Calculus 2 Workbook Solutions

Riemann sums

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SUMMATION NOTATION, FINDING THE SUM

1. Calculate the exact sum.

$$\sum_{n=1}^{6} \frac{2n^2}{3^n}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 1 \qquad \frac{2n^2}{3^n} = \frac{2(1)^2}{3^1} = \frac{2}{3}$$

$$n = 2 \qquad \frac{2n^2}{3^n} = \frac{2(2)^2}{3^2} = \frac{8}{9}$$

$$n = 3 \qquad \frac{2n^2}{3^n} = \frac{2(3)^2}{3^3} = \frac{2}{3}$$

$$n = 4 \qquad \frac{2n^2}{3^n} = \frac{2(4)^2}{3^4} = \frac{32}{81}$$

$$n = 5 \qquad \frac{2n^2}{3^n} = \frac{2(5)^2}{3^5} = \frac{50}{243}$$

$$n = 6 \qquad \frac{2n^2}{3^n} = \frac{2(6)^2}{3^6} = \frac{8}{81}$$

So the sum is

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{2}{3} + \frac{8}{9} + \frac{2}{3} + \frac{32}{81} + \frac{50}{243} + \frac{8}{81}$$

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{162}{243} + \frac{216}{243} + \frac{162}{243} + \frac{96}{243} + \frac{50}{243} + \frac{24}{243}$$

$$\sum_{n=1}^{6} \frac{2n^2}{3^n} = \frac{710}{243}$$

■ 2. Calculate the exact sum.

$$\sum_{n=1}^{5} \frac{2n}{3n+1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 1 \qquad \frac{2(1)}{3(1)+1} = \frac{2}{4} = \frac{1}{2}$$

$$n = 2 \qquad \frac{2(2)}{3(2)+1} = \frac{4}{7}$$

$$n = 3 \qquad \frac{2(3)}{3(3)+1} = \frac{6}{10} = \frac{3}{5}$$

$$n = 4 \qquad \frac{2(4)}{3(4)+1} = \frac{8}{13}$$

$$n = 5 \qquad \frac{2(5)}{3(5)+1} = \frac{10}{16} = \frac{5}{8}$$

So the sum is

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{1}{2} + \frac{4}{7} + \frac{3}{5} + \frac{8}{13} + \frac{5}{8}$$

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{1{,}820}{3{,}640} + \frac{2{,}080}{3{,}640} + \frac{2{,}184}{3{,}640} + \frac{2{,}240}{3{,}640} + \frac{2{,}275}{3{,}640}$$

$$\sum_{n=1}^{5} \frac{2n}{3n+1} = \frac{10{,}599}{3{,}640}$$

■ 3. Calculate the exact sum.

$$\sum_{n=0}^{6} 3n^2 - 5n + 7$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 0 \qquad 3(0)^2 - 5(0) + 7 = 7$$

$$n = 1 \qquad 3(1)^2 - 5(1) + 7 = 5$$

$$n = 2 \qquad 3(2)^2 - 5(2) + 7 = 9$$

$$n = 3 \qquad 3(3)^2 - 5(3) + 7 = 19$$

$$n = 4 \qquad 3(4)^2 - 5(4) + 7 = 35$$

$$n = 5 \qquad 3(5)^2 - 5(5) + 7 = 57$$

$$n = 6 \qquad 3(6)^2 - 5(6) + 7 = 85$$

So the sum is

$$\sum_{n=0}^{5} 3n^2 - 5n + 7 = 7 + 5 + 9 + 19 + 35 + 57 + 85$$

$$\sum_{n=0}^{5} 3n^2 - 5n + 7 = 217$$

SUMMATION NOTATION, EXPANDING

■ 1. Expand the sum.

$$\sum_{n=1}^{6} \frac{5n+3}{2n-1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$$n = 1 \qquad \frac{5(1)+3}{2(1)-1} = \frac{8}{1} = 8$$

$$n = 2 \qquad \frac{5(2)+3}{2(2)-1} = \frac{13}{3}$$

$$n = 3 \qquad \frac{5(3)+3}{2(3)-1} = \frac{18}{5}$$

$$n = 4 \qquad \frac{5(4)+3}{2(4)-1} = \frac{23}{7}$$

$$n = 5 \qquad \frac{5(5)+3}{2(5)-1} = \frac{28}{9}$$

$$n = 6 \qquad \frac{5(6)+3}{2(6)-1} = \frac{33}{11} = 3$$

So the sum is

$$\sum_{n=1}^{6} \frac{5n+3}{2n-1} = 8 + \frac{13}{3} + \frac{18}{5} + \frac{23}{7} + \frac{28}{9} + 3$$

2. Expand the sum.

$$\sum_{n=0}^{7} 2x^3 - 5x^2 + 9x + 3$$

Solution:

To find the sum, write each term with the value of n plugged in.

$n = 0 \qquad 2(0)^3 - 5(0)^2 + 9(0) + 3 = 3$

$n = 1 \qquad 2(1)^3 - 5(1)^2 + 9(1) + 3 = 9$

$n = 2 \qquad 2(2)^3 - 5(2)^2 + 9(2) + 3 = 17$

$n = 3 \qquad 2(3)^3 - 5(3)^2 + 9(3) + 3 = 39$

$n = 4 \qquad 2(4)^3 - 5(4)^2 + 9(4) + 3 = 87$

$n = 5 \qquad 2(5)^3 - 5(5)^2 + 9(5) + 3 = 173$

$n = 6 \qquad 2(6)^3 - 5(6)^2 + 9(6) + 3 = 309$

$n = 7 \qquad 2(7)^3 - 5(7)^2 + 9(7) + 3 = 507$

So the sum is

$$\sum_{n=0}^{7} 2x^3 - 5x^2 + 9x + 3 = 3 + 9 + 17 + 39 + 87 + 173 + 309 + 507$$

3. Expand the sum.

$$\sum_{n=0}^{8} \frac{2n-8}{n+1}$$

Solution:

To find the sum, write each term with the value of n plugged in.

$n = 0 \qquad \dfrac{2(0)-8}{0+1} = -\dfrac{8}{1} = -8$

$n = 1 \qquad \dfrac{2(1)-8}{1+1} = -\dfrac{6}{2} = -3$

$n = 2 \qquad \dfrac{2(2)-8}{2+1} = -\dfrac{4}{3}$

$n = 3 \qquad \dfrac{2(3)-8}{3+1} = -\dfrac{2}{4} = -\dfrac{1}{2}$

$n = 4 \qquad \dfrac{2(4)-8}{4+1} = \dfrac{0}{5} = 0$

$n = 5 \qquad \dfrac{2(5)-8}{5+1} = \dfrac{2}{6} = \dfrac{1}{3}$

$$n = 6 \qquad \frac{2(6)-8}{6+1} = \frac{4}{7}$$

$$n = 7 \qquad \frac{2(7)-8}{7+1} = \frac{6}{8} = \frac{3}{4}$$

$$n = 8 \qquad \frac{2(8)-8}{8+1} = \frac{8}{9}$$

So the sum is

$$\sum_{n=0}^{8} \frac{2n-8}{n+1} = -8 - 3 - \frac{4}{3} - \frac{1}{2} + 0 + \frac{1}{3} + \frac{4}{7} + \frac{3}{4} + \frac{8}{9}$$

SUMMATION NOTATION, COLLAPSING

1. Use summation notation to rewrite the sum.

$$\frac{(x+3)^2}{3-1}+\frac{(x+3)^4}{9-2}+\frac{(x+3)^6}{27-3}+\frac{(x+3)^8}{81-4}+\frac{(x+3)^{10}}{243-5}+\frac{(x+3)^{12}}{729-6}$$

Solution:

Find patterns in the sum.

$$\frac{(x+3)^2}{3-1}+\frac{(x+3)^4}{9-2}+\frac{(x+3)^6}{27-3}+\frac{(x+3)^8}{81-4}+\frac{(x+3)^{10}}{243-5}+\frac{(x+3)^{12}}{729-6}$$

$$\frac{(x+3)^{2(1)}}{3-1}+\frac{(x+3)^{2(2)}}{9-2}+\frac{(x+3)^{2(3)}}{27-3}+\frac{(x+3)^{2(4)}}{81-4}+\frac{(x+3)^{2(5)}}{243-5}+\frac{(x+3)^{2(6)}}{729-6}$$

$$\frac{(x+3)^{2(1)}}{3^1-1}+\frac{(x+3)^{2(2)}}{3^2-2}+\frac{(x+3)^{2(3)}}{3^3-3}+\frac{(x+3)^{2(4)}}{3^4-4}+\frac{(x+3)^{2(5)}}{3^5-5}+\frac{(x+3)^{2(6)}}{3^6-6}$$

$$\frac{(x+3)^{2(1)}}{3^1-1}+\frac{(x+3)^{2(2)}}{3^2-2}+\frac{(x+3)^{2(3)}}{3^3-3}+\frac{(x+3)^{2(4)}}{3^4-4}+\frac{(x+3)^{2(5)}}{3^5-5}+\frac{(x+3)^{2(6)}}{3^6-6}$$

The pattern that emerges is the sum from $n=1$ to $n=6$.

$$\sum_{n=1}^{6}\frac{(x+3)^{2n}}{3^n-n}$$

2. Use summation notation to rewrite the sum.

$$\frac{3x+1}{7x}+\frac{6x+2}{14x^2}+\frac{9x+3}{21x^3}+\frac{12x+4}{28x^4}+\frac{15x+5}{35x^5}+\frac{18x+6}{42x^6}$$

Solution:

Find patterns in the sum.

$$\frac{3x+1}{7x}+\frac{6x+2}{14x^2}+\frac{9x+3}{21x^3}+\frac{12x+4}{28x^4}+\frac{15x+5}{35x^5}+\frac{18x+6}{42x^6}$$

$$\frac{3(1)x+1}{7x}+\frac{3(2)x+2}{14x^2}+\frac{3(3)x+3}{21x^3}+\frac{3(4)x+4}{28x^4}+\frac{3(5)x+5}{35x^5}+\frac{3(6)x+6}{42x^6}$$

$$\frac{3(1)x+1}{7(1)x}+\frac{3(2)x+2}{7(2)x^2}+\frac{3(3)x+3}{7(3)x^3}+\frac{3(4)x+4}{7(4)x^4}+\frac{3(5)x+5}{7(5)x^5}+\frac{3(6)x+6}{7(6)x^6}$$

$$\frac{3(1)x+1}{7(1)x^1}+\frac{3(2)x+2}{7(2)x^2}+\frac{3(3)x+3}{7(3)x^3}+\frac{3(4)x+4}{7(4)x^4}+\frac{3(5)x+5}{7(5)x^5}+\frac{3(6)x+6}{7(6)x^6}$$

The pattern that emerges is the sum from $n=1$ to $n=6$.

$$\sum_{n=1}^{6}\frac{3nx+n}{7nx^n}$$

3. Use summation notation to rewrite the sum.

$$\frac{x^2-3x+1}{4x}+\frac{x^3-6x+2}{8x}+\frac{x^4-9x+3}{12x}+\frac{x^5-12x+4}{16x}$$

$$+\frac{x^6-15x+5}{20x}+\frac{x^7-18x+6}{24x}+\frac{x^8-21x+7}{28x}$$

Solution:

Find patterns in the sum.

$$\frac{x^2-3x+1}{4x}+\frac{x^3-6x+2}{8x}+\frac{x^4-9x+3}{12x}+\frac{x^5-12x+4}{16x}$$

$$+\frac{x^6-15x+5}{20x}+\frac{x^7-18x+6}{24x}+\frac{x^8-21x+7}{28x}$$

$$\frac{x^2-3(1)x+1}{4x}+\frac{x^3-3(2)x+2}{8x}+\frac{x^4-3(3)x+3}{12x}+\frac{x^5-3(4)x+4}{16x}$$

$$+\frac{x^6-3(5)x+5}{20x}+\frac{x^7-3(6)x+6}{24x}+\frac{x^8-3(7)x+7}{28x}$$

$$\frac{x^{1+1}-3(1)x+1}{4x}+\frac{x^{2+1}-3(2)x+2}{8x}+\frac{x^{3+1}-3(3)x+3}{12x}+\frac{x^{4+1}-3(4)x+4}{16x}$$

$$+\frac{x^{5+1}-3(5)x+5}{20x}+\frac{x^{6+1}-3(6)x+6}{24x}+\frac{x^{7+1}-3(7)x+7}{28x}$$

$$\frac{x^{1+1}-3(1)x+1}{4(1)x}+\frac{x^{2+1}-3(2)x+2}{4(2)x}+\frac{x^{3+1}-3(3)x+3}{4(3)x}+\frac{x^{4+1}-3(4)x+4}{4(4)x}$$

$$+\frac{x^{5+1}-3(5)x+5}{4(5)x}+\frac{x^{6+1}-3(6)x+6}{4(6)x}+\frac{x^{7+1}-3(7)x+7}{4(7)x}$$

The pattern that emerges is the sum from $n=1$ to $n=7$.

$$\sum_{n=1}^{7} \frac{x^{n+1} - 3nx + n}{4nx}$$

RIEMANN SUMS, LEFT ENDPOINTS

1. Use a left endpoint Riemann Sum with $n=5$ to find the area under $f(x)$ on the interval $[0,10]$.

x	0	1	2	3	4	5	6	7	8	9	10
f(x)	3	2	3	6	11	18	27	38	51	66	83

Solution:

Five equal subdivisions of $[0,10]$ gives five subintervals.

$[0,2]$, $[2,4]$, $[4,6]$, $[6,8]$, and $[8,10]$

Choose the value of the function for the left endpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(3)+2(3)+2(11)+2(27)+2(51)$$

$$190$$

2. Use a left endpoint Riemann Sum with $n=5$ to find the area under $g(x)$ on the interval $[0,20]$. Round the final answer to 2 decimal places.

$$g(x)=2\sqrt{x}+5$$

Solution:

Five equal subdivisions of $[0,20]$ gives five subintervals.

$$[0,4], [4,8], [8,12], [12,16], \text{ and } [16,20]$$

Find the value of the function for the left endpoint of each subinterval.

$$g(0) = 2\sqrt{0} + 5 = 5$$

$$g(4) = 2\sqrt{4} + 5 = 9$$

$$g(8) = 2\sqrt{8} + 5 \approx 10.6569$$

$$g(12) = 2\sqrt{12} + 5 \approx 11.9282$$

$$g(16) = 2\sqrt{16} + 5 = 13$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(5) + 4(9) + 4(10.6569) + 4(11.9282) + 4(13)$$

$$198.3404$$

The sum rounds to 198.34.

■ 3. Use a left endpoint Riemann Sum with $n = 3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 3$$

Solution:

Three equal subdivisions of $[-2,4]$ gives three subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

Find the value of the function for the left endpoint of each subinterval.

$$h(-2) = \frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 - (-2) + 3 = \frac{1}{3}$$

$$h(0) = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 - 0 + 3 = 3$$

$$h(2) = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2 + 3 = \frac{5}{3}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2\left(\frac{1}{3}\right) + 2(3) + 2\left(\frac{5}{3}\right)$$

$$\frac{2}{3} + 6 + \frac{10}{3}$$

$$\frac{2}{3} + \frac{18}{3} + \frac{10}{3}$$

$$\frac{30}{3}$$

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4. Use a left endpoint Riemann Sum with $n = 4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x) = \frac{x^2 + 4x + 4}{x^2 + 4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

Find the value of the function for the left endpoint of each subinterval.

$$k(0) = \frac{0^2 + 4(0) + 4}{0^2 + 4} = 1$$

$$k(7) = \frac{7^2 + 4(7) + 4}{7^2 + 4} = \frac{81}{53} \approx 1.5283$$

$$k(14) = \frac{14^2 + 4(14) + 4}{14^2 + 4} = \frac{32}{25} \approx 1.28$$

$$k(21) = \frac{21^2 + 4(21) + 4}{21^2 + 4} = \frac{529}{445} \approx 1.1888$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1) + 7(1.5283) + 7(1.28) + 7(1.1888)$$

$$34.9795$$

The sum rounds to 34.98.

■ 5. Use a left endpoint Riemann Sum with $n = 4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x) = 2\ln(x + 3) + 6$$

Solution:

Four equal subdivisions of $[0,2]$ gives four subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

Find the value of the function for the left endpoint of each subinterval.

$$f(0) = 2\ln(0 + 3) + 6 \approx 8.1972$$

$$f(0.5) = 2\ln(0.5 + 3) + 6 \approx 8.5055$$

$$f(1) = 2\ln(1 + 3) + 6 \approx 8.7726$$

$$f(1.5) = 2\ln(1.5 + 3) + 6 \approx 9.0082$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.1972) + 0.5(8.5055) + 1.5(8.7726) + 0.5(9.0082)$$

$$17.24175$$

The sum rounds to 17.24.

■ 6. Use a left endpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$g(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Five equal subdivisions of $[0,1]$ gives five subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

Find the value of the function for the left endpoint of each subinterval.

$$g(0) = 0^4 + 2(0)^3 - 3(0)^2 + 4(0) + 5 = 5$$

$$g(0.2) = 0.2^4 + 2(0.2)^3 - 3(0.2)^2 + 4(0.2) + 5 \approx 5.6976$$

$$g(0.4) = 0.4^4 + 2(0.4)^3 - 3(0.4)^2 + 4(0.4) + 5 \approx 6.2736$$

$$g(0.6) = 0.6^4 + 2(0.6)^3 - 3(0.6)^2 + 4(0.6) + 5 \approx 6.8816$$

$$g(0.8) = 0.8^4 + 2(0.8)^3 - 3(0.8)^2 + 4(0.8) + 5 \approx 7.7136$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5) + 0.2(5.6976) + 0.2(6.2736) + 0.2(6.8816) + 0.2(7.7136)$$

$$6.31328$$

The sum rounds to 6.31.

RIEMANN SUMS, RIGHT ENDPOINTS

■ 1. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $g(x)$ on the interval $[1,11]$.

x	1	2	3	4	5	6	7	8	9	10	11
g(x)	5	4	5	8	13	20	29	40	53	68	85

Solution:

Five equal subdivisions of $[1,11]$ gives five subintervals.

$[1,3]$, $[3,5]$, $[5,7]$, $[7,9]$, and $[9,11]$

Choose the value of the function for the right endpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(5) + 2(13) + 2(29) + 2(53) + 2(85)$$

$$370$$

■ 2. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $f(x)$ on the interval $[5,25]$. Round the final answer to 2 decimal places.

$$f(x) = \sqrt{2x} - 1$$

Solution:

Five equal subdivisions of $[5,25]$ gives five subintervals.

$[5,9]$, $[9,13]$, $[13,17]$, $[17,21]$, and $[21,25]$

Find the value of the function for the right endpoint of each subinterval.

$$f(9) = \sqrt{2(9)} - 1 \approx 3.2426$$

$$f(13) = \sqrt{2(13)} - 1 \approx 4.0990$$

$$f(17) = \sqrt{2(17)} - 1 \approx 4.8310$$

$$f(21) = \sqrt{2(21)} - 1 \approx 5.4807$$

$$f(25) = \sqrt{2(25)} - 1 \approx 6.0711$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(3.2426) + 4(4.0990) + 4(4.8310) + 4(5.4807) + 4(6.0711)$$

$$94.8976$$

The sum rounds to 94.90.

■ 3. Use a right endpoint Riemann Sum with $n = 3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 3$$

Solution:

Five equal subdivisions of $[-2,4]$ gives five subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

Find the value of the function for the right endpoint of each subinterval.

$$h(0) = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 - 0 + 3 = 3$$

$$h(2) = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - 2 + 3 = \frac{5}{3}$$

$$h(4) = \frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - 4 + 3 = \frac{37}{3}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(3) + 2\left(\frac{5}{3}\right) + 2\left(\frac{37}{3}\right)$$

$$6 + \frac{10}{3} + \frac{74}{3}$$

$$\frac{18}{3} + \frac{10}{3} + \frac{74}{3}$$

$$\frac{102}{3}$$

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■ 4. Use a right endpoint Riemann Sum with $n = 4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x) = \frac{x^2 + 4x + 4}{x^2 + 4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

Find the value of the function for the right endpoint of each subinterval.

$$k(7) = \frac{7^2 + 4(7) + 4}{7^2 + 4} = \frac{81}{53} \approx 1.5283$$

$$k(14) = \frac{14^2 + 4(14) + 4}{14^2 + 4} = \frac{32}{25} \approx 1.28$$

$$k(21) = \frac{21^2 + 4(21) + 4}{21^2 + 4} = \frac{529}{445} \approx 1.1888$$

$$k(28) = \frac{28^2 + 4(28) + 4}{28^2 + 4} = \frac{225}{197} \approx 1.1421$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1.5283) + 7(1.28) + 7(1.1888) + 7(1.1421)$$

$$35.9744$$

The sum rounds to 35.97.

■ 5. Use a right endpoint Riemann Sum with $n = 4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x) = 2\ln(x + 3) + 6$$

Solution:

Four equal subdivisions of $[0,2]$ gives four subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

Find the value of the function for the right endpoint of each subinterval.

$$f(0.5) = 2\ln(0.5 + 3) + 6 \approx 8.5055$$

$$f(1) = 2\ln(1 + 3) + 6 \approx 8.7726$$

$$f(1.5) = 2\ln(1.5 + 3) + 6 \approx 9.0082$$

$$f(2) = 2\ln(2 + 3) + 6 \approx 9.2189$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.5055) + 1.5(8.7726) + 0.5(9.0082) + 0.5(9.2189)$$

$$17.7526$$

The sum rounds to 17.75.

■ 6. Use a right endpoint Riemann Sum with $n = 5$ to find the area under $h(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$h(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Four equal subdivisions of $[0,1]$ gives four subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

Find the value of the function for the right endpoint of each subinterval.

$$g(0.2) = 0.2^4 + 2(0.2)^3 - 3(0.2)^2 + 4(0.2) + 5 \approx 5.6976$$

$$g(0.4) = 0.4^4 + 2(0.4)^3 - 3(0.4)^2 + 4(0.4) + 5 \approx 6.2736$$

$$g(0.6) = 0.6^4 + 2(0.6)^3 - 3(0.6)^2 + 4(0.6) + 5 \approx 6.8816$$

$$g(0.8) = 0.8^4 + 2(0.8)^3 - 3(0.8)^2 + 4(0.8) + 5 \approx 7.7136$$

$$g(1) = 1^4 + 2(1)^3 - 3(1)^2 + 4(1) + 5 = 9$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5.6976) + 0.2(6.2736) + 0.2(6.8816) + 0.2(7.7136) + 0.2(9)$$

$$7.11328$$

The sum rounds to 7.11.

RIEMANN SUMS, MIDPOINTS

■ 1. Use a midpoint Riemann Sum with $n = 5$ to find the area under $h(x)$ on the interval $[6,16]$.

x	6	7	8	9	10	11	12	13	14	15	16
h(x)	84	67	52	39	26	17	10	7	4	3	4

Solution:

Five equal subdivisions of $[6,16]$ gives five subintervals.

$[6,8]$, $[8,10]$, $[10,12]$, $[12,14]$, and $[14,16]$

The midpoints of the subintervals are

$x = 7, 9, 11, 13,$ and 15

Choose the value of the function for the midpoint of each subinterval. The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2(67) + 2(39) + 2(17) + 2(7) + 2(3)$$

$$266$$

2. Use a midpoint Riemann Sum with $n = 5$ to find the area under $k(x)$ on the interval $[2,22]$. Round the final answer to 2 decimal places.

$$k(x) = 3\sqrt{7x} - 8$$

Solution:

Five equal subdivisions of $[2,22]$ gives five subintervals.

$[2,6]$, $[6,10]$, $[10,14]$, $[14,18]$, and $[18,22]$

The midpoints of the subintervals are

$x = 4,\ 8,\ 12,\ 16,$ and 20

Find the value of the function for the midpoint of each subinterval.

$$k(4) = 3\sqrt{7(4)} - 8 \approx 7.8745$$

$$k(8) = 3\sqrt{7(8)} - 8 \approx 14.4499$$

$$k(12) = 3\sqrt{7(12)} - 8 \approx 19.4955$$

$$k(16) = 3\sqrt{7(16)} - 8 \approx 23.7490$$

$$k(20) = 3\sqrt{7(20)} - 8 \approx 27.4965$$

The width of each subinterval is 4 units, so the estimation of the area under the curve is

$$4(7.8745) + 4(14.4499) + 4(19.4955) + 4(23.7490) + 4(27.4965)$$

$$372.2616$$

The sum rounds to 372.26.

■ 3. Use a midpoint Riemann Sum with $n=3$ to find the area under $h(x)$ on the interval $[-2,4]$.

$$h(x)=\frac{1}{3}x^3-\frac{1}{2}x^2-x+3$$

Solution:

Three equal subdivisions of $[-2,4]$ gives three subintervals.

$[-2,0]$, $[0,2]$, and $[2,4]$

The midpoints of the subintervals are

$x=-1$, 1, and 3

Find the value of the function for the midpoint of each subinterval.

$$h(-1)=\frac{1}{3}(-1)^3-\frac{1}{2}(-1)^2-(-1)+3=\frac{19}{6}$$

$$h(1)=\frac{1}{3}(1)^3-\frac{1}{2}(1)^2-1+3=\frac{11}{6}$$

$$h(3)=\frac{1}{3}(3)^3-\frac{1}{2}(3)^2-3+3=\frac{9}{2}$$

The width of each subinterval is 2 units, so the estimation of the area under the curve is

$$2\left(\frac{19}{6}\right)+2\left(\frac{11}{6}\right)+2\left(\frac{9}{2}\right)$$

$$\frac{19}{3}+\frac{11}{3}+9$$

$$\frac{19}{3}+\frac{11}{3}+\frac{27}{3}$$

$$\frac{57}{3}$$

$$19$$

■ 4. Use a midpoint Riemann Sum with $n=4$ to find the area under $k(x)$ on the interval $[0,28]$. Round the final answer to 2 decimal places.

$$k(x)=\frac{x^2+4x+4}{x^2+4}$$

Solution:

Four equal subdivisions of $[0,28]$ gives four subintervals.

$[0,7]$, $[7,14]$, $[14,21]$, and $[21,28]$

The midpoints of the subintervals are

$$x = 3.5, 10.5, 17.5, \text{ and } 24.5$$

Find the value of the function for the midpoint endpoint of each subinterval.

$$k(3.5) = \frac{3.5^2 + 4(3.5) + 4}{3.5^2 + 4} \approx 1.8615$$

$$k(10.5) = \frac{10.5^2 + 4(10.5) + 4}{10.5^2 + 4} \approx 1.3676$$

$$k(17.5) = \frac{17.5^2 + 4(17.5) + 4}{17.5^2 + 4} \approx 1.2256$$

$$k(24.5) = \frac{24.5^2 + 4(24.5) + 4}{24.5^2 + 4} \approx 1.1622$$

The width of each subinterval is 7 units, so the estimation of the area under the curve is

$$7(1.8615) + 7(1.3676) + 7(1.2256) + 7(1.1622)$$

$$39.3183$$

The sum rounds to 39.32.

5. Use a midpoint Riemann Sum with $n = 4$ to find the area under $f(x)$ on the interval $[0,2]$. Round the final answer to 2 decimal places.

$$f(x) = 2\ln(x + 3) + 6$$

Solution:

Five equal subdivisions of $[0,2]$ gives five subintervals.

$[0,0.5]$, $[0.5,1]$, $[1,1.5]$, and $[1.5,2]$

The midpoints of the subintervals are

$x = 0.25, 0.75, 1.25$, and 1.75

Find the value of the function for the midpoint endpoint of each subinterval.

$$f(0.25) = 2\ln(0.25+3)+6 \approx 8.3573$$

$$f(0.75) = 2\ln(0.75+3)+6 \approx 8.6435$$

$$f(1.25) = 2\ln(1.25+3)+6 \approx 8.8938$$

$$f(1.75) = 2\ln(1.75+3)+6 = 9.1163$$

The width of each subinterval is 0.5 units, so the estimation of the area under the curve is

$$0.5(8.3573)+1.5(8.6435)+0.5(8.8938)+0.5(9.1163)$$

$$17.50545$$

The sum rounds to 17.51.

■ 6. Use a midpoint Riemann Sum with $n=5$ to find the area under $g(x)$ on the interval $[0,1]$. Round the final answer to 2 decimal places.

$$g(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$$

Solution:

Five equal subdivisions of $[0,1]$ gives five subintervals.

$[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, and $[0.8,1]$

The midpoints of the subintervals are

$x = 0.1, 0.3, 0.5, 0.7,$ and 0.9

Find the value of the function for the midpoint endpoint of each subinterval.

$$g(0.1) = 0.1^4 + 2(0.1)^3 - 3(0.1)^2 + 4(0.1) + 5 \approx 5.3721$$

$$g(0.3) = 0.3^4 + 2(0.3)^3 - 3(0.3)^2 + 4(0.3) + 5 \approx 5.9921$$

$$g(0.5) = 0.5^4 + 2(0.5)^3 - 3(0.5)^2 + 4(0.5) + 5 \approx 6.5625$$

$$g(0.7) = 0.7^4 + 2(0.7)^3 - 3(0.7)^2 + 4(0.7) + 5 \approx 7.2561$$

$$g(0.9) = 0.9^4 + 2(0.9)^3 - 3(0.9)^2 + 4(0.9) + 5 \approx 8.2841$$

The width of each subinterval is 0.2 units, so the estimation of the area under the curve is

$$0.2(5.3721) + 0.2(5.9921) + 0.2(6.5625) + 0.2(7.2561) + 0.2(8.2841)$$

$$6.69338$$

The sum rounds to 6.69.

MOVING FROM SUMMATION NOTATION TO THE INTEGRAL

■ 1. Convert the Riemann sum to a definite integral over the interval $[1,8]$.

$$\sum_{i=1}^{n}\left(6x_i^5 - 4x_i^{\frac{4}{3}} + 2x_i^{-3}\right)\Delta x$$

Solution:

Convert the sum to a limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(6x_i^5 - 4x_i^{\frac{4}{3}} + 2x_i^{-3}\right)\Delta x = \int f(x_i)\ dx$$

Then the integral over the interval $[1,8]$ is

$$\int_1^8 6x^5 - 4x^{\frac{4}{3}} + 2x^{-3}\ dx$$

■ 2. Convert the Riemann sum to a definite integral over the interval $[-2,4]$.

$$\sum_{i=1}^{n}\left((5x_i + 3)(2x_i^2 + x_i)^5\right)\Delta x$$

Solution:

Convert the sum to a limit.

$$\sum_{i=1}^{n} \left((5x_i + 3)(2x_i^2 + x_i)^5 \right) \Delta x = \int f(x_i)\ dx$$

Then the integral over the interval $[-2,4]$ is

$$\int_{-2}^{4} (5x + 3)(2x^2 + x)^5\ dx$$

■ 3. Convert the Riemann sum to a definite integral over the interval $[5,11]$.

$$\sum_{i=1}^{n} \left((4 - x_i)\sqrt{x_i - 5} \right) \Delta x$$

Solution:

Convert the sum to a limit.

$$\sum_{i=1}^{n} \left((4 - x_i)\sqrt{x_i - 5} \right) \Delta x = \int f(x_i)\ dx$$

Then the integral over the interval $[5,11]$ is

$$\int_{5}^{11} (4 - x)\sqrt{x - 5}\ dx$$

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