Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = \sin\theta$$

$$r = \cos\theta$$

Answer choices:

A $\dfrac{\pi + 1}{4}$

B $\dfrac{\pi - 2}{8}$

C $\dfrac{\pi + 2}{8}$

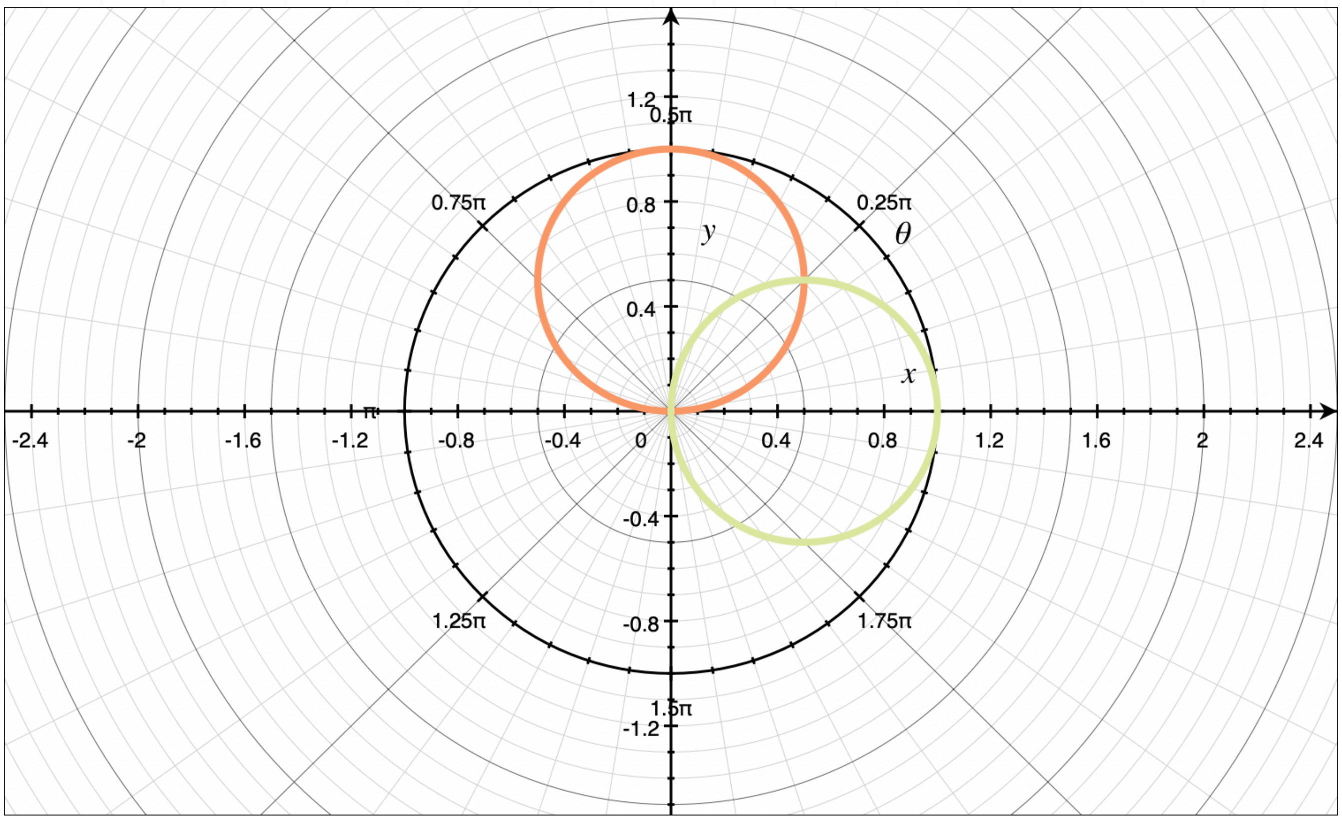
D $\dfrac{\pi - 1}{4}$

Solution: B

To find are inside a polar curve, we use the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\ d\theta$$

The best thing to do is to graph the functions we've been given.



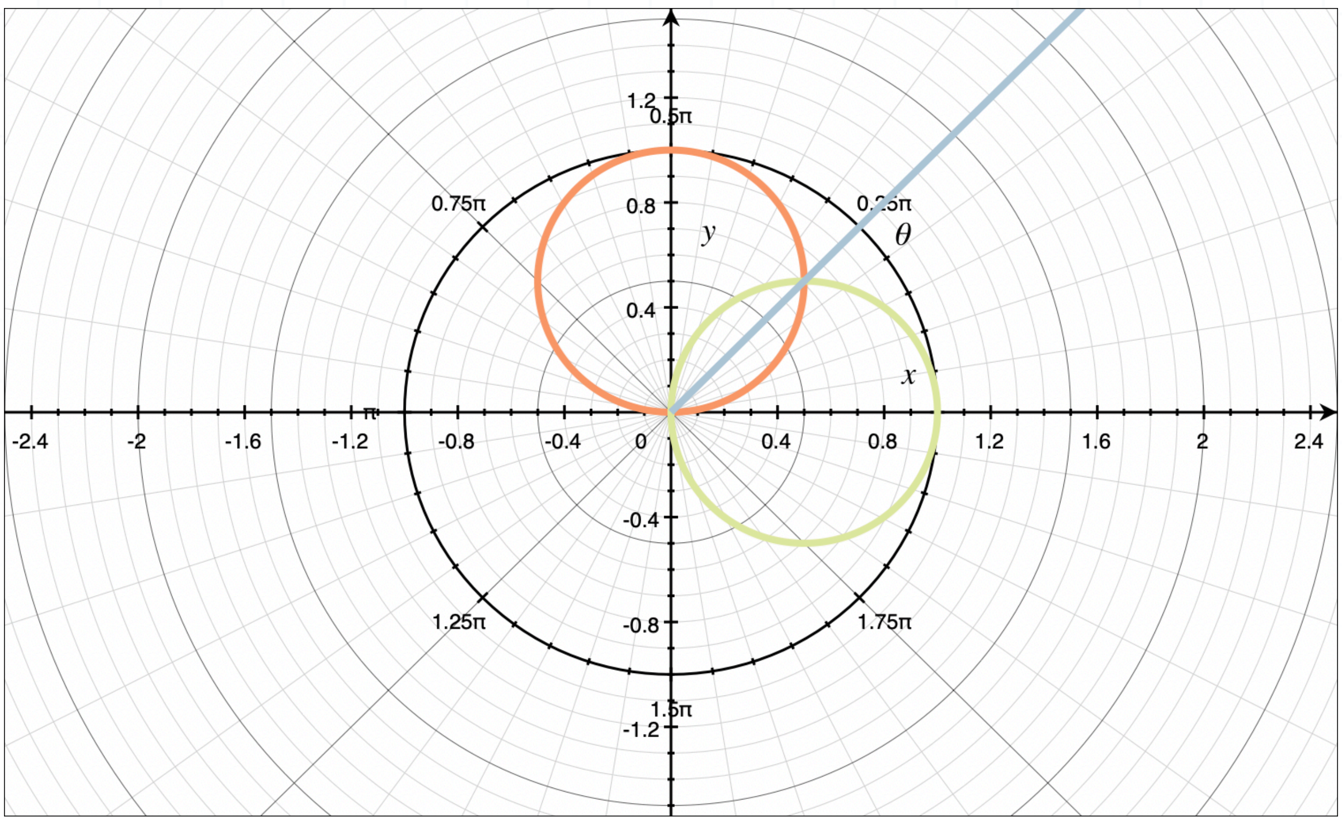
We can see that the area that's inside both curves at the same time is the single small petal where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

$$\sin\theta = \cos\theta$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta = \pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$\theta = \frac{\pi}{4}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/4$. If we sketched the $\sin\theta$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/4$. Which means the beginning of the red $\sin\theta$ curve runs underneath the line $\theta = \pi/4$.



So if we integrate $\sin\theta$ in the polar area formula, over the interval $[0,\pi/4]$, we'll only get the area that lies underneath that $\theta = \pi/4$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area that's inside both curves.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/4]$, we'll use that curve in our formula.

$$A = (2)\frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta$$

$$A=\int_0^{\frac{\pi}{4}}\sin^2\theta\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=\int_0^{\frac{\pi}{4}}\frac{1}{2}(1-\cos(2\theta))\ d\theta$$

$$A=\frac{1}{2}\int_0^{\frac{\pi}{4}}1-\cos(2\theta)\ d\theta$$

$$A=\frac{1}{2}\left(\theta-\frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\frac{\pi}{4}}$$

Evaluate over the interval.

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)\right)-\frac{1}{2}\left(0-\frac{1}{2}\sin(2(0))\right)$$

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\frac{1}{2}\left(0-\frac{1}{2}(0)\right)$$

$$A=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}(1)\right)$$

$$A=\frac{\pi}{8}-\frac{1}{4}$$

$$A = \frac{\pi}{8} - \frac{2}{8}$$

$$A = \frac{\pi - 2}{8}$$

Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = \sin(3\theta)$$

$$r = \cos(3\theta)$$

Answer choices:

A $\frac{\pi + 1}{4}$

B $\frac{\pi - 1}{4}$

C $\frac{\pi + 2}{8}$

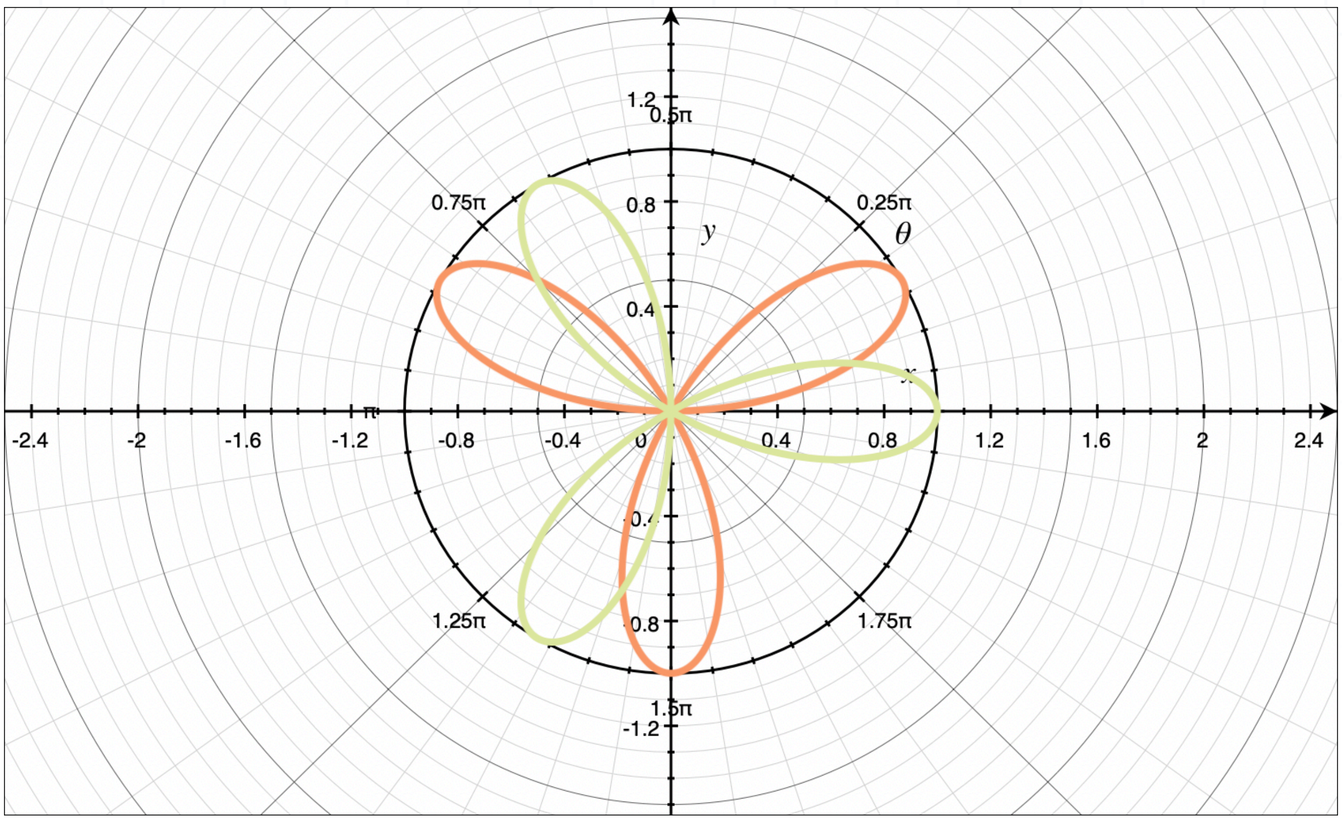
D $\frac{\pi - 2}{8}$

Solution: D

To find are inside a polar curve, we use the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

The best thing to do is to graph the functions we've been given.



We can see that the area that's inside both curves at the same time is the three small petals where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

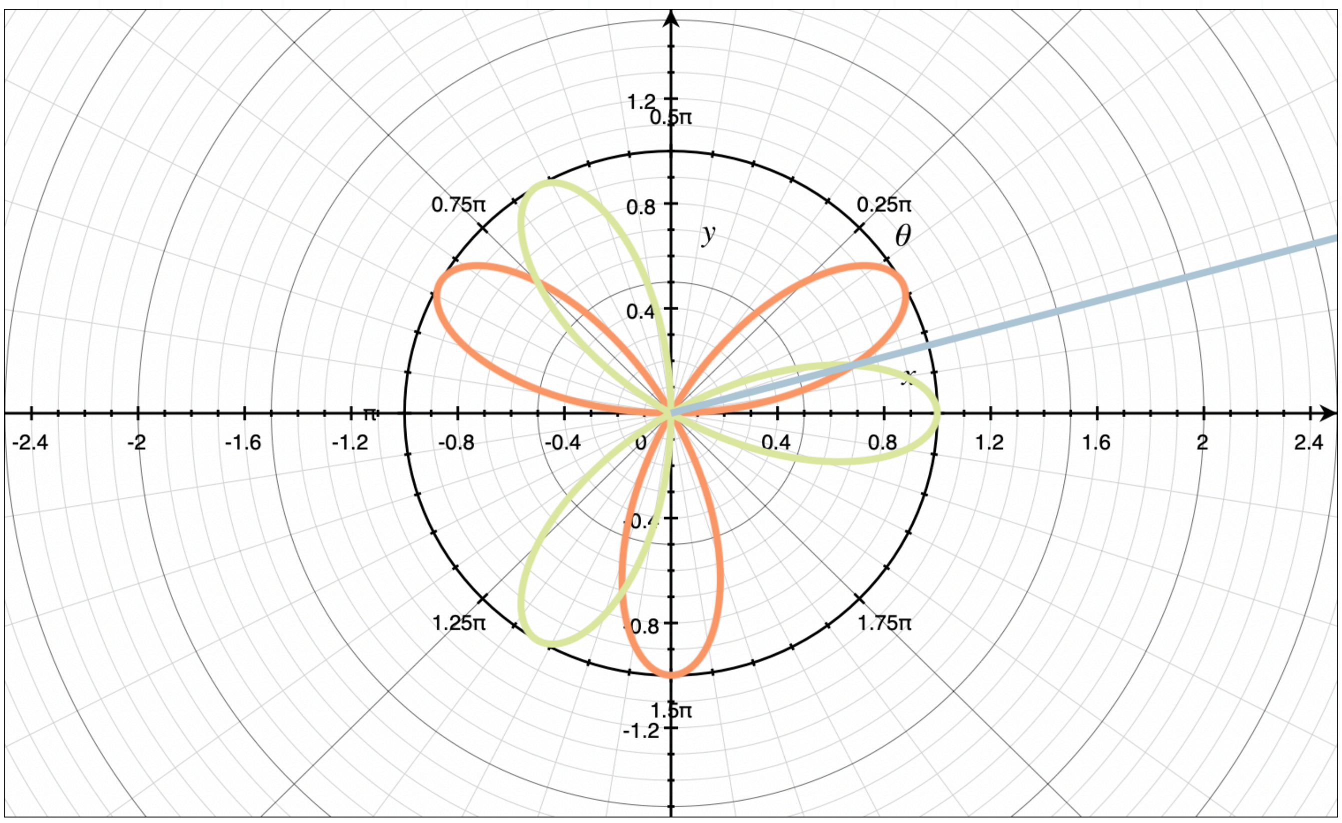
$$\sin(3\theta) = \cos(3\theta)$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta = \pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$3\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{12}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/12$. If we sketched the $\sin(3\theta)$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/12$, and out to the tip of the first red petal. Which means the beginning of the red $\sin(3\theta)$ curve runs underneath the line $\theta = \pi/12$.



So if we integrate $\sin(3\theta)$ in the polar area formula, over the interval $[0,\pi/12]$, we'll only get the area that lies underneath that $\theta = \pi/12$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area for one of the three petals we need. But there are three distinct sections of area that are inside both curves. Therefore, we'll be multiplying by 2, and then by 3, or we can just multiply by 6.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/12]$, we'll use that curve in our formula.

$$A=(6)\frac{1}{2}\int_0^{\frac{\pi}{12}}\sin^2(3\theta)\ d\theta$$

$$A=3\int_0^{\frac{\pi}{12}}\sin^2(3\theta)\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=3\int_0^{\frac{\pi}{12}}\frac{1}{2}(1-\cos(2(3\theta)))\ d\theta$$

$$A=\frac{3}{2}\int_0^{\frac{\pi}{12}}1-\cos(6\theta)\ d\theta$$

$$A=\frac{3}{2}\left(\theta-\frac{1}{6}\sin(6\theta)\right)\Bigg|_0^{\frac{\pi}{12}}$$

Evaluate over the interval.

$$A=\frac{3}{2}\left(\frac{\pi}{12}-\frac{1}{6}\sin\left(6\cdot\frac{\pi}{12}\right)\right)-\frac{3}{2}\left(0-\frac{1}{6}\sin(6(0))\right)$$

$$A = \frac{3}{2}\left(\frac{\pi}{12} - \frac{1}{6}\sin\frac{\pi}{2}\right) - \frac{3}{2}\left(0 - \frac{1}{6}(0)\right)$$

$$A = \frac{3}{2}\left(\frac{\pi}{12} - \frac{1}{6}(1)\right)$$

$$A = \frac{3\pi}{24} - \frac{3}{12}$$

$$A = \frac{3\pi}{24} - \frac{6}{24}$$

$$A = \frac{3\pi - 6}{24}$$

$$A = \frac{\pi - 2}{8}$$

Topic: Area inside both polar curves

Question: Find the area inside both curves.

$$r = 3\sin(2\theta)$$

$$r = 3\cos(2\theta)$$

Answer choices:

A $\quad \dfrac{9\pi - 18}{2}$

B $\quad \dfrac{9\pi - 9}{2}$

C $\quad \dfrac{9\pi + 9}{2}$

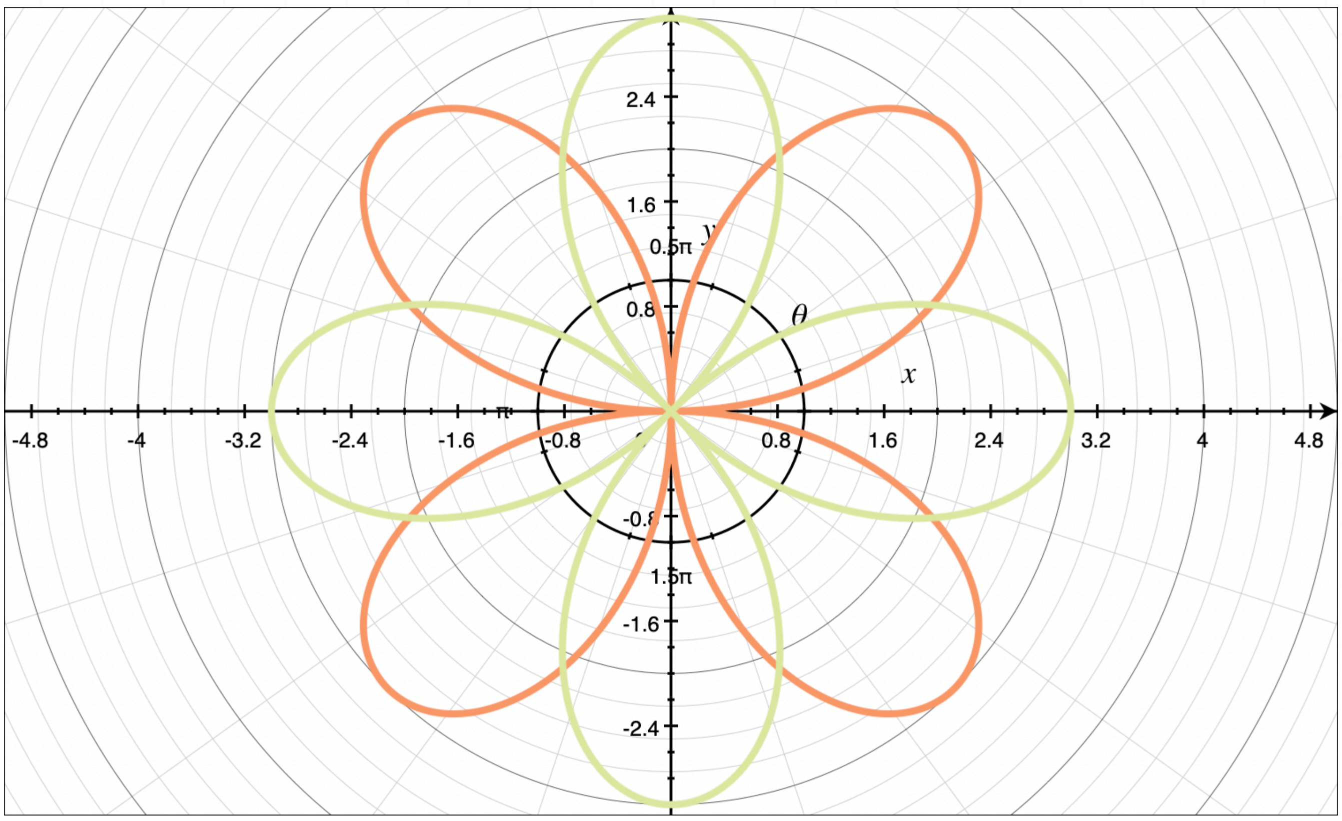
D $\quad \dfrac{9\pi + 18}{2}$

Solution: A

To find area inside a polar curve, we use the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \ d\theta$$

The best thing to do is to graph the functions we've been given.



We can see that the area that's inside both curves at the same time is the eight small petals where the curves overlap. The next step will be to find points of intersection, so we'll set the curves equal to one another.

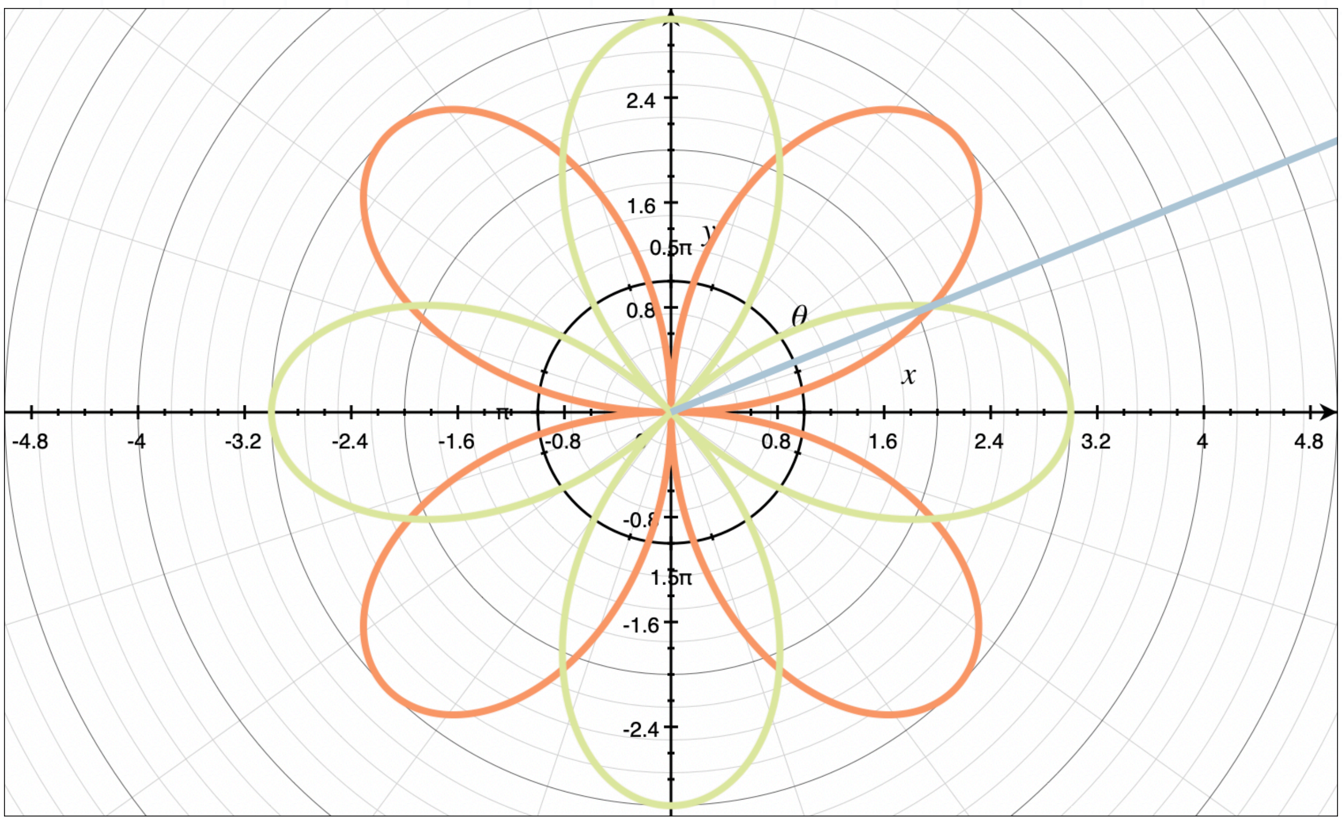
$$3\sin(2\theta) = 3\cos(2\theta)$$

From the unit circle, we know that the values of $\sin\theta$ and $\cos\theta$ are equal to each other at $\theta = \pi/4$. So if we need the angle from our functions to be $\pi/4$, then, we'll need to set our angle equal to $\pi/4$. In other words

$$2\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8}$$

We can see the intersection point of the curves that corresponds to $\theta = \pi/8$. If we sketched the $3\sin(2\theta)$ curve, we know it's the red curve that starts at the origin when $\theta = 0$, and curls up through the first quadrant, through the intersection point at $\theta = \pi/8$, and out to the tip of the first red petal. Which means the beginning of the red $3\sin(2\theta)$ curve runs underneath the line $\theta = \pi/8$.



So if we integrate $3\sin(2\theta)$ in the polar area formula, over the interval $[0,\pi/8]$, we'll only get the area that lies underneath that $\theta = \pi/8$ line. Which means that, in order to get the full area we're looking for, we'll need to multiply the area formula by 2. That'll give us the area for one of the eight petals we need. But there are eight distinct sections of area that are inside both curves. Therefore, we'll be multiplying by 2, and then by 8, or we can just multiply by 16.

Since it's the red $\sin$ curve we were talking about that corresponds to the limits of integration $[0,\pi/8]$, we'll use that curve in our formula.

$$A=(16)\frac{1}{2}\int_0^{\frac{\pi}{8}}9\sin^2(2\theta)\ d\theta$$

$$A=72\int_0^{\frac{\pi}{8}}\sin^2(2\theta)\ d\theta$$

We'll use the double-angle identity

$$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$$

to simplify the integrand.

$$A=72\int_0^{\frac{\pi}{8}}\frac{1}{2}(1-\cos(2(2\theta)))\ d\theta$$

$$A=36\int_0^{\frac{\pi}{8}}1-\cos(4\theta)\ d\theta$$

$$A=36\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_0^{\frac{\pi}{8}}$$

Evaluate over the interval.

$$A=36\left(\frac{\pi}{8}-\frac{1}{4}\sin\left(4\cdot\frac{\pi}{8}\right)\right)-36\left(0-\frac{1}{4}\sin(4(0))\right)$$

$$A = 36\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - 36\left(0 - \frac{1}{4}(0)\right)$$

$$A = 36\left(\frac{\pi}{8} - \frac{1}{4}(1)\right)$$

$$A = \frac{36\pi}{8} - \frac{36}{4}$$

$$A = \frac{36\pi}{8} - \frac{72}{8}$$

$$A = \frac{36\pi - 72}{8}$$

$$A = \frac{9\pi - 18}{2}$$