked us to round the answer to the nearest whole gallon, so the tank will have 801 gallons of solvent after leaking for 2 days.

Topic: U-substitution

Question: Use u-substitution to evaluate the integral.

$$\int x^2\sqrt{x^3+2}\ dx$$

Answer choices:

A $\quad \dfrac{3x^4}{2\left(x^3+2\right)^{\frac{1}{2}}}+2x\left(x^3+2\right)^{\frac{1}{2}}+C$

B $\quad \dfrac{2}{9}x^3\left(x^3+2\right)^{\frac{3}{2}}+C$

C $\quad \dfrac{2}{9}x^{\frac{3}{2}}+C$

D $\quad \dfrac{2}{9}\left(x^3+2\right)^{\frac{3}{2}}+C$

Solution: D

We use u-substitution to solve this integral. Letting

$$u = x^3 + 2$$

$$du = 3x^2\ dx$$

$$x^2\ dx = \frac{1}{3}\ du$$

Making these substitutions, we have

$$\int x^2\sqrt{x^3+2}\ dx$$

$$\int u^{\frac{1}{2}}\left(\frac{1}{3}\right)\ du$$

$$\frac{1}{3}\int u^{\frac{1}{2}}\ du$$

$$\frac{\frac{1}{3}}{\frac{3}{2}}u^{\frac{3}{2}} + C$$

$$\frac{2}{9}u^{\frac{3}{2}} + C$$

Back-substituting, we'll get

$$\frac{2}{9}\left(x^3+2\right)^{\frac{3}{2}} + C$$

Topic: U-substitution

Question: Use u-substitution to evaluate the integral.

$$\int x^{-2}e^{\frac{1}{x}}\ dx$$

Answer choices:

A $\quad -e^{\frac{1}{x}}+C$

B $\quad -e^{x}+C$

C $\quad e^{\frac{1}{x}}+C$

D $\quad e^{x}+C$

Solution: A

Let

$$u = \frac{1}{x} = x^{-1}$$

$$du = -x^{-2}\ dx$$

By substitution:

$$\int x^{-2} e^{\frac{1}{x}}\ dx = -\int e^{\frac{1}{x}} \left(-x^{-2}\right)\ dx$$

$$-\int e^{u}\ du$$

$$-e^{u} + C$$

$$-e^{\frac{1}{x}} + C$$

Topic: U-substitution

Question: Use u-substitution to evaluate the integral.

$$\int \csc^2 x(1-\cot x)\ dx$$

Answer choices:

A $\quad \frac{1}{2}(1-\cot x)^2+C$

B $\quad -\cot x\left(x+\csc^2 x\right)+C$

C $\quad -\csc^2 x\left(2\cot x+1+2\cot^2 x\right)+C$

D $\quad \csc^2 x\left(\csc^2 x+\cot x-2\cot^2 x\right)+C$

Solution: A

First, we see that

$$\frac{d}{dx}(1-\cot x)=\csc^2 x$$

and so we'll use u-substitution with

$$u=1-\cot x$$

$$du=\csc^2 x\ dx$$

Plugging these in, we get

$$\int \csc^2 x(1-\cot x)\ dx$$

$$\int u\ du$$

$$\frac{1}{2}u^2+C$$

$$\frac{1}{2}(1-\cot x)^2+C$$

Topic: U-substitution in definite integrals

Question: Use a u-substitution with $u = \cos t$ to evaluate the definite integral.

$$\int_0^{\frac{\pi}{2}} (1 - 2\cos^2 t + \cos^4 t)\sin t \ dt$$

Answer choices:

A $\frac{8}{15}$

B $\frac{9}{16}$

C $\frac{7}{9}$

D $\frac{13}{15}$

Solution: A

Use u-substitution, letting

$$u = \cos t$$

$$du = -\sin t \ dt$$

$$-du = \sin t \ dt$$

Plugging these into the integral, we get

$$\int_{t=0}^{t=\frac{\pi}{2}} (1 - 2u^2 + u^4)(-du)$$

$$-\int_{t=0}^{t=\frac{\pi}{2}} u^4 - 2u^2 + 1 \ du$$

$$-\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right)\Bigg|_{t=0}^{t=\frac{\pi}{2}}$$

Back-substituting for u before we evaluate over the interval, we get

$$\left(-\frac{1}{5}\cos^5 t + \frac{2}{3}\cos^3 t - \cos t\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\left(-\frac{1}{5}\cos^5 \frac{\pi}{2} + \frac{2}{3}\cos^3 \frac{\pi}{2} - \cos \frac{\pi}{2}\right) - \left(-\frac{1}{5}\cos^5 0 + \frac{2}{3}\cos^3 0 - \cos 0\right)$$

$$(-0 + 0 - 0) - \left(-\frac{1}{5}(1) + \frac{2}{3}(1) - (1)\right)$$

$$\frac{1}{5} - \frac{2}{3} + 1$$

$$\frac{3}{15} - \frac{10}{15} + \frac{15}{15}$$

$$\frac{8}{15}$$

Topic: U-substitution in definite integrals

Question: Use u-substitution to simplify the definite integral. Do not solve it.

$$\int_0^2 x^2\sqrt{x^3}\ dx$$

Answer choices:

A $\quad 3\int_0^2 \sqrt{u}\ du$

B $\quad \frac{1}{3}\int_{-2}^2 \sqrt{u}\ du$

C $\quad \frac{1}{3}\int_0^8 \sqrt{u}\ du$

D $\quad 3\int_0^8 \sqrt{u}\ du$

Solution: C

U-substitution allows us to simplify integrals that we wouldn't otherwise be able to evaluate.

When we make a substitution in our integral, it means we also need to change our limits of integration to match the new variable. This just means that we have to take our old limits of integration with respect to x, and plug each of them into our equation for u to find new limits of integration with respect to u.

If we change the limits of integration, then we can evaluate the definite integral without back-substituting. Remember, you don't absolutely have to change the limits of integration when you make a substitution. However, if you don't change the limits, it means you must back-substitute to put the function back in terms of x instead of u at the end of the problem before you evaluate over the interval.

We'll set

$$u = x^3$$

$$du = 3x^2\ dx$$

$$dx = \frac{du}{3x^2}$$

Since we're going to make a u-substitution, we'll also find limits of integration with respect to u instead of x. Next, we can solve for our u-substitution limits. Plugging our upper and lower limits of integration into our equation for u, we get

$$u=(0)^3$$

$$u=0$$

and

$$u=(2)^3$$

$$u=8$$

The old limits of integration with respect to x were $[0,2]$ and the new limits of integration with respect to u are $[0,8]$.

Making the substitution and attaching new limits of integration, we get

$$\int_0^2 x^2\sqrt{x^3}\ dx$$

$$\int_0^8 x^2\sqrt{u}\ \frac{du}{3x^2}$$

$$\int_0^8 \sqrt{u}\ \frac{du}{3}$$

$$\frac{1}{3}\int_0^8 \sqrt{u}\ du$$

Topic: U-substitution in definite integrals

Question: Use u-substitution to simplify the definite integral. Do not solve it.

$$\int_2^5 \frac{x+1}{x^2+2x}\,dx$$

Answer choices:

A $\quad \dfrac{1}{2}\displaystyle\int_8^{35} \frac{1}{u}\,du$

B $\quad \dfrac{1}{2}\displaystyle\int_0^{5} \frac{1}{u}\,du$

C $\quad 2\displaystyle\int_2^{5} \frac{1}{u}\,du$

D $\quad 2\displaystyle\int_8^{35} \frac{1}{u}\,du$

Solution: A

U-substitution allows us to simplify integrals that we wouldn't otherwise be able to evaluate.

When we make a substitution in our integral, it means we also need to change our limits of integration to match the new variable. This just means that we have to take our old limits of integration with respect to x, and plug each of them into our equation for u to find new limits of integration with respect to u.

If we change the limits of integration, then we can evaluate the definite integral without back-substituting. Remember, you don't absolutely have to change the limits of integration when you make a substitution. However, if you don't change the limits, it means you must back-substitute to put the function back in terms of x instead of u at the end of the problem before you evaluate over the interval.

We'll set

$$u = x^2 + 2x$$

$$du = 2x + 2\ dx$$

$$dx = \frac{du}{2(x+1)}$$

Since we're going to make a u-substitution, we'll also find limits of integration with respect to u instead of x. Next, we can solve for our u-substitution limits. Plugging our upper and lower limits of integration into our equation for u, we get

$$u=(2)^2+2(2)$$

$$u=8$$

and

$$u=(5)^2+2(5)$$

$$u=35$$

The old limits of integration with respect to x were $[2,5]$ and the new limits of integration with respect to u are $[8,35]$.

Making the substitution and attaching new limits of integration, we get

$$\int_2^5 \frac{x+1}{x^2+2x}\ dx$$

$$\int_8^{35} \frac{x+1}{u}\left[\frac{du}{2(x+1)}\right]$$

$$\int_8^{35} \frac{1}{u}\left(\frac{du}{2}\right)$$

$$\frac{1}{2}\int_8^{35} \frac{1}{u}\ du$$

Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\csc^2 x\ dx$$

Answer choices:

A $\frac{\sqrt{3}}{3}\pi + \ln 2$

B $\frac{\sqrt{3}}{6}\pi - \ln 2$

C $\frac{\sqrt{3}}{6}\pi + \ln 2$

D $\frac{\sqrt{3}}{3}\pi - \ln 2$

Solution: C

To use integration by parts, first identify suitable expressions for u and dv such as

$$u = x$$

$$dv = \csc^2 x \ dx$$

Take the differential of u and the integral of dv.

$$du = dx$$

$$v = \int \csc^2 x \ dx = -\cot x$$

Plug the values into the formula for integration by parts.

$$\int u \ dv = uv - \int v \ du$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx = -x \cot x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\cot x \ dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx = -x \cot x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \ dx$$

We know that $\int \cot x \ dx = \ln(\sin x) + C$, so we apply this formula and get

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \csc^2 x \ dx$$

$$-x\cot x+\ln(\sin x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\left[-\frac{\pi}{2}\cot\frac{\pi}{2}+\ln\left(\sin\frac{\pi}{2}\right)\right]-\left[-\frac{\pi}{6}\cot\frac{\pi}{6}+\ln\left(\sin\frac{\pi}{6}\right)\right]$$

$$-\frac{\pi}{2}\cot\frac{\pi}{2}+\ln\left(\sin\frac{\pi}{2}\right)+\frac{\pi}{6}\cot\frac{\pi}{6}-\ln\left(\sin\frac{\pi}{6}\right)$$

$$-\frac{\pi}{2}\cdot\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}+\ln 1+\frac{\pi}{6}\cdot\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}-\ln\frac{1}{2}$$

$$-\frac{\pi}{2}\cdot\frac{0}{1}+\ln 1+\frac{\pi}{6}\cdot\frac{\sqrt{3}/2}{1/2}-\ln\frac{1}{2}$$

$$0+0+\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}\cdot\frac{2}{1}-\ln\frac{1}{2}$$

$$\pi\frac{\sqrt{3}}{6}-\ln\frac{1}{2}$$

$$\pi\frac{\sqrt{3}}{6}-\ln 2^{-1}$$

$$\pi\frac{\sqrt{3}}{6}-(-1)\ln 2$$

$$\frac{\sqrt{3}}{6}\pi+\ln 2$$

Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int xe^x \, dx$$

Answer choices:

A $e^x(x+1)+C$

B $e^x(x-1)+C$

C xe^x+C

D $e^{2x}+C$

Solution: B

To use integration by parts, first identify suitable expressions for u and dv such as

$$u = x$$

$$dv = e^x\ dx$$

Take the differential of u and the integral of dv.

$$du = dx$$

$$v = e^x$$

Plug the values into the formula for integration by parts.

$$\int u\ dv = uv - \int v\ du$$

$$\int xe^x\ dx = xe^x - \int e^x\ dx$$

Now we can integrate, and the value of the integral is

$$xe^x - e^x + C$$

$$e^x(x - 1) + C$$

Topic: Integration by parts

Question: Use integration by parts to evaluate the integral.

$$\int x \cos x \ dx$$

Answer choices:

A $\quad x \sin x + \cos x$

B $\quad x \sin x + \cos x + C$

C $\quad x \cos x - \sin x + C$

D $\quad x \sin x - \cos x + C$

Solution: B

The question asks us to evaluate

$$\int x \cos x \ dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u \ dv = uv - \int v \ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $x\cos x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \cos x \ dx$$

$$v = \sin x$$

We are now ready to integrate by parts using the general formula.

$$\int u \ dv = uv - \int v \ du$$

$$\int x\cos x \ dx = x\sin x - \int \sin x \ dx$$

Next, let's evaluate the new integral.

$$x\sin x - \int \sin x \ dx = x\sin x - (-\cos x) + C$$

We added a "C" to accommodate the possibility of a constant in the function. The last step is to simplify the result of the integration.

$$\int x \cos x \ dx = x \sin x + \cos x + C$$

Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int e^{2x}\sin(4x)\ dx$$

Answer choices:

A $\quad \dfrac{e^{2x}\sin(4x) - 2e^{2x}\cos(4x)}{10} + C$

B $\quad \dfrac{e^{2x}\sin(4x) - e^{2x}\cos(4x)}{10} + C$

C $\quad \dfrac{e^{2x}\sin(4x) - e^{2x}\cos(4x)}{5} + C$

D $\quad \dfrac{e^{2x}\sin(4x) - 2e^{2x}\cos(4x)}{5} + C$

Solution: A

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take then integral of dv to get v. Let's do that for this integral.

$$u = \sin(4x)$$

$$du = 4\cos(4x)\ dx$$

and

$$dv = e^{2x}\ dx$$

$$v = \frac{e^{2x}}{2}$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int e^{2x}\sin(4x)\ dx = [\sin(4x)]\left(\frac{e^{2x}}{2}\right) - \int \left(\frac{e^{2x}}{2}\right)[4\cos(4x)\ dx]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\int e^{2x}\cos(4x)\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = \cos(4x)$$

$$du_2 = -4\sin(4x)\ dx$$

and

$$dv_2 = e^{2x}\ dx$$

$$v_2 = \frac{e^{2x}}{2}$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\left[\cos(4x)\left(\frac{e^{2x}}{2}\right) - \int\left(\frac{e^{2x}}{2}\right)\left[-4\sin(4x)\ dx\right]\right]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - 2\left[\frac{e^{2x}\cos(4x)}{2} + 2\int e^{2x}\sin(4x)\ dx\right]$$

$$\int e^{2x}\sin(4x)\ dx = \frac{e^{2x}\sin(4x)}{2} - e^{2x}\cos(4x) - 4\int e^{2x}\sin(4x)\ dx$$

We've simplified the right-hand side as much as we can, and our remaining integral is the same as our original integral, and the same as the integral on the left-hand side, which means we can add the integral from the right side to the left side, and then solve for our original integral.

$$\int e^{2x}\sin(4x)\ dx+4\int e^{2x}\sin(4x)\ dx=\frac{e^{2x}\sin(4x)}{2}-e^{2x}\cos(4x)+C$$

$$5\int e^{2x}\sin(4x)\ dx=\frac{e^{2x}\sin(4x)}{2}-e^{2x}\cos(4x)+C$$

$$\int e^{2x}\sin(4x)\ dx=\frac{e^{2x}\sin(4x)}{10}-\frac{e^{2x}\cos(4x)}{5}+C$$

$$\int e^{2x}\sin(4x)\ dx=\frac{e^{2x}\sin(4x)-2e^{2x}\cos(4x)}{10}+C$$

Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int x^2e^x\ dx$$

Answer choices:

A $\quad e^x\left(x^2-2x+2\right)+5$

B $\quad e^x\left(x^2-2x+2\right)+C$

C $\quad x^2e^x-2xe^x+2e^x$

D $\quad e^x\left(x^2-2x+2\right)$

Solution: B

The question asks us to evaluate

$$\int x^2 e^x \, dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u \, dv = uv - \int v \, du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is x^2e^x where we have a polynomial exponential expression and an exponential expression. In the LIPET sequence, polynomial comes before exponential, so u is the polynomial expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du and dv.

$$u_1 = x^2$$

$$du_1 = 2x\ dx$$

$$dv_1 = e^x\ dx$$

$$v_1 = e^x$$

We are now ready to integrate by parts using the general formula.

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

$$\int x^2e^x\ dx = x^2e^x - \int 2xe^x\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$\int 2xe^x\ dx$$

Using the LIPET sequence again, we again have an exponential expression and a trigonometric expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2\ dv_2 = u_2v_2 - \int v_2\ du_2$$

$$u_2 = 2x$$

$$du_2 = 2\ dx$$

$$dv_2 = e^x\ dx$$

$$v_2 = e^x$$

$$\int 2e^x\ dx = 2xe^x - \int 2e^x\ dx$$

Now, we will rewrite the equation using the original integral.

$$\int x^2e^x\ dx = x^2e^x - \left(2xe^x - \int 2e^x\ dx\right)$$

Next, we'll distribute the negative and evaluate the integral on the right side of the equation.

$$\int x^2e^x\ dx = x^2e^x - 2xe^x + \int 2e^x\ dx$$

Since we will have an indefinite integral equal the sum of terms, we will add an arbitrary constant C to accommodate the possibility of a constant term in the answer.

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

Although we have an acceptable final answer, we can factor out the greatest common factor from the terms. The arbitrary constant does not contain the GCF.

$$\int x^2e^x\,dx = e^x\left(x^2 - 2x + 2\right) + C$$

Topic: Integration by parts two times

Question: Use integration by parts to evaluate the integral.

$$\int 2e^{5x}\sin(2x)\ dx$$

Answer choices:

A $\quad -e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)+C$

B $\quad -e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)$

C $\quad \frac{4}{29}\left(-e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)\right)$

D $\quad \frac{4}{29}\left(-e^{5x}\cos(2x)+\frac{5}{2}e^{5x}\sin(2x)\right)+C$

Solution: D

The question asks us to evaluate

$$\int 2e^{5x}\sin(2x)\ dx$$

using integration by parts.

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas.

The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L   Logarithmic expression

I   Inverse trigonometric expression

P   Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $2e^{5x}\sin(2x)$ where we have an exponential expression and a trigonometric function. In the LIPET sequence, exponential comes before trigonometric function, so u is the exponential expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du and dv.

$$u_1 = 2e^{5x}$$

$$du_1 = 10e^{5x}\ dx$$

$$dv_1 = \sin(2x)\ dx$$

$$v_1 = -\frac{1}{2}\cos(2x)$$

We're now ready to integrate by parts using the general formula.

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

$$\int 2e^{5x}\sin(2x)\ dx = \left(2e^{5x}\right)\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\left(10e^{5x}\right)\ dx$$

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + 5\int e^{5x}\cos(2x)\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$5\int e^{5x}\cos(2x)\ dx$$

Using the LIPET sequence again, we have an exponential expression and a trigonometric expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2\ dv_2 = u_2v_2 - \int v_2\ du_2$$

$$u_2 = e^{5x}$$

$$du_2 = 5e^{5x}\ dx$$

$$dv_2 = \cos(2x)\ dx$$

$$v_2 = \frac{1}{2}\sin(2x)$$

$$5\int e^{5x}\cos(2x)\ dx = 5\left[\frac{1}{2}e^{5x}\sin(2x) - \int \frac{1}{2}\left(5e^{5x}\right)\sin(2x)\ dx\right]$$

$$5\int e^{5x}\cos(2x)\ dx = \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{2}\int e^{5x}\sin(2x)\ dx$$

Now, we'll rewrite the equation using the original integral.

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{2}\int e^{5x}\sin(2x)\ dx$$

Notice the similarity between the original integral and the new integral in the above equation. We'll multiply the new integral by $1/2$ on the outside and by 2 on the inside to make the two integrals the same.

$$\int 2e^{5x}\sin(2x)\ dx = -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

Next, we will combine like terms by adding

$$\frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

to both sides of the equation.

$$\int 2e^{5x}\sin(2x)\ dx + \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

$$= -e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x) - \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx + \frac{25}{4}\int 2e^{5x}\sin(2x)\ dx$$

The integrals on the left side of the equation can be combined. The integrals on the right side of the equation cancel each other. Additionally, since we have an equation that shows an indefinite integral equals the sum of expressions, we will add a constant "C" to accommodate a possible constant term in the final answer.

Our last step is to multiply both sides of the equation by the reciprocal of $29/4$.

$$\left(\frac{4}{29}\right)\left(\frac{29}{4}\right)\int 2e^{5x}\sin(2x)\ dx = \frac{4}{29}\left[-e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x)\right] + C$$

It is not necessary to multiply the "C" by this fraction because it is an arbitrary constant. The final answer is

$$\int 2e^{5x}\sin(2x)\ dx = \frac{4}{29}\left[-e^{5x}\cos(2x) + \frac{5}{2}e^{5x}\sin(2x)\right] + C$$

Topic: Integration by parts three times

Question: Use integration by parts to evaluate the integral.

$$\int 2x^3e^x\ dx$$

Answer choices:

A $2e^x\left(x^3+3x^2-6x+6\right)+C$

B $e^x\left(x^3-3x^2+6x-6\right)+C$

C $2e^x\left(x^3-3x^2+6x-6\right)+C$

D $e^x\left(x^3+3x^2-6x+6\right)+C$

Solution: C

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take the integral of dv to get v. Let's do that for this integral.

$$u = 2x^3$$

$$du = 6x^2\ dx$$

and

$$dv = e^x\ dx$$

$$v = e^x$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - \int e^x(6x^2\ dx)$$

$$\int 2x^3e^x\ dx = 2x^3e^x - \int 6x^2e^x\ dx$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\int x^2e^x\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = x^2$$

$$du_2 = 2x\ dx$$

and

$$dv_2 = e^x\ dx$$

$$v_2 = e^x$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\left[x^2e^x - \int e^x(2x\ dx)\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6\left[x^2e^x - 2\int xe^x\ dx\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12\int xe^x\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a third time. We can sense that we're getting closer, because we started with an x^3 value in our original integral, after integration by parts it became x^2, then x after the second application. So we suspect that if we apply integration by parts one more time, we'll go from x to 1, and this term will drop out completely, leaving an integral we can actually evaluate.

$$u_3 = x$$

$$du_3 = 1\ dx$$

and

$$dv_3 = e^x\ dx$$

$$v_3 = e^x$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12\left[xe^x - \int e^x(1\ dx)\right]$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12xe^x - 12\int e^x\ dx$$

$$\int 2x^3e^x\ dx = 2x^3e^x - 6x^2e^x + 12xe^x - 12e^x + C$$

$$\int 2x^3e^x\ dx = 2e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Topic: Integration by parts three times

Question: Use integration by parts to evaluate the integral.

$$\int 3x^3e^{-x}\ dx$$

Answer choices:

A $\quad -e^{-x}\left(x^3+3x^2+6x+6\right)+C$

B $\quad -e^{-x}\left(3x^3+9x^2+18x+18\right)+C$

C $\quad e^{-x}\left(3x^3+9x^2+18x+18\right)+C$

D $\quad -e^{-x}\left(-3x^3-9x^2-18x-18\right)+C$

Solution: B

Sometimes integration by parts is the correct tool to use to evaluate the integral, but using it only once doesn't simplify the integral enough, and we have to use it a second, or even a third time. However many times we use it, we always use the same integration by parts formula,

$$\int u\ dv = uv - \int v\ du$$

We need to identify a value for u and a value for dv in our original integral, and then take the derivative of u to get du, and take the integral of dv to get v. Let's do that for this integral.

$$u = 3x^3$$

$$du = 9x^2\ dx$$

and

$$dv = e^{-x}\ dx$$

$$v = -e^{-x}$$

Plugging these values into the right side of the integration by parts formula, we get

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} - \int (-e^{-x})\left(9x^2\ dx\right)$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\int x^2e^{-x}\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a second time.

$$u_2 = x^2$$

$$du_2 = 2x\ dx$$

and

$$dv_2 = e^{-x}\ dx$$

$$v_2 = -e^{-x}$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\left[-x^2e^{-x} - \int (-e^{-x})(2x\ dx)\right]$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} + 9\left[-x^2e^{-x} + 2\int xe^{-x}\ dx\right]$$

$$\int 3x^3e^{-x}\ dx = -3x^3e^{-x} - 9x^2e^{-x} + 18\int xe^{-x}\ dx$$

Since we haven't managed to simplify our integral to the point where we can evaluate it, we'll have to try integration by parts a third time. We can sense that we're getting closer, because we started with an x^3 value in our original integral, after integration by parts it became x^2, then x after the second application. So we suspect that if we apply integration by parts one more time, we'll go from x to 1, and this term will drop out completely, leaving an integral we can actually evaluate.

$$u_3 = x$$

$$du_3 = 1\ dx$$

and

$$dv_3 = e^{-x}\ dx$$

$$v_3 = -\ e^{-x}$$

Plugging these values into the right side of the integration by parts formula to replace just the remaining integral, we get

$$\int 3x^3e^{-x}\ dx = -\ 3x^3e^{-x} - 9x^2e^{-x} + 18\left[-xe^{-x} - \int (-e^{-x})(1\ dx)\right]$$

$$\int 3x^3e^{-x}\ dx = -\ 3x^3e^{-x} - 9x^2e^{-x} - 18xe^{-x} + 18\int e^{-x}\ dx$$

$$\int 3x^3e^{-x}\ dx = -\ 3x^3e^{-x} - 9x^2e^{-x} - 18xe^{-x} - 18e^{-x} + C$$

$$\int 3x^3e^{-x}\ dx = -\ e^{-x}\left(3x^3 + 9x^2 + 18x + 18\right) + C$$

Topic: Integration by parts three times

Question: Evaluate the integral using integration by parts.

$$\int 2x^3e^{2x}\ dx$$

Answer choices:

A $\quad e^{2x}\left(x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{3}{4}\right)+C$

B $\quad e^{2x}\left(x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{3}{4}\right)$

C $\quad x^3e^{2x}-\frac{3}{2}x^2e^{2x}\frac{3}{2}xe^{2x}-\frac{3}{4}e^{2x}$

D $\quad x^3e^{2x}-\frac{3}{2}x^2e^{2x}\frac{3}{2}xe^{2x}-\frac{3}{4}e^{2x}+1$

Solution: A

Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $2x^3e^{2x}$ where we have a polynomial exponential expression and an exponential expression. In the LIPET sequence, polynomial comes before exponential, so u is the polynomial expression. Let's identify the parts we need to integrate. Additionally, since this problem will take more than one integration by parts, we will use subscripts with the u, v, du, and dv.

$$u_1 = 2x^3$$

$$du_1 = 6x^2 \ dx$$

$$dv_1 = e^{2x} \ dx$$

$$v_1 = \frac{1}{2}e^{2x}$$

We're now ready to integrate by parts using the general formula.

$$\int u_1 \ dv_1 = u_1 v_1 - \int v_1 \ du_1$$

$$\int 2x^3 e^{2x} \ dx = \left(2x^3\right)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)\left(6x^2\right) \ dx$$

$$\int 2x^3 e^{2x} \ dx = x^3 e^{2x} - \int 3x^2 e^{2x} \ dx$$

Next, let's evaluate the new integral, which will require integration by parts a second time.

$$\int 3x^2 e^{2x} \ dx$$

Using the LIPET sequence again, we again have a polynomial expression and an exponential expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_2\ dv_2 = u_2v_2 - \int v_2\ du_2$$

$$u_2 = 3x^2$$

$$du_2 = 6x\ dx$$

$$dv_2 = e^{2x}\ dx$$

$$v_2 = \frac{1}{2}e^{2x}$$

$$\int 3x^2e^{2x}\ dx = (3x^2)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(6x)\ dx$$

$$\int 3x^2e^{2x}\ dx = \frac{3}{2}x^2e^{2x} - \int 3xe^{2x}\ dx$$

Now, we'll rewrite the equation using the original integral.

$$\int 2x^3e^{2x}\ dx = x^3e^{2x} - \left(\frac{3}{2}x^2e^{2x} - \int 3xe^{2x}\ dx\right)$$

$$\int 2x^3e^{2x}\ dx = x^3e^{2x} - \frac{3}{2}x^2e^{2x} + \int 3xe^{2x}\ dx$$

Next, let's evaluate the new integral, which will require integration by parts a third time.

$$\int 3xe^{2x}\ dx$$

Using the LIPET sequence again, we again have a polynomial expression and an exponential expression. The exponential expression will be the u and the trigonometric function will be the dv.

$$\int u_3\ dv_3 = u_3v_3 - \int v_3\ du_3$$

$$u_3 = 3x$$

$$du_3 = 3\ dx$$

$$dv_3 = e^{2x}\ dx$$

$$v_3 = \frac{1}{2}e^{2x}$$

$$\int 3xe^{2x}\ dx = (3x)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(3)\ dx$$

$$\int 3xe^{2x}\ dx = \frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x}\ dx$$

$$\int \frac{3}{2}e^{2x}\ dx = \frac{3}{2}\left(\frac{e^{2x}}{2}\right) + C = \frac{3}{4}e^{2x} + C$$

Since we will have an indefinite integral equal a term, we will add an arbitrary constant C to accommodate the possibility of a constant term in the answer.

Now, we will rewrite the equation, again using the original integral. Each term comes from the three integration by parts processes.

$$\int 2x^3e^{2x}\ dx = x^3e^{2x} - \frac{3}{2}x^2e^{2x} + \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + C$$

Although we have an acceptable final answer, we can factor out the greatest common factor from the terms. The arbitrary constant does not contain the GCF.

$$\int 2x^3e^{2x}\ dx = e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + C$$

Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\pi}} 3\theta^5 \sin(\theta^3)\ d\theta$$

Answer choices:

A $\pi + 1$

B $\pi - 1$

C $\sqrt[3]{\pi} - 1$

D $\sqrt[3]{\pi} + 1$

Solution: B

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^3$$

$$dx = 3\theta^2\ d\theta$$

$$d\theta = \frac{dx}{3\theta^2}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = \sqrt[3]{\frac{\pi}{2}}$$

$$\theta^3 = \left(\sqrt[3]{\frac{\pi}{2}}\right)^3$$

$$x = \left(\sqrt[3]{\frac{\pi}{2}}\right)^3$$

$$x = \frac{\pi}{2}$$

Upper limit:

$$\theta = \sqrt[3]{\pi}$$

$$\theta^3 = (\sqrt[3]{\pi})^3$$

$$x = (\sqrt[3]{\pi})^3$$

$$x = \pi$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\pi}} 3\theta^5 \sin(\theta^3)\ d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} 3\theta^5 \sin x \frac{dx}{3\theta^2}$$

$$\int_{\frac{\pi}{2}}^{\pi} \theta^3 \sin x\ dx$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $x \sin x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \sin x\ dx$$

$$v = -\cos x$$

We're now ready to integrate by parts using the general formula.

$$\int u\ dv = uv - \int v\ du$$

$$\int x \sin x\ dx = -x \cos x - \int -\cos x\ dx$$

$$\int x \sin x\ dx = -x \cos x + \int \cos x\ dx$$

$$\int x \sin x\ dx = -x \cos x + \sin x$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = (-x \cos x + \sin x)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = (-\pi \cos \pi + \sin \pi) - \left(-\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = [-\pi(-1) + (0)] - \left[-\frac{\pi}{2}(0) + (1)\right]$$

$$\int_{\frac{\pi}{2}}^{\pi} x \sin x\ dx = \pi - 1$$

Therefore, $\pi - 1$ is the value of the integral.

Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{\frac{5\pi}{2}}} 4\theta^3 \cos(\theta^2)\ d\theta$$

Answer choices:

A $8\pi - 1$

B $8\pi + 1$

C $8\pi + 2$

D 8π

Solution: D

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^2$$

$$dx = 2\theta \ d\theta$$

$$d\theta = \frac{dx}{2\theta}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = \sqrt{\frac{3\pi}{2}}$$

$$\theta^2 = \left(\sqrt{\frac{3\pi}{2}}\right)^2$$

$$x = \left(\sqrt{\frac{3\pi}{2}}\right)^2$$

$$x = \frac{3\pi}{2}$$

Upper limit:

$$\theta = \sqrt{\frac{5\pi}{2}}$$

$$\theta^2 = \left(\sqrt{\frac{5\pi}{2}}\right)^2$$

$$x = \left(\sqrt{\frac{5\pi}{2}}\right)^2$$

$$x = \frac{5\pi}{2}$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{\frac{5\pi}{2}}} 4\theta^3 \cos(\theta^2)\ d\theta$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 4\theta^3 \cos x \frac{dx}{2\theta}$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 2\theta^2 \cos x\ dx$$

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} 2x \cos x\ dx$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x \ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u \ dv = uv - \int v \ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L Logarithmic expression

I Inverse trigonometric expression

P Polynomial expression

E Exponential expression

T Trigonometric function expression

In this problem, the integrand is $x\cos x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \cos x\ dx$$

$$v = \sin x$$

We're now ready to integrate by parts using the general formula.

$$\int u\ dv = uv - \int v\ du$$

$$2\int x\cos x\ dx = 2\left[x\sin x - \int \sin x\ dx\right]$$

$$2\int x\cos x\ dx = 2[x\sin x - (-\cos x)]$$

$$2\int x\cos x\ dx = 2x\sin x + 2\cos x$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = (2x\sin x + 2\cos x)\Big|_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = \left[2\left(\frac{5\pi}{2}\right)\sin\frac{5\pi}{2} + 2\cos\frac{5\pi}{2}\right] - \left[2\left(\frac{3\pi}{2}\right)\sin\frac{3\pi}{2} + 2\cos\frac{3\pi}{2}\right]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = \left[5\pi\sin\frac{5\pi}{2} + 2\cos\frac{5\pi}{2}\right] - \left[3\pi\sin\frac{3\pi}{2} + 2\cos\frac{3\pi}{2}\right]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = [5\pi(1) + 2(0)] - [3\pi(-1) + 2(0)]$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = 5\pi + 3\pi$$

$$2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} x\cos x\ dx = 8\pi$$

Therefore, 8π is the value of the integral.

Topic: Integration by parts with u-substitution

Question: Use u-substitution and then integration by parts to evaluate the integral.

$$\int_0^{\sqrt[4]{\pi}} 6\theta^7 \sin(\theta^4)\ d\theta$$

Answer choices:

A $\dfrac{5\pi}{2}$

B $\dfrac{\pi}{2}$

C $\dfrac{3}{2}\pi$

D $\dfrac{7}{2}(\pi - 1)$

Solution: C

First, we'll use u-substitution to simplify the integral. The integrand includes a trigonometric function of another function. Typically, the function that represents the angle will be removed/changed using the u-substitution process. Since we're going to use integration by parts, we'll use x as the new variable after the substitution. Therefore, we'll make the following substitutions:

$$x = \theta^4$$

$$dx = 4\theta^3\ d\theta$$

$$d\theta = \frac{dx}{4\theta^3}$$

Since the integration limits are in terms of θ, we'll also change the integration limits to match our substitution. If we don't do this now, we'll have to do it later.

Lower limit:

$$\theta = 0$$

$$\theta^4 = 0^4$$

$$x = 0^4$$

$$x = 0$$

Upper limit:

$$\theta = \sqrt[4]{\pi}$$

$$\theta^4 = (\sqrt[4]{\pi})^4$$

$$x = (\sqrt[4]{\pi})^4$$

$$x = \pi$$

Now, let's rewrite the integral in terms of x instead of θ.

$$\int_0^{\sqrt[4]{\pi}} 6\theta^7 \sin(\theta^4)\ d\theta$$

$$\int_0^{\pi} 6\theta^7 \sin x \frac{dx}{4\theta^3}$$

$$\int_0^{\pi} 3\theta^4 \sin x \frac{dx}{2}$$

$$\int_0^{\pi} \frac{3}{2}\theta^4 \sin x\ dx$$

$$\frac{3}{2}\int_0^{\pi} x \sin x\ dx$$

We are now prepared to integrate by parts. Integration by parts is a method of evaluating an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. The general formula for integration by parts is

$$\int u\ dv = uv - \int v\ du$$

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $x \sin x$ where we have a polynomial and a trigonometric function. In the LIPET sequence, polynomial comes before trigonometric function, so u is the polynomial. Let's identify the parts we need to integrate.

$$u = x$$

$$du = dx$$

$$dv = \sin x \ dx$$

$$v = -\cos x$$

We're now ready to integrate by parts using the general formula.

$$\int u\ dv = uv - \int v\ du$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}\left(-x\cos x - \int -\cos x\ dx\right)$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}\left(-x\cos x + \int \cos x\ dx\right)$$

$$\frac{3}{2}\int x\sin x\ dx = \frac{3}{2}(-x\cos x + \sin x)$$

We didn't add a C to accommodate the possibility of a constant in the function because this problem involves a definite integral. Now we'll evaluate the result of the integration using the integration limits.

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}(-x\cos x + \sin x)\Big|_0^\pi$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}[-\pi\cos\pi + \sin\pi] - \frac{3}{2}[-(0)\cos(0) + \sin(0)]$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}[-\pi(-1) + 0] - \frac{3}{2}[-(0)(1) + 0]$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\pi - 0$$

$$\frac{3}{2}\int_0^\pi x\sin x\ dx = \frac{3}{2}\pi$$

Therefore,

$$\frac{3}{2}\pi$$

is the value of the integral.

Topic: Tabular integration

Question: Use tabular integration to evaluate the integral.

$$\int \left(3x^2 - 4x\right) e^{5x}\ dx$$

Answer choices:

A $\quad \frac{1}{125}xe^{5x}\left(75x^2 - 130x\right) + C$

B $\quad \frac{1}{125}e^{5x}\left(75x^2 - 130x + 26\right) + C$

C $\quad \frac{1}{125}xe^{5x}\left(75x^2 + 130x\right) + C$

D $\quad \frac{1}{25}e^{5x}\left(75x^2 - 130x + 26\right) + C$

Solution: B

First, split the integrand into two functions $f(x)$ and $g(x)$.

$$f(x) = 3x^2 - 4x$$

$$g(x) = e^{5x}$$

Create the table by differentiating $f(x)$ successively until the derivative is 0 or is no longer differentiable and integrating $g(x)$ successively as many times as $f(x)$ was differentiated, the table becomes

Derivatives of $f(x)$	**Integrals of $g(x)$**
$f(x) = 3x^2 - 4x$	$g(x) = e^{5x}$
$f'(x) = 6x - 4$	$\int e^{5x}\ dx = \frac{1}{5}e^{5x}$
$f''(x) = 6$	$\int \frac{1}{5}e^{5x}\ dx = \frac{1}{25}e^{5x}$
$f'''(x) = 0$	$\int \frac{1}{25}e^{5x}\ dx = \frac{1}{125}e^{5x}$

The integral is given by

$$\int \left(3x^2 - 4x\right) e^{5x}\ dx = \frac{1}{5}e^{5x}\left(3x^2 - 4x\right) - \frac{1}{25}e^{5x}(6x - 4) + \frac{1}{125}e^{5x}(6) + C$$

$$\frac{1}{125}e^{5x}\left[25\left(3x^2 - 4x\right) - 5(6x - 4) + (6)\right] + C$$

$$\frac{1}{125}e^{5x}\left(75x^2 - 100x - 30x + 20 + 6\right) + C$$

$$\frac{1}{125}e^{5x}\left(75x^2-130x+26\right)+C$$

Topic: Tabular integration

Question: Use tabular integration to evaluate the integral.

$$\int_0^{2\pi} 2x^2 \sin\left(\frac{x}{2}\right)\ dx$$

Answer choices:

A $\quad 16\pi^2$

B $\quad 16\pi^2 + 64$

C $\quad \pi^2 - 64$

D $\quad 16\pi^2 - 64$

Solution: D

First, split the integrand into two functions $f(x)$ and $g(x)$.

$$f(x) = 2x^2$$

$$g(x) = \sin\left(\frac{x}{2}\right)$$

Create the table containing the derivatives of $f(x)$ and the integrals of $g(x)$ as shown below.

Derivatives of $f(x)$	**Integrals of** $g(x)$
$f(x) = 2x^2$	$g(x) = \sin\left(\frac{x}{2}\right)$
$f'(x) = 4x$	$\int \sin\left(\frac{x}{2}\right)\ dx = -2\cos\left(\frac{x}{2}\right)$
$f''(x) = 4$	$\int -2\cos\left(\frac{x}{2}\right)\ dx = -4\sin\left(\frac{x}{2}\right)$
$f'''(x) = 0$	$\int -4\sin\left(\frac{x}{2}\right)\ dx = 8\cos\left(\frac{x}{2}\right)$

Therefore, the integration is

$$\int_0^{2\pi} 2x^2 \sin\left(\frac{x}{2}\right)\ dx = \left[2x^2\left[-2\cos\left(\frac{x}{2}\right)\right] - 4x\left[-4\sin\left(\frac{x}{2}\right)\right] + 4\left[8\cos\left(\frac{x}{2}\right)\right]\right]\Bigg|_0^{2\pi}$$

$$-4x^2\cos\left(\frac{x}{2}\right)+16x\sin\left(\frac{x}{2}\right)+32\cos\left(\frac{x}{2}\right)\Bigg|_0^{2\pi}$$

$$\left[-4(2\pi)^2\cos\left(\frac{2\pi}{2}\right)+16(2\pi)\sin\left(\frac{2\pi}{2}\right)+32\cos\left(\frac{2\pi}{2}\right)\right]-$$

$$\left[-4(0)^2\cos\left(\frac{0}{2}\right)+16(0)\sin\left(\frac{0}{2}\right)+32\cos\left(\frac{0}{2}\right)\right]$$

$$\left(-16\pi^2\cos\pi+32\pi\sin\pi+32\cos\pi\right)-(0+0+32\cos 0)$$

$$\left[-16\pi^2(-1)+32\pi(0)+32(-1)\right]-32$$

$$16\pi^2-32-32$$

$$16\pi^2-64$$

Topic: Tabular integration

Question: Evaluate the integral using tabular integration.

$$\int \left(x^2 + 3x - 4\right) e^x \, dx$$

Answer choices:

A $\quad e^x \left(x^2 + x - 5\right) + C$

B $\quad x^2 + x - 5 + C$

C $\quad e^x \left(x^2 + x - 5\right)$

D $\quad x^2 + x - 5$

Solution: A

Tabular integration is a method of integrating by parts to evaluate an integral that cannot be evaluated using normal integration techniques, by using integration by substitution, or by using integration formulas. This method of integration is particularly useful when the integral requires multiple iterations of integration by parts. It works only when the portion of the integrand that we select as u eventually differentiates to 0.

The general formula for the first iterations of integration by parts is

$$\int u_1\ dv_1 = u_1v_1 - \int v_1\ du_1$$

If we use integration by parts a second time, it looks like

$$\int u_1\ dv_1 = u_1v_1 - u_2v_2 + \int v_2\ du_2$$

Then, if we use integration by parts a third time, it looks like

$$\int u_1\ dv_1 = u_1v_1 - u_2v_2 + u_3v_3 - \int v_3\ du_3$$

The process would continue until the integration is finished.

In this formula, we separate the integrand into two parts; one part is called u and the other part is called dv. In making these two parts, we must use all of the integrand.

Although there is sometimes flexibility in choosing u, we can generally use the following sequence of choices to select the best part of the integrand to be u. This method involves the acronym LIPET, where we select the first u in the sequence of the list below. The letters mean

L	Logarithmic expression
I	Inverse trigonometric expression
P	Polynomial expression
E	Exponential expression
T	Trigonometric function expression

In this problem, the integrand is $\left(x^2+3x-4\right)e^x$ where we have a polynomial and an exponential function. In the LIPET sequence, polynomial comes before the exponential function, so u is the polynomial.

Now, when we use tabular integration, we create a table with the first column containing u and the second column containing dv.

Next, differentiate u until the derivative is 0, and integrate dv until the number of rows in each column is the same. The table for this problem looks like this:

u	dv
x^2+3x-4	e^x
$2x+3$	e^x
2	e^x
0	e^x

Now that the table is finished, multiply the first row of the first column by the second row of the second column. Next, multiply the second row of the first column by the opposite of the third row of the second column. Then, multiply the third row of the first column by the fourth row of the second column. The fourth row of the first column is zero, so the process is finished. The result is

$$\left(x^2+3x-4\right)e^x-(2x+3)e^x+2e^x$$

This could be the final answer, but we can factor out e^x, as a GCF, and combine like terms.

$$\left(x^2+3x-4\right)e^x-(2x+3)e^x+2e^x$$

$$e^x\left(x^2+3x-4-2x-3+2\right)$$

$$e^x\left(x^2+x-5\right)$$

We add a C to accommodate the possibility of a constant in the function. The last step is to simplify the result of the integration. The final answer is

$$\int\left(x^2+3x-4\right)e^x\ dx=e^x\left(x^2+x-5\right)+C$$

Topic: Distinct linear factors

Question: Use partial fractions to evaluate the integral.

$$\int \frac{5x+3}{x^2-9}\ dx$$

Answer choices:

A $3\ln|x+3|+2\ln|x-3|+C$

B $3\ln|x+3|-2\ln|x-3|+C$

C $2\ln|x+3|+3\ln|x-3|+C$

D $2\ln|x+3|-3\ln|x-3|+C$

Solution: C

First, factor the denominator.

$$\int \frac{5x+3}{x^2-9}\ dx = \int \frac{5x+3}{(x+3)(x-3)}\ dx$$

Using partial fractions with distinct linear factors since we have two, unequal, linear factors, the decomposition gives us

$$\frac{5x+3}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$5x+3 = \frac{A}{x+3}(x+3)(x-3) + \frac{B}{x-3}(x+3)(x-3)$$

$$5x+3 = A(x-3) + B(x+3)$$

$$5x+3 = Ax - 3A + Bx + 3B$$

$$5x+3 = (Ax + Bx) + (-3A + 3B)$$

$$5x+3 = (A+B)x + (-3A + 3B)$$

Equating coefficients on both sides gives

[1] $5 = A + B$

and

$$3 = -3A + 3B$$

$$1 = -A + B$$

[2] $1 + A = B$

Substituting [2] into [1] gives

$$5 = A + (1 + A)$$

$$4 = 2A$$

$$A = 2$$

Plugging this value back into [2] gives

$$1 + A = B$$

$$1 + 2 = B$$

$$B = 3$$

With values for both coefficients, we'll plug into the partial fractions decomposition.

$$\frac{5x + 3}{(x + 3)(x - 3)} = \frac{A}{x + 3} + \frac{B}{x - 3}$$

$$\frac{5x + 3}{(x + 3)(x - 3)} = \frac{2}{x + 3} + \frac{3}{x - 3}$$

Then we'll put the decomposition back into the integral in place of the original function.

$$\int \frac{5x + 3}{x^2 - 9}\ dx = \int \frac{2}{x + 3} + \frac{3}{x - 3}\ dx$$

$$2\int \frac{1}{x + 3}\ dx + 3\int \frac{1}{x - 3}\ dx$$

$$2\ln|x+3|+3\ln|x-3|+C$$

Topic: Distinct linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{2}{(x-1)(x+1)}\ dx$$

Answer choices:

A $\int \frac{1}{x+1} - \frac{1}{x+1}\ dx$

B $\int \frac{1}{x-1} + \frac{1}{x-1}\ dx$

C $\int \frac{1}{x-1} + \frac{1}{x+1}\ dx$

D $\int \frac{1}{x-1} - \frac{1}{x+1}\ dx$

Solution: D

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{2}{(x-1)(x+1)}\ dx$$

Since the factors are linear, we know the numerators are going to be A, B, C, etc. For the partial fractions decomposition, we get

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Now we'll solve for constants.

$$\left[\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}\right](x-1)(x+1)$$

$$2 = A(x+1) + B(x-1)$$

Now we need to solve for A and B. Let's start by solving for A. The easiest way to do this is to figure out what value of x will make the B term go away. In this case if $x = 1$, we'll get

$$2 = A(1+1) + B(1-1)$$

$$2 = A(2) + B(0)$$

$$A = 1$$

We set x equal to the value that would make the factor with B equal to 0, which made B disappear and allowed us to solve for A.

We'll solve for B the same way. If $x=-1$, we'll get

$$2=A(-1+1)+B(-1-1)$$

$$2=A(0)+B(-2)$$

$$B=-1$$

Plugging the values for both constants back into our partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{2}{(x-1)(x+1)}\ dx=\int \frac{1}{x-1}+\frac{-1}{x+1}\ dx$$

$$\int \frac{2}{(x-1)(x+1)}\ dx=\int \frac{1}{x-1}-\frac{1}{x+1}\ dx$$

Topic: Distinct linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{4}{(3x-1)(x+1)}\ dx$$

Answer choices:

A $\quad \displaystyle\int \frac{3}{3x-1}+\frac{1}{x+1}\ dx$

B $\quad \displaystyle\int \frac{3}{3x-1}-\frac{1}{x+1}\ dx$

C $\quad \displaystyle\int \frac{3}{x+1}+\frac{1}{3x-1}\ dx$

D $\quad \displaystyle\int \frac{3}{x+1}-\frac{1}{3x-1}\ dx$

Solution: B

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{4}{(3x-1)(x+1)}\ dx$$

Since the factors are linear, we know the numerators are going to be A, B, C, etc. For the partial fractions decomposition, we get

$$\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}$$

Now we'll solve for constants.

$$\left[\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}\right](3x-1)(x+1)$$

$$4 = A(x+1) + B(3x-1)$$

Now we need to solve for A and B. Let's start by solving for A. The easiest way to do this is to figure out what value of x will make the B term go away. In this case if $x = 1/3$, we'll get

$$4 = A\left(\frac{1}{3}+1\right) + B\left(3 \cdot \frac{1}{3} - 1\right)$$

$$4 = A\left(\frac{4}{3}\right) + B(0)$$

$$A = 3$$

We set x equal to the value that would make the factor with B equal to 0, which made B disappear and allowed us to solve for A.

We'll solve for B the same way. If $x = -1$, we'll get

$$4 = A(-1 + 1) + B\left[3(-1) - 1\right]$$

$$4 = A(0) + B(-4)$$

$$B = -1$$

Plugging the values for both constants back into our partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{4}{(3x - 1)(x + 1)}\ dx = \int \frac{3}{3x - 1} + \frac{-1}{x + 1}\ dx$$

$$\int \frac{4}{(3x - 1)(x + 1)}\ dx = \int \frac{3}{3x - 1} - \frac{1}{x + 1}\ dx$$

Topic: Distinct quadratic factors

Question: Use partial fractions to evaluate the integral.

$$\int \frac{x^3 - 8x^2 - 1}{\left(x^2 + x - 6\right)\left(x^2 + 1\right)}\ dx$$

Answer choices:

A $2\ln|x+3| - \ln|x-2| - \tan^{-1}x + C$

B $2\ln|x+3| - \ln|x-2| + \tan^{-1}x + C$

C $2\ln|x+3| - \ln|x-2| + C$

D $2\ln|x+3| + \ln|x-2| + C$

Solution: A

First, factor the denominator.

$$\int \frac{x^3-8x^2-1}{\left(x^2+x-6\right)\left(x^2+1\right)}\ dx=\int \frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$$

Now we'll solve for constants.

$$\left[\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}\right](x+3)(x-2)\left(x^2+1\right)$$

$$x^3-8x^2-1=A(x-2)\left(x^2+1\right)+B(x+3)\left(x^2+1\right)+(Cx+D)(x+3)(x-2)$$

$$x^3-8x^2-1=A\left(x^3-2x^2+x-2\right)+B\left(x^3+3x^2+x+3\right)+(Cx+D)\left(x^2+x-6\right)$$

$$x^3-8x^2-1=Ax^3-2Ax^2+Ax-2A+Bx^3+3Bx^2+Bx+3B$$

$$+Cx^3+Cx^2-6Cx+Dx^2+Dx-6D$$

$$x^3-8x^2-1=\left(Ax^3+Bx^3+Cx^3\right)+\left(-2Ax^2+3Bx^2+Cx^2+Dx^2\right)$$

$$+\left(Ax+Bx-6Cx+Dx\right)+\left(-2A+3B-6D\right)$$

$$x^3-8x^2-1=(A+B+C)x^3+(-2A+3B+C+D)x^2$$

$$+\left(A+B-6C+D\right)x+\left(-2A+3B-6D\right)$$

Equating coefficients on both sides, we get

[1] $A+B+C=1$

[2] $-2A+3B+C+D=-8$

[3] $A+B-6C+D=0$

[4] $-2A+3B-6D=-1$

Subtracting [4] from [2], we get

$$(-2A+3B+C+D=-8)-(-2A+3B-6D=-1)$$

$$-2A+2A+3B-3B+C+D+6D=-8+1$$

[5] $C+7D=-7$

Subtracting [1] from [3], we get

$$(A+B-6C+D=0)-(A+B+C=1)$$

$$A-A+B-B-6C-C+D=0-1$$

[6] $-7C+D=-1$

Multiplying [6] by 7, we get

$$7(-7C+D=-1)$$

[7] $-49C+7D=-7$

Subtracting [7] from [5], we get

$$(C + 7D = -7) - (-49C + 7D = -7)$$

$$C + 49C + 7D - 7D = -7 + 7$$

$$50C = 0$$

$$C = 0$$

Once we've solved for one constant it gets easier to solve for the others.

$$C + 7D = -7$$

$$0 + 7D = -7$$

$$D = -1$$

Plugging the values for C and D into [2] and [3], we get

$$-2A + 3B + 0 - 1 = -8$$

$$A + B - 6(0) - 1 = 0$$

and these simplify to

[8] $-2A + 3B = -7$

[9] $A + B = 1$

Multiplying [9] by 2, we get

$$A + B = 1$$

$$2(A + B = 1)$$

[10] $2A + 2B = 2$

Now we can add [10] to [8] and get

$$(-2A+3B=-7)+(2A+2B=2)$$

$$-2A+2A+3B+2B=-7+2$$

$$5B=-5$$

$$B=-1$$

Plugging $B=-1$ into $A+B=1$, we get

$$A+B=1$$

$$A-1=1$$

$$A=2$$

With values for all of the constants, we'll plug into the partial fractions decomposition.

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{2}{x+3}+\frac{-1}{x-2}+\frac{0x-1}{x^2+1}$$

$$\frac{x^3-8x^2-1}{(x+3)(x-2)\left(x^2+1\right)}=\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}$$

Then we'll put the decomposition back into the integral in place of the original function.

$$\int\frac{x^3-8x^2-1}{\left(x^2+x-6\right)\left(x^2+1\right)}\ dx=\int\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}\ dx$$

$$2\int \frac{1}{x+3}\,dx - \int \frac{1}{x-2}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$2\ln|x+3| - \ln|x-2| - \tan^{-1}x + C$$

Topic: Distinct quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{6x^3 + 2x^2 - x + 12}{(2x^2 - 1)\left(x^2 + 1\right)}\ dx$$

Answer choices:

A $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{26}{3}\int \frac{1}{2x^2 - 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

B $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{46}{3}\int \frac{1}{2x^2 - 1}\ dx - \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

C $\quad \frac{4}{3}\int \frac{x}{2x^2 + 1}\ dx + \frac{26}{3}\int \frac{1}{2x^2 + 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 - 1}\ dx + \frac{10}{3}\int \frac{1}{x^2 - 1}\ dx$

D $\quad \frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{46}{3}\int \frac{1}{2x^2 - 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$

Solution: A

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{6x^3+2x^2-x+12}{(2x^2-1)\left(x^2+1\right)}\ dx$$

Since the factors are quadratic, we know the numerators are going to be $Ax+B$, $Cx+D$, $Ex+F$, etc. For the partial fractions decomposition, we get

$$\frac{6x^3+2x^2-x+12}{(2x^2-1)\left(x^2+1\right)}=\frac{Ax+B}{2x^2-1}+\frac{Cx+D}{x^2+1}$$

Now we'll solve for constants.

$$\left[\frac{6x^3+2x^2-x+12}{\left(2x^2-1\right)\left(x^2+1\right)}=\frac{Ax+B}{2x^2-1}+\frac{Cx+D}{x^2+1}\right]\left(2x^2-1\right)\left(x^2+1\right)$$

$$6x^3+2x^2-x+12=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(2x^2-1\right)$$

$$6x^3+2x^2-x+12=Ax^3+Ax+Bx^2+B+2Cx^3-Cx+2Dx^2-D$$

$$6x^3+2x^2-x+12=Ax^3+2Cx^3+Bx^2+2Dx^2+Ax-Cx+B-D$$

$$6x^3+2x^2-x+12=(A+2C)x^3+(B+2D)x^2+(A-C)x+(B-D)$$

Equating coefficients on both sides, we get

[1] $A+2C=6$

[2] $A-C=-1$

[3] $B + 2D = 2$

[4] $B - D = 12$

We'll solve for A and C using [1] and [2], and for B and D using [3] and [4]. Subtracting [2] from [1], we get

$$(A + 2C = 6) - (A - C = -1)$$

$$A - A + 2C + C = 6 + 1$$

$$3C = 7$$

$$C = \frac{7}{3}$$

Plugging the value for C back into [2] to solve for A, we get

$$A - \frac{7}{3} = -1$$

$$A = \frac{4}{3}$$

Subtracting [4] from [3], we get

$$(B + 2D = 2) - (B - D = 12)$$

$$B - B + 2D + D = 2 - 12$$

$$3D = -10$$

$$D = -\frac{10}{3}$$

Plugging the value for D back into [4] to solve for B, we get

$$B - \left(-\frac{10}{3}\right) = 12$$

$$B = \frac{26}{3}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{6x^3 + 2x^2 - x + 12}{(2x^2 - 1)(x^2 + 1)}\ dx = \int \frac{\frac{4}{3}x + \frac{26}{3}}{2x^2 - 1} + \frac{\frac{7}{3}x - \frac{10}{3}}{x^2 + 1}\ dx$$

$$\int \frac{\frac{4}{3}x}{2x^2 - 1} + \frac{\frac{26}{3}}{2x^2 - 1} + \frac{\frac{7}{3}x}{x^2 + 1} + \frac{-\frac{10}{3}}{x^2 + 1}\ dx$$

$$\frac{4}{3}\int \frac{x}{2x^2 - 1}\ dx + \frac{26}{3}\int \frac{1}{2x^2 - 1}\ dx + \frac{7}{3}\int \frac{x}{x^2 + 1}\ dx - \frac{10}{3}\int \frac{1}{x^2 + 1}\ dx$$

Topic: Distinct quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)}\ dx$$

Answer choices:

A $\quad -\frac{13}{14}\int \frac{x}{4x^2 - 2}\ dx - \frac{5}{7}\int \frac{1}{4x^2 - 2}\ dx - \frac{3}{14}\int \frac{x}{2x^2 + 6}\ dx - \frac{1}{7}\int \frac{1}{2x^2 + 6}\ dx$

B $\quad \frac{13}{14}\int \frac{x}{4x^2 - 2}\ dx - \frac{5}{7}\int \frac{1}{4x^2 - 2}\ dx + \frac{3}{14}\int \frac{x}{2x^2 + 6}\ dx + \frac{1}{7}\int \frac{1}{2x^2 + 6}\ dx$

C $\quad \frac{13}{14}\int \frac{x}{4x^2 - 2}\ dx + \frac{5}{7}\int \frac{1}{4x^2 - 2}\ dx - \frac{3}{14}\int \frac{x}{2x^2 + 6}\ dx - \frac{1}{7}\int \frac{1}{2x^2 + 6}\ dx$

D $\quad \frac{13}{14}\int \frac{x}{4x^2 - 2}\ dx - \frac{5}{7}\int \frac{1}{4x^2 - 2}\ dx - \frac{3}{14}\int \frac{x}{2x^2 + 6}\ dx - \frac{1}{7}\int \frac{1}{2x^2 + 6}\ dx$

Solution: D

The denominator is already factored as much as it can be, which means it's a product of irreducible factors.

$$\int \frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)}\ dx$$

Since the factors are quadratic, we know the numerators are going to be $Ax + B$, $Cx + D$, $Ex + F$, etc. For the partial fractions decomposition, we get

$$\frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)} = \frac{Ax + B}{4x^2 - 2} + \frac{Cx + D}{2x^2 + 6}$$

Now we'll solve for constants.

$$\left[\frac{x^3 - 2x^2 + 6x - 4}{\left(4x^2 - 2\right)\left(2x^2 + 6\right)} = \frac{Ax + B}{4x^2 - 2} + \frac{Cx + D}{2x^2 + 6}\right]\left(4x^2 - 2\right)\left(2x^2 + 6\right)$$

$$x^3 - 2x^2 + 6x - 4 = (Ax + B)\left(2x^2 + 6\right) + (Cx + D)\left(4x^2 - 2\right)$$

$$x^3 - 2x^2 + 6x - 4 = 2Ax^3 + 6Ax + 2Bx^2 + 6B + 4Cx^3 - 2Cx + 4Dx^2 - 2D$$

$$x^3 - 2x^2 + 6x - 4 = 2Ax^3 + 4Cx^3 + 2Bx^2 + 4Dx^2 + 6Ax - 2Cx + 6B - 2D$$

$$x^3 - 2x^2 + 6x - 4 = (2A + 4C)x^3 + (2B + 4D)x^2 + (6A - 2C)x + (6B - 2D)$$

Equating coefficients on both sides, we get

[1] $2A + 4C = 1$

[2] $6A - 2C = 6$

[3] $2B + 4D = -2$

[4] $6B - 2D = -4$

We'll solve for A and C using [1] and [2], and for B and D using [3] and [4]. Multiplying [1] by 3 so that both [1] and [2] contain $6A$, and then subtracting [2] from [1], we get

$$6A + 12C - (6A - 2C) = 3 - 6$$

$$6A + 12C - 6A + 2C = 3 - 6$$

$$14C = -3$$

$$C = -\frac{3}{14}$$

Plugging the value for C back into [2] to solve for A, we get

$$6A - 2\left(-\frac{3}{14}\right) = 6$$

$$A = \frac{13}{14}$$

Multiplying [3] by 3 so that both [3] and [4] contain $6B$, and then subtracting [4] from [3], we get

$$6B + 12D - (6B - 2D) = -6 - (-4)$$

$$6B + 12D - 6B + 2D = -6 - (-4)$$

$$14D = -2$$

$$D=-\frac{2}{14}$$

Plugging the value for D back into [4] to solve for B, we get

$$6B-2\left(-\frac{1}{7}\right)=-4$$

$$B=-\frac{5}{7}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x^3-2x^2+6x-4}{\left(4x^2-2\right)\left(2x^2+6\right)}\ dx=\int \frac{\frac{13}{14}x-\frac{5}{7}}{4x^2-2}+\frac{-\frac{3}{14}x-\frac{1}{7}}{2x^2+6}\ dx$$

$$\int \frac{\frac{13}{14}x-\frac{5}{7}}{4x^2-2}-\frac{\frac{3}{14}x+\frac{1}{7}}{2x^2+6}\ dx$$

$$\int \frac{\frac{13}{14}x}{4x^2-2}+\frac{-\frac{5}{7}}{4x^2-2}-\frac{\frac{3}{14}x}{2x^2+6}-\frac{\frac{1}{7}}{2x^2+6}\ dx$$

$$\frac{13}{14}\int \frac{x}{4x^2-2}\ dx-\frac{5}{7}\int \frac{1}{4x^2-2}\ dx-\frac{3}{14}\int \frac{x}{2x^2+6}\ dx-\frac{1}{7}\int \frac{1}{2x^2+6}\ dx$$

Topic: Repeated linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x+2}{(x-1)^2}\ dx$$

Answer choices:

A $\displaystyle\int \frac{1}{(x+1)^2}+\frac{3}{(x-1)}\ dx$

B $\displaystyle\int \frac{3}{(x-1)^2}+\frac{1}{(x-1)}\ dx$

C $\displaystyle\int \frac{3}{(x-1)^2}-\frac{1}{(x-1)}\ dx$

D $\displaystyle\int \frac{1}{(x-1)^2}-\frac{3}{(x-1)}\ dx$

Solution: B

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{x+2}{(x-1)^2}\ dx = \int \frac{A}{(x-1)^2} + \frac{B}{(x-1)}\ dx$$

Using partial fractions decomposition containing a linear factor, we have

$$\frac{x+2}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)}$$

Now we'll solve for constants.

$$\frac{(x+2)(x-1)^2}{(x-1)^2} = \frac{A(x-1)^2}{(x-1)^2} + \frac{B(x-1)^2}{(x-1)}$$

$$x+2 = A + B(x-1)$$

$$x+2 = A + Bx - B$$

$$x+2 = Bx + A - B$$

$$x+2 = Bx + (A-B)$$

Equating coefficients on both sides, we get

[1] $B = 1$

[2] $A - B = 2$

We already know the value of B. Plugging [1] into [2] to solve for A, we get

$$A - 1 = 2$$

$$A = 3$$

Plugging the values for both constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x+2}{(x-1)^2}\ dx = \int \frac{3}{(x-1)^2} + \frac{1}{(x-1)}\ dx$$

Topic: Repeated linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{4x^2 + 10x + 8}{(x+1)^3} \ dx$$

Answer choices:

A $\int \frac{2}{(x+1)} + \frac{2}{(x+1)^2} + \frac{4}{(x+1)^3} \ dx$

B $\int \frac{2}{(x+1)^3} + \frac{3}{(x+1)^2} + \frac{4}{(x+1)} \ dx$

C $\int \frac{2}{(x+1)^3} - \frac{2}{(x+1)^2} + \frac{4}{(x+1)} \ dx$

D $\int \frac{2}{(x+1)^3} + \frac{2}{(x+1)^2} + \frac{4}{(x+1)} \ dx$

Solution: D

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{4x^2 + 10x + 8}{(x+1)^3}\ dx = \int \frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)}\ dx$$

Using partial fractions decomposition containing a linear factor, we have

$$\frac{4x^2 + 10x + 8}{(x+1)^3} = \frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)}$$

Now we'll solve for constants.

$$\frac{(4x^2 + 10x + 8)(x+1)^3}{(x+1)^3} = \frac{A(x+1)^3}{(x+1)^3} + \frac{B(x+1)^3}{(x+1)^2} + \frac{C(x+1)^3}{(x+1)}$$

$$4x^2 + 10x + 8 = A + B(x+1) + C(x+1)^2$$

$$4x^2 + 10x + 8 = A + Bx + B + C\left(x^2 + 2x + 1\right)$$

$$4x^2 + 10x + 8 = A + Bx + B + Cx^2 + 2Cx + C$$

$$4x^2 + 10x + 8 = Cx^2 + Bx + 2Cx + A + B + C$$

$$4x^2 + 10x + 8 = Cx^2 + (B + 2C)x + (A + B + C)$$

Equating coefficients on both sides, we get

[1] $C = 4$

[2] $B + 2C = 10$

[3] $A+B+C=8$

We already know the value of C. Plugging [1] into [2] to solve for B, we get

$$B+2(4)=10$$

$$B=2$$

Plugging the values for B and C into [3] to solve for A, we get

$$A+2+4=8$$

$$A=2$$

Plugging the values for each of the three constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{4x^2+10x+8}{(x+1)^3}\ dx=\int \frac{2}{(x+1)^3}+\frac{2}{(x+1)^2}+\frac{4}{(x+1)}\ dx$$

Topic: Repeated linear factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}\ dx$$

Answer choices:

A $\int \frac{1}{(x-2)^4} + \frac{4}{(x-2)^3} + \frac{4}{(x-2)^2} + \frac{1}{(x-2)}\ dx$

B $\int \frac{1}{(x-2)^4} - \frac{6}{(x-2)^3} + \frac{4}{(x-2)^2} - \frac{1}{(x-2)}\ dx$

C $\int \frac{1}{(x-2)^4} + \frac{6}{(x-2)^3} + \frac{4}{(x-2)^2} + \frac{1}{(x-2)}\ dx$

D $\int \frac{1}{(x-2)} + \frac{6}{(x-2)^2} + \frac{4}{(x-2)^3} + \frac{1}{(x-2)^4}\ dx$

Solution: C

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{x^3-2x^2+2x-3}{(x-2)^4}\ dx = \int \frac{A}{(x-2)^4}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)}\ dx$$

Using partial fractions decomposition containing a linear factor, we have

$$\frac{x^3-2x^2+2x-3}{(x-2)^4}=\frac{A}{(x-2)^4}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)}$$

Now we'll solve for constants.

$$\frac{(x^3-2x^2+2x-3)(x-2)^4}{(x-2)^4}=\frac{A(x-2)^4}{(x-2)^4}+\frac{B(x-2)^4}{(x-2)^3}+\frac{C(x-2)^4}{(x-2)^2}+\frac{D(x-2)^4}{(x-2)}$$

$$x^3-2x^2+2x-3=A+B(x-2)+C(x-2)^2+D(x-2)^3$$

$$x^3-2x^2+2x-3=A+Bx-2B+C\left(x^2-4x+4\right)+D\left(x^3-6x^2+12x-8\right)$$

$$x^3-2x^2+2x-3=A+Bx-2B+Cx^2-4Cx+4C+Dx^3-6Dx^2+12Dx-8D$$

$$x^3-2x^2+2x-3=Dx^3+Cx^2-6Dx^2+Bx-4Cx+12Dx+A-2B+4C-8D$$

$$x^3-2x^2+2x-3=Dx^3+(C-6D)x^2+(B-4C+12D)x+(A-2B+4C-8D)$$

Equating coefficients on both sides, we get

[1] $D=1$

[2] $C-6D=-2$

[3] $B - 4C + 12D = 2$

[4] $A - 2B + 4C - 8D = -3$

We already know the value of D. Plugging [1] into [2] to solve for C, we get

$$C - 6(1) = -2$$

$$C = 4$$

Plugging the values for C and D into [3] to solve for B, we get

$$B - 4(4) + 12(1) = 2$$

$$B = 6$$

Plugging the values for B, C and D into [4] to solve for A, we get

$$A - 2(6) + 4(4) - 8(1) = -3$$

$$A = 1$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}\ dx = \int \frac{1}{(x-2)^4} + \frac{6}{(x-2)^3} + \frac{4}{(x-2)^2} + \frac{1}{(x-2)}\ dx$$

Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3 + 2x - 1}{\left(x^2 + 1\right)^2}\, dx$$

Answer choices:

A $\displaystyle \int \frac{x - 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)}\, dx$

B $\displaystyle \int \frac{x + 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)}\, dx$

C $\displaystyle \int \frac{x - 1}{\left(x^2 + 1\right)^2} - \frac{x}{\left(x^2 + 1\right)}\, dx$

D $\displaystyle \int \frac{x - 1}{\left(x^2 + 1\right)^2} + \frac{x}{\left(x^2 + 1\right)^2}\, dx$

Solution: A

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{x^3+2x-1}{\left(x^2+1\right)^2}\ dx = \int \frac{Ax+B}{\left(x^2+1\right)^2}+\frac{Cx+D}{\left(x^2+1\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{x^3+2x-1}{\left(x^2+1\right)^2}=\frac{Ax+B}{\left(x^2+1\right)^2}+\frac{Cx+D}{\left(x^2+1\right)}$$

Now we'll solve for constants.

$$\frac{\left(x^3+2x-1\right)\left(x^2+1\right)^2}{\left(x^2+1\right)^2}=\frac{(Ax+B)\left(x^2+1\right)^2}{\left(x^2+1\right)^2}+\frac{(Cx+D)\left(x^2+1\right)^2}{\left(x^2+1\right)}$$

$$x^3+2x-1=Ax+B+(Cx+D)\left(x^2+1\right)$$

$$x^3+2x-1=Ax+B+Cx^3+Cx+Dx^2+D$$

$$x^3+2x-1=Cx^3+Dx^2+Ax+Cx+B+D$$

$$x^3+2x-1=Cx^3+Dx^2+(A+C)x+(B+D)$$

Equating coefficients on both sides, we get

[1] $C=1$

[2] $D=0$

[3] $A+C=2$

[4] $B+D=-1$

We already know the value of C and D. Plugging [1] into [3] to solve for A, we get

$$A+1=2$$

$$A=1$$

Plugging [2] into [4] to solve for B, we get

$$B+0=-1$$

$$B=-1$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{x^3+2x-1}{\left(x^2+1\right)^2}\ dx=\int \frac{1x+(-1)}{\left(x^2+1\right)^2}+\frac{1x+0}{\left(x^2+1\right)}\ dx$$

$$\int \frac{x-1}{\left(x^2+1\right)^2}+\frac{x}{\left(x^2+1\right)}\ dx$$

Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{x^3+4x^2-10}{x^2(x^2-1)}\ dx$$

Answer choices:

A $\displaystyle \int \frac{1}{x^2}+\frac{7}{x+1}-\frac{5}{x-1}\ dx$

B $\displaystyle \int \frac{1}{x^2}-\frac{7}{x+1}+\frac{5}{x-1}\ dx$

C $\displaystyle \int \frac{10}{x^2}+\frac{\frac{7}{2}}{x+1}-\frac{\frac{5}{2}}{x-1}\ dx$

D $\displaystyle \int \frac{10}{x^2}-\frac{\frac{7}{2}}{x+1}+\frac{\frac{5}{2}}{x-1}\ dx$

Solution: C

First, factor the denominator.

$$\int \frac{x^3+4x^2-10}{x^2(x+1)(x-1)}\ dx$$

Set up the partial fractions decomposition.

$$\frac{x^3+4x^2-10}{x^2(x+1)(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

Solve for the constants.

$$x^3+4x^2-10 = (Ax+B)(x+1)(x-1) + C(x^2)(x-1) + D(x^2)(x+1)$$

$$x^3+4x^2-10 = (Ax+B)(x^2-1) + Cx^2(x-1) + Dx^2(x+1)$$

$$x^3+4x^2-10 = Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + Dx^3 + Dx^2$$

$$x^3+4x^2-10 = (A+C+D)x^3 + (B-C+D)x^2 - Ax - B$$

Equating coefficients on both sides, we get

[1] $A+C+D=1$

[2] $B-C+D=4$

[3] $-A=0$

[4] $-B=-10$

From equation [3] we know $A=0$, and from equation [4] we know $B=10$. So equations [1] and [2] become

$$0 + C + D = 1$$

[5] $C + D = 1$

and

$$10 - C + D = 4$$

$$-C + D = -6$$

$$C - D = 6$$

Solve [5] for C to get $C = 1 - D$. Substituting $C = 1 - D$ into $C - D = 6$ gives

$$1 - D - D = 6$$

$$1 - 2D = 6$$

$$-2D = 5$$

$$D = -\frac{5}{2}$$

Then

$$C = 1 - \left(-\frac{5}{2}\right)$$

$$C = 1 + \frac{5}{2}$$

$$C = \frac{7}{2}$$

Plugging the values for each of the four constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{0x+10}{x^2}+\frac{\frac{7}{2}}{x+1}+\frac{-\frac{5}{2}}{x-1}\ dx$$

$$\int \frac{10}{x^2}+\frac{\frac{7}{2}}{x+1}-\frac{\frac{5}{2}}{x-1}\ dx$$

Topic: Repeated quadratic factors

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx$$

Answer choices:

A $\quad \displaystyle\int \frac{4}{x}+\frac{12x}{\left(x^2+2\right)^2}+\frac{2x}{\left(x^2+2\right)}\ dx$

B $\quad \displaystyle\int \frac{4}{x}-\frac{12x}{\left(x^2+2\right)^2}-\frac{2x}{\left(x^2+2\right)}\ dx$

C $\quad \displaystyle\int \frac{4}{x}-\frac{-12x}{\left(x^2+2\right)^2}+\frac{-2x}{\left(x^2+2\right)}\ dx$

D $\quad \displaystyle\int \frac{4}{x}+\frac{-12x}{\left(x^2+2\right)}+\frac{-2x}{\left(x^2+2\right)^2}\ dx$

Solution: B

First, factor the denominator. Since we have a repeated factor, we need to include all factors of a lesser degree.

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx = \int \frac{A}{x} + \frac{Bx+C}{\left(x^2+2\right)^2} + \frac{Dx+E}{\left(x^2+2\right)}\ dx$$

Using partial fractions decomposition containing a quadratic factor, we have

$$\frac{2x^4+16}{x\left(x^2+2\right)^2} = \frac{A}{x} + \frac{Bx+C}{\left(x^2+2\right)^2} + \frac{Dx+E}{\left(x^2+2\right)}$$

Now we'll solve for constants.

$$\frac{\left(2x^4+16\right)\left[x\left(x^2+2\right)^2\right]}{x\left(x^2+2\right)^2} = \frac{A\left[x\left(x^2+2\right)^2\right]}{x} + \frac{(Bx+C)\left[x\left(x^2+2\right)^2\right]}{\left(x^2+2\right)^2} + \frac{(Dx+E)\left[x\left(x^2+2\right)^2\right]}{\left(x^2+2\right)}$$

$$2x^4+16 = A\left(x^2+2\right)^2 + (Bx+C)(x) + (Dx+E)\left[x\left(x^2+2\right)\right]$$

$$2x^4+16 = A\left(x^4+4x^2+4\right) + Bx^2 + Cx + (Dx+E)\left(x^3+2x\right)$$

$$2x^4+16 = Ax^4 + 4Ax^2 + 4A + Bx^2 + Cx + Dx^4 + 2Dx^2 + Ex^3 + 2Ex$$

$$2x^4+16 = Ax^4 + Dx^4 + Ex^3 + 4Ax^2 + Bx^2 + 2Dx^2 + Cx + 2Ex + 4A$$

$$2x^4+16 = (A+D)x^4 + Ex^3 + (4A+B+2D)x^2 + (C+2E)x + 4A$$

Equating coefficients on both sides, we get

[1] $A + D = 2$

[2] $E = 0$

[3] $4A + B + 2D = 0$

[4] $C + 2E = 0$

[5] $4A = 16$

We already know the value of E. Plugging [2] into [4] to solve for C, we get

$$C + 2(0) = 0$$

$$C = 0$$

Solving [5] for A, we get

$$4A = 16$$

$$A = 4$$

Plugging $A = 4$ into [1] to solve for D, we get

$$4 + D = 2$$

$$D = -2$$

Plugging our values for A and D into [3] to solve for B, we get

$$4(4) + B + 2(-2) = 0$$

$$B = -12$$

Plugging the values for each of the five constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{2x^4+16}{x\left(x^2+2\right)^2}\ dx=\int \frac{4}{x}+\frac{-12x+0}{\left(x^2+2\right)^2}+\frac{-2x+0}{\left(x^2+2\right)}\ dx$$

$$\int \frac{4}{x}-\frac{12x}{\left(x^2+2\right)^2}-\frac{2x}{\left(x^2+2\right)}\ dx$$

Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+4}}{x} \ dx$$

Answer choices:

A $\quad 2\int \frac{1}{u-2} \ du - 2\int \frac{1}{u+2} \ du$

B $\quad 2\int \frac{1}{u+2} \ du - 2\int \frac{1}{u-2} \ du$

C $\quad 2u + 2\int \frac{1}{u-2} \ du - 2\int \frac{1}{u+2} \ du$

D $\quad 2u + 2\int \frac{1}{u+2} \ du - 2\int \frac{1}{u-2} \ du$

Solution: C

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u = \sqrt{x+4}$$

$$du = \frac{1}{2\sqrt{x+4}}\ dx$$

$$dx = 2\sqrt{x+4}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+4}}{x}\ dx = \int \frac{u}{x}\left(2\sqrt{x+4}\right)\ du$$

$$\int \frac{u}{x}(2u)\ du$$

$$\int \frac{2u^2}{x}\ du$$

Solving $u = \sqrt{x+4}$ for x so that we can replace the remaining x in the integral, we get

$$u = \sqrt{x+4}$$

$$u^2 = x+4$$

$$x = u^2 - 4$$

So

$$\int \frac{2u^2}{u^2-4}\ du$$

Since we've completely removed the radical and the new integral is in terms of the new variable, we're finished with our rationalizing substitution.

Normally here we would go straight to our partial fractions decomposition, but first we have to make the integral proper. Remember, in order to use partial fractions, the function must be proper, which means that the degree of the numerator must be less than the degree of the denominator. Currently we have u^2 in both the numerator and denominator, so their degrees are equal.

We'll use polynomial long division to make the function proper, dividing u^2-4 into u^2. We'll get

$$1+\frac{4}{u^2-4}$$

Plugging this into the integral, we get

$$2\int 1+\frac{4}{u^2-4}\ du$$

$$2\int 1\ du+2\int \frac{4}{u^2-4}\ du$$

$$2u+\int \frac{8}{(u-2)(u+2)}\ du$$

Use partial fractions to simplify the remaining integral.

$$\frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$\left[\frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}\right](u-2)(u+2)$$

$$8 = A(u+2) + B(u-2)$$

If we set $u = 2$, then

$$8 = A(2+2) + B(2-2)$$

$$8 = A(4) + B(0)$$

$$A = 2$$

And if we set $u = -2$, then

$$8 = A(-2+2) + B(-2-2)$$

$$8 = A(0) + B(-4)$$

$$B = -2$$

Plugging the values for both of the constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{8}{(u-2)(u+2)}\ du = \int \frac{2}{u-2} + \frac{-2}{u+2}\ du$$

$$\int \frac{2}{u-2} - \frac{2}{u+2} \ du$$

$$2u + \int \frac{2}{u-2} - \frac{2}{u+2} \ du$$

$$2u + 2\int \frac{1}{u-2} \ du - 2\int \frac{1}{u+2} \ du$$

Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+9}}{x}\ dx$$

Answer choices:

A $\quad 2u + 3\int \frac{1}{u+3}\ du - 3\int \frac{1}{u-3}\ du$

B $\quad 2u + 3\int \frac{1}{u-3}\ du - 3\int \frac{1}{u+3}\ du$

C $\quad 3\int \frac{1}{u+3}\ du - 3\int \frac{1}{u-3}\ du$

D $\quad 3\int \frac{1}{u-3}\ du - 3\int \frac{1}{u+3}\ du$

Solution: B

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u=\sqrt{x+9}$$

$$du=\frac{1}{2\sqrt{x+9}}\ dx$$

$$dx=2\sqrt{x+9}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+9}}{x}\ dx=\int \frac{u}{x}\left(2\sqrt{x+9}\right)\ du$$

$$\int \frac{u}{x}(2u)\ du$$

$$\int \frac{2u^2}{x}\ du$$

Solving $u=\sqrt{x+9}$ for x so that we can replace the remaining x in the integral, we get

$$u=\sqrt{x+9}$$

$$u^2=x+9$$

$$x=u^2-9$$

So

$$\int \frac{2u^2}{u^2-9}\ du$$

Since we've completely removed the radical and the new integral is in terms of the new variable, we're finished with our rationalizing substitution.

Normally here we would go straight to our partial fractions decomposition, but first we have to make the integral proper. Remember, in order to use partial fractions, the function must be proper, which means that the degree of the numerator must be less than the degree of the denominator. Currently we have u^2 in both the numerator and denominator, so their degrees are equal.

We'll use polynomial long division to make the function proper, dividing u^2-9 into u^2. We'll get

$$1+\frac{9}{u^2-9}$$

Plugging this into the integral, we get

$$2\int 1+\frac{9}{u^2-9}\ du$$

$$2\int 1\ du+2\int \frac{9}{u^2-9}\ du$$

$$2u+\int \frac{18}{(u-3)(u+3)}\ du$$

Use partial fractions to simplify the remaining integral.

$$\frac{18}{(u-3)(u+3)} = \frac{A}{u-3} + \frac{B}{u+3}$$

$$\left[\frac{18}{(u-3)(u+3)} = \frac{A}{u-3} + \frac{B}{u+3}\right](u-3)(u+3)$$

$$18 = A(u+3) + B(u-3)$$

If we set $u = 3$, then

$$18 = A(3+3) + B(3-3)$$

$$18 = A(6) + B(0)$$

$$A = 3$$

And if we set $u = -3$, then

$$18 = A(-3+3) + B(-3-3)$$

$$18 = A(0) + B(-6)$$

$$B = -3$$

Plugging the values for both of the constants back into the partial fractions decomposition, and putting the decomposition back into the integral, we get

$$\int \frac{18}{(u-3)(u+3)}\ du = \int \frac{3}{u-3} + \frac{-3}{u+3}\ du$$

$$\int \frac{3}{u-3} - \frac{3}{u+3} \ du$$

$$2u + \int \frac{3}{u-3} - \frac{3}{u+3} \ du$$

$$2u + 3\int \frac{1}{u-3} \ du - 3\int \frac{1}{u+3} \ du$$

Topic: Rationalizing substitutions

Question: Rewrite the integral using partial fractions. Do not solve it.

$$\int \frac{\sqrt{x+1}}{x^2}\ dx$$

Answer choices:

A $\quad \frac{1}{2}\int \frac{1}{u+1}\ du - \frac{1}{2}\int \frac{1}{u-1}\ du$

B $\quad -\frac{1}{2}\int \frac{1}{u+1}\ du + \frac{1}{2}\int \frac{1}{u-1}\ du$

C $\quad \frac{1}{2}\int \frac{1}{u+1}\ du - \frac{1}{2}\int \frac{1}{(u+1)^2}\ du - \frac{1}{2}\int \frac{1}{u-1}\ du - \frac{1}{2}\int \frac{1}{(u-1)^2}\ du$

D $\quad -\frac{1}{2}\int \frac{1}{u+1}\ du + \frac{1}{2}\int \frac{1}{(u+1)^2}\ du + \frac{1}{2}\int \frac{1}{u-1}\ du + \frac{1}{2}\int \frac{1}{(u-1)^2}\ du$

Solution: D

In order to use partial fractions to evaluate an integral, we need a rational, proper function. Rational functions can't include radicals, so we have to eliminate the radical before we can more forward. The easiest way to eliminate the radical is to use u-substitution. We'll let

$$u = \sqrt{x+1}$$

$$du = \frac{1}{2\sqrt{x+1}}\ dx$$

$$dx = 2\sqrt{x+1}\ du$$

And now we'll substitute into the integral.

$$\int \frac{\sqrt{x+1}}{x^2}\ dx = \int \frac{u}{x^2}\left(2\sqrt{x+1}\right)\ du$$

$$\int \frac{u}{x^2}(2u)\ du$$

$$\int \frac{2u^2}{x^2}\ du$$

Solving $u = \sqrt{x+1}$ for x so that we can replace the remaining x in the integral, we get

$$u = \sqrt{x+1}$$

$$u^2 = x+1$$

$$x = u^2 - 1$$

So

$$\int \frac{2u^2}{\left(u^2-1\right)^2}\ du$$

$$\int \frac{2u^2}{(u+1)^2(u-1)^2}\ du$$

Use partial fractions with repeated linear factors to simplify the remaining integral.

$$\frac{2u^2}{(u+1)^2(u-1)^2}=\frac{A}{u+1}+\frac{B}{(u+1)^2}+\frac{C}{u-1}+\frac{D}{(u-1)^2}$$

$$\left[\frac{2u^2}{(u+1)^2(u-1)^2}=\frac{A}{u+1}+\frac{B}{(u+1)^2}+\frac{C}{u-1}+\frac{D}{(u-1)^2}\right](u+1)^2(u-1)^2$$

$$2u^2=A(u+1)(u-1)^2+B(u-1)^2+C(u+1)^2(u-1)+D(u+1)^2$$

$$2u^2=A(u+1)(u^2-2u+1)+B(u^2-2u+1)$$

$$+C(u^2+2u+1)(u-1)+D(u^2+2u+1)$$

$$2u^2=A(u^3-2u^2+u+u^2-2u+1)+B(u^2-2u+1)$$

$$+C(u^3-u^2+2u^2-2u+u-1)+D(u^2+2u+1)$$

$$2u^2=A(u^3-u^2-u+1)+B(u^2-2u+1)$$

$$+C(u^3+u^2-u-1)+D(u^2+2u+1)$$

$$2u^2=Au^3-Au^2-Au+A+Bu^2-2Bu+B$$

$$+Cu^3 + Cu^2 - Cu - C + Du^2 + 2Du + D$$

$$2u^2 = (A + C)u^3 + (-A + B + C + D)u^2$$

$$+(-A - 2B - C + 2D)u + (A + B - C + D)$$

Equating coefficients on both sides, we get

[1] $A + C = 0$

[2] $-A + B + C + D = 2$

[3] $-A - 2B - C + 2D = 0$

[4] $A + B - C + D = 0$

Solve [1] for A.

$$A + C = 0$$

$$A = -C$$

Then equations [2] through [4] become

[5] $-(-C) + B + C + D = 2$, or $B + 2C + D = 2$

[6] $-(-C) - 2B - C + 2D = 0$, or $-2B + 2D = 0$

[7] $-C + B - C + D = 0$, or $B - 2C + D = 0$

Solve [6] for B.

$$-2B + 2D = 0$$

$$-2B = -2D$$

$$B = D$$

Then we can substitute into equations [5] and [7] to make a system that's in terms of C and D only.

$$D + 2C + D = 2$$

$$2C + 2D = 2$$

[8] $C + D = 1$

and

$$D - 2C + D = 0$$

$$-2C + 2D = 0$$

[9] $-C + D = 0$

Add equations [8] and [9].

$$C + D + (-C + D) = 1 + 0$$

$$C + D - C + D = 1$$

$$D + D = 1$$

$$2D = 1$$

$$D = \frac{1}{2}$$

We know $B=D$, so $B=1/2$. And $C+D=1$, so $C+(1/2)=1$, so $C=1/2$. And we know $A=-C$, so $A=-1/2$. Then we can plug the coefficients into the partial fractions decomposition.

$$\frac{2u^2}{(u+1)^2(u-1)^2}=\frac{-\frac{1}{2}}{u+1}+\frac{\frac{1}{2}}{(u+1)^2}+\frac{\frac{1}{2}}{u-1}+\frac{\frac{1}{2}}{(u-1)^2}$$

Putting this back into the integral, we could write the simplified integral as

$$\int\frac{-\frac{1}{2}}{u+1}+\frac{\frac{1}{2}}{(u+1)^2}+\frac{\frac{1}{2}}{u-1}+\frac{\frac{1}{2}}{(u-1)^2}\ du$$

$$-\frac{1}{2}\int\frac{1}{u+1}\ du+\frac{1}{2}\int\frac{1}{(u+1)^2}\ du+\frac{1}{2}\int\frac{1}{u-1}\ du+\frac{1}{2}\int\frac{1}{(u-1)^2}\ du$$

Topic: Trigonometric integrals

Question: Evaluate the trigonometric integral.

$$\int \sec^2 x + x^2 \ dx$$

Answer choices:

A $\quad \frac{1}{3}\sec^3 x + \frac{1}{3}x^3 + C$

B $\quad \tan x + \frac{1}{3}x^3 + C$

C $\quad \sec x \tan x + \frac{1}{3}x^3 + C$

D $\quad 2\sec^2 x \tan x + 2x + C$

Solution: B

In order to integrate the sum of two terms, we integrate each term and add the results.

$$\int \sec^2 x + x^2 \ dx$$

$$\int \sec^2 x \ dx + \int x^2 \ dx$$

$$\tan x + \frac{1}{3}x^3 + C$$

Topic: Trigonometric integrals

Question: Evaluate the trigonometric integral.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 - \cos x \ dx$$

Answer choices:

A $\frac{\pi}{2} + \sqrt{2}$

B $\frac{\pi}{2} - \sqrt{2}$

C $\sqrt{2}$

D $-\sqrt{2}$

Solution: B

Since

$$\frac{d}{dx}(x)=1$$

and

$$\frac{d}{dx}(\sin x)=\cos x$$

we have

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1-\cos x\ dx$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\ dx-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\ dx$$

$$x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}-\sin x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$x-\sin x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\frac{\pi}{4}-\sin\left(\frac{\pi}{4}\right)-\left[-\frac{\pi}{4}-\sin\left(-\frac{\pi}{4}\right)\right]$$

$$\frac{\pi}{4}-\frac{\sqrt{2}}{2}-\left[-\frac{\pi}{4}-\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$\frac{\pi}{2} - \sqrt{2}$$

Topic: Trigonometric integrals

Question: Evaluate the trigonometric integral.

$$\int \frac{\sin^3 x}{1 - \cos^2 x} \, dx$$

Answer choices:

A $\dfrac{-\cos x}{x - \frac{1}{3}\cos^3 x} + C$

B $\dfrac{1}{2\sin x} + C$

C $-\cos x + C$

D $\cos x + C$

Solution: C

Before we can integrate, we have to rewrite the integral to simplify it.

$$\int \frac{\sin^3 x}{1 - \cos^2 x}\ dx$$

$$\int \frac{\sin^3 x}{\sin^2 x}\ dx$$

$$\int \sin x\ dx$$

$$-\cos x + C$$

Topic: sin^m cos^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \sin^7 3x \cos^2 3x \ dx$$

Answer choices:

A $\quad -\frac{1}{9}\cos^3 3x + \frac{1}{5}\cos^5 3x - \frac{1}{7}\cos^7 3x + \frac{1}{27}\cos^9 3x + C$

B $\quad \frac{1}{9}\cos^3 3x - \frac{1}{5}\cos^5 3x + \frac{1}{7}\cos^7 3x - \frac{1}{27}\cos^9 3x + C$

C $\quad -\frac{1}{9}\cos^9 3x + \frac{1}{5}\cos^7 3x - \frac{1}{7}\cos^5 3x + \frac{1}{27}\cos^3 3x + C$

D $\quad \frac{1}{9}\cos^9 3x - \frac{1}{5}\cos^7 3x + \frac{1}{7}\cos^5 3x - \frac{1}{27}\cos^3 3x + C$

Solution: A

In the specific case where our function is the product of

an **odd** number of **sine** factors and

an **even or odd** number of **cosine** factors,

our plan is to

1. save one sine factor and use the identity $\sin^2 x = 1 - \cos^2 x$ to write the other sine factors in terms of cosine, then

2. use u-substitution with $u = \cos x$.

We'll separate a single sine factor and then replace the remaining sine factors using the identity.

$$\int \sin^7 3x \cos^2 3x \ dx$$

$$\int \sin 3x \sin^6 3x \cos^2 3x \ dx$$

$$\int \sin 3x \left(\sin^2 3x\right)^3 \cos^2 3x \ dx$$

$$\int \sin 3x \left(1 - \cos^2 3x\right)^3 \cos^2 3x \ dx$$

Using u-substitution with $u = \cos 3x$, we get

$$u = \cos 3x$$

$$du = -3\sin 3x\ dx$$

$$\sin 3x\ dx = \frac{du}{-3}$$

Substitute into the integral.

$$\int \sin 3x\left(1-u^2\right)^3 u^2\ dx$$

$$\int \left(1-u^2\right)^3 u^2\left(\sin 3x\ dx\right)$$

$$\int \left(1-u^2\right)^3 u^2\left(\frac{du}{-3}\right)$$

$$-\frac{1}{3}\int \left(1-u^2-2u^2+2u^4+u^4-u^6\right)u^2\ du$$

$$-\frac{1}{3}\int \left(1-3u^2+3u^4-u^6\right)u^2\ du$$

$$-\frac{1}{3}\int u^2-3u^4+3u^6-u^8\ du$$

$$-\frac{1}{3}\left(\frac{1}{3}u^3-\frac{3}{5}u^5+\frac{3}{7}u^7-\frac{1}{9}u^9\right)+C$$

Back-substituting for u, we get

$$-\frac{1}{3}\left(\frac{1}{3}\cos^3 3x-\frac{3}{5}\cos^5 3x+\frac{3}{7}\cos^7 3x-\frac{1}{9}\cos^9 3x\right)+C$$

$$-\frac{1}{9}\cos^3 3x+\frac{3}{15}\cos^5 3x-\frac{3}{21}\cos^7 3x+\frac{1}{27}\cos^9 3x+C$$

$$-\frac{1}{9}\cos^3 3x + \frac{1}{5}\cos^5 3x - \frac{1}{7}\cos^7 3x + \frac{1}{27}\cos^9 3x + C$$

Topic: sin^m cos^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \sin^5\theta\cos^4\theta\ d\theta$$

Answer choices:

A $\quad \frac{1}{5}\sin^5\theta-\frac{2}{7}\sin^7\theta+\frac{1}{9}\sin^9\theta+C$

B $\quad -\frac{1}{5}\sin^5\theta+\frac{2}{7}\sin^7\theta-\frac{1}{9}\sin^9\theta+C$

C $\quad -\frac{1}{5}\cos^5\theta+\frac{2}{7}\cos^7\theta-\frac{1}{9}\cos^9\theta+C$

D $\quad \frac{1}{5}\cos^5\theta-\frac{2}{7}\cos^7\theta+\frac{1}{9}\cos^9\theta+C$

Solution: C

In the specific case where our function is the product of

an **odd** number of **sine** factors and

an **even or odd** number of **cosine** factors,

our plan is to

1. save one sine factor and use the identity $\sin^2 x = 1 - \cos^2 x$ to write the other sine factors in terms of cosine, then

2. use u-substitution with $u = \cos x$.

We'll separate a single sine factor and then replace the remaining sine factors using the identity.

$$\int \sin^5 \theta \cos^4 \theta \ d\theta$$

$$\int \sin \theta \sin^4 \theta \cos^4 \theta \ d\theta$$

$$\int \sin \theta \left(\sin^2 \theta\right)^2 \cos^4 \theta \ d\theta$$

$$\int \sin \theta \left(1 - \cos^2 \theta\right)^2 \cos^4 \theta \ d\theta$$

Using u-substitution with $u = \cos \theta$, we get

$$u = \cos \theta$$

$$du = -\sin\theta\ d\theta$$

$$-du = \sin\theta\ d\theta$$

Substitute into the integral.

$$\int \sin\theta\left(1-u^2\right)^2 u^4\ d\theta$$

$$\int \left(1-u^2\right)^2 u^4\left(\sin\theta\ d\theta\right)$$

$$\int \left(1-u^2\right)^2 u^4\left(-\ du\right)$$

$$-\int \left(1-u^2\right)^2 u^4\ du$$

$$-\int \left(1-2u^2+u^4\right) u^4\ du$$

$$-\int u^4-2u^6+u^8\ du$$

$$-\left(\frac{1}{5}u^5-\frac{2}{7}u^7+\frac{1}{9}u^9\right)+C$$

$$-\frac{1}{5}u^5+\frac{2}{7}u^7-\frac{1}{9}u^9+C$$

Back-substituting for u, we get

$$-\frac{1}{5}\cos^5\theta+\frac{2}{7}\cos^7\theta-\frac{1}{9}\cos^9\theta+C$$

Topic: sin^m cos^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \sin^3\theta \cos^2\theta \ d\theta$$

Answer choices:

A $\quad \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C$

B $\quad -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$

C $\quad -\frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

D $\quad \frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

Solution: B

In the specific case where our function is the product of

an **odd** number of **sine** factors and

an **even or odd** number of **cosine** factors,

our plan is to

1. save one sine factor and use the identity $\sin^2 x = 1 - \cos^2 x$ to write the other sine factors in terms of cosine, then

2. use u-substitution with $u = \cos x$.

We'll separate a single sine factor and then replace the remaining sine factors using the identity.

$$\int \sin^3 \theta \cos^2 \theta \ d\theta$$

$$\int \sin \theta \sin^2 \theta \cos^2 \theta \ d\theta$$

$$\int \sin \theta \left(1 - \cos^2 \theta\right) \cos^2 \theta \ d\theta$$

Using u-substitution with $u = \cos \theta$, we get

$$u = \cos \theta$$

$$du = -\sin \theta \ d\theta$$

$$-du = \sin \theta \ d\theta$$

Substitute into the integral.

$$\int \sin\theta \left(1-u^2\right) u^2 \ d\theta$$

$$\int \left(1-u^2\right) u^2 \left(\sin\theta \ d\theta\right)$$

$$\int \left(1-u^2\right) u^2 \left(-\ du\right)$$

$$-\int \left(1-u^2\right) u^2 \ du$$

$$-\int u^2 - u^4 \ du$$

$$-\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$-\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

Back-substituting for u, we get

$$-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$$

Topic: sin^m cos^n, odd n

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \ dx$$

Answer choices:

A $\frac{1}{4}$

B $\frac{1}{2}$

C $\frac{1}{3}$

D $\frac{2}{3}$

Solution: C

In the specific case where our function is the product of

an **odd** number of **cosine** factors and

an **even or odd** number of **sine** factors,

our plan is to

1. save one cosine factor and use the identity $\cos^2 x = 1 - \sin^2 x$ to write the other cosine factors in terms of sine, then

2. use u-substitution with $u = \sin x$.

Since we only have one cosine factor to begin with, we don't need to separate factors and use the identity. Instead, we'll go straight to the u-substitution.

Using u-substitution with $u = \sin x$, we get

$$u = \sin x$$

$$du = \cos x \ dx$$

Because we're dealing with a definite integral, we have to either change the limits of integration when we make our substitution, or we have to indicate that the limits of integration are in terms of x until we back-substitute. Substitute into the integral.

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \ dx$$

$$\int_{x=0}^{x=\frac{\pi}{2}} u^2(\cos x\ dx)$$

$$\int_{x=0}^{x=\frac{\pi}{2}} u^2(du)$$

$$\int_{x=0}^{x=\frac{\pi}{2}} u^2\ du$$

$$\frac{1}{3}u^3\Big|_{x=0}^{x=\frac{\pi}{2}}$$

Back-substituting for u, we get

$$\frac{1}{3}\sin^3 x\Big|_0^{\frac{\pi}{2}}$$

$$\frac{1}{3}\sin^3\left(\frac{\pi}{2}\right) - \sin^3 0$$

$$\frac{1}{3}(1)^3 - (0)^3$$

$$\frac{1}{3}$$

Topic: sin^m cos^n, odd n

Question: Evaluate the trigonometric integral.

$$\int \sin^2\theta\cos^3\theta\ d\theta$$

Answer choices:

A $\frac{1}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta+C$

B $\frac{1}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta+C$

C $\frac{1}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta+C$

D $\frac{1}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta+C$

Solution: A

In the specific case where our function is the product of

an **odd** number of **cosine** factors and

an **even or odd** number of **sine** factors,

our plan is to

1. save one cosine factor and use the identity $\cos^2 x = 1 - \sin^2 x$ to write the other cosine factors in terms of sine, then

2. use u-substitution with $u = \sin x$.

We'll separate a single cosine factor and then replace the remaining cosine factors using the identity.

$$\int \sin^2\theta \cos^3\theta \ d\theta$$

$$\int \sin^2\theta \cos^2\theta \cos\theta \ d\theta$$

$$\int \sin^2\theta \left(1 - \sin^2\theta\right) \cos\theta \ d\theta$$

Using u-substitution with $u = \sin\theta$, we get

$$u = \sin\theta$$

$$du = \cos\theta \ d\theta$$

Substitute into the integral.

$$\int u^2\left(1-u^2\right)\cos\theta\ d\theta$$

$$\int u^2\left(1-u^2\right)\left(\cos\theta\ d\theta\right)$$

$$\int u^2\left(1-u^2\right)\left(du\right)$$

$$\int u^2\left(1-u^2\right)\ du$$

$$\int u^2-u^4\ du$$

$$\frac{1}{3}u^3-\frac{1}{5}u^5+C$$

Back-substituting for u, we get

$$\frac{1}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta+C$$

Topic: sin^m cos^n, odd n

Question: Evaluate the trigonometric integral.

$$\int \sin^4 \pi x \cos^3 \pi x \ dx$$

Answer choices:

A $\quad \dfrac{1}{5\pi}\cos^5 \pi x - \dfrac{1}{7\pi}\cos^7 \pi x + C$

B $\quad \dfrac{1}{5\pi}\sin^5 \pi x + \dfrac{1}{7\pi}\sin^7 \pi x + C$

C $\quad \dfrac{1}{5\pi}\sin^5 \pi x - \dfrac{1}{7\pi}\sin^7 \pi x + C$

D $\quad \dfrac{1}{5\pi}\cos^5 \pi x + \dfrac{1}{7\pi}\cos^7 \pi x + C$

Solution: C

In the specific case where our function is the product of

an **odd** number of **cosine** factors and

an **even or odd** number of **sine** factors,

our plan is to

1. save one cosine factor and use the identity $\cos^2 x = 1 - \sin^2 x$ to write the other cosine factors in terms of sine, then

2. use u-substitution with $u = \sin x$.

We'll separate a single cosine factor and then replace the remaining cosine factors using the identity.

$$\int \sin^4 \pi x \cos^3 \pi x \ dx$$

$$\int \sin^4 \pi x \cos^2 \pi x \cos \pi x \ dx$$

$$\int \sin^4 \pi x \left(1 - \sin^2 \pi x\right) \cos \pi x \ dx$$

Using u-substitution with $u = \sin \pi x$, we get

$$u = \sin \pi x$$

$$du = \pi \cos \pi x \ dx$$

$$\frac{du}{\pi} = \cos \pi x \ dx$$

Substitute into the integral.

$$\int u^4 \left(1-u^2\right) \cos \pi x \ dx$$

$$\int u^4 \left(1-u^2\right) \left(\cos \pi x \ dx\right)$$

$$\int u^4 \left(1-u^2\right) \left(\frac{du}{\pi}\right)$$

$$\frac{1}{\pi} \int u^4 \left(1-u^2\right) \ du$$

$$\frac{1}{\pi} \int u^4 - u^6 \ du$$

$$\frac{1}{\pi} \left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + C$$

$$\frac{1}{5\pi}u^5 - \frac{1}{7\pi}u^7 + C$$

Back-substituting for u, we get

$$\frac{1}{5\pi} \sin^5 \pi x - \frac{1}{7\pi} \sin^7 \pi x + C$$

Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int \sin^2\theta\cos^2\theta\ d\theta$$

Answer choices:

A $\frac{1}{8}\theta+\frac{1}{32}\sin 4\theta+C$

B $\frac{1}{8}\theta-\frac{1}{32}\sin 4\theta+C$

C $\frac{1}{8}\theta-\frac{1}{32}\cos 4\theta+C$

D $\frac{1}{8}\theta+\frac{1}{32}\cos 4\theta+C$

Solution: B

In the specific case where our function is the product of

an **even** number of **sine** factors and

an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

    a. $\sin x \cos x = \frac{1}{2}\sin 2x$,

2. then use the half-angle formulas to make substitutions,

    a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

    b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

    a. $\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int \sin^2\theta \cos^2\theta \ d\theta$$

$$\int \left(\sin\theta\cos\theta\right)^2 \ d\theta$$

$$\int \left(\frac{1}{2}\sin 2\theta\right)^2 \ d\theta$$

$$\int \frac{1}{4}\sin^2 2\theta \ d\theta$$

$$\frac{1}{4}\int \sin^2 2\theta \ d\theta$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{4}\int \frac{1}{2}\left[1-\cos 2(2\theta)\right] \ d\theta$$

$$\frac{1}{8}\int 1-\cos 4\theta \ d\theta$$

$$\frac{1}{8}\left(\theta-\frac{1}{4}\sin 4\theta\right)+C$$

$$\frac{1}{8}\theta-\frac{1}{32}\sin 4\theta+C$$

Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int \sin^6 x \cos^4 x \ dx$$

Answer choices:

A $\quad \frac{1}{256}\left(3x - \frac{1}{2}\sin 2x - \frac{1}{2}\cos 8x \sin 2x - \sin 4x + \frac{1}{8}\sin 8x + \frac{1}{5}\sin 10x\right) + C$

B $\quad -\frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x + \frac{1}{8}\sin 8x - \frac{1}{10}\sin 10x\right) + C$

C $\quad \frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x\right) + C$

D $\quad -\frac{1}{256}\left(3x + \sin^2 4x \sin 2x - \frac{5}{2}\sin 2x - \sin 4x\right) + C$

Solution: A

In the specific case where our function is the product of

an **even** number of **sine** factors and

an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

    a. $\sin x \cos x = \frac{1}{2}\sin 2x$,

2. then use the half-angle formulas to make substitutions,

    a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

    b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

    a. $\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int \sin^6 x \cos^4 x \ dx$$

$$\int \sin^2 x \left(\sin x \cos x\right)^4 \ dx$$

$$\int \sin^2 x \left(\frac{1}{2}\sin 2x\right)^4 \ dx$$

$$\frac{1}{16}\int \sin^2 x \sin^4 2x \ dx$$

$$\frac{1}{16}\int \sin^2 x \left(\sin^2 2x\right)^2 \ dx$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{16}\int \frac{1}{2}(1-\cos 2x)\left[\frac{1}{2}(1-\cos 2(2x))\right]^2 \ dx$$

$$\frac{1}{32}\int (1-\cos 2x)\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)^2 \ dx$$

$$\frac{1}{32}\int (1-\cos 2x)\left(\frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{4}\cos^2 4x\right) \ dx$$

$$\frac{1}{32}\int \frac{1}{4}-\frac{1}{2}\cos 4x+\frac{1}{4}\cos^2 4x$$

$$-\frac{1}{4}\cos 2x+\frac{1}{2}\cos 2x\cos 4x-\frac{1}{4}\cos^2 4x\cos 2x \ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int \frac{1}{4}\cos^2 4x+\frac{1}{2}\cos 2x\cos 4x-\frac{1}{4}\cos^2 4x\cos 2x \ dx$$

Using the identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

we'll simplify the integral.

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int\frac{1}{4}\cos^2 4x+\frac{1}{2}\left[\frac{1}{2}\left(\cos(4x-2x)+\cos(4x+2x)\right)\right]$$

$$-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int\frac{1}{4}\cos^2 4x+\frac{1}{4}\cos 2x+\frac{1}{4}\cos 6x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x+\frac{1}{8}\sin 2x\right)$$

$$+\frac{1}{32}\int\frac{1}{4}\cos^2 4x+\frac{1}{4}\cos 6x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x-\frac{1}{8}\sin 2x+\frac{1}{8}\sin 2x+\frac{1}{24}\sin 6x\right)$$

$$+\frac{1}{32}\int\frac{1}{4}\cos^2 4x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{32}\left(\frac{1}{4}x-\frac{1}{8}\sin 4x+\frac{1}{24}\sin 6x\right)$$

$$+\frac{1}{32}\int\frac{1}{4}\cos^2 4x-\frac{1}{4}\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int\cos^2 4x-\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int\cos^2 4x\ dx-\frac{1}{128}\int\cos^2 4x\cos 2x\ dx$$

Use the identity

$$\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos(2x)$$

to rewrite the first integral.

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{128}\int\frac{1}{2}+\frac{1}{2}\cos(2(4x))\ dx-\frac{1}{128}\int\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\int 1+\cos 8x\ dx-\frac{1}{128}\int\cos^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)-\frac{1}{128}\int\cos^2 4x\cos 2x\ dx$$

Use the identity $\cos^2 x=1-\sin^2 x$ to rewrite the second integral.

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)-\frac{1}{128}\int(1-\sin^2 4x)\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{128}\int \cos 2x-\sin^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{128}\left(\frac{1}{2}\sin 2x\right)-\frac{1}{128}\int -\sin^2 4x\cos 2x\ dx$$

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\int \sin^2 4x\cos 2x\ dx$$

Now use integration by parts with

$$u=\sin^2 4x$$

$$du=8\sin 4x\cos 4x\ dx$$

$$dv=\cos 2x\ dx$$

$$v=\frac{1}{2}\sin 2x$$

Focusing on just the remaining integral, we can say

$$\int \sin^2 4x\cos 2x\ dx=uv-\int v\ du$$

$$\int \sin^2 4x \cos 2x \ dx = (\sin^2 4x)\left(\frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\sin 2x(8\sin 4x\cos 4x \ dx)$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 4\int \sin 2x \sin 4x \cos 4x \ dx$$

Use the identity

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

to rewrite the integral.

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 4\int \cos 4x\left[\frac{1}{2}(\cos(4x-2x) - \cos(4x+2x))\right] \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 4x(\cos 2x - \cos 6x) \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 2x\cos 4x - \cos 4x\cos 6x \ dx$$

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x - 2\int \cos 2x\cos 4x \ dx + 2\int \cos 4x\cos 6x \ dx$$

Use the identity

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

to rewrite both integrals.

$$\int \sin^2 4x \cos 2x \ dx = \frac{1}{2}\sin^2 4x \sin 2x$$

$$-2\int\frac{1}{2}(\cos(4x-2x)+\cos(4x+2x))\ dx+2\int\frac{1}{2}(\cos(6x-4x)+\cos(6x+4x))\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x$$

$$-\int\cos 2x+\cos 6x\ dx+\int\cos 2x+\cos 10x\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x$$

$$-\int\cos 2x\ dx-\int\cos 6x\ dx+\int\cos 2x\ dx+\int\cos 10x\ dx$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{2}\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{2}\sin 2x+\frac{1}{10}\sin 10x$$

$$\int\sin^2 4x\cos 2x\ dx=\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{10}\sin 10x$$

Now we can plug this value back in for just the integral in

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\int\sin^2 4x\cos 2x\ dx$$

We get

$$\frac{1}{128}\left(x-\frac{1}{2}\sin 4x+\frac{1}{6}\sin 6x\right)+\frac{1}{256}\left(x+\frac{1}{8}\sin 8x\right)$$

$$-\frac{1}{256}\sin 2x+\frac{1}{128}\left[\frac{1}{2}\sin^2 4x\sin 2x-\frac{1}{6}\sin 6x+\frac{1}{10}\sin 10x\right]$$

$$\frac{1}{128}x-\frac{1}{256}\sin 4x+\frac{1}{768}\sin 6x+\frac{1}{256}x+\frac{1}{2,048}\sin 8x$$

$$-\frac{1}{256}\sin 2x+\frac{1}{256}\sin^2 4x\sin 2x-\frac{1}{768}\sin 6x+\frac{1}{1,280}\sin 10x$$

$$\frac{3}{256}x-\frac{1}{256}\sin 2x-\frac{1}{256}\sin 4x+\frac{1}{2,048}\sin 8x+\frac{1}{1,280}\sin 10x+\frac{1}{256}\sin^2 4x\sin 2x$$

$$\frac{1}{256}\left(3x+\sin^2 4x\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

To reduce the degree of the $\sin^2$ term, we could use the identity

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

to say

$$\sin^2 4x=\frac{1}{2}(1-\cos(2\cdot 4x))$$

$$\sin^2 4x=\frac{1}{2}-\frac{1}{2}\cos 8x$$

Then the expression becomes

$$\frac{1}{256}\left[3x+\left(\frac{1}{2}-\frac{1}{2}\cos 8x\right)\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right]+C$$

$$\frac{1}{256}\left(3x+\frac{1}{2}\sin 2x-\frac{1}{2}\cos 8x\sin 2x-\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

$$\frac{1}{256}\left(3x-\frac{1}{2}\sin 2x-\frac{1}{2}\cos 8x\sin 2x-\sin 4x+\frac{1}{8}\sin 8x+\frac{1}{5}\sin 10x\right)+C$$

Topic: sin^m cos^n, m and n even

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\ d\theta$$

Answer choices:

A $-\frac{\pi}{8}$

B $-\frac{\pi}{32}$

C $\frac{\pi}{8}$

D $\frac{\pi}{32}$

Solution: D

In the specific case where our function is the product of

an **even** number of **sine** factors and

an **even** number of **cosine** factors,

our plan is to

1. create sets of $\sin x \cos x$ and replace each of them with

   a. $\sin x \cos x = \frac{1}{2} \sin 2x$,

2. then use the half-angle formulas to make substitutions,

   a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

   b. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

3. remembering that we may need to use the identity

   a. $\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$

We'll create sets of $\sin x \cos x$ and then use the $\sin x \cos x$ identity to make a substitution.

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^4 \theta \ d\theta$$

$$\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2\cos^2\theta\ d\theta$$

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^2\cos^2\theta\ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\cos^2\theta\ d\theta$$

Now we'll use the $\sin^2 x$ identity to make a second substitution.

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[1-\cos 2(2\theta)\right]\cos^2\theta\ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\cos^2\theta\ d\theta$$

Now we'll use the $\cos^2 x$ identity to make a third substitution.

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\frac{1}{2}(1+\cos 2\theta)\ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{4}+\frac{1}{4}\cos 2\theta-\frac{1}{4}\cos 4\theta-\frac{1}{4}\cos 4\theta\cos 2\theta\ d\theta$$

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta-\cos 4\theta-\cos 4\theta\cos 2\theta\ d\theta$$

Using the identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

we'll simplify the integral.

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta - \cos 4\theta - \left[\frac{1}{2}\left(\cos(4\theta - 2\theta) + \cos(4\theta + 2\theta)\right)\right]\ d\theta$$

$$\frac{1}{16}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta - \cos 4\theta - \frac{1}{2}\cos 2\theta - \frac{1}{2}\cos 6\theta\ d\theta$$

$$\frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta - \frac{1}{4}\sin 2\theta - \frac{1}{12}\sin 6\theta\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\frac{1}{16}\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{1}{4}\sin\left(4\cdot\frac{\pi}{2}\right) - \frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{1}{12}\sin\left(6\cdot\frac{\pi}{2}\right)\right]$$

$$-\frac{1}{16}\left[0 + \frac{1}{2}\sin 2(0) - \frac{1}{4}\sin 4(0) - \frac{1}{4}\sin 2(0) - \frac{1}{12}\sin 6(0)\right]$$

$$\frac{1}{16}\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{1}{4}\sin 2\pi - \frac{1}{4}\sin\pi - \frac{1}{12}\sin 3\pi\right)$$

$$-\frac{1}{16}\left(0 + \frac{1}{2}\sin 0 - \frac{1}{4}\sin 0 - \frac{1}{4}\sin 0 - \frac{1}{12}\sin 0\right)$$

$$\frac{1}{16}\left[\frac{\pi}{2} + \frac{1}{2}(0) - \frac{1}{4}(0) - \frac{1}{4}(0) - \frac{1}{12}(0)\right] - \frac{1}{16}\left[0 + \frac{1}{2}(0) - \frac{1}{4}(0) - \frac{1}{4}(0) - \frac{1}{12}(0)\right]$$

$$\frac{\pi}{32}$$

Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \tan^5 x \sec x \ dx$$

Answer choices:

A $\frac{1}{5}\tan^5 x - \frac{2}{3}\tan^3 x + \tan x + C$

B $\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

C $\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

D $\frac{1}{5}\sec^5 x + \frac{2}{3}\sec^3 x + \sec x + C$

Solution: C

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int \tan^5 x \sec x \ dx$$

$$\int \tan^4 x \tan x \sec x \ dx$$

$$\int \left(\tan^2 x\right)^2 \tan x \sec x \ dx$$

$$\int \left(\sec^2 x - 1\right)^2 \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Substitute into the integral.

$$\int \left(u^2 - 1\right)^2 \ du$$

$$\int u^4 - 2u^2 + 1 \ du$$

$$\frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C$$

Back-substituting for u, we get

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$$

Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int \tan^3 x \sec x \ dx$$

Answer choices:

A $\quad \frac{1}{3}\tan^3 x + \tan x + C$

B $\quad \frac{1}{3}\tan^3 x - \tan x + C$

C $\quad \frac{1}{3}\sec^3 x + \sec x + C$

D $\quad \frac{1}{3}\sec^3 x - \sec x + C$

Solution: D

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int \tan^3 x \sec x \ dx$$

$$\int \tan^2 x \tan x \sec x \ dx$$

$$\int \left(\sec^2 x - 1\right) \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Substitute into the integral.

$$\int u^2 - 1 \ du$$

$$\frac{1}{3}u^3 - u + C$$

Back-substituting for u, we get

$$\frac{1}{3}\sec^3 x - \sec x + C$$

Topic: tan^m sec^n, odd m

Question: Evaluate the trigonometric integral.

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^5 x \sec x \ dx$$

Answer choices:

A $\frac{15}{38}$

B $\frac{38}{15}$

C $\frac{5}{18}$

D $\frac{18}{5}$

Solution: B

In the specific case where our function is the product of

an **odd** number of **tangent** factors and

an **even or odd** number of **secant** factors,

our plan is to

1. save one $\sec x \tan x$ factor and use the identity $\tan^2 x = \sec^2 x - 1$ to write the other cosine factors in terms of secant, then

2. use u-substitution with $u = \sec x$.

We'll separate a single tangent to make $\sec x \tan x$, and then replace the remaining tangent factors using the identity.

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^5 x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \tan^4 x \tan x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \left(\tan^2 x\right)^2 \tan x \sec x \ dx$$

$$\int_{\frac{2\pi}{3}}^{\pi} \left(\sec^2 x - 1\right)^2 \tan x \sec x \ dx$$

Using u-substitution with $u = \sec x$, we get

$$u = \sec x$$

$$du = \sec x \tan x \ dx$$

Because we're dealing with a definite integral, we have to either change the limits of integration when we make our substitution, or we have to indicate that the limits of integration are in terms of x until we back-substitute. Substitute into the integral.

$$\int_{x=\frac{2\pi}{3}}^{x=\pi} \left(u^2 - 1\right)^2 \ du$$

$$\int_{x=\frac{2\pi}{3}}^{x=\pi} u^4 - 2u^2 + 1 \ du$$

$$\frac{1}{5}u^5 - \frac{2}{3}u^3 + u \Big|_{x=\frac{2\pi}{3}}^{x=\pi}$$

Back-substituting for u, we get

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x \Big|_{\frac{2\pi}{3}}^{\pi}$$

$$\frac{1}{5}\sec^5 \pi - \frac{2}{3}\sec^3 \pi + \sec \pi - \left(\frac{1}{5}\sec^5 \frac{2\pi}{3} - \frac{2}{3}\sec^3 \frac{2\pi}{3} + \sec \frac{2\pi}{3}\right)$$

$$\frac{1}{5\cos^5 \pi} - \frac{2}{3\cos^3 \pi} + \frac{1}{\cos \pi} - \left(\frac{1}{5\cos^5 \frac{2\pi}{3}} - \frac{2}{3\cos^3 \frac{2\pi}{3}} + \frac{1}{\cos \frac{2\pi}{3}}\right)$$

$$\frac{1}{5(-1)^5}-\frac{2}{3(-1)^3}+\frac{1}{(-1)}-\left[\frac{1}{5\left(-\frac{1}{2}\right)^5}-\frac{2}{3\left(-\frac{1}{2}\right)^3}+\frac{1}{\left(-\frac{1}{2}\right)}\right]$$

$$-\frac{1}{5}+\frac{2}{3}-1-\left(-\frac{1}{\frac{5}{32}}+\frac{2}{\frac{3}{8}}-2\right)$$

$$-\frac{1}{5}+\frac{2}{3}-1+\frac{32}{5}-\frac{16}{3}+2$$

$$\frac{31}{5}-\frac{14}{3}+1$$

$$\frac{93}{15}-\frac{70}{15}+\frac{15}{15}$$

$$\frac{38}{15}$$

Topic: tan^m sec^n, even n

Question: Evaluate the trigonometric integral.

$$\int \tan^2 x \sec^4 x \ dx$$

Answer choices:

A $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + C$

B $\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

C $\frac{1}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$

D $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

Solution: B

In the specific case where our function is the product of

an **even** number of **secant** factors and

an **even or odd** number of **tangent** factors,

our plan is to

1. save one $\sec^2 x$ factor and use the identity $\sec^2 x = 1 + \tan^2 x$ to write the other cosine factors in terms of tangent, then

2. use u-substitution with $u = \tan x$.

We'll separate a $\sec^2 x$, and then replace the remaining secant factors using the identity.

$$\int \tan^2 x \sec^4 x \ dx$$

$$\int \tan^2 x \sec^2 x \sec^2 x \ dx$$

$$\int \tan^2 x \sec^2 x \left(1 + \tan^2 x\right) \ dx$$

Using u-substitution with $u = \tan x$, we get

$$u = \tan x$$

$$du = \sec^2 x \ dx$$

Substitute into the integral.

$$\int u^2 \sec^2 x \left(1+u^2\right) \ dx$$

$$\int u^2 \left(1+u^2\right) \left(\sec^2 x \ dx\right)$$

$$\int u^2 \left(1+u^2\right) \left(du\right)$$

$$\int u^2 \left(1+u^2\right) \ du$$

$$\int u^2 + u^4 \ du$$

$$\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

Back-substituting for u, we get

$$\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Topic: tan^m sec^n, even n

Question: Evaluate the trigonometric integral.

$$\int \tan^4 x \sec^4 x \ dx$$

Answer choices:

A $\quad \frac{1}{7}\sec^7 x + \frac{1}{5}\sec^5 x + C$

B $\quad \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

C $\quad \frac{1}{7}\tan^7 x - \frac{1}{5}\tan^5 x + C$

D $\quad \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

Solution: D

In the specific case where our function is the product of

an **even** number of **secant** factors and

an **even or odd** number of **tangent** factors,

our plan is to

1. save one $\sec^2 x$ factor and use the identity $\sec^2 x = 1 + \tan^2 x$ to write the other cosine factors in terms of tangent, then

2. use u-substitution with $u = \tan x$.

We'll separate a $\sec^2 x$, and then replace the remaining secant factors using the identity.

$$\int \tan^4 x \sec^4 x \ dx$$

$$\int \tan^4 x \sec^2 x \sec^2 x \ dx$$

$$\int \tan^4 x \sec^2 x \left(1 + \tan^2 x\right) \ dx$$

Using u-substitution with $u = \tan x$, we get

$$u = \tan x$$

$$du = \sec^2 x \ dx$$

Substitute into the integral.

$$\int u^4 \sec^2 x \left(1 + u^2\right) \ dx$$

$$\int u^4 \left(1 + u^2\right) \left(\sec^2 x \ dx\right)$$

$$\int u^4 \left(1 + u^2\right) \left(du\right)$$

$$\int u^4 \left(1 + u^2\right) \ du$$

$$\int u^4 + u^6 \ du$$

$$\frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

Back-substituting for u, we get

$$\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

Topic: tan^m sec^n, even n

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^6 x \ dx$$

Answer choices:

A $-\frac{105}{8}$

B $\frac{105}{8}$

C $-\frac{92}{105}$

D $\frac{92}{105}$

Solution: D

In the specific case where our function is the product of

an **even** number of **secant** factors and

an **even or odd** number of **tangent** factors,

our plan is to

1. save one $\sec^2 x$ factor and use the identity $\sec^2 x = 1 + \tan^2 x$ to write the other cosine factors in terms of tangent, then

2. use u-substitution with $u = \tan x$.

We'll separate a $\sec^2 x$, and then replace the remaining secant factors using the identity.

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^6 x \ dx$$

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \sec^4 x \ dx$$

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \left(\sec^2 x\right)^2 \ dx$$

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \left(1 + \tan^2 x\right)^2 \ dx$$

Using u-substitution with $u = \tan x$, we get

$$u = \tan x$$

$$du = \sec^2 x \ dx$$

Because we're dealing with a definite integral, we have to either change the limits of integration when we make our substitution, or we have to indicate that the limits of integration are in terms of x until we back-substitute. Substitute into the integral.

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 \sec^2 x \left(1 + u^2\right)^2 \ dx$$

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 \left(1 + u^2\right)^2 \left(\sec^2 x \ dx\right)$$

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 \left(1 + u^2\right)^2 \left(du\right)$$

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 \left(1 + u^2\right)^2 \ du$$

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 \left(1 + 2u^2 + u^4\right) \ du$$

$$\int_{x=0}^{x=\frac{\pi}{4}} u^2 + 2u^4 + u^6 \ du$$

$$\left(\frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7\right)\Bigg|_{x=0}^{x=\frac{\pi}{4}}$$

Back-substituting for u, we get

$$\left(\frac{1}{3}\tan^3 x+\frac{2}{5}\tan^5 x+\frac{1}{7}\tan^7 x\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$\left(\frac{1}{3}\tan^3\frac{\pi}{4}+\frac{2}{5}\tan^5\frac{\pi}{4}+\frac{1}{7}\tan^7\frac{\pi}{4}\right)-\left(\frac{1}{3}\tan^3 0+\frac{2}{5}\tan^5 0+\frac{1}{7}\tan^7 0\right)$$

$$\left(\frac{1}{3}\frac{\sin^3\frac{\pi}{4}}{\cos^3\frac{\pi}{4}}+\frac{2}{5}\frac{\sin^5\frac{\pi}{4}}{\cos^5\frac{\pi}{4}}+\frac{1}{7}\frac{\sin^7\frac{\pi}{4}}{\cos^7\frac{\pi}{4}}\right)-\left(\frac{1}{3}\frac{\sin^3 0}{\cos^3 0}+\frac{2}{5}\frac{\sin^5 0}{\cos^5 0}+\frac{1}{7}\frac{\sin^7 0}{\cos^7 0}\right)$$

$$\left[\frac{1}{3}\frac{\left(\frac{\sqrt{2}}{2}\right)^3}{\left(\frac{\sqrt{2}}{2}\right)^3}+\frac{2}{5}\frac{\left(\frac{\sqrt{2}}{2}\right)^5}{\left(\frac{\sqrt{2}}{2}\right)^5}+\frac{1}{7}\frac{\left(\frac{\sqrt{2}}{2}\right)^7}{\left(\frac{\sqrt{2}}{2}\right)^7}\right]-\left(\frac{1}{3}\frac{0^3}{1^3}+\frac{2}{5}\frac{0^5}{1^5}+\frac{1}{7}\frac{0^7}{1^7}\right)$$

$$\left[\frac{1}{3}(1)+\frac{2}{5}(1)+\frac{1}{7}(1)\right]-\left[\frac{1}{3}(0)+\frac{2}{5}(0)+\frac{1}{7}(0)\right]$$

$$\frac{1}{3}+\frac{2}{5}+\frac{1}{7}$$

$$\frac{35}{105}+\frac{42}{105}+\frac{15}{105}$$

$$\frac{92}{105}$$

Topic: sin(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int \sin 4x \cos 3x \ dx$$

Answer choices:

A $\quad \frac{1}{2}\cos x + \frac{1}{14}\cos 7x + C$

B $\quad \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

C $\quad -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C$

D $\quad -\frac{1}{2}\sin x - \frac{1}{14}\sin 7x + C$

Solution: C

In the specific case where our function is the product of

one **sine** factor and

one **cosine** factor,

our plan is to

1. use the identity $\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \sin 4x\cos 3x\ dx$$

$$\int \frac{1}{2}\left[\sin(4x-3x)+\sin(4x+3x)\right]\ dx$$

$$\frac{1}{2}\int \sin x+\sin 7x\ dx$$

$$\frac{1}{2}\left(-\cos x-\frac{1}{7}\cos 7x\right)+C$$

$$-\frac{1}{2}\cos x-\frac{1}{14}\cos 7x+C$$

Topic: sin(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int \sin 7x \cos 2x \ dx$$

Answer choices:

A $\quad \frac{1}{10}\sin 5x + \frac{1}{18}\sin 9x + C$

B $\quad -\frac{1}{10}\sin 5x - \frac{1}{18}\sin 9x + C$

C $\quad \frac{1}{10}\cos 5x + \frac{1}{18}\cos 9x + C$

D $\quad -\frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + C$

Solution: D

In the specific case where our function is the product of

one **sine** factor and

one **cosine** factor,

our plan is to

1. use the identity $\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \sin 7x\cos 2x\ dx$$

$$\int \frac{1}{2}\left[\sin(7x-2x)+\sin(7x+2x)\right]\ dx$$

$$\frac{1}{2}\int \sin 5x+\sin 9x\ dx$$

$$\frac{1}{2}\left(-\frac{1}{5}\cos 5x-\frac{1}{9}\cos 9x\right)+C$$

$$-\frac{1}{10}\cos 5x-\frac{1}{18}\cos 9x+C$$

Topic: sin(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{2}} \sin 5x \cos 2x \ dx$$

Answer choices:

A $\frac{21}{5}$

B $\frac{5}{21}$

C $-\frac{21}{5}$

D $-\frac{5}{21}$

Solution: B

In the specific case where our function is the product of

one **sine** factor and

one **cosine** factor,

our plan is to

1. use the identity $\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int_0^{\frac{\pi}{2}}\sin 5x\cos 2x\ dx$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{2}\left[\sin(5x-2x)+\sin(5x+2x)\right]\ dx$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 3x+\sin 7x\ dx$$

$$\frac{1}{2}\left(-\frac{1}{3}\cos 3x-\frac{1}{7}\cos 7x\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\left(-\frac{1}{6}\cos 3x-\frac{1}{14}\cos 7x\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\left[-\frac{1}{6}\cos 3\left(\frac{\pi}{2}\right)-\frac{1}{14}\cos 7\left(\frac{\pi}{2}\right)\right]-\left[-\frac{1}{6}\cos 3(0)-\frac{1}{14}\cos 7(0)\right]$$

$$\left[-\frac{1}{6}\cos\frac{3\pi}{2}-\frac{1}{14}\cos\frac{7\pi}{2}\right]-\left(-\frac{1}{6}\cos 0-\frac{1}{14}\cos 0\right)$$

$$\left[-\frac{1}{6}(0)-\frac{1}{14}(0)\right]-\left[-\frac{1}{6}(1)-\frac{1}{14}(1)\right]$$

$$\frac{1}{6}+\frac{1}{14}$$

$$\frac{14}{84}+\frac{6}{84}$$

$$\frac{20}{84}$$

$$\frac{5}{21}$$

Topic: sin(mx) sin(nx)

Question: Evaluate the trigonometric integral.

$$\int \sin 3x \sin 2x \ dx$$

Answer choices:

A $\quad -\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$

B $\quad \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$

C $\quad \frac{1}{2}\cos x + \frac{1}{10}\cos 5x + C$

D $\quad \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

Solution: B

In the specific case where our function is the product of

two **sine** factors,

our plan is to

1. use the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \sin 3x \sin 2x \ dx$$

$$\int \frac{1}{2}\left[\cos(3x-2x)-\cos(3x+2x)\right] \ dx$$

$$\frac{1}{2}\int \cos x - \cos 5x \ dx$$

$$\frac{1}{2}\left(\sin x - \frac{1}{5}\sin 5x\right)+C$$

$$\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$$

Topic: sin(mx) sin(nx)

Question: Evaluate the trigonometric integral.

$$\int \sin 5x \sin 2x \ dx$$

Answer choices:

A $\frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$

B $\frac{1}{6}\sin 3x + \frac{1}{14}\sin 7x + C$

C $\frac{1}{6}\cos 3x - \frac{1}{14}\cos 7x + C$

D $\frac{1}{6}\cos 3x + \frac{1}{14}\cos 7x + C$

Solution: A

In the specific case where our function is the product of

two **sine** factors,

our plan is to

1. use the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \sin 5x \sin 2x \ dx$$

$$\int \frac{1}{2}\left[\cos(5x-2x) - \cos(5x+2x)\right] \ dx$$

$$\frac{1}{2}\int \cos 3x - \cos 7x \ dx$$

$$\frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C$$

$$\frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$$

Topic: sin(mx) sin(nx)

Question: Evaluate the trigonometric integral.

$$\int_0^{\frac{\pi}{2}} \sin 4x \sin 3x \ dx$$

Answer choices:

A $-\frac{4}{7}$

B $\frac{3}{7}$

C $-\frac{3}{7}$

D $\frac{4}{7}$

Solution: D

In the specific case where our function is the product of

two **sine** factors,

our plan is to

1. use the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int_0^{\frac{\pi}{2}} \sin 4x \sin 3x \ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos(4x-3x) - \cos(4x+3x)\right] \ dx$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x - \cos 7x \ dx$$

$$\frac{1}{2}\left(\sin x - \frac{1}{7}\sin 7x\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\left(\frac{1}{2}\sin x - \frac{1}{14}\sin 7x\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\left[\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{14}\sin 7\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\sin(0) - \frac{1}{14}\sin 7(0)\right]$$

$$\left(\frac{1}{2}\sin\frac{\pi}{2}-\frac{1}{14}\sin\frac{7\pi}{2}\right)-\left[\frac{1}{2}(0)-\frac{1}{14}(0)\right]$$

$$\frac{1}{2}(1)-\frac{1}{14}(-1)$$

$$\frac{1}{2}+\frac{1}{14}$$

$$\frac{7}{14}+\frac{1}{14}$$

$$\frac{8}{14}$$

$$\frac{4}{7}$$

Topic: cos(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int \cos 4x \cos 2x \ dx$$

Answer choices:

A $\quad \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$

B $\quad \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

C $\quad \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C$

D $\quad \frac{1}{4}\cos 2x + \frac{1}{12}\cos 6x + C$

Solution: A

In the specific case where our function is the product of

two **cosine** factors,

our plan is to

1. use the identity $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \cos 4x \cos 2x \ dx$$

$$\int \frac{1}{2}\left[\cos(4x-2x)+\cos(4x+2x)\right] \ dx$$

$$\frac{1}{2}\int \cos 2x + \cos 6x \ dx$$

$$\frac{1}{2}\left(\frac{1}{2}\sin 2x + \frac{1}{6}\sin 6x\right) + C$$

$$\frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$$

Topic: cos(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int \cos 2\pi x \cos \pi x \ dx$$

Answer choices:

A $\quad \frac{1}{2\pi}\cos \pi x - \frac{1}{6\pi}\cos 3\pi x + C$

B $\quad \frac{1}{2\pi}\sin \pi x - \frac{1}{6\pi}\sin 3\pi x + C$

C $\quad \frac{1}{2\pi}\cos \pi x + \frac{1}{6\pi}\cos 3\pi x + C$

D $\quad \frac{1}{2\pi}\sin \pi x + \frac{1}{6\pi}\sin 3\pi x + C$

Solution: D

In the specific case where our function is the product of

two **cosine** factors,

our plan is to

1. use the identity $\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int \cos 2\pi x\cos \pi x\ dx$$

$$\int \frac{1}{2}\left[\cos(2\pi x-\pi x)+\cos(2\pi x+\pi x)\right]\ dx$$

$$\frac{1}{2}\int \cos \pi x+\cos 3\pi x\ dx$$

$$\frac{1}{2}\left(\frac{1}{\pi}\sin \pi x+\frac{1}{3\pi}\sin 3\pi x\right)+C$$

$$\frac{1}{2\pi}\sin \pi x+\frac{1}{6\pi}\sin 3\pi x+C$$

Topic: cos(mx) cos(nx)

Question: Evaluate the trigonometric integral.

$$\int_{\frac{\pi}{2}}^{\pi} \cos 5x \cos 3x \ dx$$

Answer choices:

A $\quad 0$

B $\quad 1$

C $\quad -1$

D $\quad \pi$

Solution: A

In the specific case where our function is the product of

two **cosine** factors,

our plan is to

1. use the identity $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

We'll use the identity to simplify the integral.

$$\int_{\frac{\pi}{2}}^{\pi} \cos 5x \cos 3x \ dx$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\left[\cos(5x-3x)+\cos(5x+3x)\right] \ dx$$

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \cos 2x + \cos 8x \ dx$$

$$\frac{1}{2}\left(\frac{1}{2}\sin 2x + \frac{1}{8}\sin 8x\right)\Bigg|_{\frac{\pi}{2}}^{\pi}$$

$$\left(\frac{1}{4}\sin 2x + \frac{1}{16}\sin 8x\right)\Bigg|_{\frac{\pi}{2}}^{\pi}$$

$$\left(\frac{1}{4}\sin 2\pi + \frac{1}{16}\sin 8\pi\right) - \left(\frac{1}{4}\sin 2\cdot\frac{\pi}{2} + \frac{1}{16}\sin 8\cdot\frac{\pi}{2}\right)$$

$$\left(\frac{1}{4}\sin 2\pi + \frac{1}{16}\sin 8\pi\right) - \left(\frac{1}{4}\sin \pi + \frac{1}{16}\sin 4\pi\right)$$

$$\frac{1}{4}\sin 2\pi + \frac{1}{16}\sin 8\pi - \frac{1}{4}\sin \pi - \frac{1}{16}\sin 4\pi$$

$$\frac{1}{4}(0) + \frac{1}{16}(0) - \frac{1}{4}(0) - \frac{1}{16}(0)$$

$$0$$

Topic: Hyperbolic integrals

Question: Evaluate the hyperbolic integral.

$$\int x \cosh\left(x^2 + 3\right)\ dx$$

Answer choices:

A $\sinh\left(x^2 - 3\right) + C$

B $\sinh\left(x^2\right) + C$

C $\frac{1}{2}\sinh\left(x^2\right) + C$

D $\frac{1}{2}\sinh\left(x^2 + 3\right) + C$

Solution: D

Let

$$u=x^2+3$$

$$du=2x\ dx$$

$$\frac{du}{2x}=dx$$

When we plug these values into the integral, we get

$$\int x\cosh\left(x^2+3\right)\ dx$$

$$\int x\cosh u\ \frac{du}{2x}$$

$$\frac{1}{2}\int\cosh u\ du$$

Knowing that

$$\int\cosh x\ dx=\sinh x+C$$

we get

$$\frac{1}{2}\sinh\left(x^2+3\right)+C$$

Topic: Hyperbolic integrals

Question: Evaluate the hyperbolic integral.

$$\int x^2 \text{sech}\left(\frac{1}{3}x^3\right)\ dx$$

Answer choices:

A $\quad \sin^{-1}\left[\sinh\left(\frac{1}{3}x^3\right)\right] + C$

B $\quad \cos^{-1}\left[\sinh\left(\frac{1}{3}x^3\right)\right] + C$

C $\quad \tan^{-1}\left[\sinh\left(\frac{1}{3}x^3\right)\right] + C$

D $\quad \cot^{-1}\left[\sinh\left(\frac{1}{3}x^3\right)\right] + C$

Solution: C

Let

$$u = \frac{1}{3}x^3$$

$$du = x^2\ dx$$

When we plug these values into the integral, we get

$$\int x^2 \text{sech}\left(\frac{1}{3}x^3\right)\ dx$$

$$\int \text{sech}\left(\frac{1}{3}x^3\right)\ \left(x^2\ dx\right)$$

$$\int \text{sech}\ u\ du$$

Knowing that

$$\int \text{sech}\ u\ du = \tan^{-1}\left(\sinh u\right) + C$$

we get

$$\tan^{-1}\left(\sinh u\right) + C$$

$$\tan^{-1}\left[\sinh\left(\frac{1}{3}x^3\right)\right] + C$$

Topic: Hyperbolic integrals

Question: Evaluate the hyperbolic integral.

$$\int_{-\ln 4}^{-\ln 12} 2e^t \sinh t \ dt$$

Answer choices:

A $\frac{2}{5} + \ln 2$

B $\ln 5 - \frac{1}{12}$

C $\frac{1}{36} - \ln 3$

D $\ln 3 - \frac{1}{36}$

Solution: D

At first, the integral may look like a hyperbolic function problem. However, the exponential expression in the integral complicates the problem. So we decompose the hyperbolic expression into its exponential definition

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

and solve the integral as an exponential function.

$$\int_{-\ln 4}^{-\ln 12} 2e^t \sinh t \ dt$$

$$\int_{-\ln 4}^{-\ln 12} 2e^t \left(\frac{e^t - e^{-t}}{2}\right) \ dt$$

$$\int_{-\ln 4}^{-\ln 12} \left(e^{2t} - e^0\right) \ dt$$

$$\int_{-\ln 4}^{-\ln 12} e^{2t} - 1 \ dt$$

$$\frac{1}{2}e^{2t} - t \Big|_{-\ln 4}^{-\ln 12}$$

$$\left[\frac{1}{2}e^{2(-\ln 12)} - (-\ln 12)\right] - \left[\frac{1}{2}e^{2(-\ln 4)} - (-\ln 4)\right]$$

$$\frac{1}{2}e^{2(-\ln 12)} + \ln 12 - \frac{1}{2}e^{2(-\ln 4)} - \ln 4$$

$$\frac{1}{2}e^{\ln 12^{-2}} + \ln 12 - \frac{1}{2}e^{\ln 4^{-2}} - \ln 4$$

$$\frac{1}{2}12^{-2} + \ln 12 - \frac{1}{2}4^{-2} - \ln 4$$

$$\frac{1}{288} - \frac{1}{32} + \ln 12 - \ln 4$$

$$\frac{1}{288} - \frac{9}{288} + \ln \frac{12}{4}$$

$$-\frac{8}{288} + \ln 3$$

$$\ln 3 - \frac{1}{36}$$

Topic: Inverse hyperbolic integrals

Question: Use inverse hyperbolic functions to evaluate the integral.

$$\int_5^8 \frac{dx}{9 - x^2}$$

Answer choices:

A $\frac{1}{6} \ln \frac{20}{11}$

B $\frac{1}{6} \ln \frac{11}{20}$

C $4 \ln 5$

D $-4 \ln 2$

Solution: B

Rewriting the integral gives

$$\int_5^8 \frac{dx}{9-x^2}$$

$$\int_5^8 \frac{1}{3^2-x^2}\ dx$$

The integral is of the form

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\coth^{-1}\frac{u}{a}+C$$

Note that when the integrated function matches the form

$$\int \frac{1}{a^2-u^2}\ du$$

the integration formula you'll use depends on the relationship between u and a. More specifically,

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\coth^{-1}\frac{u}{a}+C \qquad \text{when } u^2>a^2$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\tanh^{-1}\frac{u}{a}+C \qquad \text{when } u^2<a^2$$

Since the limits of integration are given as $[5,8]$, x will always be between 5 and 8, which means x^2, or u^2, will always be between $5^2=25$ and $8^2=64$. Since $a=3$ and $3^2=9$, we can say that $u^2>a^2$, we'll use the hyperbolic cotangent formula, and therefore the integral becomes

$$\frac{1}{3}\coth^{-1}\frac{x}{3}\Big|_5^8$$

$$\frac{1}{3}\coth^{-1}\frac{8}{3}-\frac{1}{3}\coth^{-1}\frac{5}{3}$$

Knowing that

$$\coth^{-1}x=\frac{1}{2}\ln\frac{x+1}{x-1}$$

we can apply the formula and get

$$\frac{1}{3}\left(\frac{1}{2}\ln\frac{\frac{8}{3}+1}{\frac{8}{3}-1}\right)-\frac{1}{3}\left(\frac{1}{2}\ln\frac{\frac{5}{3}+1}{\frac{5}{3}-1}\right)$$

$$\frac{1}{6}\ln\frac{\frac{8}{3}+\frac{3}{3}}{\frac{8}{3}-\frac{3}{3}}-\frac{1}{6}\ln\frac{\frac{5}{3}+\frac{3}{3}}{\frac{5}{3}-\frac{3}{3}}$$

$$\frac{1}{6}\ln\frac{\frac{11}{3}}{\frac{5}{3}}-\frac{1}{6}\ln\frac{\frac{8}{3}}{\frac{2}{3}}$$

$$\frac{1}{6}\ln\left(\frac{11}{3}\cdot\frac{3}{5}\right)-\frac{1}{6}\ln\left(\frac{8}{3}\cdot\frac{3}{2}\right)$$

$$\frac{1}{6}\ln\frac{11}{5}-\frac{1}{6}\ln\frac{8}{2}$$

$$\frac{1}{6}\ln\frac{\frac{11}{5}}{\frac{8}{2}}$$

$$\frac{1}{6}\ln\left(\frac{11}{5}\cdot\frac{2}{8}\right)$$

$$\frac{1}{6}\ln\frac{11}{20}$$

Topic: Inverse hyperbolic integrals

Question: Evaluate the integral using integration of inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2+4x+8}}\ dx$$

Answer choices:

A $\cosh^{-1}\left(\frac{x+2}{2}\right)+C$

B $\frac{1}{2}\tanh^{-1}\left(\frac{x+2}{2}\right)+C$

C $\sinh^{-1}\left(\frac{x+2}{2}\right)+C$

D $\frac{1}{2}\sinh^{-1}\left(\frac{x+2}{2}\right)$

Solution: C

An integral of inverse hyperbolic functions takes on one of these common patterns.

$$\int \frac{1}{\sqrt{1+u^2}}\ du = \sinh^{-1}u + C$$

$$\int \frac{1}{\sqrt{u^2-1}}\ du = \cosh^{-1}u + C$$

$$\int \frac{1}{1-u^2}\ du = \tanh^{-1}u + C$$

We'll start by manipulating the integrand. The first step is the change the denominator so it contains a squared binomial. If we split the 8 into two 4's, we can accomplish this. The new integral is

$$\int \frac{1}{\sqrt{4+\left(x^2+4x+4\right)}}\ dx$$

$$\int \frac{1}{\sqrt{4+(x+2)^2}}\ dx$$

Now the denominator needs to be changed to the form $1+u^2$ so if we divide the terms in the denominator by $\sqrt{4}$, we can convert it to what we want. However, if we divide the denominator by $\sqrt{4}$, we also have to divide the numerator by $\sqrt{4}$.

$$\int \frac{\frac{1}{\sqrt{4}}}{\sqrt{\frac{4}{4}+\frac{(x+2)^2}{4}}}\ dx$$

Simplify the denominator and include the 4 in the squared binomial, as a 2.

$$\int \frac{\frac{1}{2}}{\sqrt{1+\left(\frac{x+2}{2}\right)^2}}\ dx$$

Now the integral is almost in the form of the $\sinh^{-1}$ integral. The angle is the function

$$u=\frac{x+2}{2}$$

The derivative is $du/dx=1/2$, so $dx=2\ du$. The integral becomes

$$\int \frac{\frac{1}{2}}{\sqrt{1+u^2}}\cdot 2\ du$$

$$\int \frac{1}{\sqrt{1+u^2}}\ du$$

Now we can evaluate the integral.

$$\sinh^{-1} u+C$$

$$\sinh^{-1}\left(\frac{x+2}{2}\right)+C$$

Topic: Inverse hyperbolic integrals

Question: Evaluate the integral using integration of inverse hyperbolic functions.

$$\int_0^1 \frac{x}{4-x^4}\ dx$$

Answer choices:

A $\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)$

B $\frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\right)$

C $\frac{1}{4}\sinh^{-1}\left(\frac{1}{2}\right)$

D $\frac{1}{2}\sinh^{-1}\left(\frac{1}{2}\right)$

Solution: A

We should use a substitution with $u^2 = x^4$, and therefore $u = x^2$, $du/dx = 2x$, $du = 2x\ dx$, or $dx = du/2x$.

Convert the bounds $x = [0,1]$ into bounds in terms of u, using $u = x^2$.

$$u = 0^2 = 0$$

$$u = 1^2 = 1$$

Then the integral becomes

$$\int_0^1 \frac{x}{4 - u^2} \left(\frac{du}{2x} \right)$$

$$\frac{1}{2} \int_0^1 \frac{1}{4 - u^2}\ du$$

$$\frac{1}{2} \int_0^1 \frac{1}{2^2 - u^2}\ du$$

Whether this particular integrand integrates to inverse hyperbolic tangent or inverse hyperbolic cotangent depends on the relationship between a^2 and u^2.

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\text{arctanh}\left(\frac{u}{a} \right) + C \qquad \text{if } u^2 < a^2$$

$$\int \frac{1}{a^2 - u^2}\ du = \frac{1}{a}\text{arccoth}\left(\frac{u}{a} \right) + C \qquad \text{if } u^2 > a^2$$

Because the bounds on the integral are $u = [0,1]$, the largest possible value of u^2 in the interval is $u^2 = 1^2 = 1$. Then because the value of a^2 is $a^2 = 2^2 = 4$, we know $u^2 = 1 < a^2 = 4$, which means we'll evaluate this integral to inverse hyperbolic tangent, instead of inverse hyperbolic cotangent.

$$\frac{1}{2}\left(\frac{1}{2}\tanh^{-1}\left(\frac{u}{2}\right)\right)\Bigg|_0^1$$

$$\frac{1}{4}\tanh^{-1}\left(\frac{u}{2}\right)\Bigg|_0^1$$

Evaluate over the interval.

$$\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)-\frac{1}{4}\tanh^{-1}\left(\frac{0}{2}\right)$$

$$\frac{1}{4}\tanh^{-1}\left(\frac{1}{2}\right)$$

Topic: Trigonometric substitution setup

Question: Set up this integral for trigonometric substitution. Simplify, but don't evaluate the integral.

$$\int \frac{x}{\sqrt{4-9x^2}}\ dx$$

Answer choices:

A $\quad \dfrac{2}{9}\displaystyle\int \cos\theta\ d\theta$

B $\quad \displaystyle\int \sin\theta\ d\theta$

C $\quad \displaystyle\int \cos\theta\ d\theta$

D $\quad \dfrac{2}{9}\displaystyle\int \sin\theta\ d\theta$

Solution: D

The question asks us to set up this integral for trigonometric substitution.

$$\int \frac{x}{\sqrt{4-9x^2}}\ dx$$

The integral contains an expression of the form a^2-u^2 where a^2 is a number and u^2 is a function of x. This format requires the trigonometric substitution to be in the form $u=a\sin\theta$.

Now we'll use the values in the integral to find a and u.

$$a^2=4$$

$$a=2$$

and

$$u^2=9x^2$$

$$u=3x$$

This means that

$$u=a\sin\theta$$

$$3x=2\sin\theta$$

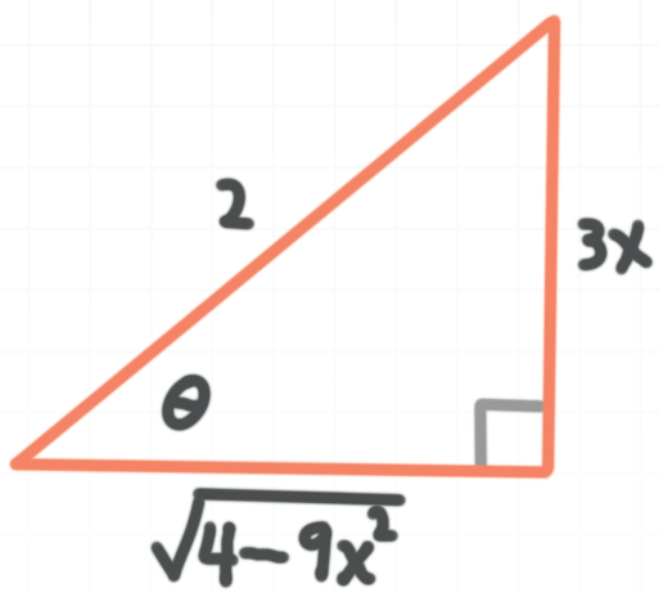
$$\sin\theta=\frac{3x}{2}$$

$$\theta=\arcsin\left(\frac{3x}{2}\right)$$

To put this in the perspective of right triangle trigonometry, recall that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

This means we're dealing with a right triangle like this



We'll solve for x in terms of θ.

$$3x = 2\sin\theta$$

$$x = \frac{2}{3}\sin\theta$$

$$dx = \frac{2}{3}\cos\theta\ d\theta$$

We're finally ready to do the trigonometric substitution.

$$\int \frac{x}{\sqrt{4-9x^2}}\ dx$$

$$\int \frac{\left(\frac{2}{3}\sin\theta\right)}{\sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2}}\left(\frac{2}{3}\cos\theta\right)\ d\theta$$

$$\int \frac{\frac{2}{3}\sin\theta}{\sqrt{4-9\left(\frac{4}{9}\sin^2\theta\right)}}\frac{2}{3}\cos\theta\ d\theta$$

We can cancel the 9 in the radical. We can also remove the fractions, simplify them, and place them in front of the integral.

$$\frac{4}{9}\int \frac{\sin\theta\cos\theta}{\sqrt{4-4\sin^2\theta}}\ d\theta$$

$$\frac{4}{9}\int \frac{\sin\theta\cos\theta}{\sqrt{4\left(1-\sin^2\theta\right)}}\ d\theta$$

$$\frac{4}{9}\int \frac{\sin\theta\cos\theta}{\sqrt{4\left(\cos^2\theta\right)}}\ d\theta$$

$$\frac{4}{9}\int \frac{\sin\theta\cos\theta}{2\left(\cos\theta\right)}\ d\theta$$

$$\frac{2}{9}\int \sin\theta\ d\theta$$

Therefore,

$$\int \frac{x}{\sqrt{4-9x^2}}\ dx = \frac{2}{9}\int \sin\theta\ d\theta$$

Topic: Trigonometric substitution setup

Question: Set up this integral for trigonometric substitution. Simplify, but don't evaluate the integral.

$$\int \sqrt{9x^2 + 16}\ dx$$

Answer choices:

A $\quad \dfrac{16}{3} \int \sec^2\theta\ d\theta$

B $\quad \dfrac{16}{3} \int \sec^3\theta\ d\theta$

C $\quad \dfrac{16}{3} \int \tan^3\theta\ d\theta$

D $\quad \dfrac{16}{3} \int \tan^2\theta\ d\theta$

Solution: B

The question asks us to set up this integral for trigonometric substitution.

$$\int \sqrt{9x^2 + 16}\ dx$$

The integral contains an expression of the form $u^2 + a^2$ where a^2 is a number and u^2 is a function of x. This format requires the trigonometric substitution to be in the form $u = a \tan \theta$.

Now we'll use the values in the integral to find a and u.

$$a^2 = 16$$

$$a = 4$$

and

$$u^2 = 9x^2$$

$$u = 3x$$

This means that

$$u = a \tan \theta$$

$$3x = 4 \tan \theta$$

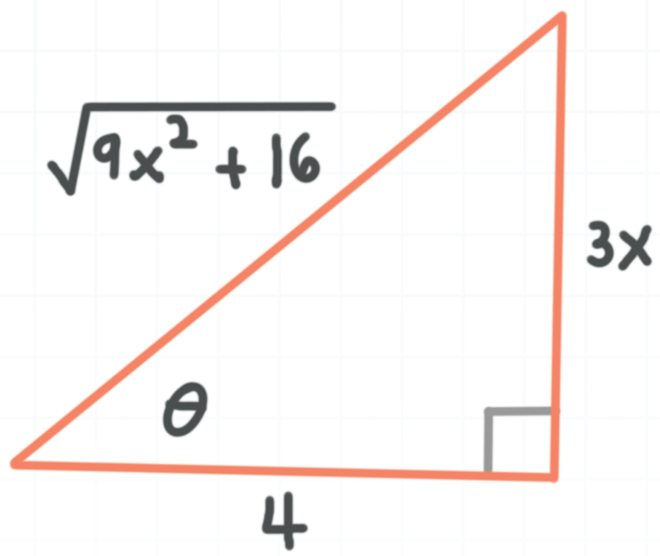
$$\tan \theta = \frac{3x}{4}$$

$$\theta = \arctan \left(\frac{3x}{4} \right)$$

To put this in the perspective of right triangle trigonometry, recall that

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

This means we're dealing with a right triangle like this



We'll solve for x in terms of θ.

$$3x = 4\tan\theta$$

$$x = \frac{4}{3}\tan\theta$$

$$dx = \frac{4}{3}\sec^2\theta\ d\theta$$

We're finally ready to do the trigonometric substitution.

$$\int \sqrt{9x^2+16}\ dx$$

$$\int \sqrt{9\left(\frac{4}{3}\tan\theta\right)^2+16}\left(\frac{4}{3}\sec^2\theta\right)\ d\theta$$

$$\frac{4}{3}\int \sec^2\theta\sqrt{9\left(\frac{16}{9}\tan^2\theta\right)+16}\ d\theta$$

$$\frac{4}{3}\int \sec^2\theta\sqrt{16\tan^2\theta+16}\ d\theta$$

$$\frac{4}{3}\int \sec^2\theta\sqrt{16\left(\tan^2\theta+1\right)}\ d\theta$$

$$\frac{4}{3}\int \sec^2\theta\sqrt{16\sec^2\theta}\ d\theta$$

$$\frac{16}{3}\int \sec^3\theta\ d\theta$$

Therefore,

$$\int \sqrt{9x^2+16}\ dx=\frac{16}{3}\int \sec^3\theta\ d\theta$$

Topic: Trigonometric substitution setup

Question: Setup this integral for trigonometric substitution.

$$\int \frac{\sqrt{4x^2 - 25}}{x} \, dx$$

Answer choices:

A $\quad 5 \int \tan^2\theta \ d\theta$

B $\quad \int \tan^2\theta \ d\theta$

C $\quad 5 \int \sec^2\theta \ d\theta$

D $\quad 5 \int \sec^2\theta \ d\theta$

Solution: A

The question asks us to set up this integral for trigonometric substitution.

$$\int \frac{\sqrt{4x^2 - 25}}{x}\ dx$$

The integral contains an expression of the form $u^2 - a^2$ where a^2 is a number and u^2 is a function of x. This format requires the trigonometric substitution to be in the form $u = a \sec \theta$.

Now we'll use the values in the integral to find a and u.

$$a^2 = 25$$

$$a = 5$$

and

$$u^2 = 4x^2$$

$$u = 2x$$

This means that

$$u = a \sec \theta$$

$$2x = 5 \sec \theta$$

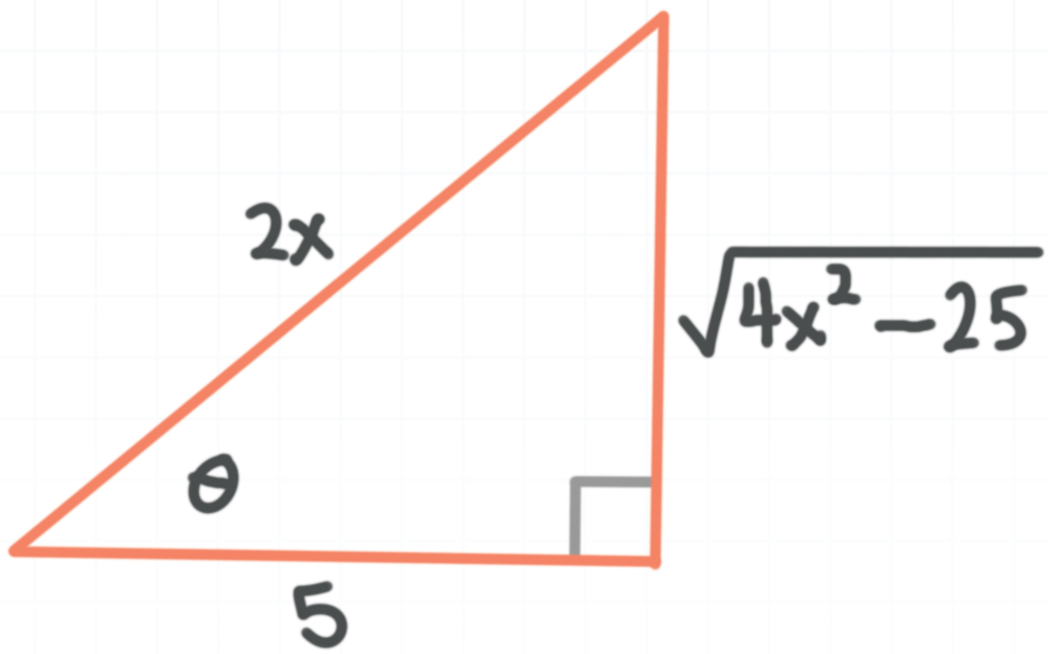
$$\sec \theta = \frac{2x}{5}$$

$$\theta = \text{arcsec} \left(\frac{2x}{5} \right)$$

To put this in the perspective of right triangle trigonometry, recall that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

This means we're dealing with a right triangle like this



We'll solve for x in terms of θ.

$$2x = 5\sec\theta$$

$$x = \frac{5}{2}\sec\theta$$

$$dx = \frac{5}{2}\sec\theta\tan\theta\ d\theta$$

We're finally ready to do the trigonometric substitution.

$$\int \frac{\sqrt{4x^2-25}}{x}\ dx$$

$$\int \frac{\sqrt{4\left(\frac{5}{2}\sec\theta\right)^2 - 25}}{\frac{5}{2}\sec\theta}\left(\frac{5}{2}\sec\theta\tan\theta\ d\theta\right)$$

$$\int \sqrt{4\left(\frac{5}{2}\sec\theta\right)^2 - 25}\ (\tan\theta\ d\theta)$$

$$\int \sqrt{4\left(\frac{25}{4}\sec^2\theta\right) - 25}\ (\tan\theta\ d\theta)$$

$$\int \sqrt{25\sec^2\theta - 25}\ (\tan\theta\ d\theta)$$

$$\int \tan\theta\sqrt{25\left(\sec^2\theta - 1\right)}\ d\theta$$

Remembering the trig identity $1 + \tan^2 x = \sec^2 x$, we can make the substitution for $\sec^2\theta - 1$.

$$\int \tan\theta\sqrt{25\tan^2\theta}\ d\theta$$

$$\int \tan\theta(5\tan\theta)\ d\theta$$

$$5\int \tan^2\theta\ d\theta$$

Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{dx}{\sqrt{x^2 - 2x}}$$

Answer choices:

A $\ln \left| x - 1 + \sqrt{x^2 - 2x} \right| + C$

B $\ln \left| \sqrt{x^2 - 2x} \right| + C$

C $\ln \left(x^2 - 2x \right) + C$

D $\ln \left(x^2 + 2x \right) + C$

Solution: A

First, complete the square to rewrite the integral as

$$\int \frac{dx}{\sqrt{x^2 - 2x}}$$

$$\int \frac{dx}{\sqrt{\left(x^2 - 2x + 1\right) - 1}}$$

$$\int \frac{dx}{\sqrt{(x-1)^2 - 1^2}}$$

We can now use trigonometric substitution to evaluate the integral. Recognizing that

$$u^2 - a^2 = (x-1)^2 - 1^2$$

we get

$$u = x - 1$$

$$a = 1$$

Knowing that

$$u = a\sec\theta$$

is the substitution we use for $u^2 - a^2$, we get

$$x - 1 = 1\sec\theta$$

$$x - 1 = \sec\theta$$

$$x = 1 + \sec\theta$$

$$dx = \sec\theta\tan\theta\ d\theta$$

$$\theta = \sec^{-1}(x - 1)$$

Plugging these into the integral we get

$$\int \frac{\sec\theta\tan\theta\ d\theta}{\sqrt{(1 + \sec\theta - 1)^2 - 1^2}}$$

$$\int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\ d\theta$$

We know that $\tan^2 x = \sec^2 x - 1$, so we'll make a substitution to simplify the integral.

$$\int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\ d\theta$$

$$\int \frac{\sec\theta\tan\theta}{\tan\theta}\ d\theta$$

$$\int \sec\theta\ d\theta$$

The formula for the integral of $\sec x$ is

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

Using the formula, the integral becomes

$$\ln|\sec\theta+\tan\theta|+C$$

Since $\theta=\sec^{-1}(x-1)$, we get

$$\ln\left|\sec\left[\sec^{-1}(x-1)\right]+\tan\left[\sec^{-1}(x-1)\right]\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{1-\frac{1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{(x-1)^2}{(x-1)^2}-\frac{1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{(x-1)^2-1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{x^2-2x+1-1}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\sqrt{\frac{x^2-2x}{(x-1)^2}}\right|+C$$

$$\ln\left|x-1+(x-1)\frac{\sqrt{x^2-2x}}{\sqrt{(x-1)^2}}\right|+C$$

$$\ln \left| x - 1 + (x - 1)\frac{\sqrt{x^2 - 2x}}{x - 1} \right| + C$$

$$\ln \left| x - 1 + \sqrt{x^2 - 2x} \right| + C$$

Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int_5^8 \frac{dx}{x^2\sqrt{x^2-16}}$$

Answer choices:

A $\quad \dfrac{5\sqrt{3}-6}{160}$

B $\quad \dfrac{\sqrt{3}}{8}-\dfrac{3}{20}$

C $\quad \dfrac{5\sqrt{3}-6}{80}$

D $\quad \dfrac{5\sqrt{3}}{16}-\dfrac{3}{80}$

Solution: A

Recognizing that we have

$$u^2 - a^2 = x^2 - 4^2$$

in the integral, we get

$$u = x$$

$$a = 4$$

Knowing that

$$u = a\sec\theta$$

is the substitution we use for $u^2 - a^2$, we get

$$x = 4\sec\theta$$

$$\frac{x}{4} = \sec\theta$$

$$dx = 4\sec\theta\tan\theta\ d\theta$$

$$\theta = \sec^{-1}\frac{x}{4}$$

Plugging these into the integral we get

$$\int_5^8 \frac{dx}{x^2\sqrt{x^2-16}}$$

$$\int_5^8 \frac{4\sec\theta\tan\theta\ d\theta}{(4\sec\theta)^2\sqrt{(4\sec\theta)^2-16}}$$

$$\int_{5}^{8} \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\ d\theta$$

$$\int_{5}^{8} \frac{\tan\theta}{4\sec\theta\sqrt{16\left(\sec^2\theta-1\right)}}\ d\theta$$

$$\frac{1}{16}\int_{5}^{8} \frac{\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\ d\theta$$

We know that $\tan^2 x = \sec^2 x - 1$, so we'll make a substitution to simplify the integral.

$$\frac{1}{16}\int_{5}^{8} \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\ d\theta$$

$$\frac{1}{16}\int_{5}^{8} \frac{\tan\theta}{\sec\theta\tan\theta}\ d\theta$$

$$\frac{1}{16}\int_{5}^{8} \frac{1}{\sec\theta}\ d\theta$$

$$\frac{1}{16}\int_{5}^{8} \cos\theta\ d\theta$$

The integral of $\cos x$ is

$$\int \cos x\ dx = \sin x + C$$

so the integral becomes

$$\frac{1}{16}\sin\theta\Big|_5^8$$

Back-substituting for x before we evaluate over the interval, we get

$$\frac{1}{16}\sin\left(\sec^{-1}\frac{x}{4}\right)\Big|_5^8$$

$$\frac{1}{16}\sqrt{1-\frac{1}{\left(\frac{x}{4}\right)^2}}\Bigg|_5^8$$

$$\frac{1}{16}\sqrt{1-\frac{16}{x^2}}\Bigg|_5^8$$

$$\frac{1}{16}\left(\sqrt{1-\frac{16}{8^2}}-\sqrt{1-\frac{16}{5^2}}\right)$$

$$\frac{1}{16}\left(\sqrt{\frac{3}{4}}-\sqrt{\frac{9}{25}}\right)$$

$$\frac{1}{16}\left(\frac{\sqrt{3}}{2}-\frac{3}{5}\right)$$

$$\frac{1}{16}\left(\frac{5\sqrt{3}}{10}-\frac{6}{10}\right)$$

$$\frac{5\sqrt{3}-6}{160}$$

Topic: Trigonometric substitution with secant

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{6}{\left(9x^2-16\right)^{\frac{3}{2}}}\ dx$$

Answer choices:

A $\quad \dfrac{3x}{8\sqrt{9x^2-16}}+C$

B $\quad -\dfrac{3x}{8\sqrt{9x^2-16}}+C$

C $\quad -\dfrac{3x}{\sqrt{9x^2-16}}+C$

D $\quad -\dfrac{x}{8\sqrt{9x^2-16}}+C$

Solution: B

Recognizing that we have

$$u^2 - a^2 = 9x^2 - 4^2$$

in the integral, we get

$$u = 3x$$

$$a = 4$$

Knowing that

$$u = a\sec\theta$$

is the substitution we use for $u^2 - a^2$, we get

$$3x = 4\sec\theta$$

$$\frac{3x}{4} = \sec\theta$$

$$x = \frac{4}{3}\sec\theta$$

$$dx = \frac{4}{3}\sec\theta\tan\theta\ d\theta$$

$$\theta = \sec^{-1}\frac{3x}{4}$$

Plugging these into the integral we get

$$\int \frac{6}{(9x^2-16)^{\frac{3}{2}}}\ dx$$

$$\int \frac{6}{\left[9\left(\frac{4}{3}\sec\theta\right)^2-16\right]^{\frac{3}{2}}}\left(\frac{4}{3}\sec\theta\tan\theta\ d\theta\right)$$

$$\int \frac{8\sec\theta\tan\theta}{\left[9\left(\frac{4}{3}\sec\theta\right)^2-16\right]^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{\left[9\left(\frac{16}{9}\sec^2\theta\right)-16\right]^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{(16\sec^2\theta-16)^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8\sec\theta\tan\theta}{\left[16\left(\sec^2\theta-1\right)\right]^{\frac{3}{2}}}\ d\theta$$

We know that $\tan^2 x = \sec^2 x - 1$, so we'll make a substitution to simplify the integral.

$$\int \frac{8\sec\theta\tan\theta}{(16\tan^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$\int \frac{8 \sec \theta \tan \theta}{\left(\sqrt{16 \tan^2 \theta}\right)^3}\ d\theta$$

$$\int \frac{8 \sec \theta \tan \theta}{(4 \tan \theta)^3}\ d\theta$$

$$\int \frac{8 \sec \theta \tan \theta}{64 \tan^3 \theta}\ d\theta$$

$$\int \frac{\sec \theta}{8 \tan^2 \theta}\ d\theta$$

Make substitutions into the integral.

$$\int \frac{\frac{1}{\cos \theta}}{8 \frac{\sin^2 \theta}{\cos^2 \theta}}\ d\theta$$

$$\frac{1}{8} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta}\ d\theta$$

$$\frac{1}{8} \int \frac{\cos \theta}{\sin^2 \theta}\ d\theta$$

$$\frac{1}{8} \int \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta}\ d\theta$$

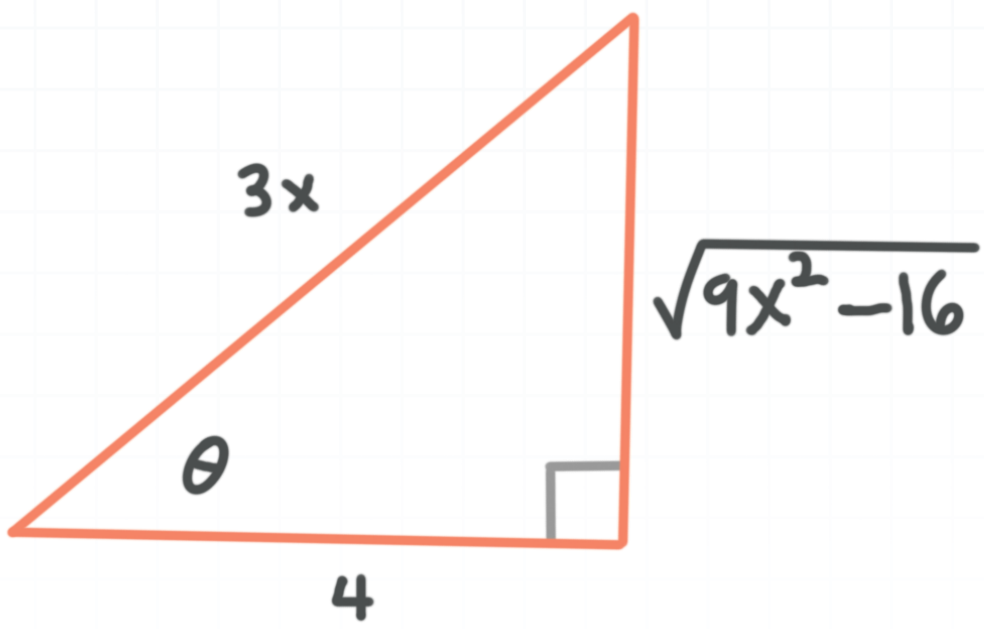
$$\frac{1}{8} \int \cot \theta \csc \theta\ d\theta$$

$$-\frac{1}{8} \csc \theta + C$$

We have successfully integrated this problem using trigonometric substitution with secant. However, to finish with an appropriate answer, we'll now put the problem back in terms of x.

$$-\frac{1}{8\sin\theta}+C$$

The reference triangle is



Because

$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta=\frac{\sqrt{9x^2-16}}{3x}$$

Therefore,

$$-\frac{1}{8\frac{\sqrt{9x^2-16}}{3x}}+C$$

$$-\frac{3x}{8\sqrt{9x^2-16}}+C$$

Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-x^2}}\, dx$$

Answer choices:

A π

B $\frac{\pi}{2}$

C $\frac{\pi}{3}$

D $\frac{\pi}{4}$

Solution: D

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 1 - x^2$$

we get

$$u = x$$

$$a = 1$$

Knowing that

$$u = a \sin \theta$$

is the substitution we use for $a^2 - u^2$, we get

$$x = 1 \sin \theta$$

$$x = \sin \theta$$

$$dx = \cos \theta \ d\theta$$

$$\theta = \sin^{-1} x$$

Plugging these into the integral we get

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1 - x^2}} \ dx$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-\left(\sin\theta\right)^2}} \cos\theta \ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} \ d\theta$$

We know that $1-\sin^2 x=\cos^2 x$, so we'll make a substitution to simplify the integral.

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\sqrt{\cos^2\theta}} \ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \frac{\cos\theta}{\cos\theta} \ d\theta$$

$$\int_{x=0}^{x=\frac{\sqrt{2}}{2}} \ d\theta$$

$$\theta \Big|_{x=0}^{x=\frac{\sqrt{2}}{2}}$$

Back-substituting for x before we evaluate over the interval, we get

$$\sin^{-1} x \Big|_{0}^{\frac{\sqrt{2}}{2}}$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)-\sin^{-1}(0)$$

$$\frac{\pi}{4} - 0$$

$$\frac{\pi}{4}$$

Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{\sqrt{1-x^2}}{x}\,dx$$

Answer choices:

A $\quad \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

B $\quad -\ln\left|\frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

C $\quad \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

D $\quad \ln\left|\frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C$

Solution: A

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 1 - x^2$$

we get

$$u = x$$

$$a = 1$$

Knowing that

$$u = a \sin \theta$$

is the substitution we use for $a^2 - u^2$, we get

$$x = 1 \sin \theta$$

$$x = \sin \theta$$

$$dx = \cos \theta \ d\theta$$

$$\theta = \sin^{-1} x$$

Plugging these into the integral we get

$$\int \frac{\sqrt{1 - x^2}}{x} \ dx$$

$$\int \frac{\sqrt{1 - (\sin \theta)^2}}{\sin \theta} \cos \theta \ d\theta$$

$$\int \frac{\cos\theta\sqrt{1-\sin^2\theta}}{\sin\theta}\ d\theta$$

We know that $1-\sin^2 x=\cos^2 x$, so we'll make a substitution to simplify the integral.

$$\int \frac{\cos\theta\sqrt{\cos^2\theta}}{\sin\theta}\ d\theta$$

$$\int \frac{\cos^2\theta}{\sin\theta}\ d\theta$$

$$\int \frac{1-\sin^2\theta}{\sin\theta}\ d\theta$$

$$\int \frac{1}{\sin\theta}-\frac{\sin^2\theta}{\sin\theta}\ d\theta$$

$$\int \csc\theta-\sin\theta\ d\theta$$

Knowing that

$$\int \csc\theta\ d\theta=\ln\left|\csc\theta-\cot\theta\right|+C$$

the integral becomes

$$\ln\left|\csc\theta-\cot\theta\right|-(-\cos\theta)+C$$

$$\ln\left|\csc\theta-\cot\theta\right|+\cos\theta+C$$

Back-substituting for x, we get

$$\ln\left|\csc\left(\sin^{-1}x\right)-\cot\left(\sin^{-1}x\right)\right|+\cos\left(\sin^{-1}x\right)+C$$

$$\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+\sqrt{1-x^2}+C$$

$$\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+\sqrt{1-x^2}+C$$

Topic: Trigonometric substitution with sine

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{x^2}{\sqrt{9-4x^2}}\ dx$$

Answer choices:

A $\quad \frac{9}{8}\arcsin\frac{2x}{3} - \frac{x\sqrt{9-4x^2}}{8} + C$

B $\quad \frac{9}{8}\arcsin\frac{2x}{3} + \frac{x\sqrt{9-4x^2}}{8} + C$

C $\quad \frac{9}{16}\arcsin\frac{2x}{3} - \frac{x\sqrt{9-4x^2}}{8} + C$

D $\quad \frac{9}{16}\arcsin\frac{2x}{3} + \frac{x\sqrt{9-4x^2}}{8} + C$

Solution: C

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 - u^2 = 9 - 4x^2$$

we get

$$u = 2x$$

$$a = 3$$

Knowing that

$$u = a\sin\theta$$

is the substitution we use for $a^2 - u^2$, we get

$$2x = 3\sin\theta$$

$$x = \frac{3}{2}\sin\theta$$

$$dx = \frac{3}{2}\cos\theta\ d\theta$$

$$\frac{2x}{3} = \sin\theta$$

$$\theta = \sin^{-1}\frac{2x}{3}$$

Plugging these into the integral we get

$$\int \frac{x^2}{\sqrt{9-4x^2}}\ dx$$

$$\int \frac{\left(\frac{3}{2}\sin\theta\right)^2}{\sqrt{9-4\left(\frac{3}{2}\sin\theta\right)^2}}\left(\frac{3}{2}\cos\theta\ d\theta\right)$$

$$\int \frac{\frac{9}{4}\sin^2\theta}{\sqrt{9-4\left(\frac{9}{4}\sin^2\theta\right)}}\left(\frac{3}{2}\cos\theta\ d\theta\right)$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9-9\sin^2\theta}}\ d\theta$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9\left(1-\sin^2\theta\right)}}\ d\theta$$

We know that $1-\sin^2 x = \cos^2 x$, so we'll make a substitution to simplify the integral.

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{\sqrt{9\cos^2\theta}}\ d\theta$$

$$\frac{27}{8}\int \frac{\sin^2\theta\cos\theta}{3\cos\theta}\ d\theta$$

$$\frac{9}{8}\int \sin^2\theta\ d\theta$$

$$\frac{9}{8}\int 1-\cos^2\theta\ d\theta$$

Because we know the trigonometric identity

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we can substitute into the integral.

$$\frac{9}{8}\int 1-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta$$

$$\frac{9}{8}\int 1-\frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta$$

$$\frac{9}{8}\int \frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta$$

$$\frac{9}{8}\int \frac{1}{2}(1-\cos 2\theta)\ d\theta$$

$$\frac{9}{16}\int 1-\cos 2\theta\ d\theta$$

Integrate.

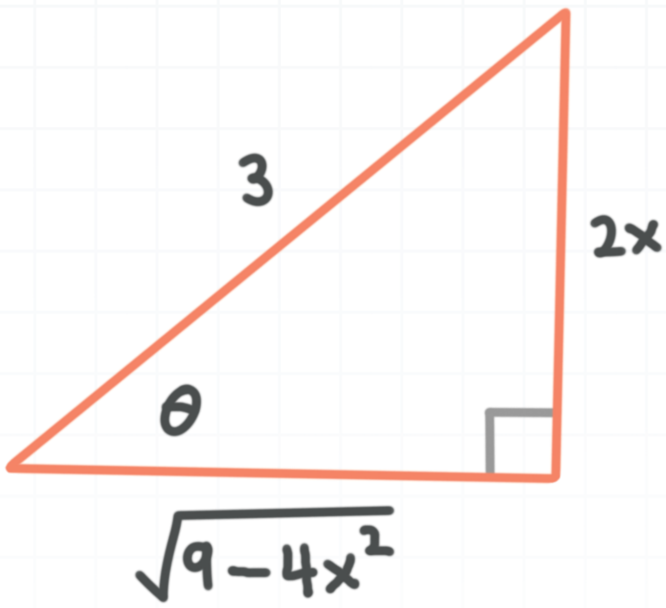
$$\frac{9}{16}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C$$

The trigonometric identity $\sin(2x)=2\sin x\cos x$ lets us rewrite this value.

$$\frac{9}{16}\left(\theta-\frac{1}{2}(2\sin\theta\cos\theta)\right)+C$$

$$\frac{9}{16}\left(\theta - \sin\theta\cos\theta\right) + C$$

We need to remember the reference triangle



Because

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{2x}{3}$$

and

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\sqrt{9-4x^2}}{3}$$

and because we already know from earlier that

$$\theta = \sin^{-1} \frac{2x}{3}$$

we can make substitutions to put the value back in terms of x.

$$\frac{9}{16} \left[\sin^{-1} \frac{2x}{3} - \left(\frac{2x}{3} \right) \left(\frac{\sqrt{9 - 4x^2}}{3} \right) \right] + C$$

$$\frac{9}{16} \arcsin \frac{2x}{3} - \frac{9}{16} \cdot \frac{2x\sqrt{9 - 4x^2}}{9} + C$$

$$\frac{9}{16} \arcsin \frac{2x}{3} - \frac{x\sqrt{9 - 4x^2}}{8} + C$$

Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{1}{1+4x^2}\, dx$$

Answer choices:

A $\quad \dfrac{1}{2}\sin^{-1}(2x)+C$

B $\quad \dfrac{1}{2}\cos^{-1}(2x)+C$

C $\quad \dfrac{1}{2}\tan^{-1}(2x)+C$

D $\quad \dfrac{1}{2}\cot^{-1}(2x)+C$

Solution: C

We can use trigonometric substitution to evaluate the integral. Recognizing that

$$a^2 + u^2 = 1 + 4x^2$$

we get

$$a = 1$$

$$u = 2x$$

Knowing that

$$u = a\tan\theta$$

is the substitution we use for $a^2 + u^2$, we get

$$2x = 1\tan\theta$$

$$2x = \tan\theta$$

$$x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\ d\theta$$

$$\theta = \tan^{-1}2x$$

Plugging these into the integral we get

$$\int \frac{1}{1 + 4x^2}\ dx$$

$$\int \frac{1}{1+4\left(\frac{1}{2}\tan\theta\right)^2}\frac{1}{2}\sec^2\theta\ d\theta$$

$$\frac{1}{2}\int \frac{\sec^2\theta}{1+4\left(\frac{1}{4}\tan^2\theta\right)}\ d\theta$$

$$\frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}\ d\theta$$

We know that $1+\tan^2 x = \sec^2 x$, so we'll make a substitution to simplify the integral.

$$\frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta}\ d\theta$$

$$\frac{1}{2}\int\ d\theta$$

$$\frac{1}{2}\theta + C$$

Back-substituting for x, we get

$$\frac{1}{2}\tan^{-1}(2x) + C$$

Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$$

Answer choices:

A $\quad 0.65$

B $\quad 0.74$

C $\quad 0.56$

D $\quad 0.99$

Solution: C

First, complete the square to rewrite the integral as

$$\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$$

$$\int_0^1 \frac{dx}{\sqrt{\left(x^2+2x+1\right)+1}}$$

$$\int_0^1 \frac{dx}{\sqrt{(x+1)^2+1}}$$

We can now use trigonometric substitution to evaluate the integral. Recognizing that

$$u^2+a^2=(x+1)^2+1$$

we get

$$u=x+1$$

$$a=1$$

Knowing that

$$u=a\tan\theta$$

is the substitution we use for u^2+a^2, we get

$$x+1=1\tan\theta$$

$$x + 1 = \tan\theta$$

$$x = \tan\theta - 1$$

$$dx = \sec^2\theta\ d\theta$$

$$\theta = \tan^{-1}(x + 1)$$

Plugging these into the integral we get

$$\int_{x=0}^{x=1} \frac{\sec^2\theta\ d\theta}{\sqrt{(\tan\theta - 1 + 1)^2 + 1}}$$

$$\int_{x=0}^{x=1} \frac{\sec^2\theta}{\sqrt{\tan^2\theta + 1}}\ d\theta$$

We know that $\tan^2 x + 1 = \sec^2 x$, so we'll make a substitution to simplify the integral.

$$\int_{x=0}^{x=1} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\ d\theta$$

$$\int_{x=0}^{x=1} \frac{\sec^2\theta}{\sec\theta}\ d\theta$$

$$\int_{x=0}^{x=1} \sec\theta\ d\theta$$

Our formula for the integral of $\sec x$ is

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

so the integral becomes

$$\ln|\sec\theta+\tan\theta|\Big|_{x=0}^{x=1}$$

Using $\theta=\tan^{-1}(x+1)$ to back-substitute, we get

$$\ln\left|\sec\left[\tan^{-1}(x+1)\right]+\tan\left[\tan^{-1}(x+1)\right]\right|\Bigg|_0^1$$

The $\tan$ and $\tan^{-1}$ functions cancel with one another. We also know that

$$\sec\left(\tan^{-1}x\right)=\sqrt{x^2+1}$$

so we can simplify to

$$\ln\left|\sqrt{(x+1)^2+1}+(x+1)\right|\Bigg|_0^1$$

$$\ln\left|\sqrt{x^2+2x+1+1}+x+1\right|\Bigg|_0^1$$

$$\ln\left|x+1+\sqrt{x^2+2x+2}\right|\Bigg|_0^1$$

Evaluate over the interval.

$$\ln\left|1+1+\sqrt{(1)^2+2(1)+2}\right|-\ln\left|0+1+\sqrt{(0)^2+2(0)+2}\right|$$

$$\ln\left|2+\sqrt{1+2+2}\right|-\ln\left|1+\sqrt{2}\right|$$

$$\ln\left|2+\sqrt{5}\right|-\ln\left|1+\sqrt{2}\right|$$

Using laws of logarithms, we get

$$\ln\frac{\left|2+\sqrt{5}\right|}{\left|1+\sqrt{2}\right|}$$

The absolute values are irrelevant, since neither the numerator or denominator could ever be negative.

$$\ln\frac{2+\sqrt{5}}{1+\sqrt{2}}$$

$$0.56$$

Topic: Trigonometric substitution with tangent

Question: Use trigonometric substitution to evaluate the integral.

$$\int \frac{x^3}{\left(x^2+4\right)^{\frac{3}{2}}}\ dx$$

Answer choices:

A $\dfrac{x^2-12}{\sqrt{x^2+4}}+C$

B $\dfrac{x^2-8}{\sqrt{x^2+4}}+C$

C $\dfrac{x^2+12}{\sqrt{x^2+4}}+C$

D $\dfrac{x^2+8}{\sqrt{x^2+4}}+C$

Solution: D

Recognizing that we have

$$u^2 + a^2 = x^2 + 2^2$$

in the integral, we get

$$u = x$$

$$a = 2$$

Knowing that

$$u = a \tan\theta$$

is the substitution we use for $u^2 + a^2$, we get

$$x = 2\tan\theta$$

$$\frac{x}{2} = \tan\theta$$

$$dx = 2\sec^2\theta\ d\theta$$

$$\theta = \tan^{-1}\frac{x}{2}$$

Plugging these into the integral we get

$$\int \frac{x^3}{\left(x^2+4\right)^{\frac{3}{2}}}\ dx$$

$$\int \frac{(2\tan\theta)^3}{\left[(2\tan\theta)^2+4\right]^{\frac{3}{2}}} 2\sec^2\theta \ d\theta$$

$$16\int \frac{\tan^3\theta\sec^2\theta}{\left(4\tan^2\theta+4\right)^{\frac{3}{2}}} \ d\theta$$

$$16\int \frac{\tan^3\theta\sec^2\theta}{\left(4^{\frac{1}{2}}\right)^3\left(\tan^2\theta+1\right)^{\frac{3}{2}}} \ d\theta$$

$$2\int \frac{\tan^3\theta\sec^2\theta}{\left(\tan^2\theta+1\right)^{\frac{3}{2}}} \ d\theta$$

We know that $\tan^2 x + 1 = \sec^2 x$, so we'll make a substitution to simplify the integral.

$$2\int \frac{\tan^3\theta\sec^2\theta}{\left[\left(\sec^2\theta\right)^{\frac{1}{2}}\right]^3} \ d\theta$$

$$2\int \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta} \ d\theta$$

$$2\int \frac{\tan^3\theta}{\sec\theta} \ d\theta$$

$$2\int \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}} \ d\theta$$

$$2\int \frac{\sin^3\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin^2\theta\sin\theta}{\cos^2\theta}\ d\theta$$

We know that $\sin^2 x = 1 - \cos^2 x$, so we'll make a substitution to simplify the integral.

$$2\int \frac{\left(1-\cos^2\theta\right)\sin\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin\theta - \sin\theta\cos^2\theta}{\cos^2\theta}\ d\theta$$

$$2\int \frac{\sin\theta}{\cos^2\theta} - \frac{\sin\theta\cos^2\theta}{\cos^2\theta}\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta - \sin\theta\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta - 2\int \sin\theta\ d\theta$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta - 2(-\cos\theta) + C$$

$$2\int \sin\theta\cos^{-2}\theta\ d\theta + 2\cos\theta + C$$

For the remaining integral, we'll use substitution and let

$$v = \cos\theta$$

$$dv = -\sin\theta\ d\theta$$

$$-\frac{dv}{\sin\theta} = d\theta$$

Plugging these back into the integral, we'll get

$$2\int \sin\theta v^{-2}\left(-\frac{dv}{\sin\theta}\right) + 2\cos\theta + C$$

$$-2\int v^{-2}\ dv + 2\cos\theta + C$$

$$-2\left(-v^{-1}\right) + 2\cos\theta + C$$

$$\frac{2}{v} + 2\cos\theta + C$$

Back-substituting for θ, we get

$$\frac{2}{\cos\theta} + 2\cos\theta + C$$

Back-substituting for x, we get

$$\frac{2}{\cos\left(\tan^{-1}\frac{x}{2}\right)} + 2\cos\left(\tan^{-1}\frac{x}{2}\right) + C$$

$$\frac{2}{\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2+1}}} + 2\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2+1}} + C$$

$$\frac{2}{\frac{1}{\sqrt{\frac{x^2+4}{4}}}}+\frac{2}{\sqrt{\frac{x^2+4}{4}}}+C$$

$$2\sqrt{\frac{x^2+4}{4}}+\frac{2}{\sqrt{\frac{x^2+4}{4}}}+C$$

$$\sqrt{x^2+4}+\frac{2}{\frac{1}{2}\sqrt{x^2+4}}+C$$

$$\sqrt{x^2+4}+\frac{4}{\sqrt{x^2+4}}+C$$

$$\frac{x^2+4}{\sqrt{x^2+4}}+\frac{4}{\sqrt{x^2+4}}+C$$

$$\frac{x^2+8}{\sqrt{x^2+4}}+C$$

Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \frac{x+1}{x^2+2x+2}\ dx$$

Answer choices:

A $\quad \frac{1}{2}\ln\left|x^2+1\right|+C$

B $\quad \frac{1}{2}\ln\left|(x+1)^2+2\right|+C$

C $\quad \frac{1}{2}\ln\left|(x-1)^2+1\right|+C$

D $\quad \frac{1}{2}\ln\left|x^2+2x+2\right|+C$

Solution: D

Quadratic functions are polynomial functions of the specific form

$$f(x) = ax^2 + bx + c$$

Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c\ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square in the denominator. Taking the coefficient 2 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \frac{x+1}{x^2+2x+2}\ dx$$

$$\int \frac{x+1}{x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+2}\ dx$$

$$\int \frac{x+1}{x^2+2x+1-1+2}\ dx$$

$$\int \frac{x+1}{\left(x^2+2x+1\right)+1}\ dx$$

$$\int \frac{x+1}{(x+1)^2+1}\ dx$$

$$\int \frac{x+1}{(x+1)^2+1^2}\ dx$$

Because the denominator is the sum of two squares, we can try trigonometric substitution to evaluate the integral. Setting up trigonometric substitution by comparing u^2+a^2 with $(x+1)^2+1^2$, we get

$$u^2+a^2=(x+1)^2+1^2$$

$$u=x+1$$

$$a=1$$

$$u=a\tan\theta$$

$$x+1=1\tan\theta$$

$$x+1=\tan\theta$$

$$x=\tan\theta-1$$

$$dx=\sec^2\theta\ d\theta$$

In the right triangle,

Adjacent side	1
Opposite side	$x+1$

Hypotenuse $\qquad \sqrt{x^2+2x+2}$

Plugging these values into the integral, we get

$$\int \frac{\tan\theta}{(\tan\theta)^2+1}\sec^2\theta\ d\theta$$

$$\int \frac{\tan\theta\sec^2\theta}{\tan^2\theta+1}\ d\theta$$

Since we know that $\tan^2\theta+1=\sec^2\theta$, we can make a substitution into the denominator.

$$\int \frac{\tan\theta\sec^2\theta}{\sec^2\theta}\ d\theta$$

$$\int \tan\theta\ d\theta$$

From here we can use a formula for the integral of $\tan\theta$, or we can solve it without the formula. In case you forget the formula, here's how you can solve it with u-substitution.

$$\int \tan\theta\ d\theta$$

$$\int \frac{\sin\theta}{\cos\theta}\ d\theta$$

$$u=\cos\theta$$

$$du=-\sin\theta\ d\theta$$

$$d\theta = \frac{du}{-\sin\theta}$$

$$\int \frac{\sin\theta}{u}\left(\frac{du}{-\sin\theta}\right)$$

$$-\int \frac{1}{u}\ du$$

$$-\ln|u| + C$$

$$-\ln|\cos\theta| + C$$

$$\ln\left|(\cos\theta)^{-1}\right| + C$$

$$\ln\left|\frac{1}{\cos\theta}\right| + C$$

Since cosine of an angle is equal to the adjacent side over the hypotenuse, we get

$$\ln\left|\frac{1}{\frac{1}{\sqrt{x^2+2x+2}}}\right| + C$$

$$\ln\left|\sqrt{x^2+2x+2}\right| + C$$

$$\ln\left|\left(x^2+2x+2\right)^{\frac{1}{2}}\right| + C$$

$$\frac{1}{2}\ln\left|x^2+2x+2\right| + C$$

Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \sqrt{x^2-2x-8}\ dx$$

Answer choices:

A $\quad \dfrac{(x-1)\sqrt{x^2-2x-8}}{2}-\dfrac{9}{2}\ln\left|\dfrac{x-1+\sqrt{x^2-2x-8}}{3}\right|$

B $\quad \dfrac{(x+1)\sqrt{x^2-2x-8}}{2}-\dfrac{9}{2}\ln\left|\dfrac{x+1+\sqrt{x^2-2x-8}}{3}\right|$

C $\quad \dfrac{(x-1)\sqrt{x^2-2x-8}}{2}+\dfrac{9}{2}\ln\left|\dfrac{x-1+\sqrt{x^2-2x-8}}{3}\right|$

D $\quad \dfrac{(x+1)\sqrt{x^2-2x-8}}{2}+\dfrac{9}{2}\ln\left|\dfrac{x+1+\sqrt{x^2-2x-8}}{3}\right|$

Solution: A

Quadratic functions are polynomial functions of the specific form

$$f(x) = ax^2 + bx + c$$

Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c\ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square inside the radical. Taking the coefficient -2 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \sqrt{x^2 - 2x - 8}\ dx$$

$$\int \sqrt{x^2 - 2x + \left(\frac{-2}{2}\right)^2 - \left(\frac{-2}{2}\right)^2 - 8}\ dx$$

$$\int \sqrt{x^2 - 2x + 1 - 1 - 8}\ dx$$

$$\int \sqrt{\left(x^2 - 2x + 1\right) - 1 - 8}\ dx$$

$$\int \sqrt{(x-1)^2 - 3^2}\ dx$$

Because the value inside the radical is the difference of two squares, we can try trigonometric substitution to evaluate the integral. Setting up trigonometric substitution by comparing $u^2 - a^2$ with $(x-1)^2 - 3^2$, we get

$$u^2 - a^2 = (x-1)^2 - 3^2$$

$$u = x - 1$$

$$a = 3$$

$$u = a \sec \theta$$

$$x - 1 = 3 \sec \theta$$

$$\frac{x-1}{3} = \sec \theta$$

$$x = 1 + 3 \sec \theta$$

$$dx = 3 \sec \theta \tan \theta\ d\theta$$

In the right triangle,

Adjacent side	3
Opposite side	$\sqrt{x^2 - 2x - 8}$
Hypotenuse	$x - 1$

Plugging these values into the integral, we get

$$\int 3\sec\theta\tan\theta\sqrt{(3\sec\theta)^2-3^2}\ d\theta$$

$$\int 3\sec\theta\tan\theta\sqrt{9\sec^2\theta-9}\ d\theta$$

$$\int 3\sec\theta\tan\theta\sqrt{9\left(\sec^2\theta-1\right)}\ d\theta$$

$$9\int \sec\theta\tan\theta\sqrt{\sec^2\theta-1}\ d\theta$$

Since we know that $\tan^2\theta = \sec^2\theta - 1$, we can make a substitution.

$$9\int \sec\theta\tan\theta\sqrt{\tan^2\theta}\ d\theta$$

$$9\int \sec\theta\tan\theta\tan\theta\ d\theta$$

$$9\int \sec\theta\tan^2\theta\ d\theta$$

$$9\int \sec\theta\left(\sec^2\theta-1\right)\ d\theta$$

$$9\int \sec^3\theta - \sec\theta\ d\theta$$

$$9\int \sec^3\theta\ d\theta - 9\int \sec\theta\ d\theta$$

Integrating both integrals, we get

$$\frac{9}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)-9\ln\left|\sec\theta+\tan\theta\right|$$

$$\frac{9}{2}\sec\theta\tan\theta+\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|-\frac{18}{2}\ln\left|\sec\theta+\tan\theta\right|$$

$$\frac{9}{2}\sec\theta\tan\theta-\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|$$

With

$$\frac{x-1}{3}=\sec\theta$$

$$\sec^{-1}\left(\frac{x-1}{3}\right)=\theta$$

we can back-substitute for θ.

$$\frac{9}{2}\left(\frac{x-1}{3}\right)\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)-\frac{9}{2}\ln\left|\frac{x-1}{3}+\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)\right|$$

We know that

$$\tan(\sec^{-1}x)=x\sqrt{1-\frac{1}{x^2}}$$

In this case, the “x” is $(x-1)/3$, so let’s make that substitution.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{1}{\left(\frac{x-1}{3}\right)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{1}{\frac{(x-1)^2}{9}}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{1-\frac{9}{(x-1)^2}}$$

Find a common denominator.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{\frac{(x-1)^2}{(x-1)^2}-\frac{9}{(x-1)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\sqrt{\frac{(x-1)^2-9}{(x-1)^2}}$$

Apply the square root to the numerator and denominator separately.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\cdot\frac{\sqrt{(x-1)^2-9}}{\sqrt{(x-1)^2}}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{x-1}{3}\cdot\frac{\sqrt{(x-1)^2-9}}{x-1}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{(x-1)^2-9}}{3}$$

Simplify the numerator.

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{x^2-2x+1-9}}{3}$$

$$\tan\left(\sec^{-1}\left(\frac{x-1}{3}\right)\right)=\frac{\sqrt{x^2-2x-8}}{3}$$

Then we can simplify the parts of the expression where we're taking the tangent of the inverse secant.

$$\frac{9}{2}\left(\frac{x-1}{3}\right)\frac{\sqrt{x^2-2x-8}}{3}-\frac{9}{2}\ln\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right|$$

$$\frac{9(x-1)\sqrt{x^2-2x-8}}{18}-\frac{9}{2}\ln\left|\frac{x-1+\sqrt{x^2-2x-8}}{3}\right|$$

$$\frac{(x-1)\sqrt{x^2-2x-8}}{2}-\frac{9}{2}\ln\left|\frac{x-1+\sqrt{x^2-2x-8}}{3}\right|$$

Topic: Quadratic functions

Question: Evaluate the integral.

$$\int \frac{1}{x^2+6x+13}\ dx$$

Answer choices:

A $\quad \frac{1}{2}\tan\left(\frac{x+3}{2}\right)$

B $\quad \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$

C $\quad \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)$

D $\quad \frac{1}{2}\tan\left(\frac{x-3}{2}\right)$

Solution: B

Quadratic functions are polynomial functions of the specific form

$$f(x) = ax^2 + bx + c$$

Integrals of simple quadratic functions, like

$$\int ax^2 + bx + c\ dx$$

can be easily evaluated using power rule, like any other polynomial function. However, if we start manipulating the quadratic function, we'll likely have to use other techniques to solve the integral. For example, when the quadratic function appears as the denominator of a rational function (fraction), we can very often use trigonometric substitution and/or u-substitution in order to evaluate the integral.

For this particular integral, we need to start by completing the square in the denominator. Taking the coefficient 6 on the first-degree x-term, we'll complete the square by dividing it by 2 and then squaring the result. This will be the number we have to add in (and subtract out) to complete the square.

$$\int \frac{1}{x^2 + 6x + 13}\ dx$$

$$\int \frac{1}{x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 13}\ dx$$

$$\int \frac{1}{\left(x^2+6x+9\right)-9+13}\ dx$$

$$\int \frac{1}{(x+3)^2+4}\ dx$$

$$\int \frac{1}{(x+3)^2+2^2}\ dx$$

$$u=x+3$$

$$du=dx$$

$$\int \frac{1}{u^2+2^2}\ du$$

We know that

$$\int \frac{1}{u^2+a^2}\ du=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

so we get

$$\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$$

Topic: Improper integrals, case 1

Question: Evaluate the improper integral.

$$\int_2^\infty \frac{dx}{x(\ln x)^2}$$

Answer choices:

A $\quad 0$

B $\quad \ln 2$

C $\quad -1$

D $\quad \dfrac{1}{\ln 2}$

Solution: D

Using an arbitrary variable b, first take the limit of the integral as $b \to \infty$.

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b \to \infty} \int_2^b \frac{dx}{x(\ln x)^2}$$

Let

$$u = \ln x$$

$$du = \frac{1}{x}\ dx$$

$$dx = x\ du$$

Plugging these values into the integral, we get

$$\lim_{b \to \infty} \int_{x=2}^{x=b} \frac{x\ du}{xu^2}$$

$$\lim_{b \to \infty} \int_{x=2}^{x=b} u^{-2}\ du$$

$$\lim_{b \to \infty} \left(-u^{-1}\right)\Big|_{x=2}^{x=b}$$

Back-substituting for x before we evaluate over the interval, we get

$$\lim_{b \to \infty} \left[-(\ln x)^{-1}\right]\Big|_2^b$$

$$\lim_{b\to\infty}\left(-\frac{1}{\ln x}\right)\Bigg|_2^b$$

Evaluating over the interval, we get

$$\lim_{b\to\infty}\left(-\frac{1}{\ln b}+\frac{1}{\ln 2}\right)$$

$$-\frac{1}{\ln\infty}+\frac{1}{\ln 2}$$

$$\frac{1}{\ln 2}$$

Topic: Improper integrals, case 1

Question: Evaluate the improper integral.

$$\int_{3}^{\infty} 5x^{-7}\ dx$$

Answer choices:

A $\dfrac{5}{4,374}$

B $-\dfrac{5}{4,374}$

C $\dfrac{1}{729}$

D $-\dfrac{1}{729}$

Solution: A

The integral in this problem is considered to be an improper integral, case 1, because the lower limit of integration is a constant and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx$$

We basically ignore the upper limit by replacing it with b and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_3^\infty 5x^{-7}\ dx = \lim_{b\to\infty}\int_3^b 5x^{-7}\ dx$$

$$5\lim_{b\to\infty}\int_3^b x^{-7}\ dx$$

$$\left[5\lim_{b\to\infty}\frac{x^{-6}}{-6}\right]_3^b$$

$$-\frac{5}{6}\lim_{b\to\infty}\left(b^{-6}-3^{-6}\right)$$

$$-\frac{5}{6}\lim_{b\to\infty}\left(\frac{1}{b^6}-\frac{1}{3^6}\right)$$

$$-\frac{5}{6}\lim_{b\to\infty}\left(\frac{1}{b^6}-\frac{1}{729}\right)$$

When we take the limit, $1/b^6$ becomes 0.

$$-\frac{5}{6}\left(0-\frac{1}{729}\right)$$

$$\frac{5}{6}\left(\frac{1}{729}\right)$$

$$\frac{5}{4,374}$$

Topic: Improper integrals, case 1

Question: Evaluate the improper integral.

$$\int_9^\infty \frac{2x-5}{x^2-5x-7}\,dx$$

Answer choices:

A $-\infty$

B 0

C ∞

D $\ln\left(\frac{13}{29}\right)$

Solution: C

The integral in this problem is considered to be an improper integral, case 1, because the lower limit of integration is a constant and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx$$

We basically ignore the upper limit by replacing it with b and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_9^\infty \frac{2x-5}{x^2-5x-7}\ dx = \lim_{b\to\infty}\int_9^b \frac{2x-5}{x^2-5x-7}\ dx$$

Now we'll change the integral using u-substitution.

$$u = x^2 - 5x - 7$$

$$du = (2x-5)\ dx$$

$$dx = \frac{du}{2x-5}$$

Substitute into the integral.

$$\lim_{b\to\infty}\int_{x=9}^{x=b} \frac{2x-5}{u}\left(\frac{du}{2x-5}\right)$$

$$\lim_{b\to\infty}\int_{x=9}^{x=b} \frac{1}{u}\ du$$

Integrate.

$$\lim_{b\to\infty} \ln|u| \Big|_{x=9}^{x=b}$$

Back-substitute to get the value in terms of x.

$$\lim_{b\to\infty} \ln\left|x^2-5x-7\right| \Big|_9^b$$

$$\lim_{b\to\infty} \left[\ln\left|b^2-5b-7\right| - \ln\left|(9)^2-5(9)-7\right|\right]$$

$$\lim_{b\to\infty} \left[\ln\left|b^2-5b-7\right| - \ln 29\right]$$

$$\lim_{b\to\infty} \ln \frac{\left|b^2-5b-7\right|}{29}$$

When we take the limit, the numerator becomes ∞. Therefore, the value of the whole limit will be ∞.

Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$$

Answer choices:

A $\frac{1}{4}$

B $-\frac{1}{4}$

C $-\frac{1}{2}$

D $\frac{1}{2}$

Solution: B

Using an arbitrary variable b, first take the limit of the integral as $b \to -\infty$.

$$\int_{-\infty}^{0} \frac{dx}{(2x-1)^3} = \lim_{b \to -\infty} \int_{b}^{0} \frac{dx}{(2x-1)^3}$$

$$\lim_{b \to -\infty} \int_{b}^{0} (2x-1)^{-3}\ dx$$

$$\lim_{b \to -\infty} \left[-\frac{1}{2}(2x-1)^{-2} \cdot \frac{1}{2} \right] \Bigg|_{b}^{0}$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4(2x-1)^2} \right] \Bigg|_{b}^{0}$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4(2(0)-1)^2} + \frac{1}{4(2(b)-1)^2} \right]$$

$$\lim_{b \to -\infty} \left[-\frac{1}{4} + \frac{1}{4(2b-1)^2} \right]$$

$$-\frac{1}{4} + \frac{1}{4(2(-\infty)-1)^2}$$

$$-\frac{1}{4} + \frac{1}{\infty}$$

$$-\frac{1}{4} + 0$$

$$-\frac{1}{4}$$

Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{2} \frac{7}{7x-16}\,dx$$

Answer choices:

A 0

B $-\infty$

C ∞

D $\ln 26$

Solution: B

The integral in this problem is considered to be an improper integral, case 2, because the lower limit of integration is $-\infty$ and the upper limit is a constant. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\ dx$$

We basically ignore the lower limit by replacing it with a and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{2} \frac{7}{7x-16}\ dx = \lim_{a\to-\infty} \int_{a}^{2} \frac{7}{7x-16}\ dx$$

Use a u-substitution on the integrand.

$$u = 7x - 16$$

$$du = 7\ dx$$

$$dx = \frac{du}{7}$$

Substitute into the integral.

$$\lim_{a\to-\infty} \int_{x=a}^{x=2} \frac{7}{u} \left(\frac{du}{7}\right)$$

$$\lim_{a\to-\infty} \int_{x=a}^{x=2} \frac{1}{u}\ du$$

Integrate and then back-substitute. Then evaluate over the interval.

$$\lim_{a \to -\infty} \ln|u| \Big|_{x=a}^{x=2}$$

$$\lim_{a \to -\infty} \ln|7x - 16| \Big|_{a}^{2}$$

$$\lim_{a \to -\infty} \left[\ln|7(2) - 16| - \ln|7(a) - 16|\right]$$

$$\lim_{a \to -\infty} \left[\ln 2 - \ln|7a - 16|\right]$$

When we take the limit, $\ln|7a - 16|$ becomes ∞. Therefore, we essentially have

$$\ln 2 - \infty$$

$$-\infty$$

Topic: Improper integrals, case 2

Question: Evaluate the improper integral.

$$\int_{-\infty}^{5} \frac{1}{e^x + e^{-x}} \, dx$$

Answer choices:

A $-\infty$

B 0

C ∞

D $\dfrac{\pi}{2}$

Solution: D

The integral in this problem is considered to be an improper integral, case 2, because the lower limit of integration is $-\infty$ and the upper limit is a constant. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\ dx$$

We basically ignore the lower limit by replacing it with a and using a limit process. Then, once we integrate, finding the anti-derivative, we use the limit to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{5} \frac{1}{e^x + e^{-x}}\ dx = \lim_{a \to -\infty} \int_{a}^{5} \frac{1}{e^x + e^{-x}}\ dx$$

Rewrite the integrand.

$$\lim_{a \to -\infty} \int_{a}^{5} \frac{1}{e^{-x}\left(e^{2x} + 1\right)}\ dx$$

$$\lim_{a \to -\infty} \int_{a}^{5} \frac{e^x}{(e^x)^2 + 1}\ dx$$

Use a u-substitution on the integral.

$$u = e^x$$

$$du = e^x\ dx$$

$$dx = \frac{du}{e^x}$$

Substitute into the integral.

$$\lim_{a \to -\infty} \int_{x=a}^{x=5} \frac{u}{u^2+1} \left(\frac{du}{e^x} \right)$$

$$\lim_{a \to -\infty} \int_{x=a}^{x=5} \frac{u}{u^2+1} \left(\frac{du}{u} \right)$$

$$\lim_{a \to -\infty} \int_{x=a}^{x=5} \frac{1}{u^2+1} \ du$$

Integrate and then back-substitute. Then evaluate over the interval.

$$\lim_{a \to -\infty} \tan^{-1} u \Big|_{x=a}^{x=5}$$

$$\lim_{a \to -\infty} \tan^{-1} e^x \Big|_{a}^{5}$$

$$\lim_{a \to -\infty} \left(\tan^{-1} e^5 - \tan^{-1} e^a \right)$$

Taking the limit essentially gives us

$$\tan^{-1} e^5 - \tan^{-1} e^{-\infty}$$

$$\tan^{-1} e^5 - \tan^{-1} \frac{1}{e^{\infty}}$$

$$\tan^{-1} e^5 - \tan^{-1} \frac{1}{\infty}$$

$$\tan^{-1} e^5 - \tan^{-1} 0$$

$$\frac{\pi}{2} - 0$$

$$\frac{\pi}{2}$$

Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\ dx$$

Answer choices:

A $\frac{\pi}{2}$

B $\frac{\pi}{3}$

C $\frac{\pi}{4}$

D $\frac{\pi}{6}$

Solution: A

First, rewrite the integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5}\ dx = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+2^2}$$

Since both limits of integration are infinite, we'll split the interval at $x = 0$ and rewrite the integral.

$$\int_{-\infty}^{0} \frac{dx}{(x+1)^2+2^2} + \int_{0}^{\infty} \frac{dx}{(x+1)^2+2^2}$$

Using arbitrary variables a and b, take the limit of the first integral as $a \to -\infty$ and the second integral as $b \to \infty$.

$$\lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{(x+1)^2+2^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{(x+1)^2+2^2}$$

To integrate, we need to use trigonometric substitution. We recognize that in the denominator of the function, we have a variable term $(x+1)^2$ plus a constant term 2^2. So we'll go through the setup process for trigonometric substitution.

$u^2 = (x+1)^2$ so $u = x+1$

$a^2 = 2^2$ so $a = 2$

$x + 1 = 2\tan\theta$

$$\tan\theta = \frac{x+1}{2}$$

$$\theta = \arctan \frac{x+1}{2}$$

$$x = 2\tan\theta - 1$$

$$\frac{dx}{d\theta} = 2\sec^2\theta \text{ so } dx = 2\sec^2\theta \ d\theta$$

Make substitutions into the integral.

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{(2\tan\theta)^2 + 2^2} + \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{(2\tan\theta)^2 + 2^2}$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\tan^2\theta + 4} + \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\tan^2\theta + 4}$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\left(\tan^2\theta + 1\right)} + \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\left(\tan^2\theta + 1\right)}$$

Knowing that $\tan^2\theta + 1 = \sec^2\theta$, we get

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} \frac{2\sec^2\theta \ d\theta}{4\sec^2\theta} + \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{2\sec^2\theta \ d\theta}{4\sec^2\theta}$$

$$\lim_{a\to-\infty}\int_{x=a}^{x=0} \frac{1}{2} \ d\theta + \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{1}{2} \ d\theta$$

$$\lim_{a\to-\infty} \frac{1}{2}\theta\Big|_{x=a}^{x=0} + \lim_{b\to\infty} \frac{1}{2}\theta\Big|_{x=0}^{x=b}$$

Back-substitute for θ.

$$\lim_{a\to-\infty}\frac{1}{2}\arctan\frac{x+1}{2}\Bigg|_a^0+\lim_{b\to\infty}\frac{1}{2}\arctan\frac{x+1}{2}\Bigg|_0^b$$

$$\lim_{a\to-\infty}\left(\frac{1}{2}\arctan\frac{0+1}{2}-\frac{1}{2}\arctan\frac{a+1}{2}\right)+\lim_{b\to\infty}\left(\frac{1}{2}\arctan\frac{b+1}{2}-\frac{1}{2}\arctan\frac{0+1}{2}\right)$$

$$\frac{1}{2}\arctan\frac{1}{2}-\frac{1}{2}\arctan\frac{-\infty+1}{2}+\frac{1}{2}\arctan\frac{\infty+1}{2}-\frac{1}{2}\arctan\frac{1}{2}$$

$$\frac{1}{2}\arctan\frac{\infty+1}{2}-\frac{1}{2}\arctan\frac{-\infty+1}{2}$$

$$\frac{1}{2}\arctan(\infty)-\frac{1}{2}\arctan(-\infty)$$

$$\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{2}\right)$$

$$\frac{\pi}{4}+\frac{\pi}{4}$$

$$\frac{\pi}{2}$$

Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} \frac{4}{9+x^2}\ dx$$

Answer choices:

A $\frac{2\pi}{3}$

B $\frac{\pi}{2}$

C $\frac{4\pi}{3}$

D ∞

Solution: C

The integral in this problem is considered to be an improper integral, case 3, because the lower limit of integration is $-\infty$ and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{a\to-\infty} \int_a^c f(x)\ dx + \lim_{b\to\infty} \int_c^b f(x)\ dx$$

We basically ignore both limits of integration by replacing them with a and b and by using a limit process instead. Then, once we integrate, finding the anti-derivative, we use the limits to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{\infty} \frac{4}{9+x^2}\ dx = \lim_{a\to-\infty} \int_a^c \frac{4}{9+x^2}\ dx + \lim_{b\to\infty} \int_c^b \frac{4}{9+x^2}\ dx$$

$$\lim_{a\to-\infty} \int_a^c \frac{\frac{4}{9}}{\frac{9}{9}+\frac{x^2}{9}}\ dx + \lim_{b\to\infty} \int_c^b \frac{\frac{4}{9}}{\frac{9}{9}+\frac{x^2}{9}}\ dx$$

$$\frac{4}{9} \lim_{a\to-\infty} \int_a^c \frac{1}{1+\frac{x^2}{9}}\ dx + \frac{4}{9} \lim_{b\to\infty} \int_c^b \frac{1}{1+\frac{x^2}{9}}\ dx$$

$$\frac{4}{9} \lim_{a\to-\infty} \int_a^c \frac{1}{1+\left(\frac{x}{3}\right)^2}\ dx + \frac{4}{9} \lim_{b\to\infty} \int_c^b \frac{1}{1+\left(\frac{x}{3}\right)^2}\ dx$$

Integrate.

$$\frac{4}{9} \lim_{a\to-\infty} 3\arctan\frac{x}{3}\Bigg|_a^c + \frac{4}{9} \lim_{b\to\infty} 3\arctan\frac{x}{3}\Bigg|_c^b$$

$$\frac{4}{3}\lim_{a\to-\infty}\arctan\frac{x}{3}\Big|_a^c+\frac{4}{3}\lim_{b\to\infty}\arctan\frac{x}{3}\Big|_c^b$$

Evaluate over the interval.

$$\frac{4}{3}\lim_{a\to-\infty}\left(\arctan\frac{c}{3}-\arctan\frac{a}{3}\right)+\frac{4}{3}\lim_{b\to\infty}\left(\arctan\frac{b}{3}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\left[\arctan\frac{c}{3}-\left(-\frac{\pi}{2}\right)\right]+\frac{4}{3}\left(\frac{\pi}{2}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\left(\arctan\frac{c}{3}+\frac{\pi}{2}\right)+\frac{4}{3}\left(\frac{\pi}{2}-\arctan\frac{c}{3}\right)$$

$$\frac{4}{3}\arctan\frac{c}{3}+\frac{2\pi}{3}+\frac{2\pi}{3}-\frac{4}{3}\arctan\frac{c}{3}$$

$$\frac{2\pi}{3}+\frac{2\pi}{3}$$

$$\frac{4\pi}{3}$$

Topic: Improper integrals, case 3

Question: Evaluate the improper integral.

$$\int_{-\infty}^{\infty} xe^{x^2}\, dx$$

Answer choices:

A The integral diverges

B 0

C $\frac{8}{3}$

D $\frac{1}{2}$

Solution: A

The integral in this problem is considered to be an improper integral, case 3, because the lower limit of integration is $-\infty$ and the upper limit is ∞. Evaluating this type of improper integral follows this general rule:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{a \to -\infty} \int_{a}^{c} f(x)\ dx + \lim_{b \to \infty} \int_{c}^{b} f(x)\ dx$$

We basically ignore both limits of integration by replacing them with a and b and by using a limit process instead. Then, once we integrate, finding the anti-derivative, we use the limits to finish the evaluation. Let's begin by rewriting the integral as a limit.

$$\int_{-\infty}^{\infty} xe^{x^2}\ dx = \lim_{a \to -\infty} \int_{a}^{c} xe^{x^2}\ dx + \lim_{b \to \infty} \int_{c}^{b} xe^{x^2}\ dx$$

Use a u-substitution.

$$u = x^2$$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

Substitute into each integral.

$$\lim_{a \to -\infty} \int_{x=a}^{x=c} xe^{u} \left(\frac{du}{2x}\right) + \lim_{b \to \infty} \int_{x=c}^{x=b} xe^{u} \left(\frac{du}{2x}\right)$$

$$\frac{1}{2} \lim_{a \to -\infty} \int_{x=a}^{x=c} e^{u}\ du + \frac{1}{2} \lim_{b \to \infty} \int_{x=c}^{x=b} e^{u}\ du$$

Integrate and then back-substitute.

$$\frac{1}{2}\lim_{a\to-\infty} e^u \Big|_{x=a}^{x=c} + \frac{1}{2}\lim_{b\to\infty} e^u \Big|_{x=c}^{x=b}$$

$$\frac{1}{2}\lim_{a\to-\infty} e^{x^2} \Big|_{a}^{c} + \frac{1}{2}\lim_{b\to\infty} e^{x^2} \Big|_{c}^{b}$$

Evaluate over the interval.

$$\frac{1}{2}\lim_{a\to-\infty} e^{c^2} - e^{a^2} + \frac{1}{2}\lim_{b\to\infty} e^{b^2} - e^{c^2}$$

$$\frac{1}{2}\left(e^{c^2} - \infty\right) + \frac{1}{2}\left(\infty - e^{c^2}\right)$$

$$\frac{1}{2}e^{c^2} - \frac{1}{2}\infty + \frac{1}{2}\infty - \frac{1}{2}e^{c^2}$$

$$-\infty + \infty$$

This doesn't converge to a real-number value, so the integral diverges.

Topic: Improper integrals, case 4

Question: Evaluate the improper integral.

$$\int_{-3}^{5} \frac{6}{x-5}\, dx$$

Answer choices:

A $6\ln 8$

B ∞

C $-\infty$

D $-6\ln 8$

Solution: C

The integral in this problem is considered to be an improper integral, case 4, because the integrand is undefined at the upper limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

Let's begin by re-writing the integral using this rule.

$$\int_{-3}^{5} \frac{6}{x-5}\ dx = \lim_{c \to 5^-} \int_{-3}^{c} \frac{6}{x-5}\ dx$$

$$6 \lim_{c \to 5^-} \int_{-3}^{c} \frac{1}{x-5}\ dx$$

Integrate.

$$6 \lim_{c \to 5^-} \ln \left| x-5 \right| \Big|_{-3}^{c}$$

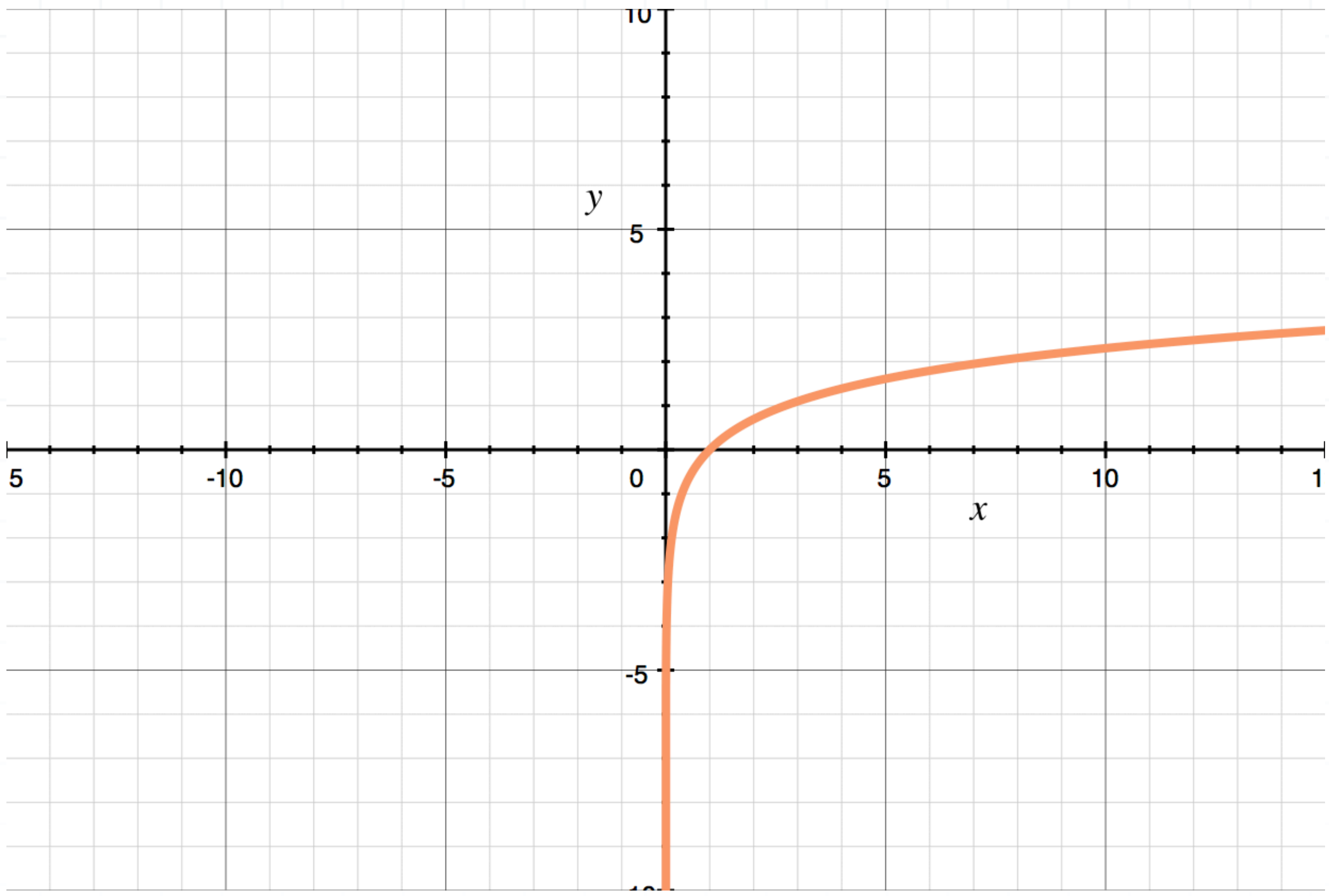
Evaluate over the interval.

$$6 \lim_{c \to 5^-} \left[\ln \left| c-5 \right| - \ln \left| -3-5 \right| \right]$$

$$6 \lim_{c \to 5^-} \left[\ln \left| c-5 \right| - \ln 8 \right]$$

$$6 \ln \left| 5-5 \right| - 6 \ln 8$$

When we look at $\ln|5-5| = \ln 0$, we know that $\ln 0$ is undefined. If we look at the graph of the natural logarithm, we can see that the value approaches $-\infty$.



Therefore, we can evaluate the limit using the graph.

$$6\ln 0 - 6\ln 8$$

$$-\infty - 6\ln 8$$

$$-\infty$$

Topic: Improper integrals, case 4

Question: Evaluate the improper integral.

$$\int_{-4}^{6} \frac{x-8}{x^2-14x+48}\, dx$$

Answer choices:

A ∞

B $-\infty$

C $\ln 10$

D $-\ln 10$

Solution: B

The integral in this problem is considered to be an improper integral, case 4, because the integrand is undefined at the upper limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

Since the denominator in the given integrand is factorable, let's factor it and see if the integrand can be simplified.

$$\int_{-4}^{6} \frac{x-8}{x^2-14x+48}\ dx = \int_{-4}^{6} \frac{x-8}{(x-6)(x-8)}\ dx$$

$$\int_{-4}^{6} \frac{\cancel{x-8}}{(x-6)\cancel{(x-8)}}\ dx$$

$$\int_{-4}^{6} \frac{1}{x-6}\ dx$$

Now let's begin by re-writing the integral using the rule above.

$$\lim_{c \to 6^-} \int_{-4}^{c} \frac{1}{x-6}\ dx$$

Integrate.

$$\lim_{c \to 6^-} \ln|x-6| \Big|_{-4}^{c}$$

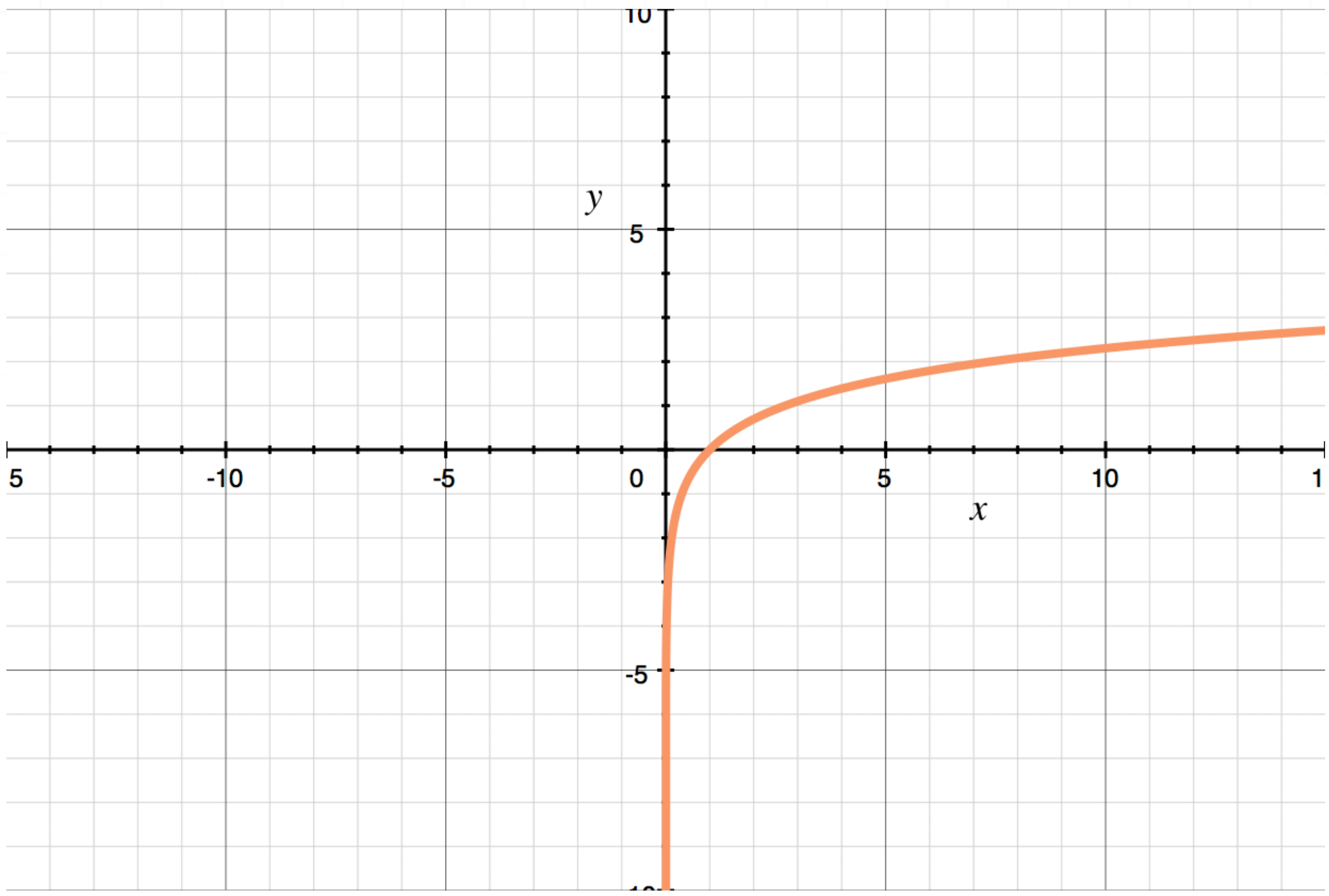
Evaluate over the interval.

$$\lim_{c \to 6^-} \left[\ln|c-6| - \ln|-4-6|\right]$$

$$\lim_{c \to 6^-} \left[\ln|c-6| - \ln 10\right]$$

$$\ln|6-6| - \ln 10$$

When we look at $\ln|6-6| = \ln 0$, we know that $\ln 0$ is undefined. If we look at the graph of the natural logarithm, we can see that the value approaches $-\infty$.



Therefore, we can evaluate the limit using the graph.

$$-\infty - \ln 10$$

$$-\infty$$

Topic: Improper integrals, case 4

Question: Evaluate the improper integral.

$$\int_{-7}^{7} \frac{15}{7-x}\, dx$$

Answer choices:

A ∞

B $-\infty$

C $15\ln 14$

D $-15\ln 14$

Solution: A

The integral in this problem is considered to be an improper integral, case 4, because the integrand is undefined at the upper limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

Let's begin by re-writing the integral using the rule.

$$\int_{-7}^{7} \frac{15}{7-x}\ dx = \lim_{c \to 7^-} \int_{-7}^{c} \frac{15}{7-x}\ dx$$

Integrate, remembering that chain rule tells us to multiply by -1, since the derivative of $7-x$ is -1.

$$-15 \lim_{c \to 7^-} \ln|7-x| \Big|_{-7}^{c}$$

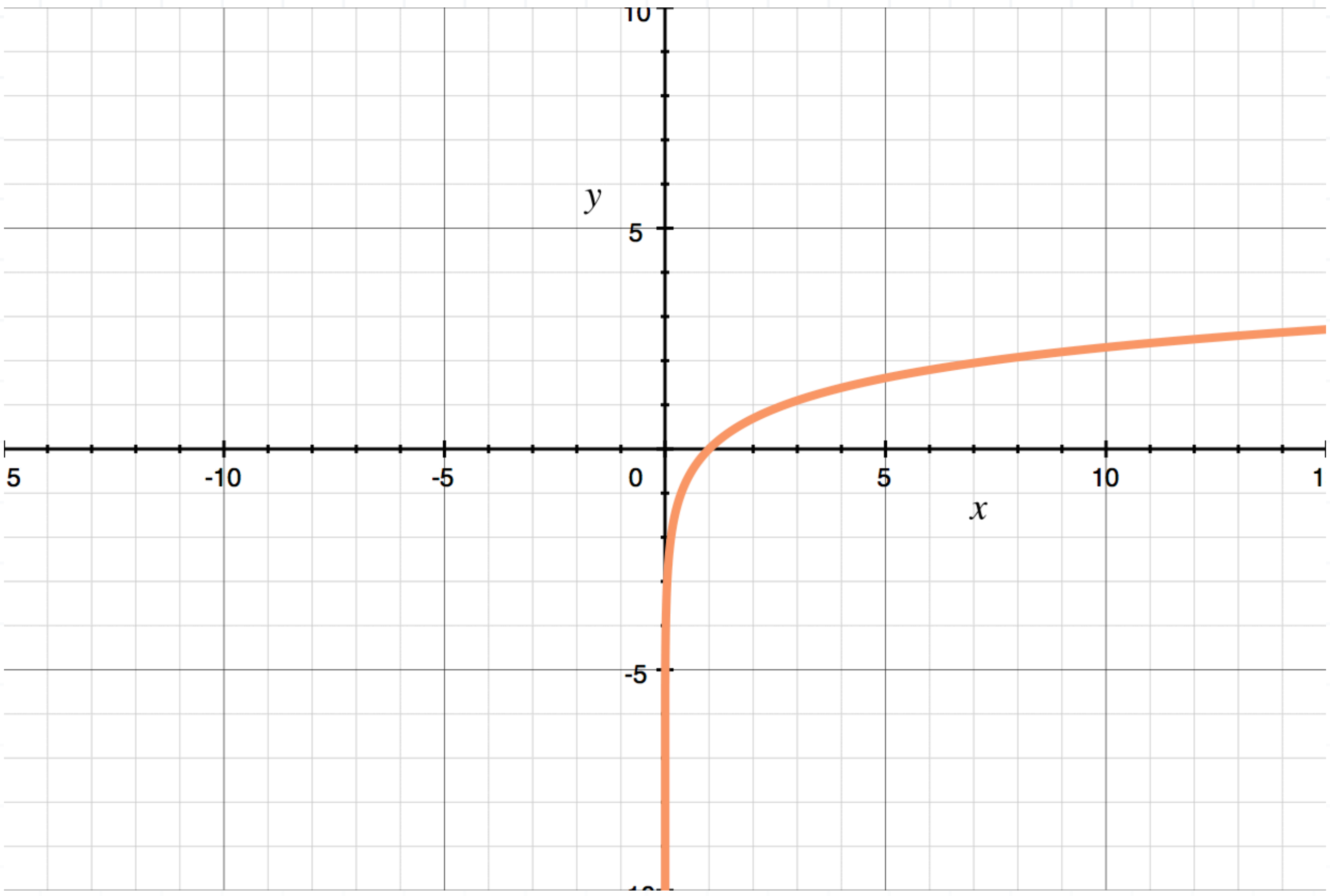
Evaluate over the interval.

$$-15 \lim_{c \to 7^-} \left[\ln|7-c| - \ln|7-(-7)|\right]$$

$$-15 \lim_{c \to 7^-} \left[\ln|7-c| - \ln 14\right]$$

$$-15 \left[\ln|7-7| - \ln 14\right]$$

When we look at $\ln|7-7| = \ln 0$, we know that $\ln 0$ is undefined. If we look at the graph of the natural logarithm, we can see that the value approaches $-\infty$.



Therefore, we can evaluate the limit using the graph.

$$-15\left[-\infty-\ln 14\right]$$

$$\infty$$

Topic: Improper integrals, case 5

Question: Evaluate the improper integral.

$$\int_0^5 \frac{3\ln x}{x}\, dx$$

Answer choices:

A $\quad \frac{3}{2}\ln 5$

B $\quad -\frac{3}{2}\ln 5$

C $\quad \infty$

D $\quad -\infty$

Solution: D

The integral in this problem is considered to be an improper integral, case 5, because the integrand is undefined at the lower limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c\to a^+}\int_c^b f(x)\ dx$$

Let's begin by re-writing the integral using the rule.

$$\int_0^5 \frac{3\ln x}{x}\ dx = \lim_{c\to 0^+}\int_c^5 \frac{3\ln x}{x}\ dx$$

Use a u-substitution.

$$u = \ln x$$

$$du = \frac{1}{x}\ dx$$

$$dx = x\ du$$

Make substitutions into the integral, then integrate and back-substitute.

$$\lim_{c\to 0^+}\int_{x=c}^{x=5} \frac{3u}{x}\left(x\ du\right)$$

$$\lim_{c\to 0^+}\int_{x=c}^{x=5} 3u\ du$$

$$\frac{3}{2}\lim_{c\to 0^+} u^2\Big|_{x=c}^{x=5}$$

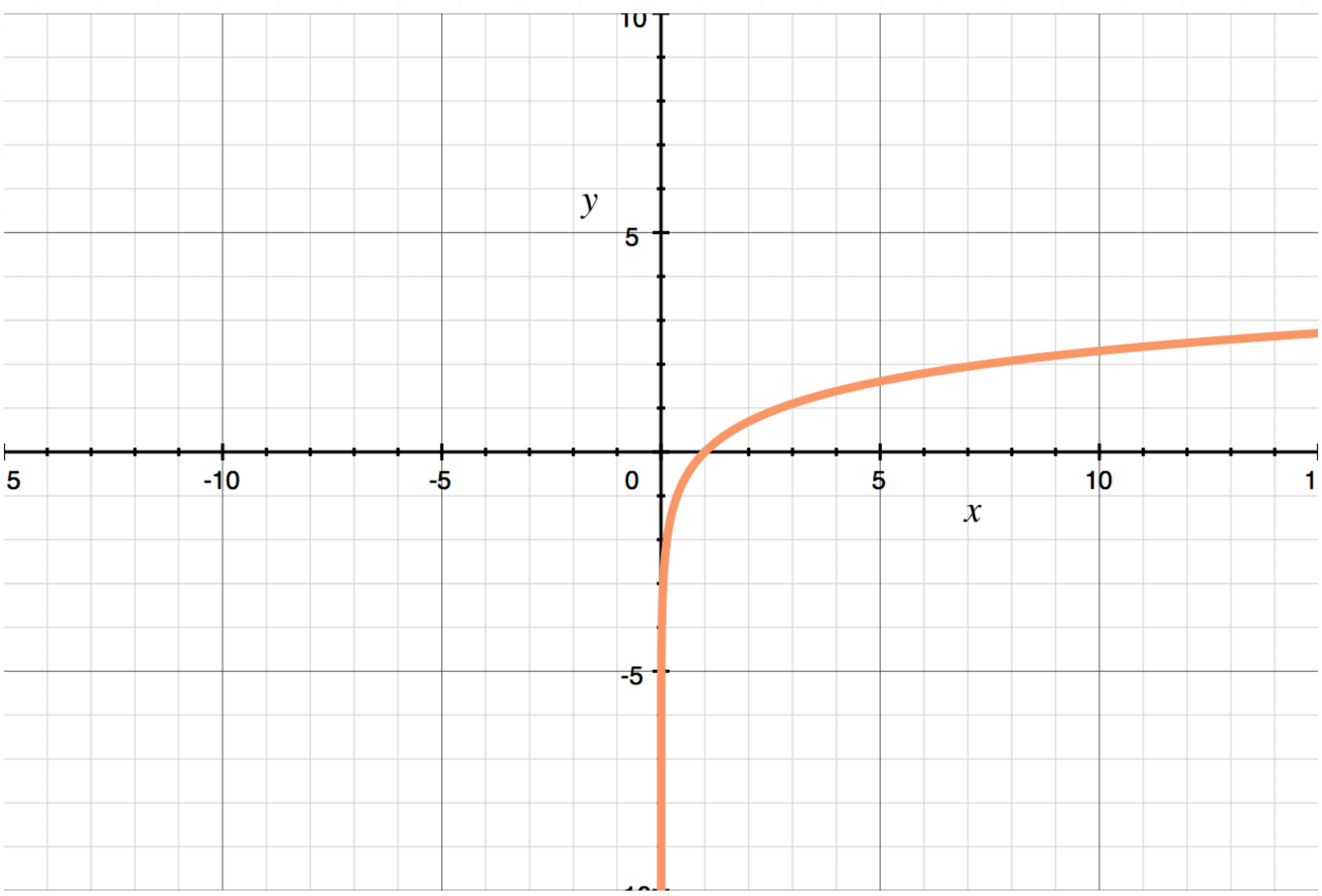
$$\frac{3}{2}\lim_{c\to 0^+}\ln^2 x\Big|_c^5$$

Evaluate over the interval.

$$\frac{3}{2}\lim_{c\to 0^+}\ln^2 5-\ln^2 c$$

$$\frac{3}{2}\left(\ln^2 5-\ln^2 0\right)$$

When we look at $\ln^2 0$, we know that $\ln 0$ is undefined. If we look at the graph of the natural logarithm, we can see that the value approaches $-\infty$.



Therefore, we can evaluate the limit using the graph.

$$\frac{3}{2}\left(\ln^2 5-(-\infty)^2\right)$$

$$\frac{3}{2}\left(\ln^2 5-\infty\right)$$

$$-\infty$$

Topic: Improper integrals, case 5

Question: Evaluate the improper integral.

$$\int_{\pi}^{\frac{5\pi}{4}} \frac{\cos\theta}{\sin\theta}\, d\theta$$

Answer choices:

A $-\infty$

B $\ln\frac{\sqrt{2}}{2}$

C ∞

D $-\ln\frac{\sqrt{2}}{2}$

Solution: C

The integral in this problem is considered to be an improper integral, case 5, because the integrand is undefined at the lower limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c \to a^+} \int_c^b f(x)\ dx$$

Let's begin by re-writing the integral using the rule.

$$\int_{\pi}^{\frac{5\pi}{4}} \frac{\cos\theta}{\sin\theta}\ dx = \lim_{c \to \pi^+} \int_c^{\frac{5\pi}{4}} \frac{\cos\theta}{\sin\theta}\ dx$$

Use a u-substitution.

$$u = \sin\theta$$

$$du = \cos\theta\ dx$$

$$dx = \frac{du}{\cos\theta}$$

Make substitutions into the integral, then integrate and back-substitute.

$$\lim_{c \to \pi^+} \int_{x=c}^{x=\frac{5\pi}{4}} \frac{\cos\theta}{u} \left(\frac{du}{\cos\theta} \right)$$

$$\lim_{c \to \pi^+} \int_{x=c}^{x=\frac{5\pi}{4}} \frac{1}{u}\ du$$

$$\lim_{c \to \pi^+} \ln|u| \Big|_{x=c}^{x=\frac{5\pi}{4}}$$

$$\lim_{c \to \pi^+} \ln|\sin\theta| \Big|_c^{\frac{5\pi}{4}}$$

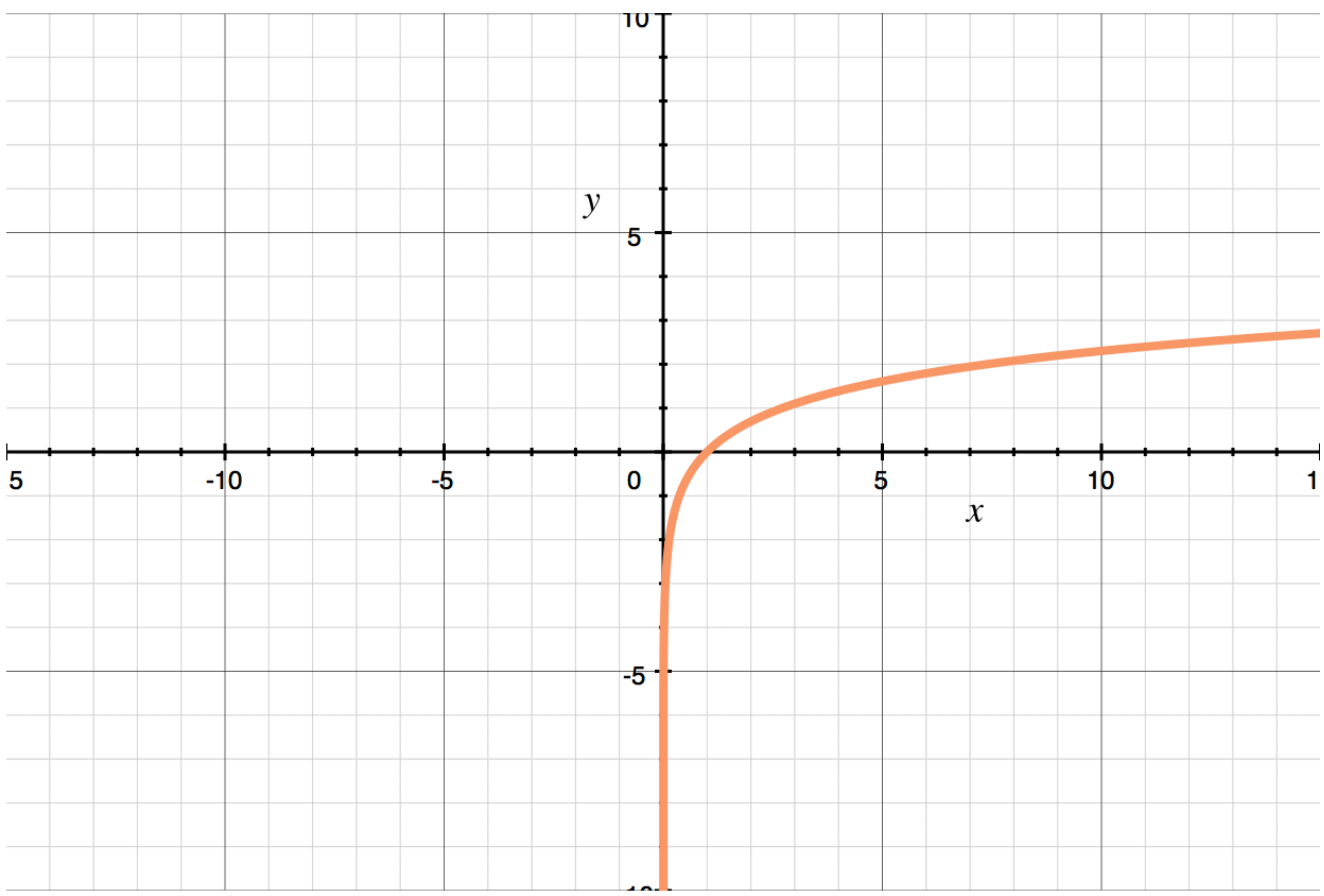
Evaluate over the interval.

$$\lim_{c \to \pi^+} \left(\ln \left| \sin \frac{5\pi}{4} \right| - \ln|\sin c| \right)$$

$$\lim_{c \to \pi^+} \left(\ln \frac{\sqrt{2}}{2} - \ln|\sin c| \right)$$

$$\ln \frac{\sqrt{2}}{2} - \ln|\sin \pi|$$

When we look at $\ln \sin \pi = \ln 0$, we know that $\ln 0$ is undefined. If we look at the graph of the natural logarithm, we can see that the value approaches $-\infty$.



Therefore, we can evaluate the limit using the graph.

$$\ln\frac{\sqrt{2}}{2}-(-\infty)$$

$$\infty$$

Topic: Improper integrals, case 5

Question: Evaluate the improper integral.

$$\int_6^{15} \frac{8}{\sqrt{x-6}}\ dx$$

Answer choices:

A 48

B $-\infty$

C -48

D ∞

Solution: A

The integral in this problem is considered to be an improper integral, case 5, because the integrand is undefined at the lower limit limit of integration. Evaluating this type of improper integral follows this general rule:

$$\int_a^b f(x)\ dx = \lim_{c\to a^+}\int_c^b f(x)\ dx$$

Let's begin by re-writing the integral using the rule.

$$\int_6^{15} \frac{8}{\sqrt{x-6}}\ dx = \lim_{c\to 6^+}\int_c^{15} \frac{8}{\sqrt{x-6}}\ dx$$

$$8\lim_{c\to 6^+}\int_c^{15} (x-6)^{-\frac{1}{2}}\ dx$$

Integrate.

$$8\lim_{c\to 6^+} 2(x-6)^{\frac{1}{2}}\Big|_c^{15}$$

$$16\lim_{c\to 6^+}\sqrt{x-6}\Big|_c^{15}$$

Evaluate over the interval.

$$16\lim_{c\to 6^+}\left(\sqrt{15-6}-\sqrt{c-6}\right)$$

$$16\lim_{c\to 6^+}\left(3-\sqrt{c-6}\right)$$

$$16\left(3-\sqrt{6-6}\right)$$

$$16(3-0)$$

$$48$$

Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_0^6 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

Answer choices:

A $-3+3\sqrt[3]{5}$

B $3-3\sqrt[3]{5}$

C ∞

D $3+3\sqrt[3]{5}$

Solution: D

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x \to c^-} \int_a^c f(x)\ dx + \lim_{x \to c^+} \int_c^b f(x)\ dx$$

We'll start by rewriting the given integral as

$$\int_0^6 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^6 (x-1)^{-\frac{2}{3}}\ dx$$

The integrand is undefined at $x = 1$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 1$.

$$\int_0^1 (x-1)^{-\frac{2}{3}}\ dx + \int_1^6 (x-1)^{-\frac{2}{3}}\ dx$$

Since the integrand has a vertical asymptote at $x = 1$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{b\to1^-}\int_0^b (x-1)^{-\frac{2}{3}}\ dx + \lim_{a\to1^+}\int_a^6 (x-1)^{-\frac{2}{3}}\ dx$$

Integrate.

$$\lim_{b\to1^-}\left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_0^b + \lim_{a\to1^+}\left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_a^6$$

$$\lim_{b\to1^-}\left[3(x-1)^{\frac{1}{3}}\right]_0^b + \lim_{a\to1^+}\left[3(x-1)^{\frac{1}{3}}\right]_a^6$$

Evaluate over the interval.

$$\lim_{b\to1^-}\left[3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}\right] + \lim_{a\to1^+}\left[3(6-1)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}\right]$$

$$\lim_{b\to1^-}\left[3(b-1)^{\frac{1}{3}} + 3\right] + \lim_{a\to1^+}\left[3(5)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}\right]$$

Take the limit.

$$\left[3(1-1)^{\frac{1}{3}} + 3\right] + \left[3(5)^{\frac{1}{3}} - 3(1-1)^{\frac{1}{3}}\right]$$

$$(0+3) + \left[3(5)^{\frac{1}{3}}\right]$$

$$3 + 3(5)^{\frac{1}{3}}$$

$$3 + 3\sqrt[3]{5}$$

Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_{-3}^{3} \frac{7}{x^6}\, dx$$

Answer choices:

A $-\infty$

B ∞

C 0

D $8 \ln 6$

Solution: B

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x \to c^-} \int_a^c f(x)\ dx + \lim_{x \to c^+} \int_c^b f(x)\ dx$$

We'll start by rewriting the given integral as

$$\int_{-3}^{3} \frac{7}{x^6}\ dx = \int_{-3}^{3} 7x^{-6}\ dx$$

The integrand is undefined at $x = 0$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 0$.

$$\int_{-3}^{0} 7x^{-6}\ dx + \int_{0}^{3} 7x^{-6}\ dx$$

Since the integrand has a vertical asymptote at $x = 0$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{c \to 0^-} \int_{-3}^{c} 7x^{-6}\ dx + \lim_{c \to 0^+} \int_{c}^{3} 7x^{-6}\ dx$$

Integrate.

$$\lim_{c \to 0^-} \left(-\frac{7}{5}x^{-5} \right) \Bigg|_{-3}^{c} + \lim_{c \to 0^+} \left(-\frac{7}{5}x^{-5} \right) \Bigg|_{c}^{3}$$

$$\lim_{c \to 0^-} \left(-\frac{7}{5x^5} \right) \Bigg|_{-3}^{c} + \lim_{c \to 0^+} \left(-\frac{7}{5x^5} \right) \Bigg|_{c}^{3}$$

Evaluate over the interval.

$$\lim_{c \to 0^-} \left(-\frac{7}{5c^5} + \frac{7}{5(-3)^5} \right) + \lim_{c \to 0^+} \left(-\frac{7}{5(3)^5} + \frac{7}{5c^5} \right)$$

$$\frac{7}{5} \lim_{c \to 0^-} \left[\frac{1}{(-3)^5} - \frac{1}{c^5} \right] + \frac{7}{5} \lim_{c \to 0^+} \left[\frac{1}{c^5} - \frac{1}{(3)^5} \right]$$

Take the limit.

$$\frac{7}{5} \left[\frac{1}{(-3)^5} + \infty \right] + \frac{7}{5} \left[\infty - \frac{1}{(3)^5} \right]$$

$$\infty$$

Topic: Improper integrals, case 6

Question: Evaluate the improper integral.

$$\int_0^4 \frac{x^2}{x^3-8}\,dx$$

Answer choices:

A $3\ln 2 + 3\ln 4$

B 0

C $-3\ln 2$

D The integral diverges

Solution: D

The integral in this problem is considered to be an improper integral, case 6, because the integrand is undefined not at either endpoint of the interval, but instead at a point inside the interval. Evaluating this type of improper integral requires us to split the integral at the point of discontinuity into two separate integrals.

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

But there's a discontinuity at $x = c$, so we rewrite the integral as

$$\int_a^b f(x)\ dx = \lim_{x \to c^-} \int_a^c f(x)\ dx + \lim_{x \to c^+} \int_c^b f(x)\ dx$$

The given integral is

$$\int_0^4 \frac{x^2}{x^3 - 8}\ dx$$

The integrand is undefined at $x = 2$, which is between the integration limits. Therefore, we'll re-write the integral as two integrals, separating them at $x = 2$.

$$\int_0^2 \frac{x^2}{x^3 - 8}\ dx + \int_2^4 \frac{x^2}{x^3 - 8}\ dx$$

Since the integrand has a vertical asymptote at $x = 2$, both integrals are still improper integrals. Therefore, we'll re-write the integrals as limits, as shown in the above rule.

$$\lim_{c \to 2^-} \int_0^c \frac{x^2}{x^3 - 8} \ dx + \lim_{c \to 2^+} \int_c^4 \frac{x^2}{x^3 - 8} \ dx$$

Use a u-substitution to integrate.

$$u = x^3 - 8$$

$$du = 3x^2 \ dx$$

$$dx = \frac{du}{3x^2}$$

Make substitutions into the integral.

$$\lim_{c \to 2^-} \int_{x=0}^{x=c} \frac{x^2}{u} \left(\frac{du}{3x^2} \right) + \lim_{c \to 2^+} \int_{x=c}^{x=4} \frac{x^2}{u} \left(\frac{du}{3x^2} \right)$$

$$\frac{1}{3} \lim_{c \to 2^-} \int_{x=0}^{x=c} \frac{1}{u} \ du + \frac{1}{3} \lim_{c \to 2^+} \int_{x=c}^{x=4} \frac{1}{u} \ du$$

Integrate and then back substitute.

$$\frac{1}{3} \lim_{c \to 2^-} \ln |u| \Big|_{x=0}^{x=c} + \frac{1}{3} \lim_{c \to 2^+} \ln |u| \Big|_{x=c}^{x=4}$$

$$\frac{1}{3} \lim_{c \to 2^-} \ln \left| x^3 - 8 \right| \Big|_0^c + \frac{1}{3} \lim_{c \to 2^+} \ln \left| x^3 - 8 \right| \Big|_c^4$$

Evaluate over the interval.

$$\frac{1}{3} \lim_{c \to 2^-} \left[\ln \left| c^3 - 8 \right| - \ln \left| 0^3 - 8 \right| \right] + \frac{1}{3} \lim_{c \to 2^+} \left[\ln \left| 4^3 - 8 \right| - \ln \left| c^3 - 8 \right| \right]$$

$$\frac{1}{3} \lim_{c \to 2^-} \left[\ln \left| c^3 - 8 \right| - \ln 8 \right] + \frac{1}{3} \lim_{c \to 2^+} \left[\ln 56 - \ln \left| c^3 - 8 \right| \right]$$

For the first limit, when $c \to 2^-$, $\ln|c^3-8|$ takes on a value of $-\infty$, so that first limit becomes

$$\frac{1}{3}\left[-\infty - \ln 8\right]$$

$$\frac{1}{3}[-\infty]$$

$$-\infty$$

Because the first integral diverges, the whole integral diverges.

Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_2^\infty \frac{\cos^2 x}{x^2}\, dx$$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: A

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_2^\infty \frac{\cos^2 x}{x^2}\ dx$$

to

$$\int_2^\infty \frac{1}{x^2}\ dx$$

and prove that the given integrand is always less than or equal to the comparison function, and then prove that the comparison function converges, then we'll have proven that the given integral converges.

First, let's confirm that

$$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

on $[2,\infty)$. We can say

$$\cos^2 x \le 1$$

We know that the value of $\cos x$ is always between -1 and 1, so when we square $\cos x$, the value will always be between 0 and 1. So we can confirm that the inequality is true. We could have also confirmed this by graphing both functions.

Now we'll test the convergence of

$$\int_2^\infty \frac{1}{x^2}\ dx$$

Convert the improper integral to a limit and convert the integrand so we can integrate using the power rule.

$$\lim_{b\to\infty} \int_2^b x^{-2}\ dx$$

$$\lim_{b\to\infty} \frac{x^{-1}}{-1} \Bigg|_2^b$$

$$-\lim_{b\to\infty} \frac{1}{x} \Bigg|_2^b$$

Evaluate over the interval.

$$-\lim_{b\to\infty}\left(\frac{1}{b}-\frac{1}{2}\right)$$

$$-\left(0-\frac{1}{2}\right)$$

$$\frac{1}{2}$$

We have just proven that the comparison integral converges to $1/2$. Therefore, by the comparison theorem, the given integral also converges.

Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_1^\infty \frac{1 + 2\sin^2(2x)}{\sqrt{x}}\, dx$$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: B

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_1^\infty \frac{1+2\sin^2(2x)}{\sqrt{x}}\ dx$$

to

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx$$

and prove that the given integrand is always greater than or equal to the comparison function, and then prove that the comparison function diverges, then we'll have proven that the given integral diverges.

First, let's confirm that

$$\frac{1+2\sin^2(2x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$$

on $[1,\infty)$. We can say

$$1+2\sin^2(2x) \geq 1$$

$$2\sin^2(2x) \geq 0$$

$$\sin^2(2x) \geq 0$$

We know that the value of $\sin x$ is always between -1 and 1, so when we square $\sin x$, the value will always be between 0 and 1. So we can confirm that the inequality is true. We could have also confirmed this by graphing both functions.

Now we'll test the convergence of

$$\int_1^\infty \frac{1}{\sqrt{x}}\ dx$$

Convert the improper integral to a limit and convert the integrand so we can integrate using the power rule.

$$\lim_{b\to\infty}\int_1^b x^{-\frac{1}{2}}\ dx$$

$$\lim_{b\to\infty}\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_1^b$$

$$2 \lim_{b \to \infty} \sqrt{x} \Big|_1^b$$

Evaluate over the interval.

$$2 \lim_{b \to \infty} \left(\sqrt{b} - \sqrt{1} \right)$$

$$\lim_{b \to \infty} \left(2\sqrt{b} - 2 \right)$$

$$\infty$$

We have just proven that the comparison integral diverges. Therefore, by the comparison theorem, the given integral also diverges.

Topic: Comparison theorem

Question: Use the comparison theorem to say whether the integral converges or diverges.

$$\int_2^\infty \frac{2-e^{-x}}{x}\ dx$$

when compared to $\displaystyle\int_2^\infty \frac{1}{2x}\ dx$

Answer choices:

A Converges

B Diverges

C Comparison theorem does not apply

D Cannot compare the integral

Solution: B

The comparison test is used to determine convergence of divergence of an improper integral that cannot be evaluated directly. In this test, we find another improper integral, and compare the two integrals to each other.

If we want to confirm that an integral converges, we find another integral that contains an integrand that always has values greater than or equal to the integrand in our integral, in the interval of integration, and converges. Using the comparison theorem, we could say that our integral also converges.

Conversely, if we want to confirm that an integral diverges, we find another integral that contains an integrand that always has values less than or equal to the integrand in our integral, in the interval of integration, and diverges. Using the comparison theorem, we could say that our integral also diverges.

If we compare the given integral

$$\int_2^\infty \frac{2-e^{-x}}{x}\ dx$$

to

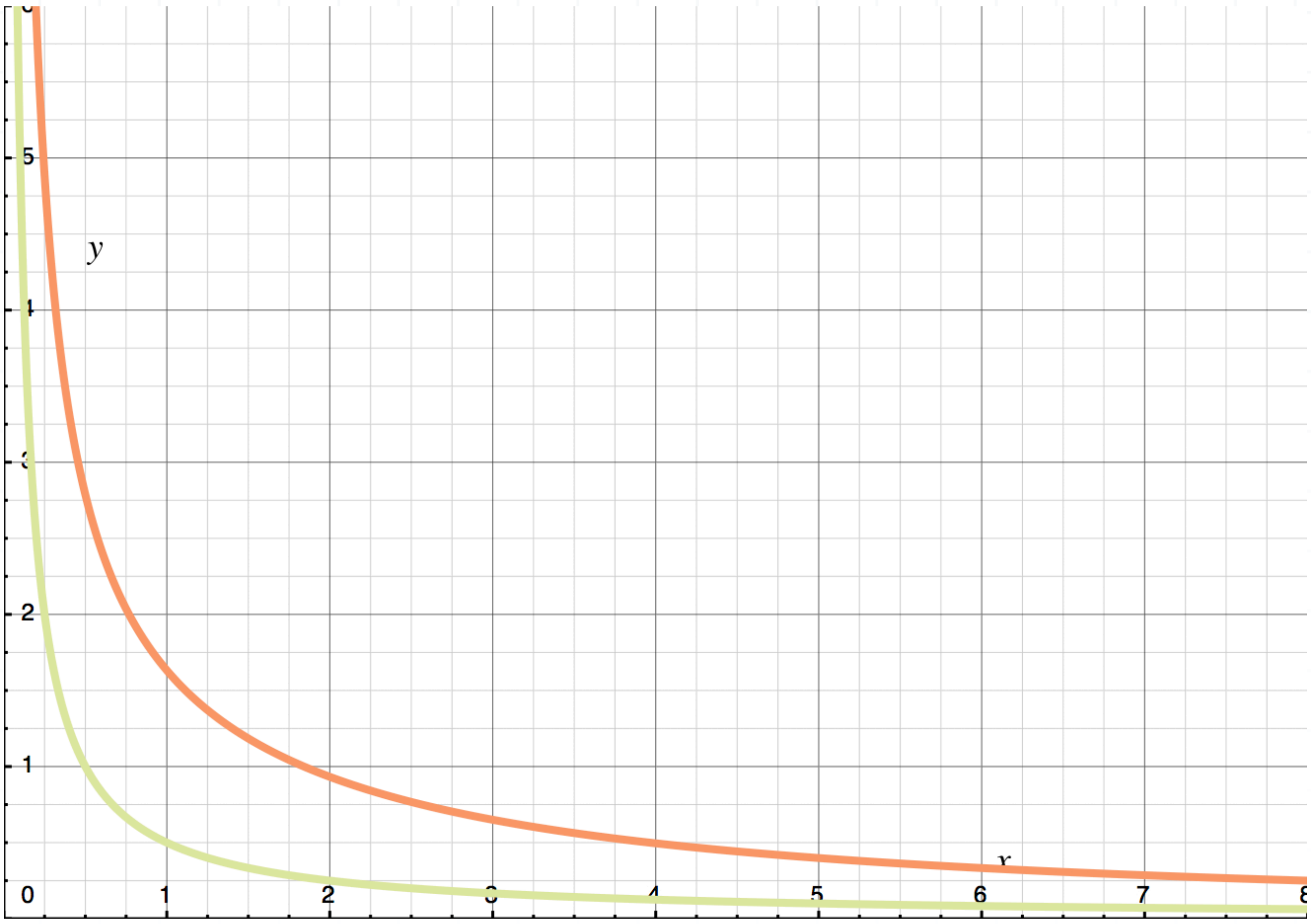
$$\int_2^\infty \frac{1}{2x}\ dx$$

and prove that the given integrand is always greater than or equal to the comparison function, and then prove that the comparison function diverges, then we'll have proven that the given integral also diverges.

First, let's confirm that

$$\frac{2-e^{-x}}{x} \geq \frac{1}{2x}$$

on $[2,\infty)$. We can do that graphically. The graph of both functions is below.



From the graphs, we can confirm that the inequality is true.

Now we'll test the convergence of

$$\int_2^\infty \frac{1}{2x}\ dx$$

Convert the improper integral to a limit, then integrate.

$$\frac{1}{2}\lim_{b\to\infty}\int_2^b \frac{1}{x}\ dx$$

$$\frac{1}{2}\lim_{b\to\infty}\ln|x|\Big|_2^b$$

Evaluate over the interval.

$$\frac{1}{2}\lim_{b\to\infty}\left(\ln|b| - \ln|2|\right)$$

$$\frac{1}{2}\left(\infty - \ln 2\right)$$

$$\infty$$

We have just proven that the comparison integral diverges. Therefore, by the comparison theorem, the given integral also diverges.

Topic: Integrals using reduction formulas

Question: Use the reduction formula to find the integral.

$$\int \tan^6 x \ dx$$

Use $\int \tan^n x \ dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x \ dx$

Answer choices:

A $\quad \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$

B $\quad \tan^5 x - \tan^3 x + \tan x - x + C$

C $\quad \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$

D $\quad \frac{1}{7}\tan^7 x + C$

Solution: A

The reduction formula

$$\int \tan^n x \ dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \ dx$$

simply gives us a method of evaluating the integral without trying to figure out how to do the integration. In our integral, $n = 6$. We use the reduction formula, repeatedly until we have the final result of the integration. The formula tells us to reduce the exponents each time.

Let's do the first iteration of the reduction formula.

$$\int \tan^6 x \ dx = \frac{1}{6-1} \tan^{6-1} x - \int \tan^{6-2} x \ dx$$

$$\int \tan^6 x \ dx = \frac{1}{5} \tan^5 x - \int \tan^4 x \ dx$$

Let's do the second iteration of the reduction formula.

$$\int \tan^6 x \ dx = \frac{1}{5} \tan^5 x - \left(\frac{1}{4-1} \tan^{4-1} x - \int \tan^{4-2} x \ dx \right)$$

$$\int \tan^6 x \ dx = \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \int \tan^2 x \ dx$$

Let's do the third iteration of the reduction formula.

$$\int \tan^6 x \ dx = \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \frac{1}{2-1} \tan^{2-1} x - \int \tan^{2-2} x \ dx$$

$$\int \tan^6 x \ dx = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - \int dx$$

Do the final integration without the reduction formula.

$$\int \tan^6 x \ dx = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$$

Topic: Integrals using reduction formulas

Question: Use the reduction formula to find the integral.

$$\int x^4 e^x \, dx$$

Use $\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$

Answer choices:

A $\quad x^4e^x - x^3e^x + x^2e^x - xe^x + e^x + C$

B $\quad x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x$

C $\quad x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + x + C$

D $\quad x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$

Solution: D

The reduction formula

$$\int x^n e^x \ dx = x^n e^x - \int n x^{n-1} e^x \ dx$$

simply gives us a method of evaluating the integral without trying to figure out how to do the integration. In our integral, $n = 4$. We use the reduction formula, repeatedly until we have the final result of the integration. The formula tells us to reduce the exponents each time.

Let's do the first iteration of the reduction formula.

$$\int x^4 e^x \ dx = x^4 e^x - \int 4x^{4-1} e^x \ dx$$

$$\int x^4 e^x \ dx = x^4 e^x - 4\int x^3 e^x \ dx$$

Let's do the second iteration of the reduction formula.

$$\int x^4 e^x \ dx = x^4 e^x - 4\left[x^3 e^x - \int 3x^{3-1} e^x \ dx\right]$$

$$\int x^4 e^x \ dx = x^4 e^x - 4x^3 e^x + 4\int 3x^{3-1} e^x \ dx$$

$$\int x^4 e^x \ dx = x^4 e^x - 4x^3 e^x + 12\int x^2 e^x \ dx$$

Let's do the third iteration of the reduction formula.

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12\left[x^2e^x - \int 2x^{2-1}e^x\ dx\right]$$

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 12\int 2x^{2-1}e^x\ dx$$

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24\int xe^x\ dx$$

Let's do the fourth iteration of the reduction formula.

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left[xe^x - \int 1x^{1-1}e^x\ dx\right]$$

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24\int e^x\ dx$$

Do the final integration without the reduction formula.

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x$$

$$\int x^4e^x\ dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$$

Topic: Integrals using reduction formulas

Question: Use the reduction formula to find the integral.

$$\int (\ln x)^4 \ dx$$

Use $\int (\ln x)^n \ dx = x(\ln x)^n - n \int (\ln x)^{n-1} \ dx$

Answer choices:

A $(\ln x)^3 - 4(\ln x)^3 + 12(\ln x)^2 - 24 \ln x + 24 + C$

B $x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x(\ln x) + 24x + C$

C $x\left[(\ln x)^3 - 4(\ln x)^2 + 12(\ln x)\right]$

D $(\ln x)^3 - 4(\ln x)^3 + 12(\ln x)^2 - 24 \ln x + 24$

Solution: B

The reduction formula

$$\int (\ln x)^n\ dx = x(\ln x)^n - n\int (\ln x)^{n-1}\ dx$$

simply gives us a method of evaluating the integral without trying to figure out how to do the integration. In our integral, $n = 4$. We use the reduction formula, repeatedly until we have the final result of the integration. The formula tells us to reduce the exponents each time.

Let's do the first iteration of the reduction formula.

$$\int (\ln x)^4\ dx = x(\ln x)^4 - 4\int (\ln x)^{4-1}\ dx$$

$$\int (\ln x)^4\ dx = x(\ln x)^4 - 4\int (\ln x)^3\ dx$$

Let's do the second iteration of the reduction formula.

$$\int (\ln x)^4\ dx = x(\ln x)^4 - 4\left[x(\ln x)^3 - 3\int (\ln x)^{3-1}\ dx\right]$$

$$\int (\ln x)^4\ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12\int (\ln x)^2\ dx$$

Let's do the third iteration of the reduction formula.

$$\int (\ln x)^4\ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2\int (\ln x)^{2-1}\ dx\right]$$

$$\int (\ln x)^4 \ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24 \int \ln x \ dx$$

Let's do the fourth iteration of the reduction formula.

$$\int (\ln x)^4 \ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24 \left[x(\ln x)^1 - 1 \int (\ln x)^{1-1} \ dx \right]$$

$$\int (\ln x)^4 \ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x(\ln x) + 24 \int \ dx$$

Do the final integration without the reduction formula.

$$\int (\ln x)^4 \ dx = x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x(\ln x) + 24x + C$$

Topic: Area between upper and lower curves

Question: Find the area between the curves.

$$y = \sec^2 x$$

$$y = 1$$

and

$$x = 0$$

$$x = \frac{\pi}{4}$$

Answer choices:

A $\frac{\pi}{4} - 1$

B $1 - \frac{\pi}{4}$

C 1

D $\sqrt{2} - \frac{\pi}{4}$

Solution: B

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for y in terms of x, it means these are upper and lower curves. If we're using these curves, then the lines $x = 0$ and $x = \pi/4$ define our interval. Remember though, just because we're given the interval doesn't mean we can skip step 2. We still need to make sure that there are no points of intersection inside the given interval.

To check for points of intersection, we'll set the curves equal to each other.

$$\sec^2 x = 1$$

$$\frac{1}{\cos^2 x} = 1$$

$$1 = \cos^2 x$$

$$\sqrt{1} = \sqrt{\cos^2 x}$$

$$1 = \cos x$$

$$x = 0$$

Since this point of intersection is the same as the left endpoint of our interval, and there are no other points of intersection inside the interval, we know that one curve is always above the other curve throughout the entire interval. Our next step is to determine which curve has a larger y-value on the x-interval $[0,\pi/4]$.

We can do this by picking an x-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(x)$, and whichever curve returns a lower value we'll call $g(x)$.

Plugging $x = \pi/6$ into both functions, we get

$$y = 1$$

and

$$y = \sec^2 x$$

$$y = \sec^2 \frac{\pi}{6}$$

$$y = \frac{1}{\cos^2 \frac{\pi}{6}}$$

$$y = \frac{1}{\left(\cos\frac{\pi}{6}\right)^2}$$

$$y = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$y = \frac{1}{\frac{3}{4}}$$

$$y = \frac{4}{3}$$

Since $y = \sec^2 x$ gives a larger value, we'll say

$$f(x) = \sec^2 x$$

and

$$g(x) = 1$$

Now we can plug these functions and the interval we found earlier into the formula for area between upper and lower curves.

$$\int_0^{\frac{\pi}{4}} \sec^2 x - 1 \ dx$$

$$\tan x - x \Big|_0^{\frac{\pi}{4}}$$

$$\tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)$$

$$\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}-\frac{\pi}{4}-\frac{\sin 0}{\cos 0}$$

$$\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}-\frac{\pi}{4}-\frac{0}{1}$$

$$1-\frac{\pi}{4}$$

Topic: Area between upper and lower curves

Question: Find the area between the curves.

$$y = x^3 + 2x^2 + 1$$

$$y = x + 3$$

Answer choices:

A $\frac{37}{12}$

B $-\frac{37}{12}$

C $-\frac{9}{4}$

D $\frac{9}{4}$

Solution: A

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for y in terms of x, it means these are upper and lower curves.

To find points of intersection, we'll set the curves equal to each other.

$$x^3 + 2x^2 + 1 = x + 3$$

$$x^3 + 2x^2 - x - 2 = 0$$

$$x^2(x + 2) - x - 2 = 0$$

$$x^2(x + 2) - (x + 2) = 0$$

$$(x + 2)\left(x^2 - 1\right) = 0$$

$$(x + 2)(x + 1)(x - 1) = 0$$

$$x = -2, \ -1, \ 1$$

These left-most and right-most points define the endpoints of our interval in terms of x. Because there's a third point of intersection inside the interval $[-2,1]$, we know that this is a point where the curves cross each other, which means our next step is to determine which curve has a larger y-value on the x-interval $[-2,-1]$. We know that they'll have the opposite orientation on the x-interval $[-1,1]$.

We can do this by picking an x-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(x)$, and whichever curve returns a lower value we'll call $g(x)$.

Plugging $x = -3/2$ into both functions, we get

$$y = x^3 + 2x^2 + 1$$

$$y = \left(-\frac{3}{2}\right)^3 + 2\left(-\frac{3}{2}\right)^2 + 1$$

$$y = -\frac{27}{8} + \frac{18}{4} + 1$$

$$y = -\frac{27}{8} + \frac{36}{8} + \frac{8}{8}$$

$$y = \frac{17}{8}$$

and

$$y = x + 3$$

$$y = -\frac{3}{2} + 3$$

$$y = -\frac{3}{2} + \frac{6}{2}$$

$$y = \frac{3}{2}$$

Since $y = x^3 + 2x^2 + 1$ gives a larger value, we'll say

$$f(x) = x^3 + 2x^2 + 1$$

and

$$g(x) = x + 3$$

Now we can plug these functions and the interval we found earlier into the formula for area between upper and lower curves. Remember, since the curves cross each other at $x = -1$, we have to use two separate integrals.

$$\int_a^b f(x) - g(x)\ dx + \int_b^c g(x) - f(x)\ dx$$

$$\int_{-2}^{-1} \left(x^3 + 2x^2 + 1\right) - (x + 3)\ dx + \int_{-1}^{1} (x + 3) - \left(x^3 + 2x^2 + 1\right)\ dx$$

$$\int_{-2}^{-1} x^3 + 2x^2 - x - 2\ dx + \int_{-1}^{1} -x^3 - 2x^2 + x + 2\ dx$$

$$\left(\frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 - 2x\right)\Bigg|_{-2}^{-1} + \left(-\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Bigg|_{-1}^{1}$$

$$\left(\frac{x^4}{4}+\frac{2x^3}{3}-\frac{x^2}{2}-2x\right)\Bigg|_{-2}^{-1}+\left(-\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x\right)\Bigg|_{-1}^{1}$$

$$\frac{(-1)^4}{4}+\frac{2(-1)^3}{3}-\frac{(-1)^2}{2}-2(-1)-\left[\frac{(-2)^4}{4}+\frac{2(-2)^3}{3}-\frac{(-2)^2}{2}-2(-2)\right]$$

$$+\left[-\frac{(1)^4}{4}-\frac{2(1)^3}{3}+\frac{(1)^2}{2}+2(1)\right]-\left[-\frac{(-1)^4}{4}-\frac{2(-1)^3}{3}+\frac{(-1)^2}{2}+2(-1)\right]$$

$$\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2-\left(\frac{16}{4}-\frac{16}{3}-\frac{4}{2}+4\right)+\left(-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}+2\right)-\left(-\frac{1}{4}+\frac{2}{3}+\frac{1}{2}-2\right)$$

$$\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2-\frac{16}{4}+\frac{16}{3}+\frac{4}{2}-4-\frac{1}{4}-\frac{2}{3}+\frac{1}{2}+2+\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2$$

$$\frac{1}{4}-\frac{16}{4}-\frac{1}{4}+\frac{1}{4}-\frac{2}{3}+\frac{16}{3}-\frac{2}{3}-\frac{2}{3}-\frac{1}{2}+\frac{4}{2}+\frac{1}{2}-\frac{1}{2}+2-4+2+2$$

$$-\frac{15}{4}+\frac{10}{3}+\frac{3}{2}+2$$

$$-\frac{45}{12}+\frac{40}{12}+\frac{18}{12}+\frac{24}{12}$$

$$\frac{37}{12}$$

Topic: Area between upper and lower curves

Question: Determine the area of the region enclosed by the curves.

$$f(x) = 1 + 4x - x^2$$

$$g(x) = 6 - 2x$$

Answer choices:

A $\frac{33}{2}$

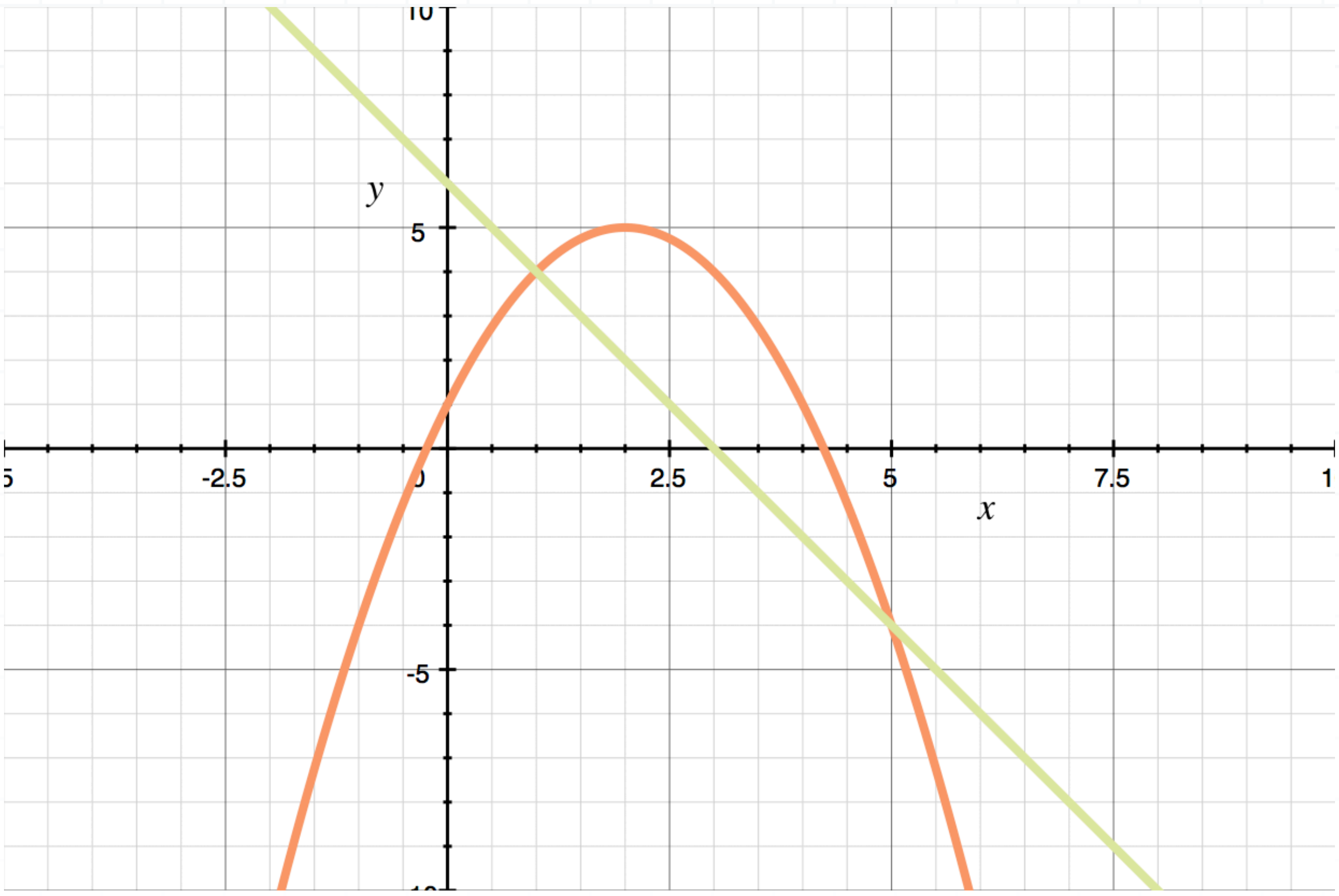
B $\frac{32}{3}$

C $-\frac{32}{3}$

D $-\frac{33}{2}$

Solution: B

The graph of the two functions is



The graph shows that the quadratic function $f(x) = 1 + 4x - x^2$ is higher than the linear function $g(x) = 6 - 2x$. The graph also shows that the two graphs intersect at the points $(1, 4)$ and $(5, -4)$. The x-values in these points give the integration limits. The integral to find the area between the two curves is

$$A = \int_1^5 \left(1 + 4x - x^2\right) - \left(6 - 2x\right) \ dx$$

$$A = \int_1^5 1 + 4x - x^2 - 6 + 2x \ dx$$

$$A = \int_1^5 -x^2 + 6x - 5 \ dx$$

Integrate, then evaluate over the interval.

$$A = -\frac{1}{3}x^3 + 3x^2 - 5x\Big|_1^5$$

$$A = -\frac{1}{3}(5)^3 + 3(5)^2 - 5(5) - \left(-\frac{1}{3}(1)^3 + 3(1)^2 - 5(1)\right)$$

$$A = -\frac{125}{3} + 75 - 25 - \left(-\frac{1}{3} + 3 - 5\right)$$

$$A = -\frac{125}{3} + 75 - 25 + \frac{1}{3} - 3 + 5$$

$$A = -\frac{124}{3} + 52$$

$$A = \frac{32}{3}$$

Topic: Area between left and right curves

Question: Find the area between the curves.

$$x = 3y^2$$

$$x = y^2 + 2$$

Answer choices:

A $\frac{8}{3}$

B $\frac{4}{3}$

C $-\frac{4}{3}$

D $-\frac{8}{3}$

Solution: A

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for x in terms of y, it means these are left and right curves.

To find points of intersection, we'll set the curves equal to each other.

$$3y^2 = y^2 + 2$$

$$2y^2 = 2$$

$$y^2 = 1$$

$$y = \pm 1$$

These two points of intersection define the endpoints of our interval in terms of y, which means our next step is to determine which curve has a larger x-value on the y-interval $[-1,1]$.

We can do this by picking a y-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(y)$, and whichever curve returns a lower value we'll call $g(y)$.

Plugging $y = 0$ into both functions, we get

$$x = 3y^2$$

$$x = 3(0)^2$$

$$x = 0$$

and

$$x = y^2 + 2$$

$$x = (0)^2 + 2$$

$$x = 2$$

Since $x = y^2 + 2$ gives a larger value, we'll say

$$g(y) = 3y^2$$

and

$$f(y) = y^2 + 2$$

Now we can plug these functions and the interval we found earlier into the formula for area between left and right curves.

$$\int_a^b f(y)-g(y)\ dy$$

$$\int_{-1}^{1} y^2+2-3y^2\ dy$$

$$\int_{-1}^{1} -2y^2+2\ dy$$

$$\frac{-2y^3}{3}+2y\Big|_{-1}^{1}$$

$$\left[\frac{-2(1)^3}{3}+2(1)\right]-\left[\frac{-2(-1)^3}{3}+2(-1)\right]$$

$$\frac{-2}{3}+2-\frac{2}{3}+2$$

$$-\frac{4}{3}+\frac{12}{3}$$

$$\frac{8}{3}$$

Topic: Area between left and right curves

Question: Find the area between the curves.

$$x = 2y^2 + 1$$

$$x = y^2 + 5$$

Answer choices:

A $\quad -\frac{32}{3}$

B $\quad \frac{16}{3}$

C $\quad \frac{32}{3}$

D $\quad -\frac{16}{3}$

Solution: C

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for x in terms of y, it means these are left and right curves.

To find points of intersection, we'll set the curves equal to each other.

$$2y^2 + 1 = y^2 + 5$$

$$y^2 = 4$$

$$y = \pm 2$$

These two points of intersection define the endpoints of our interval in terms of y, which means our next step is to determine which curve has a larger x-value on the y-interval $[-2,2]$.

We can do this by picking a y-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(y)$, and whichever curve returns a lower value we'll call $g(y)$.

Plugging $y = 1$ into both functions, we get

$$x = 2y^2 + 1$$

$$x = 2(1)^2 + 1$$

$$x = 3$$

and

$$x = y^2 + 5$$

$$x = (1)^2 + 5$$

$$x = 6$$

Since $x = y^2 + 5$ gives a larger value, we'll say

$$g(y) = 2y^2 + 1$$

and

$$f(y) = y^2 + 5$$

Now we can plug these functions and the interval we found earlier into the formula for area between left and right curves.

$$\int_a^b f(y) - g(y)\ dy$$

$$\int_{-2}^{2} y^2 + 5 - \left(2y^2 + 1\right) \ dy$$

$$\int_{-2}^{2} -y^2 + 4 \ dy$$

$$\frac{-y^3}{3} + 4y \Big|_{-2}^{2}$$

$$\left[\frac{-(2)^3}{3} + 4(2)\right] - \left[\frac{-(-2)^3}{3} + 4(-2)\right]$$

$$\left(\frac{-8}{3} + 8\right) - \left(\frac{8}{3} - 8\right)$$

$$\left(\frac{16}{3}\right) - \left(\frac{-16}{3}\right)$$

$$\frac{32}{3}$$

Topic: Area between left and right curves

Question: Find the area between the curves.

$$x = y^2 + y + 3$$

$$x = 2y^2 + 2y + 1$$

Answer choices:

A $\frac{2}{3}$

B $\frac{9}{2}$

C $-\frac{9}{2}$

D $-\frac{2}{3}$

Solution: B

In order to calculate the area between two curves, we need to follow these steps:

1. Decide whether the curves are

    a. upper and lower curves, or

    b. left and right curves.

2. Find points of intersection.

3. Determine which curve has the larger value between each point of intersection.

4. Plug everything into the appropriate formula.

Since the curves we're given are both expressed for x in terms of y, it means these are left and right curves.

To find points of intersection, we'll set the curves equal to each other.

$$2y^2 + 2y + 1 = y^2 + y + 3$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$y = -2$ and $y = 1$

These two points of intersection define the endpoints of our interval in terms of y, which means our next step is to determine which curve has a larger x-value on the y-interval $[-2,1]$.

We can do this by picking a y-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(y)$, and whichever curve returns a lower value we'll call $g(y)$.

Plugging $y = 0$ into both functions, we get

$$x = y^2 + y + 3$$

$$x = (0)^2 + 0 + 3$$

$$x = 3$$

and

$$x = 2y^2 + 2y + 1$$

$$x = 2(0)^2 + 2(0) + 1$$

$$x = 1$$

Since $x = y^2 + y + 3$ gives a larger value, we'll say

$$f(y) = y^2 + y + 3$$

and

$$g(y) = 2y^2 + 2y + 1$$

Now we can plug these functions and the interval we found earlier into the formula for area between left and right curves.

$$\int_{a}^{b} f(y) - g(y)\ dy$$

$$\int_{-2}^{1} y^2 + y + 3 - \left(2y^2 + 2y + 1\right)\ dy$$

$$\int_{-2}^{1} -y^2 - y + 2\ dy$$

$$\frac{-y^3}{3} - \frac{y^2}{2} + 2y \Big|_{-2}^{1}$$

$$\left[\frac{-(1)^3}{3} - \frac{(1)^2}{2} + 2(1)\right] - \left[\frac{-(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right]$$

$$\left(\frac{-1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)$$

$$-\frac{2}{6} - \frac{3}{6} + \frac{12}{6} - \frac{16}{6} + \frac{12}{6} + \frac{24}{6}$$

$$\frac{9}{2}$$

Topic: Sketching the area between curves

Question: Sketch the region(s) enclosed by the x-axis and the curve. Determine the best way to find total area of the regions, then calculate the total area.

$$f(x) = x^3 - 3x^2 + 2x$$

Answer choices:

A $\quad \displaystyle\int_0^1 x^3 - 3x^2 + 2x\ dx + \int_1^2 x^3 - 3x^2 + 2x\ dx = \frac{1}{2}$

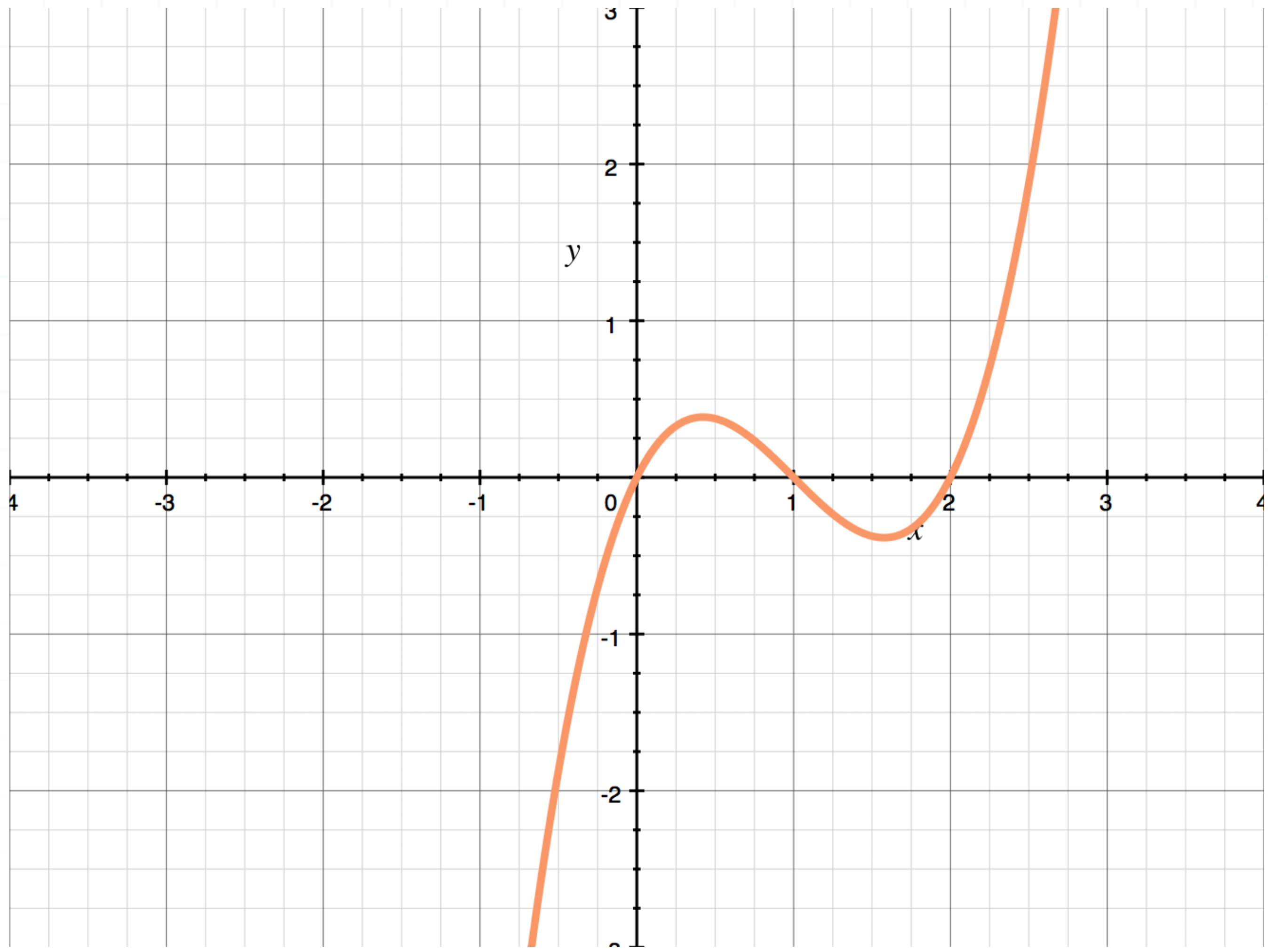
B $\quad \displaystyle\int_0^1 x^3 - 3x^2 + 2x\ dx + \left| \int_1^2 x^3 - 3x^2 + 2x\ dx \right| = \frac{1}{2}$

C $\quad \displaystyle\int_0^2 x^3 - 3x^2 + 2x\ dx = 0$

D $\quad \displaystyle\int_0^1 x^3 - 3x^2 + 2x\ dx + \int_1^2 x^3 - 3x^2 + 2x\ dx = 0$

Solution: B

The graph of $f(x) = x^3 - 3x^2 + 2x$ is the graph of a cubic function. A sketch of $f(x)$ is



Notice that the graph of $f(x)$ is above the x-axis in the interval $[0,1]$ and below the x-axis in the interval $[1,2]$.

We integrate a function to find the area between a curve and the x-axis. However, the integral of $f(x)$ over the entire interval $[0,2]$ would give us the net area, and not the total area.

Additionally, integrating a function on an interval where the function is above the x-axis gives the area between the curve and the x-axis, but integrating a function on an interval where the function is below the x-axis gives a negative value of the area between the curve and the x-axis.

Therefore, if we want the total area between $f(x)$ and the x-axis, we will have to integrate the absolute value of $f(x)$ on the interval in which $f(x)$ is below the x-axis.

To find the total area, we will integrate $f(x)$ on the interval $[0,1]$ and then integrate $|f(x)|$ on the interval $[1,2]$, and then add the results of the integration.

$$A = \int_0^1 f(x)\ dx + \int_1^2 \left|f(x)\right|\ dx$$

$$A = \int_0^1 x^3 - 3x^2 + 2x\ dx + \left|\int_1^2 x^3 - 3x^2 + 2x\ dx\right|$$

Integrate using the power rule, then evaluate over the interval.

$$A = \frac{1}{4}x^4 - x^3 + x^2\Big|_0^1 + \left|\frac{1}{4}x^4 - x^3 + x^2\Big|_1^2\right|$$

$$A = \frac{1}{4}(1)^4 - (1)^3 + (1)^2 - \left(\frac{1}{4}(0)^4 - (0)^3 + (0)^2\right) + \left|\frac{1}{4}(2)^4 - (2)^3 + (2)^2 - \left(\frac{1}{4}(1)^4 - (1)^3 + (1)^2\right)\right|$$

$$A = \frac{1}{4} - 1 + 1 + \left|\frac{1}{4}(16) - 8 + 4 - \left(\frac{1}{4} - 1 + 1\right)\right|$$

$$A = \frac{1}{4} - 1 + 1 + \left|4 - 8 + 4 - \frac{1}{4} + 1 - 1\right|$$

$$A = \frac{1}{4} + \left|-\frac{1}{4}\right|$$

$$A = \frac{1}{2}$$

Topic: Sketching the area between curves

Question: Sketch the region(s) enclosed by the curves. Determine the best integration method to find total area of the regions. Then, calculate the total area.

$$f(x) = x(x-3)$$

$$x = 0$$

$$x = 5$$

Answer choices:

A $\displaystyle \int_0^3 x^2 - 3x \ dx + \int_3^5 x^2 - 3x \ dx = \frac{25}{6}$

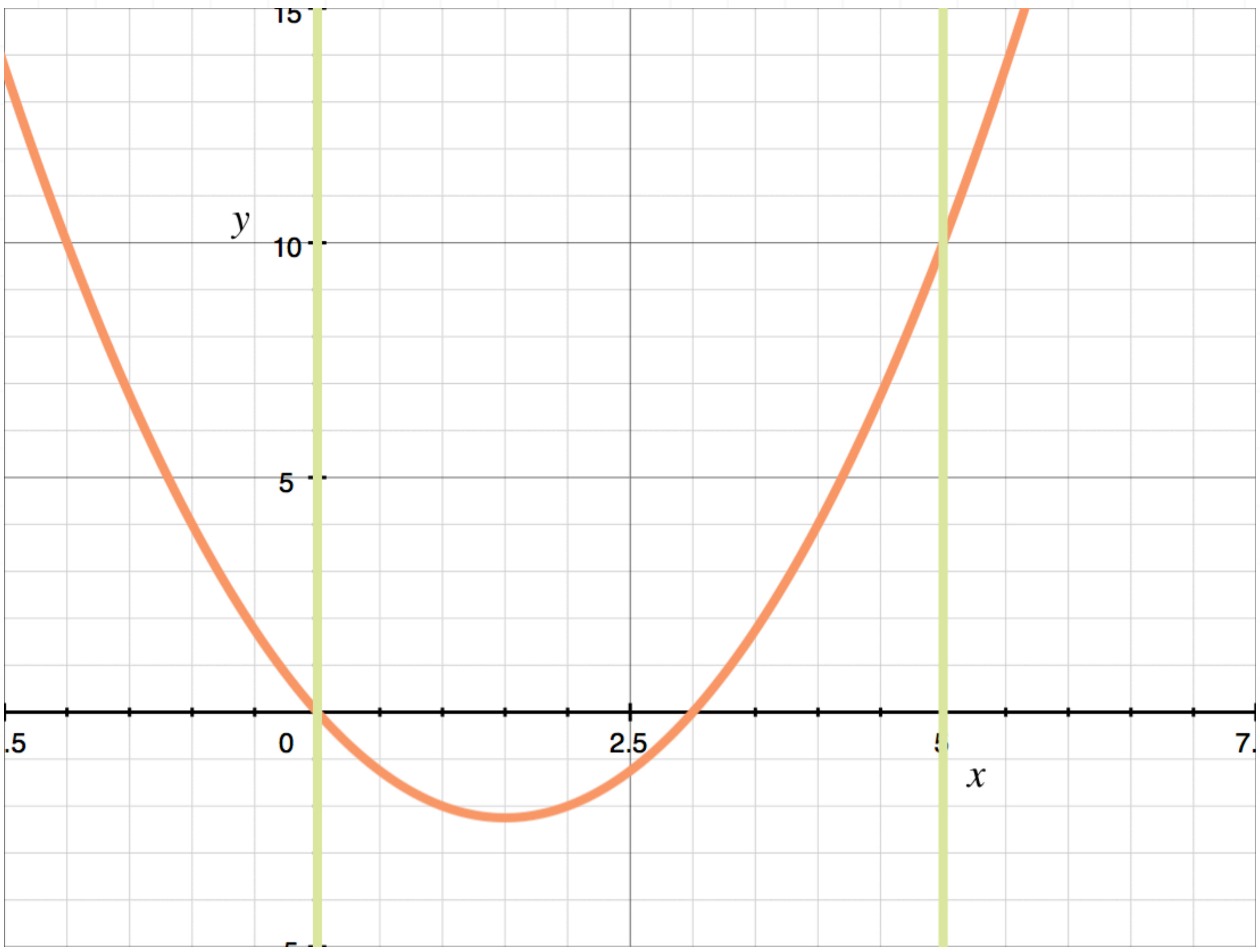
B $\displaystyle \int_0^3 x^2 - 3x \ dx + \left| \int_3^5 x^2 - 3x \ dx \right| = \frac{25}{6}$

C $\displaystyle \int_0^3 x^2 - 3x \ dx + \int_3^5 x^2 - 3x \ dx = \frac{79}{6}$

D $\displaystyle \left| \int_0^3 x^2 - 3x \ dx \right| + \int_3^5 x^2 - 3x \ dx = \frac{79}{6}$

Solution: D

The graph of $f(x) = x^2 - 3x$ is the graph of a quadratic function. A sketch of $f(x)$ between the lines $x = 0$ and $x = 5$ is



Notice that the graph of $f(x)$ is below the x-axis on the interval $[0,3]$ and above the x-axis on the interval $[3,5]$.

We integrate a function to find the area between a curve and the x-axis. However, the integral of $f(x)$ over the entire interval $[0,5]$ would give us the net area, and not the total area.

Additionally, integrating a function on an interval where the function is above the x-axis gives the area between the curve and the x-axis, but integrating a function on an interval where the function is below the x-axis gives a negative value of the area between the curve and the x-axis.

Therefore, if we want the total area between $f(x)$ and the x-axis between the lines $x = 0$ and $x = 5$, we will have to integrate the absolute value of $f(x)$ on the interval in which $f(x)$ is below the x-axis.

To find the total area, we will integrate $|f(x)|$ on the interval $[0,3]$ and then integrate $f(x)$ on the interval $[3,5]$, and then add the results of the integration.

$$\left| \int_0^3 f(x)\ dx \right| + \int_3^5 f(x)\ dx$$

$$\left| \int_0^3 x(x-3)\ dx \right| + \int_3^5 x(x-3)\ dx$$

$$\left| \int_0^3 x^2 - 3x\ dx \right| + \int_3^5 x^2 - 3x\ dx$$

Integrate.

$$\left| \frac{1}{3}x^3 - \frac{3}{2}x^2 \Big|_0^3 \right| + \frac{1}{3}x^3 - \frac{3}{2}x^2 \Big|_3^5$$

Evaluate over each interval.

$$\left| \frac{1}{3}(3)^3 - \frac{3}{2}(3)^2 - \left(\frac{1}{3}(0)^3 - \frac{3}{2}(0)^2 \right) \right| + \frac{1}{3}(5)^3 - \frac{3}{2}(5)^2 - \left(\frac{1}{3}(3)^3 - \frac{3}{2}(3)^2 \right)$$

$$\left| 9 - \frac{27}{2} \right| + \frac{125}{3} - \frac{75}{2} - \left(9 - \frac{27}{2} \right)$$

$$\left|\frac{18}{2}-\frac{27}{2}\right|+\frac{125}{3}-\frac{75}{2}-9+\frac{27}{2}$$

$$\frac{125}{3}-\frac{39}{2}-9$$

$$\frac{250}{6}-\frac{117}{6}-\frac{54}{6}$$

$$\frac{79}{6}$$

Topic: Sketching the area between curves

Question: Sketch the region enclosed by the curves. Determine the best integration method to find total area of the region. Then, calculate the total area.

$$x = y^2$$

$$x = y + 2$$

Answer choices:

A $\quad \int_{-1}^{2} y + 2 - y^2 \ dy = \frac{9}{2}$

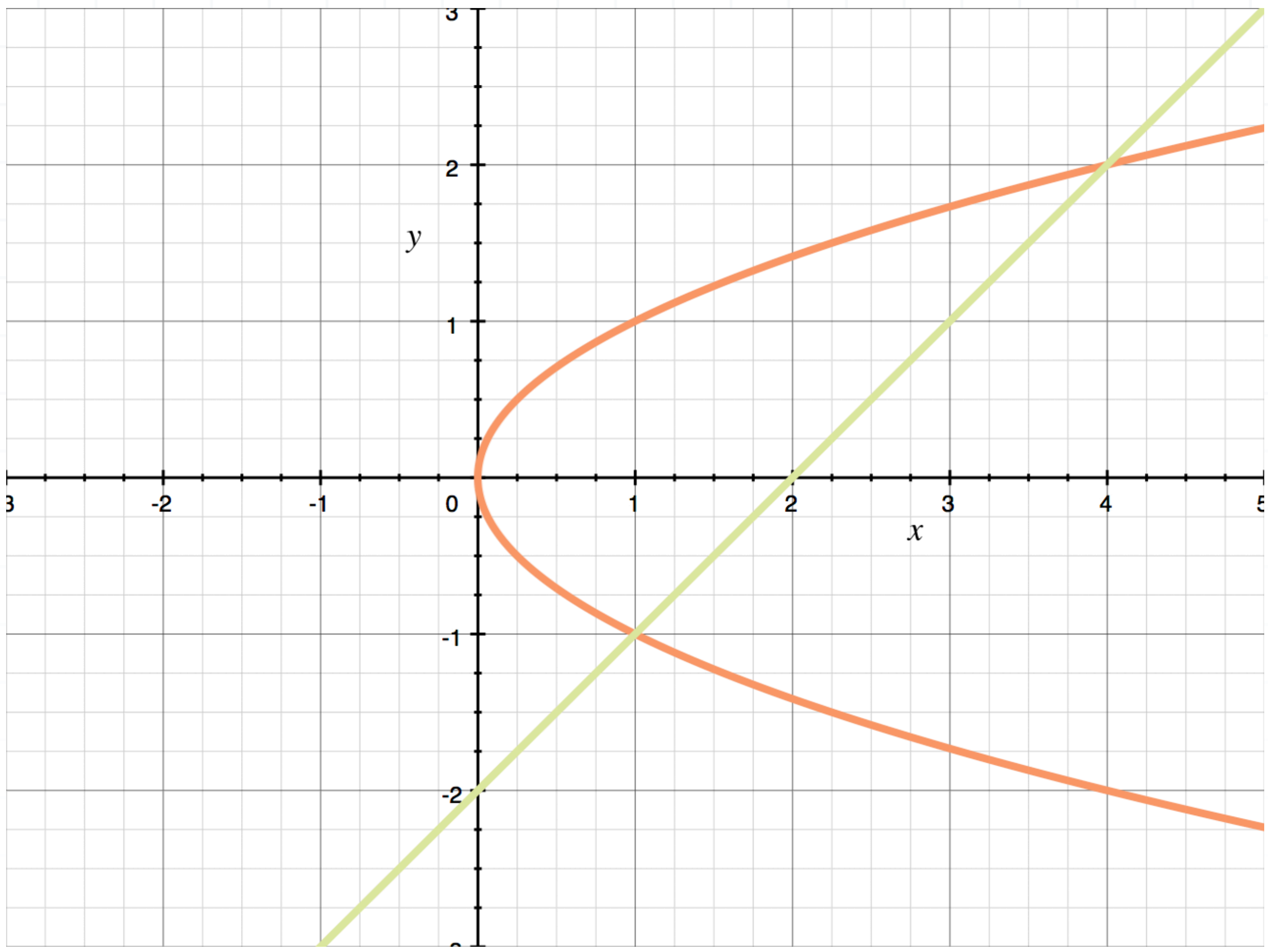
B $\quad \int_{-1}^{2} y^2 - y - 2 \ dy = \frac{9}{2}$

C $\quad \int_{0}^{5} x^2 - x - 2 \ dx = 19\frac{1}{6}$

D $\quad \int_{1}^{5} x^2 - x - 2 \ dx = 21\frac{1}{3}$

Solution: A

The graphs of $x = y^2$ and $x = y + 2$ is the graph of a quadratic curve and a linear function. A sketch is



Notice that the graph of the parabola opens toward the right and is not a function because it fails the vertical line test.

We usually integrate to find the area between the two curves with respect to x. If we do that in this problem, we will need more than one integral because one curve is not a function, and the curves intersect more than once. However, if we integrate with respect to y, we can find the area enclosed by the two curves with a single integral.

Also note, from the graph above, that the two curves intersect at the points $(1,-1)$ and $(4,2)$ which gives us our integration limits, and since we are integrating with respect to y, the integration limits are the y-values in the points of intersection. Thus, the limits of integration will be from -1 to 2.

When integrating with respect to x to find the area between two functions, we subtract the lower function from the higher function in the integrand. When we integrate with respect to y to find the area between two curves, we subtract the left curve from the right curve.

To find the area, we will use this integral

$$\int_{-1}^{2} y+2-y^2\ dy$$

Integrate.

$$\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big|_{-1}^{2}$$

Evaluate over the interval.

$$\frac{1}{2}(2)^2+2(2)-\frac{1}{3}(2)^3-\left(\frac{1}{2}(-1)^2+2(-1)-\frac{1}{3}(-1)^3\right)$$

$$2+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}$$

$$5-\frac{1}{2}$$

$$\frac{9}{2}$$

Topic: Dividing the area between curves into equal parts

Question: The line $y = k$ divides the area bounded by the curves into two equal parts. Find k.

$$f(x) = x^2$$

$$g(x) = 25$$

Answer choices:

A $\quad k = \sqrt{\dfrac{25\sqrt[3]{2}}{2}}$

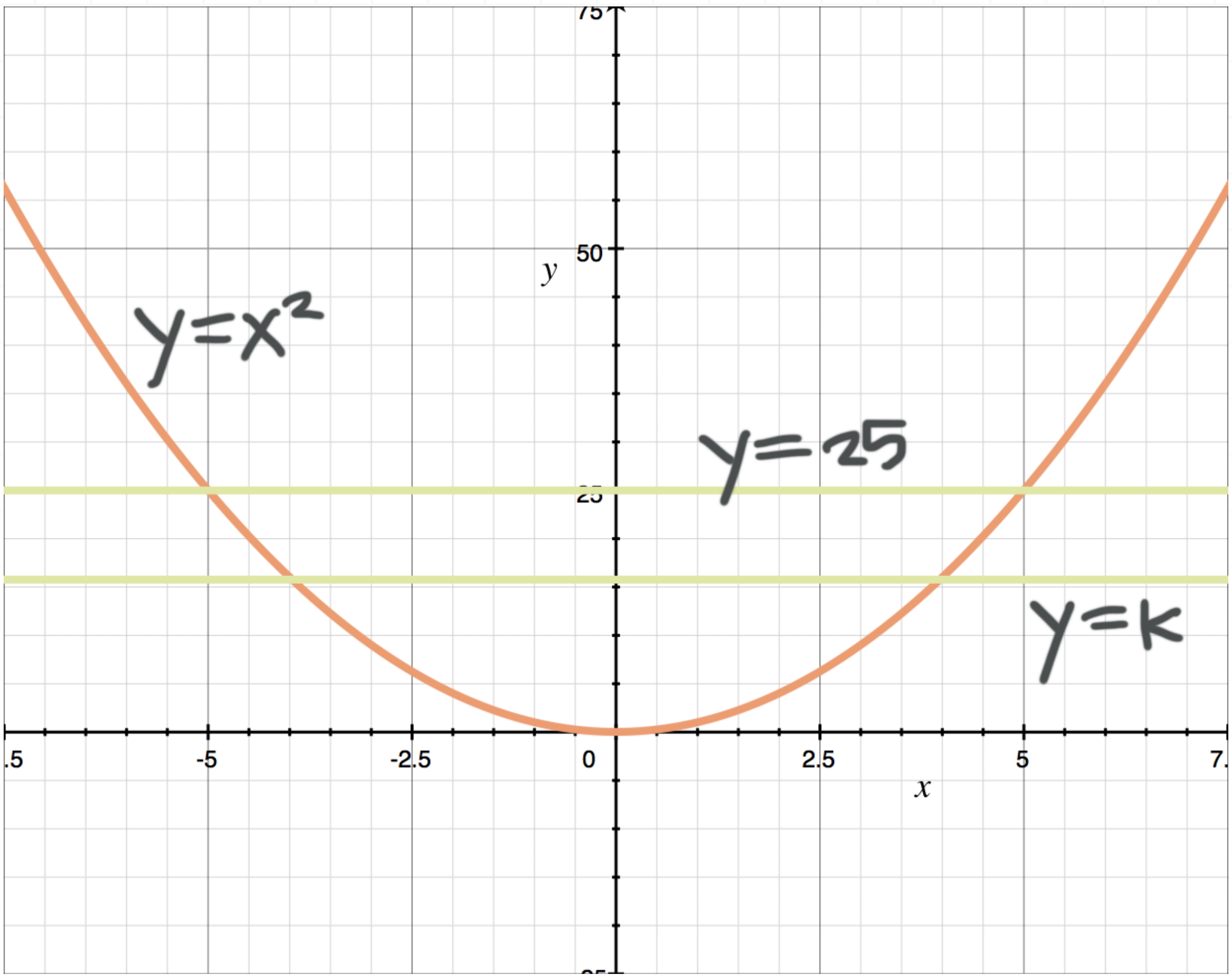
B $\quad k = -\dfrac{25\sqrt[3]{2}}{2}$

C $\quad k = -\sqrt{\dfrac{25\sqrt[3]{2}}{2}}$

D $\quad k = \dfrac{25\sqrt[3]{2}}{2}$

Solution: D

The graph of $f(x) = 25 - x^2$ with a line $y = k$ is



To answer the question, first, we will find the entire area of the bounded region. Notice from the graph that the interval on the x-axis for the bounded region is $[-5,5]$. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = 25$ is the upper function and $y = x^2$ is the lower function.

$$\int_{-5}^{5} 25 - x^2 \ dx$$

$$25x - \frac{1}{3}x^3 \Big|_{-5}^{5}$$

$$25(5)-\frac{1}{3}(5)^3-\left(25(-5)-\frac{1}{3}(-5)^3\right)$$

$$125-\frac{125}{3}-\left(-125+\frac{125}{3}\right)$$

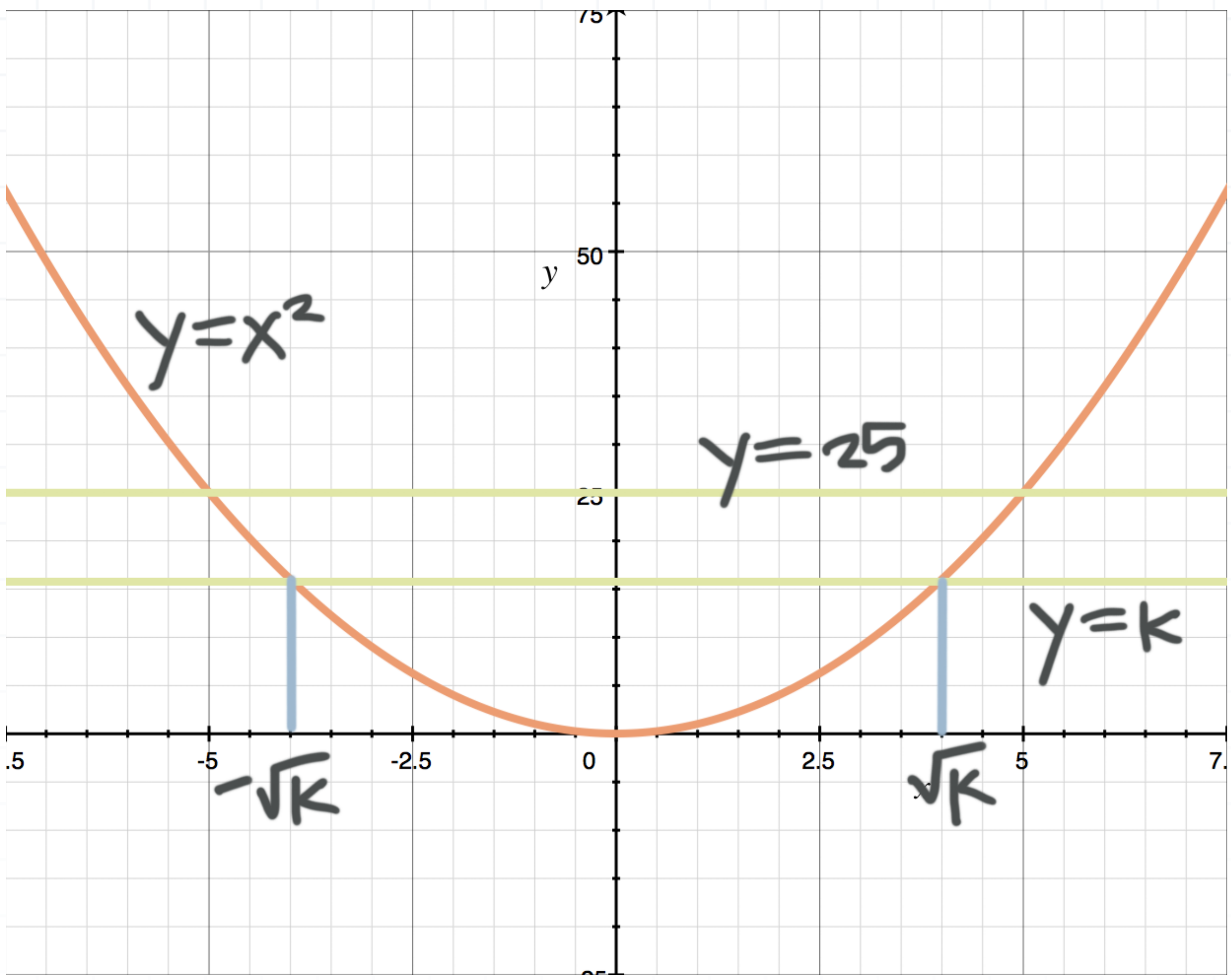
$$125-\frac{125}{3}+125-\frac{125}{3}$$

$$\frac{500}{3}$$

This is the area of the entire bounded region.

If we will find the value of k in the equation $y=k$ that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of $500/3$, which is $250/3$.

The region that represents one-half of the bounded region is the portion of the original region that is below the line $y=k$ as shown in the graph below.



Notice from the graph that since the functions are $y = x^2$ and $y = k$, the points of intersection are $\left(-\sqrt{k}, k\right)$ and $\left(\sqrt{k}, k\right)$. Therefore, the interval of the integration will be $\left[-\sqrt{k}, \sqrt{k}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = k$ is the upper function and $y = x^2$ is the lower function.

$$A = \int_{-\sqrt{k}}^{\sqrt{k}} k - x^2 \, dx$$

$$A = kx - \frac{1}{3}x^3 \Big|_{-\sqrt{k}}^{\sqrt{k}}$$

$$A = k\sqrt{k} - \frac{1}{3}\left(\sqrt{k}\right)^3 - \left(k\left(-\sqrt{k}\right) - \frac{1}{3}\left(-\sqrt{k}\right)^3\right)$$

$$A = k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}} - \left(-k^{\frac{3}{2}} + \frac{1}{3}k^{\frac{3}{2}}\right)$$

$$A = k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}} + k^{\frac{3}{2}} - \frac{1}{3}k^{\frac{3}{2}}$$

$$A = \frac{4}{3}k^{\frac{3}{2}}$$

Now recall that earlier we said that the area of this region had to be equal to $1/2$ of the area of the original bounded region. The area of the original bounded region was $500/3$. Which means the area A we found above must be $250/3$ square units.

$$\frac{4}{3}k^{\frac{3}{2}} = \frac{250}{3}$$

$$4k^{\frac{3}{2}} = 250$$

$$16k^3 = 250^2$$

$$k^3 = \frac{250^2}{16}$$

$$k = \sqrt[3]{\frac{250^2}{16}}$$

$$k = \sqrt[3]{\frac{25 \cdot 25 \cdot 10 \cdot 10}{16}}$$

$$k = \sqrt[3]{\frac{25 \cdot 25 \cdot 5 \cdot 5}{4}}$$

$$k = \frac{25}{\sqrt[3]{4}}$$

Rationalize the denominator.

$$k = \frac{25\sqrt[3]{16}}{4}$$

$$k = \frac{25\sqrt[3]{8}\sqrt[3]{2}}{4}$$

$$k = \frac{25\sqrt[3]{2}}{2}$$

Topic: Dividing the area between curves into equal parts

Question: The line $x = a$ divides the area bounded by the curves into two equal parts. Find a.

$$x = y^2$$

$$x = 4$$

Answer choices:

A $a = 2\sqrt{2}$

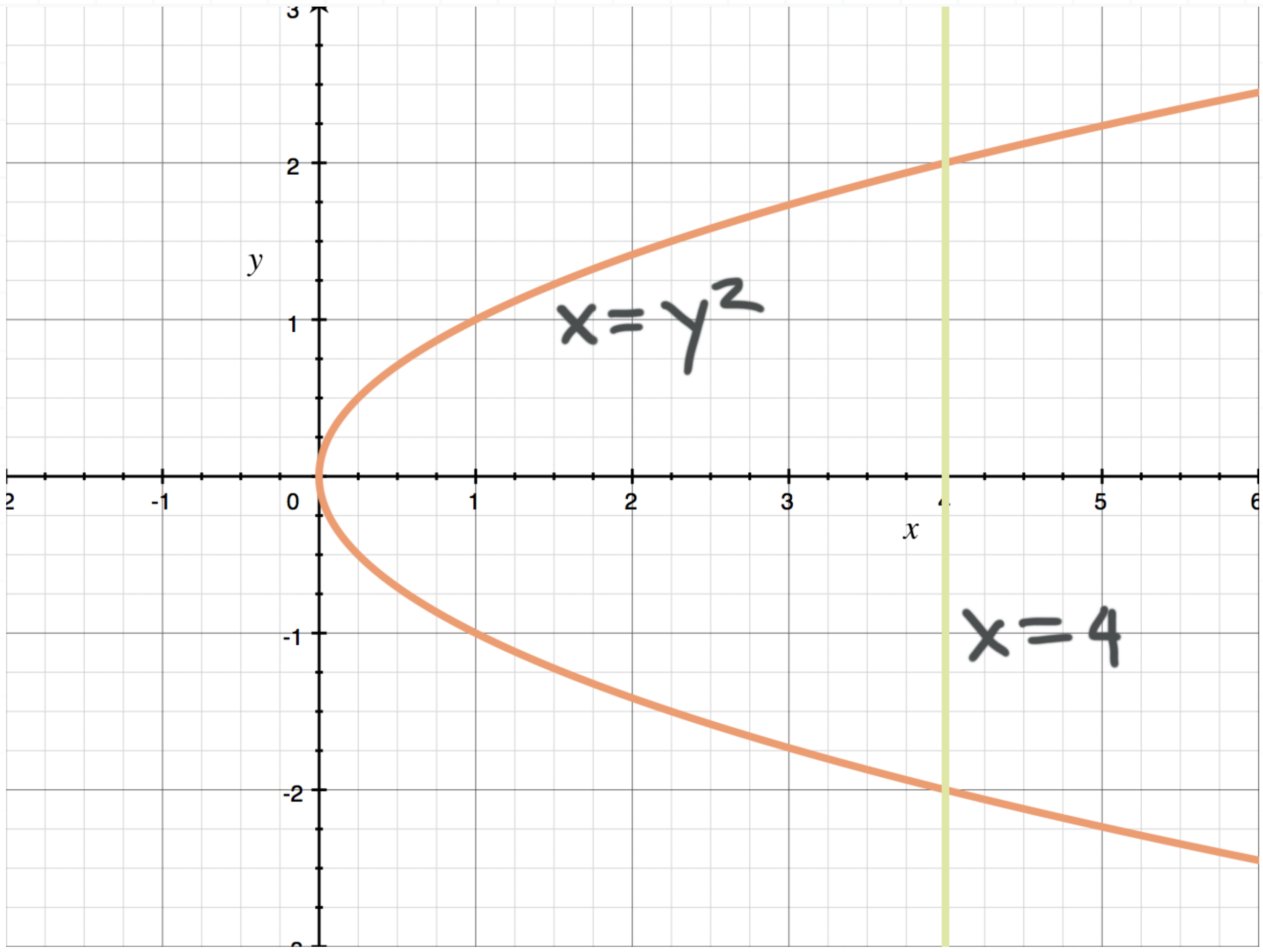
B $a = 2\sqrt[3]{2}$

C $a = 3\sqrt[3]{2}$

D $a = 3\sqrt{2}$

Solution: B

The graph of $x = y^2$ with a line $x = 4$ is



To answer the question, first, we will find the entire area of the bounded region. Since the equations in the question are not functions, and x is expressed in terms of y, we will integrate with respect to y.

Notice, from the graph that the interval on the y-axis for the bounded region is $[-2,2]$. Let's find the area of this region by integrating the right curve minus the left curve in that interval. You can see that $x = 4$ is the right curve and $x = y^2$ is the left curve.

$$\int_{-2}^{2} 4 - y^2 \, dy$$

Integrate, then evaluate over the interval.

$$4y - \frac{1}{3}y^3 \Big|_{-2}^{2}$$

$$4(2) - \frac{1}{3}(2)^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right)$$

$$8 - \frac{8}{3} + 8 - \frac{8}{3}$$

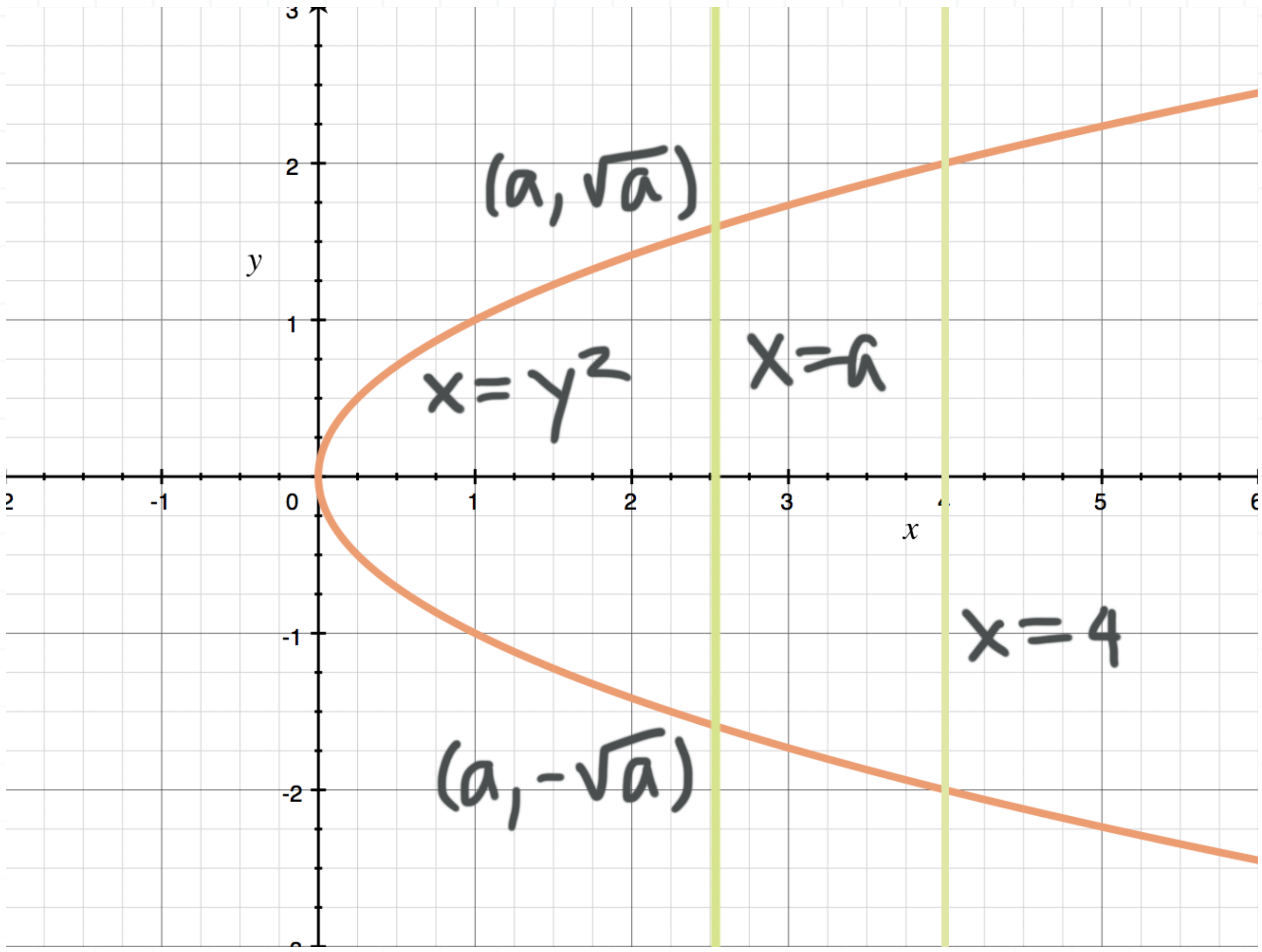
$$16 - \frac{16}{3}$$

$$\frac{32}{3}$$

This is the area of the entire bounded region.

Now, if we will find the value of a in the equation that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of $32/3$, which is $16/3$.

The region that represents $1/2$ of the bounded region is the portion of the original region that is left of the line as shown in the graph below.



Notice from the graph that since the curves are $x = y^2$ and $x = a$, the points of intersection are $\left(a, -\sqrt{a}\right)$ and $\left(a, \sqrt{a}\right)$. Therefore, the interval of the integration will be $\left[-\sqrt{a}, \sqrt{a}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the right curve minus the left curve in that interval, again, with respect to y. You can see that $x = a$ is to the right of $x = y^2$.

$$A = \int_{-\sqrt{a}}^{\sqrt{a}} a - y^2 \, dy$$

Integrate, then evaluate over the interval.

$$A = ay - \frac{1}{3}y^3 \Big|_{-\sqrt{a}}^{\sqrt{a}}$$

$$A = a\sqrt{a} - \frac{1}{3}\left(\sqrt{a}\right)^3 - \left(a\left(-\sqrt{a}\right) - \frac{1}{3}\left(-\sqrt{a}\right)^3\right)$$

$$A = a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{3}{2}} + a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{3}{2}}$$

$$A = \frac{4}{3}a^{\frac{3}{2}}$$

Now, recall that earlier we said that the area of this region has to be equal to $1/2$ of the area of the original bounded region. We stated that the area of this region is $16/3$ square units. Thus we will make the expression above equal to $16/3$.

$$\frac{4}{3}a^{\frac{3}{2}} = \frac{16}{3}$$

$$4a^{\frac{3}{2}} = 16$$

$$a^{\frac{3}{2}} = 4$$

$$a^3 = 16$$

$$a = \sqrt[3]{16}$$

$$a = \sqrt[3]{8 \cdot 2}$$

$$a = \sqrt[3]{8}\sqrt[3]{2}$$

$$a = 2\sqrt[3]{2}$$

Topic: Dividing the area between curves into equal parts

Question: The line $y = a$ divides the area bounded by the curves into two equal parts. Find a.

$$f(x) = -x^2 + 4$$

$$g(x) = -\frac{1}{4}x^2 + 1$$

Answer choices:

A $a = -1.9199$

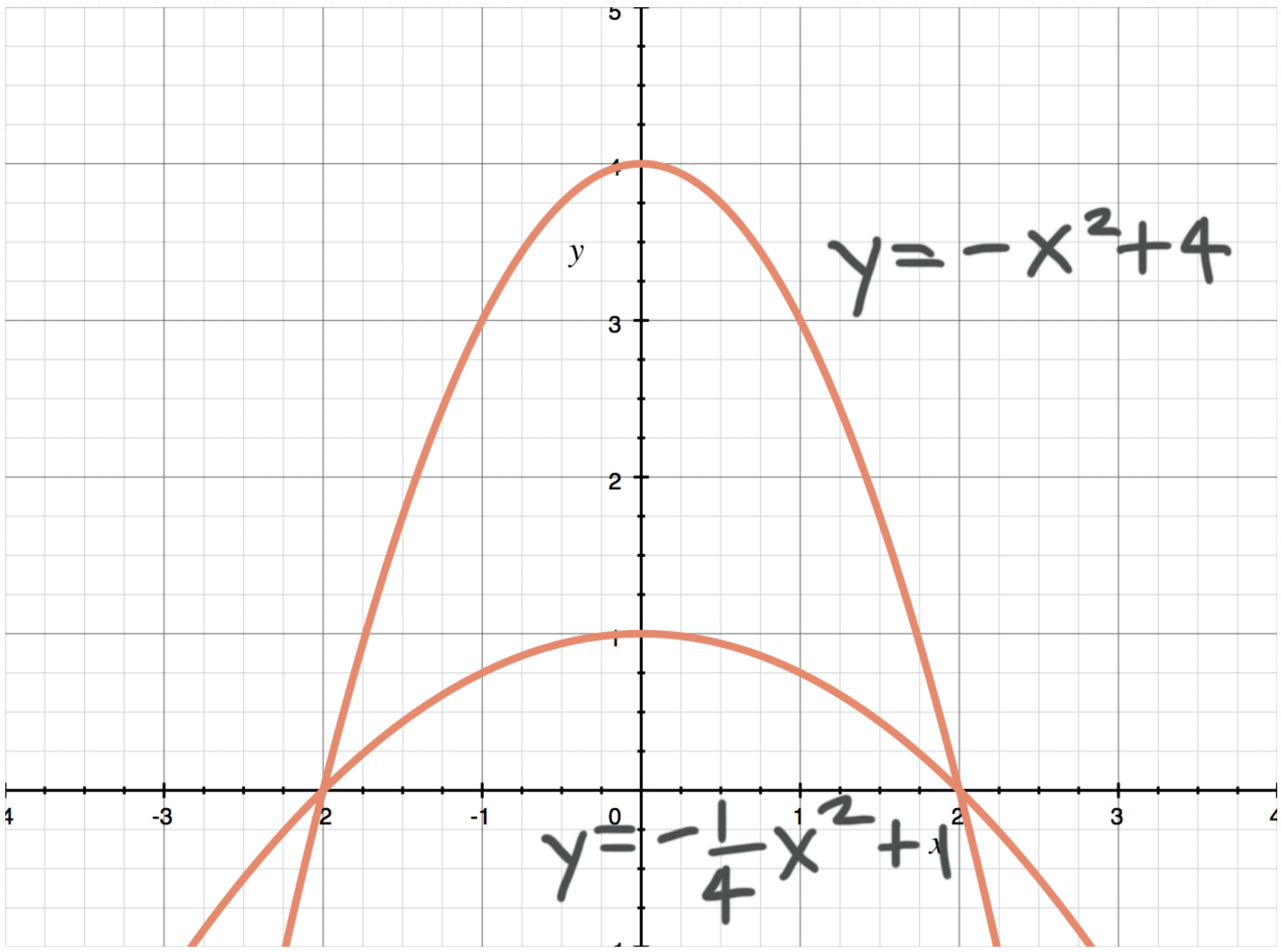
B $a = 1.4423$

C $a = 1.9199$

D $a = -1.4423$

Solution: C

The graphs of $f(x) = -x^2 + 4$ and $g(x) = -\frac{1}{4}x^2 + 1$ are shown below.



To answer the question, first, we will find the entire area of the bounded region. Notice from the graph that the interval on the x-axis for the bounded region is $[-2,2]$. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = -x^2 + 4$ is the upper function and $y = -(1/4)x^2 + 1$ is the lower function.

Let's write the integral first.

$$\int_{-2}^{2} -x^2 + 4 - \left(-\frac{1}{4}x^2 + 1\right)\ dx$$

$$\int_{-2}^{2} -x^2+4+\frac{1}{4}x^2-1\ dx$$

$$\int_{-2}^{2} 3-\frac{3}{4}x^2\ dx$$

Integrate, then evaluate over the interval.

$$3x-\frac{1}{4}x^3\Big|_{-2}^{2}$$

$$3(2)-\frac{1}{4}(2)^3-\left(3(-2)-\frac{1}{4}(-2)^3\right)$$

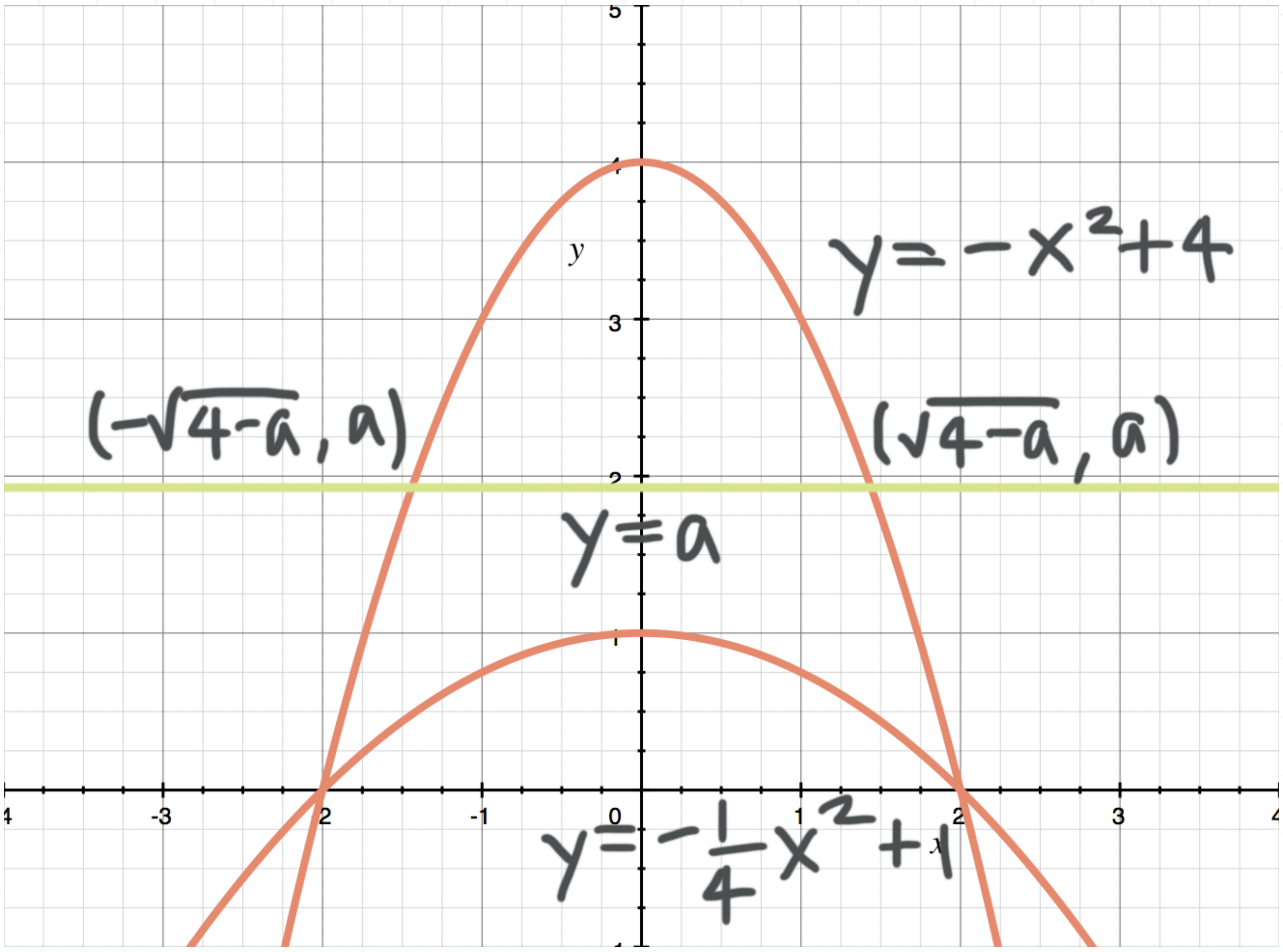
$$6-2-(-6+2)$$

$$8$$

The area of the entire bounded region is 8 square units.

Now, if we will find the value of a in the equation $y=a$ that divides the bounded region's area into two equal parts, we know that the two equal parts will have an area equal to $1/2$ of 8, which is 4.

The region that represents $1/2$ of the bounded region is the portion of the original region that is below the curve $y=-x^2+4$ and above the line $y=a$ as shown in the graph below.



Notice from the graph that the functions $y = -x^2 + 4$ and $y = a$ intersect at the points where the value of y is equal to a. To find the x-values in the points of intersection, set the function equal to a, and solve for x.

$$-x^2 + 4 = a$$

$$-x^2 = a - 4$$

$$x^2 = 4 - a$$

$$x = \pm\sqrt{4 - a}$$

Therefore, the points of intersection are $\left(-\sqrt{4-a}, a\right)$ and $\left(\sqrt{4-a}, a\right)$. Thus, the interval of the integration will be $\left[-\sqrt{4-a}, \sqrt{4-a}\right]$.

Now, let's prepare an integral. Let's find the area of this region by integrating the upper function minus the lower function in that interval. You can see that $y = -x^2 + 4$ is the upper function and $y = a$ is the lower function.

$$A = \int_{-\sqrt{4-a}}^{\sqrt{4-a}} -x^2 + 4 - a \ dx$$

Integrate, then evaluate over the interval.

$$A = -\frac{1}{3}x^3 + 4x - ax\Big|_{-\sqrt{4-a}}^{\sqrt{4-a}}$$

$$A = -\frac{1}{3}\left(\sqrt{4-a}\right)^3 + 4\sqrt{4-a} - a\sqrt{4-a} - \left(-\frac{1}{3}\left(-\sqrt{4-a}\right)^3 + 4\left(-\sqrt{4-a}\right) - a\left(-\sqrt{4-a}\right)\right)$$

$$A = -\frac{1}{3}(4-a)^{\frac{3}{2}} + 4\sqrt{4-a} - a\sqrt{4-a} - \left(\frac{1}{3}(4-a)^{\frac{3}{2}} - 4\sqrt{4-a} + a\sqrt{4-a}\right)$$

$$A = -\frac{1}{3}(4-a)^{\frac{3}{2}} + 4\sqrt{4-a} - a\sqrt{4-a} - \frac{1}{3}(4-a)^{\frac{3}{2}} + 4\sqrt{4-a} - a\sqrt{4-a}$$

$$A = -\frac{2}{3}(4-a)^{\frac{3}{2}} + 8\sqrt{4-a} - 2a\sqrt{4-a}$$

$$A = -\frac{2}{3}(4-a)^{\frac{3}{2}} + (8-2a)\sqrt{4-a}$$

$$A = -\frac{2}{3}(4-a)^{\frac{3}{2}} + 2(4-a)\sqrt{4-a}$$

$$A = -\frac{2}{3}(4-a)^{\frac{3}{2}} + 2(4-a)^{\frac{3}{2}}$$

$$A=\frac{4}{3}(4-a)^{\frac{3}{2}}$$

Now, recall that earlier we said that the area of this region had to be equal to $1/2$ of the area of the original bounded region. The area of the original bounded region was 8, which means that the area of this region is 4 square units.

$$\frac{4}{3}(4-a)^{\frac{3}{2}}=4$$

$$(4-a)^{\frac{3}{2}}=3$$

$$(4-a)^3=9$$

$$4-a=\sqrt[3]{9}$$

$$-a=-4+\sqrt[3]{9}$$

$$a=4-\sqrt[3]{9}$$

$$a=1.9199$$

Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$8y = x^4 + 2x^{-2}$$

on the interval $[1,2]$

Answer choices:

A $\frac{1}{16}$

B $\frac{33}{16}$

C $\frac{27}{16}$

D $\frac{33}{8}$

Solution: B

The formula for arc length for a curve defined as $y = f(x)$ and with limits of integration given as $x = a$ and $x = b$ is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

We already know that $a = 1$ and $b = 2$. The only other thing we need for our formula is $f'(x)$, which we'll find by taking the derivative of our original function and solving for y'.

$$8y' = 4x^3 - 4x^{-3}$$

$$y' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$$

Plugging all of these values into the formula, we get

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2}\ dx$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2}\ dx$$

Find a common denominator and combine fractions.

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^6}{2x^3} - \frac{1}{2x^3}\right)^2}\ dx$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^6 - 1}{2x^3}\right)^2}\ dx$$

$$L = \int_1^2 \sqrt{1 + \frac{\left(x^6 - 1\right)^2}{4x^6}}\ dx$$

Find a common denominator and combine fractions.

$$L = \int_1^2 \sqrt{\frac{4x^6}{4x^6} + \frac{\left(x^6 - 1\right)^2}{4x^6}}\ dx$$

$$L = \int_1^2 \sqrt{\frac{4x^6 + \left(x^6 - 1\right)^2}{4x^6}}\ dx$$

Take the square root of the numerator and denominator separately.

$$L = \frac{1}{2}\int_1^2 \frac{1}{x^3}\sqrt{4x^6 + x^{12} - 2x^6 + 1}\ dx$$

$$L = \frac{1}{2}\int_1^2 \frac{1}{x^3}\sqrt{x^{12} + 2x^6 + 1}\ dx$$

$$L = \frac{1}{2}\int_1^2 \frac{1}{x^3}\sqrt{\left(x^6 + 1\right)^2}\ dx$$

$$L = \frac{1}{2}\int_1^2 \frac{1}{x^3}\left(x^6 + 1\right)\ dx$$

$$L = \frac{1}{2}\int_1^2 \frac{x^6}{x^3} + \frac{1}{x^3}\ dx$$

$$L=\frac{1}{2}\int_{1}^{2}x^3+x^{-3}\ dx$$

Take the integral, then evaluate over the given interval.

$$L=\frac{1}{2}\left(\frac{1}{4}x^4+\frac{1}{-2}x^{-2}\right)\Bigg|_{1}^{2}$$

$$L=\frac{1}{2}\left(\frac{x^4}{4}-\frac{1}{2x^2}\right)\Bigg|_{1}^{2}$$

$$L=\frac{1}{4}\left(\frac{x^4}{2}-\frac{1}{x^2}\right)\Bigg|_{1}^{2}$$

$$L=\frac{1}{4}\left[\left(\frac{(2)^4}{2}-\frac{1}{(2)^2}\right)-\left(\frac{(1)^4}{2}-\frac{1}{(1)^2}\right)\right]$$

$$L=\frac{1}{4}\left[\left(8-\frac{1}{4}\right)-\left(\frac{1}{2}-1\right)\right]$$

$$L=\frac{1}{4}\left(8-\frac{1}{4}-\frac{1}{2}+1\right)$$

$$L=\frac{1}{4}\left(\frac{72}{8}-\frac{2}{8}-\frac{4}{8}\right)$$

$$L=\frac{66}{32}$$

$$L=\frac{33}{16}$$

Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$y = \frac{1}{3}\left(x^2 + 2\right)^{\frac{3}{2}}$$

on the interval $[0,3]$

Answer choices:

A 12

B 16

C 18

D 3

Solution: A

The formula for the arc length for a curve defined as $y = f(x)$ and with limits of integration give as $x = a$ and $x = b$ is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

We already know that $a = 0$ and $b = 3$. The only other thing we need for our formula is $f'(x)$, which we'll find by taking the derivative of our original function.

$$y' = x\left(x^2 + 2\right)^{\frac{1}{2}}$$

Plugging all of these values into the formula, we get

$$L = \int_0^3 \sqrt{1 + \left[x\left(x^2 + 2\right)^{\frac{1}{2}}\right]^2}\ dx$$

$$L = \int_0^3 \sqrt{1 + x^2\left(x^2 + 2\right)}\ dx$$

$$L = \int_0^3 \sqrt{1 + x^4 + 2x^2}\ dx$$

$$L = \int_0^3 \sqrt{x^4 + 2x^2 + 1}\ dx$$

$$L = \int_0^3 \sqrt{\left(x^2 + 1\right)^2}\ dx$$

$$L = \int_0^3 x^2 + 1\ dx$$

Take the integral, then evaluate over the given interval.

$$L=\frac{1}{3}x^3+x\Big|_0^3$$

$$L=\left[\frac{1}{3}(3)^3+3\right]-\left[\frac{1}{3}(0)^3+0\right]$$

$$L=9+3$$

$$L=12$$

Topic: Arc length of y=f(x)

Question: Find the arc length of the curve over the given interval.

$$y = 2x^{\frac{3}{2}}$$

on the interval $\left[\frac{1}{3}, 7\right]$

Answer choices:

A $\dfrac{1{,}008}{25}$

B $\dfrac{112}{3}$

C $\dfrac{1{,}008}{3}$

D $\dfrac{112}{9}$

Solution: B

Since our curve is defined as $y=f(x)$ and the limits of integration are $x=a$ and $x=b$, we know that the applicable arc length formula is

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

We already know that our limits of integration are $a=1/3$ and $b=7$. The derivative of our original function is $y'=3x^{\frac{1}{2}}$, so the arc length of the curve on the given interval is

$$L=\int_{\frac{1}{3}}^7 \sqrt{1+\left(3x^{\frac{1}{2}}\right)^2}\ dx$$

$$L=\int_{\frac{1}{3}}^7 \sqrt{1+9x}\ dx$$

Use u-substitution and let

$$u=1+9x$$

$$du=9\ dx$$

$$dx=\frac{du}{9}$$

Plugging the substitution into the integral, we get

$$L=\int_{x=\frac{1}{3}}^{x=7} \sqrt{u}\ \frac{du}{9}$$

$$L=\frac{1}{9}\int_{x=\frac{1}{3}}^{x=7}\sqrt{u}\ du$$

Take the integral, then evaluate over the given interval.

$$L=\frac{1}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x=\frac{1}{3}}^{x=7}$$

Back substitute to get the problem back in terms of x.

$$L=\frac{1}{9}\left[\frac{2}{3}(1+9x)^{\frac{3}{2}}\right]\Bigg|_{\frac{1}{3}}^{7}$$

$$L=\frac{2}{27}(1+9x)^{\frac{3}{2}}\Bigg|_{\frac{1}{3}}^{7}$$

$$L=\frac{2}{27}\left[1+9(7)\right]^{\frac{3}{2}}-\frac{2}{27}\left[1+9\left(\frac{1}{3}\right)\right]^{\frac{3}{2}}$$

$$L=\frac{2}{27}(64)^{\frac{3}{2}}-\frac{2}{27}(4)^{\frac{3}{2}}$$

$$L=\frac{2}{27}\left(64^{\frac{1}{2}}\right)^{3}-\frac{2}{27}\left(4^{\frac{1}{2}}\right)^{3}$$

$$L=\frac{2}{27}\left[\left(64^{\frac{1}{2}}\right)^{3}-\left(4^{\frac{1}{2}}\right)^{3}\right]$$

$$L=\frac{2}{27}\left(512-8\right)$$

$$L=\frac{2}{27}\left(504\right)$$

$$L = \frac{2}{3}(56)$$

$$L = \frac{112}{3}$$

Topic: Arc length of x=g(y)

Question: Find the arc length of the curve over the given interval.

$$6xy = y^4 + 3$$

on the interval $[1,2]$

Answer choices:

A $\frac{11}{8}$

B $\frac{17}{16}$

C $\frac{11}{12}$

D $\frac{17}{12}$

Solution: D

The formula for arc length for a curve defined as $x = g(y)$ and with limits of integration given as $y = c$ and $y = d$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ dy$$

We already know that $c = 1$ and $d = 2$. The only thing we need for our formula is $g'(y)$, which we'll find by taking the derivative of our original function and solving for x'.

$$x = \frac{y^4 + 3}{6y}$$

$$x = \frac{y^4}{6y} + \frac{3}{6y}$$

$$x = \frac{y^3}{6} + \frac{1}{2y}$$

$$x' = \frac{y^2}{2} - \frac{1}{2y^2}$$

Plugging all of these values into the formula, we get

$$L = \int_1^2 \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\ dy$$

Find a common denominator and combine fractions.

$$L=\int_1^2\sqrt{1+\left(\frac{y^4}{2y^2}-\frac{1}{2y^2}\right)^2}\ dy$$

$$L=\int_1^2\sqrt{1+\left(\frac{y^4-1}{2y^2}\right)^2}\ dy$$

$$L=\int_1^2\sqrt{1+\frac{\left(y^4-1\right)^2}{4y^4}}\ dy$$

Find a common denominator and combine fractions.

$$L=\int_1^2\sqrt{\frac{4y^4}{4y^4}+\frac{\left(y^4-1\right)^2}{4y^4}}\ dy$$

$$L=\int_1^2\sqrt{\frac{4y^4+\left(y^4-1\right)^2}{4y^4}}\ dy$$

Take the square root of the numerator and denominator separately.

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{4y^4+\left(y^4-1\right)^2}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{4y^4+y^8-2y^4+1}\ dy$$

$$L=\frac{1}{2}\int_1^2\frac{1}{y^2}\sqrt{y^8+2y^4+1}\ dy$$

$$L = \frac{1}{2}\int_1^2 \frac{1}{y^2}\sqrt{\left(y^4+1\right)^2}\ dy$$

$$L = \frac{1}{2}\int_1^2 \frac{1}{y^2} \cdot \left(y^4+1\right)\ dy$$

$$L = \frac{1}{2}\int_1^2 \frac{y^4+1}{y^2}\ dy$$

$$L = \frac{1}{2}\int_1^2 \frac{y^4}{y^2} + \frac{1}{y^2}\ dy$$

$$L = \frac{1}{2}\int_1^2 y^2 + y^{-2}\ dy$$

Take the integral, then evaluate over the given interval.

$$L = \frac{1}{2}\left(\frac{1}{3}y^3 - y^{-1}\right)\Bigg|_1^2$$

$$L = \frac{1}{2}\left(\frac{y^3}{3} - \frac{1}{y}\right)\Bigg|_1^2$$

$$L = \frac{1}{2}\left[\left(\frac{(2)^3}{3} - \frac{1}{(2)}\right) - \left(\frac{(1)^3}{3} - \frac{1}{(1)}\right)\right]$$

$$L = \frac{1}{2}\left[\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$L = \frac{1}{2}\left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right)$$

$$L=\frac{1}{2}\left(\frac{7}{3}-\frac{1}{2}+1\right)$$

$$L=\frac{1}{2}\left(\frac{14}{6}-\frac{3}{6}+\frac{6}{6}\right)$$

$$L=\frac{17}{12}$$

Topic: Arc length of x=g(y)

Question: Find the arc length of the curve over the given interval.

$$3x = 2\left(y^2 + 1\right)^{\frac{3}{2}}$$

on the interval $[0,3]$

Answer choices:

A 21

B 22

C 23

D 24

Solution: A

First, rewrite the given curve as

$$x = \frac{2}{3}\left(y^2+1\right)^{\frac{3}{2}}$$

Differentiating with respect to y gives us

$$\frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}\left(y^2+1\right)^{\frac{1}{2}}\cdot 2y$$

$$\frac{dx}{dy} = 2y\left(y^2+1\right)^{\frac{1}{2}}$$

The applicable formula for arc length is

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

Putting limits of integration and the derivative into the formula gives

$$L = \int_0^3 \sqrt{1+\left[2y\left(y^2+1\right)^{\frac{1}{2}}\right]^2}\ dy$$

$$L = \int_0^3 \sqrt{1+4y^2\left(y^2+1\right)}\ dy$$

$$L = \int_0^3 \sqrt{1+4y^2+4y^4}\ dy$$

$$L = \int_0^3 \sqrt{\left(1+2y^2\right)^2}\ dy$$

$$L = \int_0^3 1 + 2y^2 \ dy$$

Take the integral, then evaluate over the given interval.

$$L = y + \frac{2}{3}y^3 \Big|_0^3$$

$$L = 3 + \frac{2}{3}(3)^3 - \left[0 + \frac{2}{3}(0)^3\right]$$

$$L = 21$$

Topic: Arc length of x=g(y)

Question: Find the arc length of the function on the interval.

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

on the interval $y = [1,3]$

Answer choices:

A $\frac{33}{2}$

B $\frac{53}{6}$

C 17

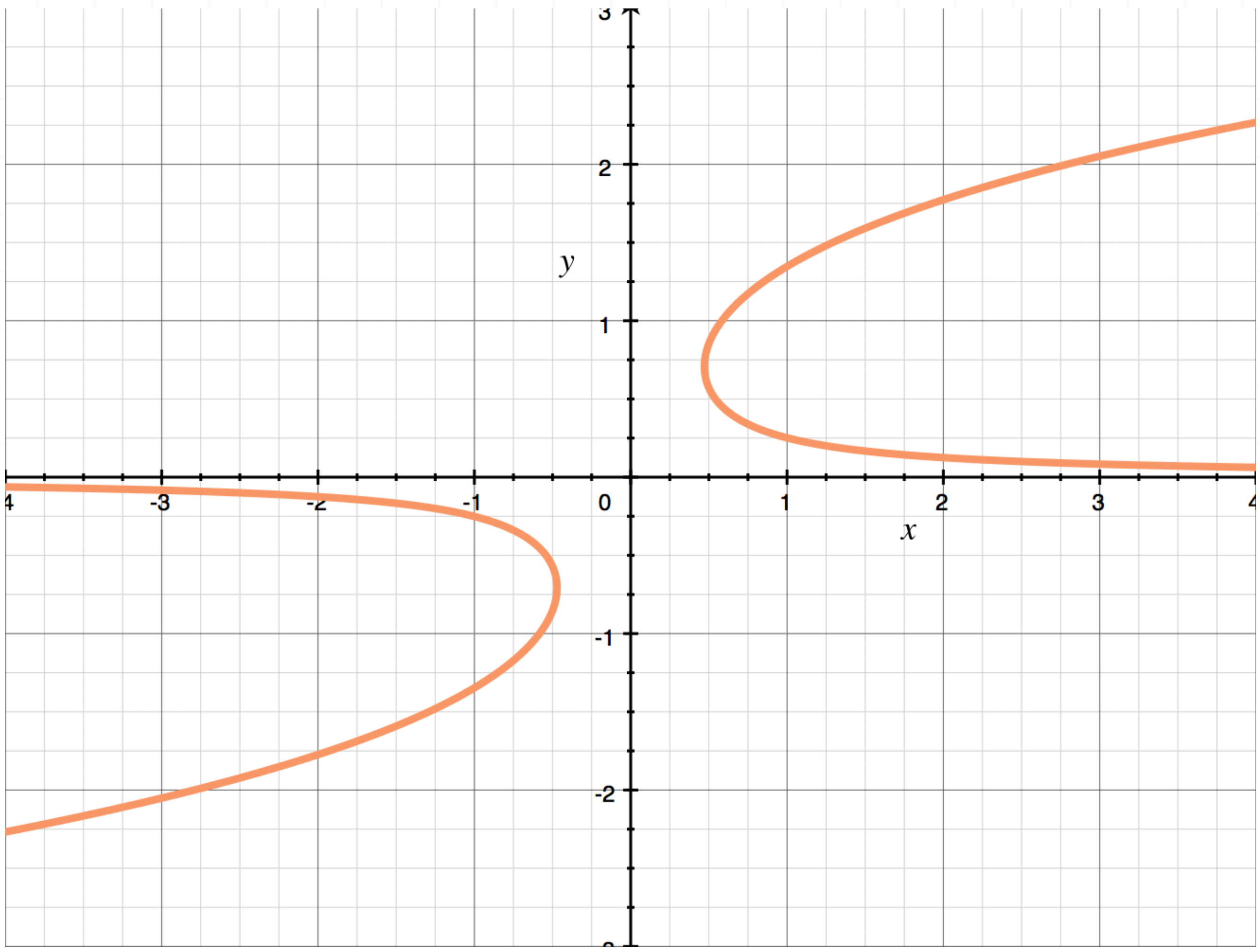
D $\frac{58}{6}$

Solution: B

The graph of

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

is shown below.



We can see that the curve is not a function, so it cannot be integrated with respect to x. We can also see that x is a smooth curve of y on the interval $y = [1,3]$. Therefore, the arc length can be given by

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \ dy$$

We'll find dx/dy.

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

$$x = \frac{1}{3}y^3 + \frac{1}{4}y^{-1}$$

$$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(3y^2\right) + \left(\frac{1}{4}\right)(-1)\left(y^{-2}\right)$$

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

Now plug everything into the integral formula.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$L = \int_1^3 \sqrt{1 + \left(y^2 - \frac{1}{4y^2}\right)^2}\ dy$$

$$L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\ dy$$

$$L = \int_1^3 \sqrt{\frac{1}{2} + y^4 + \frac{1}{16y^4}}\ dy$$

The expression inside the radical becomes a perfect square.

$$L = \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\ dy$$

$$L = \int_1^3 y^2 + \frac{1}{4y^2}\ dy$$

Integrate, then evaluate over the interval.

$$L = \frac{1}{3}y^3 - \frac{1}{4y}\Big|_1^3$$

$$L = \frac{1}{3}(3)^3 - \frac{1}{4(3)} - \left(\frac{1}{3}(1)^3 - \frac{1}{4(1)}\right)$$

$$L = 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$L = \frac{108}{12} - \frac{1}{12} - \frac{4}{12} + \frac{3}{12}$$

$$L = \frac{106}{12}$$

$$L = \frac{53}{6}$$

Topic: Average value

Question: Find the average value of the function over the given interval.

$$f(x) = 5 + 2x - x^2$$

on the interval $[-2,3]$

Answer choices:

A $\frac{11}{3}$

B -2

C -10

D $\frac{55}{3}$

Solution: A

To find average value of a function over a given interval, we use the integration formula

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

where $f(x)$ is the function for which we want the average, and $[a, b]$ is the interval we're interested in.

Plugging the given function and the interval into the formula, we get

$$\frac{1}{3-(-2)}\int_{-2}^{3} 5 + 2x - x^2\ dx$$

$$\frac{1}{5}\left(5x + x^2 - \frac{1}{3}x^3\right)\Big|_{-2}^{3}$$

$$\frac{1}{5}\left[\left(5(3) + (3)^2 - \frac{1}{3}(3)^3\right) - \left(5(-2) + (-2)^2 - \frac{1}{3}(-2)^3\right)\right]$$

$$\frac{1}{5}\left[15 + 9 - 9 - \left(-10 + 4 + \frac{8}{3}\right)\right]$$

$$\frac{1}{5}\left(15 + 10 - 4 - \frac{8}{3}\right)$$

$$\frac{1}{5}\left(21 - \frac{8}{3}\right)$$

$$\frac{21}{5} - \frac{8}{15}$$

$$\frac{63}{15} - \frac{8}{15}$$

$$\frac{55}{15}$$

$$\frac{11}{3}$$

Topic: Average value

Question: Find the average value of the function over the given interval.

$$f(x) = x^3 + 4x$$

on the interval $[2,5]$

Answer choices:

A 89

B $\dfrac{259}{4}$

C $\dfrac{89}{3}$

D $\dfrac{259}{3}$

Solution: B

To find average value of a function over a given interval, we use the integration formula

$$f_{avg}=\frac{1}{b-a}\int_a^b f(x)\ dx$$

where $f(x)$ is the function for which we want the average, and $[a,b]$ is the interval we're interested in.

Plugging the given function and the interval into the formula, we get

$$f_{avg}=\frac{1}{5-2}\int_2^5 x^3+4x\ dx$$

$$f_{avg}=\frac{1}{3}\int_2^5 x^3+4x\ dx$$

$$f_{avg}=\frac{1}{3}\left(\frac{x^4}{4}+\frac{4x^2}{2}\right)\Bigg|_2^5$$

$$f_{avg}=\left(\frac{x^4}{12}+\frac{2x^2}{3}\right)\Bigg|_2^5$$

$$f_{avg}=\left[\frac{(5)^4}{12}+\frac{2(5)^2}{3}\right]-\left[\frac{(2)^4}{12}+\frac{2(2)^2}{3}\right]$$

$$f_{avg}=\left(\frac{625}{12}+\frac{50}{3}\right)-\left(\frac{16}{12}+\frac{8}{3}\right)$$

$$f_{avg} = \frac{825}{12} - \frac{48}{12}$$

$$f_{avg} = \frac{777}{12}$$

$$f_{avg} = \frac{259}{4}$$

Topic: Average value

Question: Find the average value of the function over the given interval.

$$f(x) = 4xe^{2x^2}$$

on the interval $[1,3]$

Answer choices:

A $\dfrac{e^{18} - e^2}{2}$

B $8e^{16}$

C $\dfrac{e^{16}}{2}$

D $12e^{18} - 4e^2$

Solution: A

To find average value of a function over a given interval, we use the integration formula

$$f_{avg}=\frac{1}{b-a}\int_a^b f(x)\ dx$$

where $f(x)$ is the function for which we want the average, and $[a,b]$ is the interval we're interested in.

Plugging the given function and the interval into the formula, we get

$$f_{avg}=\frac{1}{3-1}\int_1^3 4xe^{2x^2}\ dx$$

$$f_{avg}=\frac{1}{2}\int_1^3 4xe^{2x^2}\ dx$$

In order to solve this integral, we'll need to use u-substitution, letting

$$u=2x^2$$

$$du=4x\ dx$$

Plugging these back into the integral, remembering that our limits of integration still relate to x, and not u, we get

$$f_{avg}=\frac{1}{2}\int_{x=1}^{x=3} 4xe^u\ dx$$

$$f_{avg}=\frac{1}{2}\int_{x=1}^{x=3} e^u(4x\ dx)$$

$$f_{avg} = \frac{1}{2}\int_{x=1}^{x=3} e^u \ du$$

$$f_{avg} = \frac{1}{2}e^u \Big|_{x=1}^{x=3}$$

$$f_{avg} = \frac{e^u}{2} \Big|_{x=1}^{x=3}$$

Back-substituting before we evaluate over the interval, we get

$$f_{avg} = \frac{e^{2x^2}}{2} \Big|_1^3$$

$$f_{avg} = \frac{e^{2(3)^2}}{2} - \frac{e^{2(1)^2}}{2}$$

$$f_{avg} = \frac{e^{18}}{2} - \frac{e^2}{2}$$

$$f_{avg} = \frac{e^{18} - e^2}{2}$$

Topic: Mean value theorem for integrals

Question: Choose the correct formula.

Given the function $f(x)$ over the interval $[a, b]$, where c is a point in the interval, the mean value theorem for integrals says that...

Answer choices:

A $\int_a^b f(x)\ dx = f(c)(b + a)$

B $\int_c^c f(x)\ dx = f(c)(b - a)$

C $\int_a^b f(x)\ dx = f(c)(b - a)$

D $\int_c^c f(x)\ dx = f(c)(b + a)$

Solution: C

The mean value theorem states that a point c must exist on the given interval $[a, b]$ for the function $f(x)$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

If you have trouble remembering this formula, remember that it's just a rearrangement of the average value formula

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

If we say that $f(c) = f_{avg}$, then we make a substitution and get

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

$$f(c)(b-a) = \int_a^b f(x)\ dx$$

and we're back to the mean value theorem.

Topic: Mean value theorem for integrals

Question: Apply the mean value theorem.

Use the mean value theorem to find the value of the function at an unknown point c.

$$\int_0^2 f(x)\, dx = 6$$

Answer choices:

A 3

B 0

C 6

D 2

Solution: A

The mean value theorem states that a point c must exist on the given interval $[a, b]$ for the function $f(x)$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

The given integral tells us that $a=0$, $b=2$ and $f(c)(b-a)=6$. We can plug the values of a and b into $f(c)(b-a)=6$ and then solve for $f(c)$, which is the value of the function at the unknown point c.

$$f(c)(2-0)=6$$

$$f(c)=3$$

Topic: Mean value theorem for integrals

Question: Apply the mean value theorem.

Use the mean value theorem to find the value of the function at an unknown point c.

$$\int_1^5 x^2\,dx$$

Answer choices:

A $\frac{62}{9}$

B $\frac{31}{3}$

C $\frac{31}{12}$

D $\frac{124}{3}$

Solution: B

The Mean Value Theorem states that a point c must exist on the given interval $[a, b]$ for the function $f(x)$ such that

$$\int_{a}^{b} f(x)\ dx = f(c)(b-a)$$

The given integral tells us that $a = 1$, $b = 5$. We need to solve the definite integral and then we can solve for $f(c)$.

$$\int_{1}^{5} x^2\ dx = \frac{1}{3}x^3 \Big|_{1}^{5}$$

$$\int_{1}^{5} x^2\ dx = \frac{1}{3}(5)^3 - \frac{1}{3}(1)^3$$

$$\int_{1}^{5} x^2\ dx = \frac{124}{3}$$

Using the mean value theorem this means that

$$f(c)(b-a) = \frac{124}{3}$$

Now we can solve for $f(c)$, which is the value of the function at the unknown point c.

$$f(c)(5-1) = \frac{124}{3}$$

$$f(c) = \frac{124}{12}$$

$$f(c) = \frac{31}{3}$$

Topic: Surface area of revolution

Question: Find the surface area generated by revolving the curve around the given axis over the given interval.

$$y = 3x + 1$$

on the interval $0 \leq x \leq 1$

about the x-axis

Answer choices:

A 50π

B $5\pi\sqrt{10}$

C $2\pi\sqrt{10}$

D $\pi\sqrt{50}$

Solution: B

Because our curve is defined in the form $y = f(x)$ and our limits of integration are defined as $x = 0$ and $x = 1$, the formula we use to find the surface area of revolution is

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

The derivative of our function is

$$\frac{dy}{dx} = 3$$

Plugging the derivative and our limits of integration into the surface area of revolution formula gives

$$A = \int_0^1 2\pi(3x+1)\sqrt{1+(3)^2}\ dx$$

$$A = 2\pi\sqrt{10}\int_0^1 3x + 1\ dx$$

$$A = 2\pi\sqrt{10}\left(\frac{3}{2}x^2 + x\right)\Big|_0^1$$

$$A = 2\pi\sqrt{10}\left[\left(\frac{3}{2}(1)^2 + (1)\right) - \left(\frac{3}{2}(0)^2 + (0)\right)\right]$$

$$A = 2\pi\sqrt{10}\left(\frac{3}{2} + 1\right)$$

$$A = 2\pi\sqrt{10}\left(\frac{5}{2}\right)$$

$$A = 5\pi\sqrt{10}$$

Topic: Surface area of revolution

Question: Find the surface area generated by revolving the curve around the given axis over the given interval.

$$y = \sqrt{25 - x^2}$$

on the interval $-2 \leq x \leq 3$

about the x-axis

Answer choices:

A 50

B 25

C 25π

D 50π

Solution: D

Because our curve is defined in the form $y = f(x)$ and our limits of integration are defined as $x = -2$ and $x = 3$, the formula we use to find surface area of revolution is

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

The derivative of our function is

$$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$$

Plugging the derivative, and our limits of integration into the surface area of revolution formula gives

$$A = \int_{-2}^{3} 2\pi \sqrt{25 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{25 - x^2}}\right)^2}\ dx$$

$$A = 2\pi \int_{-2}^{3} \sqrt{25 - x^2} \sqrt{1 + \frac{x^2}{25 - x^2}}\ dx$$

Find a common denominator and combine fractions.

$$A = 2\pi \int_{-2}^{3} \sqrt{25 - x^2} \sqrt{\frac{25 - x^2}{25 - x^2} + \frac{x^2}{25 - x^2}}\ dx$$

$$A = 2\pi \int_{-2}^{3} \sqrt{25 - x^2} \sqrt{\frac{25}{25 - x^2}}\ dx$$

$$A = 2\pi \int_{-2}^{3} \sqrt{25 - x^2} \cdot \frac{\sqrt{25}}{\sqrt{25 - x^2}}\ dx$$

$$A = 10\pi \int_{-2}^{3}\ dx$$

$$A = 10\pi x \Big|_{-2}^{3}$$

$$A = 10\pi(3) - 10\pi(-2)$$

$$A = 30\pi + 20\pi$$

$$A = 50\pi$$

Topic: Surface area of revolution

Question: Find the surface area generated by revolving the curve around the given axis over the given interval.

$$y = \frac{1}{2}x^2 - 1$$

on the interval $0 \le x \le 2\sqrt{2}$

about the y-axis

Answer choices:

A $\dfrac{52\pi}{3}$

B 52π

C $\dfrac{54\pi}{3}$

D 26π

Solution: A

Because our curve is defined in the form $y=f(x)$ and our limits of integration are defined as $x=0$ and $x=2\sqrt{2}$, the formula we use to find surface area of revolution is

$$A=\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

The derivative of our function is

$$\frac{dy}{dx}=x$$

Plugging the derivative and our limits of integration into the surface area of revolution formula gives us

$$A=\int_0^{2\sqrt{2}} 2\pi x\sqrt{1+x^2}\ dx$$

We'll use u-substitution.

$$u=1+x^2$$

$$du=2x\ dx$$

$$dx=\frac{du}{2x}$$

Making the substitution into the integral gives

$$A=\int_{x=0}^{x=2\sqrt{2}} 2\pi x\sqrt{u}\ \frac{du}{2x}$$

$$A=\pi\int_{x=0}^{x=2\sqrt{2}}\sqrt{u}\ du$$

Integrate.

$$A=\pi\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=2\sqrt{2}}$$

Back substitute to get the problem back in terms of x, then evaluate over the interval.

$$A=\pi\left[\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]\Bigg|_{0}^{2\sqrt{2}}$$

$$A=\pi\left[\frac{2}{3}\left(1\left(2\sqrt{2}\right)^2\right)^{\frac{3}{2}}-\frac{2}{3}\left(1+(0)^2\right)^{\frac{3}{2}}\right]$$

$$A=\pi\left[\frac{2}{3}(1+8)^{\frac{3}{2}}-\frac{2}{3}(1)^{\frac{3}{2}}\right]$$

$$A=\pi\left(\frac{54}{3}-\frac{2}{3}\right)$$

$$A=\frac{52\pi}{3}$$

Topic: Surface of revolution equation

Question: Find an equation for the surface generated by revolving the curve around the given axis.

$$x^2 + 4y^2 = 5$$

about the x-axis

Answer choices:

A $\quad x^2 + 4y^2 + z^2 = 5$

B $\quad x^2 + 4y^2 + 4z^2 = 5$

C $\quad x^2 + y^2 + 4z^2 = 5$

D $\quad x^2 + y^2 + z^2 = 5$

Solution: B

If we select a generic point $P(x,y,z)$ on the surface of revolution, and then another point on the curve with the same x-coordinate as P, like $Q(x,y_1,0)$, then for Q we get

$$x^2+4y_1^2=5$$

Remember that both P and Q are on the surface of revolution and both have the same x-coordinate. Therefore, their distances from the x-axis are the same and the squares of these distances are equal. Therefore, the squared distance from P to the x-axis is y^2+z^2, and the squared distance from Q to the x-axis is $y_1^2+0^2$ or just y_1^2.

Since the two distances are equal, we get

$$y_1^2=y^2+z^2$$

Substituting this into $x^2+4y_1^2=5$, the surface of revolution is

$$x^2+4\left(y^2+z^2\right)=5$$

$$x^2+4y^2+4z^2=5$$

Topic: Surface of revolution equation

Question: Find an equation for the surface generated by revolving the curve around the given axis.

$$x^2 = 4y^2$$

about the y-axis

Answer choices:

A $\quad x^2 + 4z^2 = 4y^2$

B $\quad x^2 + z^2 = 4y^2$

C $\quad 4x^2 + 4z^2 = 4y^2$

D $\quad x^2 + z^2 = y^2$

Solution: B

If we select a generic point $P(x, y, z)$ on the surface of revolution, and then another point on the curve with the same y-coordinate as P, like $Q(x_1, y, 0)$, then for Q we get

$$x_1^2 = 4y^2$$

Remember that both P and Q are on the surface of revolution and both have the same y-coordinate. Therefore, their distances from the y-axis are the same and the squares of these distances are equal. Therefore, the squared distance from P to the y-axis is $x^2 + z^2$ and the squared distance from Q to the y-axis is $x_1^2 + 0^2$ or just x_1^2.

Since the two distances are equal, we get

$$x_1^2 = x^2 + z^2$$

Substituting this into $x_1^2 = 4y^2$, the surface of revolution equation is

$$x^2 + z^2 = 4y^2$$

Topic: Surface of revolution equation

Question: Find an equation for the surface generated by revolving the curve around the given axis.

$$3z^2 = 4 - y^2$$

about the z-axis

Answer choices:

A $z^2 = 4 - x^2 - y^2$

B $3z^2 = x^2 + y^2$

C $3z^2 = 4 + x^2 - y^2$

D $3z^2 = 4 - x^2 - y^2$

Solution: D

If we select a generic point $P(x,y,z)$ on the surface of revolution, and then another point on the curve with the same z-coordinate as P, like $Q(0,y_1,z)$, then for Q we get

$$3z^2 = 4 - y_1^2$$

Remember that both P and Q are on the surface of revolution and both have the same z-coordinate. Therefore, their distances from the z-axis are the same and the squares of these distances are equal. Therefore, the squared distance from P to the z-axis is $x^2 + y^2$ and the squared distance from Q to the z-axis is $y_1^2 + 0^2$ or just y_1^2.

Since the two distances are equal, we get

$$y_1^2 = x^2 + y^2$$

Substituting this into $3z^2 = 4 - y_1^2$, the surface of revolution is

$$3z^2 = 4 - \left(x^2 + y^2\right)$$

$$3z^2 = 4 - x^2 - y^2$$

Topic: Disks, horizontal axis

Question: Use disks to find the volume of the solid formed by rotating the region enclosed by the curves.

$y = x^2$ and $y = 0$

$x = 0$ and $x = 3$

about the x-axis

Answer choices:

A $V = \frac{243}{5}\pi$ cubic units

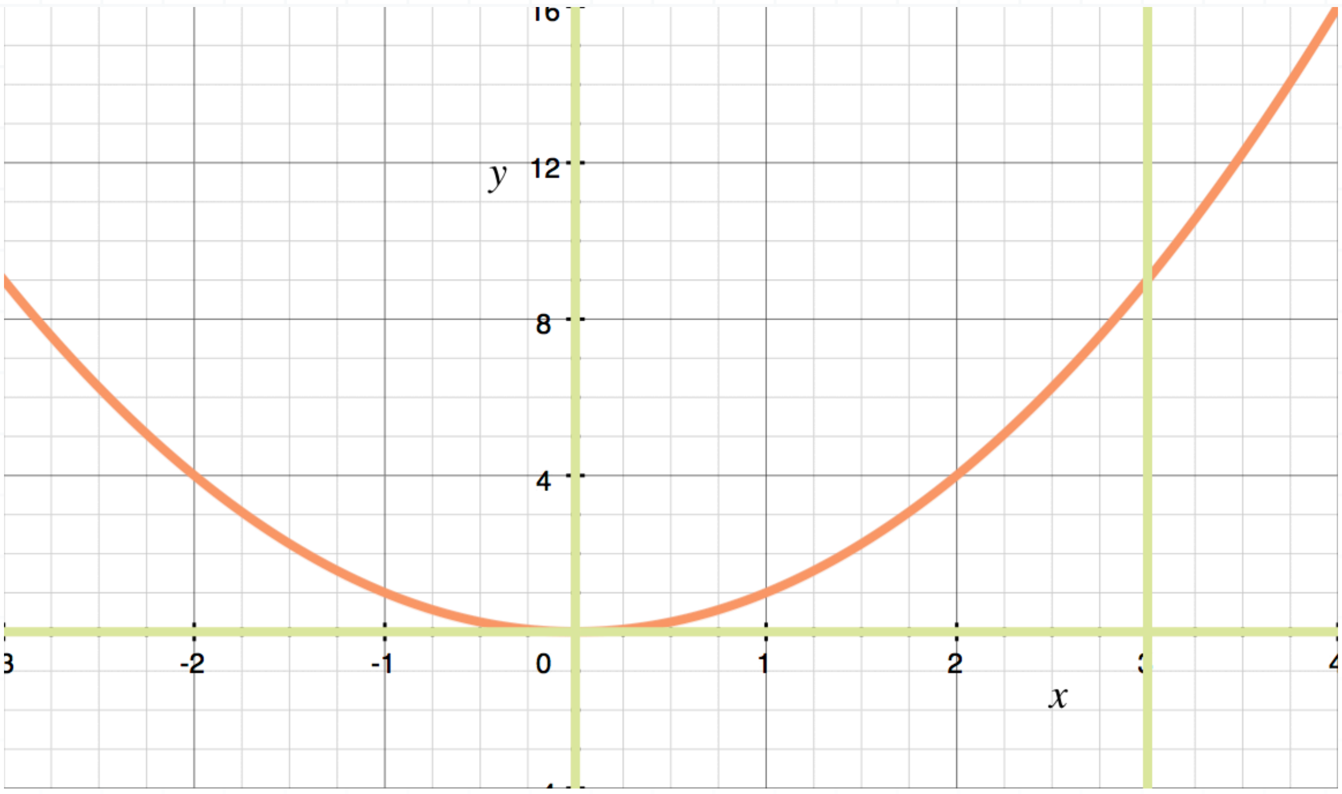
B $V = \frac{243}{5}$ cubic units

C $V = 243\pi$ cubic units

D $V = 81\pi$ cubic units

Solution: A

The region enclosed by $y = x^2$, $y = 0$, $x = 0$ and $x = 3$ is



Because we're rotating about the x-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by $x = [0,3]$. The outer radius will be defined by $y = x^2$. So the volume can be given by

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_0^3 \pi \left(x^2\right)^2 \ dx$$

$$V=\int_0^3 \pi x^4\ dx$$

Integrate, then evaluate over the interval.

$$V=\frac{1}{5}\pi x^5\Big|_0^3$$

$$V=\frac{1}{5}\pi(3)^5-\left(\frac{1}{5}\pi(0)^5\right)$$

$$V=\frac{243\pi}{5}$$

Topic: Disks, horizontal axis

Question: Use disks to find the volume of the solid formed by rotating the region enclosed by the curves.

$y = -x^2 + 5x$ and $y = 0$

about the x-axis

Answer choices:

A $V = \frac{625}{5}\pi$ cubic units

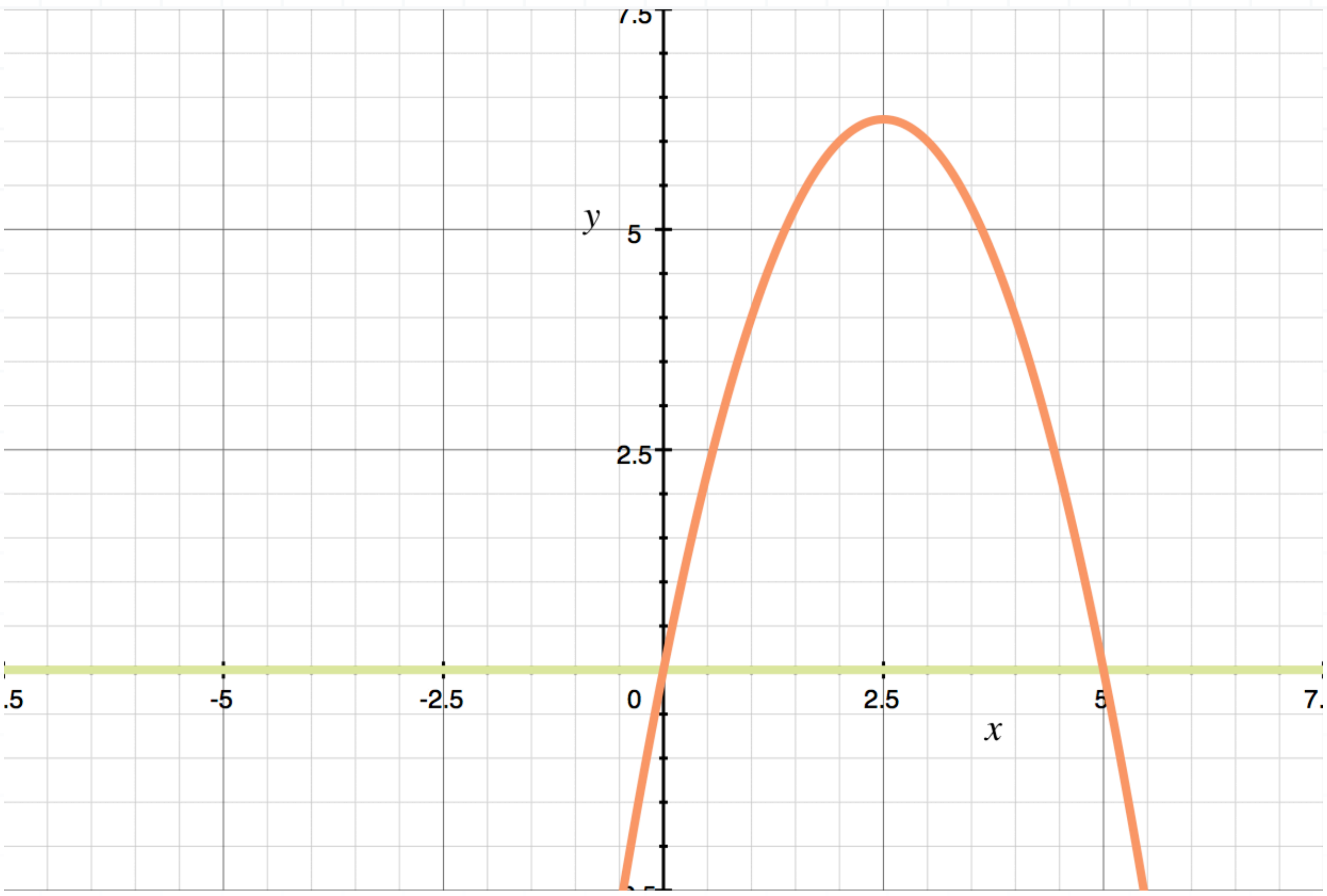
B $V = 625\pi$ cubic units

C $V = \frac{625}{6}\pi$ cubic units

D $V = \frac{626}{5}\pi$ cubic units

Solution: C

The region enclosed by $y = -x^2 + 5x$ and $y = 0$ is



Because we're rotating about the x-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by the points where $y = -x^2 + 5x$ intersects $y = 0$.

$$-x^2 + 5x = 0$$

$$x^2 - 5x = 0$$

$$x(x - 5) = 0$$

$x = 0$ and $x = 5$

The limits of integration are therefore $x = [0,5]$. The outer radius will be defined by $y = -x^2 + 5x$. So the volume can be given by

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_0^5 \pi \left(-x^2 + 5x\right)^2 \ dx$$

$$V = \int_0^5 \pi \left(x^4 - 10x^3 + 25x^2\right) \ dx$$

$$V = \pi \int_0^5 x^4 - 10x^3 + 25x^2 \ dx$$

Integrate, then evaluate over the interval.

$$V = \pi \left[\frac{1}{5}x^5 - \frac{5}{2}x^4 + \frac{25}{3}x^3\right]\Big|_0^5$$

$$V = \pi \left[\frac{1}{5}(5)^5 - \frac{5}{2}(5)^4 + \frac{25}{3}(5)^3\right] - \pi \left[\frac{1}{5}(0)^5 - \frac{5}{2}(0)^4 + \frac{25}{3}(0)^3\right]$$

$$V = \pi \left(5^4 - \frac{5^5}{2} + \frac{5^5}{3}\right)$$

$$V = \pi \left(\frac{6(5^4)}{6} - \frac{3(5^5)}{6} + \frac{2(5^5)}{6}\right)$$

$$V = \frac{6(5^4) - 5^5}{6}\pi$$

$$V = \frac{625}{6}\pi$$

Topic: Disks, horizontal axis

Question: Use disks to find the volume of the solid formed by rotating the region enclosed by the curves.

$y = \sqrt{9 - x^2}$ and $y = 0$

$x = -\frac{5}{2}$ and $x = \frac{5}{2}$

about the x-axis

Answer choices:

A $V = \frac{415}{6}\pi$ cubic units

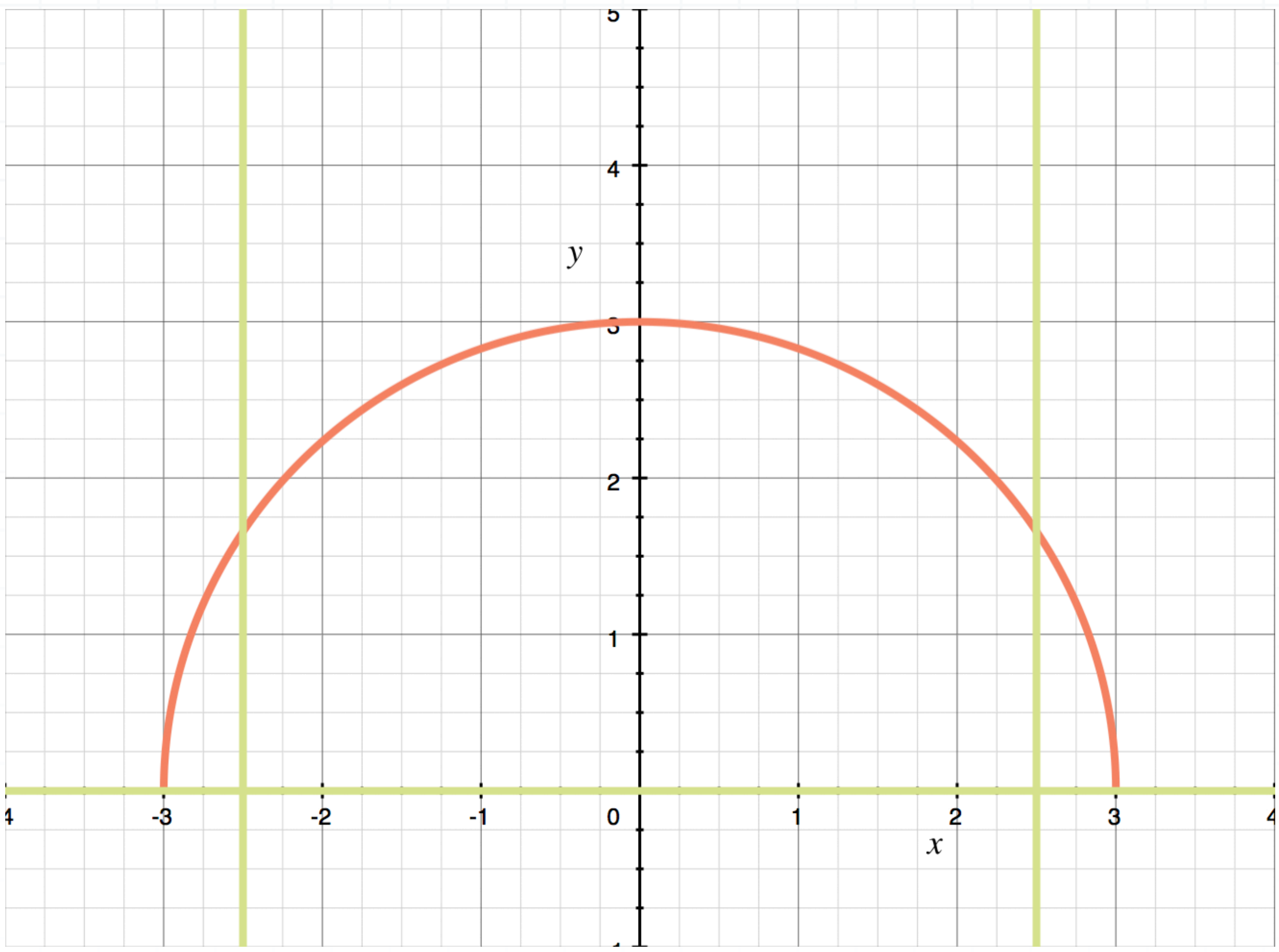
B $V = \frac{415}{24}\pi$ cubic units

C $V = \frac{415}{12}\pi$ cubic units

D $V = 415\pi$ cubic units

Solution: C

The region enclosed by $y = \sqrt{9 - x^2}$, $y = 0$, $x = -5/2$ and $x = 5/2$ is



Because we're rotating about the x-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by $x = [-5/2, 5/2]$. The outer radius will be defined by $y = \sqrt{9 - x^2}$. So the volume can be given by

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V=\int_{-\frac{5}{2}}^{\frac{5}{2}}\pi\left(\sqrt{9-x^2}\right)^2\ dx$$

$$V=\int_{-\frac{5}{2}}^{\frac{5}{2}}\pi\left(9-x^2\right)\ dx$$

$$V=\int_{-\frac{5}{2}}^{\frac{5}{2}}9\pi-\pi x^2\ dx$$

Integrate, then evaluate over the interval.

$$V=9\pi x-\frac{1}{3}\pi x^3\Big|_{-\frac{5}{2}}^{\frac{5}{2}}$$

$$V=9\pi\left(\frac{5}{2}\right)-\frac{1}{3}\pi\left(\frac{5}{2}\right)^3-\left[9\pi\left(-\frac{5}{2}\right)-\frac{1}{3}\pi\left(-\frac{5}{2}\right)^3\right]$$

$$V=\frac{45}{2}\pi-\frac{125}{24}\pi+\frac{45}{2}\pi-\frac{125}{24}\pi$$

$$V=45\pi-\frac{250}{24}\pi$$

$$V=\frac{415}{12}\pi$$

Topic: Disks, vertical axis

Question: Use disks to find the volume of the solid formed by rotating the region enclosed by the curves.

$x = \sqrt{5}y^2$ and $x = 0$

$y = -\frac{3}{2}$ and $y = \frac{3}{2}$

about the y-axis

Answer choices:

A $V = \frac{243}{16}\pi$ cubic units

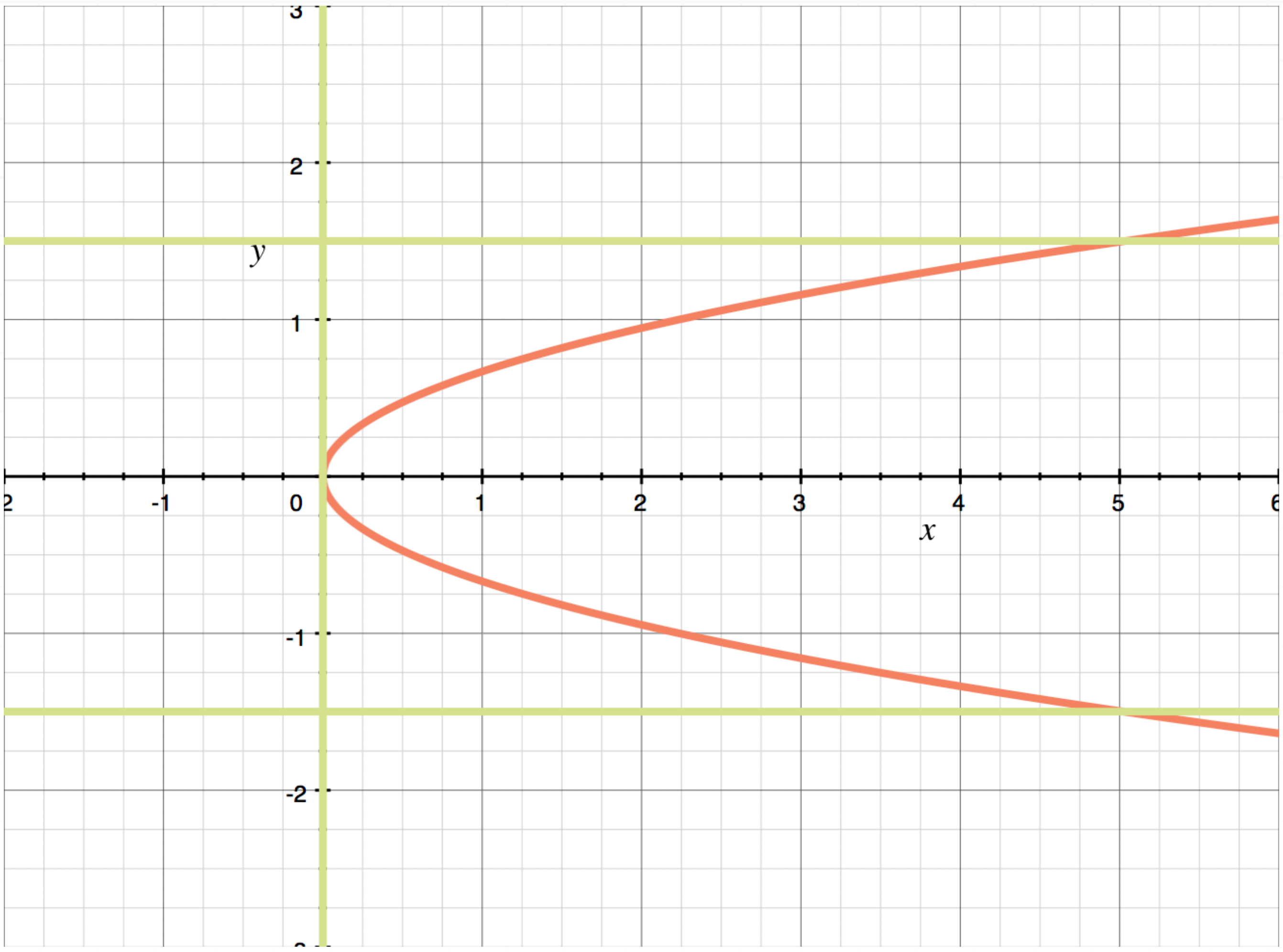
B $V = \frac{243}{16}$ cubic units

C $V = 243\pi$ cubic units

D $V = 81\pi$ cubic units

Solution: A

The region enclosed by $x = \sqrt{5}y^2$, $x = 0$, $y = -3/2$ and $y = 3/2$ is



Because we're rotating about the y-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking horizontal slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dy, which means we'll be integrating with respect to y. Therefore, the limits of integration will be given by $y = [-3/2,3/2]$. The outer radius will be defined by $x = \sqrt{5}y^2$. So the volume can be given by

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V=\int_{-\frac{3}{2}}^{\frac{3}{2}}\pi\left(\sqrt{5}y^2\right)^2\ dy$$

$$V=\int_{-\frac{3}{2}}^{\frac{3}{2}}5\pi y^4\ dy$$

Integrate, then evaluate over the interval.

$$V=\pi y^5\Big|_{-\frac{3}{2}}^{\frac{3}{2}}$$

$$V=\pi\left(\frac{3}{2}\right)^5-\left[\pi\left(-\frac{3}{2}\right)^5\right]$$

$$V=\frac{243}{32}\pi+\frac{243}{32}\pi$$

$$V=\frac{243}{16}\pi$$

Topic: Disks, vertical axis

Question: Use disks to find the volume of the solid formed by rotating the region enclosed by the curves.

$x = y^{\frac{3}{2}}$ and $x = 0$

$y = 0$ and $y = 4$

about the y-axis

Answer choices:

A $V = \frac{81}{4}\pi$ cubic units

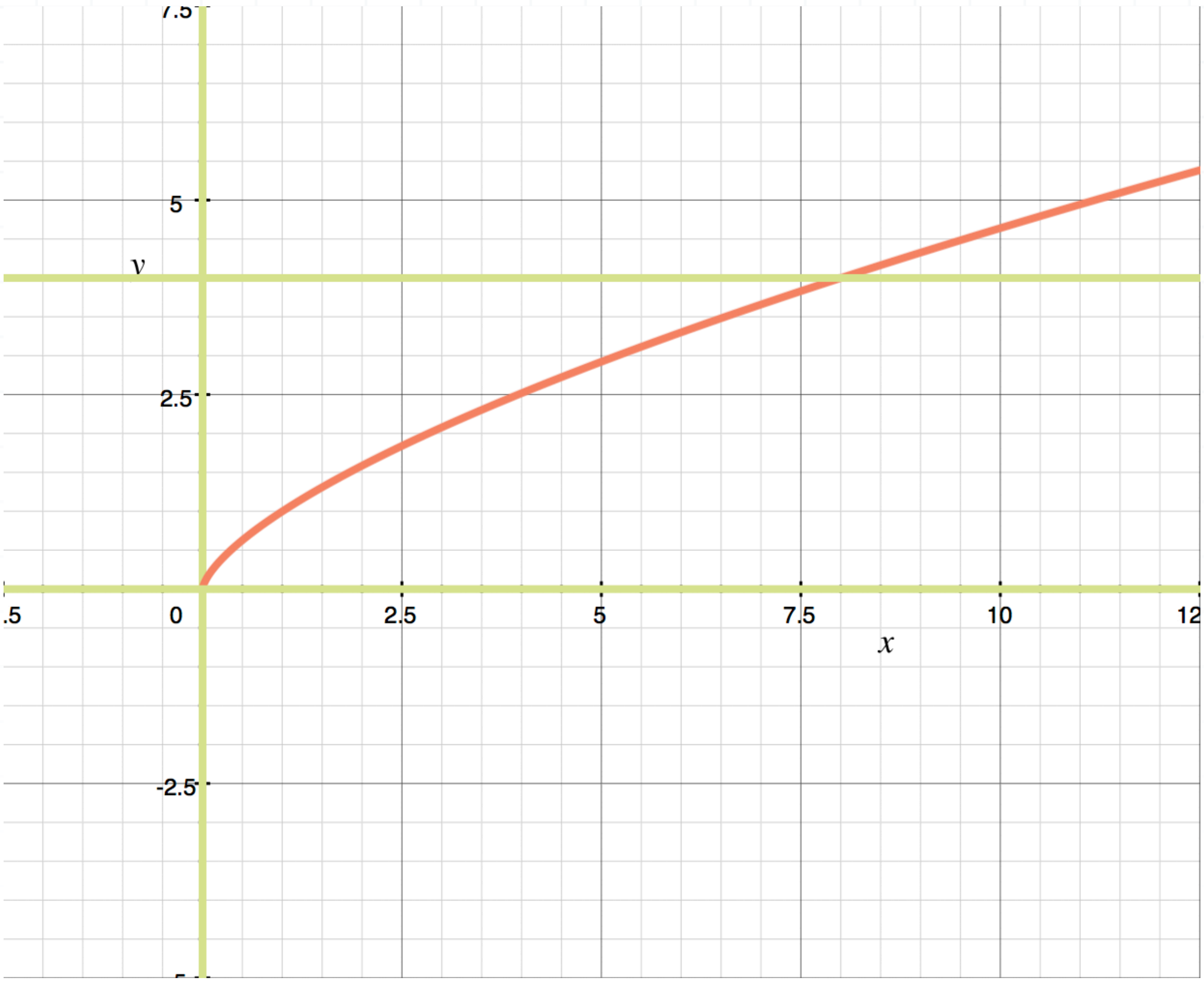
B $V = 64\pi$ cubic units

C $V = 64$ cubic units

D $V = \frac{64}{3}\pi$ cubic units

Solution: B

The region enclosed by $x = y^{\frac{3}{2}}$, $x = 0$, $y = 0$ and $y = 4$ is



Because we're rotating about the y-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking horizontal slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dy, which means we'll be integrating with respect to y. Therefore, the limits of integration will be given by $y = [0,4]$. The outer radius will be defined by $x = y^{\frac{3}{2}}$. So the volume can be given by

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V=\int_0^4 \pi \left(y^{\frac{3}{2}}\right)^2\ dy$$

$$V=\int_0^4 \pi y^3\ dy$$

Integrate, then evaluate over the interval.

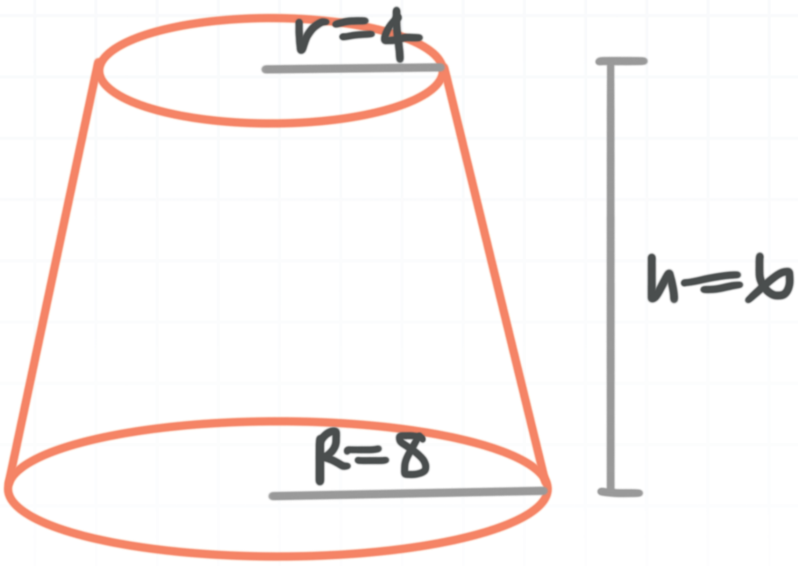
$$V=\frac{1}{4}\pi y^4\Big|_0^4$$

$$V=\frac{1}{4}\pi(4)^4-\frac{1}{4}\pi(0)^4$$

$$V=64\pi$$

Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h = 12$ inches, a lower base radius $R = 8$ inches, and an upper radius of $r = 4$ inches.



Answer choices:

A $\quad V = 224$ cubic inches

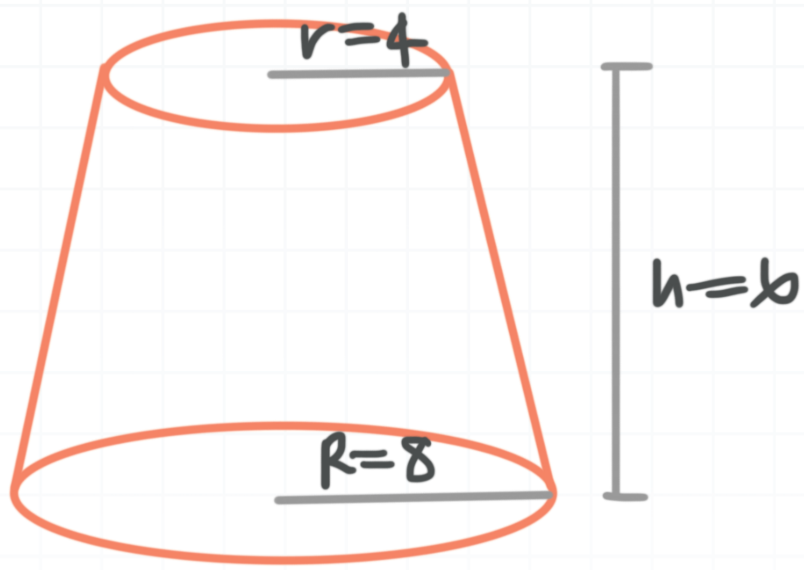
B $\quad V = 224\pi$ cubic inches

C $\quad V = 45\pi$ cubic inches

D $\quad V = 54\pi$ cubic inches

Solution: B

The frustum in this problem looks like this



We will use the y-axis as the center-height of the cone. From the problem, the lower base radius is $R = 8$. This means that the edge of the cone is at the point $(8,0)$. The height of the cone is $h = 12$, so the vertex of the cone is at the point $(0,12)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

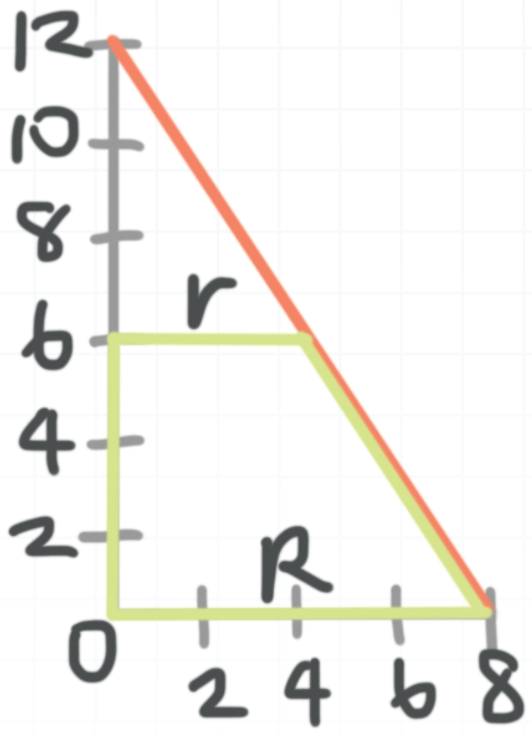
We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{0-12}{8-0} = -\frac{12}{8} = -\frac{3}{2}$$

The vertex of the cone is on the y-axis at the point $(0,12)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = -\frac{3}{2}x + 12$$

The frustum is formed by rotating the region shown below about the y-axis.



To find its volume, we'll realize that the axis of rotation is vertical. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be horizontal. Which means we'll represent the width of each infinitely thin slice as dy. Which means we'll be integrating with respect to y. Therefore, the limits of integration will be given as $y = [0,6]$, and we can write the volume integral as

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V = \int_0^6 \pi \left[f(y)\right]^2 \ dy$$

The outer radius $f(y)$ is given by

$$y = -\frac{3}{2}x + 12$$

but we need to rearrange the equation so that it's solved for x in terms of y.

$$y - 12 = -\frac{3}{2}x$$

$$2y - 24 = -3x$$

$$x = 8 - \frac{2}{3}y$$

The volume of the frustum is then

$$V = \int_0^6 \pi \left(8 - \frac{2}{3}y\right)^2 \ dy$$

$$V = \int_0^6 \pi \left(64 - \frac{32}{3}y + \frac{4}{9}y^2\right) \ dy$$

$$V = \int_0^6 64\pi - \frac{32}{3}\pi y + \frac{4}{9}\pi y^2 \ dy$$

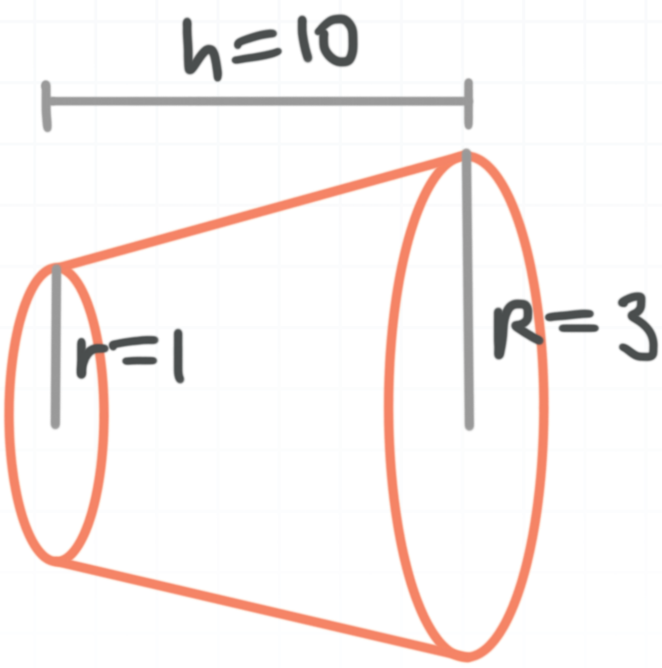
Integrate, then evaluate over the interval.

$$V = 64\pi y - \frac{16}{3}\pi y^2 + \frac{4}{27}\pi y^3 \Big|_0^6$$

$$V = 64\pi(6) - \frac{16}{3}\pi(6)^2 + \frac{4}{27}\pi(6)^3 - \left(64\pi(0) - \frac{16}{3}\pi(0)^2 + \frac{4}{27}\pi(0)^3\right)$$

Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h = 15$ cm, a lower base radius $R = 3$ cm, and an upper radius of $r = 1$ cm.



Answer choices:

A $\quad V = \dfrac{130}{3}\pi$ cubic cm

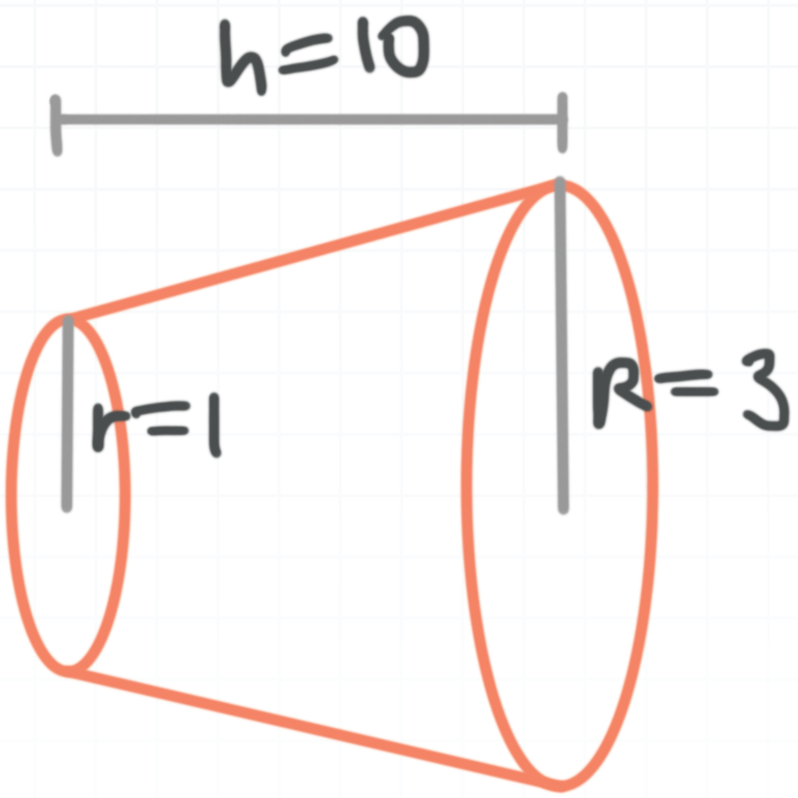
B $\quad V = 130\pi$ cubic cm

C $\quad V = \dfrac{130}{3}$ cubic cm

D $\quad V = \dfrac{130}{3}\pi$ square cm

Solution: A

The frustum in this problem looks like this



We will use the x-axis as the center-height of the cone. From the problem, the lower base radius is $R = 3$. This means that the edge of the cone is at the point $(15,3)$. The height of the cone is $h = 15$, so the vertex of the cone is at the point $(0,0)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{3-0}{15-0} = \frac{3}{15} = \frac{1}{5}$$

The vertex of the cone is on the x-axis at the point $(0,0)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = \frac{1}{5}x$$

The frustum is formed by rotating the region shown below about the x-axis.

To find its volume, we'll realize that the axis of rotation is horizontal. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be vertical. Which means we'll represent the width of each infinitely thin slice as dx. Which means we'll be integrating with respect to x. Therefore, the limits of integration will be given as $x = [5,15]$, and we can write the volume integral as

$$V = \int_a^b \pi \left[f(x)\right]^2 \ dx$$

$$V = \int_5^{15} \pi \left[f(x)\right]^2 \ dx$$

The outer radius $f(x)$ is given by

$$y = \frac{1}{5}x$$

The volume of the frustum is then

$$V = \int_{5}^{15} \pi \left(\frac{1}{5}x \right)^2 \ dx$$

$$V = \int_{5}^{15} \frac{1}{25}\pi x^2 \ dx$$

Integrate, then evaluate over the interval.

$$V = \frac{1}{75}\pi x^3 \Big|_{5}^{15}$$

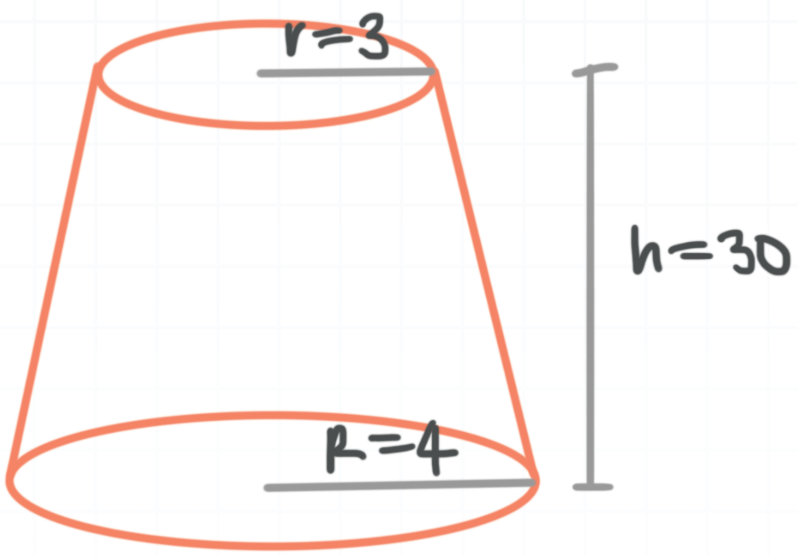
$$V = \frac{1}{75}\pi(15)^3 - \left(\frac{1}{75}\pi(5)^3 \right)$$

$$V = \frac{3,375}{75}\pi - \frac{125}{75}\pi$$

$$V = \frac{130}{3}\pi$$

Topic: Disks, volume of a frustum

Question: Use disks to find the volume of the frustum of a right circular cone with height $h = 120$ mm, a lower base radius $R = 4$ mm, and an upper radius of $r = 3$ mm.



Answer choices:

A $V = 37\pi$ cubic mm

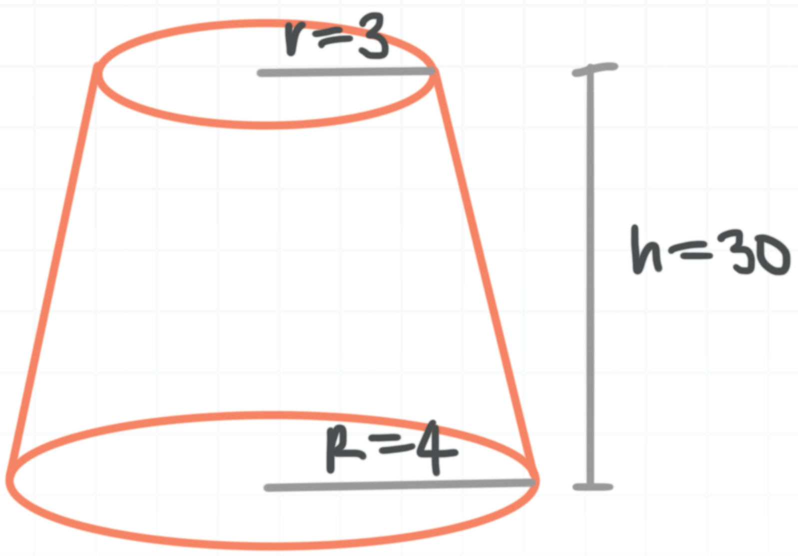
B $V = 370\pi$ square mm

C $V = 370$ cubic mm

D $V = 370\pi$ cubic mm

Solution: D

The frustum in this problem looks like this



We will use the y-axis as the center-height of the cone. From the problem, the lower base radius is $R = 4$. This means that the edge of the cone is at the point $(4,0)$. The height of the cone is $h = 120$, so the vertex of the cone is at the point $(0,120)$. Therefore, we can determine the equation of the line that is formed by the lateral surface of the cone.

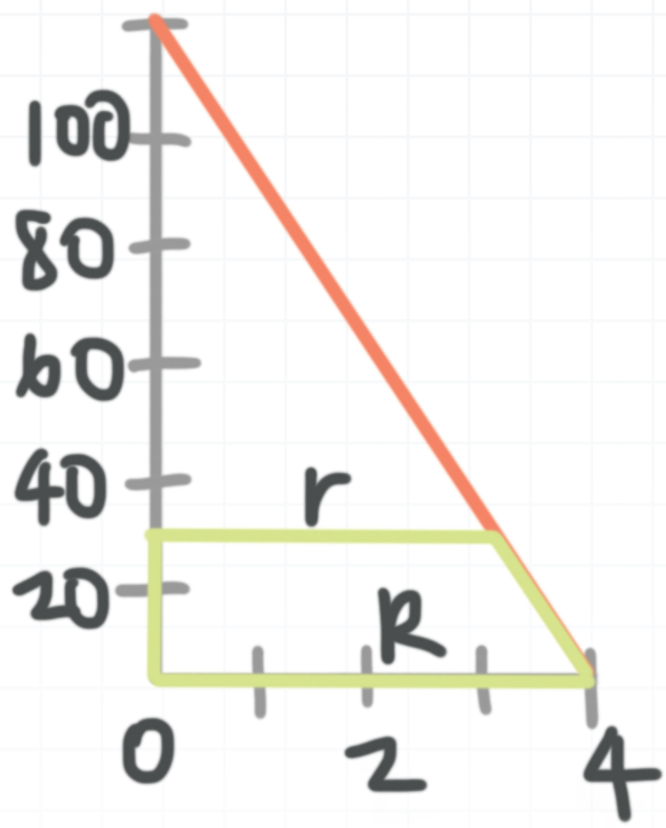
We will use the slope-intercept form to determine the equation. First, the slope is

$$\frac{0 - 120}{4 - 0} = -\frac{120}{4} = -30$$

The vertex of the cone is on the y-axis at the point $(0,120)$. Thus, the equation of the line that forms the lateral surface of the cone, as a rotation is

$$y = -30x + 120$$

The frustum is formed by rotating the region shown below about the y-axis.



To find its volume, we'll realize that the axis of rotation is vertical. Because the slices we'll use to approximate volume must always be perpendicular to the axis of rotation, that means the slices must be horizontal. Which means we'll represent the width of each infinitely thin slice as dy. Which means we'll be integrating with respect to y. Therefore, the limits of integration will be given as $y = [0,30]$, and we can write the volume integral as

$$V = \int_c^d \pi \left[f(y)\right]^2 \ dy$$

$$V = \int_0^{30} \pi \left[f(y)\right]^2 \ dy$$

The outer radius $f(y)$ needs to come from

$$y = -30x + 120$$

So we need to solve this for x in terms of y.

$$y - 120 = -30x$$

$$x = -\frac{1}{30}y + 4$$

The volume of the frustum is then

$$V = \int_0^{30} \pi \left(-\frac{1}{30}y + 4\right)^2 \ dy$$

$$V = \int_0^{30} \pi \left(\frac{1}{900}y^2 - \frac{4}{15}y + 16\right) \ dy$$

$$V = \int_0^{30} \frac{1}{900}\pi y^2 - \frac{4}{15}\pi y + 16\pi \ dy$$

Integrate, then evaluate over the interval.

$$V = \frac{1}{2{,}700}\pi y^3 - \frac{2}{15}\pi y^2 + 16\pi y \Big|_0^{30}$$

$$V = \frac{1}{2{,}700}\pi(30)^3 - \frac{2}{15}\pi(30)^2 + 16\pi(30) - \left(\frac{1}{2{,}700}\pi(0)^3 - \frac{2}{15}\pi(0)^2 + 16\pi(0)\right)$$

$$V = 10\pi - 120\pi + 480\pi$$

$$V = 370\pi$$

Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2$ and $y = (32x)^{\frac{1}{3}}$

about the x-axis

Answer choices:

A $\frac{64\pi}{5}$

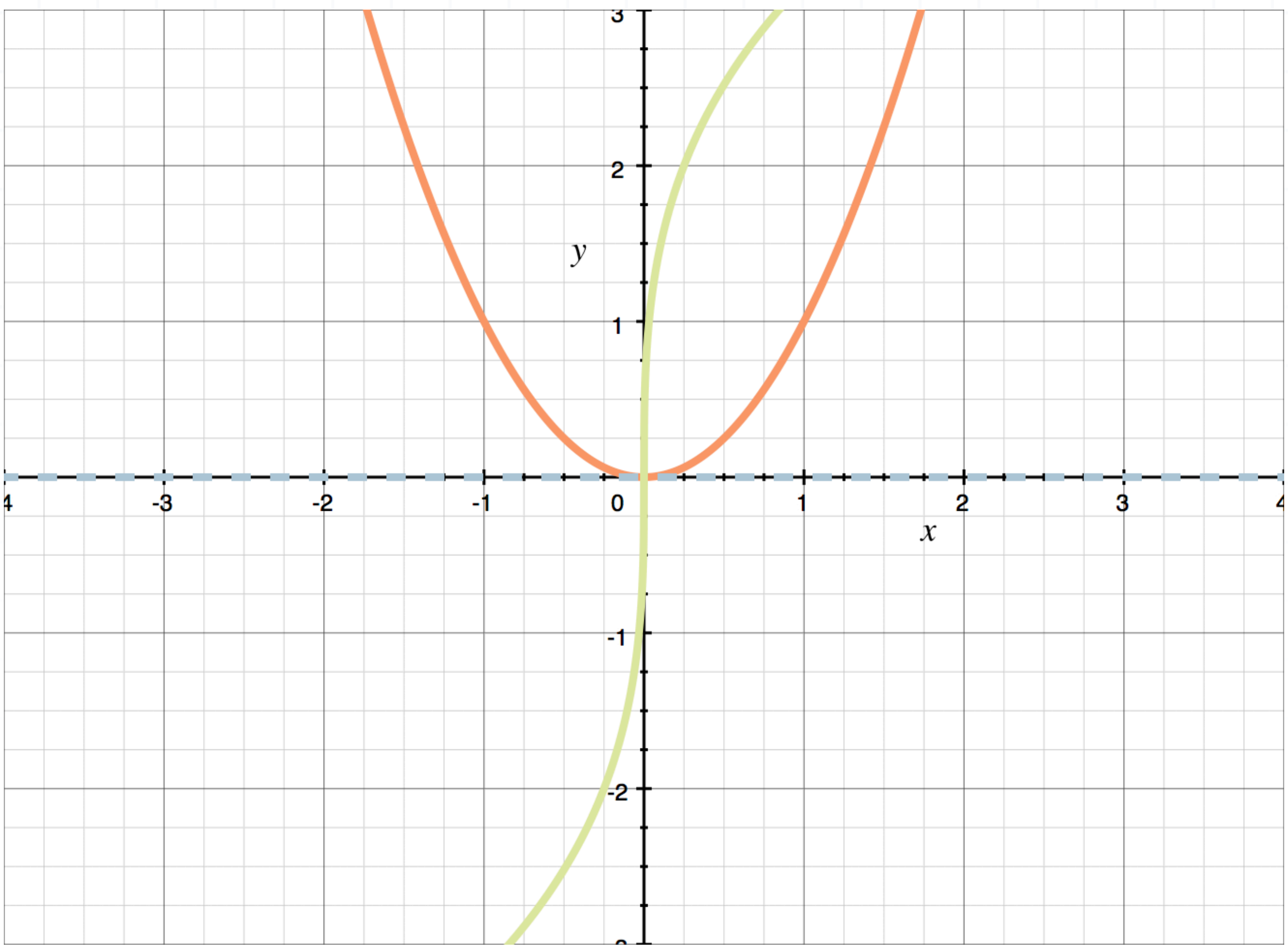
B 64π

C $\frac{64}{5}$

D $\frac{288\pi}{5}$

Solution: A

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take vertical slices of our area and rotate each of them around the axis to form washers.

Using washers around a horizontal axis, specifically the x-axis, tells us that we'll use the volume formula

$$V = \int_a^b \pi \left[f(x)\right]^2 - \pi \left[g(x)\right]^2 \ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined. Since these are just the two points of intersection, we can do this by looking at the graph, or we can set the curves equal to one another and solve for x.

$$x^2 = (32x)^{\frac{1}{3}}$$

$$x^6 = 32x$$

$$x^6 - 32x = 0$$

$$x\left(x^5 - 32\right) = 0$$

$$x = 0$$

or

$$x^5 - 32 = 0$$

$$x^5 = 32$$

$$\left(x^5\right)^{\frac{1}{5}} = (32)^{\frac{1}{5}}$$

$$x = 2$$

Now we know that our limits of integration are $a = 0$ and $b = 2$.

$f(x)$ is the radius of the curve that's further from the axis of revolution, and $g(x)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 0$ and $x = 2$) into both curves to see which function returns a larger value (this will be the further curve) and which one returns a smaller value (this will be the closer curve). Let's plug in $x = 1$ to check.

$$y = x^2$$

$$y = (1)^2$$

$$y = 1$$

and

$$y = (32x)^{\frac{1}{3}}$$

$$y = \left[32(1)\right]^{\frac{1}{3}}$$

$$y = (8 \cdot 4)^{\frac{1}{3}}$$

$$y = 8^{\frac{1}{3}}4^{\frac{1}{3}}$$

$$y = 2(4)^{\frac{1}{3}}$$

Since $y = (32x)^{\frac{1}{3}}$ returns a larger value than $y = x^2$, we can say

$$g(x) = x^2$$

and

$$f(x) = (32x)^{\frac{1}{3}}$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^2 \pi \left[(32x)^{\frac{1}{3}}\right]^2 - \pi \left(x^2\right)^2 \ dx$$

$$V=\int_0^2 \pi\left[(32x)^{\frac{1}{3}}\right]^2-\pi x^4\ dx$$

$$V=\int_0^2 \pi(32x)^{\frac{2}{3}}-\pi x^4\ dx$$

$$V=\int_0^2 \pi\left(32^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 \pi\left(8^{\frac{2}{3}}4^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(4^{\frac{2}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(16^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 4\pi\left(8^{\frac{1}{3}}2^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 8\pi\left(2^{\frac{1}{3}}x^{\frac{2}{3}}\right)-\pi x^4\ dx$$

$$V=\int_0^2 \left[8\pi(2)^{\frac{1}{3}}\right]x^{\frac{2}{3}}-\pi x^4\ dx$$

Integrate and then evaluate over the interval.

$$V=\left[\frac{3}{5}\left[8\pi(2)^{\frac{1}{3}}\right]x^{\frac{5}{3}}-\frac{\pi}{5}x^5\right]\Bigg|_0^2$$

$$V=\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}x^{\frac{5}{3}}-\frac{\pi}{5}x^5\right]\Bigg|_0^2$$

$$V=\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}(2)^{\frac{5}{3}}-\frac{\pi}{5}(2)^5\right]-\left[\frac{24\pi(2)^{\frac{1}{3}}}{5}(0)^{\frac{5}{3}}-\frac{\pi}{5}(0)^5\right]$$

$$V=\frac{24\pi(2)^{\frac{1}{3}}}{5}(32)^{\frac{1}{3}}-\frac{32\pi}{5}$$

$$V=\frac{24\pi(2)^{\frac{1}{3}}}{5}8^{\frac{1}{3}}4^{\frac{1}{3}}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(2)^{\frac{1}{3}}(4)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(2\cdot 4)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{48\pi(8)^{\frac{1}{3}}}{5}-\frac{32\pi}{5}$$

$$V=\frac{96\pi}{5}-\frac{32\pi}{5}$$

$$V=\frac{64\pi}{5}$$

Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2$ and $y = 0$ and $x = 1$

about $y = 1$

Answer choices:

A $\frac{2}{3}$

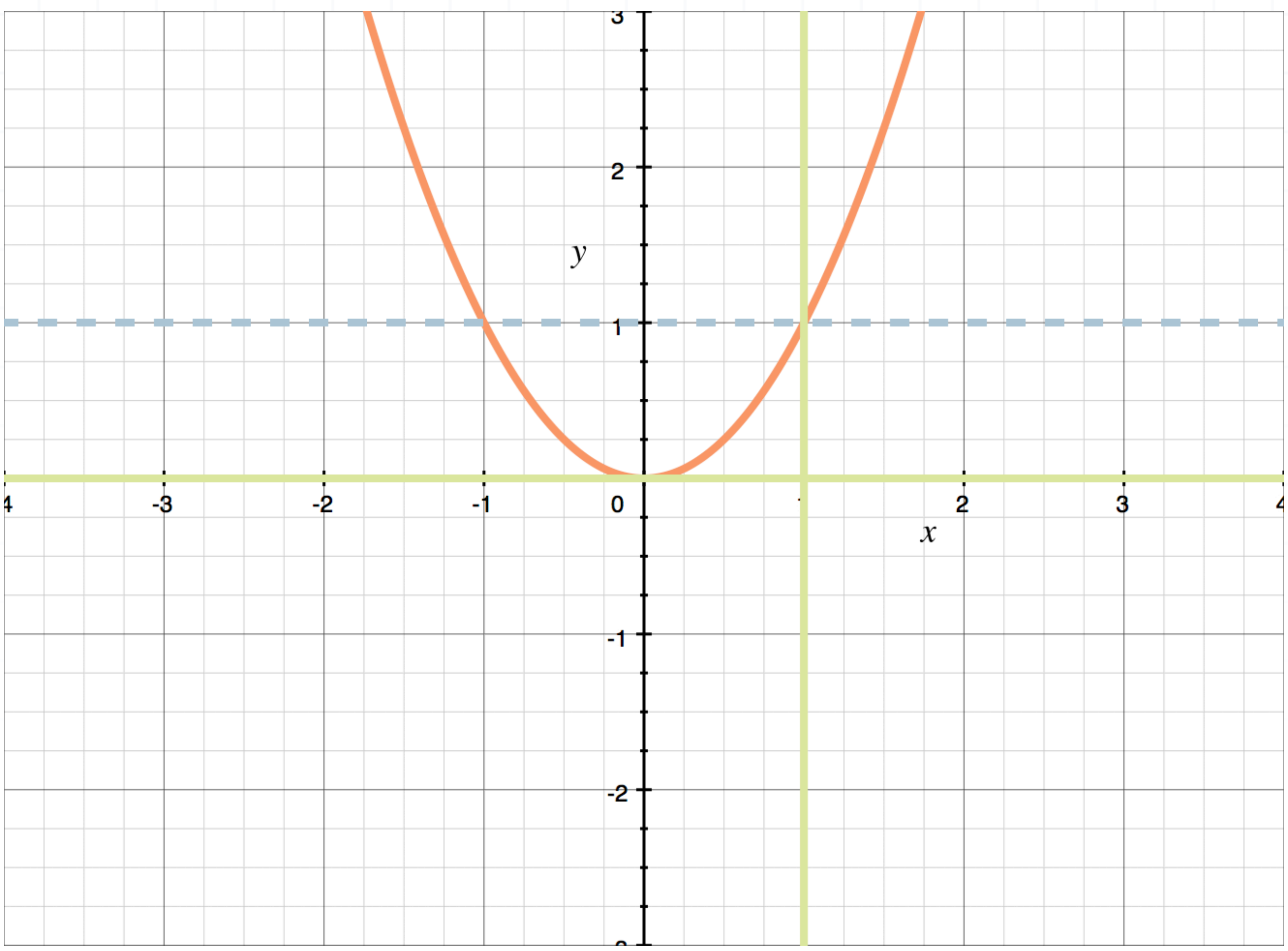
B $\frac{7\pi}{30}$

C $\frac{7\pi}{15}$

D $\frac{3}{2}$

Solution: C

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take vertical slices of our area and rotate each of them around the axis to form washers.

Using washers around a horizontal axis, specifically $y = 1$, tells us that we'll use the volume formula

$$V = \int_a^b \pi \left[k - g(x)\right]^2 - \pi \left[k - f(x)\right]^2 \ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined.

We can see from the graph that the largest x-value for which the area is defined is $x = 1$. This was given in the original problem. We can see that the smallest value for which it's defined is a point of intersection, so we can set the curves equal to one another and solve for x.

$$x^2 = 0$$

$$x = 0$$

Now we know that our limits of integration are $a = 0$ and $b = 1$.

$g(x)$ is the radius of the curve that's further from the axis of revolution, and $f(x)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 0$ and $x = 1$) into both curves to see which function returns a larger value (this will be the closer curve) and which one returns a smaller value (this will be the further curve). Let's plug in $x = 1/2$ to check.

$$y = x^2$$

$$y = \left(\frac{1}{2}\right)^2$$

$$y = \frac{1}{4}$$

and

$$y = 0$$

Since $y = x^2$ returns a larger value than $y = 0$, we can say

$$f(x) = x^2$$

and

$$g(x) = 0$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^1 \pi (1 - 0)^2 - \pi \left(1 - x^2\right)^2 \ dx$$

$$V = \int_0^1 \pi - \pi \left(1 - 2x^2 + x^4\right) \ dx$$

$$V = \int_0^1 \pi - \pi + 2\pi x^2 - \pi x^4 \ dx$$

$$V = \int_0^1 2\pi x^2 - \pi x^4 \ dx$$

Integrate and then evaluate over the interval.

$$V = \left(\frac{2\pi}{3}x^3 - \frac{\pi}{5}x^5\right)\Bigg|_0^1$$

$$V = \left[\frac{2\pi}{3}(1)^3 - \frac{\pi}{5}(1)^5\right] - \left[\frac{2\pi}{3}(0)^3 - \frac{\pi}{5}(0)^5\right]$$

$$V = \frac{2\pi}{3} - \frac{\pi}{5}$$

$$V = \frac{10\pi}{15} - \frac{3\pi}{15}$$

$$V = \frac{7\pi}{15}$$

Topic: Washers, horizontal axis

Question: Use washers to find the volume of the solid formed by rotating the region bounded by the curves.

$y = x^2$ and $y = 0$

$x = 0$ and $x = 3$

about the line $y = -1$

Answer choices:

A $\quad V = 333\pi$ cubic units

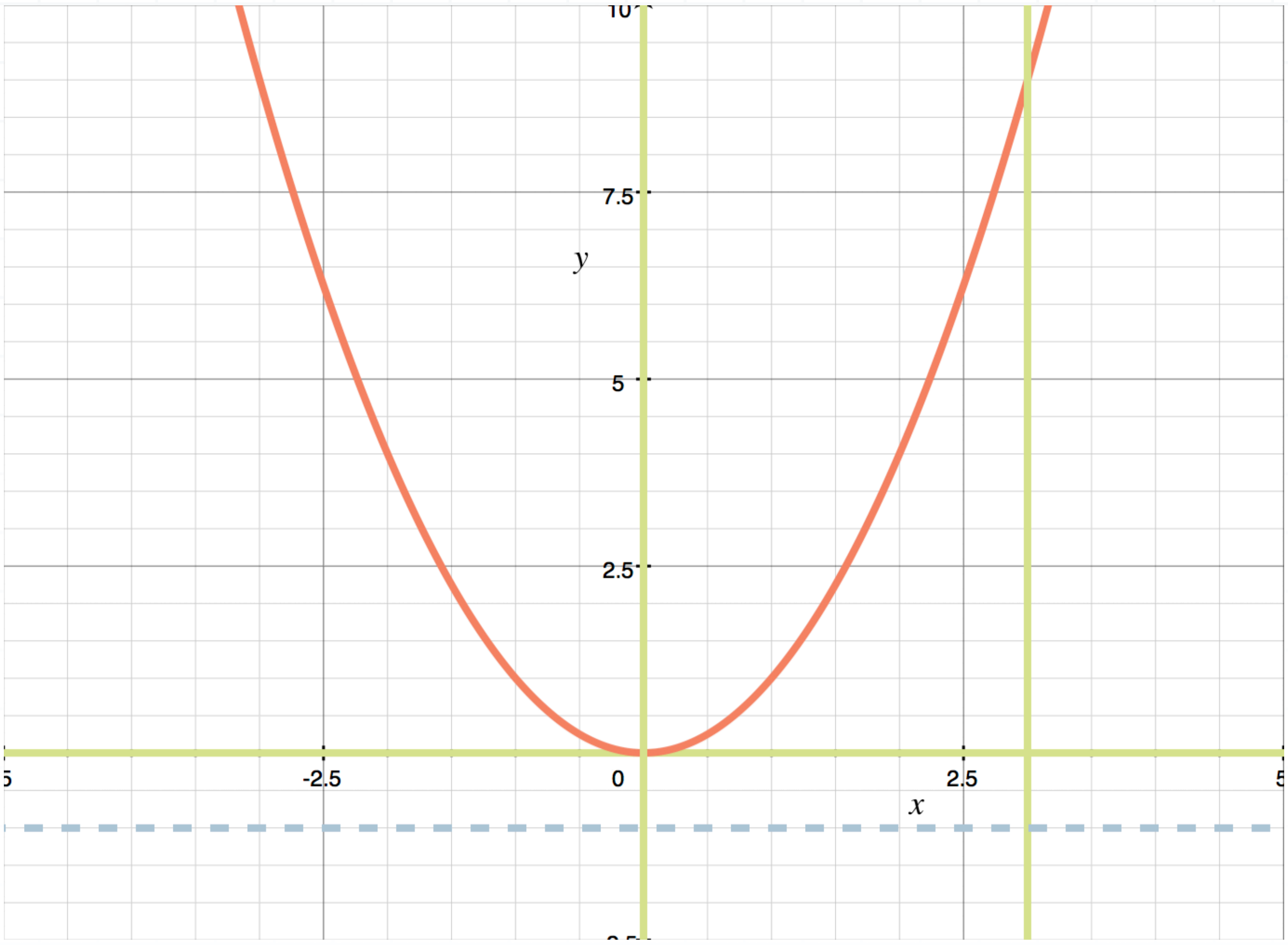
B $\quad V = \dfrac{333}{5}$ cubic units

C $\quad V = \dfrac{117}{4}\pi$ cubic units

D $\quad V = \dfrac{333}{5}\pi$ cubic units

Solution: D

The region enclosed by $y = x^2$, $y = 0$, $x = 0$ and $x = 3$ is



Because we're rotating about $y = -1$, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume. Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by $x = [0,3]$. The outer radius will be defined by $y = x^2$. So if the axis of rotation is $y = -k$, then the volume can be given by

$$V = \int_a^b \pi \left[k + f(x)\right]^2 - \pi \left[k + g(x)\right]^2 \ dx$$

$$V = \int_0^3 \pi \left(1 + x^2\right)^2 - \pi (1 + 0)^2 \ dx$$

$$V = \int_0^3 \pi \left(1 + 2x^2 + x^4\right) - \pi \ dx$$

$$V = \int_0^3 \pi + 2\pi x^2 + \pi x^4 - \pi \ dx$$

$$V = \int_0^3 2\pi x^2 + \pi x^4 \ dx$$

Integrate, then evaluate over the interval.

$$V = \frac{2}{3}\pi x^3 + \frac{1}{5}\pi x^5 \Big|_0^3$$

$$V = \frac{2}{3}\pi(3)^3 + \frac{1}{5}\pi(3)^5 - \left(\frac{2}{3}\pi(0)^3 + \frac{1}{5}\pi(0)^5\right)$$

$$V = 18\pi + \frac{243}{5}\pi$$

$$V = \frac{90}{5}\pi + \frac{243}{5}\pi$$

$$V = \frac{333}{5}\pi$$

Topic: Washers, vertical axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by $x = \sqrt{y}$, $x = 2$, and $y = 0$ about the y-axis.

Answer choices:

A 2π

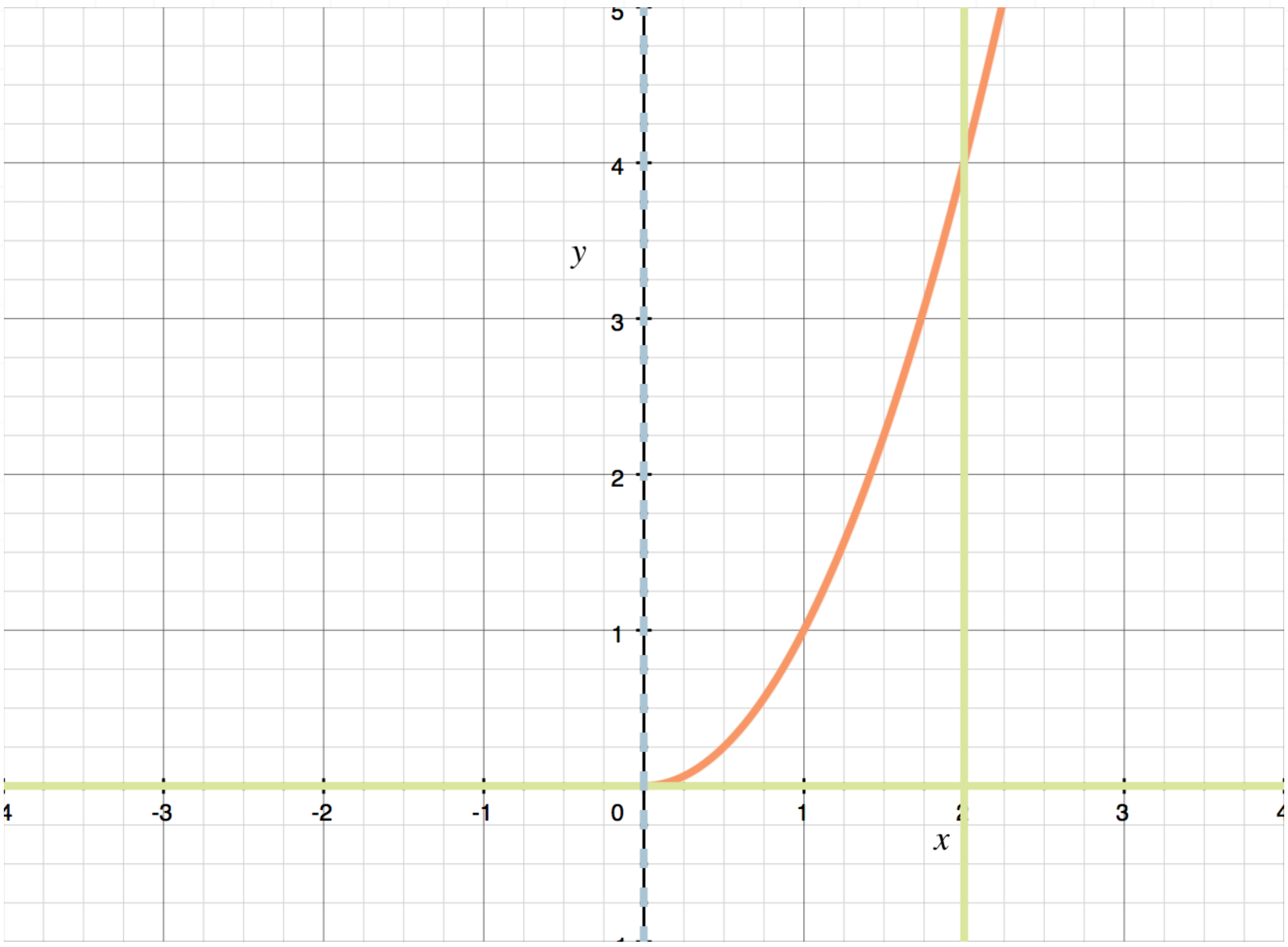
B 4π

C 8π

D 16π

Solution: C

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take horizontal slices of our area and rotate each of them around the axis to form washers.

Using washers around a vertical axis, specifically the y-axis, tells us that we'll use the volume formula

$$V=\int_c^d \pi[f(y)]^2-\pi[g(y)]^2\ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. The given curves are already defined for x in terms of y, so now we just need to find limits of integration, which will be the smallest and largest y-values for which the area is defined.

We can see from the graph that the smallest y-value for which the area is defined is $y=0$. This was given in the original problem. We can see that the largest value for which it's defined is a point of intersection, so we can set the curves equal to one another and solve for y.

$$\sqrt{y}=2$$

$$y=4$$

Now we know that our limits of integration are $c=0$ and $d=4$.

The curve $f(y)$ is the radius of the curve that's further from the axis of revolution, and $g(y)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug a y-value between the points of intersection (between $y=0$ and $y=4$) into both curves to see which function returns a larger value (this will be the further curve) and which one returns a smaller value (this will be the closer curve). Let's plug in $y=1$ to check.

$$x=\sqrt{y}$$

$$x = \sqrt{1}$$

$$x = 1$$

and

$$x = 2$$

Since $x = 2$ returns a larger value than $x = \sqrt{y}$, we can say

$$g(y) = \sqrt{y}$$

and

$$f(y) = 2$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^4 \pi(2)^2 - \pi(\sqrt{y})^2 \ dy$$

$$V = \int_0^4 4\pi - \pi y \ dy$$

$$V = \left(4\pi y - \frac{\pi}{2}y^2\right)\Big|_0^4$$

$$V = \left[4\pi(4) - \frac{\pi}{2}(4)^2\right] - \left[4\pi(0) - \frac{\pi}{2}(0)^2\right]$$

$$V = 16\pi - 8\pi$$

$$V = 8\pi$$

Topic: Washers, vertical axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about $x = -1$.

$$x = \frac{y^4}{4} - \frac{y^2}{2} \text{ and } x = \frac{y^2}{2}$$

Answer choices:

A $\quad V = \frac{1{,}984\pi}{315}$ cubic units

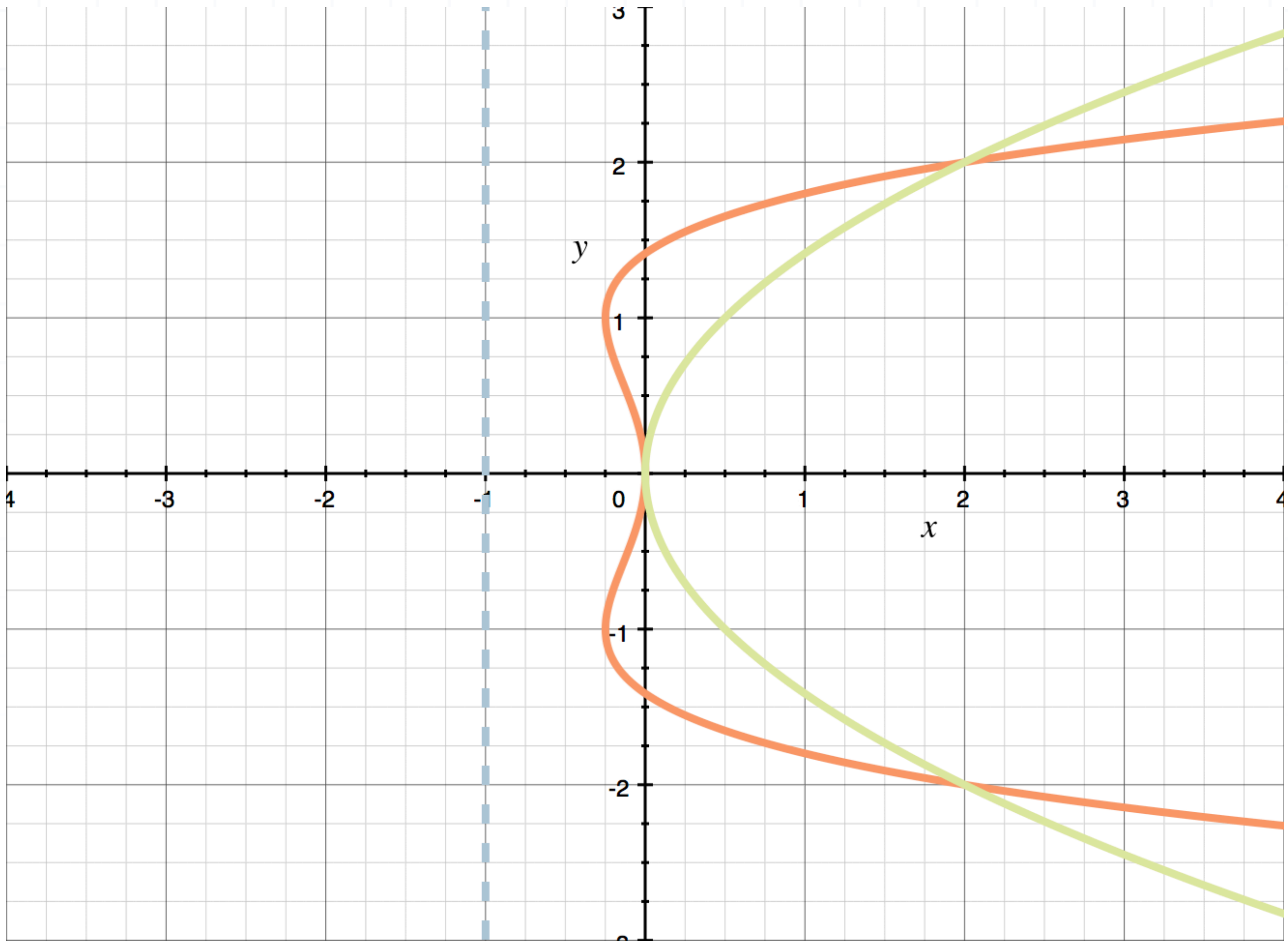
B $\quad V = \frac{1{,}984\pi}{315}$ square units

C $\quad V = \frac{986\pi}{315}$ cubic units

D $\quad V = \frac{986\pi}{315}$ square units

Solution: A

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take horizontal slices of our area and rotate each of them around the axis to form washers.

Using washers around a vertical axis, specifically the line $x = -1$, tells us that we'll use the volume formula

$$V=\int_c^d \pi\left[k+f(y)\right]^2-\pi\left[k+g(y)\right]^2\ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. The given curves are already defined for x in terms of y, so now we just need to find limits of integration, which will be the smallest and largest y-values for which the area is defined.

From the graph, it looks like the area is defined between $y=-2$ and $y=2$. To be sure, we'll set the curves equal to each other and solve for y.

$$\frac{y^2}{2}=\frac{y^4}{4}-\frac{y^2}{2}$$

$$2y^2=y^4-2y^2$$

$$4y^2=y^4$$

$$4=y^2$$

$$y=\pm 2$$

Now we know that our limits of integration are $c=-2$ and $d=2$.

The curve $f(y)$ is the radius of the curve that's further from the axis of revolution, and $g(y)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug a y-value between the points of intersection (between $y=-2$ and $y=2$) into both curves to see which function returns a larger value (this will be the further curve) and which one returns a smaller value (this will be the closer curve). Let's plug in $y=1$ to check.

$$x=\frac{y^4}{4}-\frac{y^2}{2}$$

$$x=\frac{1^4}{4}-\frac{1^2}{2}$$

$$x=-\frac{1}{4}$$

and

$$x=\frac{y^2}{2}$$

$$x=\frac{1^2}{2}$$

$$x=\frac{1}{2}$$

Since $x=y^2/2$ returns a larger value than $x=y^4/4-y^2/2$, we can say

$$g(y)=\frac{y^4}{4}-\frac{y^2}{2}$$

and

$$f(y)=\frac{y^2}{2}$$

Plugging everything we know into the volume formula, we get

$$V=\int_{-2}^{2}\pi\left(1+\frac{y^2}{2}\right)^2-\pi\left(1+\frac{y^4}{4}-\frac{y^2}{2}\right)^2\ dy$$

$$V=\int_{-2}^{2}\pi\left(1+y^2+\frac{y^4}{4}\right)-\pi\left(1+\frac{y^4}{4}-\frac{y^2}{2}+\frac{y^4}{4}+\frac{y^8}{16}-\frac{y^6}{8}-\frac{y^2}{2}-\frac{y^6}{8}+\frac{y^4}{4}\right)\ dy$$

$$V=\int_{-2}^{2}\pi\left(1+y^2+\frac{y^4}{4}\right)-\pi\left(1-y^2+\frac{3y^4}{4}-\frac{y^6}{4}+\frac{y^8}{16}\right)\ dy$$

$$V=\int_{-2}^{2}\pi+\pi y^2+\frac{\pi}{4}y^4-\left(\pi-\pi y^2+\frac{3\pi}{4}y^4-\frac{\pi}{4}y^6+\frac{\pi}{16}y^8\right)\ dy$$

$$V=\int_{-2}^{2}\pi+\pi y^2+\frac{\pi}{4}y^4-\pi+\pi y^2-\frac{3\pi}{4}y^4+\frac{\pi}{4}y^6-\frac{\pi}{16}y^8\ dy$$

$$V=\int_{-2}^{2}2\pi y^2-\frac{\pi}{2}y^4+\frac{\pi}{4}y^6-\frac{\pi}{16}y^8\ dy$$

Integrate, then evaluate over the interval.

$$V=\frac{2\pi}{3}y^3-\frac{\pi}{10}y^5+\frac{\pi}{28}y^7-\frac{\pi}{144}y^9\Big|_{-2}^{2}$$

$$V=\frac{2\pi}{3}(2)^3-\frac{\pi}{10}(2)^5+\frac{\pi}{28}(2)^7-\frac{\pi}{144}(2)^9$$

$$-\left(\frac{2\pi}{3}(-2)^3-\frac{\pi}{10}(-2)^5+\frac{\pi}{28}(-2)^7-\frac{\pi}{144}(-2)^9\right)$$

$$V=\frac{16\pi}{3}-\frac{16\pi}{5}+\frac{32\pi}{7}-\frac{32\pi}{9}-\left(-\frac{16\pi}{3}+\frac{16\pi}{5}-\frac{32\pi}{7}+\frac{32\pi}{9}\right)$$

$$V=\frac{16\pi}{3}-\frac{16\pi}{5}+\frac{32\pi}{7}-\frac{32\pi}{9}+\frac{16\pi}{3}-\frac{16\pi}{5}+\frac{32\pi}{7}-\frac{32\pi}{9}$$

$$V=\frac{32\pi}{3}-\frac{32\pi}{5}+\frac{64\pi}{7}-\frac{64\pi}{9}$$

$$V=\frac{3{,}360\pi}{315}-\frac{2{,}016\pi}{315}+\frac{2{,}880\pi}{315}-\frac{2{,}240\pi}{315}$$

$$V=\frac{1{,}984\pi}{315}$$

Topic: Washers, vertical axis

Question: Use washers to find the volume of the solid generated by revolving the region bounded by the curves about $x = 4$.

$$x = \frac{y^2}{2} \text{ and } x = 2$$

Answer choices:

A $\quad V = \frac{448}{15}$ cubic units

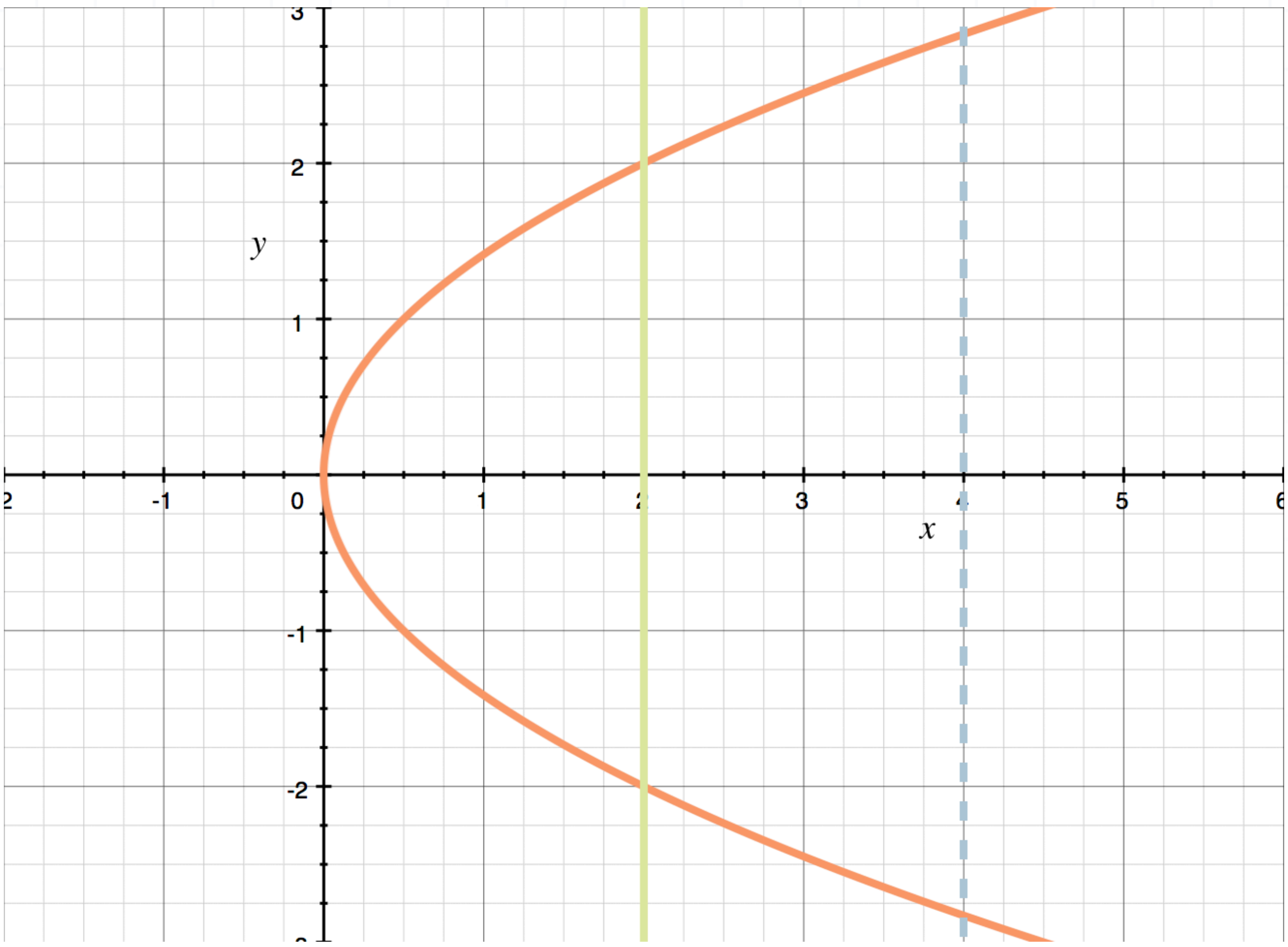
B $\quad V = \frac{1{,}552}{51}$ cubic units

C $\quad V = \frac{1{,}552}{51}\pi$ cubic units

D $\quad V = \frac{448}{15}\pi$ cubic units

Solution: D

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using washers means we'll take slices of our area that are perpendicular to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take horizontal slices of our area and rotate each of them around the axis to form washers.

Using washers around a vertical axis, specifically the line $x = 4$, tells us that we'll use the volume formula

$$V = \int_c^d \pi \left[k - f(y)\right]^2 - \pi \left[k - g(y)\right]^2 \ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. The given curves are already defined for x in terms of y, so now we just need to find limits of integration, which will be the smallest and largest y-values for which the area is defined.

From the graph, it looks like the area is defined between $y = -2$ and $y = 2$. To be sure, we'll set the curves equal to each other and solve for y.

$$\frac{y^2}{2} = 2$$

$$y^2 = 4$$

$$y = \pm 2$$

Now we know that our limits of integration are $c = -2$ and $d = 2$.

The curve $f(y)$ is the radius of the curve that's further from the axis of revolution, and $g(y)$ is the radius of the curve that's closer to the axis of revolution.

To figure out which curve is further away and which one is closer, we can look at the graph or we can plug a y-value between the points of intersection (between $y = -2$ and $y = 2$) into both curves to see which function returns a larger value (this will be the further curve) and which one returns a smaller value (this will be the closer curve). Let's plug in $y = 1$ to check.

$$x = \frac{y^2}{2}$$

$$x = \frac{1^2}{2}$$

$$x = \frac{1}{2}$$

and

$$x = 2$$

Since $x = y^2/2$ returns a smaller value than $x = 2$, we can say

$$f(y) = \frac{y^2}{2}$$

and

$$g(y) = 2$$

Plugging everything we know into the volume formula, we get

$$V = \int_{-2}^{2} \pi \left(4 - \frac{y^2}{2}\right)^2 - \pi (4 - 2)^2 \ dy$$

$$V = \int_{-2}^{2} \pi \left(16 - 4y^2 + \frac{y^4}{4}\right) - 4\pi \ dy$$

$$V = \int_{-2}^{2} 16\pi - 4\pi y^2 + \frac{\pi y^4}{4} - 4\pi \ dy$$

$$V=\int_{-2}^{2} 12\pi-4\pi y^2+\frac{\pi y^4}{4}\ dy$$

Integrate, then evaluate over the interval.

$$V=12\pi y-\frac{4\pi}{3}y^3+\frac{\pi}{20}y^5\Big|_{-2}^{2}$$

$$V=12\pi(2)-\frac{4\pi}{3}(2)^3+\frac{\pi}{20}(2)^5-\left(12\pi(-2)-\frac{4\pi}{3}(-2)^3+\frac{\pi}{20}(-2)^5\right)$$

$$V=24\pi-\frac{32\pi}{3}+\frac{8\pi}{5}-\left(-24\pi+\frac{32\pi}{3}-\frac{8\pi}{5}\right)$$

$$V=24\pi-\frac{32\pi}{3}+\frac{8\pi}{5}+24\pi-\frac{32\pi}{3}+\frac{8\pi}{5}$$

$$V=48\pi-\frac{64\pi}{3}+\frac{16\pi}{5}$$

$$V=\frac{720}{15}\pi-\frac{320\pi}{15}+\frac{48\pi}{15}$$

$$V=\frac{448}{15}\pi$$

Topic: Cylindrical shells, horizontal axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$x = \sqrt{y}$ and $x = \frac{y^3}{32}$

about the x-axis

Answer choices:

A $\frac{64\pi}{5}$

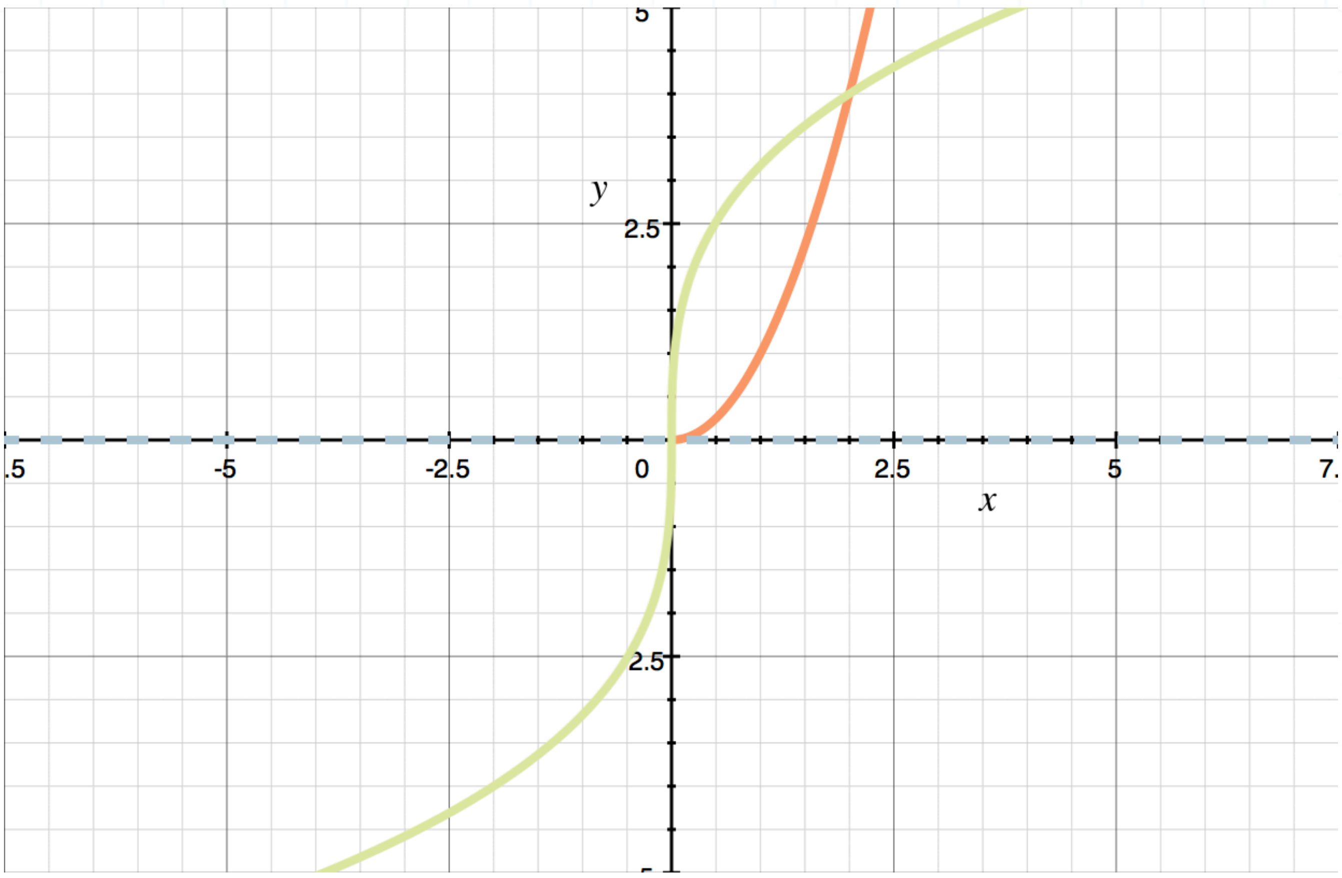
B 64π

C $\frac{5\pi}{64}$

D 5π

Solution: A

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take horizontal slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a horizontal axis, specifically the x-axis, tells us that we'll use the volume formula

$$V = \int_c^d 2\pi y \left[f(y) - g(y)\right] \ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. The given curves are already defined for x in terms of y, so now we just need to find limits of integration, which will be the smallest and largest y-values for which the area is defined. Since these are just the two points of intersection, we can do this by looking at the graph, or we can set the curves equal to one another and solve for y.

$$\sqrt{y} = \frac{y^3}{32}$$

$$32\sqrt{y} = y^3$$

$$1{,}024y = y^6$$

$$y^6 - 1{,}024y = 0$$

$$y\left(y^5 - 1{,}024\right) = 0$$

$$y = 0$$

and

$$y^5 - 1{,}024 = 0$$

$$y^5 = 1{,}024$$

$$\left(y^5\right)^{\frac{1}{5}} = 1{,}024^{\frac{1}{5}}$$

$$y = 4$$

Now we know that our limits of integration are $c = 0$ and $d = 4$. Since the axis of revolution is the x-axis, our radius is y.

$f(y)-g(y)$ is the height of the approximating cylinder, which means we need to subtract the curve on the left $g(y)$ from the curve on the right $f(y)$. To figure out which curve is on the right and which is on the left, we can look at the graph or we can plug a y-value between the points of intersection (between $y=0$ and $y=4$) into both curves to see which function returns a larger value (this will be the right curve) and which one returns a smaller value (this will be the left curve). Let's plug in $y=1$ to check.

$$x=\sqrt{y}$$

$$x=\sqrt{1}$$

$$x=1$$

and

$$x=\frac{y^3}{32}$$

$$x=\frac{(1)^3}{32}$$

$$x=\frac{1}{32}$$

Since $x=\sqrt{y}$ returns a larger value than $x=y^3/32$, we can say

$$f(y)=\sqrt{y}$$

and

$$g(y) = \frac{y^3}{32}$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^4 2\pi y \left(\sqrt{y} - \frac{y^3}{32} \right) \ dy$$

$$V = 2\pi \int_0^4 y^{\frac{3}{2}} - \frac{y^4}{32} \ dy$$

$$V = 2\pi \left(\frac{2}{5} y^{\frac{5}{2}} - \frac{y^5}{160} \right) \Bigg|_0^4$$

$$V = 2\pi \left[\left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{(4)^5}{160} \right) - \left(\frac{2}{5}(0)^{\frac{5}{2}} - \frac{(0)^5}{160} \right) \right]$$

$$V = 2\pi \left(\frac{64}{5} - \frac{1{,}024}{160} \right)$$

$$V = 2\pi \left(\frac{64}{5} - \frac{32}{5} \right)$$

$$V = \frac{64\pi}{5}$$

Topic: Cylindrical shells, horizontal axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$x = y^{\frac{1}{2}}$ and $x = 1$ and $y = 0$

about $y = 1$

Answer choices:

A $\frac{2}{3}$

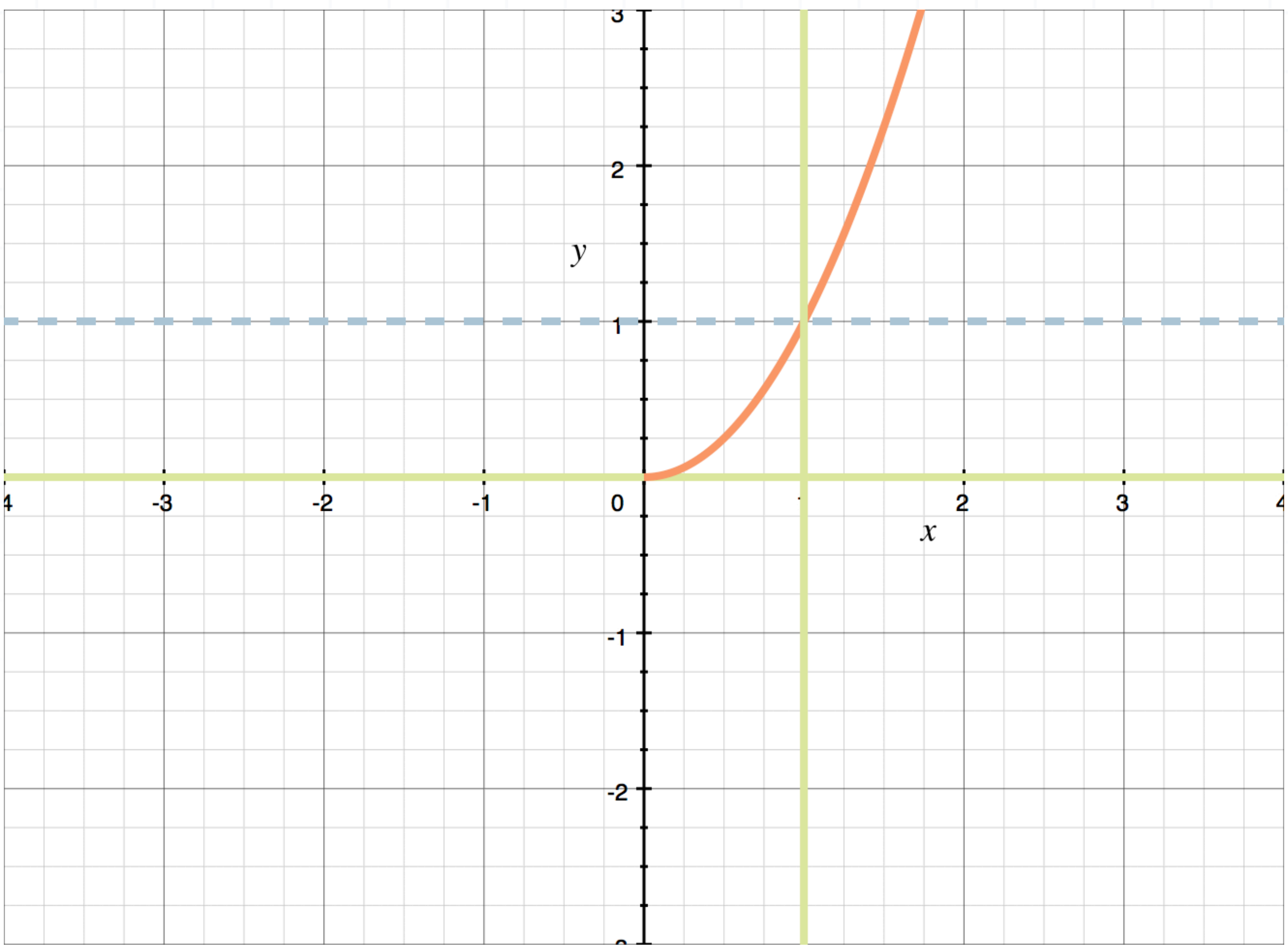
B $\frac{7\pi}{30}$

C $\frac{7\pi}{15}$

D $\frac{3}{2}$

Solution: C

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take horizontal slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a horizontal axis, specifically $y = 1$, tells us that we'll use the volume formula

$$V = \int_c^d 2\pi(k - y)\left[f(y) - g(y)\right] \ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. The given curves are already defined for x in terms of y, so now we just need to find limits of integration, which will be the smallest and largest y-values for which the area is defined.

We can see from the graph that the smallest y-value for which the area is defined is $y = 0$. This was given in the original problem. We can see that the largest value for which it's defined is a point of intersection, so we can set the curves equal to one another and solve for y.

$$y^{\frac{1}{2}} = 1$$

$$\sqrt{y} = 1$$

$$y = 1$$

Now we know that our limits of integration are $c = 0$ and $d = 1$. Since the axis of revolution is $y = 1$, our radius is $1 - y$.

$f(y) - g(y)$ is the height of the approximating cylinder, which means we need to subtract the curve on the left $g(y)$ from the curve on the right $f(y)$. To figure out which curve is on the right and which is on the left, we can look at the graph or we can plug a y-value between the points of intersection (between $y = 0$ and $y = 1$) into both curves to see which function returns a larger value (this will be the right curve) and which one returns a smaller value (this will be the left curve). Let's plug in $y = 1/4$ to check.

$$x = y^{\frac{1}{2}}$$

$$x=\sqrt{\frac{1}{4}}$$

$$x=\frac{1}{2}$$

and

$$x=1$$

Since $x=1$ returns a larger value than $x=y^{\frac{1}{2}}$, we can say

$$g(y)=y^{\frac{1}{2}}$$

and

$$f(y)=1$$

Plugging everything we know into the volume formula, we get

$$V=\int_0^1 2\pi(1-y)\left(1-y^{\frac{1}{2}}\right)\ dy$$

$$V=2\pi\int_0^1 1-y^{\frac{1}{2}}-y+y^{\frac{3}{2}}\ dy$$

$$V=2\pi\left(y-\frac{2}{3}y^{\frac{3}{2}}-\frac{1}{2}y^2+\frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_0^1$$

$$V=2\pi\left[\left((1)-\frac{2}{3}(1)^{\frac{3}{2}}-\frac{1}{2}(1)^2+\frac{2}{5}(1)^{\frac{5}{2}}\right)-\left((0)-\frac{2}{3}(0)^{\frac{3}{2}}-\frac{1}{2}(0)^2+\frac{2}{5}(0)^{\frac{5}{2}}\right)\right]$$

$$V=2\pi\left(1-\frac{2}{3}-\frac{1}{2}+\frac{2}{5}\right)$$

$$V=2\pi\left(\frac{30-20-15+12}{30}\right)$$

$$V=2\pi\left(\frac{7}{30}\right)$$

$$V=\frac{7\pi}{15}$$

Topic: Cylindrical shells, horizontal axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^3$ and $x = 2$ and $y = 0$

about the line $y = 2$

Answer choices:

A $\dfrac{28\pi + 9\pi\sqrt[3]{2}}{7}$

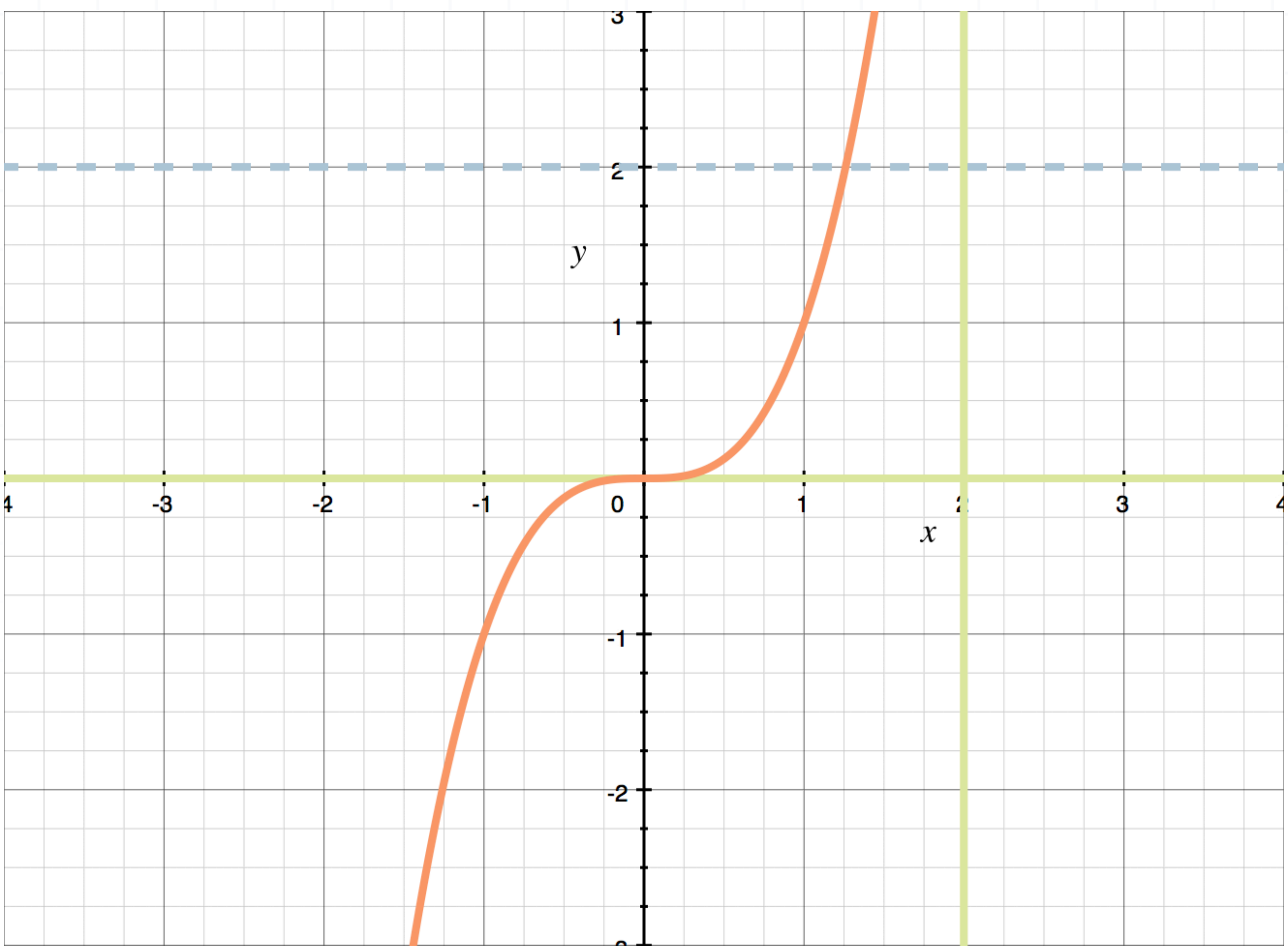
B $\dfrac{28\pi - 9\pi\sqrt[3]{2}}{7}$

C $\dfrac{56\pi + 18\pi\sqrt[3]{2}}{7}$

D $\dfrac{56\pi - 18\pi\sqrt[3]{2}}{7}$

Solution: D

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is horizontal, we'll take horizontal slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a horizontal axis, specifically $y = 2$, tells us that we'll use the volume formula

$$V = \int_c^d 2\pi(k - y)\left[f(y) - g(y)\right] \ dy$$

We can see from the formula that we need our curves and our limits of integration defined in terms of y. Our curve $x = 2$ is already defined for x in terms of y, so now we just need to define $y = x^3$ for x in terms of y.

$$y = x^3$$

$$y^{\frac{1}{3}} = \left(x^3\right)^{\frac{1}{3}}$$

$$y^{\frac{1}{3}} = x$$

$$x = y^{\frac{1}{3}}$$

Now we'll find the limits of integration, which will be the smallest and largest y-values for which the area is defined.

We can see from the graph that the smallest y-value for which the area is defined is $y = 0$. This was given in the original problem. Since the area overlaps the axis of rotation, we have to cut the area off at the axis and say that the largest y-value for which the area is defined is $y = 2$.

Now we know that our limits of integration are $c = 0$ and $d = 2$. Since the axis of revolution is $y = 2$, our radius is $2 - y$.

$f(y) - g(y)$ is the height of the approximating cylinder, which means we need to subtract the curve on the left $g(y)$ from the curve on the right $f(y)$. To figure out which curve is on the right and which is on the left, we can look at the graph or we can plug a y-value between the points of intersection (between $y = 0$ and $y = 2$) into both curves to see which function returns a larger value (this will be the right curve) and which one returns a smaller value (this will be the left curve). Let's plug in $y = 1$ to check.

$$x = y^{\frac{1}{3}}$$

$$x = (1)^{\frac{1}{3}}$$

$$x = 1$$

and

$$x = 2$$

Since $x = 2$ returns a larger value than $x = y^{\frac{1}{3}}$, we can say

$$g(y) = y^{\frac{1}{3}}$$

and

$$f(y) = 2$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^2 2\pi(2 - y)\left(2 - y^{\frac{1}{3}}\right)\ dy$$

$$V = 2\pi \int_0^2 4 - 2y^{\frac{1}{3}} - 2y + y^{\frac{4}{3}}\ dy$$

We'll integrate and then evaluate over the interval.

$$V = 2\pi\left(4y - \frac{6}{4}y^{\frac{4}{3}} - y^2 + \frac{3}{7}y^{\frac{7}{3}}\right)\Bigg|_0^2$$

$$V = 2\pi\left[\left(4(2) - \frac{6}{4}(2)^{\frac{4}{3}} - (2)^2 + \frac{3}{7}(2)^{\frac{7}{3}}\right) - \left(4(0) - \frac{6}{4}(0)^{\frac{4}{3}} - (0)^2 + \frac{3}{7}(0)^{\frac{7}{3}}\right)\right]$$

$$V = 2\pi\left[8 - \frac{3}{2}\left[(2)^4\right]^{\frac{1}{3}} - 4 + \frac{3}{7}\left[(2)^7\right]^{\frac{1}{3}}\right]$$

$$V = 2\pi\left[8 - \frac{3}{2}(16)^{\frac{1}{3}} - 4 + \frac{3}{7}(128)^{\frac{1}{3}}\right]$$

$$V = 2\pi\left[8 - \frac{3}{2}(8 \cdot 2)^{\frac{1}{3}} - 4 + \frac{3}{7}(64 \cdot 2)^{\frac{1}{3}}\right]$$

$$V = 2\pi\left[8 - \frac{3}{2}\left(8^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}\right) - 4 + \frac{3}{7}(8 \cdot 8 \cdot 2)^{\frac{1}{3}}\right]$$

$$V = 2\pi\left[8 - \frac{3}{2}\left(2 \cdot 2^{\frac{1}{3}}\right) - 4 + \frac{3}{7}\left(8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}\right)\right]$$

$$V = 2\pi\left[8 - 3\left(2^{\frac{1}{3}}\right) - 4 + \frac{3}{7}\left(2 \cdot 2 \cdot 2^{\frac{1}{3}}\right)\right]$$

$$V = 2\pi\left(8 - 3\sqrt[3]{2} - 4 + \frac{12}{7}\sqrt[3]{2}\right)$$

$$V = 2\pi\left(4 - 3\sqrt[3]{2} + \frac{12}{7}\sqrt[3]{2}\right)$$

$$V = 2\pi\left(\frac{28}{7} - \frac{21\sqrt[3]{2}}{7} + \frac{12\sqrt[3]{2}}{7}\right)$$

$$V = 2\pi\left(\frac{28 - 21\sqrt[3]{2} + 12\sqrt[3]{2}}{7}\right)$$

$$V = 2\pi\left(\frac{28 - 9\sqrt[3]{2}}{7}\right)$$

$$V = \frac{56\pi - 18\pi\sqrt[3]{2}}{7}$$

Topic: Cylindrical shells, vertical axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2 - 2x + 1$ and $y = -x^2 + 6x - 5$

about $x = -1$

Answer choices:

A 12π

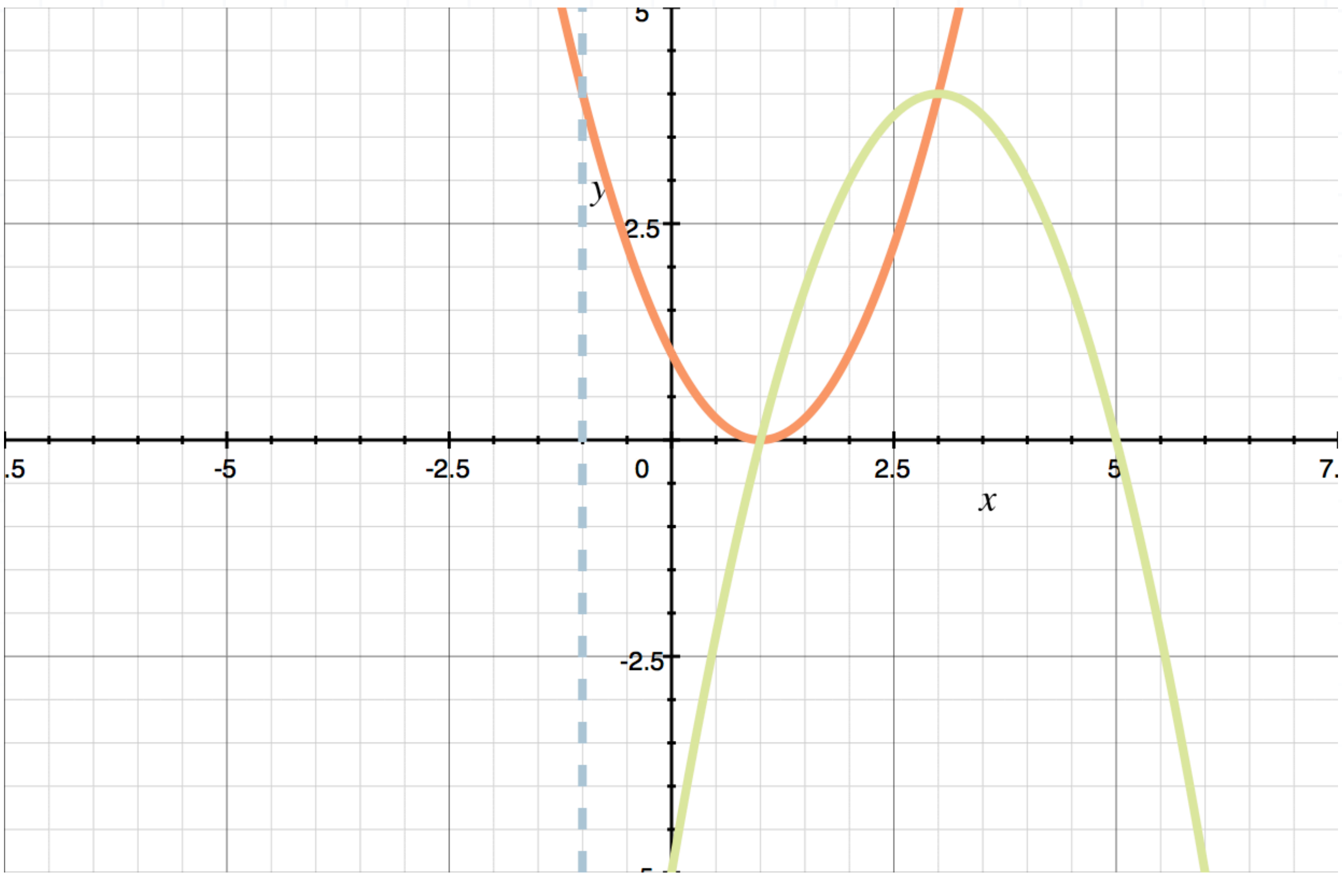
B 8π

C 16π

D 32π

Solution: C

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take vertical slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a vertical axis, specifically $x = -1$, tells us that we'll use the volume formula

$$V = \int_a^b 2\pi(x + k)\left[f(x) - g(x)\right]\ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined. Since these are just the two points of intersection, we can do this by looking at the graph, or we can set the curves equal to one another and solve for x.

$$x^2 - 2x + 1 = -x^2 + 6x - 5$$

$$2x^2 - 8x + 6 = 0$$

$$2\left(x^2 - 4x + 3\right) = 0$$

$$2(x-3)(x-1) = 0$$

$$x - 3 = 0$$

$$x = 3$$

or

$$x - 1 = 0$$

$$x = 1$$

Now we know that our limits of integration are $a = 1$ and $b = 3$. Since the axis of revolution is $x = -1$, our radius is $x + 1$.

$f(x) - g(x)$ is the height of the approximating cylinder, which means we need to subtract the lower curve $g(x)$ from the upper curve $f(x)$. To figure out which curve is above the other, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 1$ and

$x = 3$) into both curves to see which function returns a larger value (this will be the upper curve) and which one returns a smaller value (this will be the lower curve). Let's plug in $x = 2$ to check.

$$y = x^2 - 2x + 1$$

$$y = (2)^2 - 2(2) + 1$$

$$y = 1$$

and

$$y = -x^2 + 6x - 5$$

$$y = -(2)^2 + 6(2) - 5$$

$$y = 3$$

Since $y = -x^2 + 6x - 5$ returns a larger value than $y = x^2 - 2x + 1$, we can say

$$g(x) = x^2 - 2x + 1$$

and

$$f(x) = -x^2 + 6x - 5$$

Plugging everything we know into the volume formula, we get

$$V = \int_1^3 2\pi(x+1)\left[-x^2 + 6x - 5 - \left(x^2 - 2x + 1\right)\right]\ dx$$

$$V = 2\pi \int_1^3 (x+1)\left(-2x^2 + 8x - 6\right)\ dx$$

$$V = 2\pi \int_1^3 -2x^3 + 8x^2 - 6x - 2x^2 + 8x - 6 \ dx$$

$$V = 2\pi \int_1^3 -2x^3 + 6x^2 + 2x - 6 \ dx$$

Integrate and then evaluate over the interval.

$$V = 2\pi \left(-\frac{2}{4}x^4 + \frac{6}{3}x^3 + x^2 - 6x \right) \Bigg|_1^3$$

$$V = 2\pi \left(-\frac{1}{2}x^4 + 2x^3 + x^2 - 6x \right) \Bigg|_1^3$$

$$V = 2\pi \left[\left(-\frac{1}{2}x^4 + 2x^3 + x^2 - 6x \right) - \left(-\frac{1}{2}x^4 + 2x^3 + x^2 - 6x \right) \right] \Bigg|_1^3$$

$$V = 2\pi \left[\left(-\frac{1}{2}(3)^4 + 2(3)^3 + (3)^2 - 6(3) \right) - \left(-\frac{1}{2}(1)^4 + 2(1)^3 + (1)^2 - 6(1) \right) \right]$$

$$V = 2\pi \left[\left(-\frac{81}{2} + 54 + 9 - 18 \right) - \left(-\frac{1}{2} + 2 + 1 - 6 \right) \right]$$

$$V = 2\pi \left(-\frac{81}{2} + 54 + 9 - 18 + \frac{1}{2} - 2 - 1 + 6 \right)$$

$$V = 2\pi \left(-\frac{80}{2} + 48 \right)$$

$$V = 2\pi (8)$$

$$V = 16\pi$$

Topic: Cylindrical shells, vertical axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = 6 - \frac{2}{3}x^2$ and $y = 0$ and $x = 0$

for $x \geq 0$

about $x = 3$

Answer choices:

A 45π

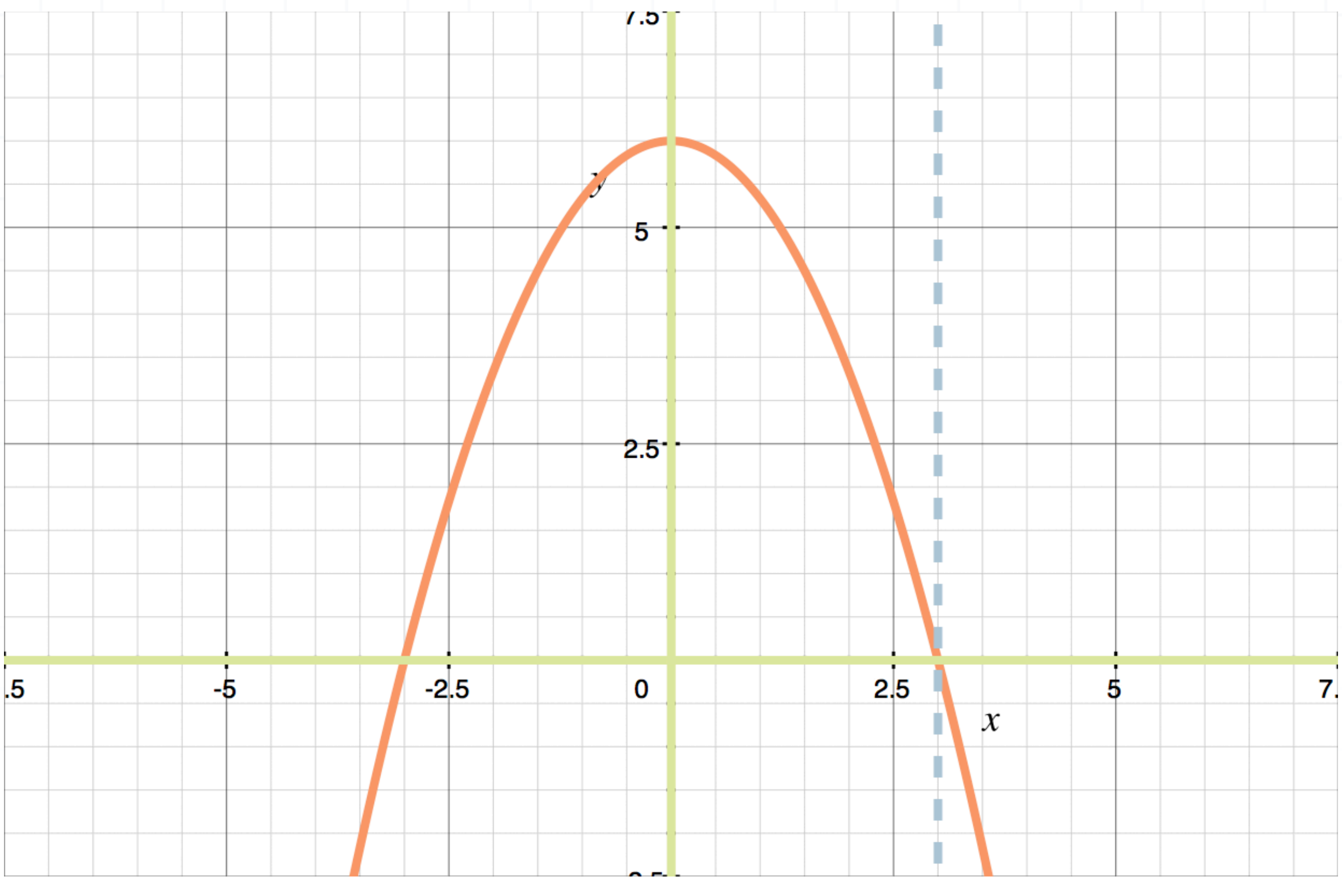
B $\frac{342\pi}{5}$

C 57π

D $\frac{57\pi}{3}$

Solution: A

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take vertical slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a vertical axis, specifically $x = 3$, tells us that we'll use the volume formula

$$V = \int_a^b 2\pi(k - x)\left[f(x) - g(x)\right] \ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined.

We can see from the graph that the smallest x-value for which the area is defined is $x = 0$. This was given in the original problem. We can see that the largest value for which it's defined is a point of intersection, so we can set the curves equal to one another and solve for x.

$$6 - \frac{2}{3}x^2 = 0$$

$$\frac{2}{3}x^2 = 6$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Since we were told that the region is defined by $x \geq 0$, and since we already know that it's bounded on one side by the line $x = 0$, we know that we can discard the solution $x = -3$ and that our limits of integration are $a = 0$ and $b = 3$. Since the axis of revolution is $x = 3$, our radius is $3 - x$.

$f(x) - g(x)$ is the height of the approximating cylinder, which means we need to subtract the lower curve $g(x)$ from the upper curve $f(x)$. To figure out which curve is above the other, we can look at the graph or we can plug an x-value between the points of intersection (between $x = 0$ and $x = 3$) into both curves to see which function returns a larger value (this will be the upper curve) and which one returns a smaller value (this will be the lower curve). Let's plug in $x = 1$ to check.

$$y = 6 - \frac{2}{3}x^2$$

$$y = 6 - \frac{2}{3}(1)^2$$

$$y = \frac{16}{3}$$

and

$$y = 0$$

Since $y = 6 - (2/3)x^2$ returns a larger value than $y = 0$, we can say

$$f(x) = 6 - \frac{2}{3}x^2$$

and

$$g(x) = 0$$

Plugging everything we know into the volume formula, we get

$$V = \int_0^3 2\pi(3 - x)\left(6 - \frac{2}{3}x^2 - 0\right)\ dx$$

$$V = 2\pi \int_0^3 18 - 2x^2 - 6x + \frac{2}{3}x^3\ dx$$

Integrate and then evaluate over the interval.

$$V = 2\pi \left[18x - \frac{2}{3}x^3 - \frac{6}{2}x^2 + \frac{2}{12}x^4 \right] \Bigg|_0^3$$

$$V = 2\pi \left[\left(18(3) - \frac{2}{3}(3)^3 - \frac{6}{2}(3)^2 + \frac{2}{12}(3)^4 \right) - \left(18(0) - \frac{2}{3}(0)^3 - \frac{6}{2}(0)^2 + \frac{2}{12}(0)^4 \right) \right]$$

$$V = 2\pi \left(54 - 18 - 27 + \frac{27}{2} \right)$$

$$V = 2\pi \left(9 + \frac{27}{2} \right)$$

$$V = 2\pi \left(\frac{18}{2} + \frac{27}{2} \right)$$

$$V = 2\pi \left(\frac{45}{2} \right)$$

$$V = 45\pi$$

Topic: Cylindrical shells, vertical axis

Question: Use cylindrical shells to find the volume of the solid generated by revolving the region bounded by the curves about the given axis.

$y = x^2$ and $y = x + 2$

about the line $x + 2 = 0$

Answer choices:

A $\frac{45\pi}{6}$

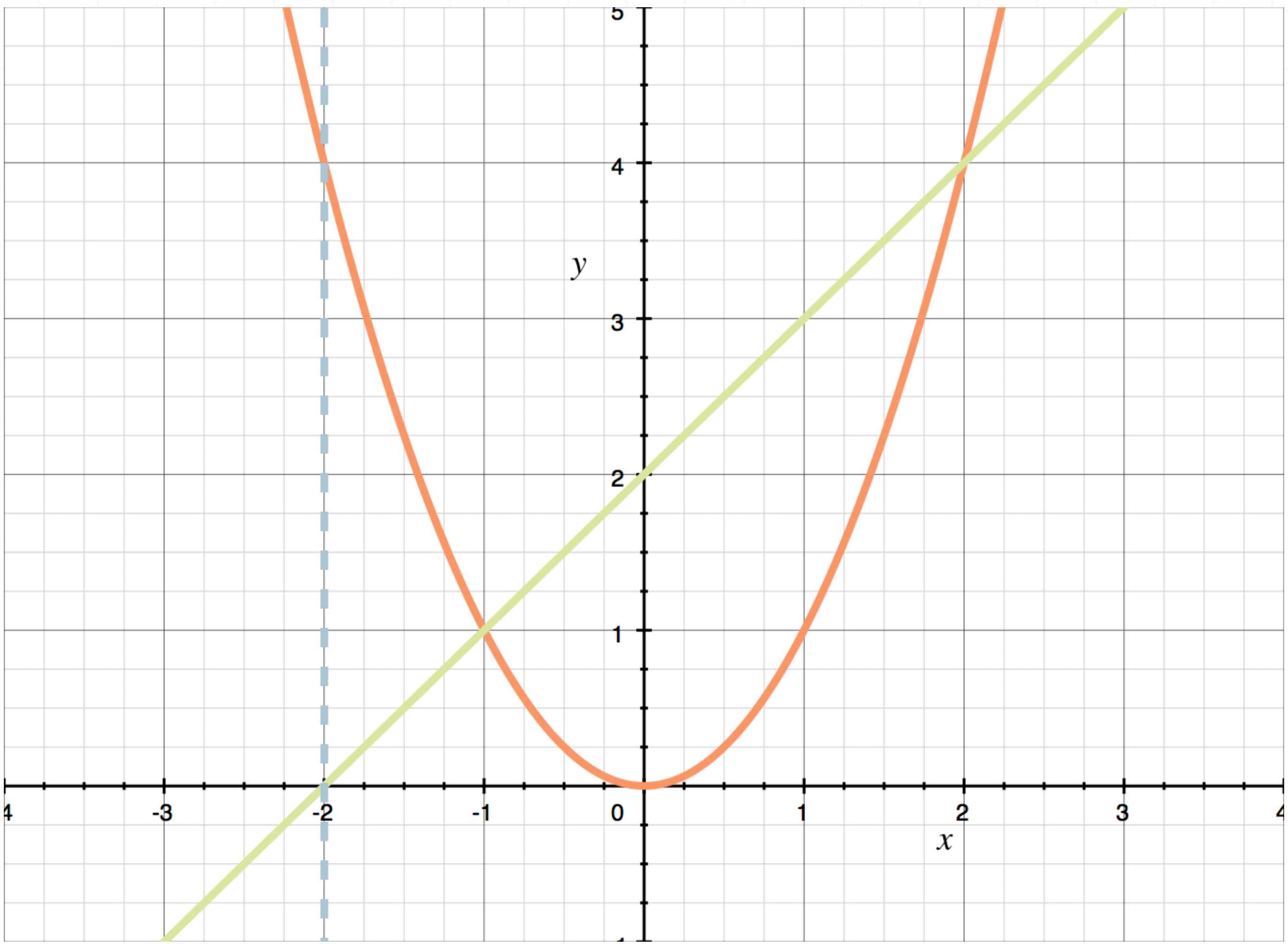
B $\frac{199\pi}{6}$

C $\frac{199\pi}{2}$

D $\frac{45\pi}{2}$

Solution: D

Before doing anything else, we always want to draw a picture of the area. If you don't know how to graph the function, just plug in values for x or y to get individual coordinate points, and plot them until you have a picture of each function.



Using cylindrical shells means we'll take slices of our area that are parallel to the axis of rotation. Therefore, since the axis of rotation is vertical, we'll take vertical slices of our area and rotate each of them around the axis to form cylindrical shells.

Using cylindrical shells around a vertical axis, specifically $x+2=0$, or $x=-2$, tells us that we'll use the volume formula

$$V=\int_a^b 2\pi(x+k)\left[f(x)-g(x)\right]\ dx$$

We can see from the formula that we need our curves and our limits of integration defined in terms of x. The given curves are already defined for y in terms of x, so now we just need to find limits of integration, which will be the smallest and largest x-values for which the area is defined. Since these are just the two points of intersection, we can do this by looking at the graph, or we can set the curves equal to one another and solve for x.

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x - 2 = 0$$

$$x = 2$$

or

$$x + 1 = 0$$

$$x = -1$$

Now we know that our limits of integration are $a = -1$ and $b = 2$. Since the axis of revolution is $x + 2 = 0$, or $x = -2$, our radius is $x + 2$.

$f(x) - g(x)$ is the height of the approximating cylinder, which means we need to subtract the lower curve $g(x)$ from the upper curve $f(x)$. To figure out which curve is above the other, we can look at the graph or we can plug an x-value between the points of intersection (between $x = -1$ and $x = 2$) into both curves to see which function returns a larger value (this will be the upper curve) and which one returns a smaller value (this will be the lower curve). Let's plug in $x = 0$ to check.

$$y = x^2$$

$$y = (0)^2$$

$$y = 0$$

and

$$y = x + 2$$

$$y = 0 + 2$$

$$y = 2$$

Since $y = x + 2$ returns a larger value than $y = x^2$, we can say

$$g(x) = x^2$$

and

$$f(x) = x + 2$$

Plugging everything we know into the volume formula, we get

$$V = \int_{-1}^{2} 2\pi(x+2)\left(x + 2 - x^2\right) \ dx$$

$$V = 2\pi \int_{-1}^{2} x^2 + 2x - x^3 + 2x + 4 - 2x^2 \ dx$$

$$V = 2\pi \int_{-1}^{2} 4 + 4x - x^2 - x^3 \ dx$$

Integrate and then evaluate over the interval.

$$V = 2\pi \left[4x + \frac{4}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right] \Bigg|_{-1}^{2}$$

$$V = 2\pi \left[\left(4(2) + \frac{4}{2}(2)^2 - \frac{1}{3}(2)^3 - \frac{1}{4}(2)^4 \right) - \left(4(-1) + \frac{4}{2}(-1)^2 - \frac{1}{3}(-1)^3 - \frac{1}{4}(-1)^4 \right) \right]$$

$$V = 2\pi \left[\left(8 + 8 - \frac{8}{3} - 4 \right) - \left(-4 + 2 + \frac{1}{3} - \frac{1}{4} \right) \right]$$

$$V = 2\pi \left(12 - \frac{8}{3} + 2 - \frac{1}{3} + \frac{1}{4} \right)$$

$$V = 2\pi \left(11 + \frac{1}{4} \right)$$

$$V = 2\pi \left(\frac{44}{4} + \frac{1}{4} \right)$$

$$V = \frac{45\pi}{2}$$

Topic: Work done to lift a weight or mass

Question: Find the work required.

A cable weighing 2 lbs/ft pulls a 200 pound load up a shaft that is 500 feet deep.

Answer choices:

A $35{,}000$ ft-lbs

B $330{,}000$ ft-lbs

C $350{,}000$ ft-lbs

D $340{,}000$ ft-lbs

Solution: C

We have to calculate the work required to lift the load, then calculate the work required to lift the rope itself, and then add them together to get total work required.

To calculate the work required to lift the load, we'll use $W = Fd$, where W is work, F is force, and d is distance. Normally to find force, we would take the product of the mass and the gravitational constant, but since we're given a weight instead of a mass, the gravitational constant is already included and we can just use the load's weight of 200 lbs as the force.

$$W_L = 200 \text{ lbs} \cdot 500 \text{ ft}$$

$$W_L = 100{,}000 \text{ ft-lbs}$$

To calculate the work required to lift the rope, we'll divide the rope into cross sections. We have to multiply the height of the cross section, Δx, by the weight, 2 lbs, by the distance between the cross section and the top of the shaft, x. Putting this into an integral to find the work required to lift the entire rope, we get

$$W_R = \int_0^{500} 2x \ dx$$

$$W_R = x^2 \Big|_0^{500}$$

$$W_R = 500^2 - 0^2$$

$$W_R = 250{,}000 \text{ ft-lbs}$$

Adding them together, we get

$$W = W_L + W_R$$

$$W = 100{,}000 + 250{,}000$$

$$W = 350{,}000 \text{ ft-lbs}$$

Topic: Work done to lift a weight or mass

Question: As a container moves along an assembly line, it's filled with material so the force to keep it moving is given in pounds by $f(x)$. Find the work needed to move the container from $x = 0$ to $x = 8$ feet.

$$f(x) = 6 + \frac{1}{4}x^3$$

Answer choices:

A $W = 304$ pounds

B $W = 320$ feet

C $W = 2{,}112$ foot-pounds

D $W = 304$ foot-pounds

Solution: D

To find the amount of work required, we use the integral formula for work. We've been told that the container moves from $x = 0$ to $x = 8$, so those become the limits on the integral. We plug the given equation into the integral as the integrand, and integrate with respect to x. So the work required is given as

$$W = \int_0^8 6 + \frac{1}{4}x^3 \ dx$$

Integrate, and then evaluate over the interval.

$$W = 6x + \frac{1}{16}x^4 \Big|_0^8$$

$$W = 6(8) + \frac{1}{16}(8)^4 - \left(6(0) + \frac{1}{16}(0)^4\right)$$

$$W = 48 + 256$$

$$W = 304$$

Recall the force was provided in pounds and the distance was provided in feet. Therefore, the work required to move the container 8 feet is 304 foot-pounds.

Topic: Work done to lift a weight or mass

Question: Two objects move along a path from $x = 4$ to $x = 9$ feet. The first object needs a constant force of 25 pounds to move and the other object needs a variable force given by $f(x)$ to move. Find the total work required to move the two objects.

$$f(x) = \frac{250}{x}$$

Answer choices:

A $\quad W = 750$ foot-pounds

B $\quad W = 125 + 250 \ln \frac{9}{4}$ foot-pounds

C $\quad W = 125 + 250 \ln \frac{9}{4}$ pounds

D $\quad W = 250 + \ln \frac{9}{4}$ feet

Solution: B

Since we are moving two objects, we can find total force required to move them by adding together the force required to move each object individually.

The first object needs a constant force of 25 pounds, so $W_1 = 25$. The second object needs a variable force, given in the problem by $f(x)$. Therefore, total force required is

$$F = 25 + \frac{250}{x}$$

The integration limits represent the distance the container is moved along the assembly line, and since we're moving both objects from $x = 4$ to $x = 9$, the work required to move both objects is given by

$$W = \int_4^9 25 + \frac{250}{x} \ dx$$

Integrate, then evaluate over the interval.

$$W = 25x + 250 \ln|x| \Big|_4^9$$

$$W = 25(9) + 250 \ln|9| - \left(25(4) + 250 \ln|4|\right)$$

$$W = 225 + 250 \ln 9 - 100 - 250 \ln 4$$

$$W = 125 + 250 \left(\ln 9 - \ln 4\right)$$

$$W = 125 + 250 \ln \frac{9}{4}$$

Topic: Work done on elastic springs

Question: Find the work required to stretch the spring by 2 feet from its natural length.

The force required to stretch a spring s feet is given by $F(s) = 11s$ lbs.

Answer choices:

A 11 ft-lbs

B 12 ft-lbs

C 21 ft-lbs

D 22 ft-lbs

Solution: D

Using Hooke's Law, $F(s) = ks$, we get

$$11s = ks$$

$$k = 11$$

The work done in stretching the spring 2 feet can be calculated using

$$W = \int_{a}^{b} F(s)\ ds$$

When the spring is at its natural length, then $a = 0$, and when it's stretched by 2 feet, $b = 2$.

$$W = \int_{0}^{2} 11s\ ds$$

$$W = 11\left(\frac{1}{2}s^2\right)\Big|_0^2$$

$$W = \left[\frac{11}{2}(2)^2\right] - \left[\frac{11}{2}(0)^2\right]$$

$$W = \frac{44}{2}$$

$$W = 22 \text{ ft-lbs}$$

Topic: Work done on elastic springs

Question: Find the work required to compress the spring from its natural length to 6 feet.

A force of 200 lbs is required to keep a spring compressed to 8 feet, when its natural length is 10 feet.

Answer choices:

A 200 ft-lbs

B 400 ft-lbs

C 600 ft-lbs

D 800 ft-lbs

Solution: D

We'll first determine the spring constant using Hooke's Law.

$$k = \frac{F(x)}{x}$$

$$k = \frac{200}{10 - 8}$$

$$k = 100$$

The work done to compress the spring can be calculated using

$$W = \int_a^b F(x)\ dx$$

When the spring is at its natural length, $a = 0$, and when it's compressed to 6 feet, or by 4 feet from its natural length, $b = 4$.

$$W = \int_0^4 100x\ dx$$

$$W = \frac{100}{2}x^2\Big|_0^4$$

$$W = 50x^2\Big|_0^4$$

$$W = 50(4)^2 - 50(0)^2$$

$$W = 800 \text{ ft-lbs}$$

The work done in compressing the spring from its natural length of 10 feet to a length of 6 feet is 800 ft-lbs.

Topic: Work done on elastic springs

Question: Find the value of the spring constant and the work done to stretch the spring 1/2 foot beyond its natural length.

A force of 8 lbs is required to keep a spring stretched $1/2$ foot beyond its natural length.

Answer choices:

A 16 lbs/ft and 2 ft-lbs

B 12 lbs/ft and 4 ft-lbs

C 8 lbs/ft and 6 ft-lbs

D 20 lbs/ft and 8 ft-lbs

Solution: A

To determine the spring constant k, we'll use Hooke's Law, which states that the force $F(x)$ necessary to keep a spring stretched (or compressed) by x units beyond (or short of) its natural length, is given by $F(x) = kx$. For the given spring where $F(x) = 8$ lbs and $x = 0.5$ ft, we'll solve for k by substituting these values into Hooke's Law.

$$k = \frac{F(x)}{x}$$

$$k = \frac{8}{0.5}$$

$$k = 16$$

The spring constant is 16 lbs/ft.

The work done to stretch the spring $1/2$ foot beyond its natural length can be calculated using

$$W = \int_a^b F(x)\ dx$$

When the spring is at its natural length, $a = 0$ and when it's stretched $1/2$ foot beyond its natural length, $b = 1/2$.

$$W = \int_0^{\frac{1}{2}} 16x\ dx$$

$$W = \frac{16}{2}x^2 \Big|_0^{\frac{1}{2}}$$

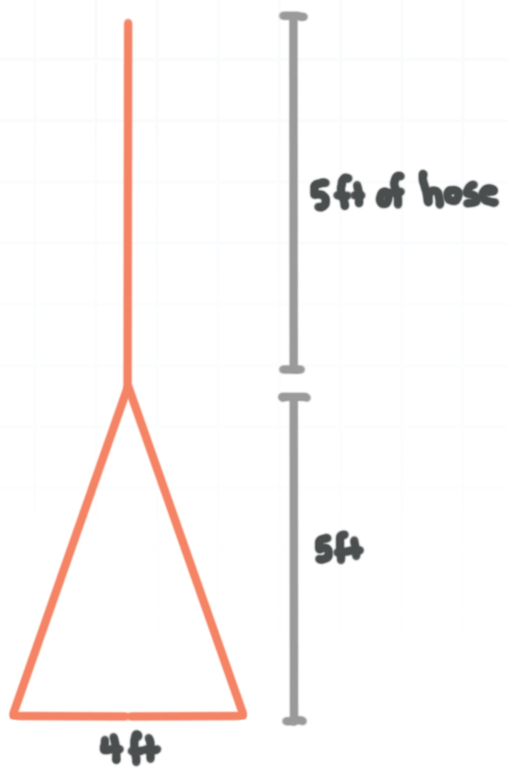
$$W = 8\left(\frac{1}{2}\right)^2 - 8(0)^2$$

$$W = 2$$

The work done in stretching the spring $1/2$ foot beyond its natural length is 2 ft-lbs.

Topic: Work done to empty a tank

Question: A triangular water tank that's 4 feet wide, 5 feet tall, and 10 feet long is completely full of water and needs to be emptied by pumping the water through a hose to a height of 5 feet above the top of the tank. Find the work required to empty the tank.



Answer choices:

A $52,000$ ft-lbs

B $68,000$ ft-lbs

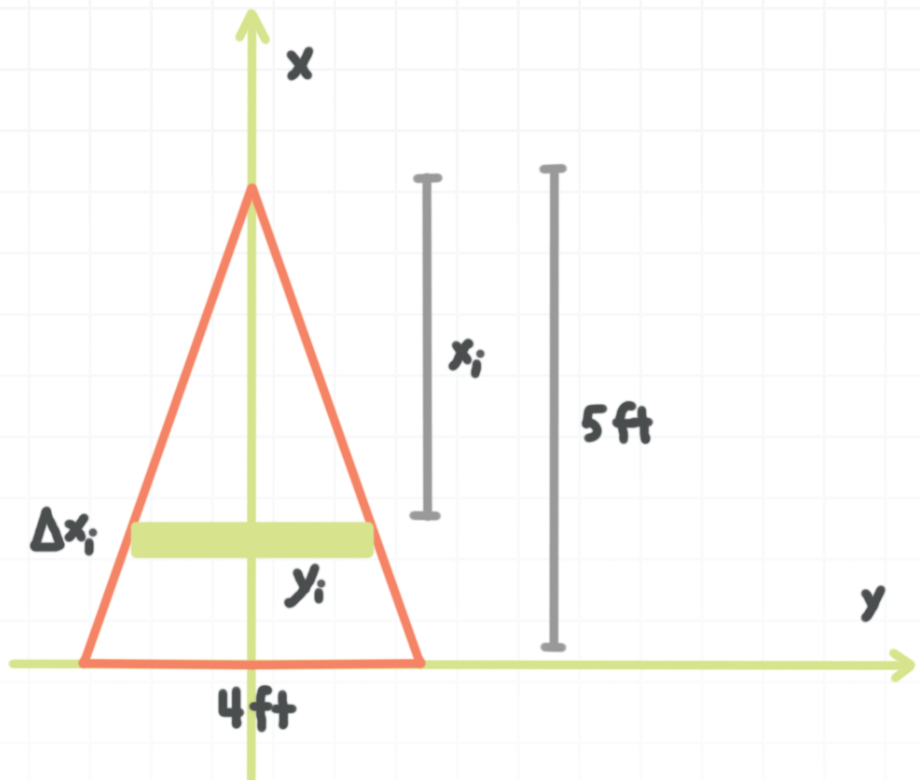
C $81,000$ ft-lbs

D $96,000$ ft-lbs

Solution: A

We can place the tank in a coordinate system, with the center of the base at the origin, and the top of the tank extending out along the x-axis. Notice that we've flipped the axes to accomplish this. The reason we sketch it this way is so that we can keep the whole problem in terms of x, instead of y.

If we divide the water in the tank into cross sections and we sketch in one cross section (which we'll call the ith cross section, where "ith" is just representing the 1st, 2nd, 3rd, 4th, cross section, etc.) then we can label the height of the cross section along the x-axis as Δx_i, the width of the cross section as y_i, and the distance between the cross section and the top of the tank as x_i.



We know that the formula for work is $W = Fd$, where W is the work done to pump out the water, F is the force required to lift the water, and d is distance traveled by the water to get out of the tank.

Since the measurements of the tank were given in feet, we'll be solving for work in terms of foot-pounds (ft-lbs), which means we need to look at the weight of the water in terms of pounds. "Weight" already accounts for gravity, so we don't need to factor gravity into our calculations.

Starting from the equation for work $W = Fd$, we can say that force F is the same as weight, and we know that weight is the product of density and volume. The weight of water in terms of feet and pounds is

$$\rho = 62.4 \ \frac{\text{lb}}{\text{ft}^3}$$

We said that weight would be the product of density and volume, and we just gave density, so now we need to find the volume of the ith cross section. Since the cross section is a rectangular slice and the tank is 10 feet long, we can say

$$V = lwh$$

$$V = 10 \cdot y_i \cdot \Delta x_i$$

We want to do this whole problem in terms of x, which means we'll need to solve for a value of y_i that's in terms of x_i, and then make a substitution into our volume equation. To do that, we'll use similar triangles and relate the dimensions of the triangular tank to the dimensions of the triangle whose base is the cross section of water.

$$\frac{\text{base of the big triangle}}{\text{height of the big triangle}} = \frac{\text{base of the little triangle}}{\text{height of the little triangle}}$$

$$\frac{4}{5} = \frac{y_i}{x_i}$$

$$y_i = \frac{4}{5}x_i$$

Now we can substitute into the volume equation for y_i.

$$V = 10 \cdot y_i \cdot \Delta x_i$$

$$V = 10\left(\frac{4}{5}x_i\right)\Delta x_i$$

$$V = 8x_i\Delta x_i$$

Now that we have volume, and we know that weight is the product of volume and density, we can say that the weight of the ith cross section is

$$\text{weight}_i = 62.4(8x_i\Delta x_i)$$

$$\text{weight}_i = 499.2x_i\Delta x_i$$

Because weight already accounts for gravity, we can make a direct substitution into the work equation for force F.

$$W = Fd$$

$$W_i = (499.2x_i\Delta x_i)d_i$$

The ith cross section must move a distance of x_i to get to the top of the tank. In addition, we know that all of the water has to be pumped 5 feet above the top of the tank, which means that the ith cross section has to travel a distance of $d_i = x_i + 5$.

Therefore,

$$W_i = (499.2x_i\Delta x_i)(x_i + 5)$$

$$W_i = (499.2x_i)(x_i + 5)\Delta x_i$$

$$W_i = 499.2(x_i)(x_i + 5)\Delta x_i$$

$$W_i = 499.2(x_i^2 + 5x_i)\Delta x_i$$

This is the equation that represents the work required to pump the water in the ith cross section out of the tank. But of course we're not just interested in this *one* cross section. We want an equation that will give us the work required to pump *all* of the water out of the tank.

The way that we do this is by pretending that we have an infinite number of cross sections, each of which is infinitely thin. We'll use a Riemann sum to add up the work required to pump out an infinite number n of infinitely thin cross sections, and we'll get

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 499.2(x_i^2 + 5x_i)\Delta x_i$$

Now we'll translate this Riemann sum into integral notation.

$$W = \int 499.2(x^2 + 5x)\ dx$$

$$W = 499.2 \int x^2 + 5x\ dx$$

We're integrating with respect to x, and we've set up our diagram so that x represents the distance from the top of the tank to any particular slice of water. Therefore, the limits of integration need to be $[0,5]$, since the water at the top of the tank has a distance of 0 ft from the top of the tank, while the water at the bottom of the tank has a distance of 5 ft from the top of the tank.

$$W = 499.2 \int_0^5 x^2 + 5x\ dx$$

$$W = 499.2 \left(\frac{1}{3}x^3 + \frac{5}{2}x^2 \right) \Bigg|_0^5$$

$$W = 499.2 \left[\left(\frac{1}{3}(5)^3 + \frac{5}{2}(5)^2 \right) - \left(\frac{1}{3}(0)^3 + \frac{5}{2}(0)^2 \right) \right]$$

$$W = 499.2\left(\frac{125}{3} + \frac{125}{2}\right)$$

$$W = 499.2\left(\frac{250}{6} + \frac{375}{6}\right)$$

$$W = 499.2\left(\frac{625}{6}\right)$$

$W = 52{,}000$ ft-lbs

Topic: Work done to empty a tank

Question: Find the work required to empty the tank.

A cylindrical tank standing on end, with base radius 5 feet and height 10 feet, is full of oil and must be emptied by pumping the oil over the edge of the tank. Assume oil density is

$$\rho = 50 \ \frac{\text{lb}}{\text{ft}^3}$$

Answer choices:

A $62{,}400\pi$ ft-lbs

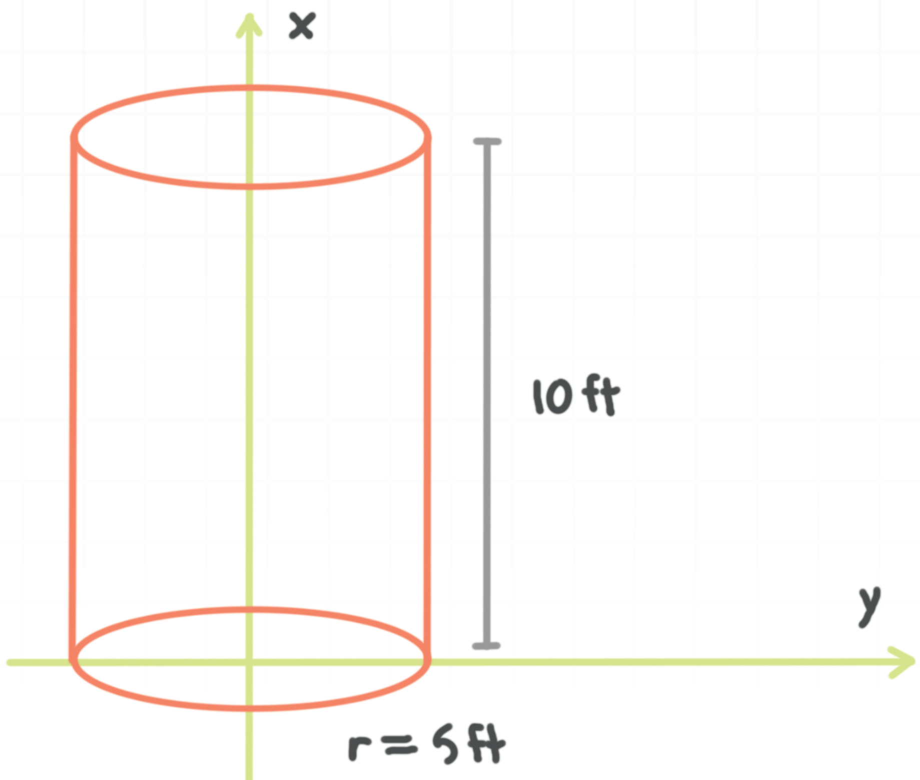
B $62{,}500\pi$ ft-lbs

C $60{,}400$ ft-lbs

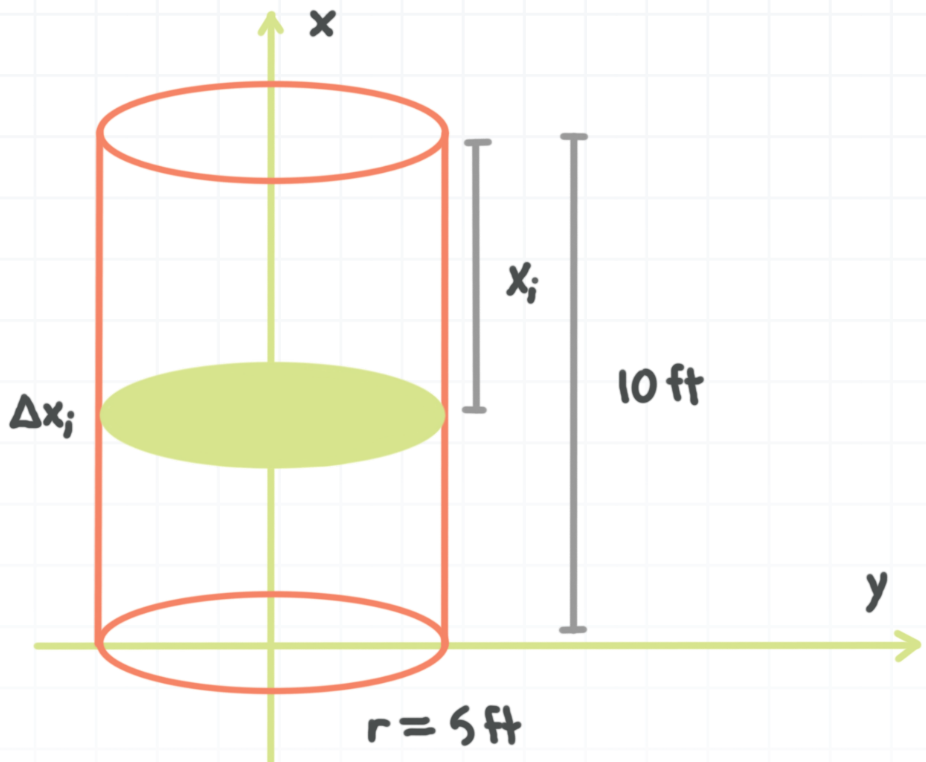
D $60{,}500$ ft-lbs

Solution: B

We can place the tank in a coordinate system, with the center of the base of the tank at the origin, and the top of the tank extending out along the x -axis. Notice that we've flipped the axes to accomplish this. The reason we sketch it this way is so that we can keep the whole problem in terms of x, instead of y.



If we divide the water in the tank into cross sections and we sketch in one cross section (which we'll call the ith cross section, where "ith" is just representing the 1st, 2nd, 3rd, 4th, cross section, etc.) then we can label the height of the cross section along the x-axis as Δx_i and the distance between the cross section and the top of the tank as x_i.



We know that the formula for work is $W = Fd$, where W is the work done to pump out the water, F is the force required to lift the water, and d is distance traveled by the water to get out of the tank.

Since the measurements of the tank were given in feet, we'll be solving for work in terms of foot-pounds (ft-lbs), which means we need to look at the weight of the water in terms of pounds. "Weight" already accounts for gravity, so we don't need to factor gravity into our calculations.

Remember that if the measurements of the tank had been given in meters instead, we'd be looking for a value for work in terms of Newtons, and we'd therefore need to look at the mass of the water in terms of kilograms, and we'd need to factor in gravity separately.

Starting from the equation for work $W = Fd$, we can say that force F is the same as weight, and we know that weight is the product of density and volume. The weight of the oil in terms of feet and pounds was given as

$$\rho = 50 \ \frac{\text{lb}}{\text{ft}^3}$$

Now we need to find the volume of the ith cross section. Since the cross section is a circular slice, and since the tank is 10 feet tall, we can say

$$V = \pi r^2 \cdot \Delta x_i$$

$$V = \pi (5)^2 \cdot \Delta x_i$$

$$V = 25\pi \cdot \Delta x_i$$

Now that we have volume, and we know that weight is the product of volume and density, we can say that the weight of the ith cross section is

$$\text{weight}_i = 50 \left(25\pi \Delta x_i\right)$$

$$\text{weight}_i = 1{,}250\pi \Delta x_i$$

Because weight already accounts for gravity, we can make a direct substitution of weight into the work equation for force F.

$$W = Fd$$

$$W_i = \left(1{,}250\pi \Delta x_i\right) d_i$$

According to the diagram we drew earlier, we know that the distance the i th cross section must move to get to the top of the tank is x_i, which means that the ith cross section has to travel a distance of $d_i = x_i$.

Therefore,

$$W_i = \left(1{,}250\pi \Delta x_i\right)(x_i)$$

Δx plays a special role here, so if it isn't already, we want to factor it out and move it to the end of the equation. We'll also simplify the equation as much as we can.

$$W_i = 1{,}250\pi \left(x_i\right) \Delta x_i$$

This is the equation that represents the work required to pump the water in the ith cross section out of the tank. But of course we're not just interested in this *one* cross section. We want an equation that will give us the work required to pump *all* of the water out of the tank.

The way that we do this is by pretending that we have an infinite number of cross sections, each of which is infinitely thin. We'll use a Riemann sum to add up the work required to pump out an infinite number n of infinitely thin cross sections, and we'll get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1{,}250\pi \left(x_i\right) \Delta x_i$$

The easy way to think about this is that you're just taking the equation for W_i and putting

$$W = \lim_{n\to\infty} \sum_{i=1}^{n}$$

in front of it. Then we realize that the summation notation becomes integral notation, and that Δx_i becomes dx. When we transition from summation notation to integral notation, all of the other x variables lose their $_i$ as well.

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}1{,}250\pi\left(x_i\right)\Delta x_i$$

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}\left[1{,}250\pi\left(x_i\right)\right]\Delta x_i$$

$$W=\int 1{,}250\pi\,(x)\ dx$$

$$W=1{,}250\pi\int x\ dx$$

Notice that all we really did here was take the equation for W_i, put it inside an integral, change Δx_i to dx and get rid of all the $_i$s. If it's easier, you can just skip the whole part with the summation notation and to straight to the integral once you find W_i.

Now we just need limits of integration for the integral. The limits of integration have to represent the distance the water has to travel to get out of the tank. If we refer back to our diagram, we can see that the water at the bottom of the tank has to travel 10 feet to get from the bottom of the tank to the top. On the other hand, the water at the top of the tank has to travel 0 feet. So the shortest distance the water travels is 0 feet; the longest distance the water travels is 10 feet. Therefore the interval we'll use for the limits of integration is $[0,10]$.

$$W=1{,}250\pi\int_0^{10}x\ dx$$

$$W=1{,}250\pi\left(\frac{1}{2}x^2\right)\Bigg|_0^{10}$$

Evaluating over the interval gives

$$W = 1{,}250\pi\left[\left(\frac{1}{2}(10)^2\right) - \left(\frac{1}{2}(0)^2\right)\right]$$

$$W = 1{,}250\pi(50 - 0)$$

$W = 62{,}500\pi$ ft-lbs

Topic: Work done to empty a tank

Question: A cylindrical tank has a height of 8 meters and a radius of 2 meters. The depth of the water in the tank is 6 meters. Find the work required to pump the water up to the level of the top of the tank and out of the tank. Use $1{,}000$ kg/m^3 for the density of the water and 9.8 m/sec^2 for acceleration due to gravity.

Answer choices:

A $W = 1{,}176{,}000$ Joules

B $W = 1{,}176{,}000\pi$ foot-pounds

C $W = 1{,}176{,}000\pi$ Joules

D $W = 1{,}176{,}000$ foot-pounds

Solution: C

We find work by integrating force times distance. Force includes the density of the water times the acceleration due to gravity times the volume of water being pumped. Distance is the distance the water is being pumped.

The formula for finding the work in this problem is

$$W = \int_a^b pgA(y)D(y)\ dy$$

In this formula,

1. W means work.

2. p is the density of the water being pumped. In this problem, the density of the water is $1{,}000$ kg/m^3.

3. g is the acceleration due to gravity. The problem states that acceleration due to gravity is 9.8 m/sec^2.

4. $A(y)$ is the surface area of the water being pumped. The problem states that the tank is shaped like a cylinder, with a radius of 2 meters, so $A(y) = \pi r^2 = \pi(2)^2 = 4\pi$.

5. $D(y)$ is the distance the water is being pumped. The problem states that the tank is 8 meters tall, and filled to a depth of 6 meters. The top layer of water needs to be moved up 2 meters, and the bottom layer of water needs to be moved up 8 meters, so the limits of integration are $[2,8]$.

6. dy is the thickness of the slice of the cross section of the water being pumped, and includes the variable of integration.

The integration limits are found using the beginning depth of the water (6 meters) and the ending depth of the water (0 meters).

Plug everything into the integral.

$$W=\int_a^b pgA(y)D(y)\ dy$$

$$W=\int_0^6 (1{,}000\text{ kg/m}^3)(9.8\text{ m/sec}^2)(4\pi\text{ m}^2)\left[(8-y)\text{ m}\right]\left(dy\text{ m}\right)$$

$$W=39{,}200\pi\frac{\text{kg m m}^2\text{ m m}}{\text{m}^3\text{ sec}^2}\int_0^6 8-y\ dy$$

$$W=39{,}200\pi\frac{\text{kg m}^2}{\text{sec}^2}\int_0^6 8-y\ dy$$

$$W=39{,}200\pi\text{ J}\int_0^6 8-y\ dy$$

Integrate, then evaluate over the interval.

$$W=39{,}200\pi\text{ J}\left(8y-\frac{1}{2}y^2\right)\Big|_0^6$$

$$W=39{,}200\pi\text{ J}\left(8(6)-\frac{1}{2}(6)^2\right)-39{,}200\pi\text{ J}\left(8(0)-\frac{1}{2}(0)^2\right)$$

$$W=39{,}200\pi\text{ J}(48-18)$$

$$W = 39{,}200\pi \text{ J}(30)$$

$$W = 1{,}176{,}000\pi \text{ J}$$

Topic: Work done by a variable force

Question: Calculate the variable force done over the interval with the given force equation.

$$F(x) = x^2$$

on the interval $[1,3]$

Answer choices:

A $\frac{28}{3}$

B $\frac{26}{3}$

C 3

D 8

Solution: B

To find the work done by a variable force, we use the work formula

$$W=\int_a^b F(x)\ dx$$

where $F(x)$ is the variable force equation, $[a,b]$ is the given interval and W is the work done.

Plugging the values we've been given into the formula, we get

$$W=\int_1^3 x^2\ dx$$

$$W=\frac{1}{3}x^3\Big|_1^3$$

$$W=\frac{1}{3}(3)^3-\frac{1}{3}(1)^3$$

$$W=\frac{27}{3}-\frac{1}{3}$$

$$W=\frac{26}{3}$$

Topic: Work done by a variable force

Question: Calculate the variable force done over the interval with the given force equation.

$$F(x) = 2\sin 4x$$

on the interval $[0,\pi]$

Answer choices:

A π

B 0

C 4

D 2

Solution: B

To find the work done by a variable force, we use the work formula

$$W=\int_a^b F(x)\ dx$$

where $F(x)$ is the variable force equation, $[a,b]$ is the given interval and W is the work done.

Plugging the values we've been given into the formula, we get

$$W=\int_0^\pi 2\sin 4x\ dx$$

$$W=-\frac{2}{4}\cos 4x\Big|_0^\pi$$

$$W=-\frac{1}{2}\cos 4x\Big|_0^\pi$$

$$W=-\frac{1}{2}\cos 4\pi-\left[-\frac{1}{2}\cos 4(0)\right]$$

$$W=-\frac{1}{2}(1)+\frac{1}{2}(1)$$

$$W=0$$

Topic: Work done by a variable force

Question: Calculate the variable force done over the interval with the given force equation.

$$F(x) = x^3 - 4e^{2x}$$

on the interval $[0,4]$

Answer choices:

A $62 - 2e^8$

B $64 - 2e^8$

C $64 - 2e^4$

D $66 - 2e^8$

Solution: D

To find the work done by a variable force, we use the work formula

$$W = \int_a^b F(x)\ dx$$

where $F(x)$ is the variable force equation, $[a, b]$ is the given interval and W is the work done.

Plugging the values we've been given into the formula, we get

$$W = \int_0^4 x^3 - 4e^{2x}\ dx$$

$$W = \frac{1}{4}x^4 - \frac{4}{2}e^{2x}\Big|_0^4$$

$$W = \frac{1}{4}x^4 - 2e^{2x}\Big|_0^4$$

$$W = \frac{1}{4}(4)^4 - 2e^{2(4)} - \left[\frac{1}{4}(0)^4 - 2e^{2(0)}\right]$$

$$W = 64 - 2e^8 + 2(1)$$

$$W = 66 - 2e^8$$

Topic: Moments of the system

Question: Calculate the moments of the system.

$$m_1 = 2$$

$$P_1(1,3)$$

and

$$m_2 = 3$$

$$P_2 = (-1,4)$$

and

$$m_3 = 5$$

$$P_3 = (3,-2)$$

Answer choices:

A $M_y = 8$ $M_x = 14$

B $M_y = 50$ $M_x = 30$

C $M_y = 14$ $M_x = 8$

D $M_y = 30$ $M_x = 50$

Solution: C

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (2)(1) + (3)(-1) + (5)(3)$$

$$M_y = 2 - 3 + 15$$

$$M_y = 14$$

and

$$M_x = (2)(3) + (3)(4) + (5)(-2)$$

$$M_x = 6 + 12 - 10$$

$$M_x = 8$$

The moments of the system are $M_y = 14$ and $M_x = 8$.

Topic: Moments of the system

Question: Calculate the moments of the system.

$$m_1 = 5$$

$$P_1 = (-2,2)$$

and

$$m_2 = 7$$

$$P_2 = (3,4)$$

and

$$m_3 = 3$$

$$P_3 = (2,3)$$

Answer choices:

A $M_y = 17$ $M_x = 47$

B $M_y = 47$ $M_x = 17$

C $M_y = 20$ $M_x = 44$

D $M_y = 37$ $M_x = 47$

Solution: A

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (5)(-2) + (7)(3) + (3)(2)$$

$$M_y = -10 + 21 + 6$$

$$M_y = 17$$

and

$$M_x = (5)(2) + (7)(4) + (3)(3)$$

$$M_x = 10 + 28 + 9$$

$$M_x = 47$$

The moments of the system are $M_y = 17$ and $M_x = 47$.

Topic: Moments of the system

Question: Calculate the moments of the system.

$$m_1 = 3$$

$$P_1 = (-3,2)$$

and

$$m_2 = 6$$

$$P_2 = (4,2)$$

and

$$m_3 = 7$$

$$p_3 = (3,3)$$

Answer choices:

A $M_y = 39$ $M_x = 36$

B $M_y = 51$ $M_x = 39$

C $M_y = 39$ $M_x = 51$

D $M_y = 36$ $M_x = 39$

Solution: D

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (3)(-3) + (6)(4) + (7)(3)$$

$$M_y = -9 + 24 + 21$$

$$M_y = 36$$

and

$$M_x = (3)(2) + (6)(2) + (7)(3)$$

$$M_x = 6 + 12 + 21$$

$$M_x = 39$$

The moments of the system are $M_y = 36$ and $M_x = 39$.

Topic: Moments of the system, x-axis

Question: Calculate the moments of the system.

Answer choices:

A $\quad M_y = 0 \qquad M_x = 101$

B $\quad M_y = 0 \qquad M_x = 20$

C $\quad M_y = 20 \qquad M_x = 0$

D $\quad M_y = 101 \qquad M_x = 0$

Solution: D

In this problem the masses are located on a real number line. However, we calculate the moments of the system in the same manner as we would if the masses were located somewhere in the quadrant plane. We consider the real number line to be the x-axis. This means that the masses are on points in which the y-coordinate is 0.

Therefore, the masses and their locations are

$$m_1 = 9$$

$$P_1 = (2,0)$$

and

$$m_2 = 4$$

$$P_2 = (5,0)$$

and

$$m_3 = 7$$

$$P_3 = (9,0)$$

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (9)(2) + (4)(5) + (7)(9)$$

$$M_y = 18 + 20 + 63$$

$$M_y = 101$$

and

$$M_x = (9)(0) + (4)(0) + (7)(0)$$

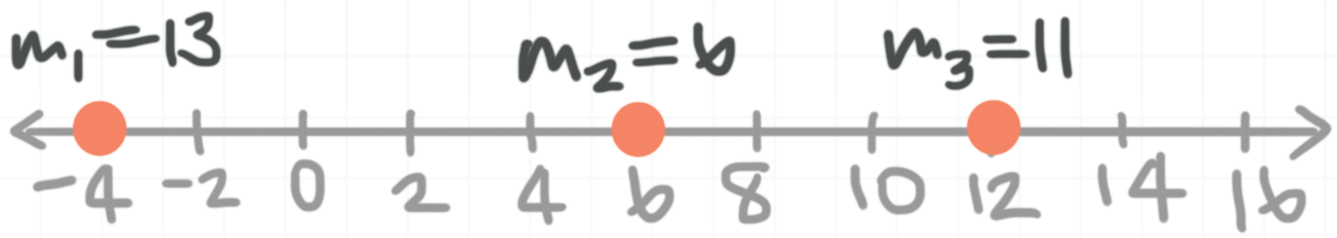
$$M_x = 0 + 0 + 0$$

$$M_x = 0$$

The moments of the system are $M_y = 101$ and $M_x = 0$.

Topic: Moments of the system, x-axis

Question: Calculate the moments of the system.



Answer choices:

A	$M_y = 14$	$M_x = 0$
B	$M_y = 116$	$M_x = 0$
C	$M_y = 0$	$M_x = 116$
D	$M_y = 0$	$M_x = 14$

Solution: B

In this problem the masses are located on a real number line. However, we calculate the moments of the system in the same manner as we would if the masses were located somewhere in the quadrant plane. We consider the real number line to be the x-axis. This means that the masses are on points in which the y-coordinate is 0.

Therefore, the masses and their locations are

$$m_1 = 13$$

$$P_1 = (-4,0)$$

and

$$m_2 = 6$$

$$P_2 = (6,0)$$

and

$$m_3 = 11$$

$$P_3 = (12,0)$$

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (13)(-4) + (6)(6) + (11)(12)$$

$$M_y = -52 + 36 + 132$$

$$M_y = 116$$

and

$$M_x = (13)(0) + (6)(0) + (11)(0)$$

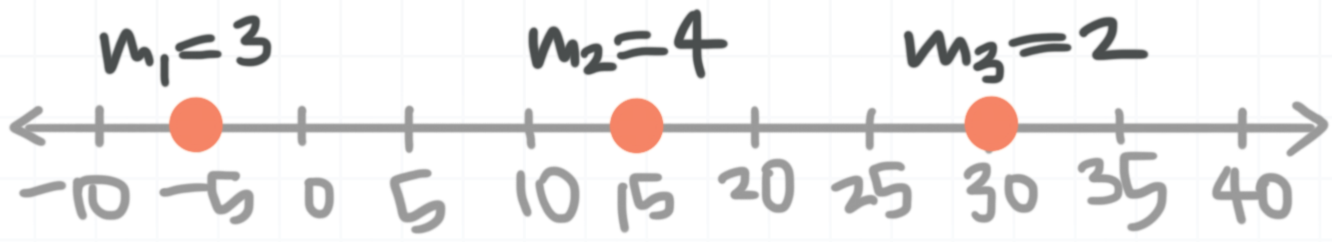
$$M_x = 0 + 0 + 0$$

$$M_x = 0$$

The moments of the system are $M_y = 116$ and $M_x = 0$.

Topic: Moments of the system, x-axis

Question: Calculate the moments of the system.



Answer choices:

A $\quad M_y = 105 \qquad M_x = 0$

B $\quad M_y = 40 \qquad M_x = 0$

C $\quad M_y = 0 \qquad M_x = 105$

D $\quad M_y = 0 \qquad M_x = 40$

Solution: A

In this problem the masses are located on a real number line. However, we calculate the moments of the system in the same manner as we would if the masses were located somewhere in the quadrant plane. We consider the real number line to be the x-axis. This means that the masses are on points in which the y-coordinate is 0.

Therefore, the masses and their locations are

$$m_1 = 3$$

$$P_1 = (-5,0)$$

and

$$m_2 = 4$$

$$P_2 = (15,0)$$

and

$$m_3 = 2$$

$$P_3 = (30,0)$$

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (3)(-5) + (4)(15) + (2)(30)$$

$$M_y = -15 + 60 + 60$$

$$M_y = 105$$

and

$$M_x = (3)(0) + (4)(0) + (2)(0)$$

$$M_x = 0 + 0 + 0$$

$$M_x = 0$$

The moments of the system are $M_y = 105$ and $M_x = 0$.

Topic: Center of mass of the system

Question: Find the center of mass of the system.

$$M_y = 12$$

$$M_x = 16$$

Total mass is $m_T = 8$

Answer choices:

A $\left(\frac{3}{2}, 2\right)$

B $\left(\frac{1}{2}, \frac{2}{3}\right)$

C $\left(2, \frac{3}{2}\right)$

D $\left(\frac{2}{3}, \frac{1}{2}\right)$

Solution: A

To find the center of mass of a system we'll use the formulas

$$\bar{x} = \frac{M_y}{m_T}$$

and

$$\bar{y} = \frac{M_x}{m_T}$$

where $(\bar{x}, \bar{y})$ is the coordinate point that represents the center of mass, where M_x and M_y are the moments of the system, and where m_T is the total mass of the system.

We'll plug the values we've been given into the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{12}{8}$$

$$\bar{x} = \frac{3}{2}$$

and

$$\bar{y} = \frac{16}{8}$$

$$\bar{y} = 2$$

The center of mass of the system is $\left(\frac{3}{2}, 2\right)$.

Topic: Center of mass of the system

Question: Find the center of mass of the system.

$$m_1 = 4$$

$$P_1(2, -1)$$

and

$$m_2 = 2$$

$$P_2 = (-1,5)$$

and

$$m_3 = 1$$

$$P_3 = (2,2)$$

Answer choices:

A $\left(\frac{7}{8}, \frac{7}{8}\right)$

B $\left(\frac{8}{7}, \frac{8}{7}\right)$

C $\left(\frac{16}{7}, \frac{12}{7}\right)$

D $\left(\frac{12}{7}, \frac{16}{7}\right)$

Solution: B

Since moments of the system are used in the formulas for center of mass, we need to calculate moments first.

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (4)(2) + (2)(-1) + (1)(2)$$

$$M_y = 8 - 2 + 2$$

$$M_y = 8$$

and

$$M_x = (4)(-1) + (2)(5) + (1)(2)$$

$$M_x = -4 + 10 + 2$$

$$M_x = 8$$

The moments of the system are $M_y = 8$ and $M_x = 8$.

We need to use these values, plus the total mass of the system, in order to find the coordinates for the center of mass. To find total mass, we'll add all three masses together.

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 4 + 2 + 1$$

$$m_T = 7$$

To find the center of mass of a system we'll use the formulas

$$\bar{x} = \frac{M_y}{m_T}$$

and

$$\bar{y} = \frac{M_x}{m_T}$$

where $(\bar{x}, \bar{y})$ is the coordinate point that represents the center of mass, where M_x and M_y are the moments of the system, and where m_T is the total mass of the system.

We'll plug the values we've been given into the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{8}{7}$$

and

$$\bar{y} = \frac{8}{7}$$

The center of mass of the system is $\left(\frac{8}{7}, \frac{8}{7}\right)$.

Topic: Center of mass of the system

Question: Find the center of mass of the system.

$m_1 = 8$ $\quad$ $m_2 = 9$ $\quad$ $m_3 = 4$

$P_1 = (4,7)$ $\quad$ $P_2 = (6,9)$ $\quad$ $P_3 = (11,15)$

Answer choices:

A $\quad$ $(197, 130)$

B $\quad$ $\left(\frac{197}{21}, \frac{130}{21}\right)$

C $\quad$ $(130, 197)$

D $\quad$ $\left(\frac{130}{21}, \frac{197}{21}\right)$

Solution: D

Since moments of the system are used in the formulas for center of mass, we need to calculate moments first.

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (8)(4) + (9)(6) + (4)(11)$$

$$M_y = 32 + 54 + 44$$

$$M_y = 130$$

and

$$M_x = (8)(7) + (9)(9) + (4)(15)$$

$$M_x = 56 + 81 + 60$$

$$M_x = 197$$

The moments of the system are $M_y = 130$ and $M_x = 197$.

We need to use these values, plus the total mass of the system, in order to find the coordinates for the center of mass. To find total mass, we'll add all three masses together.

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 8 + 9 + 4$$

$$m_T = 21$$

To find the center of mass of a system we'll use the formulas

$$\bar{x} = \frac{M_y}{m_T}$$

and

$$\bar{y} = \frac{M_x}{m_T}$$

where $(\bar{x}, \bar{y})$ is the coordinate point that represents the center of mass, where M_x and M_y are the moments of the system, and where m_T is the total mass of the system.

We'll plug the values we've been given into the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{130}{21}$$

and

$$\bar{y} = \frac{197}{21}$$

The center of mass of the system is $\left(\frac{130}{21},\frac{197}{21}\right)$.

Topic: Center of mass of the system, x-axis

Question: Find the center of mass of the system.

Answer choices:

A $(45,0)$

B $(0,2)$

C $(60,0)$

D $(2,0)$

Solution: D

Since moments of the system are used in the formulas for center of mass, we need to calculate moments first.

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

In this problem the masses are located on a real number line. However, we calculate the moments of the system in the same manner as we would if the masses were located somewhere in the quadrant plane. We consider the real number line to be the x-axis. This means that the masses are on points in which the y-coordinate is 0.

Therefore, the masses and their locations are

$$m_1 = 15$$

$$P_1 = (-3,0)$$

and

$$m_2 = 20$$

$$P_2 = (2,0)$$

and

$$m_3 = 25$$

$$P_3 = (5,0)$$

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (15)(-3) + (20)(2) + (25)(5)$$

$$M_y = -45 + 40 + 125$$

$$M_y = 120$$

and

$$M_x = (15)(0) + (20)(0) + (25)(0)$$

$$M_x = 0 + 0 + 0$$

$$M_x = 0$$

The moments of the system are $M_y = 120$ and $M_x = 0$.

We need to use these values, plus the total mass of the system, in order to find the coordinates for the center of mass. To find total mass, we'll add all three masses together.

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 15 + 20 + 25$$

$$m_T = 60$$

To find the center of mass of a system we'll use the formulas

$$\bar{x} = \frac{M_y}{m_T}$$

and

$$\bar{y} = \frac{M_x}{m_T}$$

where $(\bar{x}, \bar{y})$ is the coordinate point that represents the center of mass, where M_x and M_y are the moments of the system, and where m_T is the total mass of the system.

We'll plug the values we've been given into the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{120}{60} = 2$$

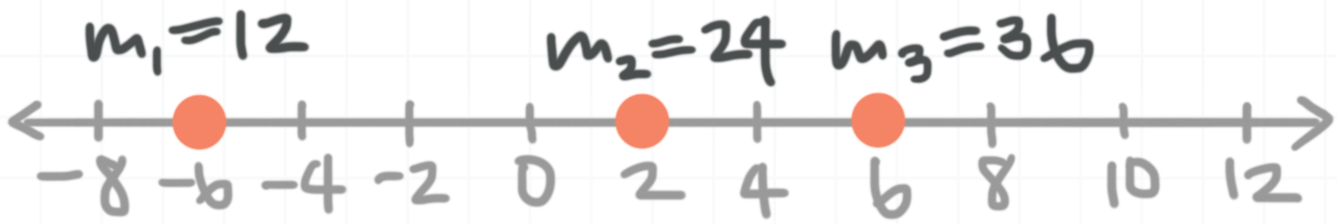
and

$$\bar{y} = \frac{0}{60} = 0$$

The center of mass of the system is $(2,0)$.

Topic: Center of mass of the system, x-axis

Question: Find the center of mass of the system.



Answer choices:

A $(8,0)$

B $\left(\frac{8}{3},0\right)$

C $\left(0,\frac{8}{3}\right)$

D $(0,8)$

Solution: B

Since moments of the system are used in the formulas for center of mass, we need to calculate moments first.

To calculate the moments of a system we'll use the formulas

$$M_y = m_1(x_1) + m_2(x_2) + m_3(x_3)$$

and

$$M_x = m_1(y_1) + m_2(y_2) + m_3(y_3)$$

where m_1, m_2 and m_3 are the given masses and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are the points associated with those masses.

In this problem the masses are located on a real number line. However, we calculate the moments of the system in the same manner as we would if the masses were located somewhere in the quadrant plane. We consider the real number line to be the x-axis. This means that the masses are on points in which the y-coordinate is 0.

Therefore, the masses and their locations are

$$m_1 = 12$$

$$P_1 = (-6,0)$$

and

$$m_2 = 24$$

$$P_2 = (2,0)$$

and

$$m_3 = 36$$

$$P_3 = (6,0)$$

We'll plug the values we've been given into the formulas for M_y and M_x.

$$M_y = (12)(-6) + (24)(2) + (36)(6)$$

$$M_y = -72 + 48 + 216$$

$$M_y = 192$$

and

$$M_x = (12)(0) + (24)(0) + (36)(0)$$

$$M_x = 0 + 0 + 0$$

$$M_x = 0$$

The moments of the system are $M_y = 192$ and $M_x = 0$.

We need to use these values, plus the total mass of the system, in order to find the coordinates for the center of mass. To find total mass, we'll add all three masses together.

$$m_T = m_1 + m_2 + m_3$$

$$m_T = 12 + 24 + 36$$

$$m_T = 72$$

To find the center of mass of a system we'll use the formulas

$$\bar{x} = \frac{M_y}{m_T}$$

and

$$\bar{y} = \frac{M_x}{m_T}$$

where $(\bar{x}, \bar{y})$ is the coordinate point that represents the center of mass, where M_x and M_y are the moments of the system, and where m_T is the total mass of the system.

We'll plug the values we've been given into the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{192}{72} = \frac{8}{3}$$

and

$$\bar{y} = \frac{0}{72} = 0$$

The center of mass of the system is $\left(\frac{8}{3}, 0\right)$.

Topic: Hydrostatic pressure

Question: Find the hydrostatic pressure at the bottom of the tank.

The tank is filled with water to a depth of 5 m. Assume the density of water is $\rho = 1{,}000$ kg/m^3.

Answer choices:

A $49{,}000$ Pa

B $98{,}000$ Pa

C $2{,}000$ Pa

D $1{,}000$ Pa

Solution: A

The formula we use to calculate hydrostatic pressure is

$$P = \rho g d$$

where ρ is fluid pressure, g is gravity and d is depth. If we're dealing with water, and not some other liquid, we can simplify the formula, knowing that the density of water is $\rho = 1{,}000$ kg/m^3.

Plugging in water's given density, the gravitational constant, and the depth of the water, we get

$$P = \left(1{,}000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8\ \frac{\text{m}}{\text{s}^2}\right)(5\ \text{m})$$

$$P = 49{,}000\ \frac{\text{kg}}{\text{ms}^2}$$

$$P = 49{,}000\ \text{Pa}$$

Topic: Hydrostatic pressure

Question: Find the hydrostatic pressure at the bottom of the tank.

The tank is 4 m deep, and completely filled to the top with water. Assume the density of water is $\rho = 1{,}000$ kg/m^3.

Answer choices:

A $P = 36{,}800$ Pa

B $P = 39{,}200$ Pa

C $P = 78{,}400$ Pa

D $P = 92{,}200$ Pa

Solution: B

The formula we use to calculate hydrostatic pressure is

$$P = \rho g d$$

where ρ is fluid pressure, g is gravity and d is depth. If we're dealing with water, and not some other liquid, we can simplify the formula, knowing that the density of water is $\rho = 1{,}000$ kg/m^3.

Plugging in water's given density, the gravitational constant, and the depth of the water, we get

$$P = \left(1{,}000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8\ \frac{\text{m}}{\text{s}^2}\right)(4\ \text{m})$$

$$P = 39{,}200\ \frac{\text{kg}}{\text{ms}^2}$$

$$P = 39{,}200\ \text{Pa}$$

Topic: Hydrostatic pressure

Question: Find the hydrostatic pressure per square foot on the bottom of this tank that's 6 m deep and filled half way to the top with water. Assume the density of water is $\rho = 1{,}000$ kg/m^3.

Answer choices:

A $P = 16{,}200$ Pa

B $P = 29{,}400$ Pa

C $P = 58{,}800$ Pa

D $P = 78{,}800$ Pa

Solution: B

The formula we use to calculate hydrostatic pressure is

$$P = \rho g d$$

where ρ is fluid pressure, g is gravity and d is depth. If we're dealing with water, and not some other liquid, we can simplify the formula, knowing that the density of water is $\rho = 1{,}000$ kg/m^3.

Plugging in water's given density, the gravitational constant, and the depth of the water, we get

$$P = \left(1{,}000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8\ \frac{\text{m}}{\text{s}^2}\right)(3\ \text{m})$$

$$P = 29{,}400\ \frac{\text{kg}}{\text{ms}^2}$$

$$P = 29{,}400\ \text{Pa}$$

Topic: Hydrostatic force

Question: A rectangular tank is 3 m tall, 4 m wide, 6 m long, and full of water. If hydrostatic pressure at the bottom of the tank is $6{,}000$ Pa, find the hydrostatic force on the bottom of the tank.

Answer choices:

A $432{,}000$ N

B $144{,}000$ N

C $78{,}000$ N

D $60{,}000$ N

Solution: B

The formula we use to calculate hydrostatic force on a horizontal surface is

$$F = PA$$

where P is the hydrostatic pressure at the bottom of the tank and A is the surface area of the bottom of the tank.

We've been told that hydrostatic pressure at the bottom of the tank is $6{,}000$ Pa, but before we can plug into our formula, we need to find the surface area of the bottom of the rectangular tank. Since we know that that the area of a rectangle is $A = l \cdot w$, we can plug in the given length and width and get

$$A = 6 \text{ m} \cdot 4 \text{ m}$$

$$A = 24 \text{ m}^2$$

Plugging everything we know into the force equation, we get

$$F = 6{,}000 \text{ Pa} \cdot 24 \text{ m}^2$$

$$F = 6{,}000 \text{ kg/ms}^2 \cdot 24 \text{ m}^2$$

$$F = 144{,}000 \text{ kg m/s}^2$$

$$F = 144{,}000 \text{ N}$$

Topic: Hydrostatic force

Question: A rectangular tank is 4 m tall, 3 m wide, and full of water. Assuming the density of water is $\rho = 1{,}000$ kg/m^3, find the hydrostatic force on the end of the tank.

Answer choices:

A $130{,}500$ N

B $180{,}000$ N

C $235{,}200$ N

D $720{,}000$ N

Solution: C

To find hydrostatic force on one end of the tank, we'll use the modified force equation

$$F = WAd$$

where W is the weight of the liquid, A is the area of the surface and d is the depth of the liquid.

Since weight is density $\times$ gravity, weight is

$$W = \left(\frac{1{,}000 \text{ kg}}{\text{m}^3}\right)\left(\frac{9.8 \text{ m}}{\text{s}^2}\right)$$

$$W = \frac{9{,}800 \text{ kg}}{\text{m}^2\text{s}^2}$$

Since we're looking for force against a vertical surface and force at deeper depths is greater than force at shallower depths, we can't use the area of the entire surface in our force equation. Instead, we have to divide the surface into small horizontal strips so that we can assume that the force against each strip is roughly the same throughout the strip.

If we divide the end of the tank into tiny slices of equal depth, then each strip is 3 m wide and Δx tall, and sitting at a depth of x_i. The area of one strip is $A_i = 3 \cdot \Delta x$. The force against one strip is

$$F = WAd$$

$$F_i = (9{,}800)(3\Delta x)(x_i)$$

In order to solve for the force against the end of the tank, instead of against a small strip of it, we need to sum together the force against all of the slices, and take the limit as the number of slices approaches infinity, $n \to \infty$. Let's put this all together and see how it looks.

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} (9{,}800)(3\Delta x)(x_i)$$

We need to remember that taking the limit as $n \to \infty$ of the sum of the force against all of the slices is the same as taking the integral of our force equation over the interval of the depth, $[0,4]$. Remember, when we move this into an integral, x_i becomes x, and Δx becomes dx. Let's put this all together and see how it looks.

$$F = \int_0^4 (9{,}800)(3\ dx)x$$

$$F = 29{,}400 \int_0^4 x\ dx$$

$$F = 29{,}400 \left(\frac{x^2}{2} \right) \Bigg|_0^4$$

$$F = 14{,}700x^2 \Big|_0^4$$

$$F = 14{,}700(4)^2 - 14{,}700(0)^2$$

$$F = 235{,}200$$

The hydrostatic force on the end is $F = 235{,}200$ N.

Topic: Hydrostatic force

Question: Find the hydrostatic force on the bottom of a tank, which is filled to the top with water, if the tank is 8 m long, 1 m wide, and 5 m tall. Assume the density of water is $\rho = 1{,}000$ kg/m^3.

Answer choices:

A $F = 392{,}000$ N

B $F = 196{,}000$ N

C $F = 98{,}000$ N

D $F = 49{,}000$ N

Solution: A

The formula we use to calculate hydrostatic force on a horizontal surface is

$$F = PA$$

where P is the hydrostatic pressure at the bottom of the tank and A is the surface area of the bottom of the tank.

The formula we use to calculate hydrostatic pressure is

$$P = \rho g d$$

where ρ is fluid pressure, g is gravity and d is depth. If we're dealing with water, and not some other liquid, we can simplify the formula, knowing that the density of water is $\rho = 1{,}000$ kg/m^3.

Plugging in water's given density, the gravitational constant, and the depth of the water, we get

$$P = \left(1{,}000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8\ \frac{\text{m}}{\text{s}^2}\right)(5\ \text{m})$$

$$P = 49{,}000\ \frac{\text{kg}}{\text{ms}^2}$$

$$P = 49{,}000\ \text{Pa}$$

This is the hydrostatic pressure at the bottom of the tank, but before we can plug into our formula, we need to find the surface area of the bottom of the rectangular tank. Since we know that that the area of a rectangle is $A = l \cdot w$, we can plug in the given length and width and get

$$A = 8 \text{ m} \cdot 1 \text{ m}$$

$$A = 8 \text{ m}^2$$

Plugging everything we know into the force equation, we get

$$F = 49{,}000 \text{ Pa} \cdot 8 \text{ m}^2$$

$$F = 392{,}000 \text{ kg/ms}^2 \cdot \text{m}^2$$

$$F = 392{,}000 \text{ kg m/s}^2$$

$$F = 392{,}000 \text{ N}$$

Topic: Vertical motion

Question: Find the particle's position function.

A particle is moving along a straight line.

Its acceleration function is $a(t) = 6t + 5$

Its velocity is 25 m/s when $t = 2$

Its position is 0 m when $t = 0$

Answer choices:

A $x(t) = t^3 + 5t^2 + 3t$

B $x(t) = t^3 + \frac{5}{2}t^2 + 3t$

C $x(t) = t^3 + \frac{5}{2}t^2 + 3t + C$

D $x(t) = t^3 - \frac{5}{2}t^2 + 3t$

Solution: B

Given the position function $x(t)$ of an object, the derivative of the position function is the velocity function, and the derivative of the velocity function is the acceleration function.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

Given the above relationship, you can also conclude that velocity is the antiderivative of acceleration, and position is the antiderivative of velocity.

$$a(t)$$

$$\int a(t) = v(t)$$

$$\int \int a(t) = \int v(t) = x(t)$$

This problem asks us to find the position function $x(t)$ given the acceleration function, so we'll need to start by integrating acceleration to find velocity.

$$v(t) = \int a(t) = \int 6t + 5 \ dt$$

$$v(t) = 3t^2 + 5t + C$$

We've been told that velocity is 25 m/s when $t = 2$, which is the initial condition $v(2) = 25$. Plugging this into our velocity function, we get

$$3(2)^2 + 5(2) + C = 25$$

$$12 + 10 + C = 25$$

$$C = 3$$

So the velocity function is

$$v(t) = 3t^2 + 5t + 3$$

To find position, we'll just integrate velocity.

$$x(t) = \int v(t) = \int 3t^2 + 5t + 3 \ dt$$

$$x(t) = t^3 + \frac{5}{2}t^2 + 3t + C$$

We've been told that position is 0 m when $t = 0$, which is the initial condition $x(0) = 0$. Plugging this into the position function, we get

$$(0)^3 + \frac{5}{2}(0)^2 + 3(0) + C = 0$$

$$C = 0$$

Therefore, the position function is

$$x(t) = t^3 + \frac{5}{2}t^2 + 3t$$

Topic: Vertical motion

Question: Find the maximum height of the ball.

A basketball is thrown straight up from the ground.

Its velocity function is $v(t) = -9.8t + 30$

Answer choices:

A 187.2 m

B 9.5 m

C 140.1 m

D 45.9 m

Solution: D

Given the position function $x(t)$ of an object, the derivative of the position function is the velocity function, and the derivative of the velocity function is the acceleration function.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

Given the above relationship, you can also conclude that velocity is the antiderivative of acceleration, and position is the antiderivative of velocity.

$$a(t)$$

$$\int a(t) = v(t)$$

$$\int \int a(t) = \int v(t) = x(t)$$

This problem asks us to find the maximum height of the ball. As the ball leaves the ground and approaches it's highest point, velocity will be positive. At the ball's maximum height, velocity is 0, and then it becomes negative as the ball starts heading back down toward the ground.

Therefore, to find the maximum height, we need to find the point at which the velocity function is equal to 0.

$$-9.8t + 30 = 0$$

$$-9.8t = -30$$

$$t = 3.1$$

The position function is the one that will tell us how high the ball is at a specific time. Since we know from the velocity function that the ball reaches maximum height at $t = 3.1$, we can plug this time into the position function, and the result will be the maximum height of the ball.

Before we can plug $t = 3.1$ into the position function, we need to find it by integrating the velocity function.

$$x(t) = \int v(t) = \int -9.8t + 30 \ dt$$

$$x(t) = -4.9t^2 + 30t + C$$

Since we know that the ball is thrown up from the ground, we know that it's initial position when $t = 0$ is 0, in other words, $x(0) = 0$. Plugging this initial condition into the position function to solve for C, we get

$$-4.9(0)^2 + 30(0) + C = 0$$

$$C = 0$$

Which means the position function is

$$x(t) = -4.9t^2 + 30t$$

To find the maximum height, we'll plug $t = 3.1$ into the position function.

$$x(3.1) = -4.9(3.1)^2 + 30(3.1)$$

$$x(3.1) = 45.9$$

The maximum height of the ball is 45.9 m.

Topic: Vertical motion

Question: How long is the rock in the air?

A rock is thrown straight from the ground.

Its velocity function is $v(t) = -9.8t + 45$

Answer choices:

A 9.2 s

B 4.9 s

C 9.8 s

D 4.6 s

Solution: A

Given the position function $x(t)$ of an object, the derivative of the position function is the velocity function, and the derivative of the velocity function is the acceleration function.

$$x(t)$$

$$x'(t) = v(t)$$

$$x''(t) = v'(t) = a(t)$$

Given the above relationship, you can also conclude that velocity is the antiderivative of acceleration, and position is the antiderivative of velocity.

$$a(t)$$

$$\int a(t) = v(t)$$

$$\int \int a(t) = \int v(t) = x(t)$$

This problem asks us to find the amount of time that the rock is in the air, which means we need to find the time at which the rock comes back down and hits the ground.

The position of the rock is 0 at two separate times: when it's on the ground before being thrown into the air, and when it's on the ground after it falls back down again. Therefore, if we can find the points at which the position function is equal to 0, we'll know how much time the rock is in the air.

To find the position function, we'll integrate the velocity function.

$$x(t) = \int v(t) = \int -9.8t + 45\ dt$$

$$x(t) = -4.9t^2 + 45t + C$$

We know the initial position at $t = 0$ is 0, since the rock is thrown up from the ground, so $x(0) = 0$. Plugging this into the position function, we get

$$-4.9(0)^2 + 45(0) + C = 0$$

$$C = 0$$

The position function is

$$x(t) = -4.9t^2 + 45t$$

We'll find the points in time when the rock is on the ground by setting the position function equal to 0.

$$-4.9t^2 + 45t = 0$$

$$-4.9t(t - 9.2) = 0$$

$t = 0$ and $t = 9.2$

We already know that $t = 0$ is associated with the rock's initial position before it was thrown into the air. That means that $t = 9.2$ must be associated with the rock's final position when it lands on the ground again. So we can conclude that the rock is in the air for 9.2 seconds.

Topic: Rectilinear motion

Question: Find the position function that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = 16t + 4$$

$$v(0) = -12 \text{ and } x(0) = 3$$

Answer choices:

A $\quad x(t) = \frac{8}{3}t^3 + 2t^2 - 12t$

B $\quad x(t) = 8t^3 + 2t^2 - 12t + 3$

C $\quad x(t) = \frac{8}{3}t^3 + 2t^2 - 12t + 3$

D $\quad x(t) = \frac{4}{3}t^3 + 2t^2 - t + 3$

Solution: C

With particle motion, acceleration, velocity and position have the following relationships.

$$a(t) = v'(t) = x''(t)$$

$$v(t) = x'(t)$$

Which means that the given acceleration function $a(t) = 16t + 4$ is the second derivative of the position function $x(t)$ that we need to find.

Let's begin by integrating $a(t)$ to find $v(t)$.

$$a(t) = 16t + 4$$

$$v(t) = \int a(t)\ dt = \int 16t + 4\ dt$$

$$v(t) = 8t^2 + 4t + C$$

Now we'll use the initial condition for the velocity function $v(0) = -12$ to find the a value for C.

$$-12 = 8(0)^2 + 4(0) + C$$

$$-12 = C$$

So the velocity function is

$$v(t) = 8t^2 + 4t - 12$$

To find $x(t)$, we'll integrate the velocity function we just found.

$$x(t) = \int v(t) \ dt = \int 8t^2 + 4t - 12 \ dt$$

$$x(t) = \frac{8}{3}t^3 + 2t^2 - 12t + D$$

Now we'll use the initial condition for the position function $x(0) = 3$ to find the a value for D.

$$3 = \frac{8}{3}(0)^3 + 2(0)^2 - 12(0) + D$$

$$3 = D$$

So the position function is

$$x(t) = \frac{8}{3}t^3 + 2t^2 - 12t + 3$$

Topic: Rectilinear motion

Question: Find the position function that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = t^2 + 7t + 2$$

$$v(0) = 6 \text{ and } x(0) = 5$$

Answer choices:

A $\quad x(t) = \frac{1}{12}t^4 + \frac{7}{6}t^3 + t^2 + 6t + 5$

B $\quad x(t) = \frac{1}{12}t^4 + \frac{7}{6}t^3 + t^2 + 6t$

C $\quad x(t) = t^4 + t^3 + t^2 + 6t + 5$

D $\quad x(t) = t^4 + 7t^3 + t^2 + 6t + 5$

Solution: A

With particle motion, acceleration, velocity and position have the following relationships.

$$a(t) = v'(t) = x''(t)$$

$$v(t) = x'(t)$$

Which means that the given acceleration function $a(t) = 16t + 4$ is the second derivative of the position function $x(t)$ that we need to find.

Let's begin by integrating $a(t)$ to find $v(t)$.

$$a(t) = t^2 + 7t + 2$$

$$v(t) = \int a(t)\ dt = \int t^2 + 7t + 2\ dt$$

$$v(t) = \frac{1}{3}t^3 + \frac{7}{2}t^2 + 2t + C$$

Now we'll use the initial condition for the velocity function $v(0) = 6$ to find the a value for C.

$$6 = \frac{1}{3}(0)^3 + \frac{7}{2}(0)^2 + 2(0) + C$$

$$6 = C$$

So the velocity function is

$$v(t) = \frac{1}{3}t^3 + \frac{7}{2}t^2 + 2t + 6$$

To find $x(t)$, we'll integrate the velocity function we just found.

$$x(t) = \int v(t)\ dt = \int \frac{1}{3}t^3 + \frac{7}{2}t^2 + 2t + 6\ dt$$

$$x(t) = \frac{1}{12}t^4 + \frac{7}{6}t^3 + t^2 + 6t + D$$

Now we'll use the initial condition for the position function $x(0) = 5$ to find the a value for D.

$$5 = \frac{1}{12}(0)^4 + \frac{7}{6}(0)^3 + (0)^2 + 6(0) + D$$

$$5 = D$$

So the position function is

$$x(t) = \frac{1}{12}t^4 + \frac{7}{6}t^3 + t^2 + 6t + 5$$

Topic: Rectilinear motion

Question: Find the position function that models the rectilinear motion of a particle moving along the x-axis.

$$a(t) = e^t + 3$$

$v(0) = 9$ and $x(0) = 14$

Answer choices:

A $\quad x(t) = e^t + \frac{3}{2}t^2 + 8t$

B $\quad x(t) = e^t + t^2 + 8t + 13$

C $\quad x(t) = e^t + \frac{3}{2}t^2 + 9t + 14$

D $\quad x(t) = e^t + \frac{3}{2}t^2 + 8t + 13$

Solution: D

With particle motion, acceleration, velocity and position have the following relationships.

$$a(t) = v'(t) = x''(t)$$

$$v(t) = x'(t)$$

Which means that the given acceleration function $a(t) = 16t + 4$ is the second derivative of the position function $x(t)$ that we need to find.

Let's begin by integrating $a(t)$ to find $v(t)$.

$$a(t) = e^t + 3$$

$$v(t) = \int a(t)\ dt = \int e^t + 3\ dt$$

$$v(t) = e^t + 3t + C$$

Now we'll use the initial condition for the velocity function $v(0) = 9$ to find the a value for C.

$$9 = e^0 + 3(0) + C$$

$$9 = 1 + C$$

$$8 = C$$

So the velocity function is

$$v(t) = e^t + 3t + 8$$

To find $x(t)$, we'll integrate the velocity function we just found.

$$x(t) = \int v(t)\ dt = \int e^t + 3t + 8\ dt$$

$$x(t) = e^t + \frac{3}{2}t^2 + 8t + D$$

Now we'll use the initial condition for the position function $x(0) = 14$ to find the a value for D.

$$14 = e^0 + \frac{3}{2}(0)^2 + 8(0) + D$$

$$14 = 1 + D$$

$$13 = D$$

So the position function is

$$x(t) = e^t + \frac{3}{2}t^2 + 8t + 13$$

Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y = 2x + 1$, $y = -x + 7$, and $x = 8$.

Answer choices:

A $(6,6)$

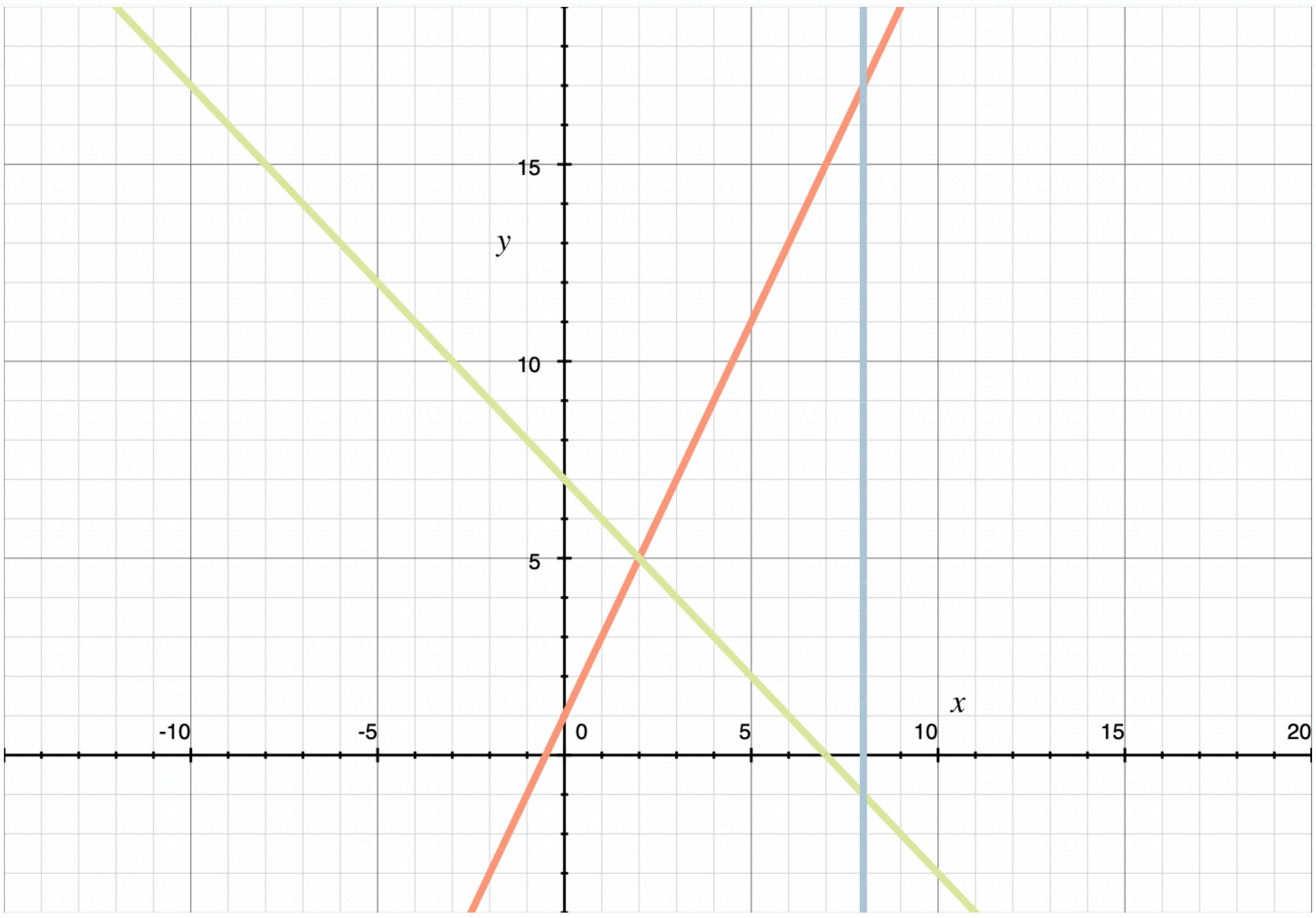
B $(6,7)$

C $(7,6)$

D $(7,7)$

Solution: B

A sketch of all three lines is



The limits of integration will be $x = [2,8]$, and the region is bounded above by $f(x) = 2x + 1$ and below by $g(x) = -x + 7$. Then the area of the region is

$$A = \int_a^b f(x) - g(x)\ dx$$

$$A = \int_2^8 2x + 1 - (-x + 7)\ dx$$

$$A = \int_2^8 3x - 6\ dx$$

$$A = \frac{3}{2}x^2 - 6x \Big|_2^8$$

$$A = \frac{3}{2}(8)^2 - 6(8) - \left[\frac{3}{2}(2)^2 - 6(2)\right]$$

$$A = 96 - 48 - (6 - 12)$$

$$A = 54$$

So the coordinates of the centroid are

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{x} = \frac{1}{54}\int_2^8 x(3x - 6)\ dx$$

$$\bar{x} = \frac{1}{54}\int_2^8 3x^2 - 6x\ dx$$

$$\bar{x} = \frac{1}{54}(x^3 - 3x^2)\Big|_2^8$$

$$\bar{x} = \frac{1}{54}\left[((8)^3 - 3(8)^2) - ((2)^3 - 3(2)^2)\right]$$

$$\bar{x} = \frac{1}{54}\left[(512 - 192) - (8 - 12)\right]$$

$$\bar{x} = \frac{1}{54}(324)$$

$$\bar{x} = 6$$

and

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\ dx$$

$$\bar{y} = \frac{1}{54}\int_2^8 \frac{1}{2}\left[(2x+1)^2 - (-x+7)^2\right]\ dx$$

$$\bar{y} = \frac{1}{108}\int_2^8 4x^2 + 4x + 1 - (x^2 - 14x + 49)\ dx$$

$$\bar{y} = \frac{1}{108}\int_2^8 4x^2 + 4x + 1 - x^2 + 14x - 49\ dx$$

$$\bar{y} = \frac{1}{108}\int_2^8 3x^2 + 18x - 48\ dx$$

$$\bar{y} = \frac{1}{108}(x^3 + 9x^2 - 48x)\Big|_2^8$$

$$\bar{y} = \frac{1}{108}\left[(8^3 + 9(8)^2 - 48(8)) - (2^3 + 9(2)^2 - 48(2))\right]$$

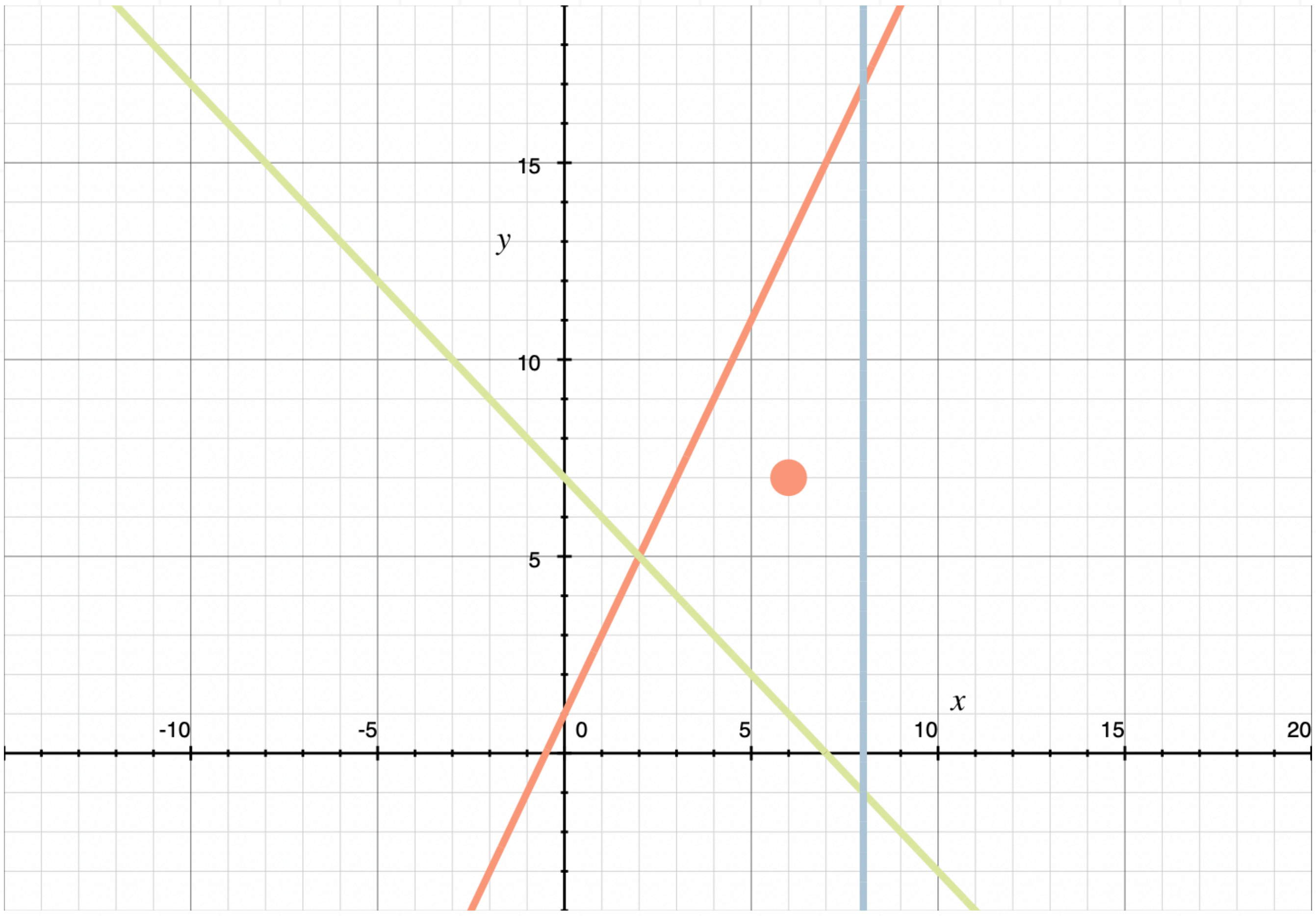
$$\bar{y} = \frac{1}{108}\left[(512 + 576 - 384) - (8 + 36 - 96)\right]$$

$$\bar{y} = \frac{1}{108}(512 + 576 - 384 - 8 - 36 + 96)$$

$$\bar{y} = \frac{1}{108}(756)$$

$$\bar{y} = 7$$

Therefore, the centroid of the region is at $(6,7)$. We can confirm this visually by graphing it in the region.



Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y = 4 - x^2$ and $y = 0$.

Answer choices:

A $\left(0, \frac{8}{5}\right)$

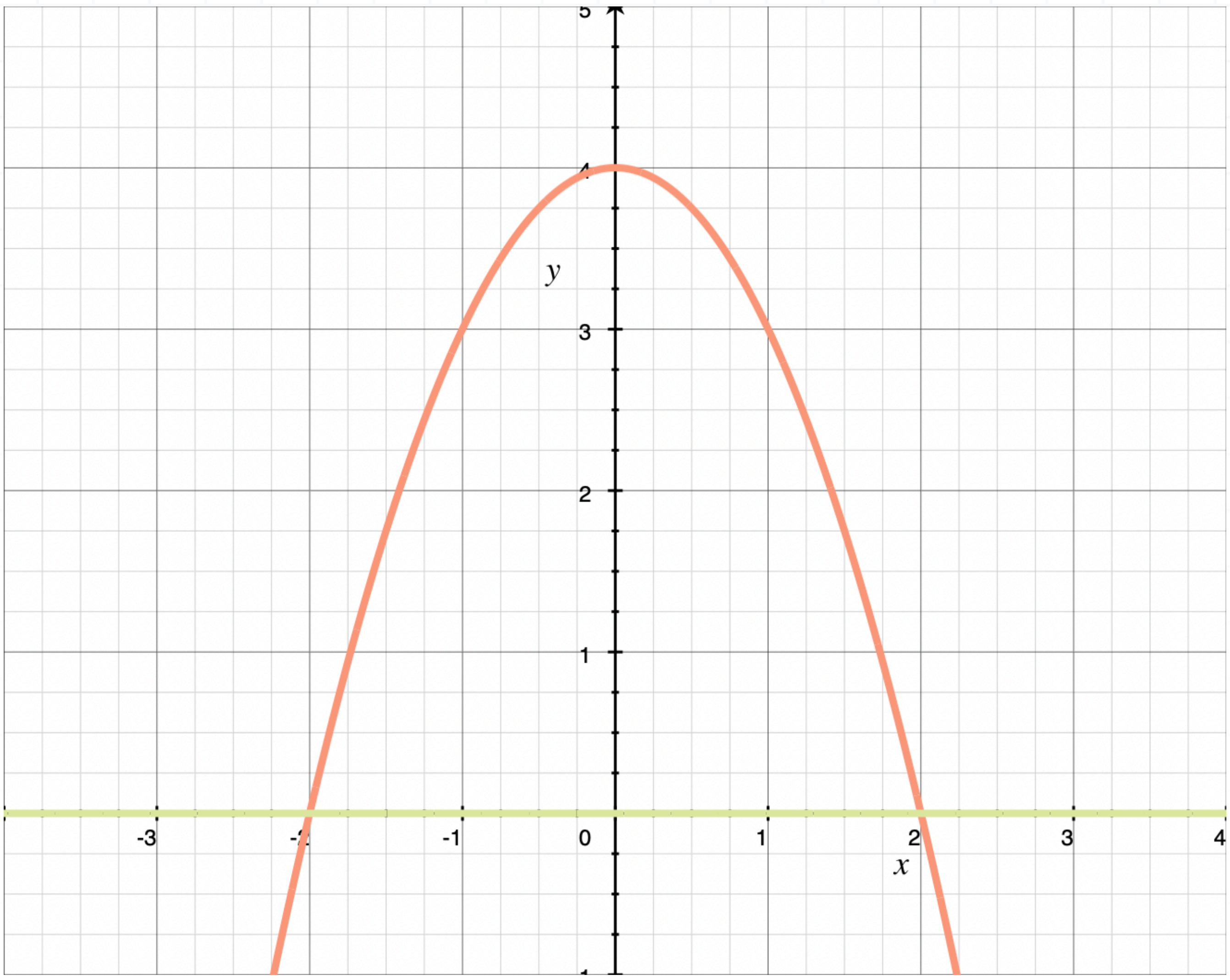
B $(0,0)$

C $\left(\frac{8}{5}, 0\right)$

D $\left(0, \frac{5}{8}\right)$

Solution: A

A sketch of both curves is



Because the region is symmetric about the y-axis, $\bar{x}=0$. To find $\bar{y}$, we'll first find the area of the region.

$$A=\int_a^b f(x)-g(x)\ dx$$

$$A=\int_{-2}^{2} 4-x^2-0\ dx$$

$$A=4x-\frac{1}{3}x^3\Big|_{-2}^{2}$$

$$A=\left[4(2)-\frac{1}{3}(2)^3\right]-\left[4(-2)-\frac{1}{3}(-2)^3\right]$$

$$A=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$A=8-\frac{8}{3}+8-\frac{8}{3}$$

$$A=16-\frac{16}{3}$$

$$A=\frac{48}{3}-\frac{16}{3}$$

$$A=\frac{32}{3}$$

Then $\bar{y}$ is

$$\bar{y}=\frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2-(g(x))^2]\ dx$$

$$\bar{y}=\frac{1}{\frac{32}{3}}\int_{-2}^{2}\frac{1}{2}(4-x^2)^2-0^2\ dx$$

$$\bar{y}=\frac{3}{32}\cdot\frac{1}{2}\int_{-2}^{2}16-8x^2+x^4\ dx$$

$$\bar{y}=\frac{3}{64}\left(16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right)\Big|_{-2}^{2}$$

$$\bar{y}=\frac{3}{64}\left[\left(16(2)-\frac{8}{3}(2)^3+\frac{1}{5}(2)^5\right)-\left(16(-2)-\frac{8}{3}(-2)^3+\frac{1}{5}(-2)^5\right)\right]$$

$$\bar{y}=\frac{3}{64}\left[\left(32-\frac{64}{3}+\frac{32}{5}\right)-\left(-32+\frac{64}{3}-\frac{32}{5}\right)\right]$$

$$\bar{y}=\frac{3}{64}\left(32-\frac{64}{3}+\frac{32}{5}+32-\frac{64}{3}+\frac{32}{5}\right)$$

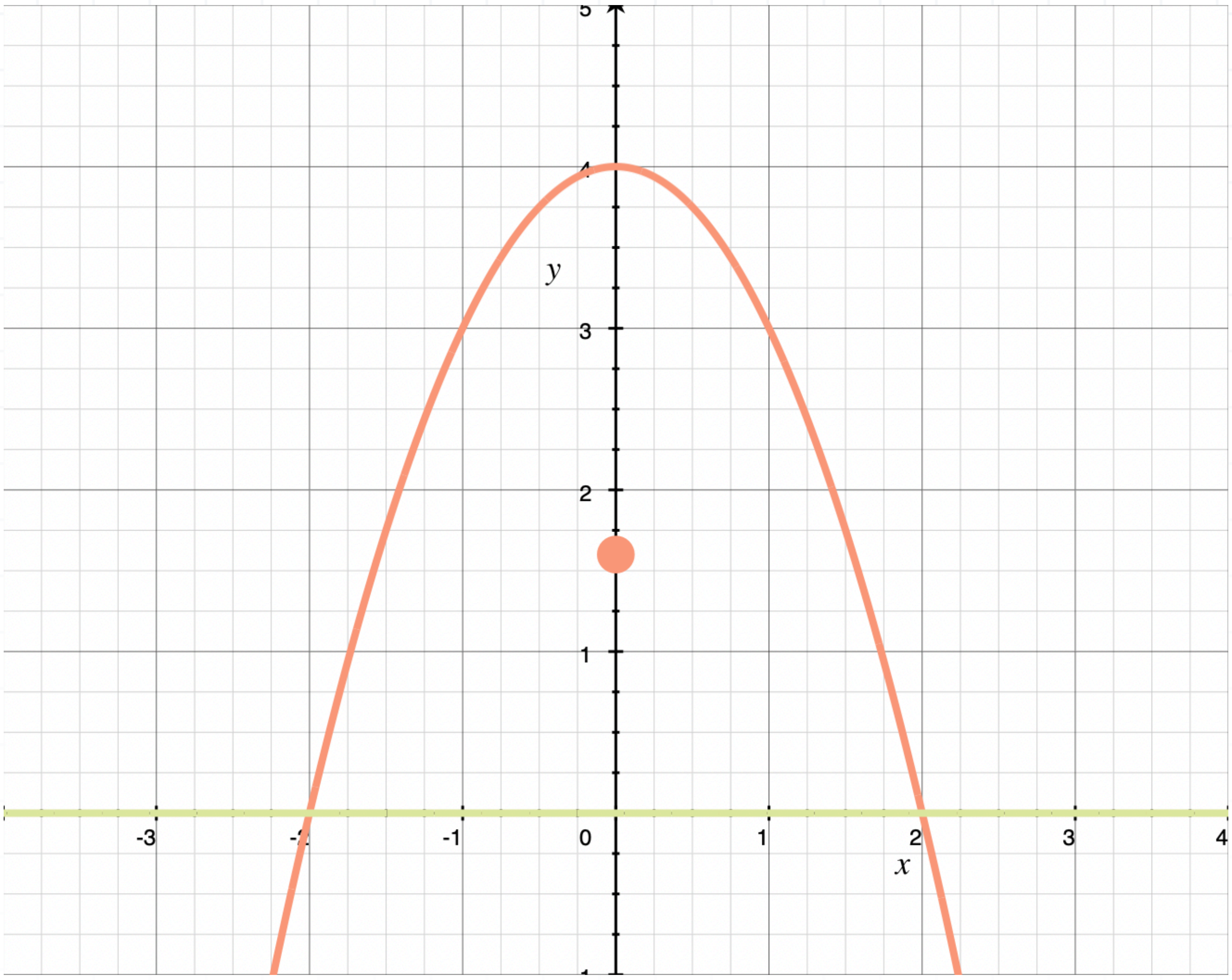
$$\bar{y}=\frac{3}{64}\left(64-\frac{128}{3}+\frac{64}{5}\right)$$

$$\bar{y}=3\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$\bar{y}=3\left(\frac{15}{15}-\frac{10}{15}+\frac{3}{15}\right)$$

$$\bar{y}=\frac{8}{5}$$

Therefore, the centroid of the region is at $(0,8/5)$. We can confirm this visually by graphing it in the region.



y
x
5
4
3
2
1
0
-3
-2
-1
1
2
3
4

Topic: Centroids of plane regions

Question: Find the centroid of the plane region bounded by $y = 2x + 4$, $y = 0$, $x = 3$, and $x = 9$.

Answer choices:

A $\left(\frac{51}{8}, \frac{67}{8}\right)$

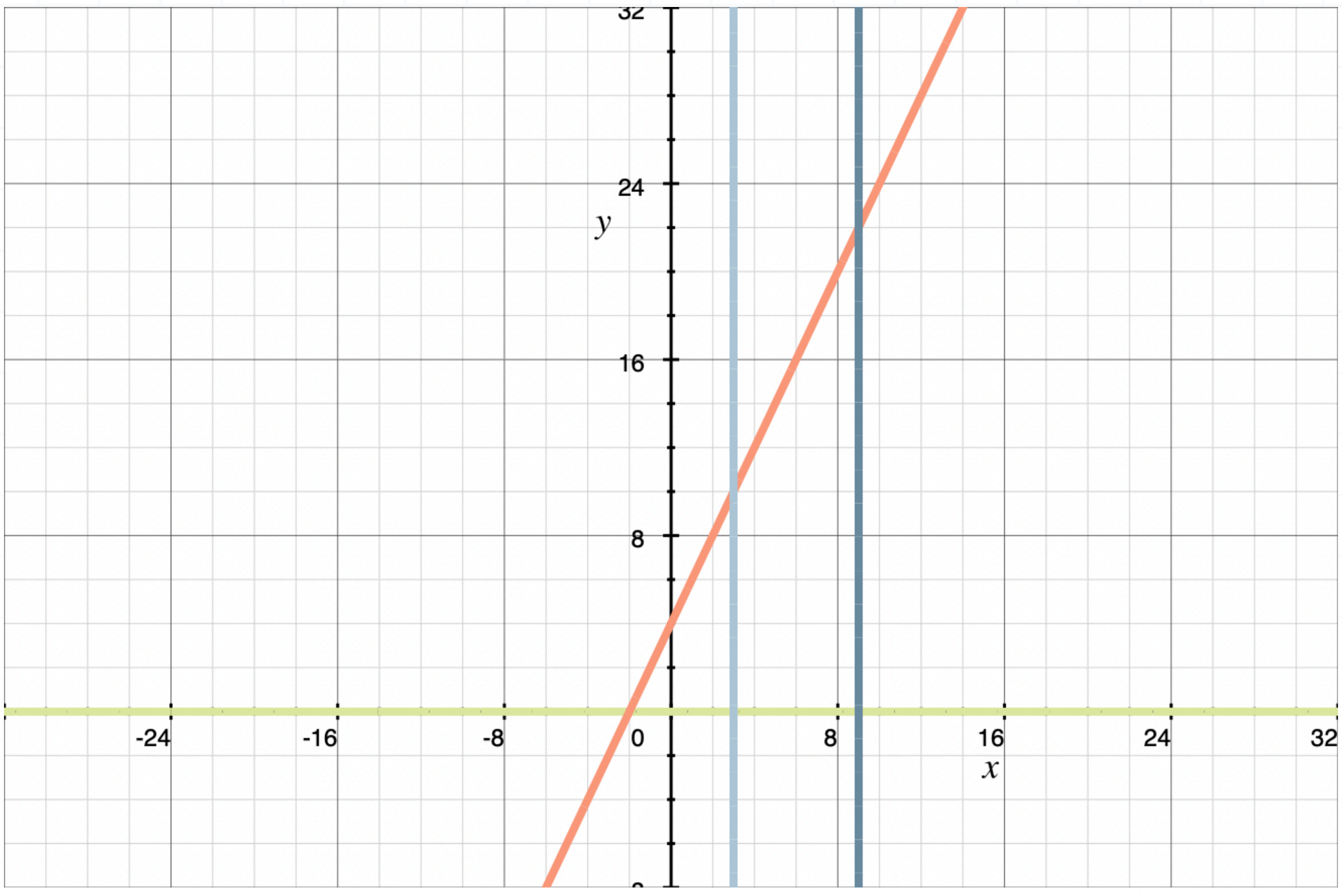
B $\left(\frac{51}{8}, \frac{67}{4}\right)$

C $\left(\frac{67}{8}, \frac{51}{8}\right)$

D $\left(\frac{51}{4}, \frac{67}{4}\right)$

Solution: A

A sketch of all the curves is



The limits of integration will be $x = [3,9]$, and the region is bounded above by $f(x) = 2x + 4$ and below by $g(x) = 0$. Then the area of the region is

$$A = \int_a^b f(x) - g(x)\ dx$$

$$A = \int_3^9 2x + 4 - 0\ dx$$

$$A = x^2 + 4x \Big|_3^9$$

$$A = 9^2 + 4(9) - (3^2 + 4(3))$$

$$A = 81 + 36 - 9 - 12$$

$$A = 96$$

So the coordinates of the centroid are

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\ dx$$

$$\bar{x} = \frac{1}{96}\int_3^9 x(2x + 4 - 0)\ dx$$

$$\bar{x} = \frac{1}{96}\int_3^9 2x^2 + 4x\ dx$$

$$\bar{x} = \frac{1}{96}\left(\frac{2}{3}x^3 + 2x^2\right)\Bigg|_3^9$$

$$\bar{x} = \frac{1}{96}\left(\frac{2}{3}(9)^3 + 2(9)^2\right) - \frac{1}{96}\left(\frac{2}{3}(3)^3 + 2(3)^2\right)$$

$$\bar{x} = \frac{1}{96}(486 + 162) - \frac{1}{96}(18 + 18)$$

$$\bar{x} = \frac{51}{8}$$

and

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]\ dx$$

$$\bar{y} = \frac{1}{96}\int_3^9 \frac{1}{2}(2x + 4)^2\ dx$$

$$\bar{y} = \frac{1}{192} \int_3^9 4x^2 + 16x + 16 \ dx$$

$$\bar{y} = \frac{1}{192} \left(\frac{4}{3}x^3 + 8x^2 + 16x \right) \Bigg|_3^9$$

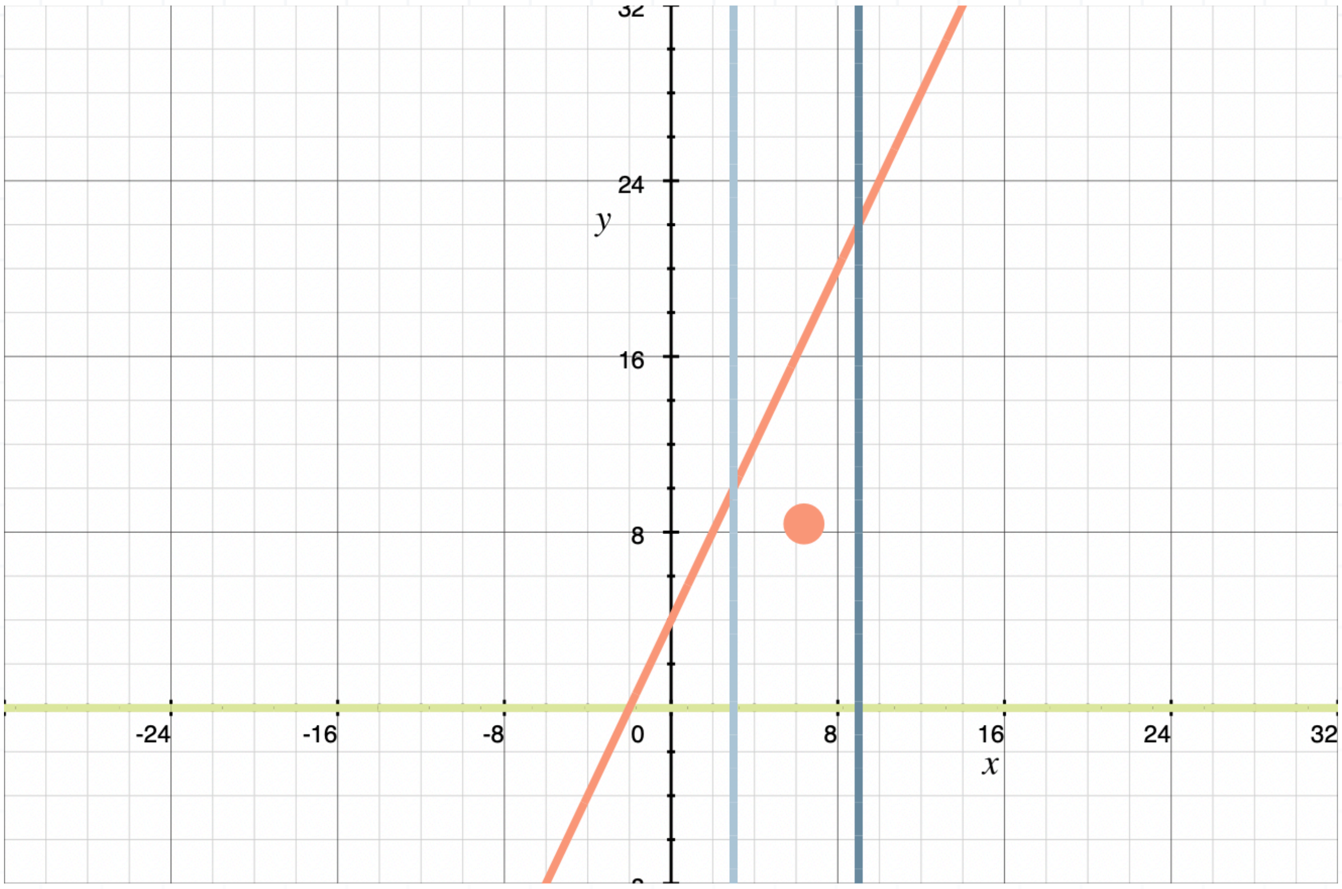
$$\bar{y} = \frac{1}{192} \left(\frac{4}{3}(9)^3 + 8(9)^2 + 16(9) \right) - \frac{1}{192} \left(\frac{4}{3}(3)^3 + 8(3)^2 + 16(3) \right)$$

$$\bar{y} = \frac{1}{192}(972 + 648 + 144) - \frac{1}{192}(36 + 72 + 48)$$

$$\bar{y} = \frac{1{,}764}{192} - \frac{156}{192}$$

$$\bar{y} = \frac{67}{8}$$

Therefore, the centroid of the region is at $(51/8, 67/8)$. We can confirm this visually by graphing it in the region.



32
24
16
8
y
-24
-16
-8
0
8
16
24
32
x

Topic: Area of a triangle with given vertices

Question: Find the area of the triangle with the given vertices.

$(-3,2)$

$(5,4)$

$(7,1)$

Answer choices:

A 70 square units

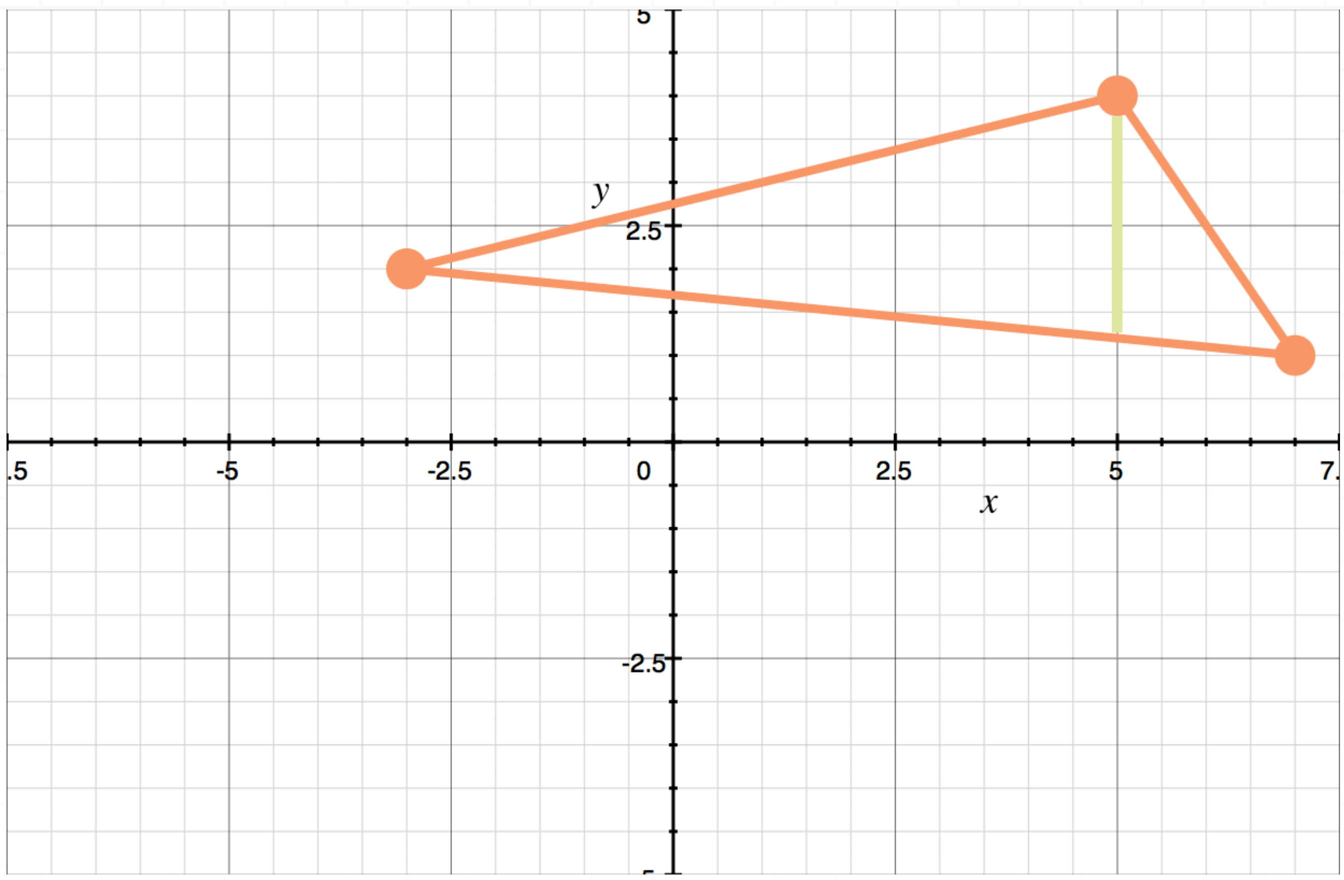
B $\frac{56}{5}$ square units

C $\frac{14}{5}$ square units

D 14 square units

Solution: D

The triangle with vertices $(-3,2)$, $(5,4)$, and $(7,1)$ is



Finding the area of the triangle is really the same as finding the area between two curves. Based on the orientation of the triangle, we will integrate the difference between the side on top of the triangle and the side on the bottom of the triangle in two different intervals. We'll deal with the area to the left of the dashed line first, then find the area to the right of the dashed line, and then add the areas together to find total area.

First, we'll find the equation of the line that defines each side of the triangle. Let's start with the line connecting $(-3,2)$ and $(5,4)$. The slope of that line is

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-2}{5-(-3)}=\frac{1}{4}$$

Use the point $(5,4)$ on that line, and the slope of the line $m=1/4$ that we just found to plug into the point-slope formula for the equation of the line.

$$y-y_1=m(x-x_1)$$

$$y-4=\frac{1}{4}(x-5)$$

$$y=\frac{1}{4}x-\frac{5}{4}+4$$

$$y=\frac{1}{4}x+\frac{11}{4}$$

Next, let's find the equation of the side connecting $(5,4)$ with $(7,1)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-4}{7-5}=-\frac{3}{2}$$

Use the point $(5,4)$ on that line, and the slope of the line $m=-3/2$ that we just found to plug into the point-slope formula for the equation of the line.

$$y-y_1=m(x-x_1)$$

$$y-4=-\frac{3}{2}(x-5)$$

$$y=-\frac{3}{2}x+\frac{15}{2}+4$$

$$y=-\frac{3}{2}x+\frac{23}{2}$$

Next, let's find the equation of the side connecting $(-3,2)$ with $(7,1)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-2}{7-(-3)}=-\frac{1}{10}$$

Use the point $(7,1)$ on that line, and the slope of the line $m=-1/10$ that we just found to plug into the point-slope formula for the equation of the line.

$$y-y_1=m(x-x_1)$$

$$y-1=-\frac{1}{10}(x-7)$$

$$y=-\frac{1}{10}x+\frac{7}{10}+1$$

$$y=-\frac{1}{10}x+\frac{17}{10}$$

Now that we have the equations for all three sides, this is simply an area between curves problem. The area to the left of the dashed line is

$$A=\int_{-3}^{5}\left(\frac{1}{4}x+\frac{11}{4}\right)-\left(-\frac{1}{10}x+\frac{17}{10}\right)\ dx$$

$$A=\int_{-3}^{5}\frac{1}{4}x+\frac{11}{4}+\frac{1}{10}x-\frac{17}{10}\ dx$$

$$A=\int_{-3}^{5}\frac{7}{20}x+\frac{21}{20}\ dx$$

$$A=\frac{7}{40}x^2+\frac{21}{20}x\Big|_{-3}^{5}$$

$$A=\frac{7}{40}(5)^2+\frac{21}{20}(5)-\left(\frac{7}{40}(-3)^2+\frac{21}{20}(-3)\right)$$

$$A=\frac{35}{8}+\frac{21}{4}-\frac{63}{40}+\frac{63}{20}$$

$$A=\frac{175}{40}+\frac{210}{40}-\frac{63}{40}+\frac{126}{40}$$

$$A=\frac{56}{5}$$

The area to the right of the dashed line is

$$A=\int_5^7\left(-\frac{3}{2}x+\frac{23}{2}\right)-\left(-\frac{1}{10}x+\frac{17}{10}\right)\ dx$$

$$A=\int_5^7-\frac{3}{2}x+\frac{23}{2}+\frac{1}{10}x-\frac{17}{10}\ dx$$

$$A=\int_5^7-\frac{7}{5}x+\frac{49}{5}\ dx$$

$$A=-\frac{7}{10}x^2+\frac{49}{5}x\Big|_5^7$$

$$A=-\frac{7}{10}(7)^2+\frac{49}{5}(7)-\left(-\frac{7}{10}(5)^2+\frac{49}{5}(5)\right)$$

$$A=-\frac{343}{10}+\frac{343}{5}+\frac{35}{2}-49$$

$$A=-\frac{343}{10}+\frac{686}{10}+\frac{175}{10}-\frac{490}{10}$$

$$A = \frac{14}{5}$$

To find total area, we'll just add these two regions together.

$$A = \frac{56}{5} + \frac{14}{5} = 14$$

Topic: Area of a triangle with given vertices

Question: Find the area of the triangle with the given vertices.

$(1,5)$

$(5,1)$

$(7,4)$

Answer choices:

A $\frac{20}{3}$ square units

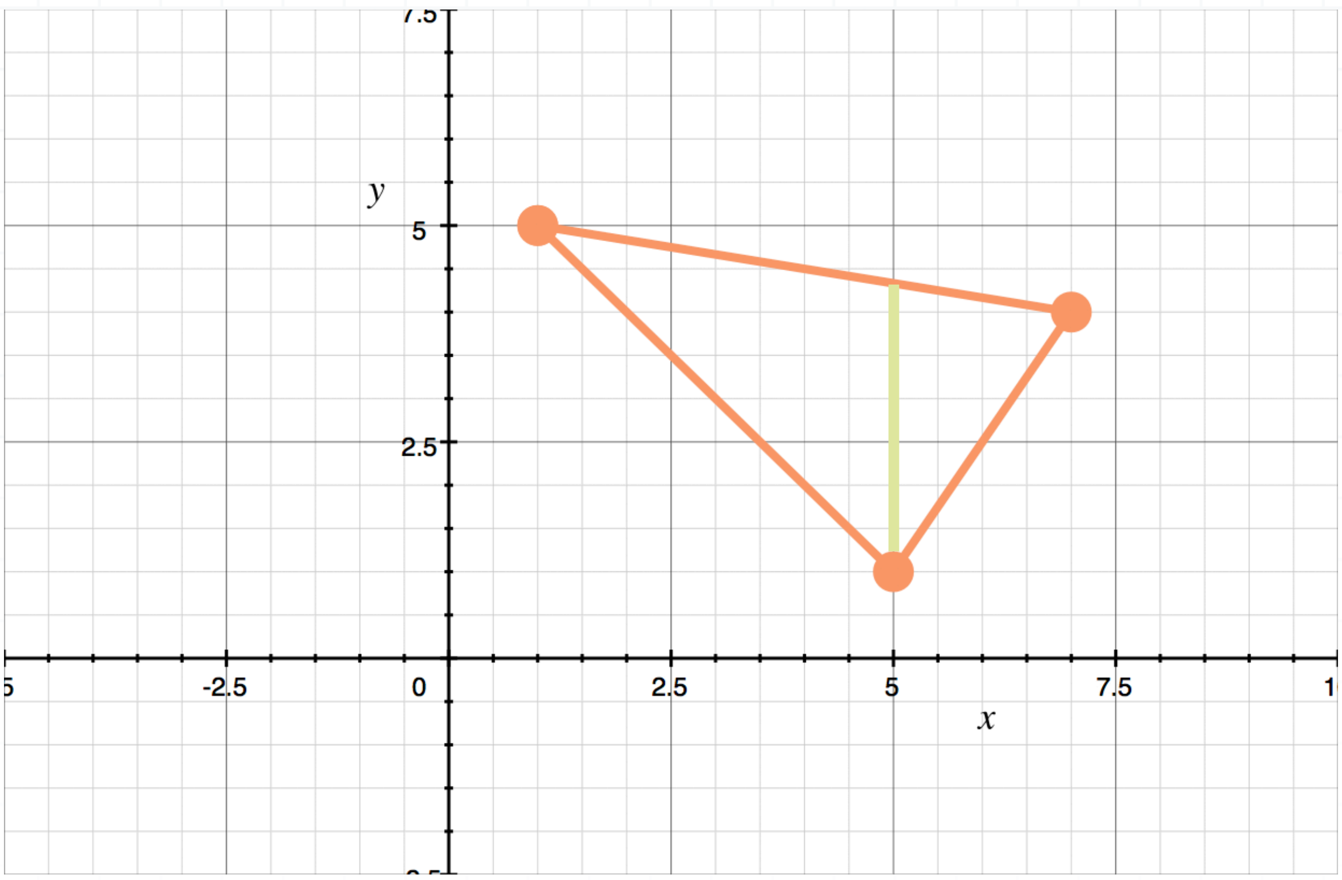
B 30 square units

C 10 square units

D $\frac{10}{3}$ square units

Solution: C

The triangle with vertices $(1,5)$, $(5,1)$, and $(7,4)$ is



Finding the area of the triangle is really the same as finding the area between two curves. Based on the orientation of the triangle, we will integrate the difference between the side on top of the triangle and the side on the bottom of the triangle in two different intervals. We'll deal with the area to the left of the dashed line first, then find the area to the right of the dashed line, and then add the areas together to find total area.

First, we'll find the equation of the line that defines each side of the triangle. Let's start with the line connecting $(1,5)$ and $(5,1)$. The slope of that line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{1-5} = -1$$

Use the point $(1,5)$ on that line, and the slope of the line $m=-1$ that we just found to plug into the point-slope formula for the equation of the line.

$$y-y_1=m(x-x_1)$$

$$y-5=-1(x-1)$$

$$y=-x+1+5$$

$$y=-x+6$$

Next, let's find the equation of the side connecting $(5,1)$ with $(7,4)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-4}{5-7}=\frac{3}{2}$$

Use the point $(5,1)$ on that line, and the slope of the line $m=3/2$ that we just found to plug into the point-slope formula for the equation of the line.

$$y-y_1=m(x-x_1)$$

$$y-1=\frac{3}{2}(x-5)$$

$$y=\frac{3}{2}x-\frac{15}{2}+1$$

$$y=\frac{3}{2}x-\frac{13}{2}$$

Next, let's find the equation of the side connecting $(7,4)$ with $(1,5)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-5}{7-1}=-\frac{1}{6}$$

Use the point $(7,4)$ on that line, and the slope of the line $m = -1/6$ that we just found to plug into the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{6}(x - 7)$$

$$y = -\frac{1}{6}x + \frac{7}{6} + 4$$

$$y = -\frac{1}{6}x + \frac{31}{6}$$

Now that we have the equations for all three sides, this is simply an area between curves problem. The area to the left of the dashed line is

$$A = \int_1^5 \left(-\frac{1}{6}x + \frac{31}{6}\right) - \left(-x + 6\right)\ dx$$

$$A = \int_1^5 -\frac{1}{6}x + \frac{31}{6} + x - 6\ dx$$

$$A = \int_1^5 \frac{5}{6}x - \frac{5}{6}\ dx$$

$$A = \frac{5}{12}x^2 - \frac{5}{6}x\Big|_1^5$$

$$A = \frac{5}{12}(5)^2 - \frac{5}{6}(5) - \left(\frac{5}{12}(1)^2 - \frac{5}{6}(1)\right)$$

$$A = \frac{125}{12} - \frac{25}{6} - \frac{5}{12} + \frac{5}{6}$$

$$A=\frac{125}{12}-\frac{50}{12}-\frac{5}{12}+\frac{10}{12}$$

$$A=\frac{20}{3}$$

The area to the right of the dashed line is

$$A=\int_5^7\left(-\frac{1}{6}x+\frac{31}{6}\right)-\left(\frac{3}{2}x-\frac{13}{2}\right)\ dx$$

$$A=\int_5^7-\frac{1}{6}x+\frac{31}{6}-\frac{3}{2}x+\frac{13}{2}\ dx$$

$$A=\int_5^7-\frac{5}{3}x+\frac{35}{3}\ dx$$

$$A=-\frac{5}{6}x^2+\frac{35}{3}x\Big|_5^7$$

$$A=-\frac{5}{6}(7)^2+\frac{35}{3}(7)-\left(-\frac{5}{6}(5)^2+\frac{35}{3}(5)\right)$$

$$A=-\frac{245}{6}+\frac{245}{3}+\frac{125}{6}-\frac{175}{3}$$

$$A=-\frac{245}{6}+\frac{490}{6}+\frac{125}{6}-\frac{350}{6}$$

$$A=\frac{10}{3}$$

To find total area, we'll just add these two regions together.

$$A=\frac{20}{3}+\frac{10}{3}=10$$

Topic: Area of a triangle with given vertices

Question: Find the area of the triangle with the given vertices.

$(-3,1)$

$(0,6)$

$(3,4)$

Answer choices:

A $\frac{21}{2}$ square units

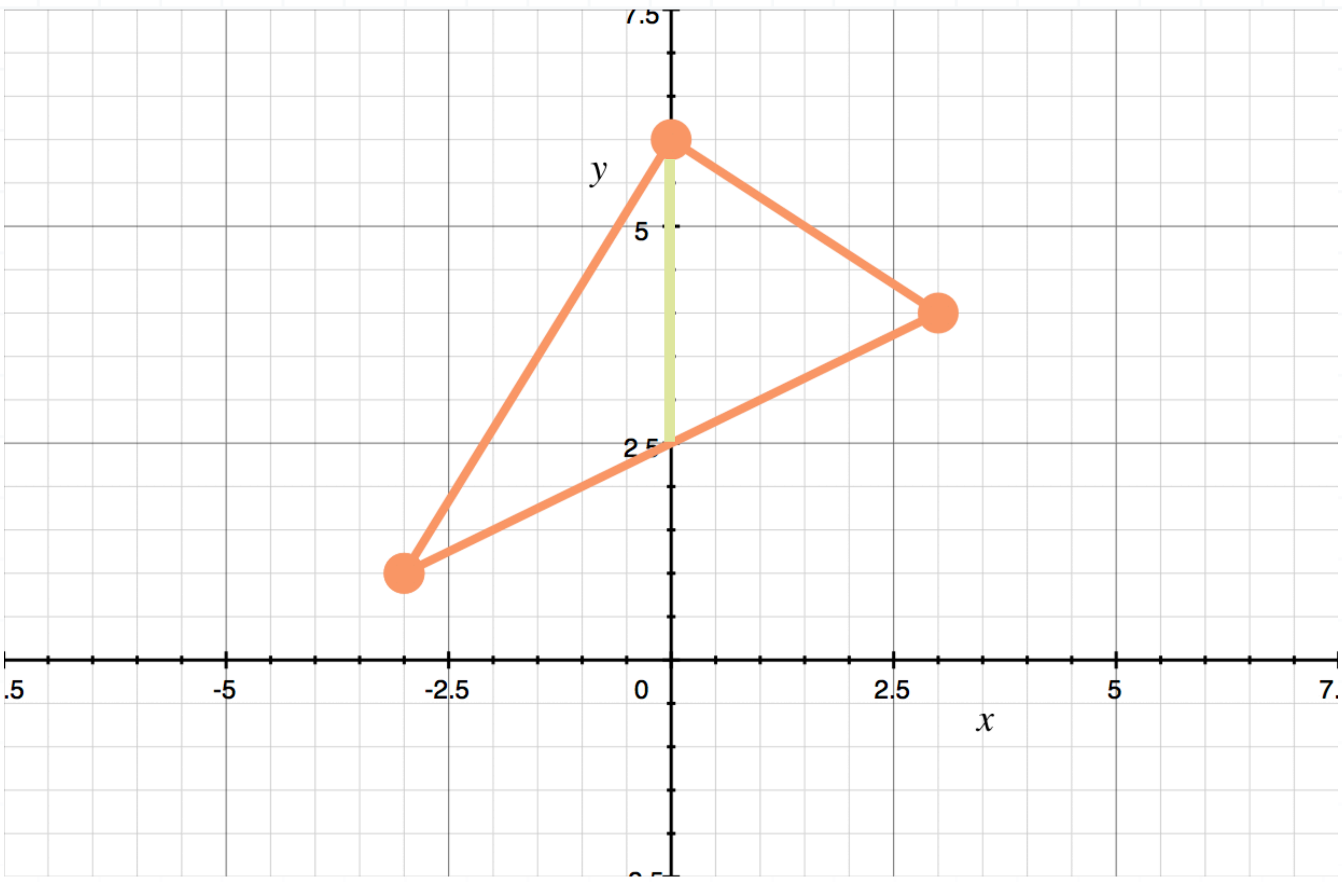
B 21 square units

C $\frac{21}{4}$ square units

D 14 square units

Solution: A

The triangle with vertices $(-3,1)$, $(0,6)$, and $(3,4)$ is



Finding the area of the triangle is really the same as finding the area between two curves. Based on the orientation of the triangle, we will integrate the difference between the side on top of the triangle and the side on the bottom of the triangle in two different intervals. We'll deal with the area to the left of the dashed line first, then find the area to the right of the dashed line, and then add the areas together to find total area.

First, we'll find the equation of the line that defines each side of the triangle. Let's start with the line connecting $(-3,1)$ and $(0,6)$. The slope of that line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-6}{-3-0} = \frac{5}{3}$$

Use the point $(0,6)$ on that line, and the slope of the line $m = 5/3$ that we just found to plug into the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - 6 = \frac{5}{3}(x - 0)$$

$$y = \frac{5}{3}x + 6$$

Next, let's find the equation of the side connecting $(-3,1)$ with $(3,4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{-3 - 3} = \frac{1}{2}$$

Use the point $(3,4)$ on that line, and the slope of the line $m = 1/2$ that we just found to plug into the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{2}(x - 3)$$

$$y = \frac{1}{2}x - \frac{3}{2} + 4$$

$$y = \frac{1}{2}x + \frac{5}{2}$$

Next, let's find the equation of the side connecting $(0,6)$ with $(3,4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 4}{0 - 3} = -\frac{2}{3}$$

Use the point $(0,6)$ on that line, and the slope of the line $m = -2/3$ that we just found to plug into the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$y - 6 = -\frac{2}{3}(x - 0)$$

$$y = -\frac{2}{3}x + 6$$

Now that we have the equations for all three sides, this is simply an area between curves problem. The area to the left of the dashed line is

$$A = \int_{-3}^{0} \left(\frac{5}{3}x + 6\right) - \left(\frac{1}{2}x + \frac{5}{2}\right)\ dx$$

$$A = \int_{-3}^{0} \frac{5}{3}x + 6 - \frac{1}{2}x - \frac{5}{2}\ dx$$

$$A = \int_{-3}^{0} \frac{7}{6}x + \frac{7}{2}\ dx$$

$$A = \frac{7}{12}x^2 + \frac{7}{2}x\Big|_{-3}^{0}$$

$$A = \frac{7}{12}(0)^2 + \frac{7}{2}(0) - \left(\frac{7}{12}(-3)^2 + \frac{7}{2}(-3)\right)$$

$$A = -\frac{21}{4} + \frac{21}{2}$$

$$A = \frac{21}{4}$$

The area to the right of the dashed line is

$$A=\int_0^3\left(-\frac{2}{3}x+6\right)-\left(\frac{1}{2}x+\frac{5}{2}\right)\ dx$$

$$A=\int_0^3-\frac{2}{3}x+6-\frac{1}{2}x-\frac{5}{2}\ dx$$

$$A=\int_0^3-\frac{7}{6}x+\frac{7}{2}\ dx$$

$$A=-\frac{7}{12}x^2+\frac{7}{2}x\Big|_0^3$$

$$A=-\frac{7}{12}(3)^2+\frac{7}{2}(3)-\left(-\frac{7}{12}(0)^2+\frac{7}{2}(0)\right)$$

$$A=-\frac{21}{4}+\frac{21}{2}$$

$$A=\frac{21}{4}$$

To find total area, we'll just add these two regions together.

$$A=\frac{21}{4}+\frac{21}{4}=\frac{21}{2}$$

Topic: Single deposit, compounded n times, future value

Question: Find the future value of a savings bond after 4 years, if the present value is $\$500$ and the bond has monthly compounded annual interest of 10%.

Answer choices:

A $\$700.00$

B $\$612.19$

C $\$744.68$

D $\$551.97$

Solution: C

Plugging the values we've been given into the future value formula, we get

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$FV = 500\left(1+\frac{0.10}{12}\right)^{(12)(4)}$$

$$FV = 500\left(1+\frac{0.10}{12}\right)^{48}$$

$$FV \approx \$744.68$$

Topic: Single deposit, compounded n times, future value

Question: Find the future value of $\$12{,}000$ after 6 years, at an annual rate of $3.5\,\%$, compounded quarterly.

Answer choices:

A $\$14{,}751.06$

B $\$14{,}790.62$

C $\$27{,}399.94$

D $\$14{,}520.00$

Solution: B

Plugging the values we've been given into the future value formula, we get

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$FV = 12{,}000\left(1+\frac{0.035}{4}\right)^{(4)(6)}$$

$$FV = 12{,}000\left(1+\frac{0.035}{4}\right)^{24}$$

$$FV \approx \$14{,}790.62$$

Topic: Single deposit, compounded n times, future value

Question: Find the future value of $\$15{,}000$ after 9 years, at an annual rate of $4.5\,\%$, compounded monthly.

Answer choices:

A $\$21{,}075.00$

B $\$15{,}513.91$

C $\$22{,}291.43$

D $\$22{,}472.51$

Solution: D

Plugging the values we've been given into the future value formula, we get

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt}$$

$$FV = 15{,}000\left(1 + \frac{0.045}{12}\right)^{(12)(9)}$$

$$FV = 15{,}000\left(1 + \frac{0.045}{12}\right)^{108}$$

$$FV \approx \$22{,}472.51$$

Topic: Single deposit, compounded n times, present value

Question: Find the present value of a single deposit that has a future value of $\$945$ after 6 years, if the account carries an annual interest of 5.5%, but compounds monthly.

Answer choices:

A $\$679.90$

B $\$824.63$

C $\$743.88$

D $\$523.81$

Solution: A

Since we've been asked to find present value, we'll solve the future value formula for PV.

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$PV = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$$

Plugging the values we've been given into this formula, remembering that $n = 12$ since there are 12 months in a year and interest is compounded monthly, we get

$$PV = \frac{945}{\left(1+\frac{0.055}{12}\right)^{(12)(6)}}$$

$$PV = \frac{945}{1.005^{72}}$$

$$PV \approx \$679.90$$

Topic: Single deposit, compounded n times, present value

Question: Find the present value of an investment that has a future value of $\$7,345.25$ after 10 years, if the weekly compounded annual interest rate is 4.3%.

Answer choices:

A $\$3,846.43$

B $\$6,486.43$

C $\$2,346.43$

D $\$4,779.00$

Solution: D

Since we've been asked to find present value, we'll solve the future value formula for PV.

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

$$PV = \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}}$$

Plugging the values we've been given into this formula, remembering that $n = 52$ since there are 52 weeks in a year and interest is compounded weekly, we get

$$PV = \frac{7{,}345.25}{\left(1+\frac{0.043}{52}\right)^{(52)(10)}}$$

$$PV = \frac{7{,}345.25}{\left(1+\frac{0.043}{52}\right)^{520}}$$

$$PV \approx \$4{,}779.00$$

Topic: Single deposit, compounded continuously, future value

Question: Find the future value of $\$9{,}500$ after 10 years, at an annual interest rate of 1.5%, compounded continuously.

Answer choices:

A $\$10{,}925.00$

B $\$11{,}036.39$

C $\$11{,}034.33$

D $\$11{,}037.43$

Solution: D

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$FV = 9{,}500e^{(0.015)(10)}$$

$$FV = 9{,}500e^{0.15}$$

$$FV \approx \$11{,}037.43$$

Topic: Single deposit, compounded continuously, future value

Question: Find the future value of $\$13{,}900$ after 5 years, at an annual interest rate of $4.5\,\%$, compounded continuously.

Answer choices:

A $\$17{,}027.50$

B $\$17{,}407.29$

C $\$17{,}321.93$

D $\$17{,}385.43$

Solution: B

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$FV = 13{,}900e^{(0.045)(5)}$$

$$FV = 13{,}900e^{0.225}$$

$$FV \approx \$17{,}407.29$$

Topic: Single deposit, compounded continuously, future value

Question: Find the future value of $\$11{,}400$ after 3 years, at an annual interest rate of $6.5\,\%$, compounded continuously.

Answer choices:

A $\$13{,}854.55$

B $\$13{,}770.63$

C $\$13{,}623.00$

D $\$13{,}832.85$

Solution: A

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$FV = 11{,}400e^{(0.065)(3)}$$

$$FV = 11{,}400e^{0.195}$$

$$FV \approx \$13{,}854.55$$

Topic: Single deposit, compounded continuously, present value

Question: Find the present value of a deposit that, after 8 years, at an annual interest rate of 3.7%, compounded continuously, will have a value of $\$9,209.62$.

Answer choices:

A $\$6,875.00$

B $\$6,850.00$

C $\$6,795.00$

D $\$6,800.00$

Solution: B

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$9{,}209.62 = PVe^{(0.037)(8)}$$

$$9{,}209.62 = PVe^{0.296}$$

Solve for PV.

$$PV = \frac{9{,}209.62}{e^{0.296}}$$

$$PV \approx \$6{,}850.00$$

Topic: Single deposit, compounded continuously, present value

Question: Find the present value of a deposit that, after 7 years, at an annual interest rate of 6.2%, compounded continuously, will have a value of $\$24,308.85$.

Answer choices:

A $\$15,875.00$

B $\$15,700.00$

C $\$15,795.00$

D $\$15,750.00$

Solution: D

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$24{,}308.85 = PVe^{(0.062)(7)}$$

$$24{,}308.85 = PVe^{0.434}$$

Solve for PV.

$$PV = \frac{24{,}308.85}{e^{0.434}}$$

$$PV \approx \$15{,}750.00$$

Topic: Single deposit, compounded continuously, present value

Question: Find the present value of a deposit that, after 5 years, at an annual interest rate of 2.75%, compounded continuously, will have a value of $\$10,240.56$.

Answer choices:

A $\$8,925.00$

B $\$8,955.00$

C $\$8,945.00$

D $\$8,835.00$

Solution: A

Plugging the values we've been given into the future value formula for continuous compounding, we get

$$FV = PVe^{rt}$$

$$10{,}240.56 = PVe^{(0.0275)(5)}$$

$$10{,}240.56 = PVe^{0.1375}$$

Solve for PV.

$$PV = \frac{10{,}240.56}{e^{0.1375}}$$

$$PV \approx \$8{,}925.00$$

Topic: Income stream, compounded continuously, future value

Question: Money is invested at a rate of $\$12{,}000$ annually and the bank pays 14.5% interest, compounded continuously. How many years will it take for the investment to reach half a million dollars?

Answer choices:

A $\quad 13.5$

B $\quad 14.5$

C $\quad 15.5$

D $\quad 16.5$

Solution: A

Plugging the values we know into the future value formula for a continuous income stream, we get

$$FV=\int_0^N S(t)e^{r(N-t)}\ dt$$

$$500{,}000=\int_0^N 12{,}000e^{0.145(N-t)}\ dt$$

$$500{,}000=12{,}000\int_0^N e^{0.145N-0.145t}\ dt$$

$$500{,}000=12{,}000\int_0^N e^{0.145N}e^{-0.145t}\ dt$$

$$500{,}000=12{,}000e^{0.145N}\int_0^N e^{-0.145t}\ dt$$

Integrate, then evaluate over the interval.

$$500{,}000=12{,}000e^{0.145N}\left(\frac{1}{-0.145}e^{-0.145t}\right)\Bigg|_0^N$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}}=e^{-0.145t}\Big|_0^N$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}}=e^{-0.145N}-e^{-0.145(0)}$$

$$\frac{500{,}000(-0.145)}{12{,}000e^{0.145N}} = e^{-0.145N} - 1$$

Multiply through by $e^{0.145N}$ to collect the N variables on one side of the equation.

$$\frac{500{,}000(-0.145)}{12{,}000} = e^{-0.145N}e^{0.145N} - 1e^{0.145N}$$

$$\frac{500{,}000(-0.145)}{12{,}000} = 1 - e^{0.145N}$$

$$e^{0.145N} = 1 - \frac{500{,}000(-0.145)}{12{,}000}$$

$$e^{0.145N} = 1 + \frac{125(0.145)}{3}$$

Take the natural log of both sides to solve for N.

$$\ln(e^{0.145N}) = \ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$0.145N = \ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$N = \frac{1}{0.145}\ln\left(1 + \frac{125(0.145)}{3}\right)$$

$$N \approx 13.5$$

Topic: Income stream, compounded continuously, future value

Question: Money is invested at $\$12{,}000$ annually and the bank pays $4.5\ \%$ interest, compounded continuously. What is the balance in the account after five years?

Answer choices:

A $\$67{,}286$

B $\$65{,}286$

C $\$65{,}000$

D $\$60{,}286$

Solution: A

Plugging the values we've been given into the future value formula for a continuous income stream, we get

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$FV = \int_0^5 12{,}000e^{0.045(5-t)}\ dt$$

$$FV = 12{,}000\int_0^5 e^{0.225}e^{-0.045t}\ dt$$

$$FV = 12{,}000e^{0.225}\int_0^5 e^{-0.045t}\ dt$$

Integrate, then evaluate over the interval.

$$FV = 12{,}000e^{0.225}\left(\frac{1}{-0.045}e^{-0.045t}\right)\Big|_0^5$$

$$FV = -\frac{12{,}000e^{0.225}}{0.045}(e^{-0.045(5)} - e^{-0.045(0)})$$

$$FV = -\frac{12{,}000e^{0.225}}{0.045}(e^{-0.225} - 1)$$

$$FV \approx \$67{,}286$$

Topic: Income stream, compounded continuously, future value

Question: Find the present and future value of an income stream given by $S(t) = \$1{,}600e^{0.02t}$, if the bank pays $4\,\%$ interest, compounded continuously, for 10 years.

Answer choices:

A $\quad PV = \$12{,}501.45$ and $FV = \$21{,}001.76$

B $\quad PV = \$14{,}501.54$ and $FV = \$21{,}633.76$

C $\quad PV = \$16{,}323.65$ and $FV = \$23{,}633.26$

D $\quad PV = \$14{,}105.54$ and $FV = \$19{,}366.75$

Solution: B

Plugging the values we've been given into the present value formula for a continuous income stream, we get

$$PV=\int_0^N S(t)e^{-rt}\ dt$$

$$PV=\int_0^{10} 1{,}600e^{0.02t}e^{-0.04t}\ dt$$

$$PV=1{,}600\int_0^{10} e^{0.02t-0.04t}\ dt$$

$$PV=1{,}600\int_0^{10} e^{-0.02t}\ dt$$

Integrate, then evaluate over the interval.

$$PV=\frac{1{,}600}{-0.02}e^{-0.02t}\Big|_0^{10}$$

$$PV=-80{,}000(e^{-0.02(10)}-e^{-0.02(0)})$$

$$PV=-80{,}000(e^{-0.2}-1)$$

$$PV\approx \$14{,}501.54$$

We'll rewrite the formula for the future value of a continuous income stream,

$$FV = \int_0^N S(t)e^{r(N-t)}\ dt$$

$$FV = \int_0^N S(t)e^{rN}e^{-rt}\ dt$$

$$FV = e^{rN}\int_0^N S(t)e^{-rt}\ dt$$

such that the remaining integral is now equivalent to the present value of the income stream.

$$FV = e^{rN}PV$$

$$FV = e^{(0.04)(10)}(14{,}501.54)$$

$$FV \approx \$21{,}633.76$$

Topic: Income stream, compounded continuously, present value

Question: If $\$25{,}000$ is deposited into an account every year for 7 years and the account pays $6.25\,\%$, compounded continuously, find the present value of the account.

Answer choices:

A $\$217{,}483.98$

B $\$219{,}532.12$

C $\$142{,}897.16$

D $\$141{,}740.59$

Solution: D

Plugging what we know into the present value formula for a continuous income stream, we get

$$PV = \int_0^T S(t)e^{-rt}\ dt$$

$$PV = \int_0^7 25{,}000e^{-0.0625t}\ dt$$

Integrate, then evaluate over the interval.

$$PV = -\frac{25{,}000}{0.0625}e^{-0.0625t}\Big|_0^7$$

$$PV = -\frac{25{,}000}{0.0625}e^{-0.0625(7)} - \left(-\frac{25{,}000}{0.0625}e^{-0.0625(0)}\right)$$

$$PV = -\frac{25{,}000}{0.0625}e^{-0.4375} + \frac{25{,}000}{0.0625}(1)$$

$$PV \approx \$141{,}740.59$$

Topic: Income stream, compounded continuously, present value

Question: Suppose that money is deposited steadily into an account at a constant rate of $\$1{,}250$ per year for 5 years. Find the present value of this income stream if the account pays $8.35\,\%$, compounded continuously.

Answer choices:

A $\$7{,}684.76$

B $\$5{,}109.41$

C $\$7{,}756.91$

D $\$5{,}151.25$

Solution: B

Plugging what we know into the present value formula for a continuous income stream, we get

$$PV = \int_0^T S(t)e^{-rt}\ dt$$

$$PV = \int_0^5 1{,}250e^{-0.0835t}\ dt$$

Integrate, then evaluate over the interval.

$$PV = -\frac{1{,}250}{0.0835}e^{-0.0835t}\Big|_0^5$$

$$PV = -\frac{1{,}250}{0.0835}e^{-0.0835(5)} - \left(-\frac{1{,}250}{0.0835}e^{-0.0835(0)}\right)$$

$$PV = -\frac{1{,}250}{0.0835}e^{-0.4175} + \frac{1{,}250}{0.0835}$$

$$PV \approx \$5{,}109.41$$

Topic: Income stream, compounded continuously, present value

Question: Suppose that money is deposited steadily into an account at a constant rate of $\$5,600$ per year for 12 years. Find the present value of this income stream if the account pays 4.9%, compounded continuously.

Answer choices:

A $\quad \$91,472.46$

B $\quad \$50,533.10$

C $\quad \$50,807.19$

D $\quad \$92,077.31$

Solution: C

Plugging what we know into the present value formula for a continuous income stream, we get

$$PV=\int_0^T S(t)e^{-rt}\ dt$$

$$PV=\int_0^{12} 5{,}600e^{-0.049t}\ dt$$

Integrate, then evaluate over the interval.

$$PV=-\frac{5{,}600}{0.049}e^{-0.049t}\Big|_0^{12}$$

$$PV=-\frac{5{,}600}{0.049}e^{-0.049(12)}-\left(-\frac{5{,}600}{0.049}e^{-0.049(0)}\right)$$

$$PV=-\frac{5{,}600}{0.049}e^{-0.588}+\frac{5{,}600}{0.049}$$

$$PV\approx \$50{,}807.19$$

Topic: Consumer and producer surplus

Question: Find equilibrium quantity and equilibrium price.

$$S(q) = 0.04q^2 + 4$$

$$D(q) = -0.3q + 11$$

Answer choices:

A $q_e = 8$ and $p_e = 10$

B $q_e = 7$ and $p_e = 9.1$

C $q_e = 9.1$ and $p_e = 7$

D $q_e = 10$ and $p_e = 8$

Solution: D

In any supply and demand system, there's an equilibrium point where the supply curve $S(q)$ and the demand curve $D(q)$ intersect each other.

The intersection point is associated with a specific *equilibrium quantity* and *equilibrium price*:

The equilibrium quantity is the x-value of the intersection point

The equilibrium price is the y-value of the intersection point

In order to find these values, we'll set the demand and supply curves equal to one another and solve for q, the equilibrium quantity. It's possible to get more then one value for q, but keep in mind that the equilibrium quantity must be a positive number, so we can discard any negative answers.

$$-0.3q+11=0.04q^2+4$$

$$0.04q^2+0.3q-7=0$$

$$\left(0.04q^2+0.3q-7=0\right)100$$

$$4q^2+30q-700=0$$

$$(4q+70)(q-10)=0$$

$$4q+70=0$$

$$4q=-70$$

$$q=-\frac{35}{2}$$

or

$$q - 10 = 0$$

$$q = 10$$

Since q must be positive, we can discard $q = -35/2$. Equilibrium quantity must be $q = 10$.

Now that we know that equilibrium quantity is $q = 10$, we can find equilibrium price by plugging $q = 10$ into either the demand function or the supply function. Remember, because the equilibrium point represents the intersection of the two functions, both functions should give us the same value for equilibrium price.

$$D(10) = -0.3(10) + 11$$

$$D(10) = 8$$

$$p = 8$$

We denote equilibrium quantity as q_e and equilibrium price as p_e, so we can say

$$q_e = 10$$

and

$$p_e = 8$$

Topic: Consumer and producer surplus

Question: Find consumer surplus.

$$S(q) = 0.02q^2 + 10$$

$$D(q) = -0.34q + 14$$

Answer choices:

A 18.08

B 10.88

C 18.80

D 10.08

Solution: B

The formula we use to find consumer surplus is

$$CS = \int_0^{q_e} D(q)\ dq - p_e q_e$$

where $D(q)$ is the demand curve, q_e is equilibrium quantity and p_e is our equilibrium price.

We've been given the demand curve, but we don't know equilibrium quantity or equilibrium price.

In any supply and demand system, there's an equilibrium point where the supply curve $S(q)$ and the demand curve $D(q)$ intersect each other.

The intersection point is associated with a specific *equilibrium quantity* and *equilibrium price*:

The equilibrium quantity is the x-value of the intersection point

The equilibrium price is the y-value of the intersection point

In order to find these values, we'll set the demand and supply curves equal to one another and solve for q, the equilibrium quantity. It's possible to get more then one value for q, but keep in mind that the equilibrium quantity must be a positive number, so we can discard any negative answers.

$$-0.34q + 14 = 0.02q^2 + 10$$

$$0.02q^2 + 0.34q - 4 = 0$$

$$\left(0.02q^2 + 0.34q - 4 = 0\right) 100$$

$$2q^2 + 34q - 400 = 0$$

$$(2q + 50)(q - 8) = 0$$

$$2q + 50 = 0$$

$$2q = -50$$

$$q = -25$$

or

$$q - 8 = 0$$

$$q = 8$$

Since q must be positive, we can discard $q = -25$. Equilibrium quantity must be $q = 8$.

Now that we know that equilibrium price is $q = 8$, we can find equilibrium price by plugging $q = 8$ into either the demand function or the supply function. Remember, because the equilibrium point represents the intersection of the two functions, both functions should give us the same value for equilibrium price.

$$D(8) = -0.34(8) + 14$$

$$D(8) = 11.28$$

$$p = 11.28$$

We denote equilibrium quantity as q_e and equilibrium price as p_e, so we can say

$$q_e = 8$$

and

$$p_e = 11.28$$

With the equilibrium point and the demand curve, we now have everything we need to solve for consumer surplus. Plugging these values into the formula, we get

$$CS = \int_0^8 -0.34q + 14\ dq - (11.28)(8)$$

$$CS = \int_0^8 -0.34q + 14\ dq - 90.24$$

$$CS = \left(-\frac{0.34}{2}q^2 + 14q \right) \Bigg|_0^8 - 90.24$$

$$CS = \left[-\frac{0.34}{2}(8)^2 + 14(8) \right] - \left[-\frac{0.34}{2}(0)^2 + 14(0) \right] - 90.24$$

$$CS = -10.88 + 112 - 90.24$$

$$CS = 10.88$$

Topic: Consumer and producer surplus

Question: Find producer surplus.

$$S(q) = 0.05q^2 + 9$$

$$D(q) = -0.1q + 15$$

Answer choices:

A $\quad 33.33$

B $\quad 30$

C $\quad 5.50$

D $\quad 5$

Solution: A

The formula we use to find consumer surplus is

$$PS = p_e q_e - \int_0^{q_e} S(q)\ dq$$

where $S(q)$ is the supply curve, q_e is equilibrium quantity and p_e is our equilibrium price.

We've been given the supply curve, but we don't know equilibrium quantity or equilibrium price.

In any supply and demand system, there's an equilibrium point where the supply curve $S(q)$ and the demand curve $D(q)$ intersect each other.

The intersection point is associated with a specific *equilibrium quantity* and *equilibrium price*:

The equilibrium quantity is the x-value of the intersection point

The equilibrium price is the y-value of the intersection point

In order to find these values, we'll set the demand and supply curves equal to one another and solve for q, the equilibrium quantity. It's possible to get more then one value for q, but keep in mind that the equilibrium quantity must be a positive number, so we can discard any negative answers.

$$-0.1q + 15 = 0.05q^2 + 9$$

$$0.05q^2 + 0.1q - 6 = 0$$

$$\left(0.05q^2 + 0.1q - 6 = 0\right) 100$$

$$5q^2 + 10q - 600 = 0$$

$$(5q + 60)(q - 10) = 0$$

$$5q + 60 = 0$$

$$5q = -60$$

$$q = -12$$

or

$$q - 10 = 0$$

$$q = 10$$

Since q must be positive, we can discard $q = -12$. Equilibrium quantity must be $q = 10$.

Now that we know that equilibrium price is $q = 10$, we can find equilibrium price by plugging $q = 10$ into either the demand function or the supply function. Remember, because the equilibrium point represents the intersection of the two functions, both functions should give us the same value for equilibrium price.

$$D(10) = -0.1(10) + 15$$

$$D(10) = 14$$

$$p = 14$$

We denote equilibrium quantity as q_e and equilibrium price as p_e, so we can say

$$q_e = 10$$

and

$$p_e = 14$$

With the equilibrium point and the supply curve, we now have everything we need to solve for producer surplus. Plugging these values into the formula, we get

$$PS = (14)(10) - \int_0^{10} 0.05q^2 + 9\ dq$$

$$PS = 140 - \int_0^{10} 0.05q^2 + 9\ dq$$

$$PS = 140 - \left(\frac{0.05}{3}q^3 + 9q\right)\Bigg|_0^{10}$$

$$PS = 140 - \left[\left[\frac{0.05}{3}(10)^3 + 9(10)\right] - \left[\frac{0.05}{3}(0)^3 + 9(0)\right]\right]$$

$$PS = 140 - \left(16.67 + 90\right)$$

$$PS = 33.33$$

Topic: Probability density functions

Question: Which of the following is a probability density function?

Answer choices:

A $\quad f(x) = x^3 \qquad -1 \le x \le 1$

B $\quad f(x) = \frac{x^3}{5{,}000}(10 - x) \qquad 0 \le x \le 10$

C $\quad f(x) = \frac{x^3}{5{,}000}(2 - x) \qquad -1 \le x \le 3$

D $\quad f(x) = \frac{x^2}{2{,}000}(4 - x) \qquad 4 \le x \le 5$

Solution: B

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

To evaluate each of answer choices as a potential probability density function, we can assess each one to see if it meet the first criterion above. The easiest way to do this is to plug in the ends of the interval.

For answer A, when we plug in the interval endpoints $-1 \le x \le 1$ we get one answer that is less than 0. This means that A is not a probability density function.

For answer B, when we plug in the interval endpoints $0 \le x \le 10$ we get two answers that are both equal to 0. This means that B is potentially a probability density function.

For answer C, when we plug in the interval endpoints $-1 \le x \le 3$ we get one answer that is less than 0. This means that C is not a probability density function.

For answer D, when we plug in the interval endpoints $4 \le x \le 5$ we get one answer that is less than 0. This means that D is not a probability density function.

Answer B is the only possibility. To be sure it’s a probability density function, we can check to make sure that it meets the second criterion.

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \int_0^{10} \frac{10}{5{,}000}x^3 - \frac{1}{5{,}000}x^4\ dx$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \frac{10}{20{,}000}x^4 - \frac{1}{25{,}000}x^5 \Big|_0^{10}$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = \left[\frac{10}{20{,}000}(10)^4 - \frac{1}{25{,}000}(10)^5\right] - \left[\frac{10}{20{,}000}(0)^4 - \frac{1}{25{,}000}(0)^5\right]$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = 5-4$$

$$\int_0^{10} \frac{x^3}{5{,}000}(10-x)\ dx = 1$$

This result verifies the second criterion, so we can say that answer choice B is a probability density function.

Topic: Probability density functions

Question: Find the probability.

$$P(x \leq 1)$$

for $f(x) = \frac{1}{4}x^3$

on the interval $0 \leq x \leq 2$

Answer choices:

A 16

B $\frac{1}{4}$

C 1

D $\frac{1}{16}$

Solution: D

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

The question gives us the function

$$f(x) = \frac{1}{4}x^3$$

and defines the interval $0 \leq x \leq 2$. The question assumes it, but let's first prove to ourselves that this is a probability density function. The first thing we need to show is that $f(x) \geq 0$ on the given interval. If we plug in the endpoints of the interval, we get

$$f(0) = \frac{1}{4}(0)^3$$

$$f(0) = 0$$

and

$$f(2) = \frac{1}{4}(2)^3$$

$$f(2) = 2$$

We've shown that $f(x) \geq 0$ at the endpoints, but what about in between the endpoints of the interval? Well, since the function starts at 0 on the left side of the interval, and works its way up to 2 on the right side of the interval, we can prove that the function is always greater than or equal to 0 if we can show that it's always increasing. To do that, we'll take the derivative, set it equal to 0 to find critical points, and then test them to show where the function is increasing and decreasing.

$$f(x) = \frac{1}{4}x^3$$

$$f'(x) = \frac{3}{4}x^2$$

$$\frac{3}{4}x^2 = 0$$

$$x = 0$$

We have one possible critical point that divides the function into the intervals

$$(-\infty,0]$$

$$[0,\infty)$$

Since the interval we're given in the problem is $0 \le x \le 2$, we're only interested in the interval to the right of the potential critical point, $[0,\infty)$, since the given interval lies entirely inside $[0,\infty)$.

We'll test the function's behavior in $[0,\infty)$ by picking one point in the interval and plugging it into the derivative. If we get a positive result, it means the function is increasing in the interval. If we get a negative result, it means the function is decreasing in the interval.

For the interval $[0,\infty)$, we'll test $x = 1$.

$$f'(1) = \frac{3}{4}(1)^2$$

$$f'(1) = \frac{3}{4}$$

Since the result is positive, it means the function is increasing in the interval $[0,\infty)$, which means we can also say that it's increasing in $0 \le x \le 2$. Therefore, we can conclude the given function is greater than or equal to 0 it its domain, which means it meets the first criterion for a probability density function.

To show that it also meets the second criterion, we need to show that the integral of the function over the given interval is equal to 1.

$$\int_0^2 \frac{1}{4}x^3\ dx$$

$$\int_0^2 \frac{1}{4}x^3\ dx = \frac{1}{16}x^4\Big|_0^2$$

$$\int_0^2 \frac{1}{4}x^3\ dx = \frac{1}{16}(2)^4 - \frac{1}{16}(0)^4$$

$$\int_0^2 \frac{1}{4}x^3\ dx = 1$$

This function also meets the second criterion, so we've proven that it represents a probability density function over the given interval.

The question is asking us to find the probability that x exists as $x \le 1$. Since we're dealing with probability, we write this as $P(x \le 1)$. Since the given interval is defined as $0 \le x \le 2$, we want to know the probability that x exists on the interval $0 \le x \le 1$.

To calculate this probability, we integrate the probability density function on the interval $0 \le x \le 1$.

$$P(x \le 1) = \int_0^1 \frac{1}{4}x^3\ dx$$

$$P(x \le 1) = \frac{1}{16}x^4 \Big|_0^1$$

$$P(x \le 1) = \frac{1}{16}(1)^4 - \frac{1}{16}(0)^4$$

$$P(x \le 1) = \frac{1}{16}$$

Topic: Probability density functions

Question: Find the probability.

$$P\left(x \leq \frac{3}{2}\right)$$

for $f(x) = \dfrac{2}{x^2}$

on the interval $1 \leq x \leq 2$

Answer choices:

A $\dfrac{3}{2}$

B $\dfrac{8}{9}$

C $\dfrac{2}{3}$

D $\dfrac{9}{8}$

Solution: C

In order for a function to be a probability density function, it must meet these two criteria:

1. The function must be greater than or equal to 0 in its entire domain.

2. The integral of the function must equal 1 in its entire domain.

The question gives us the function

$$f(x) = \frac{2}{x^2}$$

and defines the interval $1 \leq x \leq 2$. The question assumes it, but let's first prove to ourselves that this is a probability density function. The first thing we need to show is that $f(x) \geq 0$ on the given interval. If we plug in the endpoints of the interval, we get

$$f(1) = \frac{2}{(1)^2}$$

$$f(1) = 2$$

and

$$f(2) = \frac{2}{(2)^2}$$

$$f(2) = \frac{1}{2}$$

We've shown that $f(x) \geq 0$ at the endpoints, but what about in between the endpoints of the interval? Well, since the function starts at 2 on the left side of the interval, and works its way down to $1/2$ on the right side of the interval, we can prove that the function is always greater than or equal to 0 if we can show that it's always decreasing. If the function were to dip below the x-axis inside the interval, it would have to increase again to get back up to $1/2$, which is why showing that it's decreasing in the entire interval will prove that it stays positive.

To do that, we'll take the derivative, set it equal to 0 to find critical points, and then test them to show where the function is increasing and decreasing.

$$f(x) = \frac{2}{x^2}$$

$$f'(x) = -\frac{4}{x^3}$$

$$-\frac{4}{x^3} = 0$$

We can't get this function equal to 0, but it's undefined when $x = 0$. That means that $x = 0$ is a potential critical point. Since that point is outside our interval, it means that the given interval $1 \le x \le 2$ is entirely to the right of the critical point, and therefore that the function is either increasing everywhere in the interval, or decreasing everywhere in the interval. But of course, looking at the endpoints of the interval that we calculated earlier, we know that the function is decreasing throughout the interval, and therefore, $f(x) \ge 0$ throughout the interval. The function meets the first criterion.

To show that it also meets the second criterion, we need to show that the integral of the function over the given interval is equal to 1.

$$\int_1^2 \frac{2}{x^2}\ dx$$

$$\int_1^2 \frac{2}{x^2}\ dx = -\frac{2}{x}\Big|_1^2$$

$$\int_1^2 \frac{2}{x^2}\ dx = -\frac{2}{2} - \left(-\frac{2}{1}\right)$$

$$\int_1^2 \frac{2}{x^2}\ dx = -1 + 2$$

$$\int_1^2 \frac{2}{x^2}\ dx = 1$$

This function also meets the second criterion, so we've proven that it represents a probability density function over the given interval.

The question is asking us to find the probability that x exists as $x \le 3/2$. Since we're dealing with probability, we write this as $P(x \le 3/2)$. Since the given interval is defined as $1 \le x \le 2$, we want to know the probability that x exists on the interval $1 \le x \le 3/2$.

To calculate this probability, we integrate the probability density function on the interval $1 \le x \le 3/2$.

$$P\left(x \le \frac{3}{2}\right) = \int_1^{\frac{3}{2}} \frac{2}{x^2}\ dx$$

$$P\left(x \leq \frac{3}{2}\right) = -\frac{2}{x}\Big|_1^{\frac{3}{2}}$$

$$P\left(x \leq \frac{3}{2}\right) = -\frac{2}{\frac{3}{2}} - \left(-\frac{2}{1}\right)$$

$$P\left(x \leq \frac{3}{2}\right) = -\frac{4}{3} + 2$$

$$P\left(x \leq \frac{3}{2}\right) = \frac{2}{3}$$

Topic: Cardiac output

Question: Find the cardiac output if 5 mg of dye is injected into the heart and the concentration of dye remaining in the heart t seconds after the injection is modeled by $C(t) = 18te^{-0.7t}$. Assume $0 \le t \le 15$.

Answer choices:

A $\quad 0.136$ liters/second

B $\quad 8.169$ liters/second

C $\quad 0.136$ liters/minute

D $\quad 2.040$ liters/second

Solution: A

Cardiac output is given by

$$F = \frac{A}{\int_0^T C(t)\ dt}$$

where F is blood flow, A is the amount of dye that was injected into the heart, $C(t)$ is the concentration of dye injected t seconds after the injection, and T is the time at which all of the dye concentration is out of the heart.

Plugging everything we've been given into the formula gives

$$F = \frac{5}{\int_0^{15} 18te^{-0.7t}\ dt}$$

Use integration by parts to evaluate the integral.

$$u = 18t$$

$$du = 18\ dt$$

$$dv = e^{-0.7t}\ dt$$

$$v = -\frac{1}{0.7}e^{-0.7t}$$

The integration by parts formula is

$$\int u\ dv = uv - \int v\ du$$

So the cardiac output formula is now

$$F=\frac{5}{(18t)\left(-\frac{1}{0.7}e^{-0.7t}\right)\Big|_0^{15}-\int_0^{15}-\frac{1}{0.7}e^{-0.7t}(18)\ dt}$$

$$F=\frac{5}{-\frac{18t}{0.7}e^{-0.7t}\Big|_0^{15}+\int_0^{15}\frac{18}{0.7}e^{-0.7t}\ dt}$$

$$F=\frac{5}{-\frac{18t}{0.7}e^{-0.7t}-\frac{18}{0.49}e^{-0.7t}\Big|_0^{15}}$$

$$F=\frac{5}{-\frac{18(15)}{0.7}e^{-0.7(15)}-\frac{18}{0.49}e^{-0.7(15)}-\left(-\frac{18(0)}{0.7}e^{-0.7(0)}-\frac{18}{0.49}e^{-0.7(0)}\right)}$$

$$F=\frac{5}{-\frac{270}{0.7}e^{-10.5}-\frac{18}{0.49}e^{-10.5}+(0)(1)+\frac{18}{0.49}(1)}$$

$$F=\frac{5}{-\frac{270}{0.7}e^{-10.5}-\frac{18}{0.49}e^{-10.5}+\frac{18}{0.49}}$$

$$F\approx 0.136$$

Topic: Cardiac output

Question: Find the cardiac output if 7 mg of dye is injected into the heart and the concentration of dye remaining in the heart t seconds after the injection is modeled by $C(t) = 20te^{-0.5t}$. Assume $0 \leq t \leq 12$.

Answer choices:

A 5.343 liters/second

B 0.089 liters/minute

C 3.931 liters/second

D 0.089 liters/second

Solution: D

Cardiac output is given by

$$F = \frac{A}{\int_0^T C(t)\ dt}$$

where F is blood flow, A is the amount of dye that was injected into the heart, $C(t)$ is the concentration of dye injected t seconds after the injection, and T is the time at which all of the dye concentration is out of the heart.

Plugging everything we've been given into the formula gives

$$F = \frac{7}{\int_0^{12} 20te^{-0.5t}\ dt}$$

Use integration by parts to evaluate the integral.

$$u = 20t$$

$$du = 20\ dt$$

$$dv = e^{-0.5t}\ dt$$

$$v = -\frac{1}{0.5}e^{-0.5t}$$

The integration by parts formula is

$$\int u\ dv = uv - \int v\ du$$

So the cardiac output formula is now

$$F = \frac{7}{(20t)\left(-\frac{1}{0.5}e^{-0.5t}\right)\Big|_0^{12} - \int_0^{12} -\frac{1}{0.5}e^{-0.5t}(20)\ dt}$$

$$F = \frac{7}{-\frac{20t}{0.5}e^{-0.5t}\Big|_0^{12} + \int_0^{12} \frac{20}{0.5}e^{-0.5t}\ dt}$$

$$F = \frac{7}{-\frac{20t}{0.5}e^{-0.5t} - \frac{20}{0.25}e^{-0.5t}\Big|_0^{12}}$$

$$F = \frac{7}{-\frac{20(12)}{0.5}e^{-0.5(12)} - \frac{20}{0.25}e^{-0.5(12)} - \left(-\frac{20(0)}{0.5}e^{-0.5(0)} - \frac{20}{0.25}e^{-0.5(0)}\right)}$$

$$F = \frac{7}{-960e^{-6} - 80e^{-6} - \left(-(0)(1) - (80)(1)\right)}$$

$$F = \frac{7}{-960e^{-6} - 80e^{-6} + 80}$$

$$F \approx 0.089$$

Topic: Cardiac output

Question: Find the cardiac output if 10 mg of dye is injected into the heart and the concentration of dye remaining in the heart t seconds after the injection is modeled by $C(t) = 56te^{-0.8t}$. Assume $0 \le t \le 20$.

Answer choices:

A $\quad 0.114$ liters/minute

B $\quad 1.562$ liters/minute

C $\quad 0.114$ liters/second

D $\quad 1.562$ liters/second

Solution: C

Cardiac output is given by

$$F=\frac{A}{\int_0^T C(t)\ dt}$$

where F is blood flow, A is the amount of dye that was injected into the heart, $C(t)$ is the concentration of dye injected t seconds after the injection, and T is the time at which all of the dye concentration is out of the heart.

Plugging everything we've been given into the formula gives

$$F=\frac{10}{\int_0^{20} 56te^{-0.8t}\ dt}$$

Use integration by parts to evaluate the integral.

$$u=56t$$

$$du=56\ dt$$

$$dv=e^{-0.8t}\ dt$$

$$v=-\frac{1}{0.8}e^{-0.8t}$$

The integration by parts formula is

$$\int u\ dv=uv-\int v\ du$$

So the cardiac output formula is now

$$F = \frac{10}{(56t)\left(-\frac{1}{0.8}e^{-0.8t}\right)\Big|_0^{20} - \int_0^{20} -\frac{1}{0.8}e^{-0.8t}(56)\ dt}$$

$$F = \frac{10}{-\frac{56t}{0.8}e^{-0.8t} - \frac{56}{0.64}e^{-0.8t}\Big|_0^{20}}$$

$$F = \frac{10}{-\frac{56(20)}{0.8}e^{-0.8(20)} - \frac{56}{0.64}e^{-0.8(20)} - \left(-\frac{56(0)}{0.8}e^{-0.8(0)} - \frac{56}{0.64}e^{-0.8(0)}\right)}$$

$$F = \frac{10}{-1{,}400e^{-16} - 87.5e^{-16} - \left(-(0)(1) - 87.5(1)\right)}$$

$$F = \frac{10}{-1{,}400e^{-16} - 87.5e^{-16} + 87.5}$$

$$F \approx 0.114$$

Topic: Poiseuille's law

Question: Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.029$, $R = 0.009$ cm, $L = 4$ cm, $P = 3{,}800$ dynes/cm^2.

Answer choices:

A 8.44×10^{-5} cm^2/sec

B 8.44×10^{-5} cm^3/sec

C 8.44×10^{-4} cm^3/sec

D 8.44×10^{-4} cm^2/sec

Solution: B

To find blood flow using Poiseuille's law, you use the formula

$$F=\frac{\pi PR^4}{8nL}$$

where F is blood flow, P is the pressure difference in the artery between the beginning of the artery to the end of the artery, R is the radius of the artery, n is the viscosity of the blood, and L is the length of the artery.

Plugging everything we've been given into the Poiseuille's law formula gives

$$F=\frac{\pi(3{,}800)(0.009)^4}{8(0.029)(4)}$$

$$F=8.44\times10^{-5}$$

Topic: Poiseuille's law

Question: Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.031$, $R = 0.011$ cm, $L = 3.5$ cm, $P = 3{,}900$ dynes/cm^2.

Answer choices:

A $\quad 2.067 \times 10^{-4}$ cm^3/sec

B $\quad 2.067 \times 10^{-4}$ cm^2/sec

C $\quad 2.067 \times 10^{-5}$ cm^3/sec

D $\quad 2.067 \times 10^{-5}$ cm^2/sec

Solution: A

To find blood flow using Poiseuille's law, you use the formula

$$F = \frac{\pi P R^4}{8nL}$$

where F is blood flow, P is the pressure difference in the artery between the beginning of the artery to the end of the artery, R is the radius of the artery, n is the viscosity of the blood, and L is the length of the artery.

Plugging everything we've been given into the Poiseuille's law formula gives

$$F = \frac{\pi(3{,}900)(0.0011)^4}{8(0.031)(3.5)}$$

$$F = 2.067 \times 10^{-4}$$

Topic: Poiseuille's law

Question: Use Poiseuille's law to find the flow of blood in the human artery in which $n = 0.0285$, $R = 0.013$ cm, $L = 4.5$ cm, $P = 3{,}700$ dynes/cm^2.

Answer choices:

A 3.236×10^{-5} cm^3/sec

B 3.236×10^{-4} cm^2/sec

C 3.236×10^{-5} cm^2/sec

D 3.236×10^{-4} cm^3/sec

Solution: D

To find blood flow using Poiseuille's law, you use the formula

$$F = \frac{\pi PR^4}{8nL}$$

where F is blood flow, P is the pressure difference in the artery between the beginning of the artery to the end of the artery, R is the radius of the artery, n is the viscosity of the blood, and L is the length of the artery.

Plugging everything we've been given into the Poiseuille's law formula gives

$$F = \frac{\pi(3{,}700)(0.013)^4}{8(0.0285)(4.5)}$$

$$F = 3.236 \times 10^{-4}$$

Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 4 feet and height 8 feet.

Answer choices:

A $\frac{128}{3}\pi$ square feet

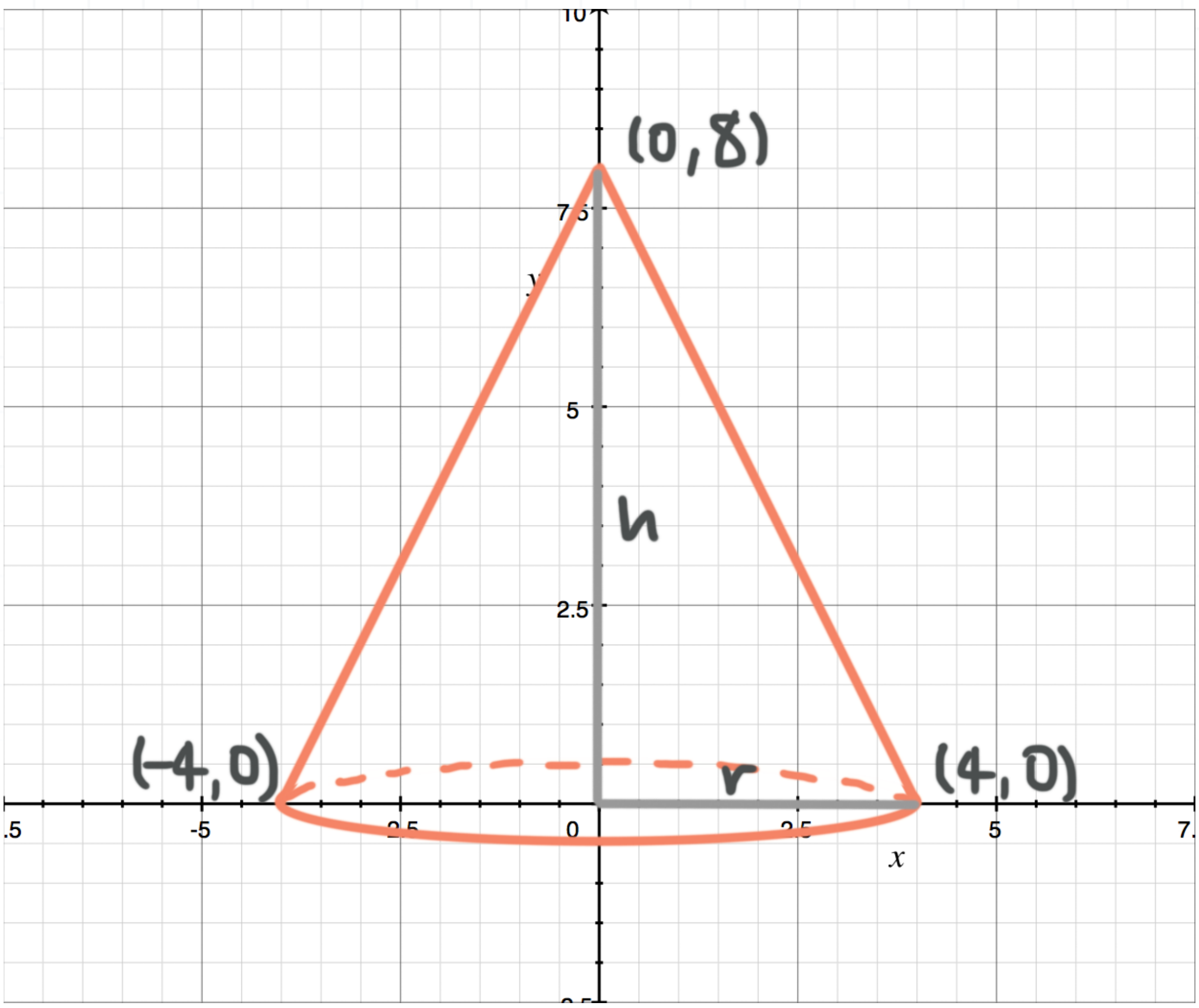
B $\frac{128}{3}$ cubic feet

C $\frac{128}{3}\pi$ cubic feet

D 128π cubic feet

Solution: C

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 4$ and $h = 8$. So

$$A = \frac{1}{2}(4)(8) = 16$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,8)$ and $(4,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 8}{4 - 0} = -\frac{8}{4} = -2$$

Use the point $(4,0)$ and the slope $m = -2$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 4)$$

$$y = -2x + 8$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

We already found that $A = 16$, $a = 0$, $b = 4$, and $y = -2x + 8$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{1}{16}\int_0^4 -2x^2 + 8x\ dx$$

$$\bar{x} = \frac{1}{16}\left(-\frac{2}{3}x^3 + 4x^2\right)\Big|_0^4$$

$$\bar{x} = \frac{1}{16}\left(-\frac{2}{3}(4)^3 + 4(4)^2\right) - \frac{1}{16}\left(-\frac{2}{3}(0)^3 + 4(0)^2\right)$$

$$\bar{x} = \frac{1}{16}\left(-\frac{128}{3} + 64\right)$$

$$\bar{x} = \frac{4}{3}$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi\left(\frac{4}{3}\right)$$

$$d = \frac{8}{3}\pi$$

Now we can find volume using

$$V = Ad$$

$$V = 16\left(\frac{8}{3}\pi\right)$$

$$V = \frac{128}{3}\pi$$

Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 5 inches and height 15 inches.

Answer choices:

A 125π square inches

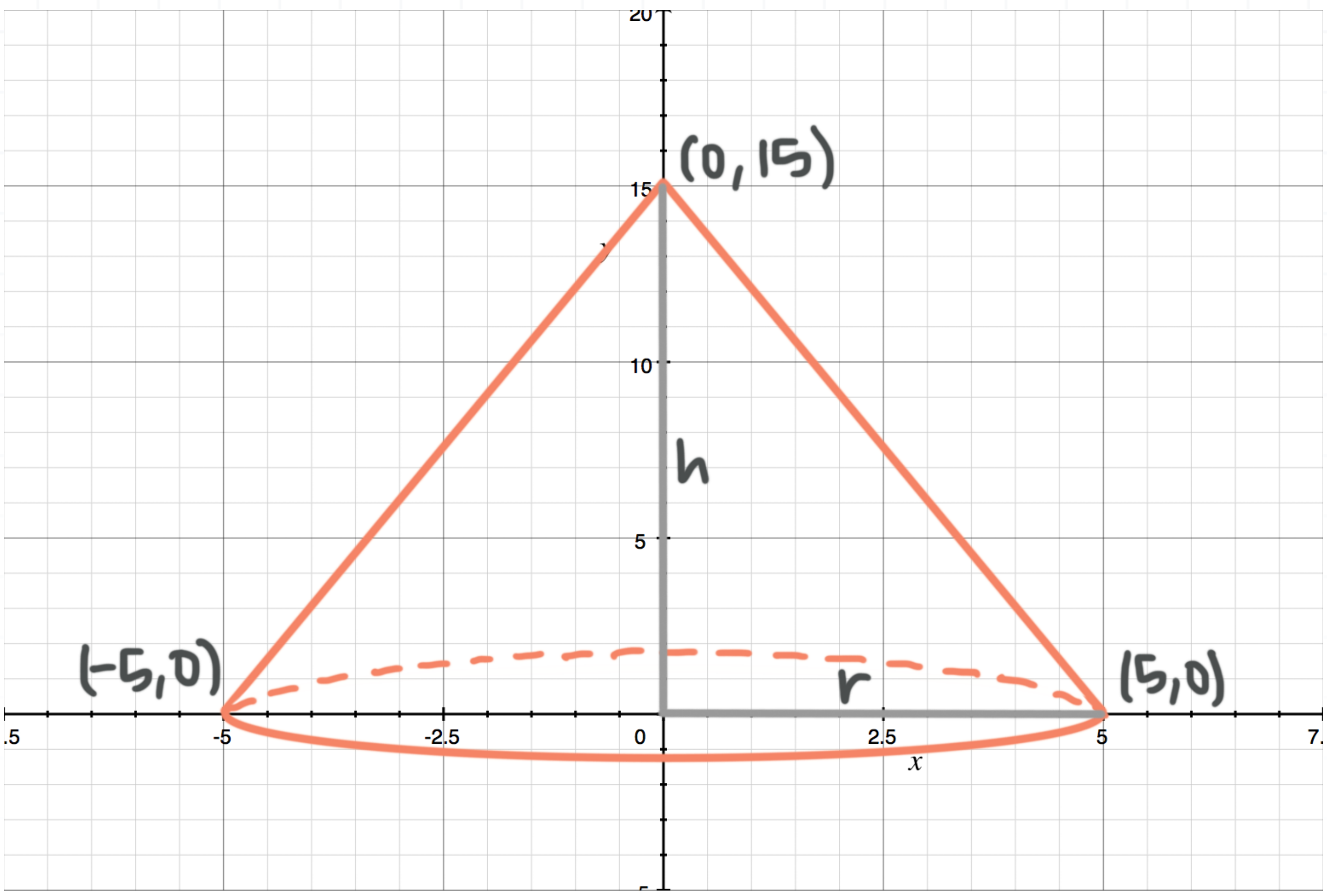
B 125π cubic inches

C $\frac{125}{3}\pi$ cubic inches

D $\frac{125}{3}\pi$ square inches

Solution: B

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 5$ and $h = 15$. So

$$A = \frac{1}{2}(5)(15) = \frac{75}{2}$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,15)$ and $(5,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 15}{5 - 0} = -\frac{15}{5} = -3$$

Use the point $(5,0)$ and the slope $m = -3$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -3(x - 5)$$

$$y = -3x + 15$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

We already found that $A = 75/2$, $a = 0$, $b = 5$, and $y = -3x + 15$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{2}{75}\int_0^5 x(-3x + 15)\ dx$$

$$\bar{x} = \frac{2}{75}\int_0^5 -3x^2 + 15x\ dx$$

$$\bar{x} = \frac{2}{75}\left(-x^3 + \frac{15}{2}x^2\right)\Bigg|_0^5$$

$$\bar{x} = \frac{2}{75}\left(-(5)^3 + \frac{15}{2}(5)^2\right) - \frac{2}{75}\left(-(0)^3 + \frac{15}{2}(0)^2\right)$$

$$\bar{x} = -\frac{250}{75} + \frac{750}{150}$$

$$\bar{x} = \frac{5}{3}$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi\left(\frac{5}{3}\right)$$

$$d = \frac{10}{3}\pi$$

Now we can find volume using

$$V = Ad$$

$$V = \frac{75}{2}\left(\frac{10}{3}\pi\right)$$

$$V = 125\pi$$

Topic: Theorem of Pappus

Question: Use the Theorem of Pappus to find the volume of a right circular cone with radius 3 meters and height 12 meters.

Answer choices:

A 36π cubic meters

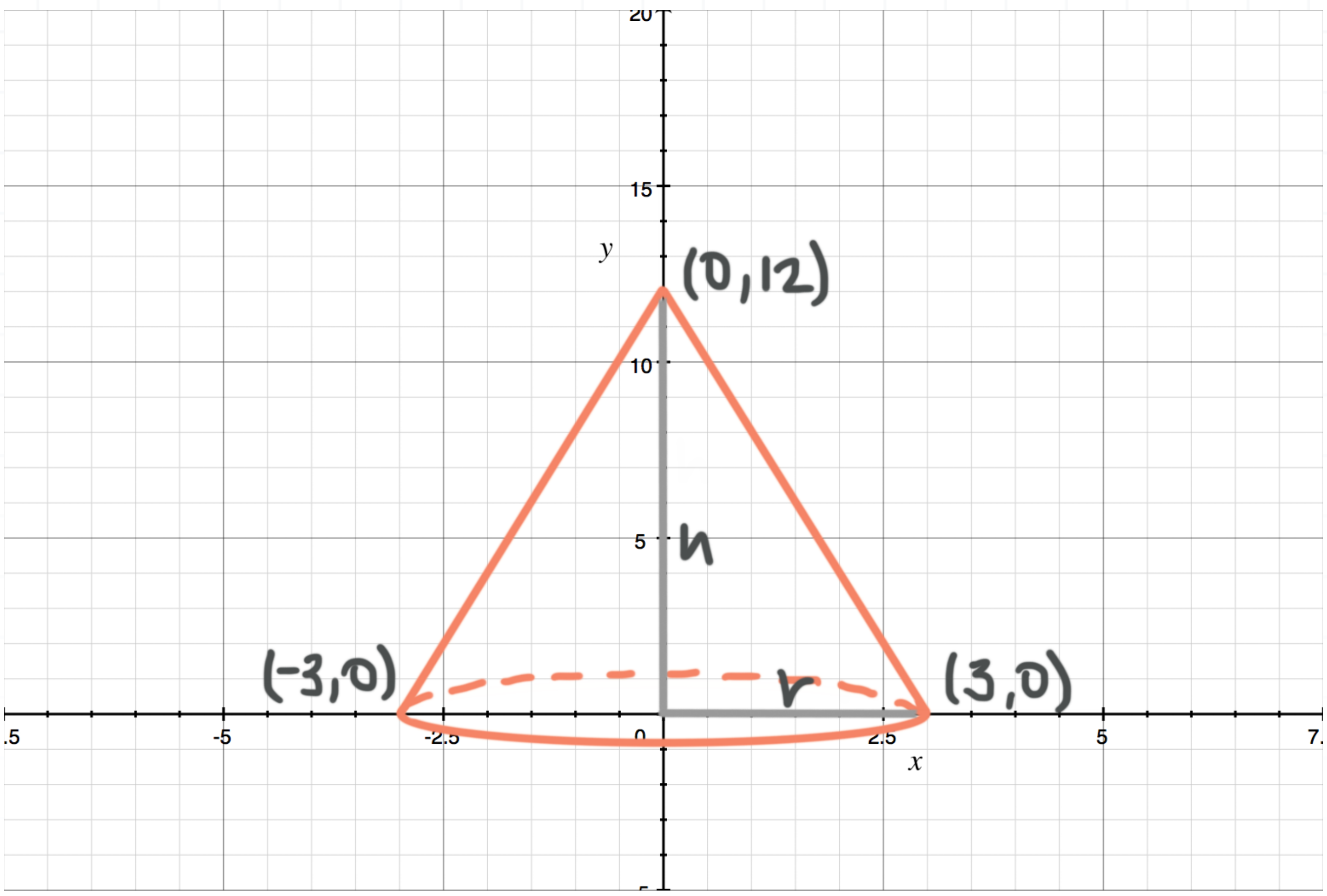
B 36π square meters

C 18π cubic meters

D 18π square meters

Solution: A

It might be helpful to visualize the right circular cone that we have in this problem, drawn with the center of the base at the origin in the coordinate plane.



The cross section that the Theorem of Pappus uses is the area of a triangle drawn from the vertex of the cone to the center of the base, and then to the edge of the cone. The area of this cross section, as in any triangle is

$$A = \frac{1}{2}bh$$

where b is the radius of the base of the cone, and h is the height of the cone. The Theorem of Pappus calculates the volume of the cone with the formula

$$V = Ad$$

where V is volume, A is area of the cross section, and d is the distance traveled by the x-value of the centroid of the cross section during an integration.

Find the area of the triangular cross section. For this cone, $b = r = 3$ and $h = 12$. So

$$A = \frac{1}{2}(3)(12) = 18$$

Next, we need to find the function in the first quadrant that contains the lateral edge of the cone. We will use it to find the x-value of the centroid of the cross section of the cone. We will begin by identifying two points, from the graph above, and write an equation of the line that contains those two points. The two points are $(0,12)$ and $(3,0)$. We will use these two points to calculate the slope of the line first.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 12}{3 - 0} = -\frac{12}{3} = -4$$

Use the point $(3,0)$ and the slope $m = -4$ we just found to write the equation of the lateral edge.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -4(x - 3)$$

$$y = -4x + 12$$

Now we'll find the x-value of the centroid of the cross section, which is $\bar{x}$.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx$$

We already found that $A = 18$, $a = 0$, $b = 3$, and $y = -4x + 12$. Substitute these values/expressions into the integral formula.

$$\bar{x} = \frac{1}{18}\int_0^3 x(-4x + 12)\ dx$$

$$\bar{x} = \frac{1}{18}\left(-\frac{4}{3}x^3 + 6x^2\right)\Big|_0^3$$

$$\bar{x} = \frac{1}{18}\left(-\frac{4}{3}(3)^3 + 6(3)^2\right) - \frac{1}{18}\left(-\frac{4}{3}(0)^3 + 6(0)^2\right)$$

$$\bar{x} = \frac{1}{18}(-36 + 54)$$

$$\bar{x} = \frac{1}{18}(18)$$

$$\bar{x} = 1$$

Now we'll find the distance traveled by the x-value of the centroid. This is given by the formula

$$d = 2\pi\bar{x}$$

$$d = 2\pi(1)$$

$$d = 2\pi$$

Now we can find volume using

$$V = Ad$$

$$V = 18(2\pi)$$

$$V = 36\pi$$

Topic: Eliminating the parameter

Question: Eliminate the parameter.

$$x = t^3 - 4t$$

$$y = t^2 - 4$$

Answer choices:

A $x^2 = y^3 + 4y^2$

B $x^2 = y^3 - 4y^2$

C $x^2 = y^2 + 4y$

D $x^2 = y^2 - 4y$

Solution: A

To eliminate the parameter from a set of equations, we have a few options. We can

1. Solve each equation for the parameter t, then set the equations equal to one another, or

2. Solve one equation for the parameter t, then plug that value into the second equation, or

3. Solve each equation for part of an identity, then plug both values into the identity.

In this case, we'll use the second option. Solving $y = t^2 - 4$ for the parameter t, we get

$$y = t^2 - 4$$

$$y + 4 = t^2$$

$$t = \sqrt{y+4}$$

Plugging this value into $x = t^3 - 4t$, we get a new equation in terms of x and y.

$$x = \left(\sqrt{y+4}\right)^3 - 4\sqrt{y+4}$$

$$x = \sqrt{y+4}\left[\left(\sqrt{y+4}\right)^2 - 4\right]$$

$$x = \sqrt{y+4}\,(y + 4 - 4)$$

$$x = y\sqrt{y+4}$$

Squaring both sides to eliminate the square root, we get

$$x^2 = y^2(y+4)$$

$$x^2 = y^3 + 4y^2$$

Topic: Eliminating the parameter

Question: Eliminate the parameter.

$$x = 2\cos t$$

$$y = 3\sin t$$

Answer choices:

A $4x^2 + 9y^2 = 36$

B $9x^2 - 4y^2 = 36$

C $9x^2 + 4y^2 = 36$

D $4x^2 - 9y^2 = 36$

Solution: C

To eliminate the parameter from a set of equations, we have a few options. We can

1. Solve each equation for the parameter t, then set the equations equal to one another, or

2. Solve one equation for the parameter t, then plug that value into the second equation, or

3. Solve each equation for part of an identity, then plug both values into the identity.

In this case, we'll use the third option. Solving each equation for the square of the trigonometric function, we get

$$x = 2\cos t$$

$$\frac{x}{2} = \cos t$$

$$\frac{x^2}{4} = \cos^2 t$$

and

$$y = 3\sin t$$

$$\frac{y}{3} = \sin t$$

$$\frac{y^2}{9} = \sin^2 t$$

Since $\sin^2 x + \cos^2 x = 1$, we can say that

$$\sin^2 t + \cos^2 t = 1$$

$$\frac{y^2}{9} + \frac{x^2}{4} = 1$$

$$9x^2 + 4y^2 = 36$$

Topic: Eliminating the parameter

Question: Eliminate the parameter.

$$x = 4t$$

$$y = \sqrt{3 - t}$$

Answer choices:

A $\quad x - 4y^2 = 12$

B $\quad x + 4y^2 = 3$

C $\quad x + 4y^2 = 12$

D $\quad x + y^2 = 3$

Solution: C

To eliminate the parameter from a set of equations, we have a few options. We can

1. solve each equation for the parameter t, then set the equations equal to one another, or

2. solve one equation for the parameter t, then plug that value into the second equation, or

3. solve each equation for part of an identity, then plug both values into the identity.

In this case, we'll use the first option. Solving each of the given equations for the parameter t, we get

$$x = 4t$$

$$t = \frac{1}{4}x$$

and

$$y = \sqrt{3 - t}$$

$$y^2 = 3 - t$$

$$t = 3 - y^2$$

Setting the resulting equations equal to one another, we get a new equation in terms of x and y.

$$\frac{1}{4}x = 3 - y^2$$

$$x = 12 - 4y^2$$

$$x + 4y^2 = 12$$

Topic: Derivative of a parametric curve

Question: Find the derivative of the parametric curve.

$$x = t^2$$

$$y = 3t^2 - t$$

Answer choices:

A $\frac{t}{t-1}$

B $\frac{2t}{6t-1}$

C $\frac{t-1}{t}$

D $\frac{6t-1}{2t}$

Solution: D

To find the derivative of a parametric curve, we will use the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt} = 6t - 1$$

and

$$\frac{dx}{dt} = 2t$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx} = \frac{6t - 1}{2t}$$

Topic: Derivative of a parametric curve

Question: Find the derivative of the parametric curve.

$$x = 4\cos 4t$$

$$y = t^2 - 4$$

Answer choices:

A $\quad -\dfrac{t}{8\sin 4t}$

B $\quad \dfrac{8\sin 4t}{t}$

C $\quad -\dfrac{8\sin 4t}{t}$

D $\quad \dfrac{t}{8\sin 4t}$

Solution: A

To find the derivative of a parametric curve, we will use the formula

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt}=2t$$

and

$$\frac{dx}{dt}=-16\sin 4t$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx}=\frac{2t}{-16\sin 4t}$$

$$\frac{dy}{dx}=-\frac{t}{8\sin 4t}$$

Topic: Derivative of a parametric curve

Question: Find the derivative of the parametric curve.

$$x = t\sin 2t$$

$$y = \frac{1}{3}\cos 9t$$

Answer choices:

A $\quad \dfrac{\sin 2t + 2t\cos 2t}{3\sin 9t}$

B $\quad -\dfrac{3\sin 9t}{\sin 2t + 2t\cos 2t}$

C $\quad -\dfrac{\sin 2t + 2t\cos 2t}{3\sin 9t}$

D $\quad \dfrac{3\sin 9t}{\sin 2t + 2t\cos 2t}$

Solution: B

To find the derivative of a parametric curve, we will use the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt} = -3\sin 9t$$

and

$$\frac{dx}{dt} = (1)(\sin 2t) + (t)(2\cos 2t)$$

$$\frac{dx}{dt} = \sin 2t + 2t\cos 2t$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx} = \frac{-3\sin 9t}{\sin 2t + 2t\cos 2t}$$

$$\frac{dy}{dx} = -\frac{3\sin 9t}{\sin 2t + 2t\cos 2t}$$

Topic: Second derivative of a parametric curve

Question: Find the second derivative of the parametric curve.

$$x = 3t$$

$$y = t^2$$

Answer choices:

A $\frac{9}{2}$

B $\frac{2}{9}$

C $\frac{2}{9}t$

D $\frac{9}{2}t$

Solution: B

Before we can find the second derivative of a parametric curve, we have to find the first derivative using the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt} = 2t$$

and

$$\frac{dx}{dt} = 3$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx} = \frac{2t}{3}$$

With this first derivative in hand, we'll use the formula

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

to find the second derivative, plugging in the values we already found for dy/dx and dx/dt.

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{2t}{3}\right)}{3}$$

The d/dt in the numerator tells us to take the derivative of our first derivative with respect to t.

$$\frac{d^2y}{dx^2}=\frac{\frac{2}{3}}{3}$$

$$\frac{d^2y}{dx^2}=\frac{2}{3}\left(\frac{1}{3}\right)$$

$$\frac{d^2y}{dx^2}=\frac{2}{9}$$

Topic: Second derivative of a parametric curve

Question: Find the second derivative of the parametric curve.

$$x = 4t^2$$

$$y = \sin t$$

Answer choices:

A $\dfrac{t\sin t - \cos t}{64t^3}$

B $\dfrac{t\sin t + \cos t}{64t^3}$

C $\dfrac{-t\sin t + \cos t}{64t^3}$

D $-\dfrac{t\sin t + \cos t}{64t^3}$

Solution: D

Before we can find the second derivative of a parametric curve, we have to find the first derivative using the formula

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt}=\cos t$$

and

$$\frac{dx}{dt}=8t$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx}=\frac{\cos t}{8t}$$

With this first derivative in hand, we'll use the formula

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

to find the second derivative, plugging in the values we already found for dy/dx and dx/dt.

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{\cos t}{8t}\right)}{8t}$$

The d/dt in the numerator tells us to take the derivative of our first derivative with respect to t. We'll use quotient rule to find the derivative of just the value in the parentheses.

$$\frac{d^2y}{dx^2}=\frac{\frac{(-\sin t)(8t)-(\cos t)(8)}{(8t)^2}}{8t}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{-8t\sin t-8\cos t}{64t^2}}{8t}$$

$$\frac{d^2y}{dx^2}=\frac{-8t\sin t-8\cos t}{64t^2}\left(\frac{1}{8t}\right)$$

$$\frac{d^2y}{dx^2}=\frac{-t\sin t-\cos t}{64t^3}$$

$$\frac{d^2y}{dx^2}=-\frac{t\sin t+\cos t}{64t^3}$$

Topic: Second derivative of a parametric curve

Question: Find the second derivative of the parametric curve.

$$x = \cos 2t$$

$$y = 3\sin t - t^2$$

Answer choices:

A $\quad \dfrac{3\sin t \sin 2t + 2\sin 2t + 6\cos t \cos 2t - 4t\cos 2t}{4\sin^3 2t}$

B $\quad \dfrac{3\sin t \sin 2t + 2\sin 2t + 6\cos t \cos 2t - 4t\cos 2t}{8\sin^3 2t}$

C $\quad -\dfrac{3\sin t \sin 2t + 2\sin 2t + 6\cos t \cos 2t - 4t\cos 2t}{4\sin^3 2t}$

D $\quad -\dfrac{3\sin t \sin 2t + 2\sin 2t + 6\cos t \cos 2t - 4t\cos 2t}{8\sin^3 2t}$

Solution: C

Before we can find the second derivative of a parametric curve, we have to find the first derivative using the formula

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

We'll find

the derivative of y with respect to t, dy/dt, and

the derivative of x with respect to t, dx/dt

and then plug both of them into the formula above. The derivatives of our separate equations are

$$\frac{dy}{dt}=3\cos t-2t$$

and

$$\frac{dx}{dt}=-2\sin 2t$$

Plugging these into the formula for the derivative of a parametric curve, we get

$$\frac{dy}{dx}=\frac{3\cos t-2t}{-2\sin 2t}$$

With this first derivative in hand, we'll use the formula

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

to find the second derivative, plugging in the values we already found for dy/dx and dx/dt.

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{3\cos t-2t}{-2\sin 2t}\right)}{-2\sin 2t}$$

The d/dt in the numerator tells us to take the derivative of our first derivative with respect to t. We'll use quotient rule to find the derivative of just the value in the parentheses.

$$\frac{d^2y}{dx^2}=\frac{\frac{(-3\sin t-2)(-2\sin 2t)-(3\cos t-2t)(-4\cos 2t)}{(-2\sin 2t)^2}}{-2\sin 2t}$$

$$\frac{d^2y}{dx^2}=\frac{(-3\sin t-2)(-2\sin 2t)-(3\cos t-2t)(-4\cos 2t)}{(-2\sin 2t)^2}\cdot\frac{1}{-2\sin 2t}$$

$$\frac{d^2y}{dx^2}=\frac{(-3\sin t-2)(-2\sin 2t)-(3\cos t-2t)(-4\cos 2t)}{(-2\sin 2t)^3}$$

$$\frac{d^2y}{dx^2}=\frac{6\sin t\sin 2t+4\sin 2t-(-12\cos t\cos 2t+8t\cos 2t)}{-8\sin^3 2t}$$

$$\frac{d^2y}{dx^2}=\frac{6\sin t\sin 2t+4\sin 2t+12\cos t\cos 2t-8t\cos 2t}{-8\sin^3 2t}$$

$$\frac{d^2y}{dx^2}=-\frac{3\sin t\sin 2t+2\sin 2t+6\cos t\cos 2t-4t\cos 2t}{4\sin^3 2t}$$

Topic: Sketching parametric curves by plotting points

Question: A parametric curve is defined by $x = 2\cos 2\theta - 3$ and $y = \sin^2 2\theta - 4$. Which statement is true about the position of the graph of the function?

Answer choices:

A The graph is a parabola that opens down around the vertex $(-3, -3)$.

B The graph is a parabola that opens down around the vertex $(-3, -4)$.

C The graph is a parabola that opens up around the vertex $(-3, -3)$.

D The graph is a parabola that opens up around the vertex $(-3, -4)$.

Solution: A

Rearrange the given equations.

$$x = 2\cos 2\theta - 3$$

$$x + 3 = 2\cos 2\theta$$

$$\frac{x+3}{2} = \cos 2\theta$$

$$\cos^2 2\theta = \frac{(x+3)^2}{4}$$

and

$$y = \sin^2 2\theta - 4$$

$$\sin^2 2\theta = y + 4$$

Now add these equations together.

$$\sin^2 2\theta + \cos^2 2\theta = \frac{(x+3)^2}{4} + y + 4$$

$$1 = \frac{(x+3)^2}{4} + y + 4$$

$$y = -\frac{(x+3)^2}{4} - 3$$

Topic: Sketching parametric curves by plotting points

Question: A parametric curve is defined by the functions $x = 3\tan t$ and $y = 2\sec t$. Which statement describes the type and position of the sketch of the given curve?

Answer choices:

A The graph of the curve is a rectangular hyperbola with vertices at $(0,-2)$ and $(0,2)$, and with asymptotes defined by $y = \pm \frac{3}{2}x$.

B The graph of the curve is a rectangular hyperbola with vertices at $(0,-2)$ and $(0,2)$, and with asymptotes defined by $y = \pm \frac{2}{3}x$.

C The graph of the curve is a rectangular hyperbola with vertices at $(0,-3)$ and $(0,3)$, and with asymptotes defined by $y = \pm \frac{2}{3}x \pm 2$.

D The graph of the curve is a rectangular hyperbola with vertices at $(0,-2)$ and $(0,2)$, and with asymptotes defined by $y = \pm \frac{2}{3}x \pm 1$.

Solution: B

Given the parametric equations $x=3\tan t$ and $y=2\sec t$, we'll square both sides, and then solve each for the trigonometric function.

$$x=3\tan t$$

$$x^2=9\tan^2 t$$

$$\frac{x^2}{9}=\tan^2 t$$

and

$$y=2\sec t$$

$$y^2=4\sec^2 t$$

$$\frac{y^2}{4}=\tan^2 t+1$$

Now subtract the first equation from the second.

$$\frac{y^2}{4}-\frac{x^2}{9}=\tan^2 t+1-\tan^2 t$$

$$\frac{y^2}{4}-\frac{x^2}{9}=1$$

The graph of the curve is a rectangular hyperbola with vertices at $(0,-2)$ and $(0,2)$, and with asymptotes defined by $y=\pm\frac{2}{3}x$.

Topic: Sketching parametric curves by plotting points

Question: The sketch of a parabola indicates that its vertex is at $(2,3)$, and it's open to the right. The graph of the line $y = 3$ divides the curve into two symmetric curves. Which pair of parametric functions represents the sketch of the given parabola?

Answer choices:

A $\quad x = (t-3)^2 + 2(2t+1)$ and $y = t + 3$

B $\quad x = (t-2)^2 + 2(2t-1)$ and $y = t - 3$

C $\quad x = (t+2)^2 + 2(3t-1)$ and $y = t + 3$

D $\quad x = (t-2)^2 + 2(2t-1)$ and $y = t + 3$

Solution: D

Choose the equations from answer choice D.

$$x = (t-2)^2 + 2(2t-1)$$

$$y = t + 3$$

Simplify the first equation.

$$x = (t-2)^2 + 2(2t-1)$$

$$x = t^2 - 4t + 4 + 4t - 2$$

$$x = t^2 + 2$$

Rearrange the second equation.

$$y = t + 3$$

$$y - 3 = t$$

$$(y-3)^2 = t^2$$

Substitute $(y-3)^2 = t^2$ into $x = t^2 + 2$.

$$x = (y-3)^2 + 2$$

The vertex of this parabola is at $(2,3)$, and it opens to the right.

Topic: Tangent line to the parametric curve

Question: Find the equation of the tangent line to the parametric curve.

$$x = t^2$$

$$y = t^3$$

$$t = 2$$

Answer choices:

A $\quad y = 3x + 4$

B $\quad y = 3x - 4$

C $\quad y = -3x - 4$

D $\quad y = -3x + 4$

Solution: B

To define the tangent line, we use the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is the point where the tangent line intersects the curve.

At $t = 2$, the slope m is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t}$$

$$m = \frac{3t}{2}$$

$$m = \frac{3(2)}{2}$$

$$m = 3$$

and the point (x_1, y_1) is

$$x_1 = 2^2 = 4$$

$$y_1 = 2^3 = 8$$

Plugging the slope and the point into our formula gives

$$y - 8 = 3(x - 4)$$

$$y - 8 = 3x - 12$$

$$y = 3x - 4$$

Topic: Tangent line to the parametric curve

Question: Find the equation of the tangent line to the parametric curve.

$$x = 2t^2 + 6$$

$$y = t^4$$

$$t = -1$$

Answer choices:

A $y = x + 7$

B $y = x - 7$

C $y = -x - 7$

D $7y = x - 1$

Solution: B

To define the tangent line, we use the point-slope formula for the equation of the line.

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is the point where the tangent line intersects the curve.

At $t = -1$, the slope m is

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t}$$

$$m = t^2$$

$$m = (-1)^2$$

$$m = 1$$

and the point (x_1, y_1) is

$$x_1 = 2(-1)^2 + 6 = 8$$

$$y_1 = (-1)^4 = 1$$

Plugging the slope and the point into our formula gives

$$y - 1 = 1(x - 8)$$

$$y - 1 = x - 8$$

$$y = x - 7$$

Topic: Tangent line to the parametric curve

Question: At which point is $9x + 7y - 126 = 0$ the equation of the tangent line to the parametric curve?

$$x = 7e^t$$

$$y = 9e^{-t}$$

Answer choices:

A $t = 0$

B $t = 1$

C $t = \frac{1}{e}$

D $t = -\frac{1}{e}$

Solution: A

Differentiate both functions.

$$\frac{dx}{dt} = 7e^t$$

$$\frac{dy}{dt} = -9e^{-t}$$

Divide dy/dt by dx/dt to get dy/dx.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-9e^{-t}}{7e^t}$$

$$\frac{dy}{dx} = -\frac{9}{7}e^{-2t}$$

At the point $t = 0$, we get:

$$x = 7e^t$$

$$x = 7e^0$$

$$x = 7(1)$$

$$x = 7$$

and

$$y = 9e^{-t}$$

$$y = 9e^{-0}$$

$$y = 9(1)$$

$$y = 9$$

and

$$\frac{dy}{dx} = -\frac{9}{7}e^{-2t}$$

$$\frac{dy}{dx} = -\frac{9}{7}e^{-2(0)}$$

$$\frac{dy}{dx} = -\frac{9}{7}(1)$$

$$\frac{dy}{dx} = -\frac{9}{7}$$

Use the information we just found to write the equation of the tangent line.

$$y - 9 = -\frac{9}{7}(x - 7)$$

$$7y - 63 = -9(x - 7)$$

$$7y - 63 = -9x + 63$$

$$9x + 7y - 126 = 0$$

This is the equation of the tangent line we were given. We found it by plugging in $t = 0$, which means answer choice A is correct.

Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$x = t - \sin t$$

$$y = 6(1 - \cos t)$$

$$0 \leq t \leq 2\pi$$

Answer choices:

A 18

B 231π

C 18π

D $18\pi - 13$

Solution: C

We'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$, but we need to find $x'(t)$ before we can plug into the area formula.

$$x(t) = t - \sin t$$

$$x'(t) = 1 - \cos t$$

Plugging into our area formula, we get

$$A = \int_{0}^{2\pi} 6(1 - \cos t)(1 - \cos t)\ dt$$

$$A = 6\int_{0}^{2\pi} 1 - 2\cos t + \cos^2 t\ dt$$

Using the power reduction formula from trigonometry,

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$$

we get

$$A = 6\int_{0}^{2\pi} 1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\ dt$$

$$A = 6 \int_0^{2\pi} \frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t \ dt$$

$$A = 6 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \right) \Bigg|_0^{2\pi}$$

$$A = 6 \left[\left(\frac{3}{2}(2\pi) - 2\sin(2\pi) + \frac{1}{4}\sin 2(2\pi) \right) - \left(\frac{3}{2}(0) - 2\sin(0) + \frac{1}{4}\sin 2(0) \right) \right]$$

$$A = 6 \left[\left(3\pi - 2(0) + \frac{1}{4}\sin 4\pi \right) - \left(0 - 2(0) + \frac{1}{4}(0) \right) \right]$$

$$A = 6 \left(3\pi + \frac{1}{4}(0) \right)$$

$$A = 6(3\pi)$$

$$A = 18\pi$$

Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$f(t) = 4t^2$$

$$g(t) = t - 4$$

$$0 \leq t \leq 4$$

Answer choices:

A 256

B $\dfrac{256}{3}$

C -256

D $-\dfrac{256}{3}$

Solution: D

We'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$ as $g(t) = t - 4$, but we need to find $x'(t)$ as $f'(t)$ before we can plug into the area formula.

$$f(t) = 4t^2$$

$$f'(t) = 8t$$

Plugging into our area formula, we get

$$A = \int_{0}^{4} (t-4)(8t)\ dt$$

$$A = \int_{0}^{4} 8t^2 - 32t\ dt$$

$$A = \frac{8}{3}t^3 - 16t^2 \Big|_0^4$$

$$A = \frac{8}{3}(4)^3 - 16(4)^2 - \left[\frac{8}{3}(0)^3 - 16(0)^2\right]$$

$$A = \frac{512}{3} - \frac{768}{3}$$

$$A = -\frac{256}{3}$$

Topic: Area under a parametric curve

Question: Find the area under the parametric curve.

$$f(t) = e^t$$

$$g(t) = 1 - t^2$$

$$0 \le t \le 4$$

Answer choices:

A $9e^4 - 1$

B $12e^4$

C $1 - 9e^4$

D $9e^4$

Solution: C

We'll find the area under the curve using the integral formula

$$A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$$

Since the question defines an interval for our parameter, we'll be able to find a real-number answer for the area.

We've already been given $y(t)$ as $g(t) = 1 - t^2$, but we need to find $x'(t)$ as $f'(t)$ before we can plug into the area formula.

$$f(t) = e^t$$

$$f'(t) = e^t$$

Plugging into our area formula, we get

$$A = \int_0^4 \left(1 - t^2\right)\left(e^t\right)\ dt$$

$$A = \int_0^4 e^t - t^2e^t\ dt$$

$$A = \int_0^4 e^t\ dt - \int_0^4 t^2e^t\ dt$$

For the second integral, we'll use integration by parts, setting

$$u = t^2$$

$$du = 2t\ dt$$

and

$$dv = e^t\ dt$$

$$v = e^t$$

Replacing the second integral using the integration by parts formula

$$\int u\ dv = uv - \int v\ du$$

we get

$$A = \int_0^4 e^t\ dt - \left[t^2e^t\Big|_0^4 - \int_0^4 2te^t\ dt\right]$$

We'll need to use integration by parts again on this new integral. Setting

$$u = 2t$$

$$du = 2\ dt$$

and

$$dv = e^t\ dt$$

$$v = e^t$$

Using the integration by parts formula to replace the integral again, we get

$$A = \int_0^4 e^t\ dt - \left[t^2e^t\Big|_0^4 - \left[2te^t\Big|_0^4 - \int_0^4 2e^t\ dt\right]\right]$$

$$A=\int_0^4 e^t\ dt-\left[t^2e^t\Big|_0^4-2te^t\Big|_0^4+\int_0^4 2e^t\ dt\right]$$

$$A=\int_0^4 e^t\ dt-t^2e^t\Big|_0^4+2te^t\Big|_0^4-\int_0^4 2e^t\ dt$$

$$A=e^t\Big|_0^4-t^2e^t\Big|_0^4+2te^t\Big|_0^4-2e^t\Big|_0^4$$

$$A=\left(e^t-t^2e^t+2te^t-2e^t\right)\Big|_0^4$$

$$A=\left[e^4-4^2e^4+2(4)e^4-2e^4\right]-\left[e^0-0^2e^0+2(0)e^0-2e^0\right]$$

$$A=e^4-16e^4+8e^4-2e^4-\left(1-0+0-2(1)\right)$$

$$A=-9e^4+1$$

$$A=1-9e^4$$

Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = \sin\theta$$

$$y = \cos\theta$$

Answer choices:

A $\quad \frac{\pi}{2}$

B $\quad 2\pi$

C $\quad \pi$

D $\quad -\pi$

Solution: C

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A = \int_a^b y(t)x'(t)\ dt$$

where $[a, b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	1	0	-1	0
y	1	0	-1	0	1

Because $(x, y) = (0,1)$ at $\theta = 0$, and we don't get back to $(0,1)$ until $\theta = 2\pi$, we know the first loop of the parametric curve is closed by $\theta = [0,2\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta) = \cos\theta$$

Plugging everything into the area formula, we get

$$A = \int_0^{2\pi} \cos\theta\cos\theta\ d\theta$$

$$A = \int_0^{2\pi} \cos^2\theta \ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2\theta$ using the identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

We'll make the substitution.

$$A = \int_0^{2\pi} \frac{1}{2}(1 + \cos(2\theta)) \ d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} 1 + \cos(2\theta) \ d\theta$$

$$A = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{2\pi}$$

$$A = \frac{1}{2}\left[2\pi + \frac{1}{2}\sin(4\pi)\right] - \frac{1}{2}\left[0 + \frac{1}{2}\sin(0)\right]$$

$$A = \frac{1}{2}\left(2\pi + \frac{1}{2}(0)\right) - \frac{1}{2}\left(0 + \frac{1}{2}(0)\right)$$

$$A = \pi$$

Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = \cos 2\theta$$

$$y = 6 + \sin 2\theta$$

Answer choices:

A $-\pi$

B $-\dfrac{\pi}{2}$

C π

D $\dfrac{\pi}{2}$

Solution: A

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A = \int_a^b y(t)x'(t)\ dt$$

where $[a, b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	1	0	-1	0	1
y	6	7	6	5	6

Because $(x, y) = (1,6)$ at $\theta = 0$, and we don't get back to $(1,6)$ until $\theta = \pi$, we know the first loop of the parametric curve is closed by $\theta = [0,\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta) = -2\sin(2\theta)$$

Plugging everything into the area formula, we get

$$A = \int_0^\pi \left[6 + \sin(2\theta)\right]\left[-2\sin(2\theta)\right]\ d\theta$$

$$A=\int_0^\pi -12\sin(2\theta)-2\sin^2(2\theta)\ d\theta$$

$$A=-12\int_0^\pi \sin(2\theta)\ d\theta-2\int_0^\pi \sin^2(2\theta)\ d\theta$$

Before we can integrate, we need to do a substitution for $\sin^2\theta$ using the identity

$$\sin^2\theta=\frac{1}{2}\left[1-\cos(2\theta)\right]$$

We'll make the substitution.

$$A=-12\int_0^\pi \sin(2\theta)\ d\theta-2\int_0^\pi \frac{1}{2}\left[1-\cos(4\theta)\right]\ d\theta$$

$$A=-12\int_0^\pi \sin(2\theta)\ d\theta-\int_0^\pi 1-\cos(4\theta)\ d\theta$$

Integrate.

$$A=-12\left(-\frac{1}{2}\cos(2\theta)\right)\Big|_0^\pi-\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_0^\pi$$

$$A=6\cos(2\theta)\Big|_0^\pi+\frac{1}{4}\sin(4\theta)-\theta\Big|_0^\pi$$

$$A=6\cos(2\theta)+\frac{1}{4}\sin(4\theta)-\theta\Big|_0^\pi$$

Evaluate over the interval.

$$A = 6\cos(2\pi) + \frac{1}{4}\sin(4\pi) - \pi - \left(6\cos(2(0)) + \frac{1}{4}\sin(4(0)) - 0\right)$$

$$A = 6(1) + \frac{1}{4}(0) - \pi - \left(6(1) + \frac{1}{4}(0)\right)$$

$$A = 6 - \pi - 6$$

$$A = -\pi$$

Topic: Area under one arc or loop of a parametric curve

Question: Find the area under one arc of the parametric curve.

$$x = 8 + \sin\theta$$

$$y = 8\cos\theta$$

Answer choices:

A $\quad -4\pi$

B $\quad -8\pi$

C $\quad 4\pi$

D $\quad 8\pi$

Solution: D

To find the area under one arc or loop of a parametric curve, we will need to use the formula

$$A=\int_a^b y(t)x'(t)\ dt$$

where $[a,b]$ is the interval that contains the loop (typically $[0,2\pi]$), and $x'(t)$ is the derivative of $x(t)$.

First we'll find the bounds we need to use by setting up a table for θ, x, and y.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	8	9	8	7	8
y	8	0	-8	0	8

Because $(x,y)=(8,8)$ at $\theta=0$, and we don't get back to $(8,8)$ until $\theta=2\pi$, we know the first loop of the parametric curve is closed by $\theta=[0,2\pi]$.

Before we can plug everything into our area formula, we'll need to find the derivative of $x(\theta)$.

$$x'(\theta)=\cos\theta$$

Plugging everything into the area formula, we get

$$A=\int_0^{2\pi}(8\cos\theta)(\cos\theta)\ d\theta$$

$$A = 8\int_0^{2\pi} \cos^2\theta \ d\theta$$

Before we can integrate, we need to do a substitution for $\cos^2\theta$ using the formula

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

We'll make the substitution.

$$A = 8\int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \ d\theta$$

$$A = 4\int_0^{2\pi} 1 + \cos(2\theta) \ d\theta$$

$$A = 4\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{2\pi}$$

$$A = 4\left[2\pi + \frac{1}{2}\sin(4\pi)\right] - 4\left[0 + \frac{1}{2}\sin(0)\right]$$

$$A = 4\left[2\pi + \frac{1}{2}(0)\right] - 4\left[0 + \frac{1}{2}(0)\right]$$

$$A = 4(2\pi)$$

$$A = 8\pi$$

Topic: Arc length of a parametric curve

Question: Find the length of the parametric curve on the given interval.

$$x = 2 - t$$

$$y = 3 + 4t$$

$$-2 \le t \le 3$$

Answer choices:

A $\sqrt{17}$

B $5\sqrt{15}$

C $\sqrt{15}$

D $5\sqrt{17}$

Solution: D

The length of a curve over an interval $a \le t \le b$ is given by

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x = 2 - t$$

$$\frac{dx}{dt} = -1$$

and

$$y = 3 + 4t$$

$$\frac{dy}{dt} = 4$$

Plugging these into our formula, we get

$$L = \int_{-2}^{3} \sqrt{(-1)^2 + (4)^2}\ dt$$

$$L = \int_{-2}^{3} \sqrt{17}\ dt$$

$$L = \sqrt{17}t\Big|_{-2}^{3}$$

$$L = \sqrt{17}(3) - \sqrt{17}(-2)$$

$$L = 3\sqrt{17} + 2\sqrt{17}$$

$$L = 5\sqrt{17}$$

Topic: Arc length of a parametric curve

Question: Find the length of the parametric curve on the given interval.

$$x = 3e^{3t} - 4t$$

$$y = 8e^{\frac{3t}{2}}$$

$$1 \leq t \leq 2$$

Answer choices:

A $3e^6$

B $6e^6$

C $6e^6 - 4$

D $3e^6 - 3e^3 + 4$

Solution: D

The length of a curve over an interval $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Let's find the derivatives of our equations so that we can plug them into our arc length formula.

$$x = 3e^{3t} - 4t$$

$$\frac{dx}{dt} = 9e^{3t} - 4$$

and

$$y = 8e^{\frac{3t}{2}}$$

$$\frac{dy}{dt} = 12e^{\frac{3t}{2}}$$

Plugging these into our formula, we get

$$L = \int_1^2 \sqrt{\left(9e^{3t} - 4\right)^2 + \left(12e^{\frac{3t}{2}}\right)^2}\ dt$$

$$L = \int_1^2 \sqrt{81e^{6t} - 72e^{3t} + 16 + 144e^{3t}}\ dt$$

$$L = \int_1^2 \sqrt{81e^{6t} + 72e^{3t} + 16}\ dt$$

$$L = \int_1^2 \sqrt{\left(9e^{3t} + 4\right)^2}\ dt$$

$$L = \int_1^2 9e^{3t} + 4 \ dt$$

$$L = 3e^{3t} + 4t \Big|_1^2$$

$$L = 3e^{3(2)} + 4(2) - \left[3e^{3(1)} + 4(1)\right]$$

$$L = 3e^6 + 8 - 3e^3 - 4$$

$$L = 3e^6 - 3e^3 + 4$$

Topic: Arc length of a parametric curve

Question: A parametric curve is defined by $x = ae^t \cos t$ and $y = be^t \sin t$, where a and b are positive real numbers. For which values of a and b between $t = 0$ and $t = \pi/2$, is this the length of the curve?

$$L = 2\sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Answer choices:

A	$a = 2$	and	$b = 2$
B	$a = 2$	and	$b = 3$
C	$a = 3$	and	$b = 2$
D	$a = 3$	and	$b = 3$

Solution: A

Choose $a=2$ and $b=2$. Then the given functions

$$x=ae^t\cos t$$

$$y=be^t\sin t$$

take on the following forms:

$$x=2e^t\cos t$$

$$y=2e^t\sin t$$

Differentiate both functions, and square the results.

$$\frac{dx}{dt}=2e^t\cos t-2e^t\sin t$$

$$\left(\frac{dx}{dt}\right)^2=\left(2e^t\cos t-2e^t\sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2=4e^{2t}\left(\cos t-\sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2=4e^{2t}\left(\cos^2 t+\sin^2 t-2\sin t\cos t\right)$$

$$\left(\frac{dx}{dt}\right)^2=4e^{2t}\left(1-2\sin t\cos t\right)$$

and

$$\frac{dy}{dt} = 2e^t \sin t + 2e^t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(\sin t + \cos t\right)^2$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(\sin^2 t + \cos^2 t + 2\sin t \cos t\right)$$

$$\left(\frac{dy}{dt}\right)^2 = 4e^{2t}\left(1 + 2\sin t \cos t\right)$$

Plug into the arc length formula. Remember that we were given the limits of integration $[0,\pi/2]$ in the problem.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{4e^{2t}\left(1 - 2\sin t \cos t\right) + 4e^{2t}\left(1 + 2\sin t \cos t\right)}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{4e^{2t}\left(1 - 2\sin t \cos t + 1 + 2\sin t \cos\right)}\ dt$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{8e^{2t}}\ dt$$

$$L = 2\sqrt{2}\int_0^{\frac{\pi}{2}} e^t\ dt$$

Integrate.

$$L = 2\sqrt{2}e^t \Big|_0^{\frac{\pi}{2}}$$

$$L = 2\sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Because this matches the arc length we were given, we know that $a = 2$ and $b = 2$ are the values we needed to choose. So answer choice A is correct.

Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = \frac{7}{4}t$$

$$y = t + 3$$

$$0 \leq t \leq 4$$

about the x-axis

Answer choices:

A 22π

B 56π

C $32\pi\sqrt{59}$

D $10\pi\sqrt{65}$

Solution: D

The formula for surface area of a parametric curve revolved about the x-axis on the given interval is

$$S=\int_0^4 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=\frac{7}{4}t$$

$$\frac{dx}{dt}=\frac{7}{4}$$

and

$$y=t+3$$

$$\frac{dy}{dt}=1$$

Plugging these into our formula, we get

$$S=\int_0^4 2\pi(t+3)\sqrt{\left(\frac{7}{4}\right)^2+(1)^2}\ dt$$

$$S=\int_0^4 2\pi(t+3)\sqrt{\frac{49}{16}+\frac{16}{16}}\ dt$$

$$S=2\pi\sqrt{\frac{65}{16}}\int_0^4 t+3\ dt$$

$$S=\frac{\pi\sqrt{65}}{2}\left(\frac{1}{2}t^2+3t\right)\Bigg|_0^4$$

$$S=\frac{\pi\sqrt{65}}{2}\left[\left(\frac{1}{2}(4)^2+3(4)\right)-\left(\frac{1}{2}(0)^2+3(0)\right)\right]$$

$$S=\frac{\pi\sqrt{65}}{2}(8+12)$$

$$S=10\pi\sqrt{65}$$

Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 3e^{3t} - 9t$$

$$y = 12e^{\frac{3t}{2}}$$

$$1 \leq t \leq 2$$

about the x-axis

Answer choices:

A $\quad 48\pi\left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$

B $\quad 48\pi\left(e^9 + 3e^3\right)$

C $\quad 48\pi\left(e^9 - 3e^3\right)$

D $\quad 32\pi\left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$

Solution: A

The formula for surface area of a parametric curve revolved about the x-axis on the given interval is

$$S=\int_1^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=3e^{3t}-9t$$

$$\frac{dx}{dt}=9e^{3t}-9$$

and

$$y=12e^{\frac{3t}{2}}$$

$$\frac{dy}{dt}=18e^{\frac{3t}{2}}$$

Plugging these into our formula, we get

$$S=\int_1^2 2\pi\left(12e^{\frac{3t}{2}}\right)\sqrt{\left(9e^{3t}-9\right)^2+\left(18e^{\frac{3t}{2}}\right)^2}\ dt$$

$$S=24\pi\int_1^2 e^{\frac{3t}{2}}\sqrt{81e^{6t}-162e^{3t}+81+324e^{3t}}\ dt$$

$$S=24\pi\int_1^2 e^{\frac{3t}{2}}\sqrt{81e^{6t}+162e^{3t}+81}\ dt$$

$$S = 24\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{81\left(e^{6t} + 2e^{3t} + 1\right)}\ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{e^{6t} + 2e^{3t} + 1}\ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \sqrt{\left(e^{3t} + 1\right)^2}\ dt$$

$$S = 216\pi \int_1^2 e^{\frac{3t}{2}} \left(e^{3t} + 1\right)\ dt$$

$$S = 216\pi \int_1^2 e^{\frac{9t}{2}} + e^{\frac{3t}{2}}\ dt$$

$$S = 216\pi \left(\frac{2}{9}e^{\frac{9t}{2}} + \frac{2}{3}e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 144\pi \left(\frac{1}{3}e^{\frac{9t}{2}} + e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 48\pi \left(e^{\frac{9t}{2}} + 3e^{\frac{3t}{2}}\right)\Bigg|_1^2$$

$$S = 48\pi \left[\left(e^{\frac{9(2)}{2}} + 3e^{\frac{3(2)}{2}}\right) - \left(e^{\frac{9(1)}{2}} + 3e^{\frac{3(1)}{2}}\right)\right]$$

$$S = 48\pi \left(e^9 + 3e^3 - e^{\frac{9}{2}} - 3e^{\frac{3}{2}}\right)$$

Topic: Surface area of revolution of a parametric curve, horizontal axis

Question: A circle is defined by the parametric functions $x = 3\cos t$ and $y = 3 + 3\sin t$. The curve is revolved around the x-axis. Which of the following pieces of surface area is the smallest?

Area A_1 is between $t = 0$ and $t = \pi$

Area A_2 is between $t = 0$ and $t = 2\pi$

Area A_3 is between $t = \pi/6$ and $t = 7\pi/6$

Area A_4 is between $t = \pi/2$ and $t = 3\pi/2$

Answer choices:

A A_1

B A_2

C A_3

D A_4

Solution: D

Differentiate both functions, and square the results.

$$x = 3\cos t$$

$$\frac{dx}{dt} = -3\sin t$$

$$\left(\frac{dx}{dt}\right)^2 = \left(-3\sin t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 = 9\sin^2 t$$

and

$$y = 3 + 3\sin t$$

$$\frac{dy}{dt} = 3\cos t$$

$$\left(\frac{dy}{dt}\right)^2 = (3\cos t)^2$$

$$\left(\frac{dy}{dt}\right)^2 = 9\cos^2 t$$

Plug the values we've found into the surface area of revolution formula:

$$S = \int_1^2 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

For A_1 between $t=0$ and $t=\pi$:

$$A_1=\int_0^{\pi}2\pi\left(3+3\sin t\right)\sqrt{9\left(\sin^2+\cos^2 t\right)}\ dt$$

$$A_1=\int_0^{\pi}6\pi\left(3+3\sin t\right)\ dt$$

$$A_1=18\pi\int_0^{\pi}\left(1+\sin t\right)\ dt$$

$$A_1=18\pi\left(t-\cos t\right)\Big|_0^{\pi}$$

$$A_1=18\pi\left[(\pi-\cos\pi)-(0-\cos 0)\right]$$

$$A_1=18\pi(\pi+1+1)$$

$$A_1=18\pi(\pi+2)$$

For A_2 between $t=0$ and $t=2\pi$:

$$A_2=\int_0^{2\pi}2\pi\left(3+3\sin t\right)\sqrt{9\left(\sin^2+\cos^2 t\right)}\ dt$$

$$A_2=18\pi\left(t-\cos t\right)\Big|_0^{2\pi}$$

$$A_2=18\pi\left[(2\pi-1)-(0-1)\right]$$

$$A_2=36\pi^2$$

For A_3 between $t=\pi/6$ and $t=7\pi/6$:

$$A_3 = \int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} 2\pi\left(3 + 3\sin t\right)\sqrt{9\left(\sin^2 + \cos^2 t\right)}\ dt$$

$$A_3 = 18\pi\left(t - \cos t\right)\Big|_{\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$A_3 = 18\pi\left[\left(\frac{7\pi}{6} - \cos\frac{7\pi}{6}\right) - \left(\frac{\pi}{6} - \cos\frac{\pi}{6}\right)\right]$$

$$A_3 = 18\pi\left(\frac{7\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}\right)$$

$$A_3 = 18\pi\left(\pi + \sqrt{3}\right)$$

For A_4 between $t = \pi/2$ and $t = 3\pi/2$:

$$A_4 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\pi\left(3 + 3\sin t\right)\sqrt{9\left(\sin^2 + \cos^2 t\right)}\ dt$$

$$A_4 = 18\pi\left(t - \cos t\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A_4 = 18\pi\left[\left(\frac{3\pi}{2} - \cos\frac{3\pi}{2}\right) - \left(\frac{\pi}{2} - \cos\frac{\pi}{2}\right)\right]$$

$$A_4 = 18\pi\left(\frac{3\pi}{2} - 0 - \frac{\pi}{2} + 0\right)$$

$$A_4 = 18\pi^2$$

If we compare the amount of area in each region,

$$A_1 = 18\pi(\pi + 2) \approx 290.75$$

$$A_2 = 36\pi^2 \approx 355.31$$

$$A_3 = 18\pi\left(\pi + \sqrt{3}\right) \approx 275.60$$

$$A_4 = 18\pi^2 \approx 177.65$$

we can see that A_4 is the smallest region of area.

Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 2t$$

$$y = 5t^2$$

$$0 \le t \le 1$$

about the y-axis

Answer choices:

A $\quad \dfrac{8\pi}{75}\left(26^{\frac{3}{2}} + 1\right)$

B $\quad \dfrac{8\pi}{75}\left(26^{\frac{3}{2}} - 1\right)$

C $\quad \dfrac{4\pi}{75}\left(26^{\frac{3}{2}} + 1\right)$

D $\quad \dfrac{4\pi}{75}\left(26^{\frac{3}{2}} - 1\right)$

Solution: B

The formula for surface area of a parametric curve revolved about the y -axis on the given interval is

$$S=\int_{0}^{1}2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=2t$$

$$\frac{dx}{dt}=2$$

and

$$y=5t^2$$

$$\frac{dy}{dt}=10t$$

Plugging these into our formula, we get

$$S=\int_{0}^{1}2\pi(2t)\sqrt{(2)^2+(10t)^2}\ dt$$

$$S=4\pi\int_{0}^{1}t\sqrt{4+100t^2}\ dt$$

$$S=4\pi\int_{0}^{1}t\sqrt{4\left(1+25t^2\right)}\ dt$$

$$S = 8\pi \int_0^1 t\sqrt{1 + 25t^2}\ dt$$

Using u-substitution with

$$u = 1 + 25t^2$$

$$du = 50t\ dt$$

$$dt = \frac{du}{50t}$$

we'll substitute and get

$$S = 8\pi \int_{t=0}^{t=1} t\sqrt{u}\ \frac{du}{50t}$$

$$S = \frac{8\pi}{50} \int_{t=0}^{t=1} \sqrt{u}\ du$$

$$S = \frac{8\pi}{50} \int_{t=0}^{t=1} u^{\frac{1}{2}}\ du$$

$$S = \frac{8\pi}{50} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=1}$$

$$S = \frac{8\pi}{25} \left(\frac{1}{3} u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=1}$$

$$S = \frac{8\pi}{75} u^{\frac{3}{2}} \Bigg|_{t=0}^{t=1}$$

Back-substituting so that we can evaluate over the interval, we get

$$S = \frac{8\pi}{75}\left(1+25t^2\right)^{\frac{3}{2}}\Big|_{t=0}^{t=1}$$

$$S = \frac{8\pi}{75}\sqrt{\left(1+25t^2\right)^3}\Big|_{t=0}^{t=1}$$

$$S = \frac{8\pi}{75}\sqrt{\left[1+25(1)^2\right]^3} - \frac{8\pi}{75}\sqrt{\left[1+25(0)^2\right]^3}$$

$$S = \frac{8\pi}{75}\sqrt{(26)^3} - \frac{8\pi}{75}\sqrt{(1)^3}$$

$$S = \frac{8\pi}{75}\left(26^{\frac{3}{2}} - 1\right)$$

Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = t$$

$$y = 2t^2 + 6$$

$$0 \leq t \leq 3$$

about the y-axis

Answer choices:

A $\frac{\pi}{6}\left(145^{\frac{3}{2}} - 1\right)$

B $\frac{\pi}{6}\left(145^{\frac{3}{2}} + 1\right)$

C $\frac{\pi}{24}\left(145^{\frac{3}{2}} - 1\right)$

D $\frac{\pi}{24}\left(145^{\frac{3}{2}} + 1\right)$

Solution: C

The formula for surface area of a parametric curve revolved about the y -axis on the given interval is

$$S=\int_0^3 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=t$$

$$\frac{dx}{dt}=1$$

and

$$y=2t^2+6$$

$$\frac{dy}{dt}=4t$$

Plugging these into our formula, we get

$$S=\int_0^3 2\pi t\sqrt{(1)^2+(4t)^2}\ dt$$

$$S=2\pi\int_0^3 t\sqrt{1+16t^2}\ dt$$

Using u-substitution with

$$u=1+16t^2$$

$$du = 32t\ dt$$

$$dt = \frac{du}{32t}$$

we'll substitute and get

$$S = 2\pi \int_{t=0}^{t=3} t\sqrt{u}\ \frac{du}{32t}$$

$$S = \frac{2\pi}{32} \int_{t=0}^{t=3} \sqrt{u}\ du$$

$$S = \frac{\pi}{16} \int_{t=0}^{t=3} u^{\frac{1}{2}}\ du$$

$$S = \frac{\pi}{16} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=3}$$

$$S = \frac{\pi}{8} \left(\frac{1}{3} u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=3}$$

$$S = \frac{\pi}{24} \left(u^{\frac{3}{2}} \right) \Bigg|_{t=0}^{t=3}$$

Back-substituting so that we can evaluate over the interval, we get

$$S = \frac{\pi}{24} \left(1 + 16t^2 \right)^{\frac{3}{2}} \Bigg|_0^3$$

$$S = \frac{\pi}{24} \left[1 + 16(3)^2 \right]^{\frac{3}{2}} - \frac{\pi}{24} \left[1 + 16(0)^2 \right]^{\frac{3}{2}}$$

$$S = \frac{\pi}{24}\left[1 + 16(9)\right]^{\frac{3}{2}} - \frac{\pi}{24}\left[1 + 16(0)\right]^{\frac{3}{2}}$$

$$S = \frac{\pi}{24}145^{\frac{3}{2}} - \frac{\pi}{24}$$

$$S = \frac{\pi}{24}\left(145^{\frac{3}{2}} - 1\right)$$

Topic: Surface area of revolution of a parametric curve, vertical axis

Question: Find the surface area of revolution of the parametric curve rotated about the given axis.

$$x = 5t^2$$

$$y = 3t^3$$

on the interval $0 \leq t \leq 4$

about the y-axis

Answer choices:

A $51{,}338.46$

B $61{,}338.46$

C $61{,}469.39$

D $81{,}338.46$

Solution: C

The formula for surface area of a parametric curve revolved about the y-axis on the given interval is

$$S=\int_0^4 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

We'll calculate the derivatives of x and y so that we can plug them into the formula.

$$x=5t^2$$

$$\frac{dx}{dt}=10t$$

and

$$y=3t^3$$

$$\frac{dy}{dt}=9t^2$$

Plugging these into our formula, we get

$$S=\int_0^4 2\pi\left(5t^2\right)\sqrt{(10t)^2+\left(9t^2\right)^2}\ dt$$

$$S=10\pi\int_0^4 t^2\sqrt{t^2\left(100+81t^2\right)}\ dt$$

$$S=10\pi\int_0^4 t^3\sqrt{100+81t^2}\ dt$$

Using u-substitution with

$$u = 100 + 81t^2$$

$$du = 162t\ dt$$

$$dt = \frac{du}{162t}$$

we'll substitute and get

$$S = 10\pi \int_{t=0}^{t=4} t^3\sqrt{u}\ \frac{du}{162t}$$

$$S = \frac{10\pi}{162} \int_{t=0}^{t=4} t^2\sqrt{u}\ du$$

$$S = \frac{5\pi}{81} \int_{t=0}^{t=4} t^2\sqrt{u}\ du$$

Since we said earlier that $u = 100 + 81t^2$, we can say

$$u = 100 + 81t^2$$

$$u - 100 = 81t^2$$

$$\frac{u - 100}{81} = t^2$$

Make this substitution.

$$S = \frac{5\pi}{81} \int_{t=0}^{t=4} \frac{u - 100}{81}\sqrt{u}\ du$$

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}\frac{u\sqrt{u}-100\sqrt{u}}{81}\ du$$

$$S=\frac{5\pi}{81}\int_{t=0}^{t=4}\frac{u^{\frac{3}{2}}-100u^{\frac{1}{2}}}{81}\ du$$

$$S=\frac{5\pi}{81^2}\int_{t=0}^{t=4}u^{\frac{3}{2}}-100u^{\frac{1}{2}}\ du$$

$$S=\frac{5\pi}{81^2}\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{200}{3}u^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=4}$$

Back-substituting so that we can evaluate over the interval, we get

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81t^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81t^2\right)^{\frac{3}{2}}\right]\Bigg|_0^4$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(4)^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(4)^2\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(0)^2\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(0)^2\right)^{\frac{3}{2}}\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+81(16)\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+81(16)\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}(100)^{\frac{5}{2}}-\frac{200}{3}(100)^{\frac{3}{2}}\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}\left(100+1{,}296\right)^{\frac{5}{2}}-\frac{200}{3}\left(100+1{,}296\right)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left[\frac{2}{5}(100{,}000)-\frac{200}{3}(1{,}000)\right]$$

$$S=\frac{5\pi}{81^2}\left[\frac{2}{5}(1{,}396)^{\frac{5}{2}}-\frac{200}{3}(1{,}396)^{\frac{3}{2}}\right]-\frac{5\pi}{81^2}\left(\frac{200{,}000}{5}-\frac{200{,}000}{3}\right)$$

$$S=\frac{10\pi}{81^2}\left[\frac{1}{5}(1{,}396)^{\frac{5}{2}}-\frac{100}{3}(1{,}396)^{\frac{3}{2}}\right]-\frac{1{,}000{,}000\pi}{81^2}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$S = \frac{10\pi}{81^2}\left[\frac{1}{5}\left(1{,}396\right)^{\frac{5}{2}} - \frac{100}{3}\left(1{,}396\right)^{\frac{3}{2}} - 100{,}000\left(\frac{1}{5} - \frac{1}{3}\right)\right]$$

$$S = \frac{10\pi}{81^2}\left[\frac{1}{5}\left(1{,}396\right)^{\frac{5}{2}} - \frac{100}{3}\left(1{,}396\right)^{\frac{3}{2}} - 100{,}000\left(-\frac{2}{15}\right)\right]$$

$$S = \frac{10\pi}{81^2}\left[\frac{1}{5}\left(1{,}396\right)^{\frac{5}{2}} - \frac{100}{3}\left(1{,}396\right)^{\frac{3}{2}} + \frac{40{,}000}{3}\right]$$

$$S = \frac{10\pi}{81^2}(14{,}562{,}754.9418 - 1{,}738{,}628.8135 + 13{,}333.3333)$$

$$S = \frac{10\pi}{81^2}(12{,}837{,}459.4616)$$

$$S \approx 61{,}469.39$$

Topic: Volume of revolution of a parametric curve

Question: Find the volume of revolution of the parametric curve.

$$x = 3e^{3t} - 9t$$

$$y = 12e^{\frac{3t}{2}}$$

$$1 \leq t \leq 2$$

about the x-axis

Answer choices:

A $216\pi e^3 \left(e^9 - 3e^3 + 2\right)$

B $216\pi e^3 \left(e^9 + 3e^3 - 2\right)$

C $342\pi \left(e^{12} - 3e^6 + 2e^3\right)$

D $342\pi \left(e^{12} + 3e^6 + 2e^3\right)$

Solution: A

Since we're rotating around the x-axis, we'll use the formula

$$V_x = \int_{\alpha}^{\beta} \pi y^2 \frac{dx}{dt}\ dt$$

The problem gave the interval $1 \leq t \leq 2$, so $\alpha = 1$ and $\beta = 2$. Now we need to find dx/dt so that we can plug it into the volume formula.

$$x = 3e^{3t} - 9t$$

$$\frac{dx}{dt} = 9e^{3t} - 9$$

Plugging everything into the volume formula, we get

$$V_x = \int_1^2 \pi \left(12e^{\frac{3t}{2}}\right)^2 \left(9e^{3t} - 9\right)\ dt$$

$$V_x = 1{,}296\pi \int_1^2 \left(e^{\frac{3t}{2}}\right)^2 \left(e^{3t} - 1\right)\ dt$$

$$V_x = 1{,}296\pi \int_1^2 e^{3t}\left(e^{3t} - 1\right)\ dt$$

$$V_x = 1{,}296\pi \int_1^2 e^{6t} - e^{3t}\ dt$$

$$V_x = 1{,}296\pi \left(\frac{1}{6}e^{6t} - \frac{1}{3}e^{3t}\right)\Bigg|_1^2$$

$$V_x = 1{,}296\pi \left[\left(\frac{1}{6}e^{6(2)} - \frac{1}{3}e^{3(2)} \right) - \left(\frac{1}{6}e^{6(1)} - \frac{1}{3}e^{3(1)} \right) \right]$$

$$V_x = 1{,}296\pi \left[\frac{e^{12}}{6} - \frac{e^6}{3} - \left(\frac{e^6}{6} - \frac{e^3}{3} \right) \right]$$

$$V_x = 1{,}296\pi \left(\frac{e^{12}}{6} - \frac{e^6}{3} - \frac{e^6}{6} + \frac{e^3}{3} \right)$$

$$V_x = 1{,}296\pi \left(\frac{e^{12}}{6} - \frac{2e^6}{6} - \frac{e^6}{6} + \frac{2e^3}{6} \right)$$

$$V_x = \frac{1{,}296}{6}\pi \left(e^{12} - 2e^6 - e^6 + 2e^3 \right)$$

$$V_x = 216\pi \left(e^{12} - 3e^6 + 2e^3 \right)$$

$$V_x = 216\pi e^3 \left(e^9 - 3e^3 + 2 \right)$$

Topic: Volume of revolution of a parametric curve

Question: Find the volume of revolution of the parametric curve.

$x = t^2$

$y = 3t^2$

$0 \leq t \leq 1$

about the x-axis

Answer choices:

A 3π

B 2π

C -3π

D -2π

Solution: A

Since we're rotating around the x-axis, we'll use the formula

$$V_x = \int_{\alpha}^{\beta} \pi y^2 \frac{dx}{dt}\ dt$$

The problem gave the interval $0 \le t \le 1$, so $\alpha = 0$ and $\beta = 1$. Now we need to find dx/dt so that we can plug it into the volume formula.

$$x = t^2$$

$$\frac{dx}{dt} = 2t$$

Plugging everything into the volume formula, we get

$$V_x = \int_0^1 \pi \left(3t^2\right)^2 (2t)\ dt$$

$$V_x = 18\pi \int_0^1 t^5\ dt$$

$$V_x = 18\pi \left(\frac{1}{6}t^6\right)\Big|_0^1$$

$$V_x = 3\pi t^6\Big|_0^1$$

$$V_x = 3\pi(1)^6 - 3\pi(0)^6$$

$$V_x = 3\pi$$

Topic: Volume of revolution of a parametric curve

Question: Find the volume of revolution of the parametric curve.

$$x = 4t^2$$

$$y = t^2 + 1$$

on the interval $0 \leq t \leq 1$

about the y-axis

Answer choices:

A $-\dfrac{16\pi}{3}$

B 6π

C $\dfrac{16\pi}{3}$

D -6π

Solution: C

Since we're rotating around the y-axis, we'll use the formula

$$V_y = \int_{\alpha}^{\beta} \pi x^2 \frac{dy}{dt}\ dt$$

The problem gave the interval $0 \le t \le 1$, so $\alpha = 0$ and $\beta = 1$. Now we need to find dy/dt so that we can plug it into the volume formula.

$$y = t^2 + 1$$

$$\frac{dy}{dt} = 2t$$

Plugging everything into the volume formula, we get

$$V_y = \int_0^1 \pi\left(4t^2\right)^2(2t)\ dt$$

$$V_y = 32\pi \int_0^1 t^5\ dt$$

$$V_y = 32\pi\left(\frac{1}{6}t^6\right)\Bigg|_0^1$$

$$V_y = \frac{16\pi}{3}t^6\Bigg|_0^1$$

$$V_y = \frac{16\pi}{3}(1)^6 - \frac{16\pi}{3}(0)^6$$

$$V_y = \frac{16\pi}{3}$$

Topic: Polar coordinates

Question: Convert the polar coordinates to rectangular coordinates.

$$\left(2, \frac{11}{6}\pi\right)$$

Answer choices:

A $\left(\sqrt{3}, -1\right)$

B $\left(\sqrt{3}, 1\right)$

C $\left(\sqrt{3}, -\frac{1}{2}\right)$

D $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Solution: A

Using the equations $x = r\cos\theta$ and $y = r\sin\theta$, the rectangular coordinates are

$$x = 2\cos\frac{11}{6}\pi$$

$$y = 2\sin\frac{11}{6}\pi$$

From the unit circle, we know that the sine and cosine of $11\pi/6$ are $-1/2$ and $\sqrt{3}/2$ respectively. Therefore,

$$x = 2\left(\frac{\sqrt{3}}{2}\right)$$

$$x = \sqrt{3}$$

and

$$y = 2\sin\frac{11}{6}\pi$$

$$y = 2\left(-\frac{1}{2}\right)$$

$$y = -1$$

The polar coordinate $\left(2, \frac{11}{6}\pi\right)$ is equal to the rectangular coordinate $\left(\sqrt{3}, -1\right)$.

Topic: Polar coordinates

Question: What are the measures of θ if $r = 4$, given that an ellipse is defined by

$$\frac{1}{r^2} = \frac{\cos^2\theta}{9} + \frac{\sin^2\theta}{16}$$

Answer choices:

A $\quad \theta = \frac{\pi}{2} \quad$ and $\quad \theta = \frac{2\pi}{3}$

B $\quad \theta = \frac{\pi}{2} \quad$ and $\quad \theta = \frac{3\pi}{2}$

C $\quad \theta = \arccos\left(\pm\frac{\sqrt{35}}{10}\right)$

D $\quad \theta = \arccos\left(\pm\frac{\sqrt{35}}{5}\right)$

Solution: B

We were asked to use $r=4$, so we'll plug that into the equation for the ellipse.

$$\frac{1}{r^2}=\frac{\cos^2\theta}{9}+\frac{\sin^2\theta}{16}$$

$$\frac{1}{4^2}=\frac{\cos^2\theta}{9}+\frac{\sin^2\theta}{16}$$

$$1=\frac{16}{9}\cos^2\theta+\sin^2\theta$$

$$9=16\cos^2\theta+9\sin^2\theta$$

$$9=16\left(1-\sin^2\theta\right)+9\sin^2\theta$$

$$9=16-16\sin^2\theta+9\sin^2\theta$$

$$-7=-7\sin^2\theta$$

$$1=\sin^2\theta$$

$$\pm 1=\sin\theta$$

$$\theta=\frac{\pi}{2},\frac{3\pi}{2}$$

Topic: Polar coordinates

Question: What is the length of r for the graph of the polar curve?

$$r = \left(\sin^6\theta + \cos^6\theta\right) - \left(\sin^4\theta + \cos^4\theta\right) - \sin^4\theta + \sin^2\theta + 1$$

Answer choices:

A $\quad r = 9$

B $\quad r = 6$

C $\quad r = 4$

D $\quad r = 1$

Solution: D

Using trigonometric identities to simplify the function.

$$r = \left(\sin^6\theta + \cos^6\theta\right) - \left(\sin^4\theta + \cos^4\theta\right) - \sin^4\theta + \sin^2\theta + 1$$

$$r = \left(\sin^2\theta + \cos^2\theta\right)\left(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta\right) - \left(\sin^4\theta + \cos^4\theta\right) - \sin^4\theta + \sin^2\theta + 1$$

$$r = \sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta - \sin^4\theta - \cos^4\theta - \sin^4\theta + \sin^2\theta + 1$$

$$r = -\sin^2\theta\cos^2\theta - \sin^4\theta + \sin^2\theta + 1$$

$$r = -\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\theta + 1$$

$$r = -\sin^2\theta + \sin^2\theta + 1$$

$$r = 1$$

Topic: Converting rectangular equations

Question: Convert the rectangular equation to a polar equation.

$$x^2 + y^2 - 4y = 0$$

Answer choices:

A $\quad r = 4 + 4\sin\theta$

B $\quad r = 4\cos\theta$

C $\quad r = 4 - 4\sin\theta$

D $\quad r = 4\sin\theta$

Solution: D

Using the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

we can convert the equation from rectangular coordinates to polar coordinates.

For this particular problem, we'll use $x = r\cos\theta$ and $y = r\sin\theta$, even though we could just as easily use $r^2 = x^2 + y^2$.

$$x^2 + y^2 - 4y = 0$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 - 4(r\sin\theta) = 0$$

$$r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta = 0$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) - 4r\sin\theta = 0$$

$$r^2(1) - 4r\sin\theta = 0$$

$$r^2 - 4r\sin\theta = 0$$

$$r - 4\sin\theta = 0$$

$$r = 4\sin\theta$$

Topic: Converting rectangular equations

Question: Convert the rectangular equation to a polar equation.

$$x^2 + y^2 = x$$

Answer choices:

A $r = -\sin\theta$

B $r = \sin\theta$

C $r = -\cos\theta$

D $r = \cos\theta$

Solution: D

Using the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

we can convert the equation from rectangular coordinates to polar coordinates.

For this particular problem, we'll use $r^2 = x^2 + y^2$ to change the left-hand side.

$$r^2 = x$$

Now we'll use $x = r\cos\theta$ to change the right-hand side.

$$r^2 = r\cos\theta$$

$$r = \cos\theta$$

Topic: Converting rectangular equations

Question: Convert the rectangular equation to a polar equation.

$$2x^2 + 2y^2 = -4y$$

Answer choices:

A $\quad r = -4\sin\theta$

B $\quad r = -2\sin\theta$

C $\quad r = 4\sin\theta$

D $\quad r = 2\sin\theta$

Solution: B

Using the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

we can convert the equation from rectangular coordinates to polar coordinates.

For this particular problem, we'll divide through by 2 to simplify the equation, then use $r^2 = x^2 + y^2$ to change the left-hand side.

$$x^2 + y^2 = -2y$$

$$r^2 = -2y$$

Now we'll use $y = r\sin\theta$ to change the right-hand side.

$$r^2 = -2r\sin\theta$$

$$r = -2\sin\theta$$

Topic: Converting polar equations

Question: Convert the polar equation to a rectangular equation.

$$r = \frac{6}{\sin\theta - 3\cos\theta}$$

Answer choices:

A $\quad y = -3x + 6$

B $\quad y = 3x + 6$

C $\quad y = 3x - 6$

D $\quad y = -3x - 6$

Solution: B

Converting a polar equation to a rectangular equation requires us to get r and θ out of the equation and get x and y into it. The following equations are needed for the conversion:

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

In this particular example, only the last two equations above are useful. Before we make substitutions, we'll simplify the given polar equation.

$$r = \frac{6}{\sin\theta - 3\cos\theta}$$

$$r(\sin\theta - 3\cos\theta) = 6$$

$$r\sin\theta - 3r\cos\theta = 6$$

Now we'll make the substitutions.

$$y - 3x = 6$$

$$y = 3x + 6$$

Topic: Converting polar equations

Question: Convert the polar equation to a rectangular equation.

$$r = (\csc\theta)2e^{3r\cos\theta}$$

Answer choices:

A $\quad y = 2e^3$

B $\quad y = 3e^{2x}$

C $\quad y = 2e^{3x}$

D $\quad y = -2e^{3x}$

Solution: C

In order to convert our polar equation to a rectangular equation, we'll need the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Using the trigonometric identity

$$\csc\theta = \frac{1}{\sin\theta}$$

the polar equation is first reduced to

$$r = \left(\frac{1}{\sin\theta}\right)2e^{3r\cos\theta}$$

Multiplying the equation by $\sin\theta$ and then using our conversion formulas gives us

$$r\sin\theta = 2e^{3r\cos\theta}$$

$$y = 2e^{3x}$$

Topic: Converting polar equations

Question: The parametric coordinates $x(t) = f(t)\cos t$ and $y(t) = g(t)\sin t$ are given, where $f(t) = t^2 - 3$ and $g(t) = \sqrt{9 - 6t^2 + t^4}$. Which statement describes the polar coordinates of the given coordinates?

Answer choices:

A The given coordinates define a circle with radius $t^2 - 3$ centered at the origin, where $t > \sqrt{3}$.

B The given coordinates define a circle with a radius $t^2 + 3$ centered at the origin, where $t > \sqrt{3}$.

C The given coordinates define a circle with a radius $t - 3$ centered at the origin, where $t > \sqrt{3}$.

D The given coordinates define a circle with a radius $t + 3$ centered at the origin, where $t < \sqrt{3}$.

Solution: A

Transform the polar coordinates to rectangular coordinates.

Replace $f(t)$ and $g(t)$ in the given coordinates and square both sides of each equation.

$$x(t) = f(t)\cos t$$

$$x(t) = \left(t^2 - 3\right)\cos t$$

$$\left[x(t)\right]^2 = \left(t^2 - 3\right)^2 \cos^2 t$$

$$x^2 = \left(t^4 - 6t^2 + 9\right)\cos^2 t$$

and

$$y(t) = g(t)\sin t$$

$$y(t) = \sqrt{9 - 6t^2 + t^4}\,\sin t$$

$$y^2 = \left(9 - 6t^2 + t^4\right)\sin^2 t$$

$$y^2 = \left(t^4 - 6t^2 + 9\right)\sin^2 t$$

Now add $x^2 = \left(t^4 - 6t + 9\right)\cos^2 t$ and $y^2 = \left(t^4 - 6t^2 + 9\right)\sin^2 t$.

$$x^2 + y^2 = \left(t^4 - 6t^2 + 9\right)\cos^2 t + \left(t^4 - 6t^2 + 9\right)\sin^2 t$$

$$x^2 + y^2 = \left(t^4 - 6t^2 + 9\right)\left(\cos^2 t + \sin^2 t\right)$$

$$x^2 + y^2 = t^4 - 6t^2 + 9$$

$$x^2 + y^2 = \left(t^2 - 3\right)^2$$

Because we know that in polar coordinates $x^2 + y^2 = r^2$, the radius of the polar coordinate is $t^2 - 3$ centered at the origin, where $t > \sqrt{3}$.

Topic: Distance between polar points

Question: Calculate the distance between the polar coordinate points.

$$(1,\pi)$$

$$\left(1,\frac{\pi}{3}\right)$$

Answer choices:

A $\sqrt{5}$

B $\sqrt{3}$

C 2

D 1

Solution: B

To find the distance between two polar coordinates, we'll use the formula

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right)}$$

where $\left(r_1, \theta_1\right)$ and $\left(r_2, \theta_2\right)$ are the given polar points. It doesn't matter which point we use for $\left(r_1, \theta_1\right)$ and $\left(r_2, \theta_2\right)$, but it's easier to make θ_1 the larger of the two θ values, since we subtract θ_2 from θ_1.

We'll set

$$\left(r_1, \theta_1\right) = (1, \pi)$$

$$\left(r_2, \theta_2\right) = \left(1, \frac{\pi}{3}\right)$$

Plugging these points into the distance formula, we get

$$D = \sqrt{(1)^2 + (1)^2 - 2(1)(1)\cos\left(\pi - \frac{\pi}{3}\right)}$$

$$D = \sqrt{2 - 2\cos\left(\frac{2\pi}{3}\right)}$$

$$D = \sqrt{2 - 2\left(-\frac{1}{2}\right)}$$

$$D = \sqrt{2 + 1}$$

$$D = \sqrt{3}$$

Topic: Distance between polar points

Question: Calculate the distance between the polar coordinate points.

$$\left(3, \frac{4\pi}{3}\right)$$

$$\left(2, \frac{\pi}{2}\right)$$

Answer choices:

A $\sqrt{13 + 6\sqrt{3}}$

B $\sqrt{13 - 6\sqrt{3}}$

C 1

D $\sqrt{19}$

Solution: A

To find the distance between two polar coordinates, we'll use the formula

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

where (r_1, θ_1) and (r_2, θ_2) are the given polar points. It doesn't matter which point we use for (r_1, θ_1) and (r_2, θ_2), but it's easier to make θ_1 the larger of the two θ values, since we subtract θ_2 from θ_1.

We'll set

$$(r_1, \theta_1) = \left(3, \frac{4\pi}{3}\right)$$

$$(r_2, \theta_2) = \left(2, \frac{\pi}{2}\right)$$

Plugging these points into the distance formula, we get

$$D = \sqrt{(3)^2 + (2)^2 - 2(3)(2)\cos\left(\frac{4\pi}{3} - \frac{\pi}{2}\right)}$$

$$D = \sqrt{9 + 4 - 12\cos\left(\frac{8\pi}{6} - \frac{3\pi}{6}\right)}$$

$$D = \sqrt{13 - 12\cos\left(\frac{5\pi}{6}\right)}$$

$$D = \sqrt{13 - 12\left(-\frac{\sqrt{3}}{2}\right)}$$

$$D = \sqrt{13 + 6\sqrt{3}}$$

Topic: Distance between polar points

Question: Calculate the distance between the polar coordinate points.

$$\left(5, \frac{\pi}{4}\right)$$

$$\left(3, \frac{3\pi}{2}\right)$$

Answer choices:

A $\sqrt{34 - 15\sqrt{2}}$

B $\sqrt{23\sqrt{2}}$

C $\sqrt{11\sqrt{2}}$

D $\sqrt{34 + 15\sqrt{2}}$

Solution: D

To find the distance between two polar coordinates, we'll use the formula

$$D=\sqrt{r_1^2+r_2^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$$

where $\left(r_1,\theta_1\right)$ and $\left(r_2,\theta_2\right)$ are the given polar points. It doesn't matter which point we use for $\left(r_1,\theta_1\right)$ and $\left(r_2,\theta_2\right)$, but it's easier to make θ_1 the larger of the two θ values, since we subtract θ_2 from θ_1.

We'll set

$$\left(r_1,\theta_1\right)=\left(3,\frac{3\pi}{2}\right)$$

$$\left(r_2,\theta_2\right)=\left(5,\frac{\pi}{4}\right)$$

Plugging these points into the distance formula, we get

$$D=\sqrt{(3)^2+(5)^2-2(3)(5)\cos\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)}$$

$$D=\sqrt{9+25-30\cos\left(\frac{6\pi}{4}-\frac{\pi}{4}\right)}$$

$$D=\sqrt{34-30\cos\left(\frac{5\pi}{4}\right)}$$

$$D=\sqrt{34-30\left(-\frac{\sqrt{2}}{2}\right)}$$

$$D = \sqrt{34 + 15\sqrt{2}}$$

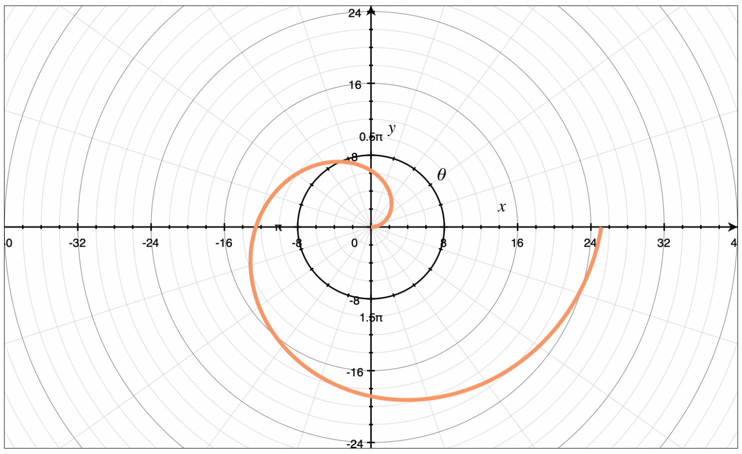
Topic: Sketching polar curves

Question: Match the polar curve to its graph.

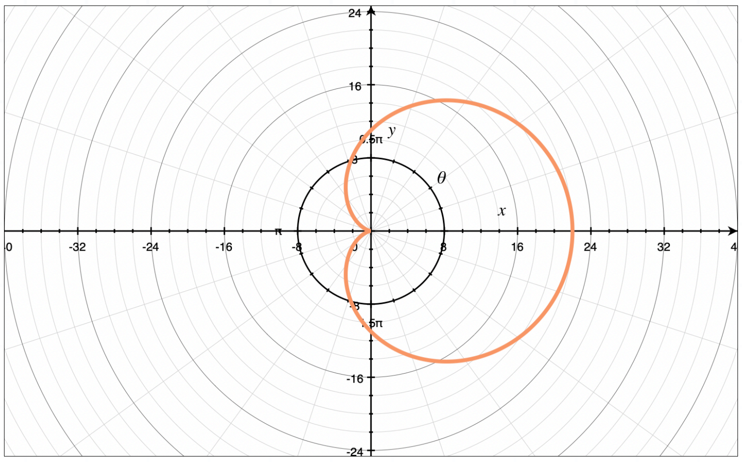
$$r = 11 + 11\cos\theta$$

Answer choices:

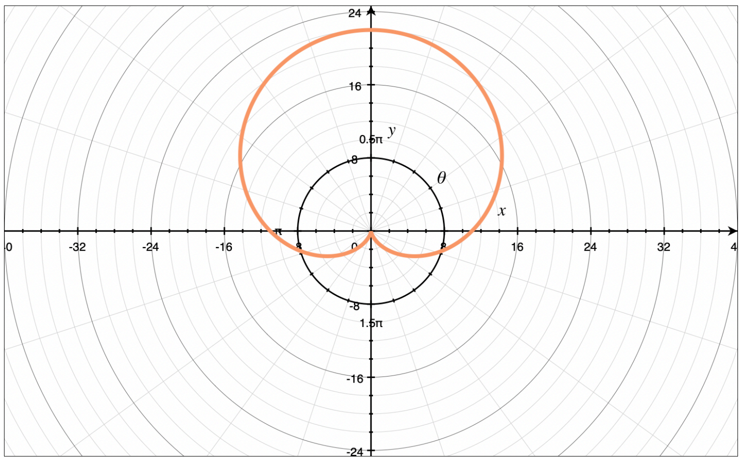
A



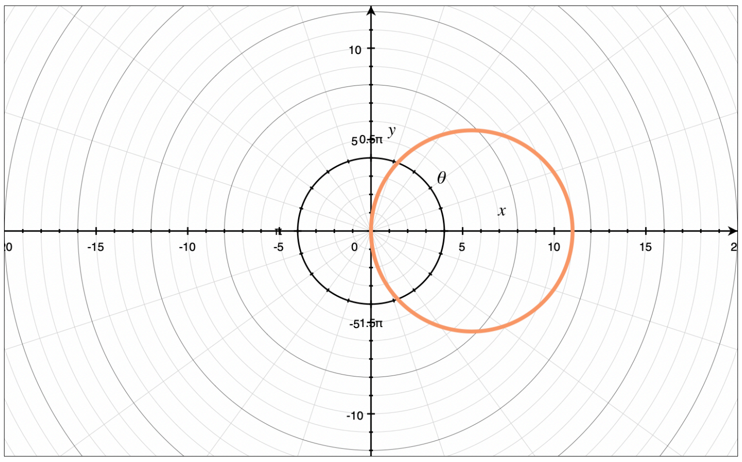
B



C



D



Solution: B

First we'll determine symmetries.

With respect to the x-axis by replacing θ with $-\theta$, and the test produces an equivalent equation.

$$r = 11 + 11\cos(-\theta)$$

$$r = 11 + 11\cos\theta$$

With respect to the y-axis by replacing θ with $\pi - \theta$, and the test does NOT produce an equivalent equation.

$$r = 11 + 11\cos(\pi - \theta)$$

$$r = 11 - 11\cos\theta$$

With respect to the origin by replacing r with $-r$, and the test does NOT produce an equivalent equation.

$$-r = 11 + 11\cos\theta$$

Therefore, the graph of the polar curve is symmetric to the x-axis only, not to the y-axis or to the origin.

Below is a table of values for θ and r and the graph of the polar curve.

θ	r
0	22
$\pi/2$	11

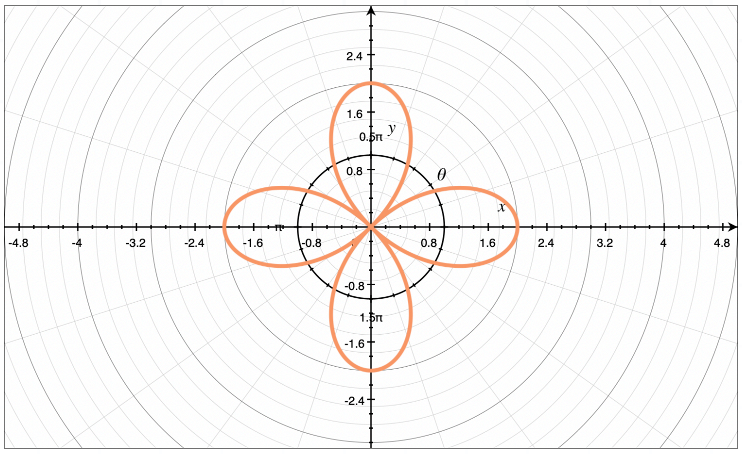
π	0
$3\pi/2$	11
2π	22

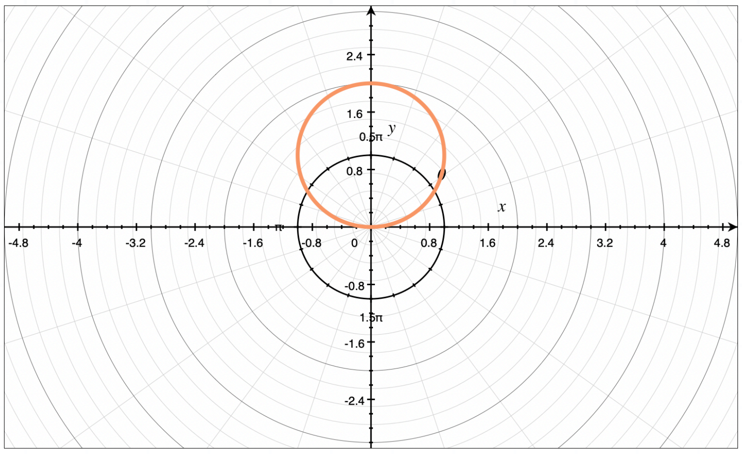
Topic: Sketching polar curves

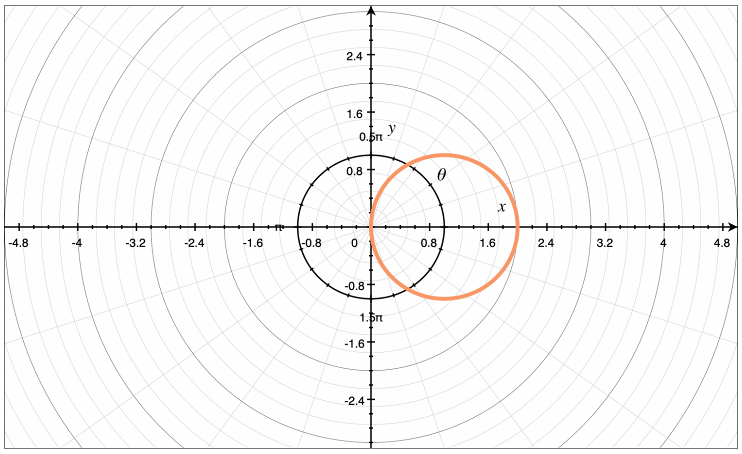
Question: Which graph represents the polar curve?

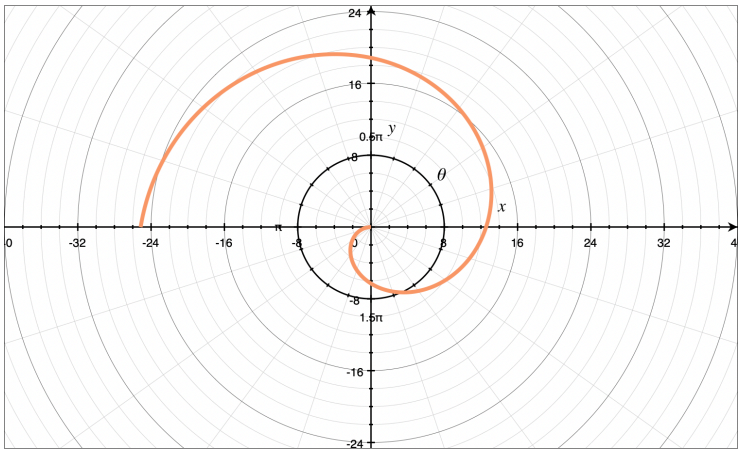
$$r = 2\sin\theta$$

Answer choices:

A 

B 

C 

D 

Solution: B

There are a number of ways to sketch a polar curve. If you are sketching directly on a polar graph and you are given a polar equation, you could simply chose a number of points for θ between 0 and 2π and solve for the associated r values. You could then take these points and plot them on your polar graph.

We can analyze the various potential graphs of $r = 2\sin\theta$ by first finding some important points.

$$\theta = 0 \qquad (0,0)$$

$$\theta = \frac{\pi}{2} \qquad \left(2,\frac{\pi}{2}\right)$$

$$\theta = \pi \qquad (0,\pi)$$

$$\theta = \frac{3\pi}{2} \qquad \left(-2,\frac{3\pi}{2}\right)$$

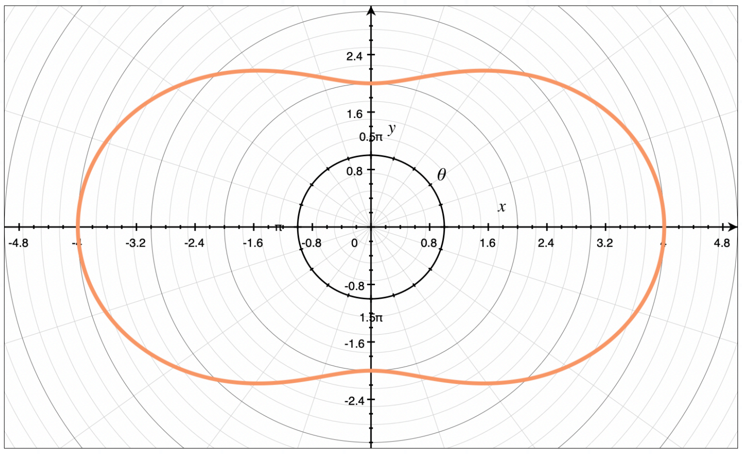
These points mean that the graph starts at $(0,0)$, then goes out to $(2,\pi/2)$, then towards $(0,\pi)$ and finally ends up at $(-2,3\pi/2)$. The only answer choice that fits this pattern is answer choice B.

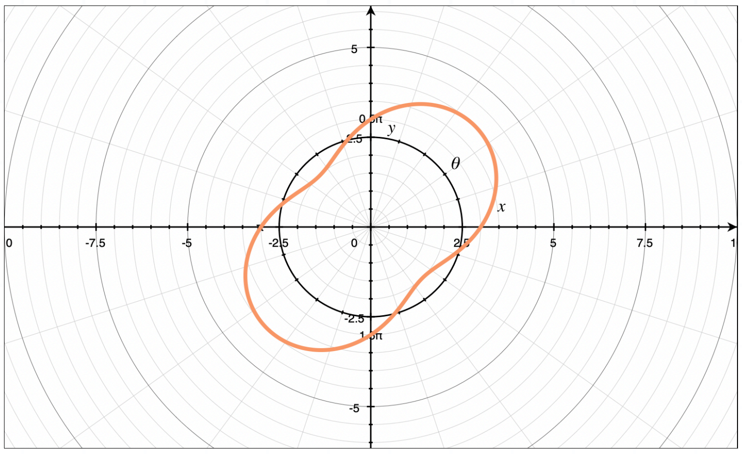
Topic: Sketching polar curves

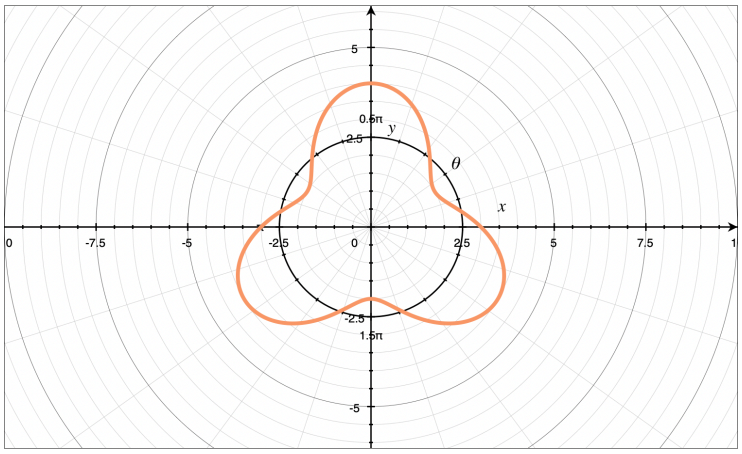
Question: Which graph represents the polar curve?

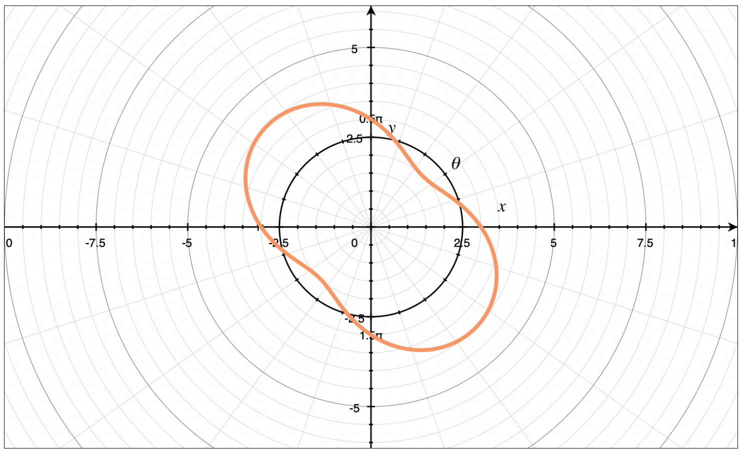
$$r = 3 + \cos 2\theta$$

Answer choices:

A


B


C


D


Solution: A

There are a number of ways to sketch a polar curve. If you are sketching directly on a polar graph and you are given a polar equation, you could simply chose a number of points for θ between 0 and 2π and solve for the associated r values. You could then take these points and plot them on your polar graph.

We can analyze the various potential graphs of $r = 3 + \cos 2\theta$ by first finding some important points.

$$\theta = 0 \qquad (4,0)$$

$$\theta = \frac{\pi}{2} \qquad \left(2,\frac{\pi}{2}\right)$$

$$\theta = \pi \qquad (4,\pi)$$

$$\theta = \frac{3\pi}{2} \qquad \left(2,\frac{3\pi}{2}\right)$$

These points mean that the graph starts at $(4,0)$, then goes out to $(2,\pi/2)$, then towards $(4,\pi)$ and finally ends up at $(2,3\pi/2)$. The only answer choice that fits this pattern is answer choice A.

Topic: Tangent line to the polar curve

Question: Find the tangent line to the polar curve at the given point.

$$r = \sin\theta$$

at $\theta = \frac{\pi}{3}$

Answer choices:

A $\quad y = \sqrt{3}x + \frac{3}{2}$

B $\quad y = -\sqrt{3}x + \frac{3}{2}$

C $\quad y = -\sqrt{3}x$

D $\quad y = \sqrt{3}x$

Solution: B

We'll find the equation of the tangent line to a polar curve by following these steps:

1. Find the **slope** of the tangent line m, using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

remembering to plug the value of θ at the tangent point into dy/dx to get a real-number value for the slope m.

2. **Find** x_1 **and** y_1 by plugging the value of θ at the tangent point into the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

3. Plug the slope m and the point (x_1, y_1) into the **point-slope formula** for the equation of a line

$$y - y_1 = m(x - x_1)$$

In order to find the slope, we need to first find $dr/d\theta$.

$$r = \sin\theta$$

$$\frac{dr}{d\theta} = \cos\theta$$

Plugging $dr/d\theta$ and the given polar equation $r = \sin\theta$ into the formula for the slope, then evaluating at $\theta = \pi/3$, we get

$$m = \frac{dy}{dx} = \frac{\cos\theta\sin\theta + \sin\theta\cos\theta}{\cos\theta\cos\theta - \sin\theta\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{\sin\theta\cos\theta + \sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$m = \frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$m = \frac{dy}{dx} = \frac{2\sin\frac{\pi}{3}\cos\frac{\pi}{3}}{\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}}$$

$$m = \frac{dy}{dx} = \frac{2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$m = \frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{4} - \frac{3}{4}}$$

$$m = \frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}}{-\frac{2}{4}}$$

$$m = \frac{dy}{dx} = \frac{\sqrt{3}}{2}\left(-\frac{4}{2}\right)$$

$$m = \frac{dy}{dx} = -\sqrt{3}$$

To find (x_1, y_1), we'll plug $\theta = \pi/3$ and the given polar equation into the conversion formulas

$$x = r\cos\theta$$

$$x_1 = \sin\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$x_1 = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$x_1 = \frac{\sqrt{3}}{4}$$

and

$$y = r\sin\theta$$

$$y_1 = \sin\frac{\pi}{3}\sin\frac{\pi}{3}$$

$$y_1 = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}$$

$$y_1 = \frac{3}{4}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{4} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{4}\right)$$

$$y = -\sqrt{3}x + \frac{3}{4} + \frac{3}{4}$$

$$y = -\sqrt{3}x + \frac{3}{2}$$

Topic: Tangent line to the polar curve

Question: Find the tangent line to the polar curve at the given point.

$$r = 4\cos 2\theta$$

at $\theta = \dfrac{2\pi}{3}$

Answer choices:

A $\quad y = \dfrac{7\sqrt{3}}{3}x - \dfrac{4\sqrt{3}}{3}$

B $\quad y = -\dfrac{7\sqrt{3}}{3}x - \dfrac{4\sqrt{3}}{3}$

C $\quad 7\sqrt{3}x - 3y = 4\sqrt{3}$

D $\quad 7\sqrt{3}x + 3y = 4\sqrt{3}$

Solution: D

We'll find the equation of the tangent line to a polar curve by following these steps:

1. Find the slope of the tangent line m, using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

remembering to plug the value of θ at the tangent point into dy/dx to get a real-number value for the slope m.

2. Find x_1 and y_1 by plugging the value of θ at the tangent point into the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

3. Plug the slope m and the point (x_1, y_1) into the point-slope formula for the equation of a line

$$y - y_1 = m(x - x_1)$$

In order to find the slope, we need to first find $dr/d\theta$.

$$r = 4\cos 2\theta$$

$$\frac{dr}{d\theta} = -8\sin 2\theta$$

Plugging $dr/d\theta$ and the given polar equation $r = 4\cos 2\theta$ into the formula for the slope, then evaluating at $\theta = \pi/3$, we get

$$m = \frac{dy}{dx} = \frac{(-8\sin 2\theta)\sin\theta + (4\cos 2\theta)\cos\theta}{(-8\sin 2\theta)\cos\theta - (4\cos 2\theta)\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{-8\sin 2\theta\sin\theta + 4\cos 2\theta\cos\theta}{-8\sin 2\theta\cos\theta - 4\cos 2\theta\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{-8\sin\left(2\cdot\frac{2\pi}{3}\right)\sin\frac{2\pi}{3} + 4\cos\left(2\cdot\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}}{-8\sin\left(2\cdot\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} - 4\cos\left(2\cdot\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}}$$

$$m = \frac{dy}{dx} = \frac{-8\sin\frac{4\pi}{3}\sin\frac{2\pi}{3} + 4\cos\frac{4\pi}{3}\cos\frac{2\pi}{3}}{-8\sin\frac{4\pi}{3}\cos\frac{2\pi}{3} - 4\cos\frac{4\pi}{3}\sin\frac{2\pi}{3}}$$

$$m = \frac{dy}{dx} = \frac{-8\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 4\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{-8\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$m = \frac{dy}{dx} = \frac{6 + 1}{-2\sqrt{3} + \sqrt{3}}$$

$$m = \frac{dy}{dx} = \frac{7}{-\sqrt{3}}$$

$$m = \frac{dy}{dx} = -\frac{7\sqrt{3}}{3}$$

To find (x_1, y_1), we'll plug $\theta = 2\pi/3$ and the given polar equation into the conversion formulas

$$x = r\cos\theta$$

$$x_1 = 4\cos\left(2\cdot\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}$$

$$x_1 = 4\cos\frac{4\pi}{3}\cos\frac{2\pi}{3}$$

$$x_1 = 4\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$x_1 = 1$$

and

$$y = r\sin\theta$$

$$y_1 = 4\cos\left(2\cdot\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

$$y_1 = 4\cos\frac{4\pi}{3}\sin\frac{2\pi}{3}$$

$$y_1 = 4\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$y_1 = -\sqrt{3}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - (-\sqrt{3}) = -\frac{7\sqrt{3}}{3}(x - 1)$$

$$3y + 3\sqrt{3} = -7\sqrt{3}(x - 1)$$

$$3y + 3\sqrt{3} = -7\sqrt{3}x + 7\sqrt{3}$$

$$7\sqrt{3}x + 3y = 4\sqrt{3}$$

Topic: Tangent line to the polar curve

Question: Find the tangent line to the polar curve at the given point.

$$r = 3 + \sin\theta$$

at $\theta = \dfrac{5\pi}{4}$

Answer choices:

A $\quad 6y - 2\left(\sqrt{2} - 3\right)x = 12 - 19\sqrt{2}$

B $\quad 6y + 2\left(\sqrt{2} - 3\right)x = 12 + 19\sqrt{2}$

C $\quad y = \dfrac{2\sqrt{2} - 6}{6}x + \dfrac{\sqrt{2} - 6}{6}$

D $\quad y = -\dfrac{2\sqrt{2} - 6}{6}x + \dfrac{\sqrt{2} - 6}{6}$

Solution: A

We'll find the equation of the tangent line to a polar curve by following these steps:

1. Find the **slope** of the tangent line m, using the formula

$$m = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

remembering to plug the value of θ at the tangent point into dy/dx to get a real-number value for the slope m.

2. **Find** x_1 **and** y_1 by plugging the value of θ at the tangent point into the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

3. Plug the slope m and the point (x_1, y_1) into the **point-slope formula** for the equation of a line

$$y - y_1 = m(x - x_1)$$

In order to find the slope, we need to first find $dr/d\theta$.

$$r = 3 + \sin\theta$$

$$\frac{dr}{d\theta} = \cos\theta$$

Plugging $dr/d\theta$ and the given polar equation $r = 3 + \sin\theta$ into the formula for the slope, then evaluating at $\theta = 5\pi/4$, we get

$$m = \frac{dy}{dx} = \frac{\cos\theta\sin\theta + (3+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (3+\sin\theta)\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{\cos\theta\sin\theta + 3\cos\theta + \cos\theta\sin\theta}{\cos\theta\cos\theta - 3\sin\theta - \sin\theta\sin\theta}$$

$$m = \frac{dy}{dx} = \frac{2\cos\theta\sin\theta + 3\cos\theta}{\cos^2\theta - 3\sin\theta - \sin^2\theta}$$

$$m = \frac{dy}{dx} = \frac{2\cos\frac{5\pi}{4}\sin\frac{5\pi}{4} + 3\cos\frac{5\pi}{4}}{\cos^2\frac{5\pi}{4} - 3\sin\frac{5\pi}{4} - \sin^2\frac{5\pi}{4}}$$

$$m = \frac{dy}{dx} = \frac{2\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 3\left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)^2 - 3\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)^2}$$

$$m = \frac{dy}{dx} = \frac{1 - \frac{3\sqrt{2}}{2}}{\frac{2}{4} + \frac{3\sqrt{2}}{2} - \frac{2}{4}}$$

$$m = \frac{dy}{dx} = \frac{\frac{2}{2} - \frac{3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}}$$

$$m = \frac{dy}{dx} = \frac{\frac{2-3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}}$$

$$m = \frac{dy}{dx} = \frac{2 - 3\sqrt{2}}{2} \cdot \frac{2}{3\sqrt{2}}$$

$$m = \frac{dy}{dx} = \frac{2 - 3\sqrt{2}}{3\sqrt{2}}$$

$$m = \frac{dy}{dx} = \frac{2 - 3\sqrt{2}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$m = \frac{dy}{dx} = \frac{2\sqrt{2} - 3 \cdot 2}{3 \cdot 2}$$

$$m = \frac{dy}{dx} = \frac{\sqrt{2} - 3}{3}$$

To find (x_1, y_1), we'll plug $\theta = 5\pi/4$ and the given polar equation into the conversion formulas

$$x = r\cos\theta$$

$$x_1 = (3 + \sin\theta)\cos\theta$$

$$x_1 = 3\cos\theta + \sin\theta\cos\theta$$

$$x_1 = 3\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\cos\frac{5\pi}{4}$$

$$x_1 = 3\left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$$

$$x_1 = -\frac{3\sqrt{2}}{2} + \frac{2}{4}$$

$$x_1 = -\frac{3\sqrt{2}}{2} + \frac{1}{2}$$

$$x_1 = \frac{1 - 3\sqrt{2}}{2}$$

and

$$y = r\sin\theta$$

$$y_1 = (3 + \sin\theta)\sin\theta$$

$$y_1 = 3\sin\theta + \sin\theta\sin\theta$$

$$y_1 = 3\sin\frac{5\pi}{4} + \sin\frac{5\pi}{4}\sin\frac{5\pi}{4}$$

$$y_1 = 3\left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$$

$$y_1 = -\frac{3\sqrt{2}}{2} + \frac{2}{4}$$

$$y_1 = -\frac{3\sqrt{2}}{2} + \frac{1}{2}$$

$$y_1 = \frac{1 - 3\sqrt{2}}{2}$$

Plugging m and (x_1, y_1) into the point-slope formula for the equation of a line, we get

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1 - 3\sqrt{2}}{2} = \frac{\sqrt{2} - 3}{3}\left(x - \frac{1 - 3\sqrt{2}}{2}\right)$$

$$y - \frac{1 - 3\sqrt{2}}{2} = \frac{\sqrt{2} - 3}{3}x - \frac{\sqrt{2} - 3 \cdot 2 - 3 + 9\sqrt{2}}{6}$$

$$y - \frac{1 - 3\sqrt{2}}{2} = \frac{\sqrt{2} - 3}{3}x - \frac{10\sqrt{2} - 9}{6}$$

$$2y - \left(1 - 3\sqrt{2}\right) = \frac{2\sqrt{2} - 6}{3}x - \frac{10\sqrt{2} - 9}{3}$$

$$6y - 3\left(1 - 3\sqrt{2}\right) = \left(2\sqrt{2} - 6\right)x - \left(10\sqrt{2} - 9\right)$$

$$6y - 3 + 9\sqrt{2} = \left(2\sqrt{2} - 6\right)x - 10\sqrt{2} + 9$$

$$6y - \left(2\sqrt{2} - 6\right)x = -10\sqrt{2} - 9\sqrt{2} + 9 + 3$$

$$6y - \left(2\sqrt{2} - 6\right)x = -19\sqrt{2} + 12$$

$$6y - 2\left(\sqrt{2} - 3\right)x = 12 - 19\sqrt{2}$$

Topic: Vertical and horizontal tangent lines to the polar curve

Question: Which are the points at which the curve has horizontal tangent lines?

$$r = 6(1 - \cos\theta)$$

Answer choices:

A $\left(1,\frac{2\pi}{3}\right)$ and $\left(1,\frac{4\pi}{3}\right)$ and $(0,0)$

B $\left(5,\frac{3\pi}{2}\right)$ and $\left(5,\frac{5\pi}{3}\right)$ and $(0,0)$

C $\left(9,\frac{2\pi}{3}\right)$ and $\left(9,\frac{4\pi}{3}\right)$ and $(0,0)$

D $\left(9,\frac{3\pi}{2}\right)$ and $\left(9,\frac{5\pi}{3}\right)$ and $(0,0)$

Solution: C

We'll use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$, and plug in the given polar equation, $r = 6(1 - \cos\theta)$.

$$x = r\cos\theta$$

$$x = 6(1 - \cos\theta)\cos\theta$$

$$x = 6\cos\theta - 6\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$

$$y = 6\sin\theta - 6\sin\theta\cos\theta$$

Take the derivative of each of these.

$$\frac{dx}{d\theta} = -6\sin\theta + 12\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 6\cos\theta + 6\sin^2\theta - 6\cos^2\theta$$

Now we'll get the derivative dy/dx.

$$\frac{dy}{dx} = \frac{6\cos\theta + 6\sin^2\theta - 6\cos^2\theta}{-6\sin\theta + 12\sin\theta\cos\theta}$$

Horizontal tangent lines exist where this derivative is equal to 0. But because the derivative is a fraction, it can only be 0 where the numerator is 0. Therefore

$$0 = 6\cos\theta + 6\sin^2\theta - 6\cos^2\theta$$

$$0 = \cos\theta + \sin^2\theta - \cos^2\theta$$

$$0 = \cos\theta + 1 - \cos^2\theta - \cos^2\theta$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$2\cos\theta + 1 = 0 \quad \text{or} \quad \cos\theta - 1 = 0$$

$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{or} \quad \theta = 0$$

If we plug these values for θ back into the original polar curve, we get

$$r = 6\left(1 - \cos\frac{2\pi}{3}\right)$$

$$r = 6\left(1 + \frac{1}{2}\right)$$

$$r = 9$$

and

$$r = 6\left(1 - \cos\frac{4\pi}{3}\right)$$

$$r = 6\left(1 + \frac{1}{2}\right)$$

$$r = 9$$

and

$$r = 6(1 - \cos 0)$$

$$r = 6(1 - 1)$$

$$r = 0$$

Thus, the horizontal tangents pass through

$$\left(9, \frac{2\pi}{3}\right)$$

$$\left(9, \frac{4\pi}{3}\right)$$

$$(0,0)$$

Topic: Vertical and horizontal tangent lines to the polar curve

Question: Where are the horizontal and vertical tangent lines to the polar curve?

$$r = 4\cos\theta$$

Answer choices:

A	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{\pi}{2}\right)$
	Horizontal tangents at	$\left(2\sqrt{2},\frac{\pi}{4}\right)$	and	$\left(-2\sqrt{2},\frac{3\pi}{4}\right)$
B	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{3\pi}{2}\right)$
	Horizontal tangents at	$\left(\frac{4}{\sqrt{2}},\frac{\pi}{4}\right)$	and	$\left(2\sqrt{2},\frac{\pi}{4}\right)$
C	Vertical tangents at	$(4,0)$	and	$\left(0,\frac{\pi}{2}\right)$
	Horizontal tangents at	$\left(\frac{4}{\sqrt{2}},\frac{3\pi}{4}\right)$	and	$\left(-\frac{4}{\sqrt{2}},\frac{3\pi}{4}\right)$
D	Vertical tangents at	$\left(2\sqrt{2},\frac{\pi}{4}\right)$	and	$\left(0,\frac{5\pi}{2}\right)$

Horizontal tangents at $\left(\frac{4}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{4}{\sqrt{2}}, \frac{3\pi}{4}\right)$

Solution: A

The function $r = 4\cos\theta$ can be described by

$$x = r\cos\theta$$

$$x = (4\cos\theta)\cos\theta$$

$$x = 4\cos^2\theta$$

and

$$y = r\sin\theta$$

$$y = (4\cos\theta)\sin\theta$$

$$y = 4\sin\theta\cos\theta$$

Take the derivative of each of these.

$$\frac{dx}{d\theta} = -8\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 4\cos^2\theta - 4\sin^2\theta$$

Horizontal tangent lines exist where $dy/d\theta = 0$, and vertical tangent lines exist where $dx/d\theta = 0$. Therefore, set the derivatives equal to 0 and solve for θ.

Vertical tangent lines at:

$$-8\sin\theta\cos\theta = 0$$

$$2\sin\theta\cos\theta = 0$$

$$\sin 2\theta = 0$$

$\theta = 0$ or $\theta = \dfrac{\pi}{2}$

and

Horizontal tangent lines at:

$$4\cos^2 - 4\sin^2\theta = 0$$

$$\cos^2 - \sin^2\theta = 0$$

$$\cos 2\theta = 0$$

$\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$

Plug each of these into the original function.

$$r = 4\cos 0$$

$$r = 4(1)$$

$$r = 4$$

and

$$r = 4\cos\frac{\pi}{2}$$

$$r = 4(0)$$

$$r = 0$$

and

$$r = 4\cos\frac{\pi}{4}$$

$$r = 4\left(\frac{\sqrt{2}}{2}\right)$$

$$r = 2\sqrt{2}$$

and

$$r = 4\cos\frac{3\pi}{4}$$

$$r = 4\left(-\frac{\sqrt{2}}{2}\right)$$

$$r = -2\sqrt{2}$$

Therefore, there are

vertical tangents at $(4,0)$ and $\left(0,\frac{\pi}{2}\right)$

horizontal tangents at $\left(2\sqrt{2}, \frac{\pi}{4}\right)$ and $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$

Topic: Vertical and horizontal tangent lines to the polar curve

Question: Which function has vertical tangent lines that pass through these points?

$$\left(3,\frac{\pi}{6}\right) \text{ and } \left(3,\frac{5\pi}{6}\right)$$

Answer choices:

A $r=3(1-\sin\theta)$

B $r=-2(1+\sin\theta)$

C $r=3(1+\sin\theta)$

D $r=2(1+\sin\theta)$

Solution: D

Starting with the function from answer choice D, $r=2(1+\sin\theta)$, we'll use the conversion formulas

$$x=r\cos\theta$$

$$y=r\sin\theta$$

and plug in the given polar curve.

$$x=2(1+\sin\theta)\cos\theta$$

$$x=2\cos\theta+2\sin\theta\cos\theta$$

and

$$y=2(1+\sin\theta)\sin\theta$$

$$y=2\sin^2\theta+2\sin\theta$$

Take the derivative of each of these.

$$\frac{dy}{d\theta}=4\sin\theta\cos\theta+2\cos\theta$$

$$\frac{dx}{d\theta}=-2\sin\theta+2\cos^2\theta-2\sin^2\theta$$

Now find the derivative dy/dx.

$$\frac{dy}{dx}=\frac{4\sin\theta\cos\theta+2\cos\theta}{-2\sin\theta+2\cos^2\theta-2\sin^2\theta}$$

Vertical tangent lines will exist where this derivative is undefined, which means we'll find vertical tangent lines wherever the denominator is equal to 0.

$$-2\sin\theta + 2\cos^2\theta - 2\sin^2\theta = 0$$

$$-\sin\theta + \cos^2\theta - \sin^2\theta = 0$$

$$-\sin\theta + 1 - \sin^2\theta - \sin^2\theta = 0$$

$$-2\sin^2\theta - \sin\theta + 1 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$2\sin\theta - 1 = 0 \qquad \text{or} \qquad \sin\theta + 1 = 0$$

$$\sin\theta = \frac{1}{2} \qquad \text{or} \qquad \sin\theta = -1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{or} \qquad \theta = \frac{3\pi}{2}$$

The angle $3\pi/2$ doesn't exist in the given points, so we can ignore this value. Plug the other θ values into the original function.

$$r = 2(1 + \sin\theta)$$

$$r = 2\left(1 + \sin\frac{\pi}{6}\right)$$

$$r = 3$$

and

$$r = 2(1 + \sin\theta)$$

$$r = 2\left(1 + \sin\frac{5\pi}{6}\right)$$

$$r = 3$$

Therefore, the vertical tangents of answer choice D pass through

$\left(3,\frac{\pi}{6}\right)$ and $\left(3,\frac{5\pi}{6}\right)$

Because these are the points we were given, we know that answer choice D is correct.

Topic: Intersection of polar curves

Question: Find the points at which $r = \sin\theta$ and $r = \cos\theta$ intersect.

Answer choices:

A $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

B $\left(-\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

C $\left(\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$

D $\left(\frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$

Solution: A

To find points of intersection, we'll set the curves equal to one another to solve for θ,

$$\sin\theta = \cos\theta$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

then find the associated values of r.

$$r = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$r = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

The polar curves intersect at

$$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$$

But we notice that these are actually identical points in space, so we'll state just a single intersection point at

$$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$$

Topic: Intersection of polar curves

Question: Find the points at which $r = \cos\theta$ and $r = \cos(2\theta)$ intersect.

Answer choices:

A $(1,0)$

B $(1,2\pi)$

C $(1,0)$, $\left(-\frac{1}{2},\frac{2\pi}{3}\right)$, and $\left(-\frac{1}{2},\frac{4\pi}{3}\right)$

D $(1,0)$ and $(1,\pi)$

Solution: C

To find points of intersection, we'll set the curves equal to one another to solve for θ,

$$\cos\theta = \cos(2\theta)$$

$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

then find the associated values of r.

$$r = \cos(0) = 1$$

$$r = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$r = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

The polar curves intersect at

$(1,0)$, $\left(-\frac{1}{2},\frac{2\pi}{3}\right)$ and $\left(-\frac{1}{2},\frac{4\pi}{3}\right)$

Topic: Intersection of polar curves

Question: Find the points at which $r = 3\cos\theta$ and $r = 1 + \cos\theta$ intersect.

Answer choices:

A $\left(1, \frac{\pi}{2}\right)$ and $\left(-1, \frac{3\pi}{2}\right)$

B $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$

C $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

D $\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3\sqrt{3}}{2}, \frac{5\pi}{3}\right)$

Solution: C

To find points of intersection, we'll set the curves equal to one another to solve for θ,

$$3\cos\theta = 1 + \cos\theta$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

then find the associated values of r.

$$r = 3\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$r = 3\cos\left(\frac{5\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

The polar curves intersect at

$\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

Topic: Area inside a polar curve

Question: Find the area bounded by the polar curve on the given interval.

$$r = 6\theta$$

$$0 \le \theta \le \pi$$

Answer choices:

A $6\pi^3$

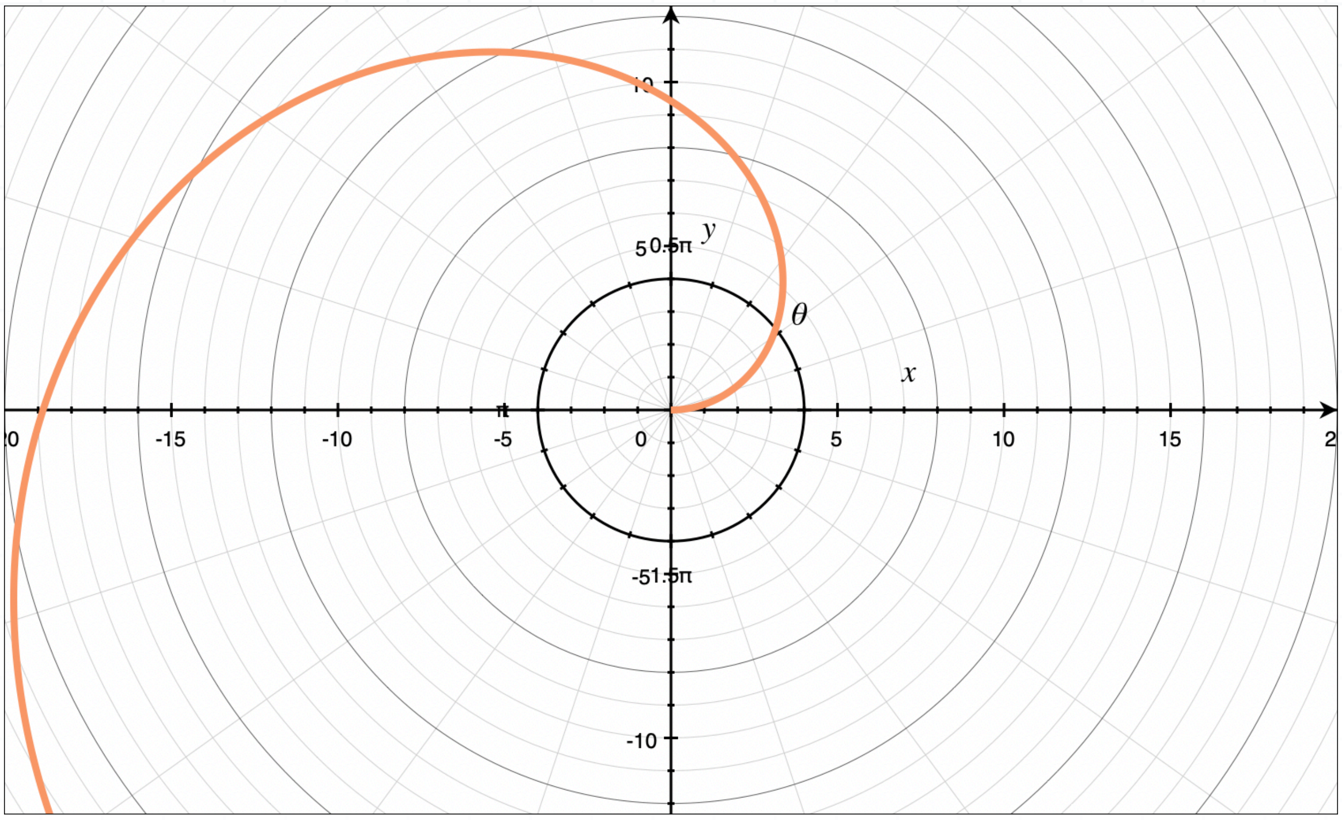
B $6\pi^2$

C $\frac{6}{5}\pi$

D $\frac{6}{5}\pi^3$

Solution: A

The graph of the polar curve looks like this:



Given the interval, the region in question is bounded by the spiral and the x-axis. The area formula is

$$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta$$

where $\alpha=0$ and $\beta=\pi$. Therefore, the area bounded by the polar curve is

$$A=\frac{1}{2}\int_{0}^{\pi} (6\theta)^2\ d\theta$$

$$A=18\int_{0}^{\pi} \theta^2\ d\theta$$

$$A = \frac{18}{3}\theta^3 \Big|_0^{\pi}$$

$$A = \frac{18}{3}\pi^3 - \frac{18}{3}(0)^3$$

$$A = 6\pi^3$$

Topic: Area inside a polar curve

Question: Find the area bounded by the polar curve on the given interval.

$$r = 5 - 5\sin\theta$$

Answer choices:

A 15π

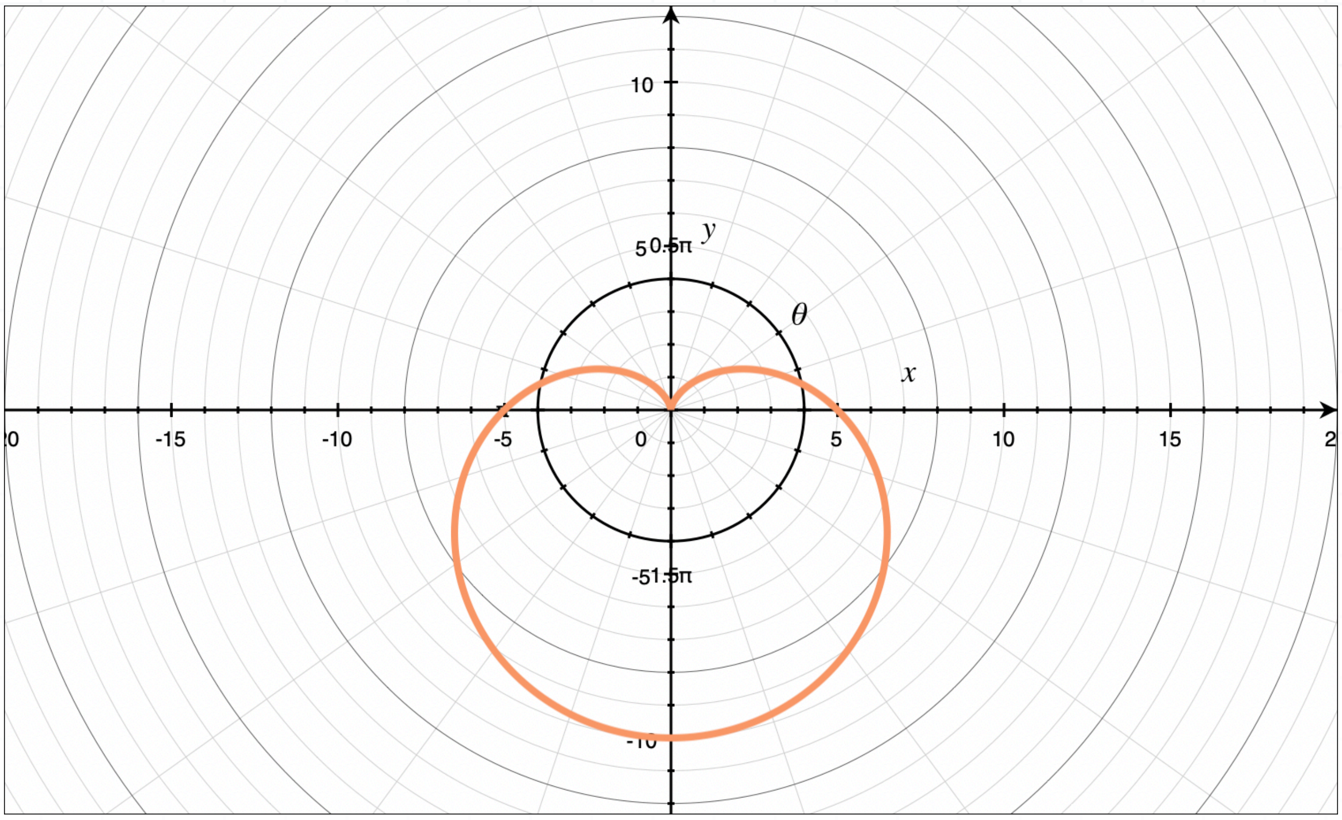
B $\frac{75\pi}{12}$

C $\frac{75\pi}{2}$

D $\frac{75}{2}$

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve is symmetric about the y-axis since $\sin\theta = \sin(\pi - \theta)$. Therefore, the area of the bounded region can be determined by doubling the integral from $\pi/2$ to $3\pi/2$. The area is given by

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta\right) = \int_{\alpha}^{\beta} r^2\ d\theta$$

where $\alpha = \pi/2$ and $\beta = 3\pi/2$.

$$A = \int_{\pi/2}^{3\pi/2} (5 - 5\sin\theta)^2\ d\theta$$

$$A = \int_{\pi/2}^{3\pi/2} 25 - 50\sin\theta + 25\sin^2\theta\ d\theta$$

$$A=\int_{\pi/2}^{3\pi/2}25\left(1-2\sin\theta+\sin^2\theta\right)\ d\theta$$

$$A=25\int_{\pi/2}^{3\pi/2}1-2\sin\theta+\sin^2\theta\ d\theta$$

Using the power reduction formula

$$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$$

we get

$$A=25\int_{\pi/2}^{3\pi/2}1-2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta$$

$$A=25\int_{\pi/2}^{3\pi/2}\frac{3}{2}-2\sin\theta-\frac{1}{2}\cos 2\theta\ d\theta$$

$$A=25\left(\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_{\pi/2}^{3\pi/2}$$

$$A=25\left[\left(\frac{3}{2}\left(\frac{3\pi}{2}\right)+2\cos\left(\frac{3\pi}{2}\right)-\frac{1}{4}\sin 2\left(\frac{3\pi}{2}\right)\right)-\left(\frac{3}{2}\left(\frac{\pi}{2}\right)+2\cos\left(\frac{\pi}{2}\right)-\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right)\right]$$

$$A=25\left[\left(\frac{9\pi}{4}+2(0)-\frac{1}{4}(0)\right)-\left(\frac{3\pi}{4}+2(0)-\frac{1}{4}(0)\right)\right]$$

$$A=25\left(\frac{9\pi}{4}-\frac{3\pi}{4}\right)$$

$$A = \frac{75\pi}{2}$$

Topic: Area inside a polar curve

Question: The x-axis is the line of symmetry for the cardioids $r = 2(1 + \cos\theta)$ and $r = 4(1 + \cos\theta)$. Assume that A_1 is the area of the first cardioid and A_2 is the area of the second cardioid. What is the ratio of A_1 to A_2?

Answer choices:

A $\quad \dfrac{1}{2}$

B $\quad \dfrac{1}{4}$

C $\quad 2$

D $\quad 4$

Solution: B

We'll first find the area A_1 of the cardioid $r = 2(1+\cos\theta)$ by plugging into the formula for polar area. Because both cardioids are symmetric about the x-axis, we can integrate over the interval $[0,\pi]$ (representing the area above the x-axis), and then multiply the integral formula by 2 to get the full area.

$$A_1 = 2 \times \frac{1}{2}\int_0^{\pi} 4(1+\cos\theta)^2 \ d\theta$$

$$A_1 = 4\int_0^{\pi} (1+\cos\theta)^2 \ d\theta$$

$$A_1 = 4\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \ d\theta$$

$$A_1 = 4\int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_1 = 4\int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \ d\theta$$

$$A_1 = 4\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi}$$

$$A_1 = 4\left(\frac{3}{2}\pi + 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - 4\left(\frac{3}{2}(0) + 2\sin(0) + \frac{1}{4}\sin 2(0)\right)$$

$$A_1 = 4\left(\frac{3}{2}\pi + 0 + 0\right) - 4(0+0+0)$$

$$A_1 = 6\pi$$

and

$$A_2 = 2 \times \frac{1}{2} \int_0^{\pi} 16 (1 + \cos \theta)^2 \ d\theta$$

$$A_2 = 16 \int_0^{\pi} (1 + \cos \theta)^2 \ d\theta$$

$$A_2 = 16 \int_0^{\pi} 1 + 2 \cos \theta + \cos^2 \theta \ d\theta$$

$$A_2 = 16 \int_0^{\pi} 1 + 2 \cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta \ d\theta$$

$$A_2 = 16 \int_0^{\pi} \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \ d\theta$$

$$A_2 = 16 \left(\frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_0^{\pi}$$

$$A_2 = 16 \left(\frac{3}{2}\pi + 2 \sin \pi + \frac{1}{4} \sin 2\pi \right) - 4 \left(\frac{3}{2}(0) + 2 \sin(0) + \frac{1}{4} \sin 2(0) \right)$$

$$A_2 = 16 \left(\frac{3}{2}\pi + 0 + 0 \right) - 4(0 + 0 + 0)$$

$$A_2 = 24\pi$$

The ratio of the areas is therefore

$$\frac{A_1}{A_2} = \frac{6\pi}{24\pi} = \frac{1}{4}$$

Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = 5\cos 3\theta$$

Answer choices:

A $\frac{25}{12}$

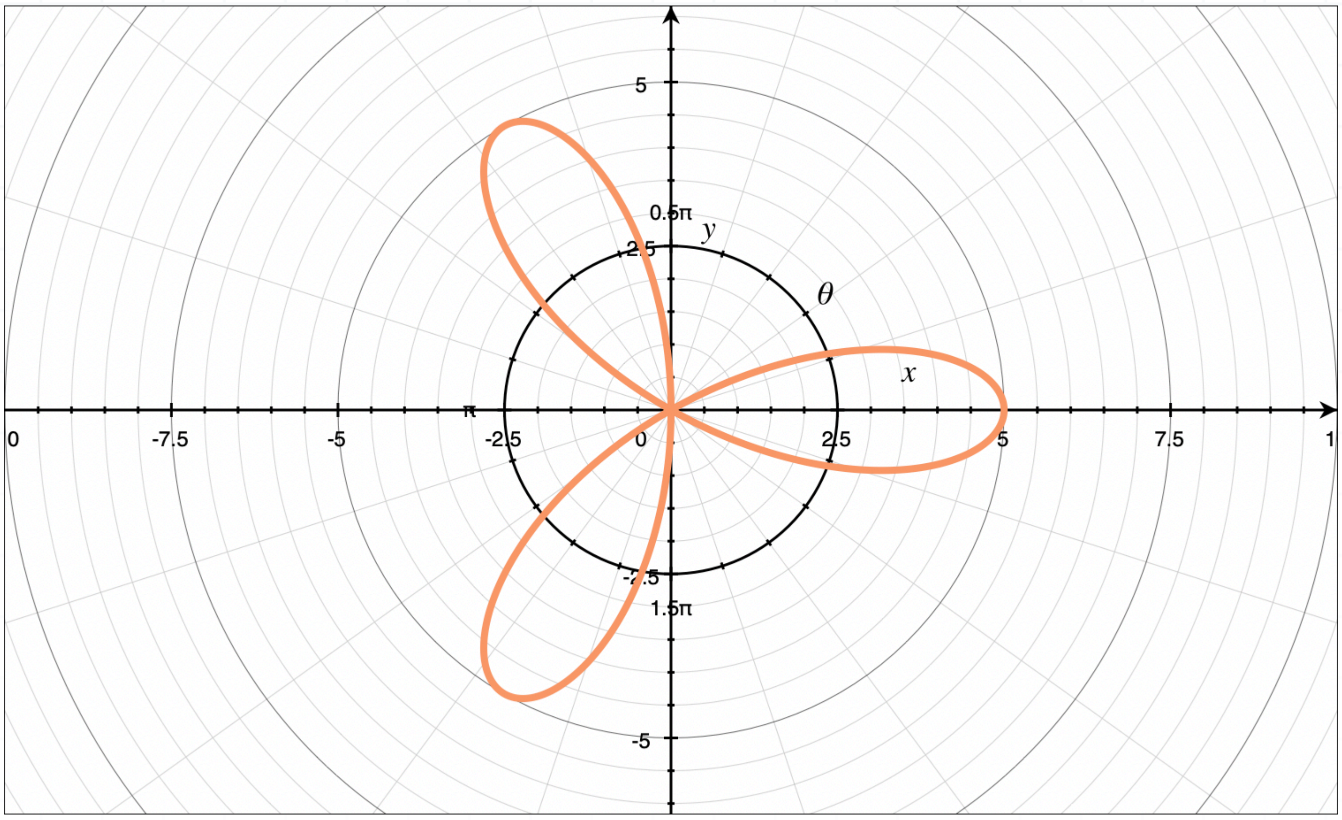
B $\frac{25\pi}{2}$

C $\frac{25\pi}{12}$

D π

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve has three loops and the best loop to consider is the one that lies on the positive x-axis. The loop is symmetric about the x-axis, so we'll consider only the top half for integration, and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta\right) = \int_{\alpha}^{\beta} r^2\ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 5$. When $\theta = \pi/6$, the polar curve loops back to the origin, so $r = 0$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/6$.

$$A=\int_0^{\frac{\pi}{6}}(5\cos 3\theta)^2\ d\theta$$

$$A=25\int_0^{\frac{\pi}{6}}\cos^2 3\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=25\int_0^{\frac{\pi}{6}}\frac{1}{2}+\frac{1}{2}\cos 6\theta\ d\theta$$

$$A=25\left(\frac{1}{2}\theta+\frac{1}{12}\sin 6\theta\right)\Bigg|_0^{\frac{\pi}{6}}$$

$$A=25\left[\left(\frac{1}{2}\left(\frac{\pi}{6}\right)+\frac{1}{12}\sin 6\left(\frac{\pi}{6}\right)\right)-\left(\frac{1}{2}(0)+\frac{1}{12}\sin 6(0)\right)\right]$$

$$A=25\left[\left(\frac{\pi}{12}+\frac{1}{12}(0)\right)-\left(0+\frac{1}{12}(0)\right)\right]$$

$$A=\frac{25\pi}{12}$$

Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = \sin(2\theta)$$

Answer choices:

A $\frac{\pi}{16}$

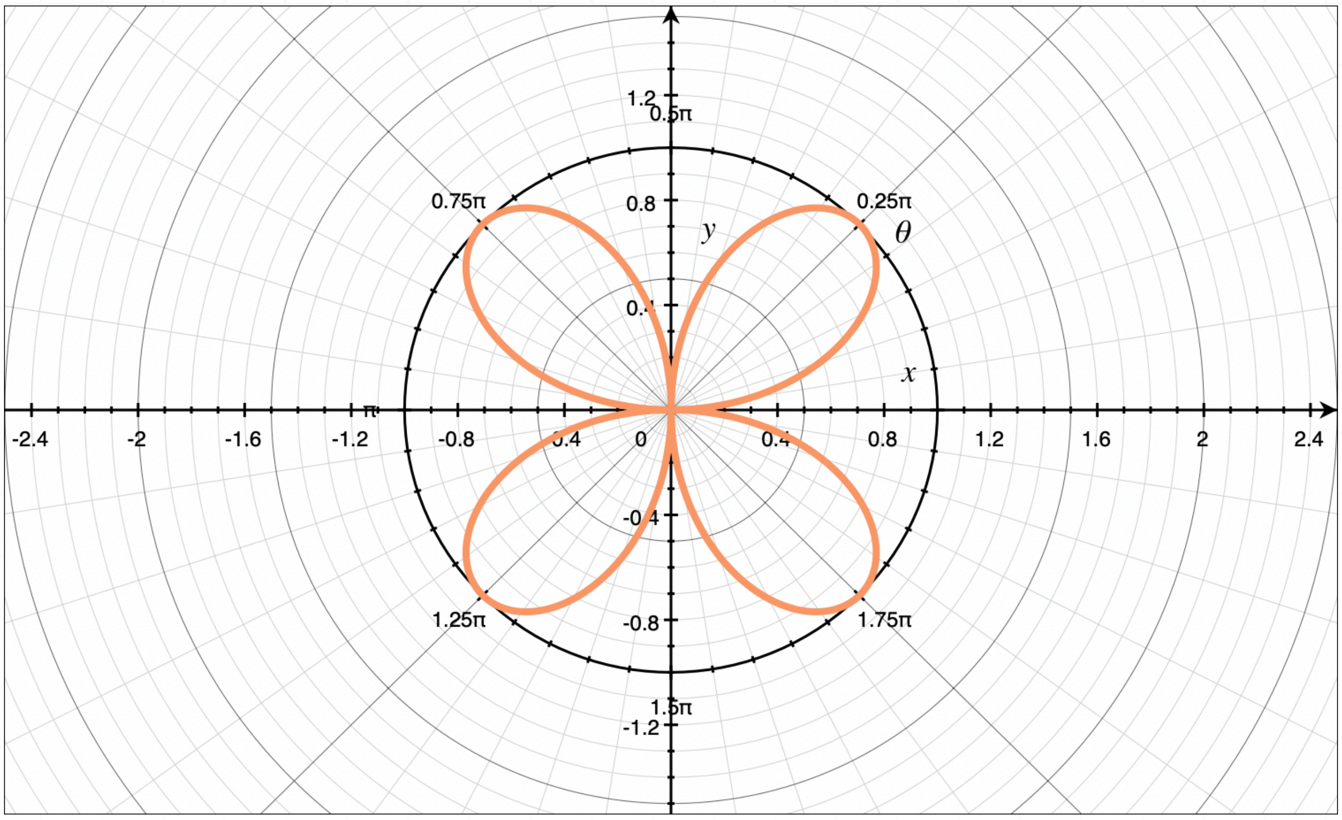
B $\frac{\pi}{2}$

C $\frac{\pi}{8}$

D $\frac{\pi}{4}$

Solution: C

The graph of the polar curve looks like this:



The graph of the polar curve has four loops and the best loop to consider is the one that lies in the first quadrant. The loop is symmetric about the line $\theta = \pi/4$, so we'll consider only the bottom half of that loop for integration (the part that lies below the line $\theta = \pi/4$), and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta\right) = \int_{\alpha}^{\beta} r^2 \ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 0$. When $\theta = \pi/4$, the polar curve loops out to the tip of the first petal, so $r = 1$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/4$.

$$A = \int_0^{\frac{\pi}{4}} (\sin 2\theta)^2 \ d\theta$$

$$A = \int_0^{\frac{\pi}{4}} \sin^2 2\theta \ d\theta$$

Using the power reduction formula

$$\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$$

we get

$$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos(2(2\theta))) \ d\theta$$

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 - \cos(4\theta) \ d\theta$$

We'll integrate, then evaluate over the interval.

$$A = \frac{1}{2} \left[\theta - \frac{1}{4} \sin(4\theta) \right] \Bigg|_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2} \left[\frac{\pi}{4} - \frac{1}{4} \sin\left(4 \cdot \frac{\pi}{4}\right) \right] - \frac{1}{2} \left[0 - \frac{1}{4} \sin(4 \cdot 0) \right]$$

$$A = \frac{1}{2} \left[\frac{\pi}{4} - \frac{1}{4}(0) \right] - \frac{1}{2} \left[0 - \frac{1}{4}(0) \right]$$

$$A = \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$A = \frac{\pi}{8}$$

Topic: Area bounded by one loop of a polar curve

Question: Find the area bounded by one loop of the polar curve.

$$r = \cos(3\theta)$$

Answer choices:

A $\dfrac{\pi}{12}$

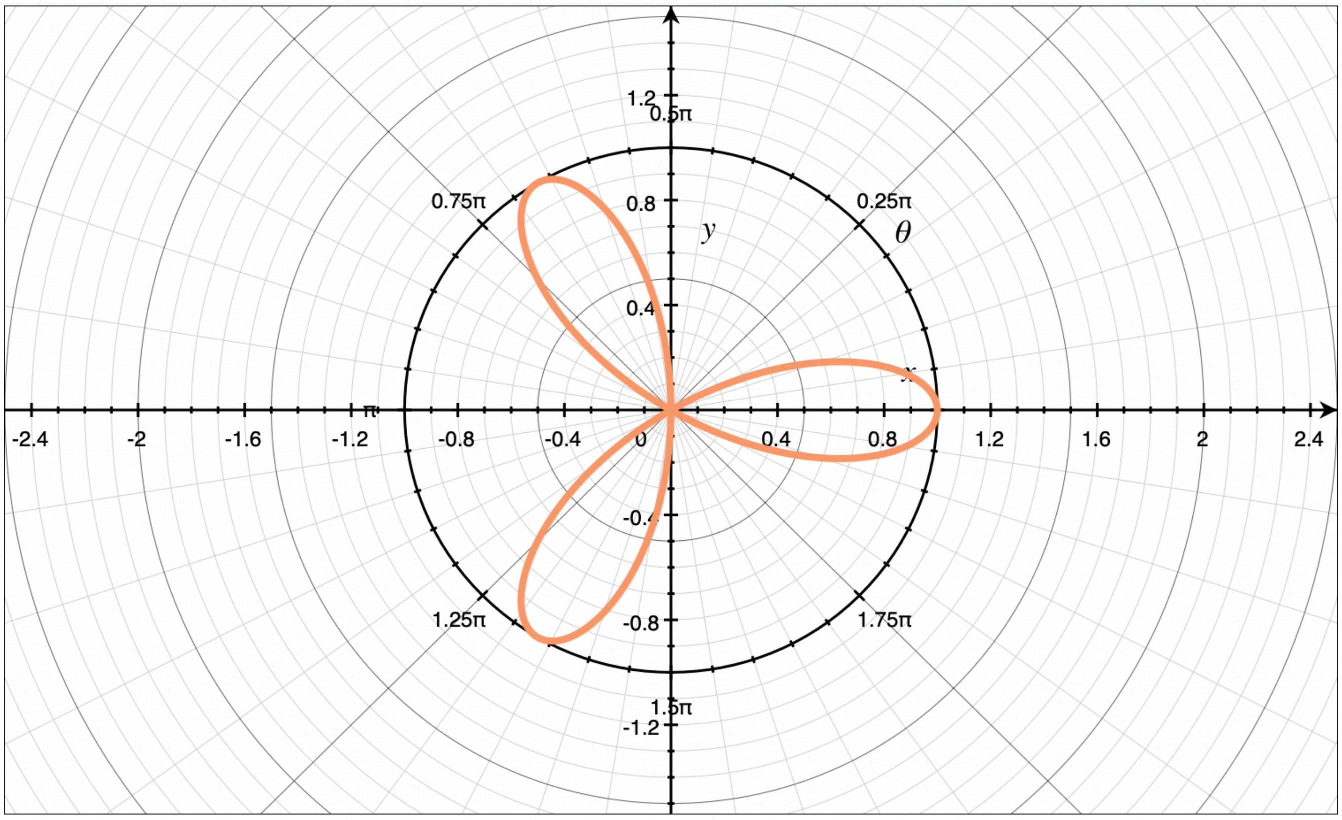
B $\dfrac{\pi}{10}$

C $\dfrac{\pi}{8}$

D $\dfrac{\pi}{14}$

Solution: A

The graph of the polar curve looks like this:



The graph of the polar curve has three loops and the best loop to consider is the one that straddles the positive side of the x-axis. The loop is symmetric about the axis, so we'll consider only the top half of that loop for integration (the part that lies above the line $\theta = 0$), and then we'll double that area. So our area formula is

$$A = 2\left(\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta\right) = \int_{\alpha}^{\beta} r^2\ d\theta$$

To get the limits of integration, we begin substituting values for θ and solve for r.

When $\theta = 0$, $r = 1$. When $\theta = \pi/6$, the polar curve loops out to the tip of the first petal, so $r = 0$. Therefore, the limits of integration are $\alpha = 0$ and $\beta = \pi/6$.

$$A = \int_0^{\frac{\pi}{6}} (\cos(3\theta))^2 \ d\theta$$

$$A = \int_0^{\frac{\pi}{6}} \cos^2(3\theta) \ d\theta$$

Using the power reduction formula

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we get

$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2} + \frac{1}{2}\cos(3(2\theta)) \ d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 + \cos(6\theta) \ d\theta$$

We'll integrate, then evaluate over the interval.

$$A = \frac{1}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\right]\Bigg|_0^{\frac{\pi}{6}}$$

$$A = \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{6}\sin\left(6 \cdot \frac{\pi}{6}\right)\right] - \frac{1}{2}\left[0 + \frac{1}{6}\sin(6(0))\right]$$

$$A = \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{6}(0)\right] - \frac{1}{2}\left[0 + \frac{1}{6}(0)\right]$$

$$A=\frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$A=\frac{\pi}{12}$$

Topic: Area between polar curves

Question: Find the area between the polar curves.

$$r = 3 - 3\cos\theta$$

$$r = 3$$

Answer choices:

A 45

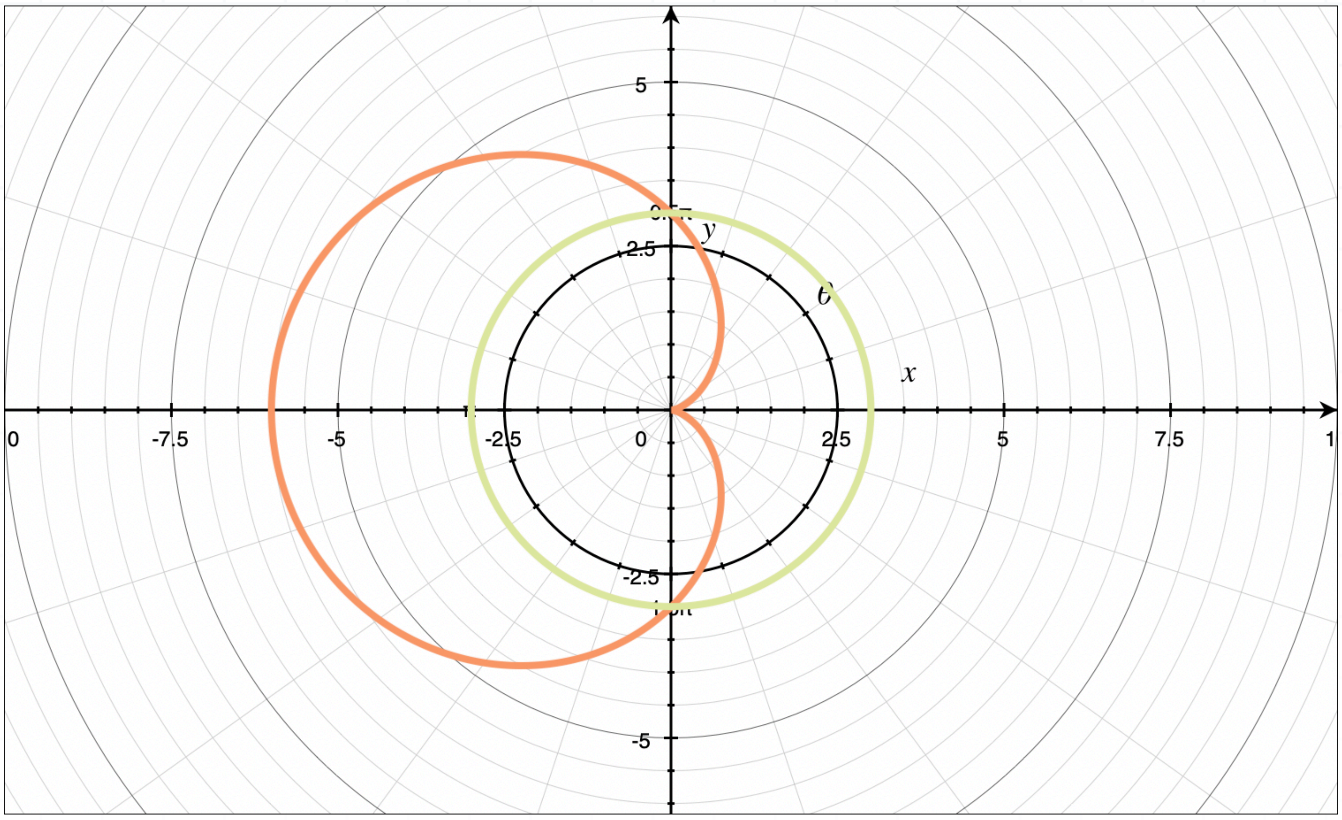
B 36

C 18

D 9

Solution: B

The area between the polar curves looks like this:



We can split the area between the curves into two parts:

1. The area outside the cardioid but inside the circle in the first and fourth quadrants, and

2. The area outside the circle but inside the cardioid in the second and third quadrants.

The curves intersect at $(3,\pi/2)$ and $(3,3\pi/2)$. We can also write $(3,3\pi/2)$ as $(3,-\pi/2)$. So our area formula will be

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2_{\text{circle}}-r^2_{\text{cardioid}}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r^2_{\text{cardioid}}-r^2_{\text{circle}}\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}3^2-(3-3\cos\theta)^2\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(3-3\cos\theta)^2-3^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}9-(9-18\cos\theta+9\cos^2\theta)\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(9-18\cos\theta+9\cos^2\theta)-9\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}9-9+18\cos\theta-9\cos^2\theta\ d\theta+\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}9-18\cos\theta+9\cos^2\theta-9\ d\theta$$

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\cos^2\theta\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^2\theta-2\cos\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos 2\theta-2\cos\theta\ d\theta$$

$$A=\frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos\theta-\frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta+\frac{9}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos 2\theta-2\cos\theta\ d\theta$$

Integrate term by term.

$$A=\frac{9}{2}\left(2\sin\theta-\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{9}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta-2\sin\theta\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$A=\frac{9}{2}\left[2\sin\frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)\right]-\frac{9}{2}\left[2\sin\left(-\frac{\pi}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\cdot-\frac{\pi}{2}\right)\right]$$

$$+\frac{9}{2}\left[\frac{1}{2}\left(\frac{3\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\frac{3\pi}{2}\right)-2\sin\frac{3\pi}{2}\right]-\frac{9}{2}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)-2\sin\frac{\pi}{2}\right]$$

$$A=\frac{9}{2}\left[2(1)-\frac{\pi}{4}-\frac{1}{4}\sin\pi\right]-\frac{9}{2}\left[2(-1)+\frac{\pi}{4}-\frac{1}{4}\sin(-\pi)\right]$$

$$+\frac{9}{2}\left[\frac{3\pi}{4}+\frac{1}{4}\sin(3\pi)-2(-1)\right]-\frac{9}{2}\left[\frac{\pi}{4}+\frac{1}{4}\sin\pi-2(1)\right]$$

$$A=\frac{9}{2}\left[2-\frac{\pi}{4}-\frac{1}{4}(0)\right]-\frac{9}{2}\left[-2+\frac{\pi}{4}-\frac{1}{4}(0)\right]$$

$$+\frac{9}{2}\left[\frac{3\pi}{4}+\frac{1}{4}(0)+2\right]-\frac{9}{2}\left[\frac{\pi}{4}+\frac{1}{4}(0)-2\right]$$

$$A=9-\frac{9\pi}{8}+9-\frac{9\pi}{8}+\frac{27\pi}{8}+9-\frac{9\pi}{8}+9$$

$$A=36-\frac{27\pi}{8}+\frac{27\pi}{8}$$

$$A=36$$

Topic: Area between polar curves

Question: Find the area between the polar curves.

$$r = \frac{1}{2} + \cos\theta$$

$$r = 1$$

Answer choices:

A $\dfrac{-2\pi - 27\sqrt{3}}{24}$

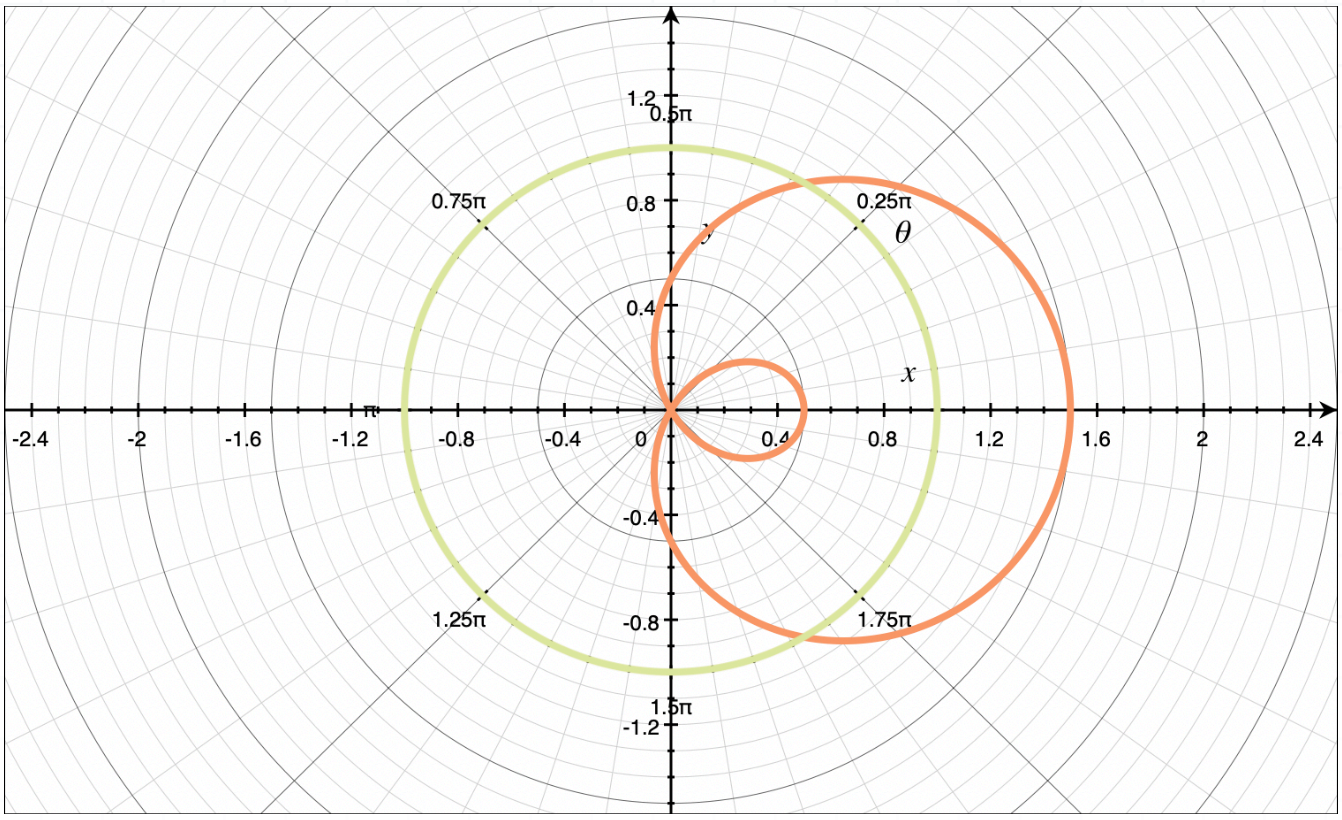
B $\dfrac{-2 + 27\sqrt{3}}{24}$

C $\dfrac{\pi - 15\sqrt{3}}{12}$

D $\dfrac{\pi + 15\sqrt{3}}{12}$

Solution: D

The area between the polar curves looks like this:



We can split the area between the curves into two parts:

1. The area outside the circle but inside the cardioid in the first and fourth quadrants, and

2. The area outside the cardioid but inside the circle (on the left).

The curves intersect at $(1,\pi/3)$ and $(1,5\pi/3)$. We can also write $(1,5\pi/3)$ as $(1,-\pi/3)$. So our area formula will be

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} r^2_{\text{cardioid}}-r^2_{\text{circle}}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} r^2_{\text{circle}}-r^2_{\text{cardioid}}\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(\frac{1}{2}+\cos\theta\right)^2-(1)^2\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1)^2-\left(\frac{1}{2}+\cos\theta\right)^2\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{4}+\cos\theta+\cos^2\theta-1\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}1-\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2\theta+\cos\theta-\frac{3}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{3}{4}-\cos\theta-\cos^2\theta\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{2}+\frac{1}{2}\cos 2\theta+\cos\theta-\frac{3}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{3}{4}-\cos\theta-\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{1}{2}\cos 2\theta+\cos\theta-\frac{1}{4}\ d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\frac{1}{4}-\cos\theta-\frac{1}{2}\cos 2\theta\ d\theta$$

Integrate term by term.

$$A=\frac{1}{2}\left(\frac{1}{4}\sin 2\theta+\sin\theta-\frac{1}{4}\theta\right)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}+\frac{1}{2}\left(\frac{1}{4}\theta-\sin\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A=\frac{1}{8}(\sin 2\theta+4\sin\theta-\theta)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}+\frac{1}{8}(\theta-4\sin\theta-\sin 2\theta)\Bigg|_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$A=\frac{1}{8}\left(\sin\frac{2\pi}{3}+4\sin\frac{\pi}{3}-\frac{\pi}{3}\right)-\frac{1}{8}\left(\sin\left(-\frac{2\pi}{3}\right)+4\sin\left(-\frac{\pi}{3}\right)+\frac{\pi}{3}\right)$$

$$+\frac{1}{8}\left(\frac{5\pi}{3}-4\sin\frac{5\pi}{3}-\sin\frac{10\pi}{3}\right)-\frac{1}{8}\left(\frac{\pi}{3}-4\sin\frac{\pi}{3}-\sin\frac{2\pi}{3}\right)$$

$$A=\frac{1}{8}\left(\frac{\sqrt{3}}{2}+4\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)-\frac{1}{8}\left(-\frac{\sqrt{3}}{2}+4\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{3}\right)$$

$$+\frac{1}{8}\left(\frac{5\pi}{3}-4\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)\right)-\frac{1}{8}\left(\frac{\pi}{3}-4\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)$$

$$A=\frac{1}{8}\left(\frac{5\sqrt{3}}{2}-\frac{\pi}{3}\right)-\frac{1}{8}\left(-\frac{5\sqrt{3}}{2}+\frac{\pi}{3}\right)+\frac{1}{8}\left(\frac{5\pi}{3}+\frac{5\sqrt{3}}{2}\right)-\frac{1}{8}\left(\frac{\pi}{3}-\frac{5\sqrt{3}}{2}\right)$$

$$A=\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\pi}{24}+\frac{5\sqrt{3}}{16}-\frac{\pi}{24}+\frac{5\sqrt{3}}{16}$$

$$A=\frac{5\sqrt{3}}{4}+\frac{\pi}{12}$$

$$A=\frac{\pi+15\sqrt{3}}{12}$$

Topic: Area between polar curves

Question: Find the area between the polar curves.

$$r = -2\cos\theta$$

$$r = 1$$

Answer choices:

A $\quad -\dfrac{4\pi + 3\sqrt{3}}{6}$

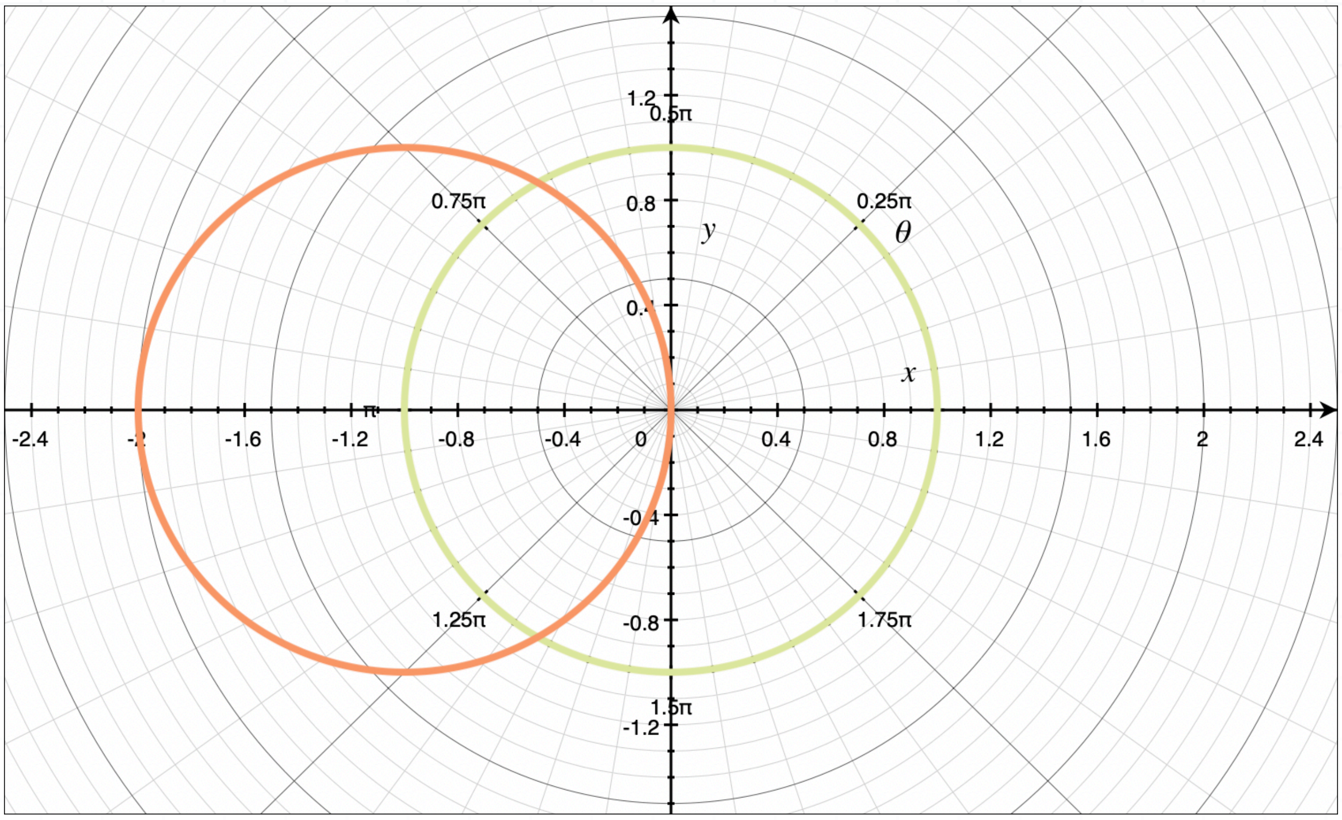
B $\quad \dfrac{4\pi + 3\sqrt{3}}{6}$

C $\quad \dfrac{-\pi + 3\sqrt{3}}{3}$

D $\quad \dfrac{\pi - 3\sqrt{3}}{3}$

Solution: C

The area between the polar curves looks like this:



We can split the area between the curves into two parts:

1. The area outside $r = -2\cos\theta$ but inside the circle (on the right), and

2. The area outside the circle but inside $r = -2\cos\theta$ (on the left).

The curves intersect at $(1,2\pi/3)$ and $(1,4\pi/3)$. We can also write $(1,4\pi/3)$ as $(1,-2\pi/3)$. So our area formula will be

$$A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1)^2 - (-2\cos\theta)^2 \ d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (-2\cos\theta)^2 - (1)^2 \ d\theta$$

$$A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} 1 - 4\cos^2\theta \ d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4\cos^2\theta - 1 \ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$A=\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}1-4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\ d\theta+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)-1\ d\theta$$

$$A=\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}1-2-2\cos 2\theta\ d\theta+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}2+2\cos 2\theta-1\ d\theta$$

$$A=-\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}1+2\cos 2\theta\ d\theta+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}1+2\cos 2\theta\ d\theta$$

Integrate term by term.

$$A=-\frac{1}{2}(\theta+\sin 2\theta)\Big|_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}+\frac{1}{2}(\theta+\sin 2\theta)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$A=-\frac{1}{2}\left(\frac{2\pi}{3}+\sin\left(2\cdot\frac{2\pi}{3}\right)\right)+\frac{1}{2}\left(-\frac{2\pi}{3}+\sin\left(2\cdot-\frac{2\pi}{3}\right)\right)$$

$$+\frac{1}{2}\left(\frac{4\pi}{3}+\sin\left(2\cdot\frac{4\pi}{3}\right)\right)-\frac{1}{2}\left(\frac{2\pi}{3}+\sin\left(2\cdot\frac{2\pi}{3}\right)\right)$$

$$A=-\frac{1}{2}\left(\frac{2\pi}{3}+\sin\frac{4\pi}{3}\right)+\frac{1}{2}\left(-\frac{2\pi}{3}+\sin\left(-\frac{4\pi}{3}\right)\right)$$

$$+\frac{1}{2}\left(\frac{4\pi}{3}+\sin\frac{8\pi}{3}\right)-\frac{1}{2}\left(\frac{2\pi}{3}+\sin\frac{4\pi}{3}\right)$$

$$A=-\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(-\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$$

$$A=-\frac{\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}+\frac{2\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}$$

$$A=-\frac{\pi}{3}+\frac{4\sqrt{3}}{4}$$

$$A=\frac{-\pi+3\sqrt{3}}{3}$$

Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = \sin\theta$$

$$r = \cos\theta$$

Answer choices:

A $\dfrac{\pi + 1}{4}$

B $\dfrac{\pi - 2}{8}$

C $\dfrac{\pi + 2}{8}$

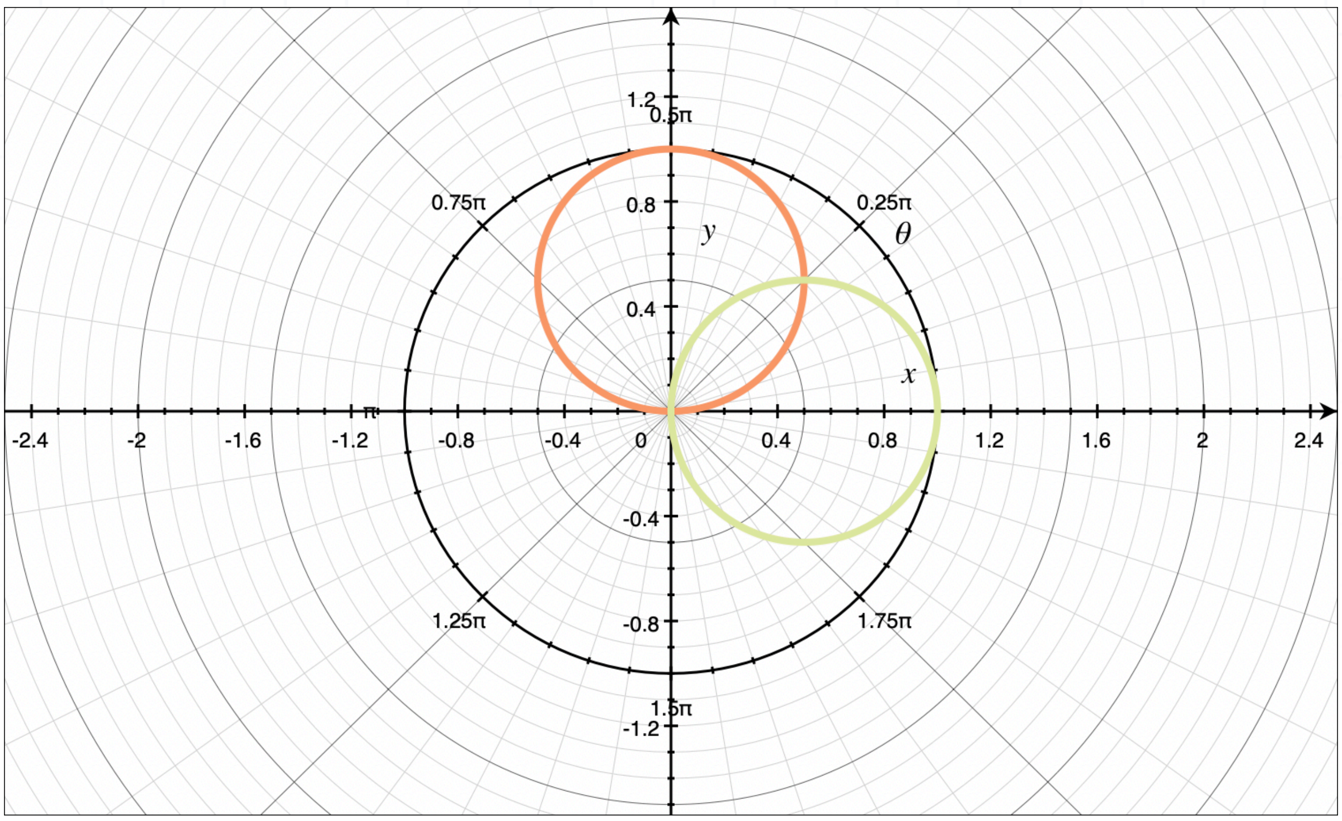
D $\dfrac{\pi - 1}{4}$

Solution: B

To find are inside a polar curve, we use the formula

$$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta$$

The best thing to do is to graph the functions we've been given.



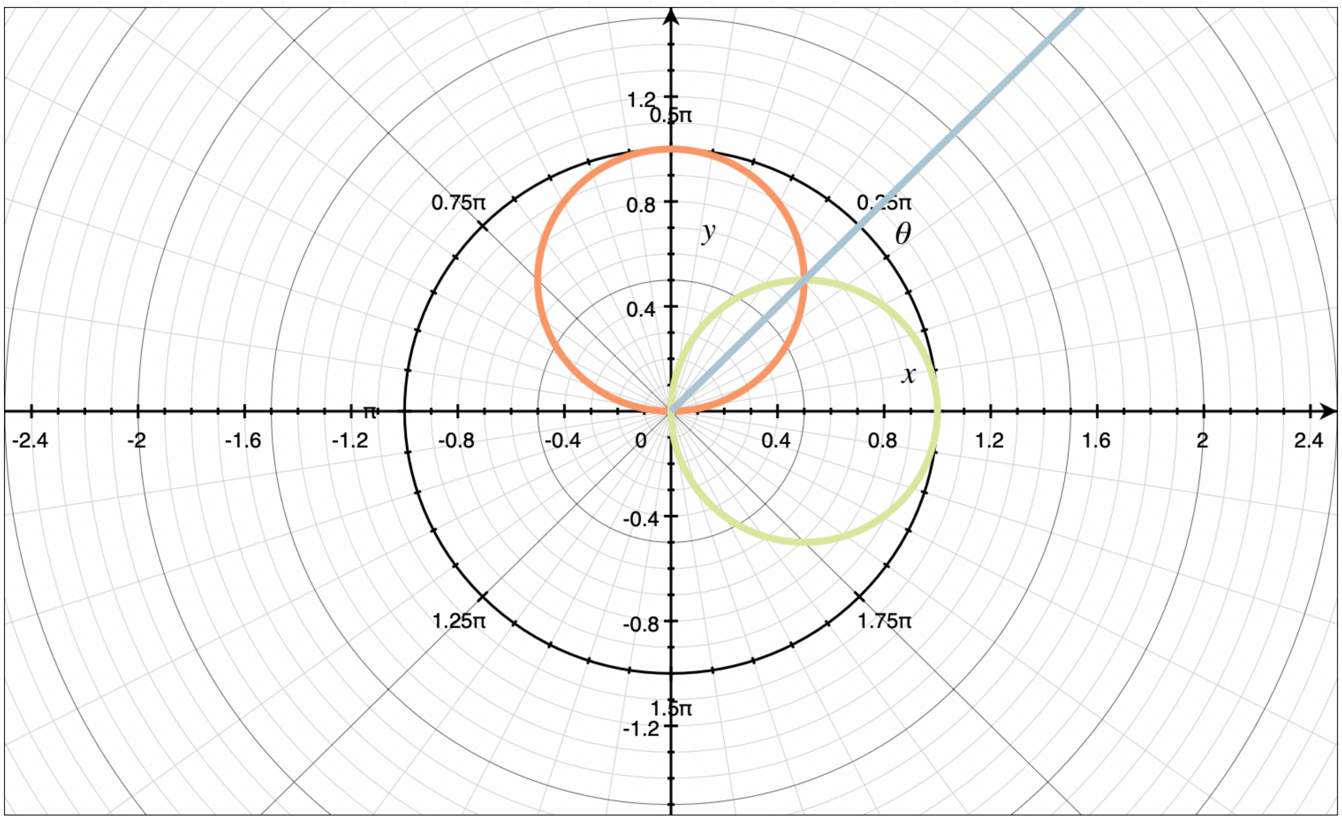
We can see that the area that's inside both curves at the same time is the single small petal where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

$$\sin\theta=\cos\theta$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta=\pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$\theta = \frac{\pi}{4}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/4$. If we sketched the $\sin\theta$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/4$. Which means the beginning of the red $\sin\theta$ curve runs underneath the line $\theta = \pi/4$.



So if we integrate $\sin\theta$ in the polar area formula, over the interval $[0,\pi/4]$, we'll only get the area that lies underneath that $\theta = \pi/4$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area that's inside both curves.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/4]$, we'll use that curve in our formula.

$$A = (2)\frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta \ d\theta$$

$$A=\int_0^{\frac{\pi}{4}}\sin^2\theta\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=\int_0^{\frac{\pi}{4}}\frac{1}{2}(1-\cos(2\theta))\ d\theta$$

$$A=\frac{1}{2}\int_0^{\frac{\pi}{4}}1-\cos(2\theta)\ d\theta$$

$$A=\frac{1}{2}\left(\theta-\frac{1}{2}\sin(2\theta)\right)\Big|_0^{\frac{\pi}{4}}$$

Evaluate over the interval.

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)\right)-\frac{1}{2}\left(0-\frac{1}{2}\sin(2(0))\right)$$

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\frac{1}{2}\left(0-\frac{1}{2}(0)\right)$$

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}(1)\right)$$

$$A=\frac{\pi}{8}-\frac{1}{4}$$

$$A=\frac{\pi}{8}-\frac{2}{8}$$

$$A=\frac{\pi-2}{8}$$

Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = \sin(3\theta)$$

$$r = \cos(3\theta)$$

Answer choices:

A $\frac{\pi + 1}{4}$

B $\frac{\pi - 1}{4}$

C $\frac{\pi + 2}{8}$

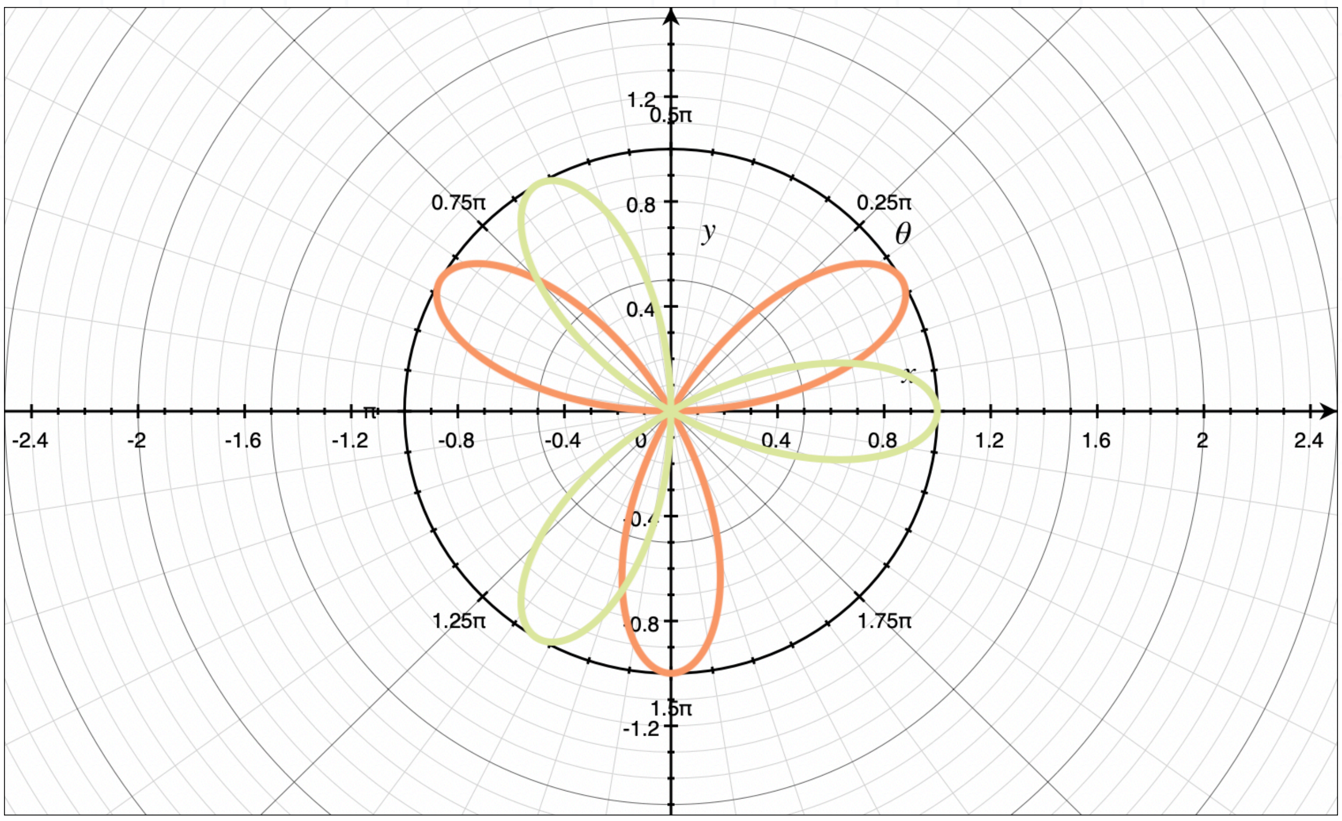
D $\frac{\pi - 2}{8}$

Solution: D

To find are inside a polar curve, we use the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta$$

The best thing to do is to graph the functions we've been given.



We can see that the area that's inside both curves at the same time is the three small petals where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

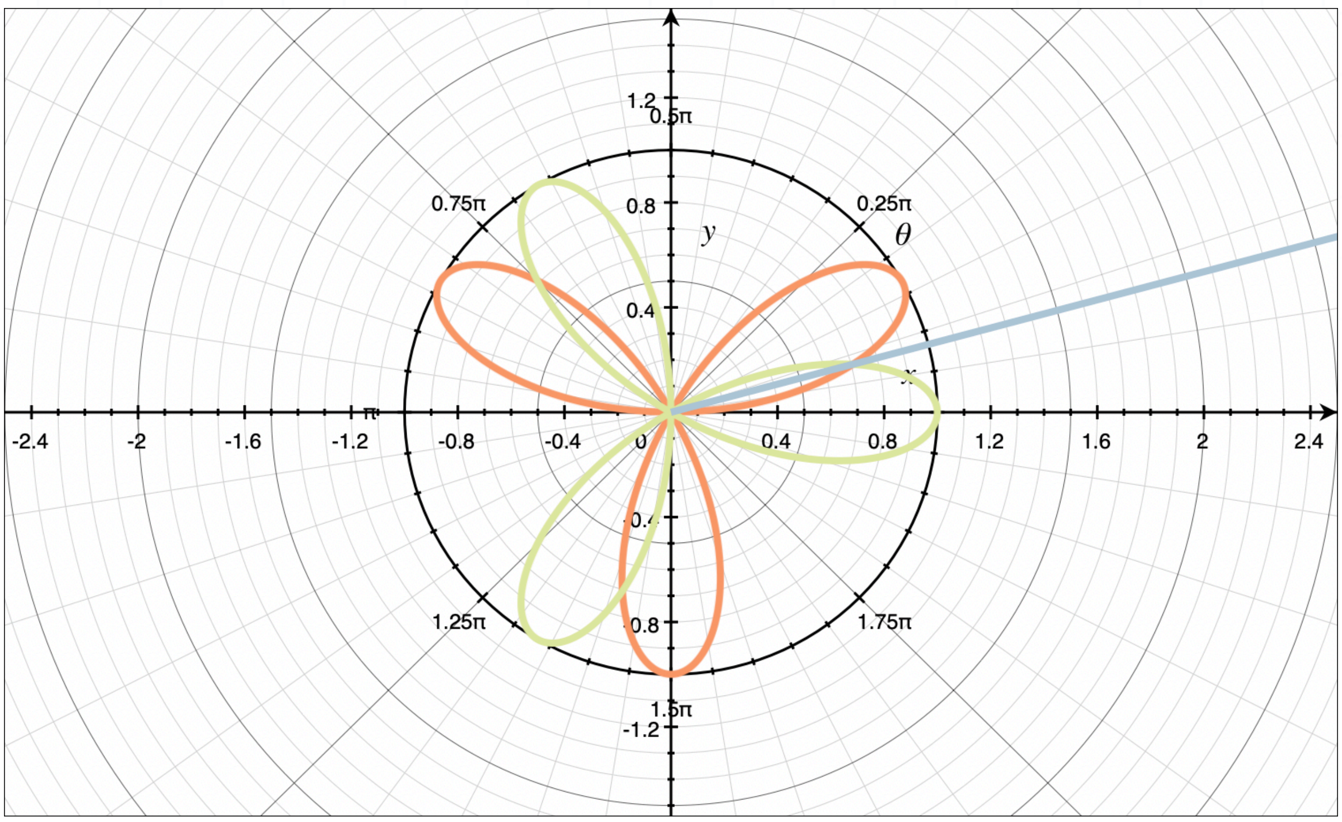
$$\sin(3\theta) = \cos(3\theta)$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta = \pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$3\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{12}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/12$. If we sketched the $\sin(3\theta)$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/12$, and out to the tip of the first red petal. Which means the beginning of the red $\sin(3\theta)$ curve runs underneath the line $\theta = \pi/12$.



So if we integrate $\sin(3\theta)$ in the polar area formula, over the interval $[0,\pi/12]$, we'll only get the area that lies underneath that $\theta = \pi/12$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area for one of the three petals we need. But there are three distinct sections of area that are inside both curves. Therefore, we'll be multiplying by 2, and then by 3, or we can just multiply by 6.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/12]$, we'll use that curve in our formula.

$$A=(6)\frac{1}{2}\int_0^{\frac{\pi}{12}}\sin^2(3\theta)\ d\theta$$

$$A=3\int_0^{\frac{\pi}{12}}\sin^2(3\theta)\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=3\int_0^{\frac{\pi}{12}}\frac{1}{2}(1-\cos(2(3\theta)))\ d\theta$$

$$A=\frac{3}{2}\int_0^{\frac{\pi}{12}}1-\cos(6\theta)\ d\theta$$

$$A=\frac{3}{2}\left(\theta-\frac{1}{6}\sin(6\theta)\right)\Bigg|_0^{\frac{\pi}{12}}$$

Evaluate over the interval.

$$A=\frac{3}{2}\left(\frac{\pi}{12}-\frac{1}{6}\sin\left(6\cdot\frac{\pi}{12}\right)\right)-\frac{3}{2}\left(0-\frac{1}{6}\sin(6(0))\right)$$

$$A=\frac{3}{2}\left(\frac{\pi}{12}-\frac{1}{6}\sin\frac{\pi}{2}\right)-\frac{3}{2}\left(0-\frac{1}{6}(0)\right)$$

$$A=\frac{3}{2}\left(\frac{\pi}{12}-\frac{1}{6}(1)\right)$$

$$A=\frac{3\pi}{24}-\frac{3}{12}$$

$$A=\frac{3\pi}{24}-\frac{6}{24}$$

$$A=\frac{3\pi-6}{24}$$

$$A=\frac{\pi-2}{8}$$

Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = 3\sin(2\theta)$$

$$r = 3\cos(2\theta)$$

Answer choices:

A $\frac{9\pi - 18}{2}$

B $\frac{9\pi - 9}{2}$

C $\frac{9\pi + 9}{2}$

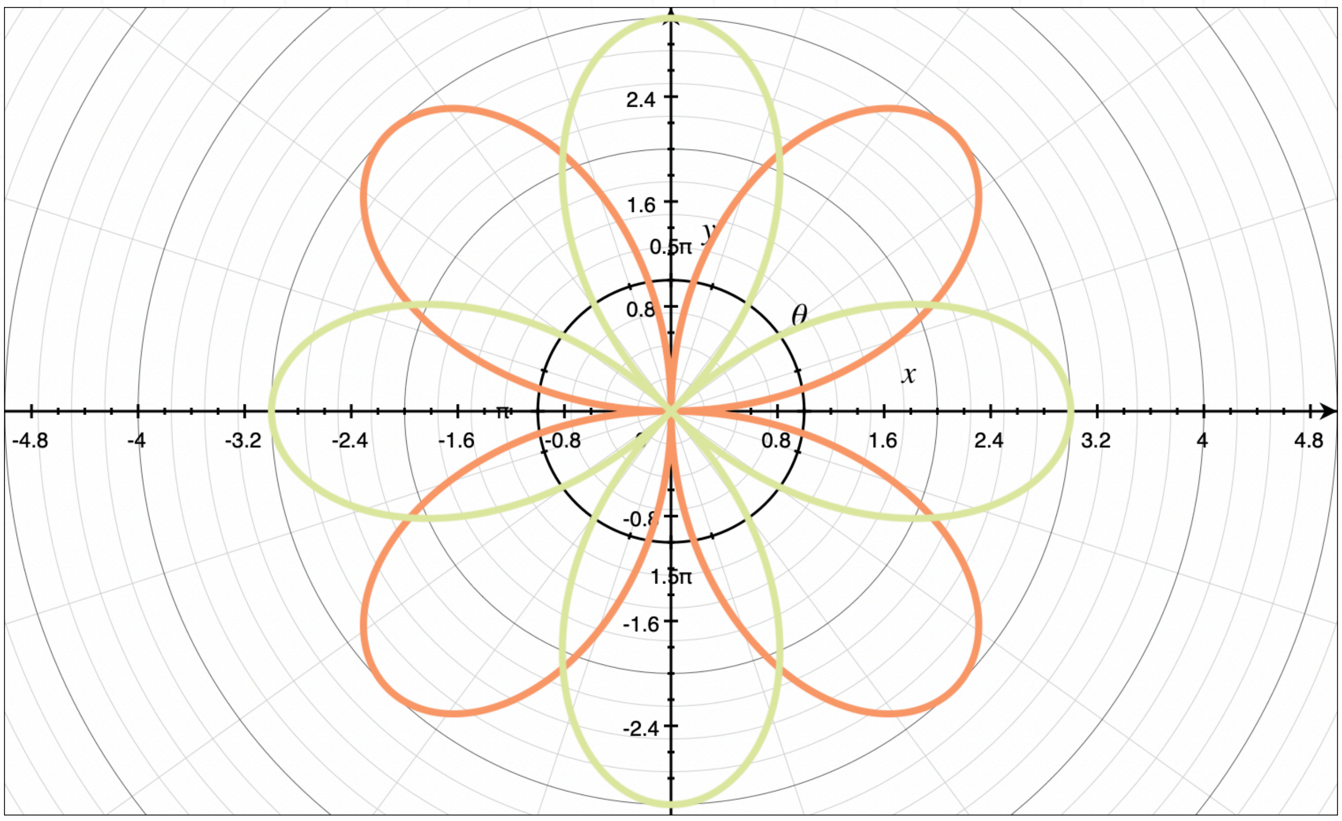
D $\frac{9\pi + 18}{2}$

Solution: A

To find area inside a polar curve, we use the formula

$$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta$$

The best thing to do is to graph the functions we've been given.



We can see that the area that's inside both curves at the same time is the eight small petals where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

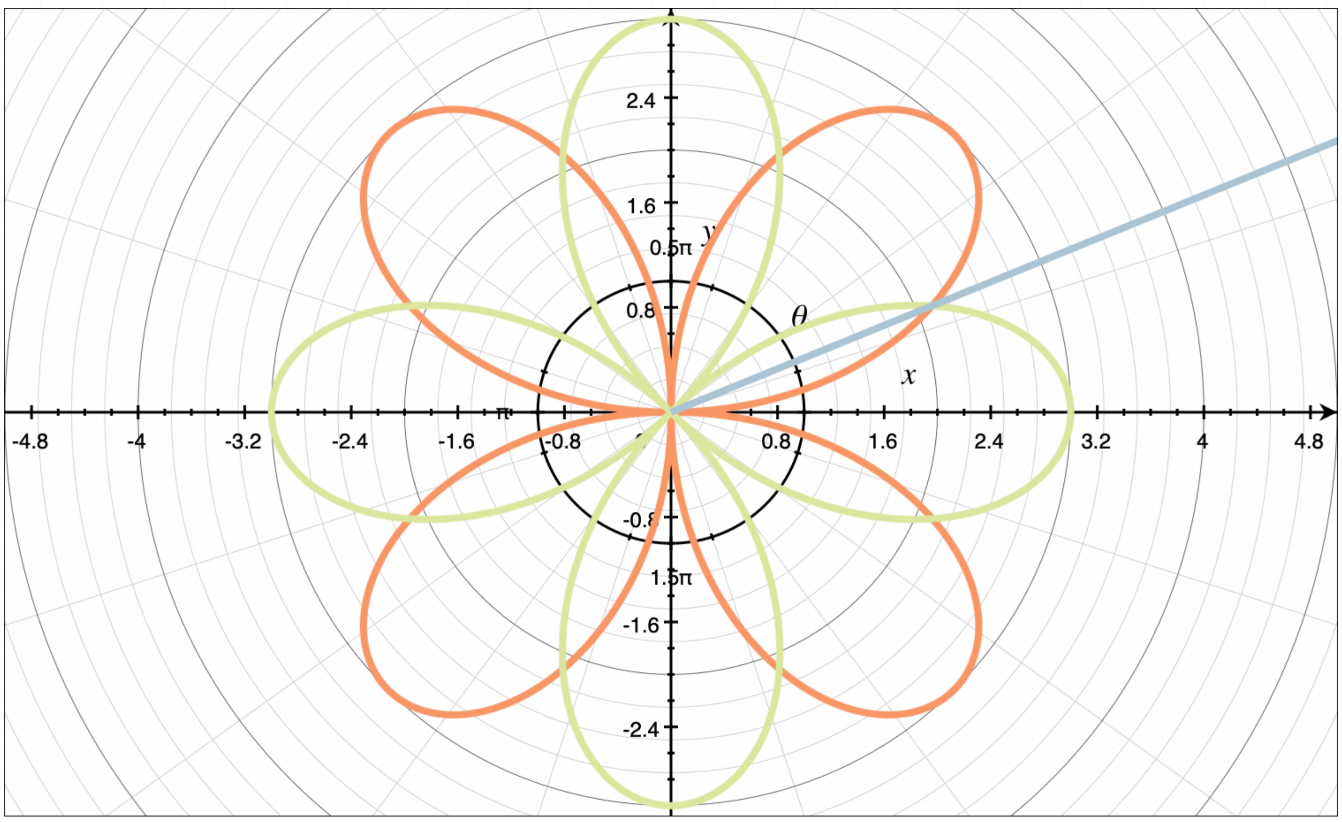
$$3\sin(2\theta)=3\cos(2\theta)$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta=\pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$2\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/8$. If we sketched the $3\sin(2\theta)$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/8$, and out to the tip of the first red petal. Which means the beginning of the red $3\sin(2\theta)$ curve runs underneath the line $\theta = \pi/8$.



So if we integrate $3\sin(2\theta)$ in the polar area formula, over the interval $[0,\pi/8]$, we'll only get the area that lies underneath that $\theta = \pi/8$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area for one of the eight petals we need. But there are eight distinct sections of area that are inside both curves. Therefore, we'll be multiplying by 2, and then by 8, or we can just multiply by 16.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/8]$, we'll use that curve in our formula.

$$A=(16)\frac{1}{2}\int_0^{\frac{\pi}{8}}9\sin^2(2\theta)\ d\theta$$

$$A=72\int_0^{\frac{\pi}{8}}\sin^2(2\theta)\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=72\int_0^{\frac{\pi}{8}}\frac{1}{2}(1-\cos(2(2\theta)))\ d\theta$$

$$A=36\int_0^{\frac{\pi}{8}}1-\cos(4\theta)\ d\theta$$

$$A=36\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_0^{\frac{\pi}{8}}$$

Evaluate over the interval.

$$A=36\left(\frac{\pi}{8}-\frac{1}{4}\sin\left(4\cdot\frac{\pi}{8}\right)\right)-36\left(0-\frac{1}{4}\sin(4(0))\right)$$

$$A=36\left(\frac{\pi}{8}-\frac{1}{4}\sin\frac{\pi}{2}\right)-36\left(0-\frac{1}{4}(0)\right)$$

$$A=36\left(\frac{\pi}{8}-\frac{1}{4}(1)\right)$$

$$A=\frac{36\pi}{8}-\frac{36}{4}$$

$$A=\frac{36\pi}{8}-\frac{72}{8}$$

$$A=\frac{36\pi-72}{8}$$

$$A=\frac{9\pi-18}{2}$$

Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = 5\theta^2$$

on the interval $0 \leq \theta \leq \sqrt{21}$

Answer choices:

A $\frac{585}{3}$

B $\frac{585\pi}{3}$

C 585

D 585π

Solution: A

The arc length for a polar curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta$$

where the limits of integration are $\alpha = 0$ and $\beta = \sqrt{21}$. Also, since $r = 5\theta^2$, then

$$\frac{dr}{d\theta} = 10\theta$$

So the length is

$$L = \int_0^{\sqrt{21}} \sqrt{\left(5\theta^2\right)^2 + (10\theta)^2} \ d\theta$$

$$L = \int_0^{\sqrt{21}} \sqrt{25\theta^4 + 100\theta^2} \ d\theta$$

$$L = \int_0^{\sqrt{21}} \sqrt{25\theta^2\left(\theta^2 + 4\right)} \ d\theta$$

$$L = 5\int_0^{\sqrt{21}} \theta\sqrt{\theta^2 + 4} \ d\theta$$

Letting

$$u = \theta^2 + 4$$

$$du = 2\theta \ d\theta$$

$$d\theta = \frac{du}{2\theta}$$

and making a substitution into our integral, we get

$$L = 5\int_{x=0}^{x=\sqrt{21}} \theta\sqrt{u}\ \frac{du}{2\theta}$$

$$L = \frac{5}{2}\int_{x=0}^{x=\sqrt{21}} \sqrt{u}\ du$$

$$L = \frac{5}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=\sqrt{21}}$$

$$L = \frac{5}{3}\left(\theta^2+4\right)^{\frac{3}{2}}\Bigg|_0^{\sqrt{21}}$$

$$L = \frac{5}{3}\left[\left(\left(\sqrt{21}\right)^2+4\right)^{\frac{3}{2}} - \left((0)^2+4\right)^{\frac{3}{2}}\right]$$

$$L = \frac{5}{3}\left[(25)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$$

$$L = \frac{5}{3}\left(125-8\right)$$

$$L = \frac{585}{3}$$

Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = \cos^3 \frac{\theta}{3}$$

on the interval $0 \leq \theta \leq \pi$

Answer choices:

A $\quad \pi + \dfrac{3\sqrt{3}}{8}$

B $\quad \dfrac{1}{2}\pi + \dfrac{3\sqrt{3}}{8}$

C $\quad \dfrac{1}{2}\pi - \dfrac{3\sqrt{3}}{8}$

D $\quad \pi - \dfrac{3\sqrt{3}}{8}$

Solution: B

The arc length of a polar curve on an interval is given by

$$L=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

where $\alpha=0$ and $\beta=\pi$. We'll find the derivative of the given polar equation so that we can plug it into the formula for arc length.

$$\frac{dr}{d\theta}=-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$$

Plugging this into the arc length formula, we get

$$L=\int_{0}^{\pi}\sqrt{\left(\cos^3\frac{\theta}{3}\right)^2+\left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2}\ d\theta$$

$$L=\int_{0}^{\pi}\sqrt{\cos^6\frac{\theta}{3}+\sin^2\frac{\theta}{3}\cos^4\frac{\theta}{3}}\ d\theta$$

$$L=\int_{0}^{\pi}\sqrt{\cos^4\frac{\theta}{3}\left(\cos^2\frac{\theta}{3}+\sin^2\frac{\theta}{3}\right)}\ d\theta$$

Using the pythagorean identity

$$\sin^2 x+\cos^2 x=1$$

we can simplify the integral to

$$L=\int_{0}^{\pi}\sqrt{\cos^4\frac{\theta}{3}(1)}\ d\theta$$

$$L=\int_0^{\pi}\cos^2\frac{\theta}{3}\ d\theta$$

Using the power reduction formula

$$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$$

we get

$$L=\int_0^{\pi}\frac{1}{2}+\frac{1}{2}\cos\frac{2\theta}{3}\ d\theta$$

$$L=\frac{1}{2}\theta+\frac{3}{4}\sin\frac{2\theta}{3}\Big|_0^{\pi}$$

$$L=\left(\frac{1}{2}\pi+\frac{3}{4}\sin\frac{2\pi}{3}\right)-\left(\frac{1}{2}(0)+\frac{3}{4}\sin\frac{2(0)}{3}\right)$$

$$L=\frac{\pi}{2}+\frac{3}{4}\cdot\frac{\sqrt{3}}{2}$$

$$L=\frac{\pi}{2}+\frac{3\sqrt{3}}{8}$$

Topic: Arc length of a polar curve

Question: Find the length of the polar curve on the given interval.

$$r = 1 - \cos\theta$$

on the interval $0 \leq \theta \leq 2\pi$

Answer choices:

A 6

B 7

C 8

D 9

Solution: C

The arc length of a polar curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta$$

According to the question, $\alpha = 0$ and $\beta = 2\pi$. Let's find the derivative of the original equation so that we can plug everything into the arc length formula.

$$r = 1 - \cos\theta$$

$$\frac{dr}{d\theta} = \sin\theta$$

Plugging into the formula, we get

$$L = \int_{0}^{2\pi} \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2} \ d\theta$$

$$L = \int_{0}^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \ d\theta$$

Knowing that $\sin^2 x + \cos^2 x = 1$, the integral simplifies to

$$L = \int_{0}^{2\pi} \sqrt{1 - 2\cos\theta + 1} \ d\theta$$

$$L = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta} \ d\theta$$

Using half-angle formulas, we can say that

$$2-2\cos\theta=4\sin^2\frac{\theta}{2}$$

and therefore that

$$L=\int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\ d\theta$$

$$L=2\int_0^{2\pi}\sin\frac{\theta}{2}\ d\theta$$

$$L=-4\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$L=-4\cos\frac{2\pi}{2}-\left(-4\cos\frac{0}{2}\right)$$

$$L=-4\cos\pi+4\cos 0$$

$$L=-4(-1)+4(1)$$

$$L=8$$

Topic: Surface area of revolution of a polar curve

Question: Find the surface area generated by revolving the polar curve about the y-axis.

$$r = 5\sqrt{\sin(2\theta)}$$

on the interval $0 \leq \theta \leq \frac{\pi}{4}$

Answer choices:

A $25\sqrt{2}$

B $2\pi\sqrt{5}$

C $25\pi\sqrt{2}$

D 25π

Solution: C

The area of the surface generated by revolving a curve about the y-axis is given by

$$S=\int_{\alpha}^{\beta}2\pi r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

We'll find the derivative of the given equation so that we can plug it into the surface area formula. We'll also calculate r^2 and hope to plug it in and avoid plugging in square roots.

$$\frac{dr}{d\theta}=5\cdot\frac{1}{2}(\sin(2\theta))^{-\frac{1}{2}}\cdot\cos(2\theta)\cdot 2$$

$$\frac{dr}{d\theta}=\frac{5\cos(2\theta)}{\sqrt{\sin(2\theta)}}$$

and

$$r=5\sqrt{\sin(2\theta)}$$

$$r^2=25\sin(2\theta)$$

To avoid plugging square roots into our formula, let's absorb the r into the square root. If it's r outside of the square root, it must have been r^2 inside the square root.

$$S=\int_{\alpha}^{\beta}2\pi\cos\theta\sqrt{r^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\ d\theta$$

$$S=\int_{\alpha}^{\beta}2\pi\cos\theta\sqrt{r^4+r^2\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Now that all of our r's are raised to even powers, we can plug in the value we found for r^2 and avoid the square roots.

$$S=\int_0^{\frac{\pi}{4}}2\pi\cos\theta\sqrt{\left(25\sin(2\theta)\right)^2+\left(25\sin(2\theta)\right)\left(\frac{5\cos(2\theta)}{\sqrt{\sin(2\theta)}}\right)^2}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\sin^2(2\theta)+25\sin(2\theta)\frac{25\cos^2(2\theta)}{\sin(2\theta)}}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\sin^2(2\theta)+625\cos^2(2\theta)}\ d\theta$$

$$S=2\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{625\left(\sin^2(2\theta)+\cos^2(2\theta)\right)}\ d\theta$$

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{\sin^2(2\theta)+\cos^2(2\theta)}\ d\theta$$

Knowing that $\sin^2 x+\cos^2 x=1$, we can simplify the integral to

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\sqrt{1}\ d\theta$$

$$S=50\pi\int_0^{\frac{\pi}{4}}\cos\theta\ d\theta$$

$$S=50\pi\sin\theta\Big|_0^{\frac{\pi}{4}}$$

$$S = 50\pi\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$S = 50\pi\left(\frac{\sqrt{2}}{2}\right)$$

$$S = 25\pi\sqrt{2}$$

Topic: Surface area of revolution of a polar curve

Question: Find surface area of revolution.

$$r = \sin\theta$$

about the x-axis

on the interval $0 \leq \theta \leq \pi$

Answer choices:

A π

B $\dfrac{\pi}{2}$

C π^2

D $\dfrac{\pi^2}{2}$

Solution: C

To find surface area of revolution when we rotate about the x-axis, we need to use the formula

$$S_x = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

To use the formula, we'll need to first find the derivative $dr/d\theta$. If $r = \sin\theta$, then

$$\frac{dr}{d\theta} = \cos\theta$$

Plugging this, and the given interval $0 \le \theta \le \pi$ into the formula, we get

$$S_x = \int_0^{\pi} 2\pi \sin\theta \sin\theta \sqrt{\sin^2\theta + \cos^2\theta}\ d\theta$$

Remembering the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute into the integral.

$$S_x = 2\pi \int_0^{\pi} \sin^2\theta \sqrt{1}\ d\theta$$

$$S_x = 2\pi \int_0^{\pi} \sin^2\theta\ d\theta$$

Using the double-angle formula

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$$

we'll rewrite the integral as

$$S_x = 2\pi \int_0^{\pi} \frac{1}{2}(1 - \cos(2\theta))\ d\theta$$

$$S_x = \pi \int_0^{\pi} 1 - \cos(2\theta)\ d\theta$$

And then we'll integrate and evaluate over the interval.

$$S_x = \pi \left(\theta - \frac{1}{2}\sin(2\theta) \right) \Bigg|_0^{\pi}$$

$$S_x = \pi\theta - \frac{\pi}{2}\sin(2\theta) \Bigg|_0^{\pi}$$

$$S_x = \pi(\pi) - \frac{\pi}{2}\sin(2(\pi)) - \left[\pi(0) - \frac{\pi}{2}\sin(2(0)) \right]$$

$$S_x = \pi^2 - \frac{\pi}{2}(0) - (0) + \frac{\pi}{2}(0)$$

$$S_x = \pi^2$$

Topic: Surface area of revolution of a polar curve

Question: Find the surface area of revolution.

$$r = 2\cos\theta$$

about the y-axis

on the interval $0 \leq \theta \leq \pi$

Answer choices:

A $4\pi^2$

B 2π

C 4π

D $2\pi^2$

Solution: A

To find surface area of revolution when we rotate about the y-axis, we need to use the formula

$$S_y = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

To use the formula, we'll need to first find the derivative $dr/d\theta$. If $r = 2\cos\theta$, then

$$\frac{dr}{d\theta} = -2\sin\theta$$

Plugging this, and the given interval $0 \le \theta \le \pi$ into the formula, we get

$$S_y = \int_0^{\pi} 2\pi(2\cos\theta)\cos\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4\cos^2\theta + 4\sin^2\theta}\ d\theta$$

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4\left(\cos^2\theta + \sin^2\theta\right)}\ d\theta$$

Remembering the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute into the integral.

$$S_y = \int_0^{\pi} 4\pi\cos^2\theta\sqrt{4(1)}\ d\theta$$

$$S_y = 8\pi\int_0^{\pi} \cos^2\theta\ d\theta$$

Using the double-angle formula

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

we'll rewrite the integral as

$$S_y = 8\pi\int_0^{\pi}\frac{1}{2}(1+\cos(2\theta))\ d\theta$$

$$S_y = 4\pi\int_0^{\pi}1+\cos(2\theta)\ d\theta$$

And then we'll integrate and evaluate over the interval.

$$S_y = 4\pi\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi}$$

$$S_y = 4\pi\theta + 2\pi\sin(2\theta)\Big|_0^{\pi}$$

$$S_y = 4\pi(\pi)+2\pi\sin(2(\pi)) - \left[4\pi(0)+2\pi\sin(2(0))\right]$$

$$S_y = 4\pi^2 + 2\pi(0) - 4\pi(0) - 2\pi(0)$$

$$S_y = 4\pi^2$$

Topic: Sequences vs. series

Question: Choose the correct definition of a sequence.

Answer choices:

A A sequence is the sum of an ordered list of elements.

B A sequence is an ordered list of elements that follows a set rule.

C A sequence is the sum of a random (non-repeating) list of elements.

D A sequence is a random (non-repeating) list of elements.

Solution: B

A sequence is an ordered list of elements that follows a set rule.

Topic: Sequences vs. series

Question: Choose the correct definition of a series.

Answer choices:

A A series is the sum of an ordered list of elements.

B A series is an ordered list of elements that follows a set rule.

C A series is the sum of a random (non-repeating) list of elements.

D A series is a random (non-repeating) list of elements.

Solution: A

A series is the sum an ordered list of elements.

Topic: Sequences vs. series

Question: How are sequences and series related?

Answer choices:

A A sequence is the sum of all of the terms of a series.

B Sequences are ordered and series are random but they are both lists of elements.

C A series is the sum of all of the terms of a sequence.

D Sequences are ordered and series are random but they are both sums of terms.

Solution: C

A series is the sum of all of the terms of a sequence.

Topic: Listing the first terms

Question: Write the first three terms of the sequence.

$$a_{n+1} = 2a_n$$

where $a_1 = 2$

Answer choices:

A $\quad 2$, 8 and 16

B $\quad 4$, 8 and 12

C $\quad 2$, 4 and 8

D $\quad 2$, 4 and 6

Solution: C

To get the first three terms of the sequence, just plug $n=1$ and $n=2$ into the formula for a_{n+1} as follows.

$$a_1 = 2$$

$$n=1 \qquad a_{1+1} = 2a_1 \qquad a_2 = 2(2) \qquad a_2 = 4$$

$$n=2 \qquad a_{2+1} = 2a_2 \qquad a_3 = 2(4) \qquad a_3 = 8$$

The first three terms of the sequence are

$$2, 4, 8$$

Topic: Listing the first terms

Question: Write the first four terms of the sequence.

$$a_{n+1} = 3a_n - 4$$

where $a_1 = 3$

Answer choices:

A $3, 5, 12$ and 32

B $3, 5, 11$ and 29

C $3, 9, 27$ and 81

D $3, 5, 7$ and 9

Solution: B

To get the first four terms of the sequence, just plug $n = 1, 2, 3$ into the formula for a_{n+1} as follows.

$$a_1 = 3$$

$$n = 1 \qquad a_{1+1} = 3a_1 - 4 \qquad a_2 = 3(3) - 4 \qquad a_2 = 5$$

$$n = 2 \qquad a_{2+1} = 3a_2 - 4 \qquad a_3 = 3(5) - 4 \qquad a_3 = 11$$

$$n = 3 \qquad a_{3+1} = 3a_3 - 4 \qquad a_4 = 3(11) - 4 \qquad a_4 = 29$$

The first four terms of the sequence are

$$3, 5, 11, 29$$

Topic: Listing the first terms

Question: Write the first five terms of the sequence.

$$a_{n+1} = (a_n)^2 + 2a_n - 1$$

where $a_1 = 1$

Answer choices:

A $1, 4, 12, 36$ and 192

B $1, 2, 7, 12$ and 42

C $1, 3, 14, 228$ and $51{,}983$

D $1, 2, 7, 62$ and $3{,}967$

Solution: D

To get the first five terms of the sequence, just plug $n = 1, 2, 3, 4$ into the formula for a_{n+1} as follows.

$$a_1 = 1$$

$$n = 1 \qquad a_{1+1} = (a_1)^2 + 2a_1 - 1 \qquad a_2 = (1)^2 + 2(1) - 1 \qquad a_2 = 2$$

$$n = 2 \qquad a_{2+1} = (a_2)^2 + 2a_2 - 1 \qquad a_3 = (2)^2 + 2(2) - 1 \qquad a_3 = 7$$

$$n = 3 \qquad a_{3+1} = (a_3)^2 + 2a_3 - 1 \qquad a_4 = (7)^2 + 2(7) - 1 \qquad a_4 = 62$$

$$n = 4 \qquad a_{4+1} = (a_4)^2 + 2a_4 - 1 \qquad a_5 = (62)^2 + 2(62) - 1 \qquad a_5 = 3{,}967$$

The first five terms of the sequence are

$$1, 2, 7, 62, 3{,}967$$

Topic: Calculating the first terms

Question: Write the first five terms of the sequence and find the limit.

$$a_n = \frac{4n^2 + 1}{n^2 - 2n + 3}$$

$$\lim_{n \to \infty} a_n$$

Answer choices:

A $\quad \frac{5}{2}, \frac{17}{3}, \frac{37}{4}, \frac{65}{5}, \frac{101}{6}$ and $\lim_{n \to \infty} a_n = 3$

B $\quad \frac{7}{2}, \frac{17}{3}, \frac{37}{6}, \frac{67}{11}, \frac{107}{18}$ and $\lim_{n \to \infty} a_n = 0$

C $\quad \frac{5}{2}, \frac{17}{3}, \frac{37}{6}, \frac{65}{11}, \frac{101}{18}$ and $\lim_{n \to \infty} a_n = 4$

D $\quad \frac{5}{2}, \frac{17}{3}, \frac{37}{6}, \frac{65}{12}, \frac{101}{18}$ and $\lim_{n \to \infty} a_n = \text{DNE}$

Solution: C

To get the first five terms of the sequence, just plug $n=1, 2, 3, 4$ into the formula for a_{n+1} as follows.

$$n=1 \qquad a_1=\frac{4(1)^2+1}{(1)^2-2(1)+3} \qquad a_1=\frac{5}{2}$$

$$n=2 \qquad a_2=\frac{4(2)^2+1}{(2)^2-2(2)+3} \qquad a_2=\frac{17}{3}$$

$$n=3 \qquad a_3=\frac{4(3)^2+1}{(3)^2-2(3)+3} \qquad a_3=\frac{37}{6}$$

$$n=4 \qquad a_4=\frac{4(4)^2+1}{(4)^2-2(4)+3} \qquad a_4=\frac{65}{11}$$

$$n=5 \qquad a_5=\frac{4(5)^2+1}{(5)^2-2(5)+3} \qquad a_5=\frac{101}{18}$$

The first five terms of the sequence are

$$\frac{5}{2}, \frac{17}{3}, \frac{37}{6}, \frac{65}{11}, \frac{101}{18}$$

To find the limit of the sequence, divide both the numerator and denominator by the highest power of n.

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{4n^2+1}{n^2-2n+3}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{4n^2}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} - \frac{2n}{n^2} + \frac{3}{n^2}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4 + \frac{1}{n^2}}{1 - \frac{2}{n} + \frac{3}{n^2}}$$

$$\lim_{n\to\infty} a_n = \frac{4+0}{1-0+0}$$

$$\lim_{n\to\infty} a_n = 4$$

Therefore, the limit of the series is 4.

Topic: Calculating the first terms

Question: Write the first five terms of the sequence and find the limit.

$$a_n = \frac{e^n}{n^2}$$

$$\lim_{n\to\infty} a_n$$

Answer choices:

A $\quad \frac{e}{2}, \frac{e^2}{4}, \frac{e^3}{9}, \frac{e^4}{16}, \frac{e^5}{25}$ and $\lim_{n\to\infty} a_n = 3$

B $\quad e, \frac{e^2}{4}, \frac{e^3}{9}, \frac{e^4}{16}, \frac{e^5}{25}$ and $\lim_{n\to\infty} a_n =$ DNE

C $\quad e, \frac{e^3}{9}, \frac{e^4}{16}, \frac{e^5}{25}, \frac{e^6}{36}$ and $\lim_{n\to\infty} a_n = 4$

D $\quad e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \frac{e^5}{5}$ and $\lim_{n\to\infty} a_n = 0$

Solution: B

To get the first five terms of the sequence, just plug $n = 1, 2, 3, 4, 5$ into the formula for a_n as follows.

$$n = 1 \qquad a_1 = \frac{e^1}{1^2} \qquad a_1 = e$$

$$n = 2 \qquad a_2 = \frac{e^2}{2^2} \qquad a_2 = \frac{e^2}{4}$$

$$n = 3 \qquad a_3 = \frac{e^3}{3^2} \qquad a_3 = \frac{e^3}{9}$$

$$n = 4 \qquad a_4 = \frac{e^4}{4^2} \qquad a_4 = \frac{e^4}{16}$$

$$n = 5 \qquad a_5 = \frac{e^5}{5^2} \qquad a_5 = \frac{e^5}{25}$$

The first five terms of the sequence are

$$e, \frac{e^2}{4}, \frac{e^3}{9}, \frac{e^4}{16}, \frac{e^5}{25}$$

To find the limit of the sequence, apply L'Hospital's rule twice by replacing the numerator and denominator with their derivatives.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{2n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{e^n}{2}$$

$$\lim_{n \to \infty} a_n = \infty$$

Therefore, the limit of the series does not exist (DNE).

Topic: Calculating the first terms

Question: Write the first three terms of the sequence.

$$a_n = 2^n$$

Answer choices:

A $\quad 2$, 8 and 16

B $\quad 2$, 4 and 8

C $\quad 4$, 8 and 12

D $\quad 2$, 4 and 6

Solution: B

To get the first three terms of the sequence, just plug $n = 1, 2, 3$ into the formula for a_n as follows.

$$n = 1 \qquad a_1 = 2^1 \qquad a_1 = 2$$

$$n = 2 \qquad a_2 = 2^2 \qquad a_2 = 4$$

$$n = 3 \qquad a_3 = 2^3 \qquad a_3 = 8$$

The first three terms of the sequence are

$$2, 4, 8$$

Topic: Formula for the general term

Question: Find a formula for the general term of the sequence.

$$1, 4, 7, 10, 13, \ldots$$

Answer choices:

A $\quad a_n = 3n - 2$

B $\quad a_n = 3n + 2$

C $\quad a_n = 5n - 4$

D $\quad a_n = 5n + 4$

Solution: A

The general term of a sequence is a single term that can represent every term in the sequence, based on the value of n that we pick. In other words, if the general term of a sequence is $1/n$, and the sequence starts at $n=1$, then we start plugging $n=1$, $n=2$, $n=3$, $n=4$, etc. into the general term, and we get the expanded sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ \ldots$$

Oftentimes we're given the expanded sequence and asked to find the general term that represents it, so it's like we're working backwards.

The easiest way to find the general term is to look at each part of our sequence, and find its relationship to the corresponding n value.

The first thing we want to do is match each term in our expanded sequence with its n value. For the sequence we've been given in this problem, we'll get

$$n=1 \qquad n=2 \qquad n=3 \qquad n=4 \qquad n=5$$

$$1 \qquad 4 \qquad 7 \qquad 10 \qquad 13$$

You want to check first to see if there's a value you can add to n, a value you can subtract from n, or a value you can subtract n from. But in this case, the difference between n and each of the terms is different.

$$n=1 \qquad 1=1+0=n$$

$$n=2 \qquad 4=2+2=n+2$$

$$n = 3 \qquad 7 = 3 + 4 = n + 4$$

$$n = 4 \qquad 10 = 4 + 6 = n + 6$$

$$n = 5 \qquad 13 = 5 + 8 = n + 8$$

Since that doesn't work, we'll look to see if we can multiply n or divide n by some value that will give us the denominators from our expanded sequence. Unfortunately, that won't work either. However, if we look at $3n$, we can see a consistent difference between $3n$ and our terms.

$$n = 1 \qquad 1 = 3(1) - 2 = 3n - 2$$

$$n = 2 \qquad 4 = 3(2) - 2 = 3n - 2$$

$$n = 3 \qquad 7 = 3(3) - 2 = 3n - 2$$

$$n = 4 \qquad 10 = 3(4) - 2 = 3n - 2$$

$$n = 5 \qquad 13 = 3(5) - 2 = 3n - 2$$

Since every term can be represented by $3n - 2$, this will be the formula for the general term.

$$a_n = 3n - 2$$

We can always double-check ourselves by testing the general term at $n = 1, 2, 3, 4, 5, \ ...$. If the formula for the general term returns the terms we were given in the original expanded sequence, then we know that our general term accurately represents the sequence.

Another thing to note: If you encounter this type of problem in a multiple choice test, keep in mind that you can always just plug $n = 1, 2, 3, 4, 5, \ ...$

into each of the answer choices, to see if the values you get match the original expanded sequence.

Topic: Formula for the general term

Question: Find a formula for the general term of the sequence.

$$-1, \frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \ldots$$

Answer choices:

A $\quad a_n = (-1)^n \frac{1}{2n-1}$

B $\quad a_n = (-1)^{n+1} \frac{1}{2n-1}$

C $\quad a_n = (-1)^n \frac{n}{2n-1}$

D $\quad a_n = (-1)^{n+1} \frac{n-1}{2n-1}$

Solution: C

The general term of a sequence is a single term that can represent every term in the sequence, based on the value of n that we pick. In other words, if the general term of a sequence is $1/n$, and the sequence starts at $n = 1$, then we start plugging $n = 1$, $n = 2$, $n = 3$, $n = 4$, etc. into the general term, and we get the expanded sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ \dots$$

Oftentimes we're given the expanded sequence and asked to find the general term that represents it, so it's like we're working backwards.

The easiest way to find the general term is to look at each part of our sequence, and find its relationship to the corresponding n value.

The first thing we want to do is match each term in our expanded sequence with its n value. For the sequence we've been given in this problem, we'll get

$$\begin{array}{lllll} n = 1 & n = 2 & n = 3 & n = 4 & n = 5 \\ \\ -1 & \dfrac{2}{3} & -\dfrac{3}{5} & \dfrac{4}{7} & -\dfrac{5}{9} \end{array}$$

We can start by noticing that the numerator of each term is equal to its n value. So we'll put n in the numerator of the formula for the general term.

$$a_n = \frac{n}{}$$

The denominator of the general term isn't quite so obvious. You want to check first to see if there's a value you can add to n, a value you can subtract from n, or a value you can subtract n from. But in this case, the difference between n and each of the denominators is different.

$$n = 1 \qquad 1 = 1 + 0 = n$$

$$n = 2 \qquad 3 = 2 + 1 = n + 1$$

$$n = 3 \qquad 5 = 3 + 2 = n + 2$$

$$n = 4 \qquad 7 = 4 + 3 = n + 3$$

$$n = 5 \qquad 9 = 5 + 4 = n + 4$$

Since that doesn't work, we'll look to see if we can multiply n or divide n by some value that will give us the denominators from our expanded sequence. Unfortunately, that won't work either. However, if we look at $2n$, we can see a consistent difference between $2n$ and our denominators.

$$n = 1 \qquad 1 = 2(1) - 1 = 2n - 1$$

$$n = 2 \qquad 3 = 2(2) - 1 = 2n - 1$$

$$n = 3 \qquad 5 = 2(3) - 1 = 2n - 1$$

$$n = 4 \qquad 7 = 2(4) - 1 = 2n - 1$$

$$n = 5 \qquad 9 = 2(5) - 1 = 2n - 1$$

Since every denominator can be represented by $2n - 1$, we'll put this into the formula for the general term, along with the numerator we already found, and we'll get the formula for the general term:

$$a_n = \frac{n}{2n-1}$$

Before we're done though, we need to analyze our alternating negative signs. When a negative sign alternates in a sequence and the first term is negative, the sequence must be multiplied by $(-1)^n$. Once we add this multiplier to the general term, we'll have the full formula:

$$a_n = (-1)^n \frac{n}{2n-1}$$

We can always double-check ourselves by testing the general term at $n = 1, 2, 3, 4, 5, \ ...$. If the formula for the general term returns the terms we were given in the original expanded sequence, then we know that our general term accurately represents the sequence.

Another thing to note: If you encounter this type of problem in a multiple choice test, keep in mind that you can always just plug $n = 1, 2, 3, 4, 5, \ ...$ into each of the answer choices, to see if the values you get match the original expanded sequence.

Topic: Formula for the general term

Question: Find a formula for the general term of the sequence.

$$\left\{\frac{1}{2},\frac{4}{2},\frac{9}{2},\frac{16}{2}\right\}$$

Answer choices:

A $\quad a_n=\dfrac{n^2}{n+1}$

B $\quad a_n=\dfrac{n^2+2}{4}$

C $\quad a_n=\dfrac{n^2}{2}$

D $\quad a_n=\dfrac{n+2}{2}$

Solution: C

The general term of a sequence is a single term that can represent every term in the sequence, based on the value of n that we pick. In other words, if the general term of a sequence is $1/n$, and the sequence starts at $n = 1$, then we start plugging $n = 1$, $n = 2$, $n = 3$, $n = 4$, etc. into the general term, and we get the expanded sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ \dots$$

Oftentimes we're given the expanded sequence and asked to find the general term that represents it, so it's like we're working backwards.

The easiest way to find the general term is to look at each part of our sequence, and find its relationship to the corresponding n value.

The first thing we want to do is match each term in our expanded sequence with its n value. For the sequence we've been given in this problem, we'll get

$$\begin{array}{cccc} n = 1 & n = 2 & n = 3 & n = 4 \\ \\ \dfrac{1}{2} & \dfrac{4}{2} & \dfrac{9}{2} & \dfrac{16}{2} \end{array}$$

We can start by noticing that the denominator of each term, regardless of its n value, is equal to 2. So we'll put a 2 in the denominator of the formula for the general term.

$$a_n = \frac{}{2}$$

Turning to the numerator, we can see that each numerator is the square of its n value.

$$n = 1 \qquad 1 = 1^2 = n^2$$

$$n = 2 \qquad 4 = 2^2 = n^2$$

$$n = 3 \qquad 9 = 3^3 = n^2$$

$$n = 4 \qquad 16 = 4^2 = n^2$$

Since every numerator can be represented by n^2, we'll put this into the formula for the general term, along with the denominator we already found, and we'll get the formula for the general term:

$$a_n = \frac{n^2}{2}$$

We can always double-check ourselves by testing the general term at $n = 1, 2, 3, 4, 5, \ \ldots$. If the formula for the general term returns the terms we were given in the original expanded sequence, then we know that our general term accurately represents the sequence.

Another thing to note: If you encounter this type of problem in a multiple choice test, keep in mind that you can always just plug $n = 1, 2, 3, 4, 5, \ \ldots$ into each of the answer choices, to see if the values you get match the original expanded sequence.

Topic: Convergence of a sequence

Question: If the sequence converges, find its limit.

$$a_n = \frac{6}{n}$$

Answer choices:

A The sequence diverges

B The sequence converges and the limit is 0

C The sequence converges and the limit is 6

D The sequence converges and the limit is 12

Solution: B

We determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$.

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Taking the limit of the sequence we've been given, we get

$$\lim_{n\to\infty} \frac{6}{n} = \frac{6}{\infty}$$

$$\lim_{n\to\infty} \frac{6}{n} = 0$$

The limit of the sequence is 0, which means the limit exists and is finite. Therefore, we can say that the sequence converges.

Topic: Convergence of a sequence

Question: If the sequence converges, find its limit.

$$a_n = \frac{n^2}{2}$$

Answer choices:

A The sequence diverges

B The sequence converges and the limit is 0

C The sequence converges and the limit is $1/2$

D The sequence converges and the limit is 1

Solution: A

We determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$.

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Taking the limit of the sequence we've been given, we get

$$\lim_{n \to \infty} \frac{n^2}{2} = \frac{\infty}{2}$$

$$\lim_{n \to \infty} \frac{n^2}{2} = \infty$$

The limit of the sequence is infinite. Therefore, we can say that the sequence diverges.

Topic: Convergence of a sequence

Question: If the sequence converges, find its limit.

$$a_n = \frac{2n^2 + 4n}{3n^2 + 3}$$

Answer choices:

A The sequence diverges

B The sequence converges and the limit is 0

C The sequence converges and the limit is $2/3$

D The sequence converges and the limit is 1

Solution: C

We determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$.

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Taking the limit of the sequence we've been given, we get

$$\lim_{n\to\infty} \frac{2n^2+4n}{3n^2+3} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we need to back up a step and simplify the function.

$$\lim_{n\to\infty} \frac{2n^2+4n}{3n^2+3}$$

$$\lim_{n\to\infty} \frac{2n^2+4n}{3n^2+3}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{2n^2}{n^2}+\frac{4n}{n^2}}{\frac{3n^2}{n^2}+\frac{3}{n^2}}$$

$$\lim_{n\to\infty} \frac{2+\frac{4}{n}}{3+\frac{3}{n^2}}$$

Evaluating our simplified function as $n \to \infty$, we get

$$\frac{2+\frac{4}{\infty}}{3+\frac{3}{\infty}}$$

$$\frac{2+0}{3+0}$$

$$\frac{2}{3}$$

The limit of the sequence is $2/3$, which means the limit exists and is finite. Therefore, we can say that the sequence converges.

Topic: Limit of a convergent sequence

Question: Find the limit of the convergent sequence.

$$a_n = \frac{2}{n^2}$$

Answer choices:

A $\sqrt{2}$

B 0

C 2

D ∞

Solution: B

We've already been told in the problem that this sequence converges.

We normally determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$, and we know that

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Based on this definition, we should expect a finite answer when we take the limit of our sequence, since we know already that our sequence converges.

$$\lim_{n\to\infty} \frac{2}{n^2} = \frac{2}{\infty}$$

$$\lim_{n\to\infty} \frac{2}{n^2} = 0$$

The limit of the sequence is 0. Therefore, we can say that the sequence converges and that the limit is 0.

Topic: Limit of a convergent sequence

Question: Say whether the sequence converges. If it does, find its limit.

$$a_n = \frac{n^2}{2n^2 - 1}$$

Answer choices:

A -1

B 0

C $\frac{1}{2}$

D ∞

Solution: C

We determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$.

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Taking the limit of the sequence we've been given, we get

$$\lim_{n\to\infty} \frac{n^2}{2n^2-1} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we need to back up a step and simplify the function.

$$\lim_{n\to\infty} \frac{n^2}{2n^2-1}$$

$$\lim_{n\to\infty} \frac{n^2}{2n^2-1}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{2n^2}{n^2}-\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{1}{2-\frac{1}{n^2}}$$

Evaluating our simplified function as $n \to \infty$, we get

$$\frac{1}{2 - \frac{1}{\infty}}$$

$$\frac{1}{2 - 0}$$

$$\frac{1}{2}$$

The limit of the sequence is $1/2$, which means the limit exists and is finite. Therefore, we can say that the sequence converges and that its limit is $1/2$.

Topic: Limit of a convergent sequence

Question: Say whether the sequence converges. If it does, find its limit.

$$a_n = \ln\left(6n^3 - n\right) - \ln\left(2n^3 + 4\right)$$

Answer choices:

A $\ln 3$

B 0

C $-\ln \frac{1}{4}$

D ∞

Solution: A

We determine the convergence or divergence of a sequence by taking the limit of the sequence as $n \to \infty$.

The sequence **converges** if the limit exists and is finite

The sequence **diverges** if the limit does not exist or is infinite

Taking the limit of the sequence we've been given, we get

$$\lim_{n \to \infty} \ln \frac{6n^3 - n}{2n^3 + 4} = \ln \frac{\infty}{\infty}$$

Since we get an indeterminate form, we need to back up a step and simplify the function.

$$\lim_{n \to \infty} \ln \frac{6n^3 - n}{2n^3 + 4}$$

$$\lim_{n \to \infty} \ln \frac{6n^3 - n}{2n^3 + 4} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right)$$

$$\lim_{n \to \infty} \ln \frac{\frac{6n^3}{n^3} - \frac{n}{n^3}}{\frac{2n^3}{n^3} + \frac{4}{n^3}}$$

$$\lim_{n \to \infty} \ln \frac{6 - \frac{1}{n^2}}{2 + \frac{4}{n^3}}$$

Evaluating our simplified function as $n \to \infty$, we get

$$\ln \frac{6 - \frac{1}{\infty}}{2 + \frac{4}{\infty}}$$

$$\ln \frac{6 + 0}{2 + 0}$$

$$\ln 3$$

The limit of the sequence is $\ln 3$, which means the limit exists and is finite. Therefore, we can say that the sequence converges and that its limit is $\ln 3$.

Topic: Increasing, decreasing, not monotonic

Question: Say whether the sequence is increasing, decreasing, or not monotonic.

$$a_n = \frac{3}{n^2 + 1}$$

Answer choices:

A The sequence is increasing

B The sequence is not monotonic

C The sequence is decreasing and monotonic

D The sequence is increasing and not monotonic

Solution: C

Monotonic sequences are those which head in the same direction throughout the entire sequence. In other words, they increase everywhere, or they decrease everywhere. If a sequence increases in some places and decreases in other places, then it's not monotonic.

To get an idea about what's happening with our sequence, we'll calculate its first few terms.

$$n = 1 \qquad a_1 = \frac{3}{(1)^2 + 1} = \frac{3}{2}$$

$$n = 2 \qquad a_2 = \frac{3}{(2)^2 + 1} = \frac{3}{5}$$

$$n = 3 \qquad a_3 = \frac{3}{(3)^2 + 1} = \frac{3}{10}$$

$$n = 4 \qquad a_4 = \frac{3}{(4)^2 + 1} = \frac{3}{17}$$

If we look at the first few terms, we can see that the terms of the sequence are getting smaller as n gets larger, which means the sequence is decreasing, and therefore it's also monotonic.

Topic: Increasing, decreasing, not monotonic

Question: Say whether the sequence is increasing, decreasing, or not monotonic.

$$a_n = \frac{2n^2 - 1}{n + 4}$$

Answer choices:

A The sequence is decreasing

B The sequence is not monotonic

C The sequence is decreasing and not monotonic

D The sequence is increasing and monotonic

Solution: D

Monotonic sequences are those which head in the same direction throughout the entire sequence. In other words, they increase everywhere, or they decrease everywhere. If a sequence increases in some places and decreases in other places, then it's not monotonic.

To get an idea about what's happening with our sequence, we'll calculate its first few terms.

$$n = 1 \qquad a_1 = \frac{2(1)^2 - 1}{1 + 4} = \frac{1}{5}$$

$$n = 2 \qquad a_2 = \frac{2(2)^2 - 1}{2 + 4} = \frac{7}{6}$$

$$n = 3 \qquad a_3 = \frac{2(3)^2 - 1}{3 + 4} = \frac{17}{7}$$

$$n = 4 \qquad a_4 = \frac{2(4)^2 - 1}{4 + 4} = \frac{31}{8}$$

If we look at the first few terms, we can see that the terms of the sequence are getting larger as n gets larger, which means the sequence is increasing, and therefore it's also monotonic.

Topic: Increasing, decreasing, not monotonic

Question: Say whether the sequence is increasing, decreasing, or not monotonic.

$$a_n = (-1)^{n+1}\frac{2}{n+2}$$

Answer choices:

A The sequence is decreasing

B The sequence is not monotonic

C The sequence is decreasing and not monotonic

D The sequence is increasing and monotonic

Solution: B

Monotonic sequences are those which head in the same direction throughout the entire sequence. In other words, they increase everywhere, or they decrease everywhere. If a sequence increases in some places and decreases in other places, then it's not monotonic.

To get an idea about what's happening with our sequence, we'll calculate its first few terms.

$$n = 1 \qquad a_1 = (-1)^{1+1}\frac{2}{1+2} = \frac{2}{3}$$

$$n = 2 \qquad a_2 = (-1)^{2+1}\frac{2}{2+2} = -\frac{2}{4} = -\frac{1}{2}$$

$$n = 3 \qquad a_3 = (-1)^{3+1}\frac{2}{3+2} = \frac{2}{5}$$

$$n = 4 \qquad a_4 = (-1)^{4+1}\frac{2}{4+2} = -\frac{2}{6} = -\frac{1}{3}$$

If we look at the first few terms, we can see that the terms of the sequence are neither consistently increasing or consistently decreasing as n gets larger, which means the sequence is not monotonic.

Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = \frac{n+2}{n^2}$$

Answer choices:

A The sequence is bounded below at 0 and not bounded above.

B The sequence is bounded below at 0 and bounded above at 3.

C The sequence is not bounded.

D The sequence is only bounded above at 3 and not bounded below.

Solution: B

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = \frac{1+2}{1^2} = 3$$

$$n = 2 \qquad a_2 = \frac{2+2}{2^2} = 1$$

$$n = 3 \qquad a_3 = \frac{3+2}{3^2} = \frac{5}{9}$$

$$n = 4 \qquad a_4 = \frac{4+2}{4^2} = \frac{3}{8}$$

We can see that the sequence is decreasing, which means it's also monotonic. Knowing that it's a decreasing monotonic sequence, we know that the sequence is bounded above by its first term, $a_1 = 3$.

To figure out whether or not the sequence is bounded below at its end, we'll take the limit of the sequence as $n \to \infty$.

> If the answer is finite, then the sequence is bounded below by that value.
>
> If the answer is infinite or doesn't exist, then the sequence isn't bounded below.

Evaluating the limit of the sequence as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{n+2}{n^2} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we'll back up a step and simplify the function.

$$\lim_{n\to\infty} \frac{n+2}{n^2}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}}{1}$$

$$\frac{\frac{1}{\infty}+\frac{2}{\infty^2}}{1}$$

$$\frac{0+0}{1}$$

$$0$$

Since this is a finite answer, we can say that the sequence is bounded below by 0.

Pulling our conclusions together, we'll say that the given sequence is

decreasing and monotonic, and

bounded above by $a_n \leq 3$

bounded below by $a_n > 0$

Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = \frac{e^n}{n}$$

Answer choices:

A The sequence is bounded below at e and not bounded above.

B The sequence is bounded below at e and bounded above at ∞.

C The sequence is not bounded.

D The sequence is only bounded above at ∞ and not bounded below.

Solution: A

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = \frac{e^1}{1} = e$$

$$n = 2 \qquad a_2 = \frac{e^2}{2}$$

$$n = 3 \qquad a_3 = \frac{e^3}{3}$$

$$n = 4 \qquad a_4 = \frac{e^4}{4}$$

We can see that the sequence is increasing, which means it's also monotonic. Knowing that it's an increasing monotonic sequence, we know that the sequence is bounded below by its first term, $a_1 = e$.

To figure out whether or not the sequence is bounded above at its end, we'll take the limit of the sequence as $n \to \infty$.

> If the answer is finite, then the sequence is bounded above by that value.
>
> If the answer is infinite or doesn't exist, then the sequence isn't bounded above.

Evaluating the limit of the sequence as $n \to \infty$, we get

$$\lim_{n \to \infty} \frac{e^n}{n} = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we'll back up a step and use L'Hospital's rule simplify the function by replacing the numerator and denominator with their derivatives.

$$\lim_{n\to\infty}\frac{e^n}{n}$$

$$\lim_{n\to\infty}\frac{e^n}{1}$$

$$e^\infty$$

$$\infty$$

Since this is an infinite answer, we can say that the sequence isn't bounded above.

Pulling our conclusions together, we'll say that the given sequence is

increasing and monotonic, and

bounded below by $a_n \geq e$

not bounded above

Topic: Bounded sequences

Question: Describe how the sequence is bounded.

$$a_n = (-1)^n \frac{3}{n^2}$$

Answer choices:

A The sequence is bounded below at -3 and not bounded above.

B The sequence is bounded below at -3 and bounded above at 0.

C The sequence is not bounded.

D The sequence is only bounded above at 0 and not bounded below.

Solution: C

Only monotonic sequences can be bounded, because bounded sequences must be either increasing or decreasing, and monotonic sequences are sequences that are always increasing or always decreasing. Bounded sequences can be

bounded above by the largest value of the sequence

bounded below by the smallest value of the sequence

bounded both above and below

The smallest value of an increasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \geq a_1$, so we know that **increasing monotonic sequences are bounded below**.

The largest value of a decreasing monotonic sequence will be its first term, where $n = 1$. In this case, $a_n \leq a_1$, so we know that **decreasing monotonic sequences are bounded above**.

To determine if the end of the monotonic sequence is bounded, we'll need to take the limit of the sequence as $n \to \infty$. If we obtain a real-number answer for the limit, then the sequence is bounded at the end as well as at the beginning.

Because we're using the limit as $n \to \infty$ to solve for any possible end of sequence bounding, our end bounds will be in the form $a_n < a_\infty$ for an increasing sequence and $a_n > a_\infty$ for a decreasing sequence if end bounds exist.

Before we can say whether or not the given sequence is bounded, we need to first prove that it's monotonic, which we'll do by expanding the sequence through the first few terms.

$$n = 1 \qquad a_1 = (-1)^1\frac{3}{1^2} = -3$$

$$n = 2 \qquad a_2 = (-1)^2\frac{3}{2^2} = \frac{3}{4}$$

$$n = 3 \qquad a_3 = (-1)^3\frac{3}{3^2} = -\frac{1}{3}$$

$$n = 4 \qquad a_4 = (-1)^4\frac{3}{4^2} = \frac{3}{16}$$

We can see that the sequence is not consistently increasing or consistently decreasing, which means it's not monotonic.

Since the sequence is not monotonic, we can't say that it's bounded above or below.

Topic: Calculating the first terms of a sequence of partial sums

Question: Approximate to three decimal places the first four terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$$

Answer choices:

A $1.000,\ 1.800,\ 1.600,\ 1.471$

B $1.000,\ 0.200,\ -0.400,\ -0.871$

C $1.000,\ 1.800,\ 2.400,\ 2.871$

D $1.000,\ 0.800,\ 0.600,\ 0.471$

Solution: C

If we're given a series a_n, and we find the first few terms of a_n, the terms will be

$$a_1$$

$$a_2$$

$$a_3$$

$$a_4$$

...

If instead we want to calculate the first few terms of the sequence of partial sums s_n, we simply add our terms together, such that

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

...

To calculate the first four terms of the sequence of partial sums, we'll calculate each term of the series a_n individually, then add that term to any previous terms to get the term of the partial sum sequence.

$$n = 1 \qquad a_1 = \frac{2(1)}{(1)^2 + 1} = \frac{2}{2} = 1.000 \qquad s_1 = a_1$$

$$s_1 = 1.000$$

$$n = 2 \qquad a_2 = \frac{2(2)}{(2)^2 + 1} = \frac{4}{5} = 0.800 \qquad s_2 = a_1 + a_2$$

$$s_2 = 1.000 + 0.800$$

$$s_2 = 1.800$$

$$n = 3 \qquad a_3 = \frac{2(3)}{(3)^2 + 1} = \frac{6}{10} = 0.600 \qquad s_3 = a_1 + a_2 + a_3$$

$$s_3 = 1.000 + 0.800 + 0.600$$

$$s_3 = 2.400$$

$$n = 4 \qquad a_4 = \frac{2(4)}{(4)^2 + 1} = \frac{8}{17} = 0.471 \qquad s_4 = a_1 + a_2 + a_3 + a_4$$

$$s_4 = 1.000 + 0.800 + 0.600 + 0.471$$

$$s_4 = 2.871$$

The first four terms of the sequence of partial sums, approximated to three decimal places, are

$$1.000, \ 1.800, \ 2.400, \ 2.871$$

Topic: Calculating the first terms of a sequence of partial sums

Question: Approximate to three decimal places the first four terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{3-n}{n^3}$$

Answer choices:

A $2.000,\ 2.125,\ 2.125,\ 2.016$

B $2.000,\ 2.125,\ 2.000,\ 1.984$

C $2.000,\ 0.125,\ 0.000,\ -0.016$

D $2.000,\ 2.125,\ 2.125,\ 2.109$

Solution: D

If we're given a series a_n, and we find the first few terms of a_n, the terms will be

$$a_1$$

$$a_2$$

$$a_3$$

$$a_4$$

$$\dots$$

If instead we want to calculate the first few terms of the sequence of partial sums s_n, we simply add our terms together, such that

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$\dots$$

To calculate the first four terms of the sequence of partial sums, we'll calculate each term of the series a_n individually, then add that term to any previous terms to get the term of the partial sum sequence.

$$n = 1 \qquad a_1 = \frac{3-1}{(1)^3} = \frac{2}{1} = 2.000 \qquad s_1 = a_1$$

$$s_1 = 2.000$$

$$n = 2 \qquad a_2 = \frac{3-2}{(2)^3} = \frac{1}{8} = 0.125 \qquad s_2 = a_1 + a_2$$

$$s_2 = 2.000 + 0.125$$

$$s_2 = 2.125$$

$$n = 3 \qquad a_3 = \frac{3-3}{(3)^3} = \frac{0}{27} = 0.000 \qquad s_3 = a_1 + a_2 + a_3$$

$$s_3 = 2.000 + 0.125 + 0.000$$

$$s_3 = 2.125$$

$$n = 4 \qquad a_4 = \frac{3-4}{(4)^3} = \frac{-1}{64} = -0.016 \qquad s_4 = a_1 + a_2 + a_3 + a_4$$

$$s_4 = 2.000 + 0.125 + 0.000 - 0.016$$

$$s_4 = 2.109$$

The first four terms of the sequence of partial sums, approximated to three decimal places, are

$$2.000,\ 2.125,\ 2.125,\ 2.109$$

Topic: Calculating the first terms of a sequence of partial sums

Question: Approximate to three decimal places the first four terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

Answer choices:

A $\quad 0.368,\ 0.271,\ 0.149,\ 0.073$

B $\quad 0.368,\ 0.639,\ 0.788,\ 0.873$

C $\quad 0.368,\ 0.639,\ 0.517,\ 0.441$

D $\quad 0.368,\ 0.639,\ 0.788,\ 0.861$

Solution: D

If we're given a series a_n, and we find the first few terms of a_n, the terms will be

$$a_1$$

$$a_2$$

$$a_3$$

$$a_4$$

$$...$$

If instead we want to calculate the first few terms of the series of partial sums s_n, we simply add our terms together, such that

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$...$$

To calculate the first four terms of the sequence of partial sums, we'll calculate each term of the series a_n individually, then add that term to any previous terms to get the term of the partial sum sequence.

$n=1$ $\quad a_1=\frac{1}{e^1}=\frac{1}{e}=0.368$ $\quad s_1=a_1$

$s_1=0.368$

$n=2$ $\quad a_2=\frac{2}{e^2}=0.271$ $\quad s_2=a_1+a_2$

$s_2=0.368+0.271$

$s_2=0.639$

$n=3$ $\quad a_3=\frac{3}{e^3}=0.149$ $\quad s_3=a_1+a_2+a_3$

$s_3=0.368+0.271+0.149$

$s_3=0.788$

$n=4$ $\quad a_4=\frac{4}{e^4}=0.073$ $\quad s_4=a_1+a_2+a_3+a_4$

$s_4=0.368+0.271+0.149+0.073$

$s_4=0.861$

The first four terms of the sequence of partial sums, approximated to three decimal places, are

$$0.368,\ 0.639,\ 0.788,\ 0.861$$

Topic: Sum of the sequence of partial sums

Question: Use the partial sums equation to find the sum of the series.

$$s_n = 4 + \frac{8}{n}$$

Answer choices:

A 2

B 0

C 8

D 4

Solution: D

Given a series a_n, we can find its partial sum s_n using the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

But when we're given the sequence of partial sums s_n, we can use the same formula to go backwards and get the sum of the series a_n.

Plugging in the partial sum sequence s_n, we get

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} 4 + \frac{8}{n}$$

$$\sum_{n=1}^{\infty} a_n = 4 + \frac{8}{\infty}$$

$$\sum_{n=1}^{\infty} a_n = 4 + 0$$

$$\sum_{n=1}^{\infty} a_n = 4$$

The sum of the series a_n is 4.

Topic: Sum of the sequence of partial sums

Question: Use the partial sums equation to find the sum of the series.

$$s_n = \frac{6n^2 + 4n}{n^2 - 9}$$

Answer choices:

A $\frac{5}{2}$

B $\frac{4}{9}$

C 6

D 0

Solution: C

Given a series a_n, we can find its partial sum s_n using the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

But when we're given the sequence of partial sums s_n, we can use the same formula to go backwards and get the sum of the series a_n.

Plugging in the partial sum sequence s_n, we get

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{6n^2+4n}{n^2-9}$$

$$\sum_{n=1}^{\infty} a_n = \frac{\infty}{\infty}$$

Since we get an indeterminate form, we'll go back a step and manipulate our function.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{6n^2+4n}{n^2-9}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{\frac{6n^2}{n^2}+\frac{4n}{n^2}}{\frac{n^2}{n^2}-\frac{9}{n^2}}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{6+\frac{4}{n}}{1-\frac{9}{n^2}}$$

$$\sum_{n=1}^{\infty} a_n = \frac{6+\frac{4}{\infty}}{1-\frac{9}{\infty}}$$

$$\sum_{n=1}^{\infty} a_n = \frac{6+0}{1-0}$$

$$\sum_{n=1}^{\infty} a_n = 6$$

The sum of the series a_n is 6.

Topic: Sum of the sequence of partial sums

Question: Use the partial sums equation to find the sum of the series.

$$s_n = 5 + \frac{4}{e^n}$$

Answer choices:

A 5

B 4

C e

D $\frac{1}{e}$

Solution: A

Given a series a_n, we can find its partial sum s_n using the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

But when we're given the sequence of partial sums s_n, we can use the same formula to go backwards and get the sum of the series a_n.

Plugging in the partial sum sequence s_n, we get

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} 5 + \frac{4}{e^n}$$

$$\sum_{n=1}^{\infty} a_n = 5 + \frac{4}{\infty}$$

$$\sum_{n=1}^{\infty} a_n = 5 + 0$$

$$\sum_{n=1}^{\infty} a_n = 5$$

The sum of the series a_n is 5.

Topic: Geometric series test

Question: Use the geometric series test to say whether the geometric series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1^{n-1}}{2^n}$$

Answer choices:

A The series is convergent and $r = \frac{1}{4}$.

B The series is divergent and $r = 1$.

C The series is convergent and $r = \frac{1}{2}$.

D The series is divergent and $r = 2$.

Solution: C

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n = 0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{1^{n-1}}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{1^n 1^{-1}}{2^n}$$

$$\sum_{n=0}^{\infty} 1^{-1} \cdot \frac{1^n}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{1}\left(\frac{1}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

Alternatively, instead of manipulating the series directly into standard form, we could have written out the first few terms,

$$\sum_{n=0}^{\infty} \frac{1^{n-1}}{2^n}$$

$$\frac{1^{0-1}}{2^0}+\frac{1^{1-1}}{2^1}+\frac{1^{2-1}}{2^2}+\frac{1^{3-1}}{2^3}+\frac{1^{4-1}}{2^4}+\dots$$

$$\frac{1^{-1}}{1}+\frac{1^0}{2}+\frac{1^1}{4}+\frac{1^2}{8}+\frac{1^3}{16}+\dots$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$$

$$1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\dots$$

Either way, we can identify $a=1$ and $r=1/2$. We'll use the geometric series test to determine whether this geometric series converges or diverges. Since

$$\left|\frac{1}{2}\right|=\frac{1}{2}<1$$

we can say that $|r|<1$ and therefore that the series converges.

Topic: Geometric series test

Question: Use the geometric series test to say whether the geometric series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{8^{n-1}}{2^n}$$

Answer choices:

A The series is convergent and $r = \frac{1}{2}$.

B The series is divergent and $r = 2$.

C The series is convergent and $r = \frac{1}{8}$.

D The series is divergent and $r = 4$.

Solution: D

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n = 0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{8^{n-1}}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{8^n 8^{-1}}{2^n}$$

$$\sum_{n=0}^{\infty} 8^{-1} \cdot \frac{8^n}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{8} \left(\frac{8}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{1}{8}(4^n)$$

Alternatively, instead of manipulating the series directly into standard form, we could have written out the first few terms,

$$\sum_{n=0}^{\infty} \frac{8^{n-1}}{2^n}$$

$$\frac{8^{0-1}}{2^0} + \frac{8^{1-1}}{2^1} + \frac{8^{2-1}}{2^2} + \frac{8^{3-1}}{2^3} + \frac{8^{4-1}}{2^4} + \dots$$

$$\frac{8^{-1}}{1} + \frac{8^0}{2} + \frac{8^1}{4} + \frac{8^2}{8} + \frac{8^3}{16} + \dots$$

$$\frac{1}{8} + \frac{1}{2} + \frac{8}{4} + \frac{64}{8} + \frac{512}{16} + \dots$$

$$\frac{1}{8}(1 + 4 + 16 + 64 + 256 + \dots)$$

Either way, we can identify $a = 1/8$ and $r = 4$. We'll use the geometric series test to determine whether this geometric series converges or diverges. Since

$$|4| = 4 \geq 1$$

we can say that $|r| \geq 1$ and therefore that the series diverges.

Topic: Geometric series test

Question: Use the geometric series test to say whether the geometric series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{3^n}$$

Answer choices:

A The series is convergent and $r = -\frac{1}{3}$.

B The series is divergent and $r = -1$.

C The series is convergent and $r = \frac{1}{3}$.

D The series is divergent and $r = 1$.

Solution: A

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n = 0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{-1}}{3^n}$$

$$\sum_{n=0}^{\infty} (-1)^{-1} \cdot \frac{(-1)^n}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{(-1)} \left(\frac{-1}{3}\right)^n$$

$$\sum_{n=0}^{\infty} -\left(-\frac{1}{3}\right)^n$$

Alternatively, instead of manipulating the series directly into standard form, we could have written out the first few terms,

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{3^n}$$

$$\frac{(-1)^{0-1}}{3^0}+\frac{(-1)^{1-1}}{3^1}+\frac{(-1)^{2-1}}{3^2}+\frac{(-1)^{3-1}}{3^3}+\frac{(-1)^{4-1}}{3^4}+\dots$$

$$\frac{(-1)^{-1}}{1}+\frac{(-1)^{0}}{3}+\frac{(-1)^{1}}{9}+\frac{(-1)^{2}}{27}+\frac{(-1)^{3}}{81}+\dots$$

$$-1+\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\dots$$

$$-1\left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\dots\right)$$

Either way, we can identify $a=-1$ and $r=-1/3$. We'll use the geometric series test to determine whether this geometric series converges or diverges. Since

$$\left|-\frac{1}{3}\right|=\frac{1}{3}<1$$

we can say that $|r|<1$ and therefore that the series converges.

Topic: Sum of the geometric series

Question: Find the sum of the geometric series.

$$\sum_{n=0}^{\infty} \frac{3}{4^n}$$

Answer choices:

A 4

B 3

C 2

D 1

Solution: A

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n = 0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{3}{4^n}$$

$$\sum_{n=0}^{\infty} 3\frac{1}{4^n}$$

$$\sum_{n=0}^{\infty} 3\frac{1^n}{4^n}$$

$$\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$$

Comparing this to the standard form, we'll say that

$$a = 3$$

and

$$r = \frac{1}{4}$$

Since

$$\left|\frac{1}{4}\right| = \frac{1}{4} < 1$$

the series converges by the geometric series test for convergence, which means we can find the sum of the series using

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{3}{1-\frac{1}{4}}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{3}{\frac{4}{4}-\frac{1}{4}}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{3}{\frac{3}{4}}$$

$$\sum_{n=0}^{\infty} ar^n = 3 \cdot \frac{4}{3}$$

$$\sum_{n=0}^{\infty} ar^n = 4$$

The sum of the series is 4.

Topic: Sum of the geometric series

Question: Find the sum of the geometric series.

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}$$

Answer choices:

A 2

B 1

C 4

D $\frac{1}{2}$

Solution: C

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n = 1$, standard form is

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Comparing our series to this standard form, we'll say that

$$a = 2$$

and

$$r = \frac{1}{2}$$

Since

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

the series converges by the geometric series test for convergence, which means we can find the sum of the series using

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{2}{1-\frac{1}{2}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{2}{\frac{2}{2} - \frac{1}{2}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{2}{\frac{1}{2}}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = 2 \cdot \frac{2}{1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = 4$$

The sum of the series is 4.

Topic: Sum of the geometric series

Question: Find the sum of the geometric series.

$$\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n}$$

Answer choices:

A $-\frac{1}{2}$

B 1

C $\frac{1}{2}$

D The sum can't be found because the series diverges.

Solution: D

We need to get the series into standard form for a geometric series to make sure the series is geometric. Since the index starts at $n=0$, standard form is

$$\sum_{n=0}^{\infty} ar^n$$

so we'll rewrite the series as

$$\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{3^n 3^{-1}}{2^n}$$

$$\sum_{n=0}^{\infty} 3^{-1} \cdot \frac{3^n}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{3}{2}\right)^n$$

Comparing this to the standard form, we'll say that

$$a = \frac{1}{3}$$

and

$$r = \frac{3}{2}$$

Since

$$\left|\frac{3}{2}\right| = \frac{3}{2} \geq 1$$

the series diverges by the geometric series test for convergence, which means we can't find the sum.

Topic: Values for which the series converges

Question: Find the values for which the geometric series converges.

$$\sum_{n=1}^{\infty} 3x^n$$

Answer choices:

A $\quad -1 < x < 1$

B $\quad -\frac{1}{3} < x < \frac{1}{3}$

C $\quad -3 < x < 3$

D $\quad -\sqrt{3} < x < \sqrt{3}$

Solution: A

From the expanded form of a geometric series,

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

we can use the value of r and the geometric series test for convergence to determine the interval over which the geometric series converges.

The geometric series test says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Therefore, in order to find the values for which the geometric series converges, we just expand the series to identify the value of r and then use it in the geometric series test.

We'll start by expanding the series, calculating its first few terms.

$n = 1 \qquad a_1 = 3x^1 = 3x$

$n = 2 \qquad a_2 = 3x^2$

$n = 3 \qquad a_3 = 3x^3$

$n = 4 \qquad a_4 = 3x^4$

Writing these terms into our expanded series, we get

$$\sum_{n=1}^{\infty} 3x^n = 3x + 3x^2 + 3x^3 + 3x^4 + \dots$$

The first term in a geometric series is always 1, which means this series is only geometric if we can factor out $3x$.

$$\sum_{n=1}^{\infty} 3x^n = 3x\left(1 + x + x^2 + x^3 + \dots\right)$$

Comparing this to the expanded form of the general geometric series, we can see that

$$a = 3x$$

$$r = x$$

Since the geometric series test tells us that the series converges when $|r| < 1$, we plug the value we found for r into this inequality, and we get

$$|x| < 1$$

$$-1 < x < 1$$

The series converges on the interval $-1 < x < 1$.

Topic: Values for which the series converges

Question: Find the values for which the geometric series converges.

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n}$$

Answer choices:

A $-1 < x < 1$

B $-\frac{1}{2} < x < \frac{1}{2}$

C $-2 < x < 2$

D $-\sqrt{2} < x < \sqrt{2}$

Solution: C

From the expanded form of a geometric series,

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

we can use the value of r and the geometric series test for convergence to determine the interval over which the geometric series converges.

The geometric series test says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Therefore, in order to find the values for which the geometric series converges, we just expand the series to identify the value of r and then use it in the geometric series test.

We'll start by expanding the series, calculating its first few terms.

$$n = 1 \qquad a_1 = \frac{x^{1-1}}{2^1} = \frac{1}{2}$$

$$n = 2 \qquad a_2 = \frac{x^{2-1}}{2^2} = \frac{x}{4}$$

$$n = 3 \qquad a_3 = \frac{x^{3-1}}{2^3} = \frac{x^2}{8}$$

$$n = 4 \qquad a_4 = \frac{x^{4-1}}{2^4} = \frac{x^3}{16}$$

Writing these terms into our expanded series, we get

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

The first term in a geometric series is always 1, which means this series is only geometric if we can factor out $1/2$.

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \dots\right)$$

Comparing this to the expanded form of the general geometric series, we can see that

$$a = \frac{1}{2}$$

$$r = \frac{x}{2}$$

Since the geometric series test tells us that the series converges when $|r| < 1$, we plug the value we found for r into this inequality, and we get

$$\left|\frac{x}{2}\right| < 1$$

$$-1 < \frac{x}{2} < 1$$

$$-2 < x < 2$$

The series converges on the interval $-2 < x < 2$.

Topic: Values for which the series converges

Question: Find the values for which the geometric series converges.

$$\sum_{n=1}^{\infty} \frac{(5x)^{n-1}}{3^n}$$

Answer choices:

A $-\frac{3}{25} < x < \frac{3}{25}$

B $-\frac{5}{3} < x < \frac{5}{3}$

C $-\frac{25}{3} < x < \frac{25}{3}$

D $-\frac{3}{5} < x < \frac{3}{5}$

Solution: D

From the expanded form of a geometric series,

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

we can use the value of r and the geometric series test for convergence to determine the interval over which the geometric series converges.

The geometric series test says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

Therefore, in order to find the values for which the geometric series converges, we just expand the series to identify the value of r and then use it in the geometric series test.

We'll start by expanding the series, calculating its first few terms.

$$n = 1 \qquad a_1 = \frac{(5x)^{1-1}}{3^1} = \frac{1}{3}$$

$$n = 2 \qquad a_2 = \frac{(5x)^{2-1}}{3^2} = \frac{5x}{9}$$

$$n = 3 \qquad a_3 = \frac{(5x)^{3-1}}{3^3} = \frac{25x^2}{27}$$

$$n = 4 \qquad a_4 = \frac{(5x)^{4-1}}{3^4} = \frac{125x^3}{81}$$

Writing these terms into our expanded series, we get

$$\sum_{n=1}^{\infty} \frac{(5x)^{n-1}}{3^n} = \frac{1}{3} + \frac{5x}{9} + \frac{25x^2}{27} + \frac{125x^3}{81} + \dots$$

The first term in a geometric series is always 1, which means this series is only geometric if we can factor out $1/3$.

$$\sum_{n=1}^{\infty} \frac{(5x)^{n-1}}{3^n} = \frac{1}{3}\left(1 + \frac{5x}{3} + \frac{25x^2}{9} + \frac{125x^3}{27} + \dots\right)$$

Comparing this to the expanded form of the general geometric series, we can see that

$$a = \frac{1}{3}$$

$$r = \frac{5x}{3}$$

Since the geometric series test tells us that the series converges when $|r| < 1$, we plug the value we found for r into this inequality, and we get

$$\left|\frac{5x}{3}\right| < 1$$

$$-1 < \frac{5x}{3} < 1$$

$$-3 < 5x < 3$$

$$-\frac{3}{5} < x < \frac{3}{5}$$

The series converges on the interval $-\frac{3}{5} < x < \frac{3}{5}$.

Topic: Geometric series for repeating decimals

Question: Express the repeating decimal as a ratio of integers.

$$0.\overline{013}$$

Answer choices:

A $\dfrac{13}{1{,}000}$

B $\dfrac{13}{998}$

C $\dfrac{13}{999}$

D $\dfrac{13}{1{,}001}$

Solution: C

We can use the formula for the sum of a geometric series to quickly and accurately convert a repeating decimal into a ratio of integers, in other words, into a fraction with whole numbers in the numerator and denominator.

We'll follow these steps:

1. Separate the non-repeating part from the repeating part of the decimal.

2. In a table, match each repeated part with its last decimal place.

3. Create a sum of each part, dividing the repeated parts by their ending decimal places.

4. Identify the geometric series within the sum and set values for a and r from the formula for the sum of a geometric series.

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

5. Use the formula to convert the series into one fraction.

6. Add this geometric series fraction to the fraction for the non-repeated part.

In this problem, the bar over the $.013$ indicates that this is the portion of the decimal that repeats. This tells us that the decimal looks like

$$0.013013013013013...$$

We'll separate each part of the repeated sequence into its own row of the table below, replacing the decimal places before it with 0s. Once we've built out the left column, we'll put the corresponding place in the second column.

013	$.013$	ends at the $1{,}000$s place
013	$.000013$	ends at the $1{,}000{,}000$s place
013	$.000000013$	ends at the $1{,}000{,}000{,}000$s place

We'll create a sum of each of the repeated parts, dividing each repeated part by its ending decimal place.

$$0.\overline{013} = \frac{013}{1{,}000} + \frac{013}{1{,}000{,}000} + \frac{013}{1{,}000{,}000{,}000} + \dots$$

$$0.\overline{013} = \frac{13}{1{,}000} + \frac{13}{1{,}000{,}000} + \frac{13}{1{,}000{,}000{,}000} + \dots$$

The sum we just created is a geometric series, so we can use the formula for the sum of a geometric series

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

to turn the sum into just one fraction. We'll factor out the first fraction from the repeated part.

$$0.\overline{013} = \frac{13}{1{,}000}\left(1 + \frac{1}{1{,}000} + \frac{1}{1{,}000{,}000} + \dots\right)$$

With our series in this form, we can identify a and r from the formula for the sum of a geometric series. a is always the value we factored that's sitting right in front of the parentheses. r is the always the second term inside the parentheses, the value immediately following the 1.

$$a=\frac{13}{1{,}000}$$

$$r=\frac{1}{1{,}000}$$

Plugging these into the formula for the sum of a geometric series, we get

$$\sum_{n=1}^{\infty}\frac{13}{1{,}000}\left(\frac{1}{1{,}000}\right)^n=\frac{\frac{13}{1{,}000}}{1-\frac{1}{1{,}000}}$$

$$\sum_{n=1}^{\infty}\frac{13}{1{,}000}\left(\frac{1}{1{,}000}\right)^n=\frac{\frac{13}{1{,}000}}{\frac{1{,}000}{1{,}000}-\frac{1}{1{,}000}}$$

$$\sum_{n=1}^{\infty}\frac{13}{1{,}000}\left(\frac{1}{1{,}000}\right)^n=\frac{\frac{13}{1{,}000}}{\frac{999}{1{,}000}}$$

$$\sum_{n=1}^{\infty}\frac{13}{1{,}000}\left(\frac{1}{1{,}000}\right)^n=\frac{13}{1{,}000}\cdot\frac{1{,}000}{999}$$

$$\sum_{n=1}^{\infty}\frac{13}{1{,}000}\left(\frac{1}{1{,}000}\right)^n=\frac{13}{999}$$

If you have access to a calculator, you can always double-check yourself. In this case, just use your calculator to divide 13 by 999. If you did this correctly, you should get the original repeating decimal, $0.\overline{013}$.

Topic: Geometric series for repeating decimals

Question: Express the repeating decimal as a ratio of integers.

$$2.2\overline{16}$$

Answer choices:

A $\dfrac{2{,}179}{990}$

B $\dfrac{1{,}081}{495}$

C $\dfrac{1{,}097}{495}$

D $\dfrac{11}{5}$

Solution: C

We can use the formula for the sum of a geometric series to quickly and accurately convert a repeating decimal into a ratio of integers, in other words, into a fraction with whole numbers in the numerator and denominator.

We'll follow these steps:

1. Separate the non-repeating part from the repeating part of the decimal.

2. In a table, match each repeated part with its last decimal place.

3. Create a sum of each part, dividing the repeated parts by their ending decimal places.

4. Identify the geometric series within the sum and set values for a and r from the formula for the sum of a geometric series.

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

5. Use the formula to convert the series into one fraction.

6. Add this geometric series fraction to the fraction for the non-repeated part.

In this problem, the bar over the $.016$ indicates that this is the portion of the decimal that repeats. This tells us that the decimal looks like

$$2.216161616161616\ldots$$

We've been asked to convert this decimal value into a fraction with a real-number numerator and denominator.

Our first step is to separate the non-repeating part from the repeating part of the decimal.

$$2.2 + .016161616161616...$$

We add a 0 in the tenths place of our repeating part because it's holding the place of the $.2$ we pulled out into the non-repeating part. The repeating sequence starts with the first 1 in the hundredths place, and we need to keep it there when we separate the decimals, so it's critical to put in the 0.

Next, we'll separate each part of the repeated sequence into its own row of the table below, replacing the decimal places before it with 0s. Once we've built out the left column, we'll put the corresponding place in the second column.

2.2		
16	$.016$	ends at the $1{,}000$s place
16	$.00016$	ends at the $100{,}000$s place
16	$.0000016$	ends at the $10{,}000{,}000$s place

We'll create a sum of the non-repeated part and each of the repeated parts, dividing each repeated part by its ending decimal place.

$$2.2\overline{16} = 2.2 + \frac{16}{1{,}000} + \frac{16}{100{,}000} + \frac{16}{10{,}000{,}000} + \ldots$$

The sum we just created is a geometric series, so we can use the formula for the sum of a geometric series

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

to turn the sum into just one fraction. We'll factor out the first fraction from the repeated part.

$$2.2\overline{16} = 2.2 + \frac{16}{1{,}000}\left(1 + \frac{1}{100} + \frac{1}{10{,}000} + \dots\right)$$

With our series in this form, we can identify a and r from the formula for the sum of a geometric series. a is always the value we factored that's sitting right in front of the parentheses. r is the always the second term inside the parentheses, the value immediately following the 1.

$$a = \frac{16}{1{,}000}$$

$$r = \frac{1}{100}$$

Plugging these into the formula for the sum of a geometric series, remembering to keep the non-repeated part of our decimal, 2.2, we get

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1{,}000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{\frac{16}{1{,}000}}{1 - \frac{1}{100}}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1{,}000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{\frac{16}{1{,}000}}{\frac{100}{100} - \frac{1}{100}}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{\frac{16}{1,000}}{\frac{99}{100}}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{16}{1,000} \cdot \frac{100}{99}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{16}{10} \cdot \frac{1}{99}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = 2.2 + \frac{16}{990}$$

Now we just need to change the non-repeated part of our original decimal into a fraction and then combine these two fractions.

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = \left(2 + \frac{2}{10}\right) + \frac{16}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = \left(\frac{20}{10} + \frac{2}{10}\right) + \frac{16}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = \frac{22}{10} + \frac{16}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = \frac{99}{99}\left(\frac{22}{10}\right) + \frac{16}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1,000}\left(\frac{1}{100}\right)^n = \frac{2,178}{990} + \frac{16}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1{,}000} \left(\frac{1}{100} \right)^n = \frac{2{,}194}{990}$$

$$2.2 + \sum_{n=1}^{\infty} \frac{16}{1{,}000} \left(\frac{1}{100} \right)^n = \frac{1{,}097}{495}$$

If you have access to a calculator, you can always double-check yourself. In this case, just use your calculator to divide $1{,}097$ by 495. If you did this correctly, you should get the original repeating decimal, $2.2\overline{16}$.

Topic: Geometric series for repeating decimals

Question: Express the repeating decimal as a ratio of integers.

$$1.34\overline{25}$$

Answer choices:

A $\dfrac{13{,}291}{9{,}900}$

B $\dfrac{13{,}241}{990}$

C $\dfrac{13{,}241}{9{,}900}$

D $\dfrac{13{,}266}{9{,}900}$

Solution: A

We can use the formula for the sum of a geometric series to quickly and accurately convert a repeating decimal into a ratio of integers, in other words, into a fraction with whole numbers in the numerator and denominator.

We'll follow these steps:

1. Separate the non-repeating part from the repeating part of the decimal.

2. In a table, match each repeated part with its last decimal place.

3. Create a sum of each part, dividing the repeated parts by their ending decimal places.

4. Identify the geometric series within the sum and set values for a and r from the formula for the sum of a geometric series.

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

5. Use the formula to convert the series into one fraction.

6. Add this geometric series fraction to the fraction for the non-repeated part.

In this problem, the bar over the $.0025$ indicates that this is the portion of the decimal that repeats. This tells us that the decimal looks like

$$1.3425252525252525\ldots$$

We've been asked to convert this decimal value into a fraction with a real-number numerator and denominator.

Our first step is to separate the non-repeating part from the repeating part of the decimal.

$$1.34 + .0025252525252525\ldots$$

We add a $.00$ in the first two decimal places of our repeating part because they're holding the place of the $.34$ we pulled out into the non-repeating part. The repeating sequence starts with the first 2 in the thousandths place, and we need to keep it there when we separate the decimals, so it's critical to put in the $.00$.

Next, we'll separate each part of the repeated sequence into its own row of the table below, replacing the decimal places before it with 0s. Once we've built out the left column, we'll put the corresponding place in the second column.

1.34		
25	$.0025$	ends at the $10{,}000$s place
25	$.000025$	ends at the $1{,}000{,}000$s place
25	$.00000025$	ends at the $100{,}000{,}000$s place

We'll create a sum of the non-repeated part and each of the repeated parts, dividing each repeated part by its ending decimal place.

$$1.34\overline{25} = 1.34 + \frac{25}{10{,}000} + \frac{25}{1{,}000{,}000} + \frac{25}{100{,}000{,}000} + \ldots$$

The sum we just created is a geometric series, so we can use the formula for the sum of a geometric series

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

to turn the sum into just one fraction. We'll factor out the first fraction from the repeated part.

$$1.34\overline{25} = 1.34 + \frac{25}{10{,}000}\left(1 + \frac{1}{100} + \frac{1}{10{,}000} + \dots\right)$$

With our series in this form, we can identify a and r from the formula for the sum of a geometric series. a is always the value we factored that's sitting right in front of the parentheses. r is the always the second term inside the parentheses, the value immediately following the 1.

$$a = \frac{25}{10{,}000}$$

$$r = \frac{1}{100}$$

Plugging these into the formula for the sum of a geometric series, remembering to keep the non-repeated part of our decimal, 1.34, we get

$$1.34 + \sum_{n=1}^{\infty} \frac{25}{10{,}000}\left(\frac{1}{100}\right)^n = 1.34 + \frac{\frac{25}{10{,}000}}{1 - \frac{1}{100}}$$

$$1.34 + \sum_{n=1}^{\infty} \frac{25}{10{,}000}\left(\frac{1}{100}\right)^n = 1.34 + \frac{\frac{25}{10{,}000}}{\frac{100}{100} - \frac{1}{100}}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=1.34+\frac{\frac{25}{10,000}}{\frac{99}{100}}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=1.34+\frac{25}{10,000}\cdot\frac{100}{99}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=1.34+\frac{25}{100}\cdot\frac{1}{99}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=1.34+\frac{25}{9,900}$$

Now we just need to change the non-repeated part of our original decimal into a fraction and then combine these two fractions.

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=\left(1+\frac{34}{100}\right)+\frac{25}{9,900}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=\left(\frac{100}{100}+\frac{34}{100}\right)+\frac{25}{9,900}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=\frac{134}{100}+\frac{25}{9,900}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=\frac{99}{99}\left(\frac{134}{100}\right)+\frac{25}{9,900}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10,000}\left(\frac{1}{100}\right)^n=\frac{13,266}{9,900}+\frac{25}{9,900}$$

$$1.34+\sum_{n=1}^{\infty}\frac{25}{10{,}000}\left(\frac{1}{100}\right)^{n}=\frac{13{,}291}{9{,}900}$$

If you have access to a calculator, you can always double-check yourself. In this case, just use your calculator to divide $13{,}291$ by $9{,}900$. If you did this correctly, you should get the original repeating decimal, $1.34\overline{25}$.

Topic: Convergence of a telescoping series

Question: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} 2^n - 2^{n+1}$$

Answer choices:

A The series diverges $s_n = 2 - 2^{n-1}$

B The series converges $s_n = 2 - 2^{n+1}$

C The series converges $s_n = 2 - 2^{n-1}$

D The series diverges $s_n = 2 - 2^{n+1}$

Solution: D

To see whether or not a telescoping series converges or diverges, we'll find the nth partial sum of the series s_n, and then take the limit as $n \to \infty$ of s_n, or

$$s = \lim_{n\to\infty} s_n$$

If we get a real-number value for s, then s_n converges, and therefore so does a_n.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums s_n.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$n = 1$	$2^1 - 2^2$	$2 - 4$
$n = 2$	$2^2 - 2^3$	$4 - 8$
$n = 3$	$2^3 - 2^4$	$8 - 16$
$n = 4$	$2^4 - 2^5$	$16 - 32$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} 2^n - 2^{n+1} = (2-4) + (4-8) + \left(8-16\right) + \left(16-32\right) + \ldots + \left(2^n - 2^{n+1}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = 2 - 2^{n+1}$$

Remember that we're looking for a real-number value for s, where

$$s = \lim_{n\to\infty} s_n$$

so we'll plug s_n into this equation and get

$$s = \lim_{n\to\infty} 2 - 2^{n+1}$$

If we find the limit as $n \to \infty$, the exponent in this equation will become infinitely large. Raising 2 to an infinitely large exponent will make the term 2^{n+1} infinitely large. Subtracting an infinitely large value from 2 will give us an infinitely negative value, so we can say

$$s = -\infty$$

Because the value of s is not a real number, the sum of the series does not exist, s_n diverges, and therefore a_n also diverges.

Topic: Convergence of a telescoping series

Question: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$$

Answer choices:

A The series diverges $s_n = \frac{1}{\ln 1} + \frac{1}{\ln(n+1)}$

B The series converges $s_n = \frac{1}{\ln 1} + \frac{1}{\ln(n+1)}$

C The series converges $s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$

D The series diverges $s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$

Solution: D

To see whether or not a telescoping series converges or diverges, we'll find the nth partial sum of the series s_n, and then take the limit as $n \to \infty$ of s_n, or

$$s = \lim_{n \to \infty} s_n$$

If we get a real-number value for s, then s_n converges, and therefore so does a_n.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums s_n.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{1}{\ln 1} - \frac{1}{\ln(1+1)} \qquad \frac{1}{\ln 1} - \frac{1}{\ln 2}$$

$$n = 2 \qquad \frac{1}{\ln 2} - \frac{1}{\ln(2+1)} \qquad \frac{1}{\ln 2} - \frac{1}{\ln 3}$$

$$n = 3 \qquad \frac{1}{\ln 3} - \frac{1}{\ln(3+1)} \qquad \frac{1}{\ln 3} - \frac{1}{\ln 4}$$

$$n = 4 \qquad \frac{1}{\ln 4} - \frac{1}{\ln(4+1)} \qquad \frac{1}{\ln 4} - \frac{1}{\ln 5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\ln n} - \frac{1}{\ln(n+1)} = \left(\frac{1}{\ln 1} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 3} - \frac{1}{\ln 4}\right)$$

$$+\left(\frac{1}{\ln 4} - \frac{1}{\ln 5}\right) + \ldots + \left(\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$$

Remember that we're looking for a real-number value for s, where

$$s = \lim_{n \to \infty} s_n$$

so we'll plug s_n into this equation and get

$$s = \lim_{n \to \infty} \frac{1}{\ln 1} - \frac{1}{\ln(n+1)}$$

If we find the limit as $n \to \infty$, then $n + 1$ is an infinitely large value, and $\ln(n+1)$ will be infinitely large. When the denominator of a fraction becomes infinitely large, the fraction itself tends toward 0, so

$$s = \lim_{n\to\infty} \frac{1}{\ln 1} - 0$$

$$s = \lim_{n\to\infty} \frac{1}{\ln 1}$$

There is no n value remaining in the expression, and the limit only effects n, which means we can remove it.

$$s = \frac{1}{\ln 1}$$

Because $\ln 1 = 0$, we'll get a 0 in the denominator of this fraction, which makes the fraction undefined. Therefore, the value of s is undefined, so the sum of the series does not have a real-number value.

Because the value of s is not a real number, the sum of the series does not exist, s_n diverges, and therefore a_n also diverges.

Topic: Convergence of a telescoping series

Question: Say whether or not the telescoping series converges.

$$\sum_{n=1}^{\infty} \frac{4}{n} - \frac{4}{n+1}$$

Answer choices:

A The series diverges $s_n = 4 + \frac{4}{n+1}$

B The series converges $s_n = 4 + \frac{4}{n+1}$

C The series diverges $s_n = 4 - \frac{4}{n+1}$

D The series converges $s_n = 4 - \frac{4}{n+1}$

Solution: D

To see whether or not a telescoping series converges or diverges, we'll find the nth partial sum of the series s_n, and then take the limit as $n \to \infty$ of s_n, or

$$s = \lim_{n \to \infty} s_n$$

If we get a real-number value for s, then s_n converges, and therefore so does a_n.

We'll expand the telescoping series by calculating the first few terms, making sure to also include the last term of the series, then simplify the sum by canceling all of the terms in the middle. The remaining series will be the series of partial sums s_n.

In order to show that the series is telescoping, we'll need to start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad \frac{4}{1} - \frac{4}{1+1} \qquad \frac{4}{1} - \frac{4}{2}$$

$$n = 2 \qquad \frac{4}{2} - \frac{4}{2+1} \qquad \frac{4}{2} - \frac{4}{3}$$

$$n = 3 \qquad \frac{4}{3} - \frac{4}{3+1} \qquad \frac{4}{3} - \frac{4}{4}$$

$$n = 4 \qquad \frac{4}{4} - \frac{4}{4+1} \qquad \frac{4}{4} - \frac{4}{5}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{4}{n} - \frac{4}{n+1} = \left(\frac{4}{1} - \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{4}{3}\right) + \left(\frac{4}{3} - \frac{4}{4}\right) + \left(\frac{4}{4} - \frac{4}{5}\right)$$

$$+ \dots + \left(\frac{4}{n} - \frac{4}{n+1}\right)$$

When we look at our expanded series, we see that the second half of the first term will cancel with the first half of the second term, that the second half of the second term will cancel with the first half of the third term, and so on, so we can say that the series is telescoping.

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{4}{1} - \frac{4}{n+1}$$

$$s_n = 4 - \frac{4}{n+1}$$

Remember that we're looking for a real-number value for s, where

$$s = \lim_{n \to \infty} s_n$$

so we'll plug s_n into this equation and get

$$s = \lim_{n \to \infty} 4 - \frac{4}{n+1}$$

$$s = \lim_{n \to \infty} 4 - \frac{\frac{4}{n}}{\frac{n}{n} + \frac{1}{n}}$$

$$s = \lim_{n \to \infty} 4 - \frac{\frac{4}{n}}{1 + \frac{1}{n}}$$

If we find the limit as $n \to \infty$, then we get

$$s = 4 - \frac{0}{1 + 0}$$

$$s = 4 - 0$$

$$s = 4$$

Because the value of s is a real number, the sum of the series is $s = 4$, s_n converges, and therefore a_n also converges.

Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{3}{9n^2 - 3n - 2}$$

Answer choices:

A 3

B 2

C 1

D 0

Solution: C

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

Before we get to the telescoping series, we'll first decompose the rational expression into simpler fractions using partial fractions.

$$\frac{3}{9n^2 - 3n - 2} = \frac{3}{(3n-2)(3n+1)}$$

Because $(3n-2)$ and $(3n+1)$ are distinct linear factors, the decomposition will be

$$\frac{3}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}$$

$$\left[\frac{3}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}\right](3n-2)(3n+1)$$

$$3 = A(3n+1) + B(3n-2)$$

$$3 = 3An + A + 3Bn - 2B$$

$$3 = (3An + 3Bn) + (A - 2B)$$

$$3 = (3A + 3B)n + (A - 2B)$$

Equating coefficients on both sides, we get the simultaneous equations

$$3A + 3B = 0 \qquad \text{and} \qquad A - 2B = 3$$

$$A = 3 + 2B$$

$$3(3 + 2B) + 3B = 0$$

$$9 + 6B + 3B = 0$$

$$9 + 9B = 0$$

$$9B = -9$$

$$B = -1$$

$$A = 3 + 2(-1)$$

$$A = 3 - 2$$

$$A = 1$$

Having solved for our constants A and B, we'll plug them into our partial fractions decomposition.

$$\frac{3}{9n^2 - 3n - 2} = \frac{A}{3n - 2} + \frac{B}{3n + 1}$$

$$\frac{3}{9n^2 - 3n - 2} = \frac{1}{3n - 2} + \frac{-1}{3n + 1}$$

$$\frac{3}{9n^2 - 3n - 2} = \frac{1}{3n - 2} - \frac{1}{3n + 1}$$

Plugging the decomposition back into the summation notation, we get

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1}$$

Now we'll start working on the convergence of this series by first expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = \frac{1}{3(1)-2} - \frac{1}{3(1)+1} = \frac{1}{1} - \frac{1}{4}$$

$$n = 2 \qquad a_2 = \frac{1}{3(2)-2} - \frac{1}{3(2)+1} = \frac{1}{4} - \frac{1}{7}$$

$$n = 3 \qquad a_3 = \frac{1}{3(3)-2} - \frac{1}{3(3)+1} = \frac{1}{7} - \frac{1}{10}$$

$$n = 4 \qquad a_4 = \frac{1}{3(4)-2} - \frac{1}{3(4)+1} = \frac{1}{10} - \frac{1}{13}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right)$$

$$+\left(\frac{1}{10} - \frac{1}{13}\right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = \frac{1}{1} - \frac{1}{3n+1}$$

$$s_n = 1 - \frac{1}{3n+1}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = \lim_{n\to\infty} 1 - \frac{1}{3n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{3(\infty)+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{\infty+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - \frac{1}{\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1 - 0$$

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} = 1$$

The sum of the series is 1.

Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}}$$

Answer choices:

A $\frac{2}{3}$

B 1

C 0

D $\frac{1}{2}$

Solution: D

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

We'll start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = \frac{1}{\sqrt{1+3}} - \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}$$

$$n = 2 \qquad a_2 = \frac{1}{\sqrt{2+3}} - \frac{1}{\sqrt{2+4}} = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}$$

$$n = 3 \qquad a_3 = \frac{1}{\sqrt{3+3}} - \frac{1}{\sqrt{3+4}} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}$$

$$n = 4 \qquad a_4 = \frac{1}{\sqrt{4+3}} - \frac{1}{\sqrt{4+4}} = \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}} = \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}}\right)$$

$$+\left(\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{8}}\right)+\ldots+\left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n=\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{n+4}}$$

$$s_n=\frac{1}{2}-\frac{1}{\sqrt{n+4}}$$

To find the sum of the telescoping series, we'll take the limit as $n\to\infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}s_n$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}=\lim_{n\to\infty}\frac{1}{2}-\frac{1}{\sqrt{n+4}}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}=\frac{1}{2}-\frac{1}{\sqrt{\infty+4}}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}=\frac{1}{2}-\frac{1}{\sqrt{\infty}}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+4}}=\frac{1}{2}-\frac{1}{\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}} = \frac{1}{2} - 0$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+4}} = \frac{1}{2}$$

The sum of the series is $1/2$.

Topic: Sum of a telescoping series

Question: Calculate the sum of the telescoping series.

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}}$$

Answer choices:

A $\quad 2$

B $\quad 3$

C $\quad \infty$

D $\quad$ The series diverges

Solution: A

The sum of a telescoping series is given by the formula

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

We know that s_n is the series of partial sums, so we can say that the sum of the telescoping series a_n is the limit as $n \to \infty$ of its corresponding series of partial sums s_n.

We'll start by expanding the series. Let's use $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$n = 1 \qquad a_1 = 3^{\frac{1}{1}} - 3^{\frac{1}{1+1}} = 3 - \sqrt{3}$$

$$n = 2 \qquad a_2 = 3^{\frac{1}{2}} - 3^{\frac{1}{2+1}} = \sqrt{3} - \sqrt[3]{3}$$

$$n = 3 \qquad a_3 = 3^{\frac{1}{3}} - 3^{\frac{1}{3+1}} = \sqrt[3]{3} - \sqrt[4]{3}$$

$$n = 4 \qquad a_4 = 3^{\frac{1}{4}} - 3^{\frac{1}{4+1}} = \sqrt[4]{3} - \sqrt[5]{3}$$

Writing these terms into our expanded series and including the last term of the series, we get

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = \left(3 - \sqrt{3}\right) + \left(\sqrt{3} - \sqrt[3]{3}\right) + \left(\sqrt[3]{3} - \sqrt[4]{3}\right)$$

$$+ \left(\sqrt[4]{3} - \sqrt[5]{3}\right) + \ldots + \left(3^{\frac{1}{n}} - 3^{\frac{1}{n+1}}\right)$$

Canceling everything but the first half of the first term and the second half of the last term gives an expression for the series of partial sums.

$$s_n = 3 - 3^{\frac{1}{n+1}}$$

To find the sum of the telescoping series, we'll take the limit as $n \to \infty$ of the series or partial sums s_n.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = \lim_{n \to \infty} 3 - 3^{\frac{1}{n+1}}$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 3^{\frac{1}{\infty+1}}$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 3^0$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 3 - 1$$

$$\sum_{n=1}^{\infty} 3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} = 2$$

The sum of the series is 2.

Topic: Limit vs. sum of the series

Question: How is the limit of a series different from its sum?

Answer choices:

A They aren't different. They both indicate the size of the series from its start to ∞.

B They aren't different. They both indicate how the series behaves as it approaches ∞.

C The limit indicates how the series is behaving as it approaches ∞ and the sum indicates the size of the series from its start to ∞.

D The sum indicates how the series is behaving as it approaches ∞ and the limit indicates the size of the series from its start to ∞.

Solution: C

The limit of a series (at ∞) indicates how a series is behaving as it approaches ∞.

The sum of a series indicates the size of the series from its start to ∞.

Since these two values are different, it's useful to be able to find both for the same series.

Topic: Limit vs. sum of the series

Question: Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} e^{-n}$$

Answer choices:

A $\quad \lim a_n = -1 \quad$ and $\quad \sum a_n = \dfrac{e}{e+1}$

B $\quad \lim a_n = 0 \quad$ and $\quad \sum a_n = \dfrac{1}{e-1}$

C $\quad \lim a_n = 1 \quad$ and $\quad \sum a_n = e-1$

D $\quad \lim a_n = \text{DNE} \quad$ and $\quad \sum a_n = \dfrac{1}{e-1}$

Solution: B

The limit of the series is given by

$$\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{e^n}$$

Notice that as $n \to \infty$, the denominator gets bigger and bigger. With a constant numerator, the value of the fraction approaches 0. Therefore,

$$\lim_{n\to\infty} e^{-n} = 0$$

Because the limit is equal to 0, it means that the series is convergent and its graph will approach 0 as the value of n is increased. To evaluate the sum of the series, rewrite it as

$$\sum_{n=1}^{\infty} e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{1}{e^n}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

If we let $a_n = \left(\frac{1}{e}\right)^n$ and we write out the first three terms of the series as

$$a_1 = \left(\frac{1}{e}\right)^1 = \frac{1}{e}$$

$$a_2 = \left(\frac{1}{e}\right)^2 = \frac{1}{e^2}$$

$$a_3 = \left(\frac{1}{e}\right)^3 = \frac{1}{e^3}$$

then its sum is equal to

$$s = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \dots + \frac{1}{e^n}$$

$$s = \frac{1}{e}\left(1 + \frac{1}{e} + \frac{1}{e^2} + \dots + \frac{1}{e^n}\right)$$

This is a geometric series with $a = 1/e$ and $r = 1/e$. The sum of a geometric series is given by

$$s = \frac{a}{1 - r}$$

so the sum of the series is

$$s = \frac{\frac{1}{e}}{1 - \frac{1}{e}}$$

$$s = \frac{\frac{1}{e}}{\frac{e}{e} - \frac{1}{e}}$$

$$s = \frac{\frac{1}{e}}{\frac{e-1}{e}}$$

$$s = \frac{1}{e} \cdot \frac{e}{e - 1}$$

$$s = \frac{1}{e - 1}$$

Topic: Limit vs. sum of the series

Question: Find the limit of the series, and if it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{81^{\frac{n}{2}}}$$

Answer choices:

A $\lim a_n = 0$ and $\sum a_n = \frac{1}{3}$

B $\lim a_n = \frac{1}{3}$ and $\sum a_n = 1$

C $\lim a_n = \infty$ and $\sum a_n = 0$

D $\lim a_n = 1$ and $\sum a_n = \infty$

Solution: D

The limit of the series is given by

$$\lim_{n\to\infty}\frac{3^{2n}}{81^{\frac{n}{2}}}$$

$$\lim_{n\to\infty}\frac{\left(3^2\right)^n}{\left(81^{\frac{1}{2}}\right)^n}$$

$$\lim_{n\to\infty}\frac{9^n}{\left(\sqrt{81}\right)^n}$$

$$\lim_{n\to\infty}\frac{9^n}{9^n}$$

$$\lim_{n\to\infty}1$$

$$1$$

The limit of the series as $n\to\infty$ is 1. Therefore,

$$\lim_{n\to\infty}\frac{3^{2n}}{81^{\frac{n}{2}}}=1$$

Because the limit is equal to 1, it means that the series is convergent and its graph will approach 1 as the value of n is increased. To evaluate the sum of the series, we'll expand it through its first few terms.

$$\sum_{n=1}^{\infty}\frac{3^{2n}}{81^{\frac{n}{2}}}$$

$$\frac{3^{2(1)}}{81^{\frac{1}{2}}}+\frac{3^{2(2)}}{81^{\frac{2}{2}}}+\frac{3^{2(3)}}{81^{\frac{3}{2}}}+\dots$$

$$\frac{3^2}{\sqrt{81}}+\frac{3^4}{81}+\frac{3^6}{\left(81^{\frac{1}{2}}\right)^3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3^6}{9^3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3\cdot3\cdot3\cdot3\cdot3\cdot3}{9\cdot9\cdot9}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3\cdot3\cdot3\cdot3\cdot3\cdot3}{3\cdot3\cdot3\cdot3\cdot3\cdot3}+\dots$$

$$\frac{9}{9}+\frac{81}{81}+\frac{3^6}{3^6}+\dots$$

$$1+1+1+\dots$$

$$\infty$$

The sum of the series is therefore

$$\sum_{n=1}^{\infty}\frac{3^{2n}}{81^{\frac{n}{2}}}=\infty$$

Topic: Integral test

Question: Use the integral test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2e^{3n}}{1+e^{6n}}$$

Answer choices:

A Convergent because the value of the integral is $\frac{1}{3}\left(\pi - 2\tan^{-1}e^3\right)$

B Convergent because the value of the integral is $\frac{1}{3}\left(\pi + 2\tan^{-1}e^3\right)$

C Divergent because the value of the integral is ∞

D Divergent because the value of the integral is $-\infty$

Solution: A

The integral test for convergence is only valid for series that are

Positive: all of the terms in the series are positive

Decreasing: every term is less than the one before it, $a_{n-1} > a_n$

Continuous: the series is defined everywhere in its domain

If the given series meets these three criteria, then we can use the integral test for convergence to integrate the series and say whether the series is converging or diverging.

Given the series

$$\sum_{n=1}^{\infty} a_n$$

we set $f(x) = a_n$ and evaluate the integral

$$\int_1^{\infty} f(x)\ dx$$

According to the integral test, the series and the integral always have the same result, meaning that they either both converge or they both diverge. This means that if the value of the of the integral

converges to a **real number**, then the series also **converges**

diverges to **infinity**, then the series also **diverges**

If we expand the series through the first few terms, we can see that the series is always positive, decreasing, and continuous.

$$\sum_{n=1}^{\infty} \frac{2e^{3n}}{1+e^{6n}} = \frac{2e^{3(1)}}{1+e^{6(1)}} + \frac{2e^{3(2)}}{1+e^{6(2)}} + \frac{2e^{3(3)}}{1+e^{6(3)}} + \frac{2e^{3(4)}}{1+e^{6(4)}} + \frac{2e^{3(5)}}{1+e^{6(5)}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2e^{3n}}{1+e^{6n}} = \frac{2e^{3}}{1+e^{6}} + \frac{2e^{6}}{1+e^{12}} + \frac{2e^{9}}{1+e^{18}} + \frac{2e^{12}}{1+e^{24}} + \frac{2e^{15}}{1+e^{30}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{2e^{3n}}{1+e^{6n}} = 0.099328 + 0.004957 + 0.000247 + 0.000012 + 0.000000 + \dots$$

Now that we know we can use the integral test to say whether or not the series converges, we'll set $f(x) = a_n$ for our series, and we'll get

$$f(x) = \frac{2e^{3n}}{1+e^{6n}}$$

Now we'll plug $f(x)$ into the integral above, and use u-substitution to evaluate the integral.

$$\int_1^{\infty} \frac{2e^{3x}}{1+e^{6x}} \ dx$$

$$u = e^{3x}$$

$$du = 3e^{3x} \ dx$$

$$dx = \frac{du}{3e^{3x}}$$

$$\int_{x=1}^{x=\infty} \frac{2u}{1+u^2} \cdot \frac{du}{3e^{3x}}$$

$$\int_{x=1}^{x=\infty} \frac{2u}{1+u^2} \cdot \frac{du}{3u}$$

$$\frac{2}{3} \int_{x=1}^{x=\infty} \frac{1}{1+u^2} \ du$$

$$\frac{2}{3} \tan^{-1} u \Big|_{x=1}^{x=\infty}$$

$$\frac{2}{3} \tan^{-1} e^{3x} \Big|_{1}^{\infty}$$

$$\lim_{b \to \infty} \left(\frac{2}{3} \tan^{-1} e^{3x} \right) \Big|_{1}^{b}$$

$$\lim_{b \to \infty} \left[\frac{2}{3} \tan^{-1} e^{3(b)} - \frac{2}{3} \tan^{-1} e^{3(1)} \right]$$

$$\lim_{b \to \infty} \left(\frac{2}{3} \tan^{-1} e^{3b} - \frac{2}{3} \tan^{-1} e^{3} \right)$$

$$\frac{2}{3} \cdot \frac{\pi}{2} - \frac{2}{3} \tan^{-1} e^{3}$$

$$\frac{\pi}{3} - \frac{2}{3} \tan^{-1} e^{3}$$

$$\frac{1}{3} \left(\pi - 2 \tan^{-1} e^{3} \right)$$

Since we got a finite value for the integral, it means that the integral converges, which proves that the series also converges.

Topic: Integral test

Question: Use the integral test to say whether the series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\ln(n+3)}{n+3}$$

Answer choices:

A Convergent because the value of the integral is 3

B Convergent because the value of the integral is -3

C Divergent because the value of the integral is ∞

D Divergent because the value of the integral is $-\infty$

Solution: C

The integral test for convergence is only valid for series that are

Positive: all of the terms in the series are positive

Decreasing: every term is less than the one before it, $a_{n-1} > a_n$

Continuous: the series is defined everywhere in its domain

If the given series meets these three criteria, then we can use the integral test for convergence to integrate the series and say whether the series is converging or diverging.

Given the series

$$\sum_{n=1}^{\infty} a_n$$

we set $f(x) = a_n$ and evaluate the integral

$$\int_1^{\infty} f(x) \ dx$$

According to the integral test, the series and the integral always have the same result, meaning that they either both converge or they both diverge. This means that if the value of the of the integral

converges to a **real number**, then the series also **converges**

diverges to **infinity**, then the series also **diverges**

If we expand the series through the first few terms, we can see that the series is always positive, decreasing, and continuous.

$$\sum_{n=3}^{\infty} \frac{\ln(n+3)}{n+3} = \frac{\ln(3+3)}{3+3} + \frac{\ln(4+3)}{4+3} + \frac{\ln(5+3)}{5+3} + \frac{\ln(6+3)}{6+3} + \frac{\ln(7+3)}{7+3} + \dots$$

$$\sum_{n=3}^{\infty} \frac{\ln(n+3)}{n+3} = \frac{\ln(6)}{6} + \frac{\ln(7)}{7} + \frac{\ln(8)}{8} + \frac{\ln(9)}{9} + \frac{\ln(10)}{10} + \dots$$

$$\sum_{n=3}^{\infty} \frac{\ln(n+3)}{n+3} = 0.299 + 0.278 + 0.260 + 0.244 + 0.230 + \dots$$

Now that we know we can use the integral test to say whether or not the series converges, we'll set $f(x) = a_n$ for our series, and we'll get

$$f(x) = \frac{\ln(x+3)}{x+3}$$

Now we'll plug $f(x)$ into the integral above, and use u-substitution to evaluate the integral.

$$\int_1^{\infty} \frac{\ln(x+3)}{x+3}\ dx$$

$$u = \ln(x+3)$$

$$du = \frac{1}{x+3}\ dx$$

$$dx = (x+3)\ du$$

$$\int_{x=3}^{x=\infty} \frac{u}{x+3}(x+3)\ du$$

$$\int_{x=3}^{x=\infty} u \ du$$

$$\frac{1}{2}u^2\Big|_{x=3}^{x=\infty}$$

$$\frac{1}{2}\left[\ln(x+3)\right]^2\Big|_3^\infty$$

$$\lim_{b\to\infty}\frac{1}{2}\left[\ln(x+3)\right]^2\Big|_3^b$$

$$\lim_{b\to\infty}\frac{1}{2}\left[\ln(b+3)\right]^2-\frac{1}{2}\left[\ln(3+3)\right]^2$$

$$\frac{1}{2}\left[\ln(\infty+3)\right]^2-\frac{1}{2}\left[\ln(3+3)\right]^2$$

$$\frac{1}{2}\left[\ln(\infty)\right]^2-\frac{1}{2}\left[\ln(6)\right]^2$$

$$\frac{1}{2}(\infty)-\frac{1}{2}\left[\ln(6)\right]^2$$

$$\infty$$

Since we got an infinite value for the integral, it means that the integral diverges, which proves that the series also diverges.

Topic: Integral test

Question: Use the integral test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (0.1)^n$$

Answer choices:

A Convergent because the value of the integral is $\frac{1}{10\ln(10)}$

B Divergent because the value of the integral is $\frac{1}{10\ln(10)}$

C Convergent because the value of the integral is $10\ln(10)$

D Divergent because the value of the integral is $10\ln(10)$

Solution: A

The integral test for convergence is only valid for series that are

Positive: all of the terms in the series are positive

Decreasing: every term is less than the one before it, $a_{n-1} > a_n$

Continuous: the series is defined everywhere in its domain

If the given series meets these three criteria, then we can use the integral test for convergence to integrate the series and say whether the series is converging or diverging.

Given the series

$$\sum_{n=1}^{\infty} a_n$$

we set $f(x) = a_n$ and evaluate the integral

$$\int_{1}^{\infty} f(x)\ dx$$

According to the integral test, the series and the integral always have the same result, meaning that they either both converge or they both diverge. This means that if the value of the of the integral

converges to a **real number**, then the series also **converges**

diverges to **infinity**, then the series also **diverges**

If we expand the series through the first few terms, we can see that the series is always positive, decreasing, and continuous.

$$\sum_{n=1}^{\infty} (0.1)^n = (0.1)^1 + (0.1)^2 + (0.1)^3 + (0.1)^4 + (0.1)^5 + \dots$$

$$\sum_{n=1}^{\infty} (0.1)^n = 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + \dots$$

Now that we know we can use the integral test to say whether or not the series converges, we'll set $f(x) = a_n$ for our series, and we'll get

$$f(x) = (0.1)^x$$

$$f(x) = \frac{1}{10^x}$$

Now we'll plug $f(x)$ into the integral above, and evaluate using substitution, with $u = -x$ and $-du = dx$.

$$\int_1^{\infty} \frac{1}{10^x}\ dx$$

$$-\int_{x=1}^{x=\infty} 10^u\ du$$

$$-\frac{10^u}{\ln 10}\Bigg|_{x=1}^{x=\infty}$$

$$-\frac{10^{-x}}{\ln 10}\Bigg|_1^{\infty}$$

$$-\frac{1}{10^x \ln 10}\Bigg|_1^\infty$$

$$\lim_{x\to\infty}\left(-\frac{1}{10^x \ln 10}\right)+\frac{1}{10^1 \ln 10}$$

$$\frac{1}{10 \ln 10}$$

Since we got a finite value for the integral, it means that the integral converges, which proves that the series also converges.

Topic: p-series test

Question: Use the p-series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: C

If we have a series a_n in the form

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

then we can use the p-series test for convergence to say whether or not a_n will converge. The p-series test says that

a_n will converge when $p > 1$

a_n will diverge when $p \leq 1$

The key is to make sure that the given series matches the format above for a p-series, and then to look at the value of p to determine convergence.

For the series we're given in this problem, $p = 1/2$ since

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

Because

$$p = \frac{1}{2} \leq 1$$

the p-series test proves that this series diverges.

Topic: p-series test

Question: Use the p-series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: C

If we have a series a_n in the form

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

then we can use the p-series test for convergence to say whether or not a_n will converge. The p-series test says that

a_n will converge when $p > 1$

a_n will diverge when $p \leq 1$

The key is to make sure that the given series matches the format above for a p-series, and then to look at the value of p to determine convergence.

For the series we're given in this problem, $p = 1$ since

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^1}$$

Because

$$p = 1 \leq 1$$

the p-series test proves that this series diverges.

Topic: p-series test

Question: Use the p-series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^8}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

If we have a series a_n in the form

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

then we can use the p-series test for convergence to say whether or not a_n will converge. The p-series test says that

a_n will converge when $p > 1$

a_n will diverge when $p \leq 1$

The key is to make sure that the given series matches the format above for a p-series, and then to look at the value of p to determine convergence.

For the series we're given in this problem, $p = 8$.

Because

$$p = 8 > 1$$

the p-series test proves that this series converges.

Topic: nth term test

Question: Use the nth term test to say whether or not the series divergence.

$$\sum_{n=1}^{\infty} 2 + \frac{1}{n}$$

Answer choices:

A The series converges.

B The series diverges.

C The test is inconclusive.

D The series is infinite.

Solution: B

The nth term test, also called the divergence test, or the zero test,

says that a series a_n **diverges** if $\lim_{n\to\infty} a_n \neq 0$

is **inconclusive** if $\lim_{n\to\infty} a_n = 0$

The nth term test can't tell us that a series converges, only that it diverges. Otherwise, the test is inconclusive.

To use it, we just take the limit as $n \to \infty$ of the series a_n that we've been given.

$$\lim_{n\to\infty} 2 + \frac{1}{n}$$

$$2 + \frac{1}{\infty}$$

$$2 + 0$$

$$2$$

Since $2 \neq 0$, the nth term test tells us that the series diverges.

Topic: nth term test

Question: Use the nth term test to say whether or not the series diverges.

$$\sum_{n=1}^{\infty} \frac{3n^2 - 2}{5n^2 + 8}$$

Answer choices:

A The series converges.

B The test is inconclusive.

C The series diverges.

D The series is infinite.

Solution: C

The nth term test, also called the divergence test, or the zero test,

says that a series a_n **diverges** if $\lim_{n\to\infty} a_n \neq 0$

is **inconclusive** if $\lim_{n\to\infty} a_n = 0$

The nth term test can't tell us that a series converges, only that it diverges. Otherwise, the test is inconclusive.

To use it, we just take the limit as $n \to \infty$ of the series a_n that we've been given.

$$\lim_{n\to\infty} \frac{3n^2-2}{5n^2+8}$$

$$\frac{\infty}{\infty}$$

Since we can an indeterminate form, we'll go back a step and manipulate our function.

$$\lim_{n\to\infty} \frac{3n^2-2}{5n^2+8}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$\lim_{n\to\infty} \frac{\frac{3n^2}{n^2}-\frac{2}{n^2}}{\frac{5n^2}{n^2}+\frac{8}{n^2}}$$

$$\lim_{n\to\infty} \frac{3 - \frac{2}{n^2}}{5 + \frac{8}{n^2}}$$

$$\frac{3 - \frac{2}{\infty^2}}{5 + \frac{8}{\infty^2}}$$

$$\frac{3 - 0}{5 + 0}$$

$$\frac{3}{5}$$

Since $3/5 \neq 0$, the nth term test tells us that the series diverges.

Topic: nth term test

Question: Use the nth term test to say whether or not the series diverges.

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

Answer choices:

A The series converges.

B The series is infinite.

C The series diverges.

D The test is inconclusive.

Solution: D

The nth term test, also called the divergence test, or the zero test,

says that a series a_n **diverges** if $\lim_{n\to\infty} a_n \neq 0$

is **inconclusive** if $\lim_{n\to\infty} a_n = 0$

The nth term test can't tell us that a series converges, only that it diverges. Otherwise, the test is inconclusive.

To use it, we just take the limit as $n \to \infty$ of the series a_n that we've been given.

$$\lim_{n\to\infty} \frac{n^2}{e^n}$$

$$\frac{\infty}{\infty}$$

Since we can an indeterminate form, we'll go back a step and use L'Hospital's rule to simplify our function by replacing both the numerator and the denominator with their derivatives.

$$\lim_{n\to\infty} \frac{2n}{e^n}$$

$$\frac{\infty}{\infty}$$

We'll back up a step and use L'Hospital's rule again.

$$\lim_{n\to\infty} \frac{2}{e^n}$$

$$\frac{2}{\infty}$$

$$0$$

Since our answer is 0, the nth term test is inconclusive.

Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+6} \right)^n$$

Answer choices:

A $\quad a_n$ converges

B $\quad a_n$ diverges

C $\quad b_n$ converges but you can't verify $0 \le b_n \le a_n$ so the test is inconclusive

D $\quad b_n$ diverges but you can't verify $a_n \ge b_n \ge 0$ so the test is inconclusive

Solution: A

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the $2n$ carries a lot more weight and will affect the series more than the 6, so we'll use only the $2n$ in the denominator of the comparison series.

$$b_n = \left(\frac{n}{2n}\right)^n$$

$$b_n = \left(\frac{1}{2}\right)^n$$

The comparison series is a geometric series. The geometric series test for convergence says that

if $|r| < 1$ then the series converges

if $|r| \geq 1$ then the series diverges

when we're pulling r from the expanded form of the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a\left\{1 + r + r^2 + r^3 + \dots\right\}$$

Expanding b_n until it matches this expanded form of a geometric series, we get

$$b_n = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \dots$$

$$b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$b_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right)$$

We'll pull r from the term immediately following the 1 inside the parentheses, so $r = 1/2$. Applying the geometric series test, we see that

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

which means that the comparison series converges.

Knowing that the comparison series converges, we need to show that

$$0 \le a_n \le b_n$$

in order to prove that the original series a_n is also converging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \left(\frac{1}{2(1)+6}\right)^1 = \frac{1}{8}$	$b_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$
$n = 2$	$a_2 = \left(\frac{2}{2(2)+6}\right)^2 = \frac{1}{25}$	$b_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$$n = 3 \qquad a_3 = \left(\frac{3}{2(3)+6}\right)^3 = \frac{1}{64} \qquad b_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Looking at just these few terms, we can see that $0 \le a_n \le b_n$ for all n, which means we can conclude that a_n converges.

Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but you can't verify $0 \leq b_n \leq a_n$ so the test is inconclusive

D b_n diverges but you can't verify $a_n \geq b_n \geq 0$ so the test is inconclusive

Solution: D

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series

b_n. Let's combine the given series into one fraction before creating the comparison series.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\left[\frac{4}{n}\left(\frac{n}{n}\right)-\frac{4}{n^2}\right]$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\left(\frac{4n}{n^2}-\frac{4}{n^2}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n^2}\right)=\sum_{n=1}^{\infty}\frac{4n-4}{n^2}$$

We'll use the denominator from a_n for b_n, since the denominator is already pretty simple. In the numerator, the $4n$ carries a lot more weight and will affect the series more than the -4, so we'll use only the $4n$ in the numerator of the comparison series.

$$b_n=\frac{4n}{n^2}$$

$$b_n=\frac{4}{n}$$

$$b_n=4\left(\frac{1}{n}\right)$$

$$b_n=4\left(\frac{1}{n^1}\right)$$

The comparison series is a p-series. Since the p-series test tells us that the series will converge when $p > 1$

diverge when $p \le 1$

we can say that $1 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$$a_n \ge b_n \ge 0$$

in order to prove that the original series a_n is also diverging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \frac{4(1) - 4}{1^2} = 0$	$b_1 = \frac{4}{1} = 4$
$n = 2$	$a_2 = \frac{4(2) - 4}{2^2} = 1$	$b_2 = \frac{4}{2} = 2$
$n = 3$	$a_3 = \frac{4(3) - 4}{3^2} = \frac{8}{9}$	$b_3 = \frac{4}{3}$

Looking at just these few terms, we can see that $0 \le a_n \le b_n$. This is the opposite of what we were looking for, $a_n \ge b_n \ge 0$, which means the test is inconclusive.

Topic: Comparison test

Question: Use the comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but you can't verify $0 \le b_n \le a_n$ so the test is inconclusive

D b_n diverges but you can't verify $a_n \ge b_n \ge 0$ so the test is inconclusive

Solution: A

The comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the comparison test to show that

the original series a_n is **diverging** if

the original series a_n is greater than or equal to the comparison series b_n and both series are positive, $a_n \geq b_n \geq 0$, and

the comparison series b_n is diverging

Note: If $a_n < b_n$, the test is inconclusive

the original series is **converging** if

the original series a_n is less than or equal to the comparison series b_n and both series are positive, $0 \leq a_n \leq b_n$, and

the comparison series b_n is converging

Note: If $b_n < a_n$, the test is inconclusive

Before we can use the comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the n^2 carries a lot more weight and will affect the series more than the 1, so we'll use only the n^2 in the denominator of the comparison series.

$$b_n = \frac{\sqrt{n}}{n^2}$$

$$b_n = \frac{n^{\frac{1}{2}}}{n^2}$$

$$b_n = \frac{1}{n^{2-\frac{1}{2}}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

we can say that $3/2 > 1$ and therefore that b_n converges.

Knowing that the comparison series converges, we need to show that

$$0 \leq a_n \leq b_n$$

in order to prove that the original series a_n is also converging. If we can't verify this inequality, then the comparison test will be inconclusive. To verify the inequality, we'll compare a few points from a_n and b_n. Since we've got a square root in a_n, let's use squares, like $n = 1$, $n = 4$ and $n = 9$.

	a_n	b_n
$n = 1$	$a_1 = \frac{\sqrt{1}}{1^2 + 1} = \frac{1}{2}$	$b_1 = \frac{1}{1^{\frac{3}{2}}} = 1$
$n = 4$	$a_4 = \frac{\sqrt{4}}{4^2 + 1} = \frac{2}{17}$	$b_4 = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{8}$
$n = 9$	$a_9 = \frac{\sqrt{9}}{9^2 + 1} = \frac{3}{82}$	$b_9 = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{27}$

Looking at just these few terms, we can see that $0 \le a_n \le b_n$ for all n, which means we can conclude that a_n converges.

Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt{n} + \sqrt[3]{n}}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but the test is inconclusive

D b_n diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple. In the denominator, the $4\sqrt{n}$ carries a lot more weight and will affect the series more than the $\sqrt[3]{n}$ when n gets very large, so we'll use only the $4\sqrt{n}$ in the denominator of the comparison series.

$$b_n = \frac{1}{4\sqrt{n}}$$

$$b_n = \frac{1}{4}\left(\frac{1}{\sqrt{n}}\right)$$

$$b_n = \frac{1}{4}\left(\frac{1}{n^{\frac{1}{2}}}\right)$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \le 1$

we can say that $1/2 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive.

To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n. Since we've got a square root in a_n, let's use squares, like $n = 1$, $n = 4$ and $n = 9$.

$$\begin{array}{lll} & a_n & b_n \\ n = 1 & a_1 = \dfrac{1}{4\sqrt{1} + \sqrt[3]{1}} = \dfrac{1}{5} & b_1 = \dfrac{1}{4\sqrt{1}} = \dfrac{1}{4} \\ n = 4 & a_4 = \dfrac{1}{4\sqrt{4} + \sqrt[3]{4}} = \dfrac{1}{8 + \sqrt[3]{4}} & b_4 = \dfrac{1}{4\sqrt{4}} = \dfrac{1}{8} \\ n = 9 & a_9 = \dfrac{1}{4\sqrt{9} + \sqrt[3]{9}} = \dfrac{1}{12 + \sqrt[3]{9}} & b_9 = \dfrac{1}{4\sqrt{9}} = \dfrac{1}{12} \end{array}$$

Looking at just these few terms, we can see that $a_n > 0$ and $b_n > 0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

Plugging our series into this formula, we get

$$L = \lim_{n \to \infty} \frac{\frac{1}{4\sqrt{n} + \sqrt[3]{n}}}{\frac{1}{4\sqrt{n}}}$$

$$L = \lim_{n \to \infty} \frac{1}{4\sqrt{n} + \sqrt[3]{n}} \left(\frac{4\sqrt{n}}{1} \right)$$

$$L = \lim_{n \to \infty} \frac{4\sqrt{n}}{4\sqrt{n} + \sqrt[3]{n}}$$

$$L = \lim_{n \to \infty} \frac{4n^{\frac{1}{2}}}{4n^{\frac{1}{2}} + n^{\frac{1}{3}}}$$

$$L = \lim_{n \to \infty} \frac{4n^{\frac{1}{2}}}{4n^{\frac{1}{2}} + n^{\frac{1}{3}}} \left(\frac{\frac{1}{n^{\frac{1}{2}}}}{\frac{1}{n^{\frac{1}{2}}}} \right)$$

$$L = \lim_{n \to \infty} \frac{\frac{4n^{\frac{1}{2}}}{n^{\frac{1}{2}}}}{\frac{4n^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{n^{\frac{1}{3}}}{n^{\frac{1}{2}}}}$$

$$L = \lim_{n \to \infty} \frac{4}{4 + \frac{1}{n^{\frac{1}{2} - \frac{1}{3}}}}$$

$$L = \lim_{n \to \infty} \frac{4}{4 + \frac{1}{n^{\frac{1}{6}}}}$$

$$L = \frac{4}{4 + \frac{1}{\infty^{\frac{1}{6}}}}$$

$$L = \frac{4}{4 + \frac{1}{\infty}}$$

$$L = \frac{4}{4 + 0}$$

$$L = 1$$

Since

$$L = 1 > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.

Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt[n]{n}}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but the test is inconclusive

D b_n diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple, but we'll leave out the 3 since it'll have less impact on our series as $n \to \infty$ (which means we'll just use 1). In the denominator, the n carries a lot more weight and will affect the series more than the $\sqrt[n]{n}$ (since $\sqrt[n]{n}$ approaches 1 when n gets very large), so we'll use only the n in the denominator of the comparison series.

$$b_n = \frac{1}{n}$$

$$b_n = \frac{1}{n^1}$$

The comparison series is a p-series (it's also the harmonic series). Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \le 1$

we can say that $1 \le 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive. To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n.

Since we've got a square root in a_n, let's use squares, like $n=1$, $n=4$ and $n=9$.

	a_n	b_n
$n=1$	$a_1=\frac{3}{1\sqrt[1]{1}}=\frac{3}{1^{\frac{1}{1}}}=3$	$b_1=\frac{1}{1}=1$
$n=4$	$a_4=\frac{3}{4\sqrt[4]{4}}=\frac{3}{4(4)^{\frac{1}{4}}}=\frac{3}{4^{\frac{5}{4}}}$	$b_4=\frac{1}{4}$
$n=9$	$a_9=\frac{3}{9\sqrt[9]{9}}=\frac{3}{9(9)^{\frac{1}{9}}}=\frac{3}{9^{\frac{10}{9}}}$	$b_9=\frac{1}{9}$

Looking at just these few terms, we can see that $a_n>0$ and $b_n>0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L>0$$

Plugging our series into this formula, we get

$$L=\lim_{n\to\infty}\frac{\frac{3}{n\sqrt[n]{n}}}{\frac{1}{n}}$$

$$L=\lim_{n\to\infty}\frac{3}{n\sqrt[n]{n}}\left(\frac{n}{1}\right)$$

$$L = \lim_{n \to \infty} \frac{3}{\sqrt[n]{n}}$$

$$L = \lim_{n \to \infty} \frac{3}{n^{\frac{1}{n}}}$$

$$L = \frac{3}{\infty^{\frac{1}{\infty}}}$$

$$L = \frac{3}{\infty^0}$$

$$L = \frac{3}{1}$$

$$L = 3$$

Since

$$L = 3 > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.

Topic: Limit comparison test

Question: Use the limit comparison test to say whether or not the series converges.

$$\sum_{n=1}^{\infty} \frac{2n}{3n^2 + 1}$$

Answer choices:

A a_n converges

B a_n diverges

C b_n converges but the test is inconclusive

D b_n diverges but the test is inconclusive

Solution: B

The limit comparison test for convergence lets us determine the convergence or divergence of the given series a_n by *comparing* it to a similar, but simpler comparison series b_n.

We're usually trying to find a comparison series that's a geometric or p-series, since it's very easy to determine the convergence of a geometric or p-series.

We can use the limit comparison test to show that

the original series a_n is **diverging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is diverging

the original series a_n is **converging** if

$a_n > 0$ and $b_n > 0$,

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, and

the comparison series b_n is converging

Before we can use the limit comparison test with the series a_n that we're given in this problem, we need to create a similar, but simpler comparison series b_n.

We'll use the numerator from a_n for b_n, since the numerator is already pretty simple, but we'll leave out the 6 since it'll have less impact on our series as $n \to \infty$. In the denominator, the $3n^2$ carries a lot more weight and will affect the series more than the 1, so we'll use only the $3n^2$ in the denominator of the comparison series, leaving out the 3 since it'll have less effect than the n^2.

$$b_n = \frac{n}{n^2}$$

$$b_n = \frac{1}{n}$$

$$b_n = \frac{1}{n^1}$$

The comparison series is a p-series. Since the p-series test tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

we can say that $1 \leq 1$ and therefore that b_n diverges.

Knowing that the comparison series diverges, we need to show that

$a_n > 0$ and $b_n > 0$, and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

in order to prove that the original series a_n is also diverging. If we can't verify these inequalities, then the limit comparison test will be inconclusive.

To verify that $a_n > 0$ and $b_n > 0$, we'll compare a few points from a_n and b_n. Let's use $n = 1$, $n = 2$ and $n = 3$.

	a_n	b_n
$n = 1$	$a_1 = \frac{2(1)}{3(1)^2 + 1} = \frac{1}{2}$	$b_1 = \frac{1}{1} = 1$
$n = 2$	$a_2 = \frac{2(2)}{3(2)^2 + 1} = \frac{4}{13}$	$b_2 = \frac{1}{2}$
$n = 3$	$a_3 = \frac{2(3)}{3(3)^2 + 1} = \frac{3}{14}$	$b_3 = \frac{1}{3}$

Looking at just these few terms, we can see that $a_n > 0$ and $b_n > 0$ for all n, so our series satisfy the first two inequalities.

Now we need to verify that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$$

Plugging our series into this formula, we get

$$L = \lim_{n \to \infty} \frac{\frac{2n}{3n^2 + 1}}{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \frac{2n}{3n^2 + 1} \left(\frac{n}{1} \right)$$

$$L = \lim_{n \to \infty} \frac{2n^2}{3n^2 + 1}$$

$$L = \lim_{n\to\infty} \frac{2n^2}{3n^2+1}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$L = \lim_{n\to\infty} \frac{\frac{2n^2}{n^2}}{\frac{3n^2}{n^2}+\frac{1}{n^2}}$$

$$L = \lim_{n\to\infty} \frac{2}{3+\frac{1}{n^2}}$$

$$L = \frac{2}{3+\frac{1}{\infty}}$$

$$L = \frac{2}{3+0}$$

$$L = \frac{2}{3}$$

Since

$$L = \frac{2}{3} > 0$$

we've shown that the limit comparison test is valid for this problem, and therefore that the original series a_n diverges since b_n diverges.

Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first three terms.

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+1}$$

Answer choices:

A $R_3 \leq 0.0333$

B $R_3 \leq 0.3000$

C $R_3 \leq 0.2500$

D $R_3 \leq 0.3333$

Solution: D

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first three terms s_3 of our original series a_n.

$$n = 1 \qquad a_1 = \frac{1}{3(1)^2 + 1} \qquad a_1 = \frac{1}{4}$$

$$n = 2 \qquad a_2 = \frac{1}{3(2)^2 + 1} \qquad a_2 = \frac{1}{13}$$

$$n = 3 \qquad a_3 = \frac{1}{3(3)^2 + 1} \qquad a_3 = \frac{1}{28}$$

The sum of the first three terms of the series a_n is

$$s_3 = \frac{1}{4} + \frac{1}{13} + \frac{1}{28}$$

$$s_3 = 0.2500 + 0.0769 + 0.0357$$

$$s_3 = 0.3626$$

Since we've rounded our decimals, we'll say

$$s_3 \approx 0.3626$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already simple. For the denominator, we can use n^2, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{1}{n^2}$$

The comparison series b_n is a p-series where $p = 2$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \leq 1$

Since $p = 2$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \leq a_n \leq b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$$n = 1 \qquad b_1 = \frac{1}{(1)^2} \qquad b_1 = 1$$

$$n = 2 \qquad b_2 = \frac{1}{(2)^2} \qquad b_2 = \frac{1}{4}$$

$$n = 3 \qquad b_3 = \frac{1}{(3)^2} \qquad b_3 = \frac{1}{9}$$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \le a_n \le b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first three terms, R_3. We'll call the remainder of the comparison series b_n after the first three terms, T_3. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_3 \le T_3$, which will be less than the total area under b_n.

$$R_3 \le T_3 \le \int_3^\infty b_n \ dx = \int_3^\infty f(x) \ dx$$

$$R_3 \le T_3 \le \int_3^\infty b_n \ dx = \int_3^\infty \frac{1}{x^2} \ dx$$

$$R_3 \le T_3 \le \int_3^\infty b_n \ dx = \int_3^\infty x^{-2} \ dx$$

$$R_3 \le \frac{x^{-1}}{-1}\Bigg|_3^\infty$$

$$R_3 \le \lim_{b \to \infty} \frac{x^{-1}}{-1}\Bigg|_3^b$$

$$R_3 \le \lim_{b \to \infty} -\frac{1}{x}\Bigg|_3^b$$

$$R_3 \le \lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{3}\right)$$

$$R_3 \le \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{3}$$

$$R_3 \le -\frac{1}{\infty} + \frac{1}{3}$$

$$R_3 \le 0 + \frac{1}{3}$$

$$R_3 \le \frac{1}{3}$$

$$R_3 \le 0.3333$$

The third partial sum of the series a_n is $s_3 \approx 0.3626$, with error $R_3 \le 0.3333$.

Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first five terms.

$$\sum_{n=1}^{\infty} \frac{n}{5n^4 + 2}$$

Answer choices:

A $R_5 \leq 0.0800$

B $R_5 \leq 0.2500$

C $R_5 \leq 0.0200$

D $R_5 \leq 0.2000$

Solution: C

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first five terms s_5 of our original series a_n.

$$n=1 \qquad a_1=\frac{(1)}{5(1)^4+2} \qquad a_1=\frac{1}{7}$$

$$n=2 \qquad a_2=\frac{(2)}{5(2)^4+2} \qquad a_2=\frac{1}{41}$$

$$n=3 \qquad a_3=\frac{(3)}{5(3)^4+2} \qquad a_3=\frac{3}{407}$$

$$n=4 \qquad a_4=\frac{(4)}{5(4)^4+2} \qquad a_4=\frac{2}{641}$$

$$n=5 \qquad a_5=\frac{(5)}{5(5)^4+2} \qquad a_5=\frac{5}{3{,}127}$$

The sum of the first five terms of the series a_n is

$$s_5=\frac{1}{7}+\frac{1}{41}+\frac{3}{407}+\frac{2}{641}+\frac{5}{3{,}127}$$

$$s_5 = 0.1429 + 0.0244 + 0.0074 + 0.0031 + 0.0016$$

$$s_5 = 0.1794$$

Since we've rounded our decimals, we'll say

$$s_5 \approx 0.1794$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already pretty simple. For the denominator, we can use n^4, since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{n}{n^4}$$

$$b_n = \frac{1}{n^3}$$

The comparison series b_n is a p-series where $p = 3$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \leq 1$

Since $p = 3$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \leq a_n \leq b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n = 1, 2, 3$.

$$n = 1 \qquad b_1 = \frac{1}{(1)^3} \qquad b_1 = 1$$

$$n = 2 \qquad b_2 = \frac{1}{(2)^3} \qquad b_2 = \frac{1}{8}$$

$$n = 3 \qquad b_3 = \frac{1}{(3)^3} \qquad b_3 = \frac{1}{27}$$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \le a_n \le b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first five terms, R_5. We'll call the remainder of the comparison series b_n after the first five terms, T_5. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_5 \le T_5$, which will be less than the total area under b_n.

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty f(x) \ dx$$

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty \frac{1}{x^3} \ dx$$

$$R_5 \le T_5 \le \int_5^\infty b_n \ dx = \int_5^\infty x^{-3} \ dx$$

$$R_5 \le \frac{x^{-2}}{-2} \Bigg|_5^\infty$$

$$R_5 \leq \lim_{b\to\infty} \frac{x^{-2}}{-2}\Bigg|_5^b$$

$$R_5 \leq \lim_{b\to\infty} -\frac{1}{2x^2}\Bigg|_5^b$$

$$R_5 \leq \lim_{b\to\infty} -\frac{1}{2(b)^2} - \left[-\frac{1}{2(5)^2}\right]$$

$$R_5 \leq \lim_{b\to\infty} -\frac{1}{2b^2} + \frac{1}{50}$$

$$R_5 \leq -\frac{1}{\infty} + \frac{1}{50}$$

$$R_5 \leq 0 + \frac{1}{50}$$

$$R_5 \leq \frac{1}{50}$$

$$R_5 \leq 0.0200$$

The fifth partial sum of the series a_n is $s_5 \approx 0.1794$, with error $R_5 \leq 0.0200$.

Topic: Error or remainder of a series

Question: Estimate the remainder of the series using the first seven terms.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^4 + 1}}$$

Answer choices:

A $R_7 \leq 0.1429$

B $R_7 \leq 0.0204$

C $R_7 \leq 0.2858$

D $R_7 \leq 0.0408$

Solution: A

To find the remainder of the series, we'll need to

Estimate the total sum by calculating a **partial sum** for the series.

Use the **comparison test** to say whether the series converges or diverges.

Use the **integral test** to solve for the remainder.

The first thing we need to do is to find the sum of the first seven terms s_7 of our original series a_n.

$$n = 1 \qquad a_1 = \frac{1}{\sqrt{2(1)^4 + 1}} \qquad a_1 = \frac{1}{\sqrt{3}}$$

$$n = 2 \qquad a_2 = \frac{1}{\sqrt{2(2)^4 + 1}} \qquad a_2 = \frac{1}{\sqrt{33}}$$

$$n = 3 \qquad a_3 = \frac{1}{\sqrt{2(3)^4 + 1}} \qquad a_3 = \frac{1}{\sqrt{163}}$$

$$n = 4 \qquad a_4 = \frac{1}{\sqrt{2(4)^4 + 1}} \qquad a_4 = \frac{1}{\sqrt{513}}$$

$$n = 5 \qquad a_5 = \frac{1}{\sqrt{2(5)^4 + 1}} \qquad a_5 = \frac{1}{\sqrt{1{,}251}}$$

$$n = 6 \qquad a_6 = \frac{1}{\sqrt{2(6)^4 + 1}} \qquad a_6 = \frac{1}{\sqrt{2{,}593}}$$

$$n = 7 \qquad a_7 = \frac{1}{\sqrt{2(7)^4 + 1}} \qquad a_7 = \frac{1}{\sqrt{4{,}803}}$$

The sum of the first seven terms of the series a_n is

$$s_7 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{33}} + \frac{1}{\sqrt{163}} + \frac{1}{\sqrt{513}} + \frac{1}{\sqrt{1{,}251}} + \frac{1}{\sqrt{2{,}593}} + \frac{1}{\sqrt{4{,}803}}$$

$$s_7 = 0.5774 + 0.1741 + 0.0783 + 0.0442 + 0.0283 + 0.0196 + 0.0144$$

$$s_7 = 0.9363$$

Since we've rounded our decimals, we'll say

$$s_7 \approx 0.9363$$

Next, we need to use the comparison test to figure out whether a_n converges or diverges. We will need to create a similar but simpler comparison series b_n. We can use the same numerator in b_n as the numerator from a_n, since it's already simple. For the denominator, we can use n^2 (the square root of n^4), since it's the element of the denominator that has the most impact on the series. The comparison series b_n will be

$$b_n = \frac{1}{n^2}$$

The comparison series b_n is a p-series where $p = 2$. The p-series test tells us that the series

will converge when $p > 1$

will diverge when $p \le 1$

Since $p=2$, we know that b_n converges.

To use the comparison test to show that a_n also converges, we have to show that $0 \le a_n \le b_n$. We'll find some of the first few values of the comparison series b_n and compare them to a_n. Let's use $n=1, 2, 3$.

$$n=1 \qquad b_1 = \frac{1}{1^2} \qquad b_1 = 1$$

$$n=2 \qquad b_2 = \frac{1}{2^2} \qquad b_2 = \frac{1}{4}$$

$$n=3 \qquad b_3 = \frac{1}{3^2} \qquad b_3 = \frac{1}{9}$$

Looking at these three terms, we can see that all of our answers have $b_n > a_n$ as well as $a_n > 0$. Since we have verified $0 \le a_n \le b_n$, we can state that a_n converges.

Now that we know that the series converges, we'll use the integral test to find the remainder of the series a_n after the first seven terms, R_7. We'll call the remainder of the comparison series b_n after the first seven terms, T_7. Since we know that $0 \le a_n \le b_n$, and that a_n and b_n converge, we can say that $R_7 \le T_7$, which will be less than the total area under b_n.

$$R_7 \le T_7 \le \int_7^\infty b_n\ dx = \int_7^\infty f(x)\ dx$$

$$R_7 \le T_7 \le \int_7^\infty b_n\ dx = \int_7^\infty \frac{1}{x^2}\ dx$$

$$R_7 \le T_7 \le \int_7^\infty b_n \ dx = \int_7^\infty x^{-2} \ dx$$

$$R_7 \le \frac{x^{-1}}{-1}\Bigg|_7^\infty$$

$$R_7 \le -\frac{1}{x}\Bigg|_7^\infty$$

$$R_7 \le \lim_{b\to\infty} -\frac{1}{x}\Bigg|_7^b$$

$$R_7 \le \lim_{b\to\infty} -\frac{1}{b} - \left(-\frac{1}{7}\right)$$

$$R_7 \le -\frac{1}{\infty} + \frac{1}{7}$$

$$R_7 \le 0 + \frac{1}{7}$$

$$R_7 \le \frac{1}{7}$$

$$R_7 \le 0.1429$$

The seventh partial sum of the series a_n is $s_7 \approx 0.9363$, with error $R_7 \le 0.1429$.

Topic: Ratio test

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

The ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{n}{2^n}$$

$$a_{n+1} = \frac{n+1}{2^{n+1}}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \frac{2^n}{2^n 2^1} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \frac{1}{2} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n+1}{2n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n}{2n} + \frac{1}{2n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{2} + \frac{1}{2n} \right|$$

$$L = \left| \frac{1}{2} + 0 \right|$$

$$L = \frac{1}{2}$$

or

$$L = \frac{1}{2} < 1$$

Therefore, the series is convergent for all $x \in R$.

Topic: Ratio test

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: C

The ratio test for convergence lets us calculate L as

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{3^n}{n^2}$$

$$a_{n+1} = \frac{3^{n+1}}{(n+1)^2}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot 3^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot 3^1 \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{n^2}{n^2+2n+1} \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{n^2}{n^2+2n+1} \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right) \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}} \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} \right|$$

$$L = 3 \left| \frac{1}{1+0+0} \right|$$

$$L = 3 \left| 1 \right|$$

$$L = 3$$

or

$$L = 3 > 1$$

Therefore, the series is divergent for all $x \in R$.

Topic: Ratio test

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{2n^2}{10^n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

The ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{2n^2}{10^n}$$

$$a_{n+1} = \frac{2(n+1)^2}{10^{n+1}}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{2(n+1)^2}{10^{n+1}}}{\frac{2n^2}{10^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{2(n+1)^2}{10^{n+1}} \cdot \frac{10^n}{2n^2} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{10^n}{10^{n+1}} \cdot \frac{2(n+1)^2}{2n^2} \right|$$

$$L = \lim_{n \to \infty} \left| 10^{n-(n+1)} \cdot \frac{2(n^2+2n+1)}{2n^2} \right|$$

$$L = \lim_{n \to \infty} \left| 10^{n-n-1} \cdot \frac{n^2+2n+1}{n^2} \right|$$

$$L = \lim_{n \to \infty} \left| 10^{-1} \cdot \frac{n^2+2n+1}{n^2} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{10} \cdot \frac{n^2+2n+1}{n^2} \right|$$

$$L = \frac{1}{10} \lim_{n \to \infty} \left| \frac{n^2+2n+1}{n^2} \right|$$

$$L = \frac{1}{10} \lim_{n \to \infty} \left| \frac{n^2+2n+1}{n^2} \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right) \right|$$

$$L = \frac{1}{10} \lim_{n \to \infty} \left| \frac{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2}} \right|$$

$$L = \frac{1}{10} \lim_{n \to \infty} \left| \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1} \right|$$

$$L = \frac{1}{10} \lim_{n \to \infty} \left| 1 + \frac{2}{n} + \frac{1}{n^2} \right|$$

$$L = \frac{1}{10} \left| 1 + 0 + 0 \right|$$

$$L = \frac{1}{10} \left| 1 \right|$$

$$L = \frac{1}{10}$$

or

$$L = \frac{1}{10} < 1$$

Therefore, the series is convergent for all $x \in R$.

Topic: Ratio test with factorials

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

The ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{x^n}{n!}$$

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$$

Expanding the factorials so that we can get an idea of what we can cancel, and then canceling terms, we get

$$L = \lim_{n \to \infty} \left| x^{n+1-n} \cdot \frac{n(n-1)(n-2)(n-3)\dots}{(n+1)(n+1-1)(n+1-2)(n+1-3)(n+1-4)\dots} \right|$$

$$L = \lim_{n \to \infty} \left| x \cdot \frac{n(n-1)(n-2)(n-3)\dots}{(n+1)(n)(n-1)(n-2)(n-3)\dots} \right|$$

$$L = \lim_{n \to \infty} \left| x \cdot \frac{1}{n+1} \right|$$

Since the limit only effects n, we can pull x outside of the limit.

$$L = x \lim_{n \to \infty} \left| \frac{1}{n+1} \right|$$

$$L = x(0)$$

$$L = 0$$

or

$$L = 0 < 1$$

Therefore, the series is convergent for all $x \in R$.

Topic: Ratio test with factorials

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

The ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{3^n}{n!}$$

$$a_{n+1} = \frac{3^{n+1}}{(n+1)!}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \right|$$

Expanding the factorials so that we can get an idea of what we can cancel, and then canceling terms, we get

$$L = \lim_{n\to\infty} \left| 3^{n+1-n} \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n+1-1)(n+1-2)(n+1-3)(n+1-4)\ldots} \right|$$

$$L = \lim_{n\to\infty} \left| 3^1 \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n)(n-1)(n-2)(n-3)\ldots} \right|$$

$$L = 3 \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$$

$$L = 3(0)$$

$$L = 0$$

or

$$L = 0 < 1$$

Therefore, the series is convergent for all $x \in R$.

Topic: Ratio test with factorials

Question: Use the ratio test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{n!}{4^n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: C

The ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

The test is inconclusive if $L = 1$.

To find L, we'll need a_n and a_{n+1}.

$$a_n = \frac{n!}{4^n}$$

$$a_{n+1} = \frac{(n+1)!}{4^{n+1}}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{(n+1)!}{4^{n+1}}}{\frac{n!}{4^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)!}{4^{n+1}} \cdot \frac{4^n}{n!} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n\to\infty} \left| \frac{4^n}{4^{n+1}} \cdot \frac{(n+1)!}{n!} \right|$$

Expanding the factorials so that we can get an idea of what we can cancel, and then canceling terms, we get

$$L = \lim_{n\to\infty} \left| 4^{n-(n+1)} \cdot \frac{(n+1)(n+1-1)(n+1-2)(n+1-3)(n+1-4)\dots}{n(n-1)(n-2)(n-3)\dots} \right|$$

$$L = \lim_{n\to\infty} \left| 4^{n-n-1} \cdot \frac{(n+1)(n)(n-1)(n-2)(n-3)\dots}{n(n-1)(n-2)(n-3)\dots} \right|$$

$$L = \lim_{n\to\infty} \left| 4^{-1} \cdot \frac{n+1}{1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{1}{4} \cdot (n+1) \right|$$

$$L = \frac{1}{4} \lim_{n\to\infty} \left| n+1 \right|$$

$$L = \frac{1}{4}(\infty + 1)$$

$$L = \frac{\infty}{4}$$

$$L = \infty$$

or

$$L = \infty > 1$$

Therefore, the series is divergent for all $x \in R$.

Topic: Root test

Question: Use the root test to determine the convergence of the series.

$$\sum_{n=2}^{\infty}\left(\frac{3n^3+4n^2-7}{\sqrt{4n^6+9n^4-10}}\right)^n$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: C

Let

$$a_n = \left(\frac{3n^3+4n^2-7}{\sqrt{4n^6+9n^4-10}}\right)^n$$

Then by the root test,

$$R = \lim_{n\to\infty} \left|a_n\right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left|\left(\frac{3n^3+4n^2-7}{\sqrt{4n^6+9n^4-10}}\right)^n\right|^{\frac{1}{n}}$$

Taking the nth root gives us

$$R = \lim_{n\to\infty} \left|\frac{3n^3+4n^2-7}{\sqrt{4n^6+9n^4-10}}\right|$$

Dividing both the numerator and denominator by the highest power of n,

$$R = \lim_{n\to\infty} \left|\frac{3n^3+4n^2-7}{\sqrt{4n^6+9n^4-10}}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)\right|$$

$$R = \lim_{n\to\infty} \left|\frac{3+\frac{4}{n}-\frac{7}{n^3}}{\sqrt{4n^6+9n^4-10}}\cdot\frac{1}{\frac{1}{\sqrt{n^6}}}\right|$$

$$R = \lim_{n\to\infty} \left| \frac{3 + \frac{4}{n} - \frac{7}{n^3}}{\frac{\sqrt{4n^6 + 9n^4 - 10}}{\sqrt{n^6}}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{3 + \frac{4}{n} - \frac{7}{n^3}}{\sqrt{\frac{4n^6 + 9n^4 - 10}{n^6}}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{3 + \frac{4}{n} - \frac{7}{n^3}}{\sqrt{4 + \frac{9}{n^2} - \frac{10}{n^6}}} \right|$$

$$R = \left| \frac{3 + 0 - 0}{\sqrt{4 + 0 - 0}} \right|$$

$$R = \left| \frac{3}{\sqrt{4}} \right|$$

$$R = \frac{3}{2}$$

Since

$$R = \frac{3}{2} > 1$$

the series is divergent.

Topic: Root test

Question: Use the root test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

Let

$$a_n = \frac{n}{e^{n^2}}$$

Then by the root test,

$$R = \lim_{n\to\infty} \left| a_n \right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left| \frac{n}{e^{n^2}} \right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left| \frac{n^{\frac{1}{n}}}{\left(e^{n^2}\right)^{\frac{1}{n}}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{n^{\frac{1}{n}}}{e^n} \right|$$

$$R = \left| \frac{1}{\infty} \right| = 0 < 1$$

Since

$$R = 0 < 1$$

the series is convergent.

Topic: Root test

Question: Use the root test to determine the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D None of these

Solution: A

Let

$$a_n = \frac{2^n}{(n+1)^n}$$

Then by the root test,

$$R = \lim_{n\to\infty} \left| a_n \right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left| \frac{2^n}{(n+1)^n} \right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left| \left(\frac{2}{n+1} \right)^n \right|^{\frac{1}{n}}$$

$$R = \lim_{n\to\infty} \left| \left(\frac{2}{n+1} \right)^{n \cdot \frac{1}{n}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{2}{n+1} \right|$$

$$R = \left| \frac{2}{\infty + 1} \right|$$

$$R = \left| \frac{2}{\infty} \right|$$

$$R = |0|$$

$$R = 0$$

Since

$$R = 0 < 1$$

the series is convergent.

Topic: Absolute and conditional convergence

Question: Determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

Both the ratio and root tests can determine absolute vs. conditional convergence of a series.

The series converges absolutely if $a_n = |a_n|$ for all possible values of n

The series converges conditionally if $a_n \neq |a_n|$ for all possible values of n

Since all terms in the given series

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

are raised to the power of n, we should use the root test to determine convergence.

Let

$$a_n = \left(\frac{n}{2n+1} \right)^n$$

Then by the root test,

$$R = \lim_{n \to \infty} \left| \left(\frac{n}{2n+1} \right)^n \right|^{\frac{1}{n}}$$

$$R = \lim_{n \to \infty} \left| \left(\frac{n}{2n+1} \right)^{n \cdot \frac{1}{n}} \right|$$

$$R = \lim_{n \to \infty} \left| \frac{n}{2n+1} \right|$$

$$R = \lim_{n \to \infty} \left| \frac{n}{2n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$R = \lim_{n \to \infty} \left| \frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}} \right|$$

$$R = \lim_{n \to \infty} \left| \frac{1}{2 + \frac{1}{n}} \right|$$

$$R = \left| \frac{1}{2 + \frac{1}{\infty}} \right|$$

$$R = \left| \frac{1}{2+0} \right|$$

$$R = \left| \frac{1}{2} \right|$$

$$R = \frac{1}{2}$$

Since

$$R = \frac{1}{2} < 1$$

the series converges absolutely.

Topic: Absolute and conditional convergence

Question: Use the ratio test to determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

Both the ratio and root tests can determine absolute vs. conditional convergence of a series.

The series converges absolutely if $a_n = |a_n|$ for all possible values of n

The series converges conditionally if $a_n \neq |a_n|$ for all possible values of n

Since the given series

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

would be easier to evaluate with the ratio test than the root test, and the ratio test for convergence lets us calculate L as

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and then says that the series

converges if $L < 1$

diverges if $L > 1$

we'll find L, by starting with a_n and a_{n+1}.

$$a_n = \frac{n+1}{2^n}$$

$$a_{n+1} = \frac{n+1+1}{2^{n+1}} = \frac{n+2}{2^{n+1}}$$

Plugging these into the formula for L from the ratio test, we get

$$L = \lim_{n\to\infty} \left| \frac{\frac{n+2}{2^{n+1}}}{\frac{n+1}{2^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{n+2}{2^{n+1}} \cdot \frac{2^n}{n+1} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n-(n+1)} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n-n-1} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{-1} \cdot \frac{n+2}{n+1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{1}{2} \cdot \frac{n+2}{n+1} \right|$$

$$L = \frac{1}{2} \lim_{n\to\infty} \left| \frac{n+2}{n+1} \right|$$

$$L = \frac{1}{2} \lim_{n \to \infty} \left| \frac{n+2}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = \frac{1}{2} \lim_{n \to \infty} \left| \frac{\frac{n}{n} + \frac{2}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = \frac{1}{2} \lim_{n \to \infty} \left| \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \right|$$

$$L = \frac{1}{2} \left| \frac{1 + \frac{2}{\infty}}{1 + \frac{1}{\infty}} \right|$$

$$L = \frac{1}{2} \left| \frac{1+0}{1+0} \right|$$

$$L = \frac{1}{2} \left| 1 \right|$$

$$L = \frac{1}{2}$$

or

$$L = \frac{1}{2} < 1$$

Therefore, the series converges absolutely for all $x \in R$.

Topic: Absolute and conditional convergence

Question: Determine the convergence (absolute or conditional) of the series.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Answer choices:

A The series converges absolutely

B The series converges conditionally

C The series diverges

D The test was inconclusive

Solution: A

The given series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a p-series with $p = 3$.

The p-series test for convergence tells us that the series will

converge when $p > 1$

diverge when $p \leq 1$

Since $3 > 1$, the given series converges by the p-series test.

Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{3n+1}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{2}{3n+1}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{2}{3(1)+1}$	$\frac{2}{4}$	$\frac{2}{3(2)+1}$	$\frac{2}{7}$
$n = 2$	$\frac{2}{3(2)+1}$	$\frac{2}{7}$	$\frac{2}{3(3)+1}$	$\frac{2}{10}$

$n=3$	$\frac{2}{3(3)+1}$	$\frac{2}{10}$	$\frac{2}{3(4)+1}$	$\frac{2}{13}$
$n=4$	$\frac{2}{3(4)+1}$	$\frac{2}{13}$	$\frac{2}{3(5)+1}$	$\frac{2}{16}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convinced by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{2}{3x+1}\right)$$

Using the quotient rule, we get

$$\frac{0(3x+1)-2(3)}{(3x+1)^2}$$

$$\frac{-6}{(3x+1)^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n=1$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3n+1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2}{n}}{3+\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n = \frac{0}{3+0}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{\ln n}{n}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{\ln 1}{1}$	0	$\frac{\ln 2}{2}$	0.3465736
$n = 2$	$\frac{\ln 2}{2}$	0.3465736	$\frac{\ln 3}{3}$	0.3662041

$n=3$	$\frac{\ln 3}{3}$	0.3662041	$\frac{\ln 4}{4}$	0.3465736
$n=4$	$\frac{\ln 4}{4}$	0.3465736	$\frac{\ln 5}{5}$	0.3218876

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. However, even when we look at the terms to eight decimal places, it's unclear whether or not $0 < a_{n+1} < a_n$. To double-check, we'll take the derivative of a_n. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

Using the quotient rule, we get

$$\frac{\left(\frac{1}{x}\right)x-(\ln x)(1)}{x^2}$$

$$\frac{1-\ln x}{x^2}$$

For $n > 2$, the derivative is negative and the series is decreasing. The final step is to verify that $\lim_{n\to\infty} a_n = 0$. Using L'Hospital's rule, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn}n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Topic: Alternating series test

Question: Use the alternating series test to say whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Answer choices:

A The series converges

B The series conditionally converges

C The series diverges

D The test was inconclusive

Solution: A

The alternating series test for convergence tells us that

an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ where } a_n > 0$$

converges if

$0 < a_{n+1} < a_n$ for all values of n, and

$$\lim_{n \to \infty} a_n = 0$$

When we use the alternating series test, we need to make sure that we separate the series a_n from the $(-1)^n$ part that makes it alternating.

$$a_n = \frac{1}{n}$$

Now we need to show that $0 < a_{n+1} < a_n$. Remembering that this series starts at $n = 1$, let's check the first few terms of the series to see if it looks like $0 < a_{n+1} < a_n$.

		a_n		a_{n+1}
$n = 1$	$\frac{1}{1}$	1	$\frac{1}{1+1}$	$\frac{1}{2}$
$n = 2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2+1}$	$\frac{1}{3}$

$n=3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3+1}$	$\frac{1}{4}$
$n=4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4+1}$	$\frac{1}{5}$

We can see that the terms of a_n and a_{n+1} will always be positive, because there's no value of n, when $n \geq 1$, that will make either series negative. We can also see that a_{n+1} is always going to be smaller than a_n. If you're not convinced by their fractional values in the table, compute the decimal values on your calculator to be sure.

If you can't be sure that $0 < a_{n+1} < a_n$ just by looking at the table, you can always take the derivative of a_n to double-check. If the derivative is negative, then you know the series is decreasing, which means that a_{n+1} will always be less than a_n.

$$\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}\left(x^{-1}\right)$$

$$-x^{-2}$$

$$-\frac{1}{x^2}$$

Looking at the derivative, we can see that for all values of the series (remember, the series starts at $n=1$), the derivative is negative because the numerator will be negative and the denominator will be positive. This confirms that the series is decreasing.

The final step is to verify that $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \frac{1}{\infty}$$

$$\lim_{n\to\infty} a_n = 0$$

Since we've shown that $0 < a_{n+1} < a_n$ and that $\lim_{n\to\infty} a_n = 0$, we can say that the series converges.

Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{10^n}$$

Answer choices:

A $\quad s_3 \approx 0.083 \quad \left| R_3 \right| \leq 0.004$

B $\quad s_3 \approx 0.083 \quad \left| R_3 \right| \leq 0.0004$

C $\quad s_3 \approx 0.083 \quad \left| R_3 \right| \leq 0.0002$

D $\quad s_3 \approx 0.083 \quad \left| R_3 \right| \leq 0.0020$

Solution: B

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left| R_n \right| = \left| s - s_n \right| \leq b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)}{10^1} \qquad a_1 = 0.1$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)}{10^2} \qquad a_2 = -0.02$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)}{10^3} \qquad a_3 = 0.003$$

$$n = 4 \qquad a_4 = \frac{(-1)^{4-1}(4)}{10^4} \qquad a_4 = -0.0004$$

$$n = 5 \qquad a_5 = \frac{(-1)^{5-1}(5)}{10^5} \qquad a_5 = 0.00005$$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.02)$$

$$a_1 + a_2 = 0.1 - 0.02$$

$$a_1 + a_2 = 0.08$$

$$s_2 = 0.08$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.1 + (-0.02) + 0.003$$

$$a_1 + a_2 + a_3 = 0.1 - 0.02 + 0.003$$

$$a_1 + a_2 + a_3 = 0.083$$

$$s_3 = 0.083$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.02) + 0.003 + (-0.0004)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.02 + 0.003 - 0.0004$$

$$a_1 + a_2 + a_3 + a_4 = 0.0826$$

$$s_4 = 0.0826$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_3 \approx 0.083$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \leq b_n$$

and

$$\lim_{n\to\infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{10^n}$$

we can say that

$$b_n = \frac{n}{10^n}$$

So

$$b_{n+1} = \frac{n+1}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

		b_n		b_{n+1}
$n = 1$	$\frac{1}{10^1}$	$\frac{1}{10}$	$\frac{1+1}{10^{1+1}}$	$\frac{1}{50}$
$n = 2$	$\frac{2}{10^2}$	$\frac{1}{50}$	$\frac{2+1}{10^{2+1}}$	$\frac{3}{1{,}000}$
$n = 3$	$\frac{3}{10^3}$	$\frac{3}{1{,}000}$	$\frac{3+1}{10^{3+1}}$	$\frac{1}{2{,}500}$

Looking at our results we can verify that $b_{n+1} \leq b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n}{10^n} = 0$$

Since both rules are true for our series, we can use

$$\left|R_n\right| = \left|s - s_n\right| \le b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_3 \approx 0.083$).

$$\left|R_3\right| = \left|s - s_3\right| \le b_{3+1}$$

We're looking for the remainder, so we'll use

$$\left|R_3\right| \le b_4$$

$$\left|R_3\right| \le \frac{4}{10^4}$$

$$\left|R_3\right| \le \frac{1}{2{,}500}$$

$$\left|R_3\right| \le 0.0004$$

We can now say that our approximation of the sum of the alternating series ($s_3 \approx 0.083$) has an error of $\left|R_3\right| \le 0.0004$.

Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$$

Answer choices:

A $\quad s_3 \approx 0.086 \qquad |R_5| \le 0.000036$

B $\quad s_3 \approx 0.068 \qquad |R_5| \le 0.00036$

C $\quad s_5 \approx 0.086 \qquad |R_5| \le 0.00036$

D $\quad s_5 \approx 0.068 \qquad |R_5| \le 0.000036$

Solution: D

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left| R_n \right| = \left| s - s_n \right| \leq b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)^2}{10^1} \qquad a_1 = 0.1$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)^2}{10^2} \qquad a_2 = -0.04$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)^2}{10^3} \qquad a_3 = 0.009$$

$n = 4 \qquad a_4 = \frac{(-1)^{4-1}(4)^2}{10^4} \qquad a_4 = -0.0016$

$n = 5 \qquad a_5 = \frac{(-1)^{5-1}(5)^2}{10^5} \qquad a_5 = 0.00025$

$n = 6 \qquad a_6 = \frac{(-1)^{6-1}(6)^2}{10^6} \qquad a_6 = -0.000036$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.1 + (-0.04)$$

$$a_1 + a_2 = 0.1 - 0.04$$

$$a_1 + a_2 = 0.06$$

$$s_2 = 0.06$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.1 + (-0.04) + 0.009$$

$$a_1 + a_2 + a_3 = 0.1 - 0.04 + 0.009$$

$$a_1 + a_2 + a_3 = 0.069$$

$$s_3 = 0.069$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.1 + (-0.04) + 0.009 + (-0.0016)$$

$$a_1 + a_2 + a_3 + a_4 = 0.1 - 0.04 + 0.009 - 0.0016$$

$$a_1 + a_2 + a_3 + a_4 = 0.0674$$

$$s_4 = 0.0674$$

The fourth decimal place didn't cause the third decimal place to round up, but the fourth term changed the third decimal place, and we need to find a consistent answer to three decimal places.

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.1 + (-0.04) + 0.009 + (-0.0016) + 0.00025$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.1 - 0.04 + 0.009 - 0.0016 + 0.00025$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0.06765$$

$$s_5 = 0.06765$$

Now we know that the fourth decimal place is going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_5 \approx 0.068$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \leq b_n$$

and

$$\lim_{n\to\infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$$

we can say that

$$b_n = \frac{n^2}{10^n}$$

So

$$b_{n+1} = \frac{(n+1)^2}{10^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

b_n b_{n+1}

$n=1$	$\frac{1^2}{10^1}$	$\frac{1}{10}$	$\frac{(1+1)^2}{10^{1+1}}$	$\frac{1}{25}$
$n=2$	$\frac{2^2}{10^2}$	$\frac{1}{25}$	$\frac{(2+1)^2}{10^{2+1}}$	$\frac{9}{1,000}$
$n=3$	$\frac{3^2}{10^3}$	$\frac{9}{1,000}$	$\frac{(3+1)^2}{10^{3+1}}$	$\frac{1}{625}$

Looking at our results we can verify that $b_{n+1} \le b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n^2}{10^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n^2}{10^n} = 0$$

Since both rules are true for our series, we can use

$$\left|R_n\right| = \left|s - s_n\right| \le b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_5 \approx 0.068$).

$$\left|R_5\right| = \left|s - s_5\right| \le b_{5+1}$$

We're looking for the remainder, so we'll use

$$|R_5| \le b_6$$

$$|R_5| \le \frac{6^2}{10^6}$$

$$|R_5| \le \frac{36}{1,000,000}$$

$$|R_5| \le 0.000036$$

We can now say that our approximation of the sum of the alternating series ($s_5 \approx 0.068$) has an error of $|R_5| \le 0.000036$.

Topic: Alternating series estimation theorem

Question: Approximate the sum of the alternating series to three decimal places, then find the remainder of the approximation.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{4^n}$$

Answer choices:

A $\quad s_8 \approx 0.096 \quad\quad \left| R_8 \right| \leq 0.00031$

B $\quad s_8 \approx 0.099 \quad\quad \left| R_8 \right| \leq 0.00031$

C $\quad s_3 \approx 0.096 \quad\quad \left| R_8 \right| \leq 0.00013$

D $\quad s_3 \approx 0.069 \quad\quad \left| R_8 \right| \leq 0.00031$

Solution: A

The alternating series estimation theorem gives us a way to approximate the sum of an alternating series and calculate the error in our approximation. We can only use the theorem if

$$b_{n+1} \le b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

where b_n is the alternating series.

If the alternating series meets these two conditions, then the error in the estimation of the sum is

$$\left| R_n \right| = \left| s - s_n \right| \le b_{n+1}$$

We've been asked to use the alternating series estimation theorem to approximate the sum of the given series to three decimal places.

We'll calculate the first few terms of the series.

$$n = 1 \qquad a_1 = \frac{(-1)^{1-1}(1)^2}{4^1} \qquad a_1 = 0.25$$

$$n = 2 \qquad a_2 = \frac{(-1)^{2-1}(2)^2}{4^2} \qquad a_2 = -0.25$$

$$n = 3 \qquad a_3 = \frac{(-1)^{3-1}(3)^2}{4^3} \qquad a_3 = 0.1406$$

$n = 4$ $\quad a_4 = \frac{(-1)^{4-1}(4)^2}{4^4}$ $\quad a_4 = -0.0625$

$n = 5$ $\quad a_5 = \frac{(-1)^{5-1}(5)^2}{4^5}$ $\quad a_5 = 0.0244$

$n = 6$ $\quad a_6 = \frac{(-1)^{6-1}(6)^2}{4^6}$ $\quad a_6 = -0.0088$

$n = 7$ $\quad a_7 = \frac{(-1)^{7-1}(7)^2}{4^7}$ $\quad a_7 = 0.0030$

$n = 8$ $\quad a_8 = \frac{(-1)^{8-1}(8)^2}{4^8}$ $\quad a_8 = -0.0010$

$n = 9$ $\quad a_9 = \frac{(-1)^{9-1}(9)^2}{4^9}$ $\quad a_9 = 0.0003$

Next, we need to sum these terms until we can see that the third decimal place isn't changing.

Adding the first two terms together, we get

$$a_1 + a_2 = 0.25 + (-0.25)$$

$$a_1 + a_2 = 0.25 - 0.25$$

$$a_1 + a_2 = 0$$

$$s_2 = 0$$

Since we're not to three decimal places, we'll add another term to the sum.

$$a_1 + a_2 + a_3 = 0.25 + (-0.25) + 0.1406$$

$$a_1 + a_2 + a_3 = 0.25 - 0.25 + 0.1406$$

$$a_1 + a_2 + a_3 = 0.1406$$

$$s_3 = 0.1406$$

We've made it to three decimal places, but we need to make sure that the third decimal place doesn't change, or that the fourth decimal place won't cause the third decimal place to round up.

$$a_1 + a_2 + a_3 + a_4 = 0.25 + (-0.25) + 0.1406 + (-0.0625)$$

$$a_1 + a_2 + a_3 + a_4 = 0.25 - 0.25 + 0.1406 - 0.0625$$

$$a_1 + a_2 + a_3 + a_4 = 0.0781$$

$$s_4 = 0.0781$$

We need to keep going until the first three decimal places aren't changing.

For s_5:

$$s_5 = 0.0781 + 0.0244$$

$$s_5 = 0.1025$$

For s_6:

$$s_6 = 0.1025 + (-0.0088)$$

$$s_6 = 0.1025 - 0.0088$$

$$s_6 = 0.0937$$

For s_7:

$$s_7 = 0.0937 + 0.0030$$

$$s_7 = 0.0967$$

For s_8:

$$s_8 = 0.0967 + (-0.0010)$$

$$s_8 = 0.0967 - 0.0010$$

$$s_8 = 0.0957$$

For s_9:

$$s_9 = 0.0957 + 0.0003$$

$$s_9 = 0.0960$$

Now we know that the fourth decimal place isn't going to cause us to round up the third decimal place, and our approximation to three decimal places is

$$s_8 \approx 0.096$$

We've got an estimation of the sum of the alternating series, so our next step is to calculate the error in our estimation. If we want to use the alternating series estimation theorem, we'll need to verify that

$$b_{n+1} \le b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

Since the sum of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

and the given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{4^n}$$

we can say that

$$b_n = \frac{n^2}{4^n}$$

So

$$b_{n+1} = \frac{(n+1)^2}{4^{n+1}}$$

Now we can calculate the first three terms for both b_n and b_{n+1}.

		b_n		b_{n+1}
$n = 1$	$\frac{1^2}{4^1}$	$\frac{1}{4}$	$\frac{(1+1)^2}{4^{1+1}}$	$\frac{1}{4}$

$n=2$	$\frac{2^2}{4^2}$	$\frac{1}{4}$	$\frac{(2+1)^2}{4^{2+1}}$	$\frac{9}{64}$
$n=3$	$\frac{3^2}{4^3}$	$\frac{9}{64}$	$\frac{(3+1)^2}{4^{3+1}}$	$\frac{1}{16}$

Looking at our results we can verify that $b_{n+1} \le b_n$.

Next, we need to show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} \frac{n^2}{4^n}$$

When we evaluate b_n as it approaches infinity, we can see that the denominator will increase much more quickly than the numerator, so

$$\lim_{n\to\infty} \frac{n^2}{4^n} = 0$$

Since both rules are true for our series, we can use

$$\left|R_n\right| = \left|s - s_n\right| \le b_{n+1}$$

to estimate the error in our approximation of the sum of the series. n will be taken from the approximate sum we already calculated ($s_8 \approx 0.096$).

$$\left|R_8\right| = \left|s - s_8\right| \le b_{8+1}$$

We're looking for the remainder, so we'll use

$$\left|R_8\right| \le b_9$$

$$\left|R_8\right| \leq \frac{9^2}{4^9}$$

$$\left|R_8\right| \leq \frac{81}{262,144}$$

$$\left|R_8\right| \leq 0.00031$$

We can now say that our approximation of the sum of the alternating series ($s_8 \approx 0.096$) has an error of $\left|R_8\right| \leq 0.00031$.

Topic: Power series representation

Question: Find the power series representation of the function.

$$f(x) = \frac{2x}{4 + x^2}$$

Answer choices:

A $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^{n+1}}$

B $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}}$

C $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^{2n+1}}$

D $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{n+1}}$

Solution: B

To find the power series representation of the given function

$$f(x) = \frac{2x}{4+x^2}$$

we'll use the standard form of a power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

We'll manipulate the given series until it's in the form

$$\frac{1}{1-x}$$

We'll get

$$\frac{1}{1-x} = \frac{2x}{4+x^2}$$

$$\frac{1}{1-x} = (2x)\frac{1}{4+x^2}$$

$$\frac{1}{1-x} = (2x)\frac{1}{4\left(1+\frac{x^2}{4}\right)}$$

$$\frac{1}{1-x} = \left(\frac{2x}{4}\right)\frac{1}{1+\frac{x^2}{4}}$$

$$\frac{1}{1-x} = \left(\frac{x}{2}\right)\frac{1}{1+\frac{x^2}{4}}$$

$$\frac{1}{1-x}=\left(\frac{x}{2}\right)\frac{1}{1-\left(-\frac{x^2}{4}\right)}$$

Matching the new form of the given series to $1/(1-x)$, we can see that $-\left(x^2\right)/4$ from the given series is going to represent x from the standard power series. Plugging this value into the standard form of a power series, we get

$$\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n$$

But don't forget about the $x/2$ that we factored out of the given series! We'll need to multiply the sum by this term.

$$\frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{x^1}{2^1}\left[\frac{(-1)x^2}{4}\right]^n$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{2^1}\left(\frac{x^{2n}}{4^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^1(-1)^n}{2^1}\left(\frac{x^{2n}}{2^{2n}}\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}}$$

This is the power series representation of the given function.

Topic: Power series representation

Question: Which statement is true about the behavior of the series when $n \to \infty$?

$$\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

Answer choices:

A Series diverges for $-1 \leq x < 1$.

B Series converges for $-1 \leq x < 1$.

C Series diverges for $0 \leq x < 2$.

D Series converges for $-3 \leq x < 0$.

Solution: B

Given the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

apply the ratio test,

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{\ln(n+1+1)}}{\frac{x^n}{\ln(n+1)}} \right|$$

then simplify the expression.

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\ln(n+1+1)} \cdot \frac{\ln(n+1)}{x^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{\ln(n+1)}{\ln(n+1+1)} \right|$$

$$L = \lim_{n\to\infty} \left| x^{n+1-n} \cdot \frac{\ln(n+1)}{\ln(n+2)} \right|$$

$$L = \lim_{n\to\infty} \left| x \cdot \frac{\ln(n+1)}{\ln(n+2)} \right|$$

The limit affects n, not x, so we can pull $|x|$ out in front of the limit.

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n+2)} \right|$$

Rewrite the fraction.

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln\left(\frac{1}{n}(n+1)n\right)}{\ln\left(\frac{1}{n}(n+2)n\right)} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln\left(\left(1+\frac{1}{n}\right)n\right)}{\ln\left(\left(1+\frac{2}{n}\right)n\right)} \right|$$

Use laws of logarithms to simplify.

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n + \ln\left(1+\frac{2}{n}\right)} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n + \ln\left(1+\frac{2}{n}\right)} \cdot \frac{\frac{1}{\ln n}}{\frac{1}{\ln n}} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{\frac{\ln n}{\ln n} + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}{\frac{\ln n}{\ln n} + \frac{\ln\left(1+\frac{2}{n}\right)}{\ln n}} \right|$$

$$L = |x| \lim_{n\to\infty} \left| \frac{1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}}{1 + \frac{\ln\left(1+\frac{2}{n}\right)}{\ln n}} \right|$$

If we now evaluate the limit as $n \to \infty$, the $1/n$ in the numerator and the $2/n$ in the denominator both go to 0.

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{\ln(1+0)}{\ln n}}{1 + \frac{\ln(1+0)}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{\ln 1}{\ln n}}{1 + \frac{\ln 1}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1 + \frac{0}{\ln n}}{1 + \frac{0}{\ln n}} \right|$$

$$L = |x| \lim_{n \to \infty} \left| \frac{1+0}{1+0} \right|$$

$$L = |x| \lim_{n \to \infty} \left| 1 \right|$$

$$L = |x|(1)$$

$$L = |x|$$

This will converge when $L < 1$, so

$$|x| < 1$$

$$-1 \leq x \leq 1$$

But if we test the endpoints of the interval, $x = -1$ and $x = 1$, we can see that the series converges at $x = -1$, but not at $x = 1$. For $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

Given the (-1^n) in the numerator, this is an alternating series, so you want to use the alternating series test. Therefore, pull the (-1^n) out in front of the sum.

$$(-1)^n \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

By the alternating series test, the remaining series, $a_n = 1/\ln(n+1)$ will converge if 1) a_n is decreasing and 2) $\lim_{n \to \infty} a_n = 0$. We can tell that the series is decreasing, because as n gets larger, the $n+1$ in the denominator gets larger, and as the argument of $\ln$ gets larger, the value for the log gets larger and larger. And as the denominator of a fraction gets larger, the value of the fraction in general gets smaller. So we know the series is decreasing. Now we just need to show that $\lim_{n \to \infty} a_n = 0$.

$$\lim_{n \to \infty} \frac{1}{\ln(n+1)} = 0$$

$$\frac{1}{\ln(\infty + 1)} = 0$$

$$\frac{1}{\ln(\infty)} = 0$$

$$\frac{1}{\infty} = 0$$

$$0 = 0$$

We've now shown that $a_n = 1/\ln(n+1)$ is decreasing and that $\lim_{n\to\infty} a_n = 0$. Therefore, we have proof that $a_n = 1/\ln(n+1)$ converges by the alternating series test, which means the series converges at $x = -1$.

For $x = 1$,

$$\sum_{n=1}^{\infty} \frac{(1)^n}{\ln(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

For this series $a_n = \frac{1}{\ln(n+1)}$, we can use the comparison test to determine convergence. We'll use the harmonic series $b_n = 1/n$, which we know diverges. Since we know it diverges, we want to see if we can prove that $a_n > b_n$. If a_n is always greater than b_n, and b_n diverges, then we'll know that a_n also diverges. Let's compare some values of n in both series.

For $n = 1$, $a_n = 1/\ln 2 = 1.44$ and $b_n = 1/1 = 1.00$

For $n = 2$, $a_n = 1/\ln 3 = 0.91$ and $b_n = 1/2 = 0.50$

For $n = 3$, $a_n = 1/\ln 4 = 0.72$ and $b_n = 1/3 = 0.33$

For $n = 4$, $a_n = 1/\ln 5 = 0.62$ and $b_n = 1/4 = 0.25$

For $n = 5$, $a_n = 1/\ln 6 = 0.56$ and $b_n = 1/5 = 0.20$

We can see that a_n is always going to be greater than b_n, and since we know that b_n diverges, this proves by the comparison test that a_n must also diverge.

Therefore, the original series in this problem converges at $x = -1$, but diverges at $x = 1$. So the series converges for $-1 \leq x < 1$.

Topic: Power series representation

Question: Which series represents the expression?

$$e^{\frac{1}{2x^2}}$$

Answer choices:

A $\sum_{n=0}^{\infty} \frac{1}{2^n n! x^{2n}}$

B $\sum_{n=0}^{\infty} \frac{1}{n! x^{2n-1}}$

C $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n-1} n!}$

D $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{2^n n!}$

Solution: A

We know that the power series expression of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replacing e^x with $e^{\frac{1}{x}}$ gives us

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{x}\right)^n}{n!}$$

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{\frac{1^n}{x^n}}{n!}$$

$$e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{1}{x^n n!}$$

Now if we substitute $2x^2$ for x, we'll arrive at the power series representation for $e^{\frac{1}{2x^2}}$.

$$e^{\frac{1}{2x^2}} = \sum_{n=0}^{\infty} \frac{1}{(2x^2)^n n!}$$

$$e^{\frac{1}{2x^2}} = \sum_{n=0}^{\infty} \frac{1}{2^n n! x^{2n}}$$

Topic: Power series multiplication

Question: Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = \sin(2x)e^{2x}$$

Answer choices:

A $\quad \sin(2x)e^{2x} = 2x - 4x^2 + \frac{8x^3}{3} - \frac{16x^5}{5}$

B $\quad \sin(2x)e^{2x} = 2x + 4x^2 - \frac{8x^3}{3} - \frac{16x^5}{5}$

C $\quad \sin(2x)e^{2x} = 2x + 4x^2 + \frac{8x^3}{3} + \frac{16x^5}{5}$

D $\quad \sin(2x)e^{2x} = 2x + 4x^2 + \frac{8}{3}x^3 - \frac{16}{15}x^5$

Solution: D

When we multiply two power series together, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the product of the series will be like multiplying polynomials.

We need to recognize that the given series is the product of two other series

$$y = \sin(2x)$$

$$y = e^{2x}$$

There are common Maclaurin series that are similar to each of these.

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5{,}040}x^7 + \frac{1}{362{,}880}x^9 - \dots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$

We want to modify each of these common series to match the given series.

For $y = \sin(2x)$, we'll substitute $2x$ for x:

$$\sin(2x) = 2x - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 - \frac{1}{5{,}040}(2x)^7 + \frac{1}{362{,}880}(2x)^9 - \dots$$

$$\sin(2x) = 2x - \frac{1}{6}(8x^3) + \frac{1}{120}(32x^5) - \frac{1}{5{,}040}(128x^7) + \frac{1}{362{,}880}(512x^9) - \dots$$

$$\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \frac{4}{2{,}835}x^9 - \dots$$

For $y = e^{2x}$, we'll substitute $2x$ for x:

$$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 + \frac{1}{120}(2x)^5 + \dots$$

$$e^{2x} = 1 + 2x + \frac{1}{2}(4x^2) + \frac{1}{6}(8x^3) + \frac{1}{24}(16x^4) + \frac{1}{120}(32x^5) + \dots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \dots$$

Multiplying the modified series together, we get

$$\sin(2x)e^{2x} = \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \frac{4}{2{,}835}x^9 - \dots\right)$$

$$\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \dots\right)$$

We need to multiply every term in the first series by every term in the second series.

$$\sin(2x)e^{2x} = 2x\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \dots\right)$$

$$-\frac{4}{3}x^3\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \dots\right)$$

$$+\frac{4}{15}x^5\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \dots\right)$$

$$-\frac{8}{315}x^7\left(1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\frac{4}{15}x^5+\dots\right)$$

$$+\frac{4}{2,835}x^9\left(1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\frac{4}{15}x^5+\dots\right)$$

$$\sin(2x)e^{2x}=2x+4x^2+4x^3+\frac{8}{3}x^4+\frac{4}{3}x^5+\frac{8}{15}x^6$$

$$-\frac{4}{3}x^3-\frac{8}{3}x^4-\frac{8}{3}x^5-\frac{16}{9}x^6-\frac{8}{9}x^7-\frac{16}{45}x^8$$

$$+\frac{4}{15}x^5+\frac{8}{15}x^6+\frac{8}{15}x^7+\frac{16}{45}x^8+\frac{8}{45}x^9+\frac{16}{225}x^{10}$$

$$-\frac{8}{315}x^7-\frac{16}{315}x^8-\frac{16}{315}x^9-\frac{32}{945}x^{10}-\frac{16}{945}x^{11}-\frac{32}{4,725}x^{12}$$

$$+\frac{4}{2,835}x^9+\frac{8}{2,835}x^{10}+\frac{8}{2,835}x^{11}+\frac{16}{8,505}x^{12}+\frac{8}{8,505}x^{13}+\frac{16}{42,525}x^{14}$$

Group like terms together.

$$\sin(2x)e^{2x}=2x+4x^2+\left(4x^3-\frac{4}{3}x^3\right)+\left(\frac{8}{3}x^4-\frac{8}{3}x^4\right)$$

$$+\left(\frac{4}{3}x^5-\frac{8}{3}x^5+\frac{4}{15}x^5\right)+\left(\frac{8}{15}x^6-\frac{16}{9}x^6+\frac{8}{15}x^6\right)$$

$$+\left(-\frac{8}{9}x^7+\frac{8}{15}x^7-\frac{8}{315}x^7\right)+\left(-\frac{16}{45}x^8+\frac{16}{45}x^8-\frac{16}{315}x^8\right)$$

$$+\left(\frac{8}{45}x^9-\frac{16}{315}x^9+\frac{4}{2,835}x^9\right)+\left(\frac{16}{225}x^{10}-\frac{32}{945}x^{10}+\frac{8}{2,835}x^{10}\right)$$

$$+\left(-\frac{16}{945}x^{11}+\frac{8}{2,835}x^{11}\right)+\left(-\frac{32}{4,725}x^{12}+\frac{16}{8,505}x^{12}\right)$$

$$+\frac{8}{8,505}x^{13}+\frac{16}{42,525}x^{14}$$

Since we only need the first four non-zero terms, we can at least simplify to

$$\sin(2x)e^{2x}=2x+4x^2+\left(\frac{12}{3}x^3-\frac{4}{3}x^3\right)+\left(\frac{8}{3}x^4-\frac{8}{3}x^4\right)$$

$$+\left(\frac{20}{15}x^5-\frac{40}{15}x^5+\frac{4}{15}x^5\right)+\left(\frac{24}{45}x^6-\frac{80}{45}x^6+\frac{24}{45}x^6\right)$$

$$\sin(2x)e^{2x}=2x+4x^2+\frac{8}{3}x^3+0-\frac{16}{15}x^5-\frac{32}{45}x^6$$

Take just the first four non-zero terms, and then the answer is

$$\sin(2x)e^{2x}=2x+4x^2+\frac{8}{3}x^3-\frac{16}{15}x^5$$

Topic: Power series multiplication

Question: Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = 3\tan(3x)\ln(1+2x)$$

Answer choices:

A $\quad 3\tan(3x)\ln(1+2x) = 6x^2 + 6x^3 + 26x^4 + 30x^5$

B $\quad 3\tan(3x)\ln(1+2x) = 18x^2 + 18x^3 + 78x^4 + 90x^5$

C $\quad 3\tan(3x)\ln(1+2x) = 18x^2 - 18x^3 + 78x^4 - 90x^5$

D $\quad 3\tan(3x)\ln(1+2x) = 6x^2 - 6x^3 + 26x^4 - 30x^5$

Solution: C

When we multiply two power series together, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the product of the series will be like multiplying polynomials.

We need to recognize that the given series is the product of two other series

$$y = 3\tan(3x)$$

$$y = \ln(1 + 2x)$$

There are common Maclaurin series that are similar to each of these.

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

We want to modify each of these common series to match the given series.

For $y = 3\tan(3x)$, we'll substitute $3x$ for x and then multiply by 3:

$$\tan(3x) = 3x + \frac{1}{3}(3x)^3 + \frac{2}{15}(3x)^5 + \dots$$

$$\tan(3x) = 3x + \frac{27x^3}{3} + \frac{486x^5}{15} + \dots$$

$$3\tan(3x) = 3\left(3x + \frac{27x^3}{3} + \frac{486x^5}{15} + \dots\right)$$

$$3\tan(3x) = 9x + 27x^3 + \frac{486x^5}{5} + \dots$$

For $y = \ln(1 + 2x)$, we'll substitute $2x$ for x:

$$\ln(1 + 2x) = 2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + \dots$$

$$\ln(1 + 2x) = 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots$$

Multiplying the modified series together, we get

$$3\tan(3x)\ln(1 + 2x) = \left(9x + 27x^3 + \frac{486x^5}{5} + \dots\right)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots\right)$$

We need to multiply every term in the first series by every term in the second series.

$$3\tan(3x)\ln(1 + 2x) = 9x\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots\right)$$

$$+27x^3\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots\right)$$

$$+\frac{486x^5}{5}\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots\right)$$

$$3\tan(3x)\ln(1 + 2x) = 18x^2 - 18x^3 + 24x^4 - 36x^5 + 54x^4 - 54x^5 + 72x^6 - 108x^7$$

$$+\frac{972x^6}{5}-\frac{972x^7}{5}+\frac{3,888x^8}{15}-\frac{1,944x^9}{5}$$

Group like terms together.

$$3\tan(3x)\ln(1+2x)=18x^2-18x^3+\left(24x^4+54x^4\right)+\left(-36x^5-54x^5\right)$$

$$+\left(72x^6+\frac{972x^6}{5}\right)+\left(-108x^7-\frac{972x^7}{5}\right)+\frac{3,888x^8}{15}-\frac{1,944x^9}{5}$$

$$3\tan(3x)\ln(1+2x)=18x^2-18x^3+78x^4-90x^5$$

$$+\left(72x^6+\frac{972x^6}{5}\right)+\left(-108x^7-\frac{972x^7}{5}\right)+\frac{3,888x^8}{15}-\frac{1,944x^9}{5}$$

Since we only need the first four non-zero terms, our answer will be

$$3\tan(3x)\ln(1+2x)=18x^2-18x^3+78x^4-90x^5$$

Topic: Power series multiplication

Question: Use power series multiplication to find the first four non-zero terms of the Maclaurin series.

$$y = \sin^{-1}(3x)e^{3x}$$

Answer choices:

A $\sin^{-1}(3x)e^{3x} = 3x + 9x^2 + 18x^3 + 27x^4$

B $\sin^{-1}(3x)e^{3x} = x - 3x^2 + 9x^3 - 18x^4$

C $\sin^{-1}(3x)e^{3x} = -x + 3x^2 - 9x^3 + 18x^4$

D $\sin^{-1}(3x)e^{3x} = x - 3x^2 + 9x^3 - 9x^4$

Solution: A

When we multiply two power series together, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the product of the series will be like multiplying polynomials.

We need to recognize that the given series is the product of two other series

$$y = \sin^{-1}(3x)$$

$$y = e^{3x}$$

There are common Maclaurin series that are similar to each of these.

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

We want to modify each of these common series to match the given series.

For $y = \sin^{-1}(3x)$, we'll substitute $3x$ for x:

$$\sin^{-1}(3x) = 3x + \frac{1}{6}(3x)^3 + \frac{3}{40}(3x)^5 + \dots$$

$$\sin^{-1}(3x) = 3x + \frac{9x^3}{2} + \frac{729x^5}{40} + \dots$$

For $y = e^{3x}$, we'll substitute $3x$ for x:

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots$$

Multiplying the modified series together, we get

$$\sin^{-1}(3x)e^{3x} = \left(3x + \frac{9x^3}{2} + \frac{729x^5}{40} + \dots\right)\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots\right)$$

We need to multiply every term in the first series by every term in the second series.

$$\sin^{-1}(3x)e^{3x} = 3x\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots\right) + \frac{9}{2}x^3\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots\right)$$

$$+\frac{729x^5}{40}\left(1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots\right)$$

$$\sin^{-1}(3x)e^{3x} = 3x + 9x^2 + \frac{27x^3}{2} + \frac{27x^4}{2} + \frac{9x^3}{2} + \frac{27x^4}{2} + \frac{81x^5}{4} + \frac{81x^6}{4}$$

$$+\frac{729x^5}{40} + \frac{2,187x^6}{40} + \frac{6,561x^7}{80} + \frac{6,561x^8}{80}$$

Group like terms together.

$$\sin^{-1}(3x)e^{3x} = 3x + 9x^2 + \left(\frac{27x^3}{2} + \frac{9x^3}{2}\right) + \left(\frac{27x^4}{2} + \frac{27x^4}{2}\right)$$

$$+\left(\frac{81x^5}{4}+\frac{729x^5}{40}\right)+\left(\frac{81x^6}{4}+\frac{2,187x^6}{40}\right)+\frac{6,561x^7}{80}+\frac{6,561x^8}{80}$$

$$\sin^{-1}(3x)e^{3x}=3x+9x^2+18x^3+27x^4$$

$$+\left(\frac{81x^5}{4}+\frac{729x^5}{40}\right)+\left(\frac{81x^6}{4}+\frac{2,187x^6}{40}\right)+\frac{6,561x^7}{80}+\frac{6,561x^8}{80}$$

Since we only need the first four non-zero terms, our answer will be

$$\sin^{-1}(3x)e^{3x}=3x+9x^2+18x^3+27x^4$$

Topic: Power series division

Question: Use power series division to find the first three non-zero terms of the Maclaurin series.

$$y = \frac{x}{e^{2x}}$$

Answer choices:

A $\quad \dfrac{x}{e^{2x}} = x + 2x^2 + 2x^3$

B $\quad \dfrac{x}{e^{2x}} = x - 2x^2 + 2x^3$

C $\quad \dfrac{x}{e^{2x}} = x - 2x^2 - 2x^3$

D $\quad \dfrac{x}{e^{2x}} = -x - 2x^2 - 2x^3$

Solution: B

When we divide one power series by another, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the quotient of the series will be like dividing polynomials.

The numerator of the given function, x, is already in polynomial form, so we just need a series expansion for the denominator.

There's a common Maclaurin series that's similar to e^{2x}.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

We want to modify this common series to match the given series.

For $y = e^{2x}$, we'll substitute $2x$ for x:

$$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \dots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$$

Dividing the numerator, x, by the modified series, we get

$$
\begin{array}{r}
x-2x^2+2x^3-\frac{4x^4}{3} \\
1+2x+2x^2+\frac{4x^3}{3}\ \overline{\big)\, x+0x^2+0x^3+0x^4} \\
-\left(x+2x^2+2x^3+\frac{4x^4}{3}\right) \\
\hline
-2x^2-2x^3-\frac{4x^4}{3} \\
-\left(-2x^2-4x^3-4x^4\right) \\
\hline
2x^3+\frac{8x^4}{3} \\
-\left(2x^3+4x^4\right) \\
\hline
-\frac{4x^4}{3} \\
-\left(-\frac{4x^4}{3}\right) \\
\hline
0
\end{array}
$$

Since we only need the first three non-zero terms, our answer will be

$$\frac{x}{e^{2x}} = x - 2x^2 + 2x^3$$

Topic: Power series division

Question: Use power series division to find the first three non-zero terms of the Maclaurin series.

$$y=\frac{2x}{\ln(1+2x)}$$

Answer choices:

A $\quad \dfrac{2x}{\ln(1+2x)}=1-x+\dfrac{x^2}{3}$

B $\quad \dfrac{2x}{\ln(1+2x)}=1+x+\dfrac{x^2}{3}$

C $\quad \dfrac{2x}{\ln(1+2x)}=1+x-\dfrac{x^2}{3}$

D $\quad \dfrac{2x}{\ln(1+2x)}=1-x-\dfrac{x^2}{3}$

Solution: C

When we divide one power series by another, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the quotient of the series will be like dividing polynomials.

The numerator of the given function, $2x$, is already in polynomial form, so we just need a series expansion for the denominator.

There's a common Maclaurin series that's similar to $\ln(1+2x)$.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

We want to modify this common series to match the given series.

For $y = \ln(1+2x)$, we'll substitute $2x$ for x:

$$\ln(1+2x) = 2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 + \ldots$$

$$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} + \ldots$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} + \ldots$$

Dividing the numerator, $2x$, by the modified series, we get

$$
\begin{array}{r|l}
 & 1 + x - \dfrac{x^2}{3} \\
\hline
2x - 2x^2 + \dfrac{8x^3}{3} & 2x + 0x^2 + 0x^3 + 0x^4 \\
 & -\left(2x - 2x^2 + \dfrac{8x^3}{3}\right) \\
\cline{2-2}
 & 2x^2 - \dfrac{8x^3}{3} \\
 & -\left(2x^2 - 2x^3 + \dfrac{8x^4}{3}\right) \\
\cline{2-2}
 & -\dfrac{2x^3}{3} - \dfrac{8x^4}{3} \\
 & -\left(-\dfrac{2x^3}{3} + \dfrac{2x^4}{3} - \dfrac{8x^5}{9}\right) \\
\cline{2-2}
 & -\dfrac{10x^4}{3} + \dfrac{8x^5}{9}
\end{array}
$$

Since we only need the first three non-zero terms, our answer will be

$$\frac{2x}{\ln(1+2x)} = 1 + x - \frac{x^2}{3}$$

Topic: Power series division

Question: Use power series division to find the first three non-zero terms of the Maclaurin series.

$$y = \frac{e^{3x}}{\frac{1}{1-3x}}$$

Answer choices:

A $\quad \dfrac{e^{3x}}{\frac{1}{1-3x}} = 1 + \dfrac{9x}{2} + 9x^2$

B $\quad \dfrac{e^{3x}}{\frac{1}{1-3x}} = 1 + \dfrac{9x^2}{2} - 9x^3$

C $\quad \dfrac{e^{3x}}{\frac{1}{1-3x}} = 1 - \dfrac{9x}{2} + 9x^2$

D $\quad \dfrac{e^{3x}}{\frac{1}{1-3x}} = 1 - \dfrac{9x^2}{2} - 9x^3$

Solution: D

When we divide one power series by another, we want to find the expansion of the sum of each series, so that we essentially have polynomial representations. Then finding the quotient of the series will be like dividing polynomials.

We need to recognize that the given series is the quotient of two other series

$$y = e^{3x}$$

$$y = \frac{1}{1-3x}$$

There are common Maclaurin series that are similar to each of these.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

We want to modify each of these common series to match the given series.

For $y = e^{3x}$, we'll substitute $3x$ for x:

$$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + \dots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots$$

For $y = 1/(1-3x)$, we'll substitute $3x$ for x:

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \dots$$

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \dots$$

Dividing these modified series, we get

$$\begin{array}{r l}
 & 1-\frac{9x^2}{2}-9x^3 \\
1+3x+9x^2+27x^3 & \overline{\big)\,1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}} \\
 & -(1+3x+9x^2+27x^3) \\ \hline
 & \qquad -\frac{9x^2}{2}-\frac{45x^3}{2} \\
 & \qquad -\left(-\frac{9x^2}{2}-\frac{27x^3}{2}\right) \\ \hline
 & \qquad\qquad -9x^3 \\
 & \qquad\qquad -(-9x^3) \\ \hline
 & \qquad\qquad\quad 0
\end{array}$$

Since we only need the first three non-zero terms, our answer will be

$$\frac{e^{3x}}{\frac{1}{1-3x}} = 1 - \frac{9x^2}{2} - 9x^3$$

Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(2-x)^2}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty} (-1)^n n(x-1)^n$

B $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$

C $\displaystyle\sum_{n=0}^{\infty} (n+1)(x-1)^n$

D $\displaystyle\sum_{n=0}^{\infty} (-1)^n + 2(n+1)(x-1)^n$

Solution: C

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(2-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(2-x)^2}\ dx$$

$$u = 2 - x$$

$du = -\ dx$, so $dx = -\ du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2}\ du$$

$$u^{-1} + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{2-x} + C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{2-x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{2-x}$$

we'll substitute $x-1$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1-(x-1)} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots = \sum_{n=0}^{\infty} (x-1)^n$$

$$\frac{1}{1-x+1} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots = \sum_{n=0}^{\infty} (x-1)^n$$

$$\frac{1}{2-x} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots = \sum_{n=0}^{\infty} (x-1)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(2-x)-(1)(-1)}{(2-x)^2} = 1 + 2(x-1) + 3(x-1)^2 + \dots = \sum_{n=0}^{\infty} n(x-1)^{n-1}$$

$$\frac{1}{(2-x)^2} = 1 + 2(x-1) + 3(x-1)^2 + \dots = \sum_{n=0}^{\infty} n(x-1)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(2-x)^2} = \sum_{n+1=0}^{\infty} (n+1)(x-1)^{n+1-1}$$

$$\frac{1}{(2-x)^2} = \sum_{n=-1}^{\infty} (n+1)(x-1)^n$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(2-x)^2} = \sum_{n=0}^{\infty} (n+1)(x-1)^n$$

This is the power series representation of the function.

Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(-2-x)^2}$$

Answer choices:

A $\sum_{n=0}^{\infty} (-1)^n(n+1)(x+2)^n$

B $\sum_{n=0}^{\infty} (n+1)(x+3)^n$

C $\sum_{n=0}^{\infty} n(x+3)^n$

D $\sum_{n=0}^{\infty} (-1)^n(n+1)(x-3)^n$

Solution: B

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(-2-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(-2-x)^2}\ dx$$

$$u = -2 - x$$

$du = -\,dx$, so $dx = -\,du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2}\ du$$

$$u^{-1}+C$$

$$\frac{1}{u}+C$$

$$\frac{1}{-2-x}+C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{-2-x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{-2-x}$$

we'll substitute $x+3$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1-(x+3)}=1+(x+3)+(x+3)^2+(x+3)^3+\ldots=\sum_{n=0}^{\infty}(x+3)^n$$

$$\frac{1}{1-x-3} = 1 + (x+3) + (x+3)^2 + (x+3)^3 + \ldots = \sum_{n=0}^{\infty} (x+3)^n$$

$$\frac{1}{-2-x} = 1 + (x+3) + (x+3)^2 + (x+3)^3 + \ldots = \sum_{n=0}^{\infty} (x+3)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(-2-x)-(1)(-1)}{(-2-x)^2} = 1 + 2(x+3) + 3(x+3)^2 + \ldots = \sum_{n=0}^{\infty} n(x+3)^{n-1}$$

$$\frac{1}{(-2-x)^2} = 1 + 2(x+3) + 3(x+3)^2 + \ldots = \sum_{n=0}^{\infty} n(x+3)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(-2-x)^2} = \sum_{n+1=0}^{\infty} (n+1)(x+3)^{n+1-1}$$

$$\frac{1}{(-2-x)^2} = \sum_{n=-1}^{\infty} (n+1)(x+3)^{n}$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(-2-x)^2} = \sum_{n=0}^{\infty} (n+1)(x+3)^n$$

This is the power series representation of the function.

Topic: Power series differentiation

Question: Differentiate to find the power series representation of the function.

$$f(x) = \frac{1}{(4-x)^2}$$

Answer choices:

A $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)(x-4)^n$

B $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)(x+3)^n$

C $\displaystyle\sum_{n=0}^{\infty} (-1)^n n(x+3)^n$

D $\displaystyle\sum_{n=0}^{\infty} (n+1)(x-3)^n$

Solution: D

When we use differentiation to find the power series representation of a function like the given function

$$f(x) = \frac{1}{(4-x)^2}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with this standard form, and then manipulate it until the value on the far left is equal to the integral of the given function. This way, when we take the derivative of the manipulated standard form, the far left will become the given function, and we can use the sum of the series on the far right as the power series representation of the given series.

To figure out what we need the far left side to become, we'll integrate the given function.

$$\int \frac{1}{(4-x)^2} \ dx$$

$$u = 4 - x$$

$du = -\ dx$, so $dx = -\ du$

$$\int \frac{1}{u^2}(-du)$$

$$\int -u^{-2}\ du$$

$$u^{-1} + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{4-x} + C$$

This tells us that we need to get the far left side of the standard form of a power series to equal

$$\frac{1}{4-x}$$

We'll start to manipulate the standard form in this direction, but we have to remember that, in order to keep the equation balanced, whenever we make a change on the far left, we have to make the same change to the expanded form of the series in the middle and to the sum of the power series on the right.

To get the far left side of the standard form of a power series to

$$\frac{1}{4-x}$$

we'll substitute $x-3$ in for x. We need to make sure we make the same change to the middle and to the right.

$$\frac{1}{1-(x-3)} = 1 + (x-3) + (x-3)^2 + (x-3)^3 + \ldots = \sum_{n=0}^{\infty} (x-3)^n$$

$$\frac{1}{1-x+3}=1+(x-3)+(x-3)^2+(x-3)^3+\ldots=\sum_{n=0}^{\infty}(x-3)^n$$

$$\frac{1}{4-x}=1+(x-3)+(x-3)^2+(x-3)^3+\ldots=\sum_{n=0}^{\infty}(x-3)^n$$

Now, if we differentiate (take the derivative) the value on the far left, the result will be the given function, which is exactly what we want. So now we'll take the derivative of the left, middle, and right sides of this manipulated standard form.

Using quotient rule to take the derivative of the left side, power rule and chain rule to take the derivative of the middle, and power rule and chain rule to take the derivative of the right side, we get

$$\frac{(0)(4-x)-(1)(-1)}{(4-x)^2}=1+2(x-3)+3(x-3)^2+\ldots=\sum_{n=0}^{\infty}n(x-3)^{n-1}$$

$$\frac{1}{(4-x)^2}=1+2(x-3)+3(x-3)^2+\ldots=\sum_{n=0}^{\infty}n(x-3)^{n-1}$$

Since the left side is now equal to the given function, we'll use the right side as our power series representation. Whenever possible though, we want the power on the x variable to be n instead of $n-1$. To accomplish this, we'll do an index shift and substitute $n+1$ in for n.

$$\frac{1}{(4-x)^2}=\sum_{n+1=0}^{\infty}(n+1)(x-3)^{n+1-1}$$

$$\frac{1}{(4-x)^2}=\sum_{n=-1}^{\infty}(n+1)(x-3)^n$$

Since plugging in $n = -1$ yields a 0 term, we can shift the index back to $n = 0$ without effecting the value of the series.

$$\frac{1}{(4-x)^2} = \sum_{n=0}^{\infty} (n+1)(x-3)^n$$

This is the power series representation of the function.

Topic: Radius of convergence

Question: Find the radius of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(x-1)^n}{2^n}$$

Answer choices:

A $\quad 2$

B $\quad 1$

C $\quad \frac{1}{2}$

D $\quad -1$

Solution: A

We can use the ratio test for convergence to find the radius of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n = \frac{(-1)^n n(x-1)^n}{2^n}$$

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)(x-1)^{n+1}}{2^{n+1}}$$

Plugging these into the formula for L, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(n+1)(x-1)^{n+1}}{2^{n+1}}}{\frac{(-1)^n n(x-1)^n}{2^n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+1)(x-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n(x-1)^n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{(-1)^{n+1}(n+1)(x-1)^{n+1}}{(-1)^n n(x-1)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(x-1)^{n+1}}{(x-1)^n} \cdot \frac{n+1}{n} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n-(n+1)} \cdot (-1)^{n+1-n} \cdot (x-1)^{n+1-n} \cdot \frac{n+1}{n} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{n-n-1} \cdot (-1) \cdot (x-1) \cdot \frac{n+1}{n} \right|$$

$$L = \lim_{n\to\infty} \left| 2^{-1} \cdot (-1) \cdot (x-1) \cdot \frac{n+1}{n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{1}{2} \cdot (-1) \cdot (x-1) \cdot \frac{n+1}{n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{-(x-1)}{2} \cdot \frac{n+1}{n} \right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L = \left| \frac{x-1}{2} \right| \lim_{n\to\infty} \left| -\frac{n+1}{n} \right|$$

$$L = \left| \frac{x-1}{2} \right| \lim_{n\to\infty} \left| -\frac{n+1}{n} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L=\left|\frac{x-1}{2}\right|\lim_{n\to\infty}\left|-\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}}\right|$$

$$L=\left|\frac{x-1}{2}\right|\lim_{n\to\infty}\left|-\frac{1+\frac{1}{n}}{1}\right|$$

$$L=\left|\frac{x-1}{2}\right|\lim_{n\to\infty}\left|-1-\frac{1}{n}\right|$$

$$L=\left|\frac{x-1}{2}\right|\left|-1-\frac{1}{\infty}\right|$$

$$L=\left|\frac{x-1}{2}\right|\left|-1-0\right|$$

$$L=\left|\frac{x-1}{2}\right|\left|-1\right|$$

$$L=\left|\frac{x-1}{2}\right|(1)$$

$$L=\left|\frac{x-1}{2}\right|$$

The ratio test tells us that the series converges when $L<1$. Plugging L into this inequality, we get

$$\left|\frac{x-1}{2}\right|<1$$

$$\frac{1}{2}\left|x-1\right|<1$$

$$\left|x-1\right|<2$$

With the interval of convergence in the form $|x-a|<R$, the radius of convergence is R. Therefore, we can say that the radius of convergence is $R=2$.

Topic: Radius of convergence

Question: Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$$

Answer choices:

A $-\frac{1}{3}$

B -3

C 3

D $\frac{1}{3}$

Solution: C

We can use the root test for convergence to find the radius of convergence of a series. The root test tells us that, for a series a_n, if

$$L = \lim_{n\to\infty} \sqrt[n]{\left| a_n \right|}$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

Plugging a_n into the formula for L, we get

$$L = \lim_{n\to\infty} \sqrt[n]{\left| \frac{(-1)^n x^n}{3^n} \right|}$$

$$L = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-x}{3} \right)^n \right|}$$

$$L = \lim_{n\to\infty} \left| \left(\frac{-x}{3} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n\to\infty} \left| \left(\frac{-x}{3} \right)^{n \cdot \frac{1}{n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{-x}{3} \right|$$

Since there are no ns remaining, and the limit only effects n, we can eliminate the limit.

$$L = \left| \frac{-x}{3} \right|$$

$$L = \left| \frac{x}{3} \right|$$

The root test tells us that the series converges when $L < 1$. Plugging L into this inequality, we get

$$\left| \frac{x}{3} \right| < 1$$

$$\frac{1}{3} \left| x \right| < 1$$

$$\left| x \right| < 3$$

$$\left| x - 0 \right| < 3$$

With the interval of convergence in the form $|x - a| < R$, the radius of convergence is R. Therefore, we can say that the radius of convergence is $R = 3$.

Topic: Radius of convergence

Question: Find the radius of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(x+1)^n}{5^n}$$

Answer choices:

A 1

B 5

C $\frac{1}{5}$

D -1

Solution: B

We can use the ratio test for convergence to find the radius of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

then the series converges absolutely if $L<1$. Therefore, we'll find the value of L for the given series, plug it into $L<1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n=\frac{(-1)^n n(x+1)^n}{5^n}$$

$$a_{n+1}=\frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{5^{n+1}}$$

Plugging these into the formula for L, we get

$$L=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{5^{n+1}}}{\frac{(-1)^n n(x+1)^n}{5^n}}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(-1)^n n(x+1)^n}\right|$$

Pairing similar numerators and denominators together, we get

$$L=\lim_{n\to\infty}\left|\frac{5^n}{5^{n+1}}\cdot\frac{(-1)^{n+1}(n+1)(x+1)^{n+1}}{(-1)^n n(x+1)^n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{5^n}{5^{n+1}}\cdot\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(x+1)^{n+1}}{(x+1)^n}\cdot\frac{n+1}{n}\right|$$

$$L=\lim_{n\to\infty}\left|5^{n-(n+1)}\cdot(-1)^{n+1-n}\cdot(x+1)^{n+1-n}\cdot\frac{n+1}{n}\right|$$

$$L=\lim_{n\to\infty}\left|5^{n-n-1}\cdot(-1)\cdot(x+1)\cdot\frac{n+1}{n}\right|$$

$$L=\lim_{n\to\infty}\left|5^{-1}\cdot(-1)\cdot(x+1)\cdot\frac{n+1}{n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{1}{5}\cdot(-1)\cdot(x+1)\cdot\frac{n+1}{n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{-(x+1)}{5}\cdot\frac{n+1}{n}\right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L=\left|\frac{x+1}{5}\right|\lim_{n\to\infty}\left|-\frac{n+1}{n}\right|$$

$$L=\left|\frac{x+1}{5}\right|\lim_{n\to\infty}\left|-\frac{n+1}{n}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right|$$

$$L = \left| \frac{x+1}{5} \right| \lim_{n \to \infty} \left| -\frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n}} \right|$$

$$L = \left| \frac{x+1}{5} \right| \lim_{n \to \infty} \left| -\frac{1 + \frac{1}{n}}{1} \right|$$

$$L = \left| \frac{x+1}{5} \right| \lim_{n \to \infty} \left| -1 - \frac{1}{n} \right|$$

$$L = \left| \frac{x+1}{5} \right| \left| -1 - \frac{1}{\infty} \right|$$

$$L = \left| \frac{x+1}{5} \right| \left| -1 - 0 \right|$$

$$L = \left| \frac{x+1}{5} \right| \left| -1 \right|$$

$$L = \left| \frac{x+1}{5} \right| (1)$$

$$L = \left| \frac{x+1}{5} \right|$$

The ratio test tells us that the series converges when $L < 1$. Plugging L into this inequality, we get

$$\left| \frac{x+1}{5} \right| < 1$$

$$\frac{1}{5}\left|x+1\right| < 1$$

$$\left|x+1\right| < 5$$

$$\left|x-(-1)\right| < 5$$

With the interval of convergence in the form $|x-a| < R$, the radius of convergence is R. Therefore, we can say that the radius of convergence is $R = 5$.

Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{8^n}$$

Answer choices:

A $2 \leq x \leq 14$

B $2 < x < 14$

C $-2 \leq x \leq 14$

D $-2 < x < 14$

Solution: D

We can use the ratio test for convergence to find the interval of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n = \frac{(x-6)^n}{8^n}$$

$$a_{n+1} = \frac{(x-6)^{n+1}}{8^{n+1}}$$

Plugging these into the formula for L, we get

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-6)^{n+1}}{8^{n+1}}}{\frac{(x-6)^n}{8^n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(x-6)^{n+1}}{8^{n+1}} \cdot \frac{8^n}{(x-6)^n} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n\to\infty} \left| \frac{(x-6)^{n+1}}{(x-6)^n} \cdot \frac{8^n}{8^{n+1}} \right|$$

$$L = \lim_{n\to\infty} \left| (x-6)^{n+1-n} \cdot 8^{n-(n+1)} \right|$$

$$L = \lim_{n\to\infty} \left| (x-6)^{1} \cdot 8^{n-n-1} \right|$$

$$L = \lim_{n\to\infty} \left| (x-6) \cdot 8^{-1} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{x-6}{8} \right|$$

Since there are no ns remaining, and the limit only effects n, we can eliminate the limit.

$$L = \left| \frac{x-6}{8} \right|$$

The ratio test tells us that the series converges when $L < 1$. Plugging L into this inequality, we get

$$\left| \frac{x-6}{8} \right| < 1$$

$$-1 < \frac{x-6}{8} < 1$$

$$-8 < x-6 < 8$$

$$-2 < x < 14$$

We always have to check both endpoints of the interval before we can give a final answer for the interval of convergence. For this particular series, we can use the nth-term test to check the convergence of the endpoints. By the nth-term test,

if $\lim_{n\to\infty} a_n = 0$, the test is inconclusive

if $\lim_{n\to\infty} a_n \neq 0$, the series diverges

We'll check the endpoints.

At $x = -2$:

$$\lim_{n\to\infty} \frac{(-2-6)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{(-8)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{(-1)^n 8^n}{8^n}$$

$$\lim_{n\to\infty} (-1)^n$$

No limit

At $x = 14$:

$$\lim_{n\to\infty} \frac{(14-6)^n}{8^n}$$

$$\lim_{n\to\infty} \frac{8^n}{8^n}$$

$$\lim_{n\to\infty} 1$$

1

Since both endpoints returned a non-zero value, by the nth-term test the series diverges at both endpoints. Therefore, the interval of convergence is the open interval $-2 < n < 14$.

Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{5x^{2n+1}}{n!}$$

Answer choices:

A $-1 < x < 1$

B $-1 \leq x \leq 1$

C $-\infty < x < \infty$

D $0 < x < \infty$

Solution: C

We can use the ratio test for convergence to find the interval of convergence of a series. The ratio test tells us that, for a series a_n, if

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

In order to get L, we'll need a_n and a_{n+1}.

$$a_n = \frac{5x^{2n+1}}{n!}$$

$$a_{n+1} = \frac{5x^{2(n+1)+1}}{(n+1)!} = \frac{5x^{2n+2+1}}{(n+1)!} = \frac{5x^{2n+3}}{(n+1)!}$$

Plugging these into the formula for L, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{5x^{2n+3}}{(n+1)!}}{\frac{5x^{2n+1}}{n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{5x^{2n+3}}{(n+1)!} \cdot \frac{n!}{5x^{2n+1}} \right|$$

Pairing similar numerators and denominators together, we get

$$L = \lim_{n \to \infty} \left| \frac{5x^{2n+3}}{5x^{2n+1}} \cdot \frac{n!}{(n+1)!} \right|$$

Expanding the factorials so that we can get an idea of what we can cancel, and then canceling terms, we get

$$L = \lim_{n \to \infty} \left| x^{2n+3-(2n+1)} \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n+1-1)(n+1-2)(n+1-3)(n+1-4)\ldots} \right|$$

$$L = \lim_{n \to \infty} \left| x^{2n+3-2n-1} \cdot \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)(n)(n-1)(n-2)(n-3)\ldots} \right|$$

$$L = \lim_{n \to \infty} \left| x^2 \cdot \frac{1}{n+1} \right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L = \left| x^2 \right| \lim_{n \to \infty} \left| \frac{1}{n+1} \right|$$

$$L = \left| x^2 \right| \left| \frac{1}{\infty + 1} \right|$$

$$L = \left| x^2 \right| \left| \frac{1}{\infty} \right|$$

$$L = \left| x^2 \right| \left| 0 \right|$$

$$L = 0$$

Since the limit is 0 and $0 < 1$ is always true regardless of the value of n, the series converges for all values of n. Therefore, we don't need to check any endpoints, and the interval of convergence is $-\infty < n < \infty$.

Topic: Interval of convergence

Question: Find the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^n}$$

Answer choices:

A $0 < x < 3$

B $0 < x < \infty$

C $-\infty < x < \infty$

D $-3 < x < 3$

Solution: C

We can use the root test for convergence to find the interval of convergence of a series. The root test tells us that, for a series a_n, if

$$L = \lim_{n \to \infty} \sqrt[n]{\left| a_n \right|}$$

then the series converges absolutely if $L < 1$. Therefore, we'll find the value of L for the given series, plug it into $L < 1$, and then solve for the variable.

Plugging a_n into the formula for L, we get

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \frac{(x-3)^n}{n^n} \right|}$$

$$L = \lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{x-3}{n} \right)^n \right|}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{x-3}{n} \right)^n \right|^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left| \left(\frac{x-3}{n} \right)^{n \cdot \frac{1}{n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x-3}{n} \right|$$

Since the limit only effects n, we can pull x out in front of the limit, as long as we keep it inside absolute value brackets.

$$L = |x-3| \lim_{n\to\infty} \left|\frac{1}{n}\right|$$

$$L = |x-3| \left|\frac{1}{\infty}\right|$$

$$L = |x-3| \ |0|$$

$$L = 0$$

Since the limit is 0 and $0 < 1$ is always true regardless of the value of n, the series converges for all values of n. Therefore, we don't need to check any endpoints, and the interval of convergence is $-\infty < n < \infty$.

Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_0^1 2e^x \, dx$$

Answer choices:

A $\quad 5.44$

B $\quad 3.24$

C $\quad 3.42$

D $\quad 5.40$

Solution: C

When we use power series to integrate a function like the given function

$$f(x) = 2e^x$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll just multiply by 2.

$$2\left(e^x\right) = 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = 2\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$2e^x = 2 + \frac{2x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{2x^n}{n!}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty} \frac{2x^n}{n!}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^1 2e^x \ dx = \int_0^1 2 + \frac{2x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \dots \ dx$$

$$\int_0^1 2e^x \ dx = 2x + \frac{2x^2}{(2)1!} + \frac{2x^3}{(3)2!} + \frac{2x^4}{(4)3!} + \dots \Big|_0^1$$

$$\int_0^1 2e^x \ dx = 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \dots \Big|_0^1$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^1 2e^x \ dx = \left[2(1) + (1)^2 + \frac{(1)^3}{3} + \frac{(1)^4}{12} + \dots\right]$$

$$-\left[2(0) + (0)^2 + \frac{(0)^3}{3} + \frac{(0)^4}{12} + \dots\right]$$

Then we'll pair like-terms together.

$$\int_0^1 2e^x \ dx = [2(1) - 2(0)] + [(1)^2 - (0)^2] + \left[\frac{(1)^3}{3} - \frac{(0)^3}{3}\right] + \left[\frac{(1)^4}{12} - \frac{(0)^4}{12}\right] + \dots$$

$$\int_0^1 2e^x\,dx = (2-0) + (1-0) + \left(\frac{1}{3} - 0\right) + \left(\frac{1}{12} - 0\right) + \dots$$

$$\int_0^1 2e^x\,dx = 2 + 1 + \frac{1}{3} + \frac{1}{12} + \dots$$

$$\int_0^1 2e^x\,dx = 3 + \frac{1}{3} + \frac{1}{12} + \dots$$

$$\int_0^1 2e^x\,dx = 3.0000 + 0.3333 + 0.0833 + \dots$$

$$\int_0^1 2e^x\,dx \approx 3.42$$

This is the estimate of the definite integral using a power series.

Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_{0}^{0.1} \sin(2x)\ dx$$

Answer choices:

A $\quad -0.299$

B $\quad 0.01$

C $\quad 0.199$

D $\quad 0.299$

Solution: B

When we use power series to integrate a function like the given function

$$f(x) = \sin(2x)$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll substitute $2x$ for x.

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^{0.1} \sin(2x)\ dx = \int_0^{0.1} 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots\ dx$$

$$\int_0^{0.1} \sin(2x)\ dx = \int_0^{0.1} 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots\ dx$$

$$\int_0^{0.1} \sin(2x)\ dx = \frac{2x^2}{2} - \frac{8x^4}{(4)3!} + \frac{32x^6}{(6)5!} - \frac{128x^8}{(8)7!} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{2x^4}{3!} + \frac{16x^6}{(3)5!} - \frac{16x^8}{7!} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{2x^4}{6} + \frac{16x^6}{360} - \frac{16x^8}{5{,}040} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \dots \Big|_0^{0.1}$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^{0.1} \sin(2x)\ dx = \left[(0.1)^2 - \frac{(0.1)^4}{3} + \frac{2(0.1)^6}{45} - \frac{(0.1)^8}{315} + \dots\right]$$

$$-\left[(0)^2 - \frac{(0)^4}{3} + \frac{2(0)^6}{45} - \frac{(0)^8}{315} + \dots\right]$$

Then we'll pair like-terms together.

$$\int_0^{0.1} \sin(2x)\ dx = [(0.1)^2 - (0)^2] + \left[-\frac{(0.1)^4}{3} + \frac{(0)^4}{3}\right]$$

$$+\left[\frac{2(0.1)^6}{45} - \frac{2(0)^6}{45}\right] + \left[-\frac{(0.1)^8}{315} + \frac{(0)^8}{315}\right] + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = (0.01 - 0) + \left[-\frac{0.0001}{3} + 0\right]$$

$$+\left[\frac{2(0.000001)}{45} - 0\right] + \left[-\frac{(0.00000001)}{315} + 0\right] + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = 0.01 - \frac{0.0001}{3} + \frac{0.000002}{45} - \frac{0.00000001}{315} + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = 0.0100000000000000$$

$$-0.0000333333333333$$

$$+0.0000000444444444$$

$$-0.0000000000317460 3 + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx \approx 0.01$$

This is the estimate of the definite integral using a power series.

Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_0^{0.2} 2\arctan(2x)\ dx$$

Answer choices:

A -0.07799

B 0.03779

C -0.03779

D 0.07799

Solution: D

When we use power series to integrate a function like the given function

$$f(x) = 2\arctan(2x)$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll substitute $2x$ for x.

$$\arctan(2x) = 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

Then we'll multiply through by 2.

$$2\arctan(2x) = 2\left[2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \dots\right] = 2\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

$$2\arctan(2x) = 4x - \frac{2(2x)^3}{3} + \frac{2(2x)^5}{5} - \frac{2(2x)^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{2(-1)^n(2x)^{2n+1}}{2n+1}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty} \frac{2(-1)^n(2x)^{2n+1}}{2n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^{0.2} 2\arctan(2x)\ dx = \int_0^{0.2} 4x - \frac{2(2x)^3}{3} + \frac{2(2x)^5}{5} - \frac{2(2x)^7}{7} + \dots\ dx$$

$$\int_0^{0.2} 2\arctan(2x)\ dx = \int_0^{0.2} 4x - \frac{16x^3}{3} + \frac{64x^5}{5} - \frac{256x^7}{7} + \dots\ dx$$

$$\int_0^{0.2} 2\arctan(2x)\ dx = \frac{4x^2}{2} - \frac{16x^4}{(4)3} + \frac{64x^6}{(6)5} - \frac{256x^8}{(8)7} + \dots \Bigg|_0^{0.2}$$

$$\int_0^{0.2} 2\arctan(2x)\ dx = 2x^2 - \frac{4x^4}{3} + \frac{32x^6}{15} - \frac{32x^8}{7} + \dots \Bigg|_0^{0.2}$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^{0.2} 2\arctan(2x)\ dx = \left[2(0.2)^2 - \frac{4(0.2)^4}{3} + \frac{32(0.2)^6}{15} - \frac{32(0.2)^8}{7} + \dots\right]$$

$$-\left[2(0)^2-\frac{4(0)^4}{3}+\frac{32(0)^6}{15}-\frac{32(0)^8}{7}+\ldots\right]$$

Then we'll pair like-terms together.

$$\int_0^{0.2} 2\arctan(2x)\ dx=\left[2(0.2)^2-2(0)^2\right]+\left[-\frac{4(0.2)^4}{3}+\frac{4(0)^4}{3}\right]+\left[\frac{32(0.2)^6}{15}-\frac{32(0)^6}{15}\right]$$

$$+\left[-\frac{32(0.2)^8}{7}+\frac{32(0)^8}{7}\right]+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=2(0.2)^2-\frac{4(0.2)^4}{3}+\frac{32(0.2)^6}{15}-\frac{32(0.2)^8}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=2(0.04)-\frac{4(0.0016)}{3}+\frac{32(0.000064)}{15}-\frac{32(0.00000256)}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=0.08-\frac{0.0064}{3}+\frac{0.002048}{15}-\frac{0.00008192}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=0.08000000-0.00213333+0.00013653-0.00001170+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx\approx 0.07799150$$

$$\int_0^{0.2} 2\arctan(2x)\ dx\approx 0.07799$$

This is the estimate of the definite integral using a power series.

Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1-r^2}\ dr$$

Answer choices:

A $\displaystyle \sum_{n=0}^{\infty} \frac{r^{2n+1}}{2n+1} + C$

B $\displaystyle \sum_{n=0}^{\infty} \frac{r^{2n+2}}{2n+2} + C$

C $\displaystyle \sum_{n=0}^{\infty} \frac{r^{n+2}}{n+2} + C$

D $\displaystyle \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1} + C$

Solution: B

When we use power series to integrate a function like the given function

$$f(r) = \frac{r}{1 - r^2}$$

we use the standard form of a power series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1 - r^2} = r\left(\frac{1}{1 - r^2}\right)$$

Matching this to the standard form, we can say that $x = r^2$. Making this substitution throughout the standard form, we get

$$\frac{1}{1 - r^2} = 1 + r^2 + \left(r^2\right)^2 + \left(r^2\right)^3 + \dots = \sum_{n=0}^{\infty} \left(r^2\right)^n$$

$$\frac{1}{1 - r^2} = 1 + r^2 + r^4 + r^6 + \dots = \sum_{n=0}^{\infty} r^{2n}$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1-r^2}\right) = r\left(1 + r^2 + r^4 + r^6 + \dots\right) = r\sum_{n=0}^{\infty} r^{2n}$$

$$\frac{r}{1-r^2} = r + r^3 + r^5 + r^7 + \dots = \sum_{n=0}^{\infty} r^{2n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty} r^{2n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int \frac{r}{1-r^2}\ dr = \int r + r^3 + r^5 + r^7 + \dots + r^{2n+1}\ dr$$

$$\int \frac{r}{1-r^2}\ dr = \frac{r^2}{2} + \frac{r^4}{4} + \frac{r^6}{6} + \frac{r^8}{8} + \dots + \frac{r^{2n+2}}{2n+2} + C$$

Then we take the last term and say that

$$\int \frac{r}{1-r^2}\ dr = \sum_{n=0}^{\infty} \frac{r^{2n+2}}{2n+2} + C$$

Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1+r}\,dr$$

Answer choices:

A $\displaystyle \sum_{n=0}^{\infty} (-1)^n \frac{r^{2n+1}}{2n+1} + C$

B $\displaystyle \sum_{n=0}^{\infty} (-1)^n \frac{r^{2n+2}}{2n+2} + C$

C $\displaystyle \sum_{n=0}^{\infty} (-1)^n \frac{r^{n+2}}{n+2} + C$

D $\displaystyle \sum_{n=0}^{\infty} (-1)^n \frac{r^{n+1}}{n+1} + C$

Solution: C

When we use power series to integrate a function like the given function

$$f(r) = \frac{r}{1+r}$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1+r} = r\left(\frac{1}{1+r}\right)$$

$$\frac{r}{1+r} = r\left[\frac{1}{1-(-r)}\right]$$

Matching this to the standard form, we can say that $x = -r$. Making this substitution throughout the standard form, we get

$$\frac{1}{1-(-r)} = 1 + (-r) + (-r)^2 + (-r)^3 + \ldots = \sum_{n=0}^{\infty} (-r)^n$$

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n r^n$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1+r}\right)=r\left(1-r+r^2-r^3+\ldots\right)=r\sum_{n=0}^{\infty}(-1)^n r^n$$

$$\frac{r}{1+r}=r-r^2+r^3-r^4+\ldots=\sum_{n=0}^{\infty}(-1)^n r^{n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty}(-1)^n r^{n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int\frac{r}{1+r}\ dr=\int r-r^2+r^3-r^4+\ldots+(-1)^n r^{n+1}\ dr$$

$$\int\frac{r}{1+r}\ dr=\frac{r^2}{2}-\frac{r^3}{3}+\frac{r^4}{4}-\frac{r^5}{5}+\ldots+(-1)^n\frac{r^{n+2}}{n+2}+C$$

Then we take the last term and say that

$$\int\frac{r}{1-r^2}\ dr=\sum_{n=0}^{\infty}(-1)^n\frac{r^{n+2}}{n+2}+C$$

Topic: Estimating indefinite integrals

Question: Evaluate the indefinite integral as a power series.

$$\int \frac{r}{1-r^3}\, dr$$

Answer choices:

A $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{3n+2}}{3n+2} + C$

B $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{2n+3}}{2n+3} + C$

C $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{n+3}}{n+3} + C$

D $\quad \displaystyle\sum_{n=0}^{\infty} \frac{r^{3n+1}}{3n+1} + C$

Solution: A

When we use power series to integrate a function like the given function

$$f(r) = \frac{r}{1 - r^3}$$

we use the standard form of a power series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Our goal will be to start with the given function, and then manipulate it until it matches the far left side of the standard form. To get the given function to match the left side of the standard form, we'll first factor out an r.

$$\frac{r}{1 - r^3} = r\left(\frac{1}{1 - r^3}\right)$$

Matching this to the standard form, we can say that $x = r^3$. Making this substitution throughout the standard form, we get

$$\frac{1}{1 - r^3} = 1 + r^3 + \left(r^3\right)^2 + \left(r^3\right)^3 + \dots = \sum_{n=0}^{\infty} \left(r^3\right)^n$$

$$\frac{1}{1 - r^3} = 1 + r^3 + r^6 + r^9 + \dots = \sum_{n=0}^{\infty} r^{3n}$$

We can't forget about the r that we factored out of the given function. We have to add that back in.

$$r\left(\frac{1}{1-r^3}\right)=r\left(1+r^3+r^6+r^9+\dots\right)=r\sum_{n=0}^{\infty}r^{3n}$$

$$\frac{r}{1-r^3}=r+r^4+r^7+r^{10}+\dots=\sum_{n=0}^{\infty}r^{3n+1}$$

Now that the far left of the manipulated standard form matches the given function, we can say that the far right of the manipulated standard form,

$$\sum_{n=0}^{\infty}r^{3n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int\frac{r}{1-r^3}\ dr=\int r+r^4+r^7+r^{10}+\dots+r^{3n+1}\ dr$$

$$\int\frac{r}{1-r^3}\ dr=\frac{r^2}{2}+\frac{r^5}{5}+\frac{r^8}{8}+\frac{r^{11}}{11}+\dots+\frac{r^{3n+2}}{3n+2}+C$$

Then we take the last term and say that

$$\int\frac{r}{1-r^3}\ dr=\sum_{n=0}^{\infty}\frac{r^{3n+2}}{3n+2}+C$$

Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (2 + x)^4$$

Answer choices:

A $\quad 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x + 1)^n$

B $\quad 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (4 - n)}{n!}(x + 1)^n$

C $\quad 1 + \sum_{n=0}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (5 - n)}{n!}(x + 1)^n$

D $\quad 1 + \sum_{n=0}^{\infty} \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot \ldots \cdot (4 - n)}{n!}(x + 1)^n$

Solution: A

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left side match our function, we'll replace x with $x+1$ and k with 4.

$$(1+x+1)^k = \sum_{n=0}^{\infty} \binom{k}{n} (x+1)^n = 1 + k(x+1) + \frac{k(k-1)}{2!}(x+1)^2 + \frac{k(k-1)(k-2)}{3!}(x+1)^3 + \dots$$

$$(2+x)^4 = \sum_{n=0}^{\infty} \binom{4}{n} (x+1)^n = 1 + 4(x+1) + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1 + 4(x+1) + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

$$1(x+1)^0 + 4(x+1)^1 + \frac{4(4-1)}{2!}(x+1)^2 + \frac{4(4-1)(4-2)}{3!}(x+1)^3 + \dots$$

$$\frac{1}{0!}(x+1)^0 + \frac{4}{1!}(x+1)^1 + \frac{4 \cdot 3}{2!}(x+1)^2 + \frac{4 \cdot 3 \cdot 2}{3!}(x+1)^3 + \dots$$

When we match up these terms with their corresponding n-values, we get

$n = 0$ $\qquad\qquad$ $\frac{1}{0!}(x+1)^0$

$$n=1 \qquad \frac{4}{1!}(x+1)^1$$

$$n=2 \qquad \frac{4\cdot 3}{2!}(x+1)^2$$

$$n=3 \qquad \frac{4\cdot 3\cdot 2}{3!}(x+1)^3$$

The $n=0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by 4, so we'll pull it out in front of the power series representation and shift the index from $n=0$ to $n=1$. For all the terms starting with $n=1$, we can see that the pattern is

$$\frac{4\cdot 3\cdot 2\cdot 1\cdot \ldots \cdot \left[4-(n-1)\right]}{n!}(x+1)^n$$

$$\frac{4\cdot 3\cdot 2\cdot 1\cdot \ldots \cdot (4-n+1)}{n!}(x+1)^n$$

$$\frac{4\cdot 3\cdot 2\cdot 1\cdot \ldots \cdot (5-n)}{n!}(x+1)^n$$

So the power series representation of the function is

$$1+\sum_{n=1}^{\infty}\frac{4\cdot 3\cdot 2\cdot 1\cdot \ldots \cdot (5-n)}{n!}(x+1)^n$$

Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (1 + 2x)^2$$

Answer choices:

A $\quad 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (2-n)}{n!}(2x)^n$

B $\quad 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3-n)}{n!}(2x)^n$

C $\quad 1 + \sum_{n=0}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3-n)}{n!}(2x)^n$

D $\quad 1 + \sum_{n=0}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (2-n)}{n!}(2x)^n$

Solution: B

We'll start with the binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

To make the left side match our function, we'll replace x with $2x$ and k with 2.

$$(1+2x)^k = \sum_{n=0}^{\infty} \binom{k}{n} (2x)^n = 1 + k(2x) + \frac{k(k-1)}{2!}(2x)^2 + \frac{k(k-1)(k-2)}{3!}(2x)^3 + \dots$$

$$(1+2x)^2 = \sum_{n=0}^{\infty} \binom{2}{n} (2x)^n = 1 + 2(2x) + \frac{2(2-1)}{2!}(2x)^2 + \frac{2(2-1)(2-2)}{3!}(2x)^3 + \dots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1 + 2(2x) + \frac{2(2-1)}{2!}(2x)^2 + \frac{2(2-1)(2-2)}{3!}(2x)^3 + \dots$$

$$1(2x)^0 + 2(2x)^1 + \frac{2(2-1)}{2!}(2x)^2 + \frac{2(2-1)(2-2)}{3!}(2x)^3 + \dots$$

$$\frac{1}{0!}(2x)^0 + \frac{2}{1!}(2x)^1 + \frac{2(2-1)}{2!}(2x)^2 + \frac{2(2-1)(2-2)}{3!}(2x)^3 + \dots$$

When we match up these terms with their corresponding n-values, we get

$$n = 0 \qquad \frac{1}{0!}(2x)^0$$

$$n = 1 \qquad \frac{2}{1!}(2x)^1$$

$$n = 2 \qquad \frac{2(2-1)}{2!}(2x)^2$$

$$n = 3 \qquad \frac{2(2-1)(2-2)}{3!}(2x)^3$$

The $n = 0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by 2, so we'll pull it out in front of the power series representation and shift the index from $n = 0$ to $n = 1$. For all the terms starting with $n = 1$, we can see that the pattern is

$$\frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot [2 - (n-1)]}{n!}(2x)^n$$

$$\frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (2 - n + 1)}{n!}(2x)^n$$

$$\frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3 - n)}{n!}(2x)^n$$

So the power series representation of the function is

$$1 + \sum_{n=1}^{\infty} \frac{2 \cdot 1 \cdot 0 \cdot \ldots \cdot (3 - n)}{n!}(2x)^n$$

Topic: Binomial series

Question: Use the binomial series to expand the function as a power series.

$$f(x) = (1+3x)^{\frac{1}{2}}$$

Answer choices:

A $\quad 1+\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n\right)}{n!}(3x)^n$

B $\quad 1+\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$

C $\quad 1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n\right)}{n!}(3x)^n$

D $\quad 1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$

Solution: D

We'll start with the binomial series.

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\ldots$$

To make the left side match our function, we'll replace x with $3x$ and k with $1/2$.

$$(1+3x)^k=\sum_{n=0}^{\infty}\binom{k}{n}(3x)^n=1+k(3x)+\frac{k(k-1)}{2!}(3x)^2+\frac{k(k-1)(k-2)}{3!}(3x)^3+\ldots$$

$$(1+3x)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(3x)^n=1+\frac{1}{2}(3x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$$

Now that the left side matches the given function, we can use the series expansion on the right side to find its power series representation. We just have to find the pattern in the expansion. We'll identify the pattern by rewriting the expansion as

$$1+\frac{1}{2}(3x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$$

$$1(3x)^0+\frac{1}{2}(3x)^1+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$$

$$\frac{1}{0!}(3x)^0+\frac{\frac{1}{2}}{1!}(3x)^1+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$$

When we match up these terms with their corresponding n-values, we get

$$n=0 \qquad \frac{1}{0!}(3x)^0$$

$$n=1 \qquad \frac{\frac{1}{2}}{1!}(3x)^1$$

$$n=2 \qquad \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2$$

$$n=3 \qquad \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3$$

The $n=0$ term doesn't follow the same pattern as the rest of the series, since it's not multiplied by $1/2$, so we'll pull it out in front of the power series representation and shift the index from $n=0$ to $n=1$. For all the terms starting with $n=1$, we can see that the pattern is

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left[\frac{1}{2}-(n-1)\right]}{n!}(3x)^n$$

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}(3x)^n$$

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$$

So the power series representation of the function is

$$1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(\frac{3}{2}-n\right)}{n!}(3x)^n$$

Topic: Taylor series

Question: Find the Taylor polynomial and use it to approximate the given value.

$$e^x$$

when $n = 3$ and $a = 2$

Find $e^{1.23}$

Answer choices:

A $\quad (x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 \quad$ and $\quad e^{1.23} = 3.427744$

B $\quad e\left[(x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right] \quad$ and $\quad e^{1.23} = 3.327744$

C $\quad e^2\left[(x-1) + (x-2)^2 + (x-2)^3\right] \quad$ and $\quad e^{1.23} = 4.427744$

D $\quad e^2\left[(x-1) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right] \quad$ and $\quad e^{1.23} = 3.327744$

Solution: D

The formula for the Taylor polynomial of f at a is

$$f^{(n)}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

To find the third-degree Taylor polynomial, we need the original function, plus its first three derivatives.

$f(x) = e^x$ and $f(2) = e^2$

$f'(x) = e^x$ and $f'(2) = e^2$

$f''(x) = e^x$ and $f''(2) = e^2$

$f'''(x) = e^x$ and $f'''(2) = e^2$

Therefore, the third-degree Taylor polynomial is

$$f^{(3)}(x) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3$$

$$f^{(3)}(x) = e^2\left[1 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$$

$$f^{(3)}(x) = e^2\left[x - 1 + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]$$

Using the series to estimate $e^{1.23}$, we get

$$e^{1.23} \approx f^{(3)}(1.23) \approx e^2\left[1.23 - 1 + \frac{1}{2}(1.23-2)^2 + \frac{1}{6}(1.23-2)^3\right]$$

$$e^{1.23} \approx f^{(3)}(1.23) \approx 3.327744$$

Topic: Taylor series

Question: Find the Taylor polynomial and use it to approximate the given value.

$$\tan x$$

when $n = 3$ and $a = \frac{\pi}{4}$

Find $\tan \frac{\pi}{8}$

Answer choices:

A $\quad 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.361536$

B $\quad 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.414214$

C $\quad 1 + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.365136$

D $\quad 2 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ and $\tan \frac{\pi}{8} = 0.414214$

Solution: A

The formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

To find the third-degree Taylor polynomial, we need the original function, plus its first three derivatives.

$f(x) = \tan x$ and $f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ and $f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x$ and $f''\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = 2\sec^2 x\left(\sec^2 x + 2\tan^2 x\right)$ and $f'''\left(\frac{\pi}{4}\right) = 16$

Therefore, the third-degree Taylor polynomial is

$$f^{(3)}(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$f^{(3)}(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Using the series to estimate $\tan \pi/8$, we get

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 1 + 2\left(\frac{\pi}{8} - \frac{\pi}{4}\right) + 2\left(\frac{\pi}{8} - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(\frac{\pi}{8} - \frac{\pi}{4}\right)^3$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 1 + 2\left(-\frac{\pi}{8}\right) + 2\left(-\frac{\pi}{8}\right)^2 + \frac{8}{3}\left(-\frac{\pi}{8}\right)^3$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 1 - \frac{\pi}{4} + \frac{\pi^2}{32} - \frac{\pi^3}{192}$$

$$\tan\left(\frac{\pi}{8}\right) \approx f^{(3)}\left(\frac{\pi}{8}\right) \approx 0.361536$$

Topic: Taylor series

Question: Find the Taylor polynomial.

$$f(x) = 2x^2 - x + 4$$

when $n = 2$ and $a = 1$

Answer choices:

A $5 + 4(x + 1) + 3(x + 1)^2$

B $5 + 3(x - 1) + 2(x - 1)^2$

C $5 + 4(x - 1) + 3(x - 1)^2$

D $5 + 3(x + 1) + 2(x + 1)^2$

Solution: B

The formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

To find the second-degree Taylor polynomial, we need the original function, plus its first two derivatives.

$f(x) = 2x^2 - x + 4$ and $f(1) = 2(1)^2 - (1) + 4 = 5$

$f'(x) = 4x - 1$ and $f'(1) = 4(1) - 1 = 3$

$f''(x) = 4$ and $f''(1) = 4$

Therefore, the second-degree Taylor polynomial is

$$f^{(2)}(x) = 5 + 3(x-1) + \frac{4}{2!}(x-1)^2$$

$$f^{(2)}(x) = 5 + 3(x-1) + 2(x-1)^2$$

Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$2 + 2(x-1) + 4(x-1)^2 + 6(x-1)^3$$

Answer choices:

A 2

B 1

C 4

D ∞

Solution: B

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$2 + 2(x-1) + 4(x-1)^2 + 6(x-1)^3$$

$$2(x-1)^0 + 2(x-1)^1 + 4(x-1)^2 + 6(x-1)^3$$

$$2(x-1)^0 + 2(1)(x-1)^1 + 2(2)(x-1)^2 + 2(3)(x-1)^3$$

The first term doesn't follow the same pattern as the second and third terms, so we can pull it out in front of the sum and represent the series as

$$2 + \sum_{n=1}^{\infty} 2n(x-1)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = 2n(x-1)^n$$

$$a_{n+1} = 2(n+1)(x-1)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n \to \infty} \left| \frac{2(n+1)(x-1)^{n+1}}{2n(x-1)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(x-1)^{n+1-n}}{n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(n+1)(x-1)}{n} \right|$$

The limit only effects n, not x, so we can pull $(x-1)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = \left| x-1 \right| \lim_{n \to \infty} \left| \frac{n+1}{n} \right|$$

$$L = |x-1| \lim_{n\to\infty} \left| \frac{n+1}{n} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |x-1| \lim_{n\to\infty} \left| \frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n}} \right|$$

$$L = |x-1| \lim_{n\to\infty} \left| \frac{1 + \frac{1}{n}}{1} \right|$$

$$L = |x-1| \left| \frac{1 + \frac{1}{\infty}}{1} \right|$$

$$L = |x-1| \left| \frac{1+0}{1} \right|$$

$$L = |x-1| \, |1|$$

$$L = |x-1|$$

Now we can set $L < 1$ to find the radius of convergence.

$$|x-1| < 1$$

Since this inequality is already in the form

$$|x-a| < R$$

and since R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1$.

Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$3 - 3(x-1) + 9(x-1)^2 - 27(x-1)^3$$

Answer choices:

A $\frac{1}{3}$

B 3

C $\frac{1}{2}$

D 1

Solution: A

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^n(a)}{n!}(x - a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x - a)^n$$

We can rewrite the given series as

$$3 - 3(x - 1) + 9(x - 1)^2 - 27(x - 1)^3$$

$$3(x - 1)^0 - 3(x - 1)^1 + 9(x - 1)^2 - 27(x - 1)^3$$

$$3(x - 1)^0 - 3^1(x - 1)^1 + 3^2(x - 1)^2 - 3^3(x - 1)^3$$

The first term doesn't follow the same pattern as the other terms, so we can pull it out in front of the sum and represent the series as

$$3 + \sum_{n=1}^{\infty} (-3)^n(x - 1)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = (-3)^n(x-1)^n$$

$$a_{n+1} = (-3)^{n+1}(x-1)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x-1)^{n+1}}{(-3)^n(x-1)^n} \right|$$

$$L = \lim_{n\to\infty} \left| (-3)^{n+1-n}(x-1)^{n+1-n} \right|$$

$$L = \lim_{n\to\infty} \left| -3(x-1) \right|$$

The limit only effects n, not x, so we can pull $(x-1)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = \left| x-1 \right| \lim_{n\to\infty} \left| -3 \right|$$

$$L = \left| x-1 \right| \left| -3 \right|$$

$$L = 3\left| x-1 \right|$$

Now, we can set $L < 1$ to find the radius of convergence.

$$3\left|x-1\right| < 1$$

$$\left|x-1\right| < \frac{1}{3}$$

Comparing this to

$$|x-a| < R$$

where R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1/3$.

Topic: Radius and interval of convergence of a Taylor series

Question: Find the radius of convergence of the Taylor series.

$$2 + (x - 2) + \frac{1}{2}(x - 2)^2 + \frac{1}{3}(x - 2)^3$$

Answer choices:

A $\frac{1}{3}$

B 2

C $\frac{1}{2}$

D 1

Solution: D

To find the radius of convergence of the given Taylor series, we'll need to transform it into its power series representation, and then use the ratio test to find the radius of convergence of that power series.

To get a power series representation of the series, we need to remember that the formula for the Taylor polynomial of f at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Which means that the power series representation of the function will be the last term,

$$\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

We can rewrite the given series as

$$2 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

$$2(x-2)^0 + (x-2)^1 + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

$$2(x-2)^0 + \frac{1}{1}(x-2)^1 + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

The first term doesn't follow the same pattern as the other terms, so we can pull it out in front of the sum and represent the series as

$$2 + \sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$$

To find the radius of convergence of this series, we'll first identify a_n and a_{n+1}.

$$a_n = \frac{1}{n}(x-2)^n$$

$$a_{n+1} = \frac{1}{n+1}(x-2)^{n+1}$$

Now we can use the ratio test to find the radius of convergence. The ratio test tells us that a series converges if $L < 1$ when

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Plugging the values we found for a_n and a_{n+1} into this formula for L, we get

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{n+1}(x-2)^{n+1}}{\frac{1}{n}(x-2)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{n+1}(x-2)^{n+1-n}}{\frac{1}{n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}}(x-2) \right|$$

$$L = \lim_{n \to \infty} \left| \frac{1}{n+1} \cdot \frac{n}{1} \cdot (x-2) \right|$$

$$L = \lim_{n \to \infty} \left| \frac{n}{n+1} \cdot (x-2) \right|$$

The limit only effects n, not x, so we can pull $(x-2)$ out in front of the limit, as long as we keep it inside absolute value bars.

$$L = |x-2| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{n}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = |x-2| \lim_{n \to \infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = |x-2| \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$L = |x-2| \left| \frac{1}{1+0} \right|$$

$$L = |x-2| \, |1|$$

$$L = |x-2|$$

Now we can set $L < 1$ to find the radius of convergence.

$$\left| x - 2 \right| < 1$$

Comparing this to

$$\left| x - a \right| < R$$

where R is the radius of convergence, we can say that the radius of convergence of the series is $R = 1$.

Topic: Taylor's inequality

Question: Taylor’s inequality can be applied to which of the following series?

Answer choices:

A Taylor series only

B Any series

C Maclaurin series only

D Taylor or Maclaurin series

Solution: D

Taylor's inequality states that

If	$\left\lvert f^{n+1}(x)\right\rvert \le M$	for	$\lvert x-a\rvert \le d$
then	$\left\lvert R_n(x)\right\rvert \le \frac{M}{(n+1)!}\lvert x-a\rvert^{n+1}$	for	$\lvert x-a\rvert \le d$

and it can be applied to both Taylor series and Maclaurin series. Remember that a Maclaurin series is just a Taylor series in which $a=0$.

Topic: Taylor's inequality

Question: Which of these is an accurate statement of Taylor's inequality?

Answer choices:

A	If	$\left\lvert f^{n+1}(x) \right\rvert > M$	for	$\lvert x-a \rvert \le d$
	then	$\left\lvert R_n(x) \right\rvert > \frac{M}{(n+1)!}\lvert x-a \rvert^{n+1}$	for	$\lvert x-a \rvert \le d$
B	If	$\left\lvert f^{n+1}(x) \right\rvert < M$	for	$\lvert x-a \rvert \le d$
	then	$\left\lvert R_n(x) \right\rvert < \frac{M}{(n+1)!}\lvert x-a \rvert^{n+1}$	for	$\lvert x-a \rvert \le d$
C	If	$\left\lvert f^{n+1}(x) \right\rvert \ge M$	for	$\lvert x-a \rvert \le d$
	then	$\left\lvert R_n(x) \right\rvert \ge \frac{M}{(n+1)!}\lvert x-a \rvert^{n+1}$	for	$\lvert x-a \rvert \le d$
D	If	$\left\lvert f^{n+1}(x) \right\rvert \le M$	for	$\lvert x-a \rvert \le d$
	then	$\left\lvert R_n(x) \right\rvert \le \frac{M}{(n+1)!}\lvert x-a \rvert^{n+1}$	for	$\lvert x-a \rvert \le d$

Solution: D

Taylor's inequality states that

If $\left|f^{n+1}(x)\right| \leq M$ for $|x-a| \leq d$

then $\left|R_n(x)\right| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$

The inequality just means that, if we can show that a Taylor or Maclaurin series has no remainder $R_n(x)$ in its power series representation, then the representation is a true and accurate reflection of the original series.

Topic: Taylor's inequality

Question: Which of these does Taylor's inequality prove?

Answer choices:

A If a Taylor or Maclaurin series has a remainder $R_n(x)$ in its power series representation, then the representation is a true and accurate reflection of the original series.

B If a Taylor or Maclaurin series has no remainder $R_n(x)$ in its power series representation, then the representation is a true and accurate reflection of the original series.

C If a Taylor or Maclaurin series has no remainder $R_n(x)$ in its power series representation, then the representation is not a true and accurate reflection of the original series.

D If a Taylor or Maclaurin series has no remainder $R_n(x)$ in its power series representation, then the representation might be a true and accurate reflection of the original series but needs further testing.

Solution: B

Taylor's inequality states that

If	$\left\lvert f^{n+1}(x) \right\rvert \leq M$	for	$\lvert x-a \rvert \leq d$
then	$\left\lvert R_n(x) \right\rvert \leq \dfrac{M}{(n+1)!} \lvert x-a \rvert^{n+1}$	for	$\lvert x-a \rvert \leq d$

The inequality just means that, if we can show that a Taylor or Maclaurin series has no remainder $R_n(x)$ in its power series representation, then the representation is a true and accurate reflection of the original series.

Topic: Maclaurin series

Question: Write the first four terms of the maclaurin series and use it to estimate the given value.

$$f(x) = \sqrt{1 + x}$$

Find $\sqrt{10}$

Answer choices:

A $\quad 1 + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{6}x^3$ and $\sqrt{10} = 3.17$

B $\quad 1 + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{16}x^4$ and $\sqrt{10} = 3.15$

C $\quad 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{8}x^3$ and $\sqrt{10} = 3.20$

D $\quad 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$ and $\sqrt{10} = 3.16$

Solution: D

The Maclaurin formula is

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^n(0)x^n}{n!}$$

To find the first four terms, we need the original function, plus the first three derivatives.

$f(x) = (1+x)^{\frac{1}{2}}$ and $f(0) = (1+0)^{\frac{1}{2}} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ and $f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ and $f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$ and $f'''(0) = \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$

Therefore, the first four terms of the maclaurin series are

$$f^{(3)}(x) = \frac{(1)x^0}{0!} + \frac{\left(\frac{1}{2}\right)x^1}{1!} + \frac{\left(-\frac{1}{4}\right)x^2}{2!} + \frac{\left(\frac{3}{8}\right)x^3}{3!}$$

$$f^{(3)}(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Using the series to estimate $\sqrt{10}$, we get

$$\sqrt{10} = \sqrt{1+9}$$

$$\sqrt{10} = \sqrt{9\left(\frac{1}{9} + 1\right)}$$

$$\sqrt{10} = 3\sqrt{1 + \frac{1}{9}}$$

Now that we have a format similar to the original function, we can say that $x = 1/9$. Because we have a 3 in front of the square root, we need to remember to multiply our series by 3.

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3\left[1 + \frac{1}{2}\left(\frac{1}{9}\right) - \frac{1}{8}\left(\frac{1}{9}\right)^2 + \frac{1}{16}\left(\frac{1}{9}\right)^3\right]$$

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3\left(1 + \frac{1}{18} - \frac{1}{648} + \frac{1}{11{,}664}\right)$$

$$\sqrt{10} \approx 3\left[f^{(3)}\left(\frac{1}{9}\right)\right] \approx 3.16$$

Topic: Maclaurin series

Question: Find the maclaurin series.

Find the maclaurin series of $f(x) = \sin x$ that includes x^7. Then use the ratio test to show that the series converges in its domain.

Answer choices:

A $\quad \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

B $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

C $\quad x^2 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

D $\quad x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

Solution: B

The Maclaurin formula is

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^n(0)x^n}{n!}$$

To find the maclaurin series that includes x^7, we need the original function, plus the first seven derivatives.

$f(x) = \sin x$ and $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ and $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ and $f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ and $f'''(0) = -\cos 0 = -1$

$f^{(4)}(x) = \sin x$ and $f^{(4)}(0) = \sin 0 = 0$

$f^{(5)}(x) = \cos x$ and $f^{(5)}(0) = \cos 0 = 1$

$f^{(6)}(x) = -\sin x$ and $f^{(6)}(0) = -\sin 0 = 0$

$f^{(7)}(x) = -\cos x$ and $f^{(7)}(0) = -\cos 0 = -1$

Therefore, the maclaurin series containing x^7 is

$$f^{(7)}(x) = \sin x = \frac{(0)x^0}{0!} + \frac{(1)x^1}{1!} + \frac{(0)x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{(1)x^5}{5!} + \frac{(0)x^6}{6!} + \frac{(-1)x^7}{7!}$$

$$f^{(7)}(x) = \sin x = 0 + \frac{x}{1} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$$

$$f^{(7)}(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

The powers and coefficients on each term are in arithmetic series. We can use the formula for the general term of an arithmetic series $T_n = a + (n-1)d$ for the powers and coefficients to generate a_n. The signs also alternate $+, -, +, -, \dots$ and to achieve this alternation, the general term must be multiplied by $(-1)^{n-1}$.

So for the powers and coefficients $(1,3,5,\dots)$, the arithmetic series with $a = 1$ and $d = 2$ may be used and

$$T_n = 1 + (n-1)2$$

$$T_n = 1 + 2n - 2$$

$$T_n = 2n - 1$$

So the general term of $\sin x$ is

$$a_n = (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Using the ratio test to show that the series is convergent, we get

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{(n+1)-1} \frac{x^{2(n+1)-1}}{(2(n+1)-1)!}}{(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}} \right|$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-1)^n x^{2n}}{(2n+1)!}}{\frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}} \right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n}}{(2n+1)!}\cdot\frac{(2n-1)!}{(-1)^{n-1}x^{2n-1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|(-1)^{n-(n-1)}x^{2n-(2n-1)}\cdot\frac{(2n-1)!}{(2n+1)!}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|(-1)^{n-n+1}x^{2n-2n+1}\cdot\frac{(2n-1)(2n-1-1)(2n-1-2)(2n-1-3)\dots}{(2n+1)(2n+1-1)(2n+1-2)(2n+1-3)\dots}\right|$$

Since the limit only effects n, we can pull x outside of the limit.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = -x\lim_{n\to\infty}\left|\frac{(2n-1)(2n-2)(2n-3)(2n-4)\dots}{(2n+1)(2n)(2n-1)(2n-2)\dots}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = -x\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n)}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = -x\lim_{n\to\infty}\left|\frac{1}{4n^2+2n}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = -x(0)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

Therefore, the series is convergent for all $x \in R$.

Topic: Maclaurin series

Question: Find the first four non-zero terms of the function's Maclaurin series.

$$f(x) = \frac{1}{1 - 2x}$$

Answer choices:

A $1 + 2x + 4x^2 + 8x^3$

B $1 + x + x^2 + x^3$

C $2 + 2x + 4x^2 + 8x^3$

D $1 + 2x + 4x^2 + 6x^3$

Solution: A

The Maclaurin series is a specific Taylor series where $a = 0$.

The Maclaurin series for

$$f(x) = \frac{1}{1-x}$$

is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1 + x + x^2 + x^3 + \dots$$

Since this is extremely close to the given series, we can manipulate it until they match. We'll just replace each x with $2x$, and then simplify.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 1 + 2x + (2x)^2 + (2x)^3 + \dots$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 1 + 2x + 4x^2 + 8x^3 + \dots$$

Since we've been asked for the first four terms of the series, we can say that the first four terms are

$$1 + 2x + 4x^2 + 8x^3$$

Topic: Sum of the Maclaurin series

Question: Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{5(x+2)^n}{n!}$$

Answer choices:

A $5e^{-x}$

B $5e^{x-2}$

C $5e^{x}$

D $5e^{x+2}$

Solution: D

The easiest way to find the sum of the series of a Maclaurin series is to identify a similar Maclaurin series with a known sum, and then manipulate the given series until it matches the known series.

In this case, the given series is similar to the known series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We'll manipulate this series until it matches the given series. We'll start by replacing x with $x+2$.

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$$

Then we'll multiply both sides by 5.

$$5e^{x+2} = \sum_{n=0}^{\infty} \frac{5(x+2)^n}{n!}$$

Since the right side of this manipulation now matches the given series, we can say that the sum of the given series is $5e^{x+2}$.

Topic: Sum of the Maclaurin series

Question: Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} 3(2)^n$$

Answer choices:

A 3

B 1

C -3

D ∞

Solution: D

The first thing we notice is that the 2^n term will only get larger and larger as n increases. Multiplying the result of 2^n by 3 only makes each term bigger. Which means that, when we add up larger and larger terms, the sum of the series will diverge to ∞.

n	$3(2)^n$	sum
0	3	3
1	6	9
2	12	21
3	24	45
4	48	93
5	96	189
...	...	...

Topic: Sum of the Maclaurin series

Question: Find the sum of the Maclaurin series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 9^n \pi^{2n}}{(2n)!}$$

Answer choices:

A -1

B 0

C 1

D ∞

Solution: A

The easiest way to find the sum of the series of a Maclaurin series is to identify a similar Maclaurin series with a known sum, and then manipulate the given series until it matches the known series.

In this case, the given series is similar to the known series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

We'll manipulate this series until it matches the given series. But first we need to work on the given series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 9^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(3^2\right)^n \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} \pi^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3\pi)^{2n}}{(2n)!}$$

Now we'll just replace x with 3π, and then simplify.

$$\cos(3\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n (3\pi)^{2n}}{(2n)!}$$

$$-1 = \sum_{n=0}^{\infty} \frac{(-1)^n (3\pi)^{2n}}{(2n)!}$$

Since the right side of this manipulation now matches the given series, we can say that the sum of the given series is -1.

Topic: Radius and interval of convergence of a Maclaurin series

Question: Find the radius of convergence of the Maclaurin series.

$$f(x) = \frac{3}{1 - x^2}$$

Answer choices:

A -1

B 3

C 1

D -3

Solution: C

To find the radius of convergence of a Maclaurin series, we'll use the ratio test to say where the power series representation of the function converges. But first, we'll have to find the power series representation of the given function.

The easiest way to find the power series representation of the given series is to identify a similar Maclaurin series and then manipulate the known series until it matches the given series.

In this case, the given series is similar to the known series

$$\frac{1}{1-x}=\sum_{n=1}^{\infty}x^n$$

We'll manipulate this series until it matches the given series. We'll just replace each x with x^2, then multiply the series by 3, and then simplify.

$$\frac{1}{1-x^2}=\sum_{n=1}^{\infty}\left(x^2\right)^n$$

$$\frac{1}{1-x^2}=\sum_{n=1}^{\infty}x^{2n}$$

$$\frac{3}{1-x^2}=\sum_{n=1}^{\infty}3x^{2n}$$

Since the left side of this manipulation now matches the given series, we can say that the power series representation of the function is $a_n=3x^{2n}$.

To use the ratio test tells us that a series converges when $L < 1$, where

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

In order to find L, we'll need to identify a_n and a_{n+1}.

$$a_n = 3x^{2n}$$

$$a_{n+1} = 3x^{2(n+1)} = 3x^{2n+2}$$

Now we'll plug these values into the formula for L.

$$L = \lim_{n \to \infty} \left| \frac{3x^{2n+2}}{3x^{2n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{x^{2n+2}}{x^{2n}} \right|$$

$$L = \lim_{n \to \infty} \left| x^{2n+2-2n} \right|$$

$$L = \lim_{n \to \infty} \left| x^2 \right|$$

Since the limit only effects n, and there are no terms involving n, we can eliminate the limit.

$$L = \left| x^2 \right|$$

The ratio test tells us that the series converges when $L < 1$.

$$\left|x^2\right| < 1$$

We need to get this into the form $|x - a| < R$, so that we can identify R as the radius of convergence.

$$-1 < x < 1$$

$$-1 < x - 0 < 1$$

Now we can say that the radius of convergence of the Maclaurin series is $R = 1$.

Topic: Radius and interval of convergence of a Maclaurin series

Question: Find the radius of convergence of the Maclaurin series.

$$f(x) = 2e^{3x-1}$$

Answer choices:

A 0

B ∞

C $\frac{1}{3}$

D 1

Solution: B

To find the radius of convergence of a Maclaurin series, we'll use the ratio test to say where the power series representation of the function converges. But first, we'll have to find the power series representation of the given function.

The easiest way to find the power series representation of the given series is to identify a similar Maclaurin series and then manipulate the known series until it matches the given series.

In this case, the given series is similar to the known series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We'll manipulate this series until it matches the given series. We'll just replace each x with $3x - 1$, then multiply the series by 2, and then simplify.

$$e^{3x-1} = \sum_{n=0}^{\infty} \frac{(3x-1)^n}{n!}$$

$$2e^{3x-1} = \sum_{n=0}^{\infty} \frac{2(3x-1)^n}{n!}$$

Since the left side of this manipulation now matches the given series, we can say that the power series representation of the function is

$$a_n = \frac{2(3x-1)^n}{n!}$$

To use the ratio test tells us that a series converges when $L < 1$, where

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

In order to find L, we'll need to identify a_n and a_{n+1}.

$$a_n = \frac{2(3x-1)^n}{n!}$$

$$a_{n+1} = \frac{2(3x-1)^{n+1}}{(n+1)!}$$

Now we'll plug these values into the formula for L.

$$L = \lim_{n \to \infty} \left| \frac{\frac{2(3x-1)^{n+1}}{(n+1)!}}{\frac{2(3x-1)^n}{n!}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{2(3x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{2(3x-1)^n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{2(3x-1)^{n+1}}{2(3x-1)^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$L = \lim_{n \to \infty} \left| (3x-1)^{n+1-n} \cdot \frac{n!}{(n+1)!} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(3x-1)(n)(n-1)(n-2)\ldots}{(n+1)(n)(n-1)(n-2)\ldots} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{3x-1}{n+1} \right|$$

Since the limit only effects n, we can pull out $3x-1$, as long as we keep it inside absolute value brackets.

$$L = \left|3x-1\right| \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$$

$$L = \left|3x-1\right| \lim_{n\to\infty} \left| \frac{1}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = \left|3x-1\right| \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = \left|3x-1\right| \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{1 + \frac{1}{n}} \right|$$

$$L = \left|3x-1\right| \left| \frac{\frac{1}{\infty}}{1 + \frac{1}{\infty}} \right|$$

$$L = \left|3x-1\right| \left| \frac{0}{1+0} \right|$$

$$L = \left|3x-1\right| \left|0\right|$$

$$L = 0$$

The ratio test tells us that the series converges when $L < 1$. Since 0 is always less than 1, it means that the series converges everywhere, which means that the radius of convergence is infinite and $R = \infty$.

Topic: Radius and interval of convergence of a Maclaurin series

Question: Find the radius of convergence of the Maclaurin series.

$$f(x) = \ln(1 + 3x)$$

Answer choices:

A $\frac{1}{3}$

B 1

C ∞

D 3

Solution: A

To find the radius of convergence of a Maclaurin series, we'll use the ratio test to say where the power series representation of the function converges. But first, we'll have to find the power series representation of the given function.

The easiest way to find the power series representation of the given series is to identify a similar Maclaurin series and then manipulate the known series until it matches the given series.

In this case, the given series is similar to the known series

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

We'll manipulate this series until it matches the given series. We'll just replace each x with $3x$, and then simplify.

$$\ln(1+3x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(3x)^n$$

Since the left side of this manipulation now matches the given series, we can say that the power series representation of the function is

$$a_n=\frac{(-1)^{n+1}}{n}(3x)^n$$

To use the ratio test tells us that a series converges when $L<1$, where

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

In order to find L, we'll need to identify a_n and a_{n+1}.

$$a_n = \frac{(-1)^{n+1}}{n}(3x)^n$$

$$a_{n+1} = \frac{(-1)^{n+1+1}}{n+1}(3x)^{n+1}$$

$$a_n = \frac{(-1)^{n+2}}{n+1}(3x)^{n+1}$$

Now we'll plug these values into the formula for L.

$$L = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(3x)^{n+1}}{n+1}}{\frac{(-1)^{n+1}(3x)^n}{n}}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(3x)^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}(3x)^n}\right|$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(3x)^{n+1}}{(-1)^{n+1}(3x)^n}\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-1)^{n+2-(n+1)}(3x)^{n+1-n}\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-1)^{n+2-n-1}(3x)\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n\to\infty}\left|(-1)(3x)\cdot\frac{n}{n+1}\right|$$

$$L = \lim_{n \to \infty} \left| (-3x) \cdot \frac{n}{n+1} \right|$$

Since the limit only effects n, we can pull out $-3x$, as long as we keep it inside absolute value brackets.

$$L = \left| -3x \right| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$L = \left| -3x \right| \lim_{n \to \infty} \left| \frac{n}{n+1} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right) \right|$$

$$L = \left| -3x \right| \lim_{n \to \infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$L = \left| -3x \right| \lim_{n \to \infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$L = \left| -3x \right| \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$L = \left| -3x \right| \left| \frac{1}{1 + 0} \right|$$

$$L = \left| -3x \right| \left| 1 \right|$$

$$L = \left| -3x \right|$$

$$L = |3x|$$

The ratio test tells us that the series converges when $L < 1$.

$$|3x| < 1$$

We need to get this into the form $|x - a| < R$, so that we can identify R as the radius of convergence.

$$|x| < \frac{1}{3}$$

$$|x - 0| < \frac{1}{3}$$

Now we can say that the radius of convergence of the Maclaurin series is $R = 1/3$.

Topic: Indefinite integral as infinite series

Question: Use an infinite series to evaluate the indefinite integral.

$$\int xe^{2x}\ dx$$

Answer choices:

A $\quad C+\sum_{n=0}^{\infty}\frac{2^nx^{n+2}}{n!(n+2)}$

B $\quad C+\sum_{n=0}^{\infty}\frac{2^nx^{n+2}}{n(n+2)}$

C $\quad C+\sum_{n=0}^{\infty}\frac{2x^{n+2}}{n!(n+2)}$

D $\quad C+\sum_{n=0}^{\infty}\frac{2^nx^{n+1}}{n!(n+1)}$

Solution: A

When we're asked to use an infinite series to evaluate an indefinite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll replace x with $2x$, and then multiply by x.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \cdot x$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n x^1}{n!}$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$$

Now we can integrate the power series instead of the original function.

$$\int xe^{2x}\ dx = \int \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int xe^{2x}\ dx = \sum_{n=0}^{\infty} \frac{2^n}{n!} \int x^{n+1}\ dx$$

$$\int xe^{2x}\ dx = \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot \frac{x^{n+1+1}}{n+1+1} + C$$

$$\int xe^{2x}\ dx = \sum_{n=0}^{\infty} \frac{2^n x^{n+2}}{n!(n+2)} + C$$

Topic: Indefinite integral as infinite series

Question: Use an infinite series to evaluate the indefinite integral.

$$\int \frac{2x}{1-3x}\,dx$$

Answer choices:

A $\quad C + \sum_{n=0}^{\infty} \frac{6x^{n+2}}{n+2}$

B $\quad C + \sum_{n=0}^{\infty} \frac{2(3^n)x^{n+1}}{n+1}$

C $\quad C + \sum_{n=0}^{\infty} \frac{2(3^n)x^{n+2}}{n+2}$

D $\quad C + \sum_{n=0}^{\infty} \frac{6^n x^{n+2}}{n+2}$

Solution: C

When we're asked to use an infinite series to evaluate an indefinite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll replace x with $3x$, and then multiply by $2x$.

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n$$

$$\frac{2x}{1-3x} = \sum_{n=0}^{\infty} 2x(3x)^n$$

$$\frac{2x}{1-3x} = \sum_{n=0}^{\infty} 2x3^n x^n$$

$$\frac{2x}{1-3x} = \sum_{n=0}^{\infty} 2(3^n)x^{n+1}$$

Now we can integrate the power series instead of the original function.

$$\int \frac{2x}{1-3x}\ dx = \int \sum_{n=0}^{\infty} 2(3^n)x^{n+1}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \int x^{n+1}\ dx$$

$$\int \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \frac{x^{n+1+1}}{n+1+1}$$

$$\int \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} \frac{2(3^n)x^{n+2}}{n+2} + C$$

Topic: Indefinite integral as infinite series

Question: Use an infinite series to evaluate the indefinite integral.

$$\int x^2 \ln(1+4x)\ dx$$

Answer choices:

A $\quad C+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n x^{n+3}}{n!(n+3)}$

B $\quad C+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n x^{n+3}}{n(n+3)}$

C $\quad C+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n x^{n+2}}{n(n+2)}$

D $\quad C+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n x^{n+2}}{n!(n+2)}$

Solution: B

When we're asked to use an infinite series to evaluate an indefinite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll replace x with $4x$, and then multiply by x^2.

$$\ln(1+4x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(4x)^n$$

$$x^2\ln(1+4x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(4x)^n\cdot x^2$$

$$x^2\ln(1+4x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot 4^nx^nx^2$$

$$x^2\ln(1+4x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^nx^{n+2}}{n}$$

Now we can integrate the power series instead of the original function.

$$\int x^2 \ln(1+4x)\ dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n x^{n+2}}{n}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int x^2 \ln(1+4x)\ dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n}{n} \int x^{n+2}\ dx$$

$$\int x^2 \ln(1+4x)\ dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n}{n} \cdot \frac{x^{n+2+1}}{n+2+1} + C$$

$$\int x^2 \ln(1+4x)\ dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n x^{n+3}}{n(n+3)} + C$$

Topic: Maclaurin series to estimate an indefinite integral

Question: Use a Maclaurin series to estimate the indefinite integral.

$$\int \frac{e^x - 1}{2x}\ dx$$

Answer choices:

A $\quad C + \sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

B $\quad C + \sum_{n=1}^{\infty} \frac{x^n}{n(n!)}$

C $\quad C + \sum_{n=1}^{\infty} \frac{x^n}{2n(n!)}$

D $\quad C + \sum_{n=1}^{\infty} \frac{x^n}{2n!}$

Solution: C

When we're asked to use Maclaurin series to estimate an indefinite integral, it means we're supposed to substitute the Maclaurin series expansion for part of the function we've been asked to integrate, simplify the polynomial expression, and then integrate that polynomial instead of the original function.

We know that the Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

Plugging the expansion into the given function in place of e^x, we get

$$\int \frac{e^x - 1}{2x}\ dx = \int \frac{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots - 1}{2x}\ dx$$

$$\int \frac{e^x - 1}{2x}\ dx = \int \frac{x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots}{2x}\ dx$$

$$\int \frac{e^x - 1}{2x}\ dx = \int \frac{x}{2x} + \frac{\frac{1}{2}x^2}{2x} + \frac{\frac{1}{6}x^3}{2x} + \dots\ dx$$

$$\int \frac{e^x - 1}{2x}\ dx = \int \frac{1}{2} + \frac{1}{4}x + \frac{1}{12}x^2 + \dots\ dx$$

Now we can integrate the polynomial expansion instead of the original function.

$$\int \frac{e^x - 1}{2x}\ dx = \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{36}x^3 + \dots + C$$

Since the right side is still an infinite series, we need to give our answer in terms of an infinite sum. We'll rewrite the right side as

$$\frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{36}x^3 + \ldots + C$$

$$\frac{1}{2}\left(x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \ldots\right) + C$$

$$\frac{1}{2}\left[\frac{1}{1(1)}x^1 + \frac{1}{2(2)}x^2 + \frac{1}{3(6)}x^3 + \ldots\right] + C$$

$$\frac{1}{2}\left[\frac{1}{1(1!)}x^1 + \frac{1}{2(2!)}x^2 + \frac{1}{3(3!)}x^3 + \ldots\right] + C$$

Therefore, the integral can be represented as

$$\int \frac{e^x - 1}{2x}\ dx = C + \frac{1}{2}\sum_{n=1}^{\infty} \frac{x^n}{n(n!)}$$

$$\int \frac{e^x - 1}{2x}\ dx = C + \sum_{n=1}^{\infty} \frac{x^n}{2n(n!)}$$

Topic: Maclaurin series to estimate an indefinite integral

Question: Use a Maclaurin series to estimate the indefinite integral.

$$\int \sin x - x \ dx$$

Answer choices:

A $\quad C + \sum_{n=1}^{\infty} \frac{x^{2n+2}}{2^{n+2}(n+1)}$

B $\quad C + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$

C $\quad C + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2^{n+2}(2n+1)}$

D $\quad C + \sum_{n=1}^{\infty} \frac{x^{2n+2}}{2^{n+1}(2n+1)}$

Solution: B

When we're asked to use Maclaurin series to estimate an indefinite integral, it means we're supposed to substitute the Maclaurin series expansion for part of the function we've been asked to integrate, simplify the polynomial expression, and then integrate that polynomial instead of the original function.

We know that the Maclaurin series expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Plugging the expansion into the given function in place of $\sin x$, we get

$$\int \sin x - x\ dx = \int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots - x\ dx$$

$$\int \sin x - x\ dx = \int -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\ dx$$

Now we can integrate the polynomial expansion instead of the original function.

$$\int \sin x - x\ dx = -\frac{x^4}{4(3!)} + \frac{x^6}{6(5!)} - \frac{x^8}{8(7!)} + \dots + C$$

Since the right side is still an infinite series, we need to give our answer in terms of an infinite sum. We'll rewrite the right side as

$$-\frac{x^4}{4(3!)} + \frac{x^6}{6(5!)} - \frac{x^8}{8(7!)} + \dots + C$$

$$-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\ldots+C$$

$$-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\ldots+C$$

$$-\frac{x^{2(1)+2}}{(2(1)+2)!}+\frac{x^{2(2)+2}}{(2(2)+2)!}-\frac{x^{2(3)+2}}{(2(3)+2)!}+\ldots+C$$

Therefore, the integral can be represented as

$$\int \sin x - x\ dx = C + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

Topic: Maclaurin series to estimate an indefinite integral

Question: Use a Maclaurin series to estimate the indefinite integral.

$$\int x \ln(1+2x)\ dx$$

Answer choices:

A $\quad C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+2}}{n(n+1)}$

B $\quad C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+2)}$

C $\quad C + \sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n+2}}{n(n+2)}$

D $\quad C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n x^{n+2}}{n(n+2)}$

Solution: D

When we're asked to use Maclaurin series to estimate an indefinite integral, it means we're supposed to substitute the Maclaurin series expansion for part of the function we've been asked to integrate, simplify the polynomial expression, and then integrate that polynomial instead of the original function.

We know that the Maclaurin series expansion of $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

In the case of our function though, we have $\ln(1+2x)$ instead of $\ln(1+x)$, which means we need to replace x with $2x$ everywhere.

$$\ln(1+2x) = 2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 - \dots$$

$$\ln(1+2x) = 2x - \frac{1}{2}\left(4x^2\right) + \frac{1}{3}\left(8x^3\right) - \dots$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - \dots$$

Plugging the expansion into the given function in place of $\ln(1+2x)$, we get

$$\int x\ln(1+2x)\ dx = \int x\left(2x - 2x^2 + \frac{8}{3}x^3 - \dots\right)\ dx$$

$$\int x\ln(1+2x)\ dx = \int 2x^2 - 2x^3 + \frac{8}{3}x^4 - \dots\ dx$$

Now we can integrate the polynomial expansion instead of the original function.

$$\int x\ln(1+2x)\ dx=\frac{2}{3}x^3-\frac{2}{4}x^4+\frac{8}{15}x^5-\ldots+C$$

Since the right side is still an infinite series, we need to give our answer in terms of an infinite sum. We'll rewrite the right side as

$$\frac{2}{3}x^3-\frac{2}{4}x^4+\frac{8}{15}x^5-\ldots+C$$

$$\frac{2}{3}x^{1+2}-\frac{2}{4}x^{2+2}+\frac{8}{15}x^{3+2}-\ldots+C$$

$$(-1)^{1+1}\frac{2}{3}x^{1+2}+(-1)^{2+1}\frac{2}{4}x^{2+2}+(-1)^{3+1}\frac{8}{15}x^{3+2}+\ldots+C$$

$$(-1)^{1+1}\frac{2}{3}x^{1+2}+(-1)^{2+1}\frac{4}{8}x^{2+2}+(-1)^{3+1}\frac{8}{15}x^{3+2}+\ldots+C$$

$$(-1)^{1+1}\frac{2^1}{3}x^{1+2}+(-1)^{2+1}\frac{2^2}{8}x^{2+2}+(-1)^{3+1}\frac{2^3}{15}x^{3+2}+\ldots+C$$

$$(-1)^{1+1}\frac{2^1}{1(1+2)}x^{1+2}+(-1)^{2+1}\frac{2^2}{2(2+2)}x^{2+2}+(-1)^{3+1}\frac{2^3}{3(3+2)}x^{3+2}+\ldots+C$$

Therefore, the integral can be represented as

$$\int x\ln(1+2x)\ dx=C+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^nx^{n+2}}{n(n+2)}$$

Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^2 xe^x \, dx$$

Answer choices:

A $\quad 8.0$

B $\quad 1.0$

C $\quad 0.8$

D $\quad 2.0$

Solution: A

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll multiply both sides by x.

$$xe^x = x\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Now we can integrate the power series instead of the original function.

$$\int_0^2 xe^x \, dx = \int_0^2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \, dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^2 x^{n+1}\ dx$$

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{x^{n+2}}{n+2} \Bigg|_0^2$$

$$\int_0^2 xe^x\ dx = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!(n+2)} \Bigg|_0^2$$

Now we'll expand the power series through its first eight terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^2 xe^x\ dx = \frac{x^{0+2}}{0!(0+2)} + \frac{x^{1+2}}{1!(1+2)} + \frac{x^{2+2}}{2!(2+2)} + \frac{x^{3+2}}{3!(3+2)} + \frac{x^{4+2}}{4!(4+2)}$$

$$+ \frac{x^{5+2}}{5!(5+2)} + \frac{x^{6+2}}{6!(6+2)} + \frac{x^{7+2}}{7!(7+2)} + \frac{x^{8+2}}{8!(8+2)} + \dots \Bigg|_0^2$$

$$\int_0^2 xe^x\ dx = \frac{x^2}{1(2)} + \frac{x^3}{1(3)} + \frac{x^4}{2(4)} + \frac{x^5}{6(5)} + \frac{x^6}{24(6)}$$

$$+ \frac{x^7}{120(7)} + \frac{x^8}{720(8)} + \frac{x^9}{5{,}040(9)} + \frac{x^{10}}{40{,}320(10)} \Bigg|_0^2$$

$$\int_0^2 xe^x\ dx = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + \frac{x^6}{144} + \frac{x^7}{840} + \frac{x^8}{5{,}760} + \frac{x^9}{45{,}360} + \frac{x^{10}}{403{,}200} \Bigg|_0^2$$

Now we can evaluate the expansion over the interval. Since plugging in $x=0$ will give 0 for every term, we only need to evaluate at $x=2$.

$$\int_0^2 xe^x\ dx = \frac{2^2}{2}+\frac{2^3}{3}+\frac{2^4}{8}+\frac{2^5}{30}+\frac{2^6}{144}+\frac{2^7}{840}+\frac{2^8}{5,760}+\frac{2^9}{45,360}+\frac{2^{10}}{403,200}$$

$$\int_0^2 xe^x\ dx = \frac{4}{2}+\frac{8}{3}+\frac{16}{8}+\frac{32}{30}+\frac{64}{144}+\frac{128}{840}+\frac{256}{5,760}+\frac{512}{45,360}+\frac{1,024}{403,200}$$

$$\int_0^2 xe^x\ dx \approx 2.00000+2.66667+2.00000+1.06667+0.44444$$

$$+0.15238+0.04444+0.01128+0.00254$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$2.00000+2.66667=4.66667$$

$$4.66667+2.00000=6.66667$$

$$6.66667+1.06667=7.73334$$

$$7.73334+0.04444=7.77778$$

$$7.77778+0.15238=7.93016$$

$$7.93016+0.04444=7.97460$$

$$7.97460+0.01128=7.98588$$

$$7.98588+0.00254=7.98842$$

Since we got a 9 in the tenths place four times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 8.0 as an approximation of the definite integral.

Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\, dx$$

Answer choices:

A $\quad 1.1$

B $\quad 1.0$

C $\quad 0.1$

D $\quad 2.0$

Solution: C

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll substitute $3x$ for x and then multiply by $2x$.

$$\frac{1}{1-3x}=\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{2x}{1-3x}=2x\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{2x}{1-3x}=\sum_{n=0}^{\infty}2x3^nx^n$$

$$\frac{2x}{1-3x}=\sum_{n=0}^{\infty}2(3^n)x^{n+1}$$

Now we can integrate the power series instead of the original function.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} 2(3^n)x^{n+1}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \int_0^{\frac{1}{4}} x^{n+1}\ dx$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} 2(3^n) \cdot \frac{x^{n+2}}{n+2} \Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \sum_{n=0}^{\infty} \frac{2(3^n)x^{n+2}}{n+2} \Bigg|_0^{\frac{1}{4}}$$

Now we'll expand the power series through its first eight terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \frac{2(3^0)x^{0+2}}{0+2} + \frac{2(3^1)x^{1+2}}{1+2} + \frac{2(3^2)x^{2+2}}{2+2} + \frac{2(3^3)x^{3+2}}{3+2} + \frac{2(3^4)x^{4+2}}{4+2}$$

$$+ \frac{2(3^5)x^{5+2}}{5+2} + \frac{2(3^6)x^{6+2}}{6+2} + \frac{2(3^7)x^{7+2}}{7+2} + \frac{2(3^8)x^{8+2}}{8+2} \Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = \frac{2(1)x^2}{2} + \frac{2(3)x^3}{3} + \frac{2(9)x^4}{4} + \frac{2(27)x^5}{5} + \frac{2(81)x^6}{6}$$

$$+\frac{2(243)x^7}{7}+\frac{2(729)x^8}{8}+\frac{2(2,187)x^9}{9}+\frac{2(6,561)x^{10}}{10}\Bigg|_0^{\frac{1}{4}}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2x^2}{2}+\frac{6x^3}{3}+\frac{18x^4}{4}+\frac{54x^5}{5}+\frac{162x^6}{6}$$

$$+\frac{486x^7}{7}+\frac{1,458x^8}{8}+\frac{4,374x^9}{9}+\frac{13,122x^{10}}{10}\Bigg|_0^{\frac{1}{4}}$$

Now we can evaluate the expansion over the interval. Since plugging in $x=0$ will give 0 for every term, we only need to evaluate at $x=1/4$.

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2\left(\frac{1}{4}\right)^2}{2}+\frac{6\left(\frac{1}{4}\right)^3}{3}+\frac{18\left(\frac{1}{4}\right)^4}{4}+\frac{54\left(\frac{1}{4}\right)^5}{5}+\frac{162\left(\frac{1}{4}\right)^6}{6}$$

$$+\frac{486\left(\frac{1}{4}\right)^7}{7}+\frac{1,458\left(\frac{1}{4}\right)^8}{8}+\frac{4,374\left(\frac{1}{4}\right)^9}{9}+\frac{13,122\left(\frac{1}{4}\right)^{10}}{10}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{2\left(\frac{1}{16}\right)}{2}+\frac{6\left(\frac{1}{64}\right)}{3}+\frac{18\left(\frac{1}{256}\right)}{4}+\frac{54\left(\frac{1}{1,024}\right)}{5}+\frac{162\left(\frac{1}{4,096}\right)}{6}$$

$$+\frac{486\left(\frac{1}{16,384}\right)}{7}+\frac{1,458\left(\frac{1}{65,536}\right)}{8}+\frac{4,374\left(\frac{1}{262,144}\right)}{9}+\frac{13,122\left(\frac{1}{1,048,576}\right)}{10}$$

$$\int_0^{\frac{1}{4}}\frac{2x}{1-3x}\ dx=\frac{1}{16}+\frac{1}{32}+\frac{9}{512}+\frac{27}{2,560}+\frac{27}{4,096}$$

$$+\frac{243}{57,344}+\frac{729}{262,144}+\frac{243}{131,072}+\frac{6,561}{5,242,880}$$

$$\int_0^{\frac{1}{4}} \frac{2x}{1-3x}\ dx = 0.06250 + 0.03125 + 0.01758 + 0.01055 + 0.00659$$

$$+0.00424 + 0.00278 + 0.00185 + 0.00125$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$0.06250 + 0.03125 = 0.09375$$

$$0.09375 + 0.01758 = 0.11133$$

$$0.11133 + 0.01055 = 0.12188$$

$$0.12188 + 0.00659 = 0.12847$$

$$0.12847 + 0.00424 = 0.13271$$

$$0.13271 + 0.00278 = 0.13549$$

$$0.13549 + 0.00185 = 0.13734$$

$$0.13734 + 0.00125 = 0.13859$$

Since we got a 1 in the tenths place four times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 0.1 as an approximation of the definite integral.

Topic: Maclaurin series to estimate a definite integral

Question: Use a Maclaurin series to estimate the definite integral.

$$\int_0^1 \frac{\cos x}{2}\, dx$$

Answer choices:

A 0.1

B 0.4

C -0.2

D 0.2

Solution: B

When we're asked to use a Maclaurin series to estimate a definite integral, it means we're supposed to find a power series representation for the function we've been asked to integrate, and then integrate that power series instead of the original function.

To find the power series representation of the given function, we'll start with the known Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and then manipulate it until it matches the given series. To get it to match the given series, we'll multiply by $1/2$.

$$\frac{1}{2}\cos x = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{\cos x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2(2n)!}$$

Now we can integrate the power series instead of the original function.

$$\int_0^1 \frac{\cos x}{2}\ dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2(2n)!}\ dx$$

Since we're integrating with respect to x, we can remove from the integral on the right any term that doesn't involve x.

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!} \int_0^1 x^{2n}\ dx$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2(2n+1)(2n)!} \Big|_0^1$$

Now we'll expand the power series through its first five terms. That should be enough to make sure that our answer is stable to one decimal place, which is the number of decimal places given in the answer choices.

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{(-1)^0 x^{2(0)+1}}{2(2(0)+1)(2(0))!} + \frac{(-1)^1 x^{2(1)+1}}{2(2(1)+1)(2(1))!} + \frac{(-1)^2 x^{2(2)+1}}{2(2(2)+1)(2(2))!}$$

$$+ \frac{(-1)^3 x^{2(3)+1}}{2(2(3)+1)(2(3))!} + \frac{(-1)^4 x^{2(4)+1}}{2(2(4)+1)(2(4))!} + \frac{(-1)^5 x^{2(5)+1}}{2(2(5)+1)(2(5))!} \Big|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{x^1}{2(1)0!} + \frac{-x^3}{2(3)2!} + \frac{x^5}{2(5)4!} + \frac{-x^7}{2(7)6!} + \frac{x^9}{2(9)8!} + \frac{-x^{11}}{2(11)10!} \Big|_0^1$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{x^1}{(2)0!} - \frac{x^3}{(6)2!} + \frac{x^5}{(10)4!} - \frac{x^7}{(14)6!} + \frac{x^9}{(18)8!} - \frac{x^{11}}{(22)10!} \Big|_0^1$$

Now we can evaluate the expansion over the interval. Since plugging in $x = 0$ will give 0 for every term, we only need to evaluate at $x = 1$.

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{1^1}{(2)0!} - \frac{1^3}{(6)2!} + \frac{1^5}{(10)4!} - \frac{1^7}{(14)6!} + \frac{1^9}{(18)8!} - \frac{1^{11}}{(22)10!}$$

$$\int_0^1 \frac{\cos x}{2}\ dx = \frac{1}{2} - \frac{1}{12} + \frac{1}{240} - \frac{1}{10{,}080} + \frac{1}{933{,}120} - \frac{1}{102{,}643{,}200}$$

$$\int_0^1 \frac{\cos x}{2}\ dx = 0.50000 - 0.08333 + 0.00417 - 0.00010 + 0.00000 - 0.00000$$

Now we need to add the terms together until we get an answer that's stable to one decimal place.

$$0.50000 - 0.08333 = 0.41667$$

$$0.41667 + 0.00417 = 0.42084$$

$$0.42084 - 0.00010 = 0.42074$$

$$0.42074 + 0.00000 = 0.42074$$

$$0.42074 - 0.00000 = 0.42074$$

Since we got a 4 in the tenths place five times in a row, we know the answer will be stable to this value. We'll round the last answer to the tenths place and get 0.4 as an approximation of the definite integral.

Topic: Maclaurin series to evaluate a limit

Question: Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Answer choices:

A 2

B $-\frac{1}{2}$

C -2

D $\frac{1}{2}$

Solution: D

When we're asked to use a Maclaurin series to evaluate a limit, we're supposed to use a known Maclaurin series expansion in place of part of the function, such that we turn the function into a polynomial expression.

The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

so we'll substitute the first few terms of this expansion into the limit we've been given.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots - 1 - x}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{1}{2} + \frac{1}{6}x + \frac{1}{24}x^2 + \dots$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{1}{6}(0) + \frac{1}{24}(0)^2 + \dots$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

The limit of the function is $1/2$.

Topic: Maclaurin series to evaluate a limit

Question: Use a Maclaurin series to evaluate the limit.

$$\lim_{x\to 0} \frac{\ln(1+2x)-2x}{x^2}$$

Answer choices:

A 2

B $-\frac{1}{2}$

C -2

D $\frac{1}{2}$

Solution: C

When we're asked to use a Maclaurin series to evaluate a limit, we're supposed to use a known Maclaurin series expansion in place of part of the function, such that we turn the function into a polynomial expression.

The Maclaurin series expansion of $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

This is really similar to the part of the function we've been given, $\ln(1+2x)$. We just have to substitute $2x$ for x.

$$\ln(1+2x) = 2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + \dots$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \dots$$

Now we'll substitute the first few terms of this expansion into the limit we've been given.

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2} = \lim_{x\to 0} \frac{2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \dots - 2x}{x^2}$$

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2} = \lim_{x\to 0} \frac{-2x^2 + \frac{8}{3}x^3 - 4x^4 + \dots}{x^2}$$

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2} = \lim_{x\to 0} -2 + \frac{8}{3}x - 4x^2 + \dots$$

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2} = -2 + \frac{8}{3}(0) - 4(0)^2 + \dots$$

$$\lim_{x \to 0} \frac{\ln(1+2x) - 2x}{x^2} = -2$$

The limit of the function is -2.

Topic: Maclaurin series to evaluate a limit

Question: Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \frac{2 \sin x - 2x}{x^3}$$

Answer choices:

A $\frac{1}{3}$

B $-\frac{1}{2}$

C $-\frac{1}{3}$

D $\frac{1}{2}$

Solution: C

When we're asked to use a Maclaurin series to evaluate a limit, we're supposed to use a known Maclaurin series expansion in place of part of the function, such that we turn the function into a polynomial expression.

The Maclaurin series expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

This is really similar to the part of the function we've been given, $2\sin x$. We just have to multiply by 2.

$$2\sin x = 2x - \frac{2x^3}{3!} + \frac{2x^5}{5!} - \dots$$

$$2\sin x = 2x - \frac{x^3}{3} + \frac{x^5}{60} - \dots$$

Now we'll substitute the first few terms of this expansion into the limit we've been given.

$$\lim_{x \to 0} \frac{2\sin x - 2x}{x^3} = \lim_{x \to 0} \frac{2x - \frac{x^3}{3} + \frac{x^5}{60} - \dots - 2x}{x^3}$$

$$\lim_{x \to 0} \frac{2\sin x - 2x}{x^3} = \lim_{x \to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{60} - \dots}{x^3}$$

$$\lim_{x \to 0} \frac{2\sin x - 2x}{x^3} = \lim_{x \to 0} -\frac{1}{3} + \frac{x^2}{60} - \dots$$

$$\lim_{x \to 0} \frac{2 \sin x - 2x}{x^3} = -\frac{1}{3} + \frac{0^2}{60} - \dots$$

$$\lim_{x \to 0} \frac{2 \sin x - 2x}{x^3} = -\frac{1}{3}$$

The limit of the function is $-1/3$.

W W W . K R I S T A K I N G M A T H . C O M