Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_0^1 2e^x \ dx$$

Answer choices:

A $\quad 5.44$

B $\quad 3.24$

C $\quad 3.42$

D $\quad 5.40$

Solution: C

When we use power series to integrate a function like the given function

$$f(x) = 2e^x$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll just multiply by 2.

$$2\left(e^x\right) = 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = 2\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$2e^x = 2 + \frac{2x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{2x^n}{n!}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty} \frac{2x^n}{n!}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^1 2e^x\ dx = \int_0^1 2 + \frac{2x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \dots\ dx$$

$$\int_0^1 2e^x\ dx = 2x + \frac{2x^2}{(2)1!} + \frac{2x^3}{(3)2!} + \frac{2x^4}{(4)3!} + \dots \Bigg|_0^1$$

$$\int_0^1 2e^x\ dx = 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \dots \Bigg|_0^1$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^1 2e^x\ dx = \left[2(1) + (1)^2 + \frac{(1)^3}{3} + \frac{(1)^4}{12} + \dots\right]$$

$$-\left[2(0) + (0)^2 + \frac{(0)^3}{3} + \frac{(0)^4}{12} + \dots\right]$$

Then we'll pair like-terms together.

$$\int_0^1 2e^x\ dx = \left[2(1) - 2(0)\right] + \left[(1)^2 - (0)^2\right] + \left[\frac{(1)^3}{3} - \frac{(0)^3}{3}\right] + \left[\frac{(1)^4}{12} - \frac{(0)^4}{12}\right] + \dots$$

$$\int_0^1 2e^x \, dx = (2-0) + (1-0) + \left(\frac{1}{3} - 0\right) + \left(\frac{1}{12} - 0\right) + \dots$$

$$\int_0^1 2e^x \, dx = 2 + 1 + \frac{1}{3} + \frac{1}{12} + \dots$$

$$\int_0^1 2e^x \, dx = 3 + \frac{1}{3} + \frac{1}{12} + \dots$$

$$\int_0^1 2e^x \, dx = 3.0000 + 0.3333 + 0.0833 + \dots$$

$$\int_0^1 2e^x \, dx \approx 3.42$$

This is the estimate of the definite integral using a power series.

Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_0^{0.1} \sin(2x)\ dx$$

Answer choices:

A -0.299

B 0.01

C 0.199

D 0.299

Solution: B

When we use power series to integrate a function like the given function

$$f(x)=\sin(2x)$$

we use the standard form of a power series

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty}x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll substitute $2x$ for x.

$$\sin(2x)=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^{0.1} \sin(2x)\ dx = \int_0^{0.1} 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots\ dx$$

$$\int_0^{0.1} \sin(2x)\ dx = \int_0^{0.1} 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots\ dx$$

$$\int_0^{0.1} \sin(2x)\ dx = \frac{2x^2}{2} - \frac{8x^4}{(4)3!} + \frac{32x^6}{(6)5!} - \frac{128x^8}{(8)7!} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{2x^4}{3!} + \frac{16x^6}{(3)5!} - \frac{16x^8}{7!} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{2x^4}{6} + \frac{16x^6}{360} - \frac{16x^8}{5{,}040} + \dots \Big|_0^{0.1}$$

$$\int_0^{0.1} \sin(2x)\ dx = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \dots \Big|_0^{0.1}$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^{0.1} \sin(2x)\ dx = \left[(0.1)^2 - \frac{(0.1)^4}{3} + \frac{2(0.1)^6}{45} - \frac{(0.1)^8}{315} + \dots\right]$$

$$-\left[(0)^2 - \frac{(0)^4}{3} + \frac{2(0)^6}{45} - \frac{(0)^8}{315} + \dots\right]$$

Then we'll pair like-terms together.

$$\int_0^{0.1} \sin(2x)\ dx = [(0.1)^2 - (0)^2] + \left[-\frac{(0.1)^4}{3} + \frac{(0)^4}{3}\right]$$

$$+\left[\frac{2(0.1)^6}{45} - \frac{2(0)^6}{45}\right] + \left[-\frac{(0.1)^8}{315} + \frac{(0)^8}{315}\right] + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = (0.01 - 0) + \left[-\frac{0.0001}{3} + 0\right]$$

$$+\left[\frac{2(0.000001)}{45} - 0\right] + \left[-\frac{(0.00000001)}{315} + 0\right] + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = 0.01 - \frac{0.0001}{3} + \frac{0.000002}{45} - \frac{0.00000001}{315} + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx = 0.01000000000000000$$

$$-0.00003333333333333$$

$$+0.00000004444444444$$

$$-0.00000000003174603 + \dots$$

$$\int_0^{0.1} \sin(2x)\ dx \approx 0.01$$

This is the estimate of the definite integral using a power series.

Topic: Estimating definite integrals

Question: Evaluate the definite integral as a power series.

$$\int_0^{0.2} 2\arctan(2x)\ dx$$

Answer choices:

A -0.07799

B 0.03779

C -0.03779

D 0.07799

Solution: D

When we use power series to integrate a function like the given function

$$f(x) = 2\arctan(2x)$$

we use the standard form of a power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

or if the standard form doesn't match the given series closely enough, we'll use a different common power series.

The given series is most similar to the common form

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Our goal will be to start with the far left side of the common form, and the manipulate it until it matches the given function. To get the left side of the common form to match the given function, we'll substitute $2x$ for x.

$$\arctan(2x) = 2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

Then we'll multiply through by 2.

$$2\arctan(2x) = 2\left[2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \ldots\right] = 2\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$$

$$2\arctan(2x)=4x-\frac{2(2x)^3}{3}+\frac{2(2x)^5}{5}-\frac{2(2x)^7}{7}+\ldots=\sum_{n=0}^{\infty}\frac{2(-1)^n(2x)^{2n+1}}{2n+1}$$

Now that the far left of the manipulated common form matches the given function, we can say that the far right of the manipulated common form,

$$\sum_{n=0}^{\infty}\frac{2(-1)^n(2x)^{2n+1}}{2n+1}$$

is the power series representation of the given function.

And instead of evaluating the integral of the given function directly, we can use the expanded sum in its place. So the integral becomes

$$\int_0^{0.2}2\arctan(2x)\ dx=\int_0^{0.2}4x-\frac{2(2x)^3}{3}+\frac{2(2x)^5}{5}-\frac{2(2x)^7}{7}+\ldots\ dx$$

$$\int_0^{0.2}2\arctan(2x)\ dx=\int_0^{0.2}4x-\frac{16x^3}{3}+\frac{64x^5}{5}-\frac{256x^7}{7}+\ldots\ dx$$

$$\int_0^{0.2}2\arctan(2x)\ dx=\frac{4x^2}{2}-\frac{16x^4}{(4)3}+\frac{64x^6}{(6)5}-\frac{256x^8}{(8)7}+\ldots\Bigg|_0^{0.2}$$

$$\int_0^{0.2}2\arctan(2x)\ dx=2x^2-\frac{4x^4}{3}+\frac{32x^6}{15}-\frac{32x^8}{7}+\ldots\Bigg|_0^{0.2}$$

With the right side simplified, we'll evaluate over the interval.

$$\int_0^{0.2}2\arctan(2x)\ dx=\left[2(0.2)^2-\frac{4(0.2)^4}{3}+\frac{32(0.2)^6}{15}-\frac{32(0.2)^8}{7}+\ldots\right]$$

$$-\left[2(0)^2-\frac{4(0)^4}{3}+\frac{32(0)^6}{15}-\frac{32(0)^8}{7}+\ldots\right]$$

Then we'll pair like-terms together.

$$\int_0^{0.2} 2\arctan(2x)\ dx=\left[2(0.2)^2-2(0)^2\right]+\left[-\frac{4(0.2)^4}{3}+\frac{4(0)^4}{3}\right]+\left[\frac{32(0.2)^6}{15}-\frac{32(0)^6}{15}\right]$$

$$+\left[-\frac{32(0.2)^8}{7}+\frac{32(0)^8}{7}\right]+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=2(0.2)^2-\frac{4(0.2)^4}{3}+\frac{32(0.2)^6}{15}-\frac{32(0.2)^8}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=2(0.04)-\frac{4(0.0016)}{3}+\frac{32(0.000064)}{15}-\frac{32(0.00000256)}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=0.08-\frac{0.0064}{3}+\frac{0.002048}{15}-\frac{0.00008192}{7}+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx=0.08000000-0.00213333+0.00013653-0.00001170+\ldots$$

$$\int_0^{0.2} 2\arctan(2x)\ dx\approx 0.07799150$$

$$\int_0^{0.2} 2\arctan(2x)\ dx\approx 0.07799$$

This is the estimate of the definite integral using a power series.