A quick take on metric spaces and topology

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1 Introduction

In Calculus, whose object of study are functions, we say that f is continuous at $a \in \mathbb{R}$ if we can make $f(x)$ and $f(a)$ as close as we want when taking x sufficiently close to a. More precisely, we say that f is continuous at a if for any $\epsilon > 0$ there is $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Also, $f : \mathbb{R} \supseteq U \to \mathbb{R}$ is continuous if it is continuous at all points in U.

It is natural for us to try and generalize this definition of continuity for spaces other than $\mathbb{R}$. The first case that comes to mind is $\mathbb{R}^n$. We do this generalization in Multivariable Calculus in a straightforward manner: $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ is continuous if for any $\epsilon > 0$ there is $\delta > 0$ such that if $\| x - a \|_n < \delta$ then $\| f(x) - f(a) \|_m < \epsilon$, where $\| \cdot \|_n$ and $\| \cdot \|_m$ are the norms in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. f is continuous if it is continuous at all points in U.

For complex functions, another natural generalization reveals itself: $f : \mathbb{C} \to \mathbb{C}$ is continuous at a if for any $\epsilon > 0$ there is $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x)-f(a)| < \epsilon$, where $| \cdot |$ is the absolute value for complex numbers. f is continuous if it is continuous at all points in U.

Hopefully you have noticed a pattern: all these spaces in which we defined continuity have a norm associated (absolute values are norms, of course: just check the axioms for a norm), and we explicitly use that fact in our definitions of continuity. So maybe we should expand our definition of continuity for normed spaces.

While that is definitely a start, we can generalize it even further. It turns out that every normed space is a *metric space*, which is a set with a a *distance* (we will define these notions precisely in the next section), and this indicates that we can generalize the definition of continuity to metric spaces. Intuitively, if we have a notion of distance between elements of a set, it is reasonable to define something like continuity, which has everything to do with how close things are.

Hence, our first mission is to define metric spaces and related concepts (including continuity), and to study a few important results and examples.

As we shall see, we can generalize most definitions in metric spaces to *topological spaces*, a topological space being a set X with a *topology* (*i.e.* a set of subsets of X (that we call *open sets of* X) with certain properties). Our second mission is to define topological spaces and related concepts (including continuity), and to study a few important results and examples.

2 Metric spaces - basic concepts

We would like to have a notion of *distance* in a set. This can be accomplished in several ways: as we discussed above, normed spaces like $\mathbb{R}$ and $\mathbb{C}$ have a notion of distance closely related with its norm. In the case of $\mathbb{R}^n$, we could say that the distance between two elements x, y is $\| x - y \|$. We have learned this long ago. What if we define a distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d(x, y) = \| x - y \|$? We get a function with some properties that are clearly what we are looking for when defining something that must resemble what we mean by a distance in everyday life: it is greater or equal to zero, being zero when $x = y$, and so on. We shall define a *metric space* as a set equipped with a function with such properties. This is useful because it has great generality - there are many weird metric spaces which seem nothing like $\mathbb{R}^n$ at first - and yet we *can* use our intuition when dealing with such strange spaces, exactly because of the properties of the distance function, which are so intuitive.

Definition 1. A *metric space* is a non-empty set X together with a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ one has

(D1) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.

(D2) $d(x,y) = d(y,x)$

(D3) $d(x,z) \leq d(x,y) + d(y,z)$

Example 2. $(\mathbb{R}^n, d_2)$ is a metric space, where d_2 is the usual distance, given by

$$d_2\left((x_1, ..., x_n), (y_1, ..., y_n)\right) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \tag{1}$$

Example 3. $(\mathbb{R}^n, d_\infty)$ is a metric space, with

$$d_\infty(x,y) = \max_{i \in \{1,...,n\}} \{|x_i - y_i|\} \tag{2}$$

Example 4. $(\mathbb{R}^n, d_1)$ is a metric space, with

$$d_1\left((x_1, ..., x_n), (y_1, ..., y_n)\right) = \sum_{i=1}^{n} |x_i - y_i| \tag{3}$$

Example 5. Let X be a non-empty set and define d by

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \tag{4}$$

Let us check explicitly that (X,d) is a metric space: (D1) and (D2) are clear. Now, let $x, y, z \in X$. If $x = z$, then (D3) holds trivially. Suppose $x \neq z$. If $y = x$ or $y = z$, then $d(x,y) + d(y,z) = 1$. If $y \neq x$ and $y \neq z$, then $d(x,y) + d(y,z) = 2$. Either way, (D3) holds. $\square$

Hopefully this example convinced you of the generality of the concept of metric space. In order to convince you of its usefulness, the next one may suffice: functional analysis use metric space theory and topology all the time.

Example 6. Let $C[a,b]$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$ and d be such that

$$d(f,g) = \int_a^b |f(x) - g(x)| dx \tag{5}$$

Definition 7. Let X, Y be metric spaces and $f : X \to Y$.

f is said to be *continuous at* $a \in X$ if for all $\epsilon > 0$, there is $\delta > 0$ such that $\forall x \in X\ (d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon)$.

f is *continuous* if it is continuous at all points of X.

Definition 8. Let X be a metric space. A subset S of X is *bounded* if there are $D \in \mathbb{R}$ and $a \in X$ such that $\forall x \in S$, $d(x,a) \leq D$.

We will now define open ball, open and closed set, closure and boundary of a set. Notice how these are defined in a way that coincides with our intuition.

Definition 9. Let X be a metric space with metric d, $x \in X$ and $\epsilon \in \mathbb{R}^+$. The *open ball in X of radius ϵ centered at x is*

$$B^d(x,\epsilon) \coloneqq \{y \in X : d(y,x) < \epsilon\} \tag{6}$$

Definition 10. Let X be a metric space. A subset U of X is said to be *open* if $\forall x \in X, \exists \epsilon > 0 : B(x,\epsilon) \subseteq U$.

Example 11. Let X be a set with the metric of example 5 (called the *discrete metric*). Then, any subset U of X is an open set: let $x \in U$. If we take $\epsilon = 1$ (for example), we get $B(x,1) = \{y \in X : d(y,x) < 1\} = \{x\} \subseteq U$.

Proposition 12. *Let $f : X \to Y$ be a map between metric spaces. f is continuous iff $\forall U$ open in Y, $f^{\dashv}(U)$ is open in X.*

Proof. Omitted. □

Definition 13. Let X be a metric space. A subset F of X is said to be *closed* if $X \backslash U$ is open.

Example 14. X and $\emptyset$ are both closed and open sets.

3 Topology - basic concepts

We first define the topological concepts involved in the definition of topological manifold.

Definition 15. A *topological space* is a set X with a set τ (called the *topology of X*) of subsets of X (called the *open sets of X*) such that

1. $\emptyset, X \in \tau$
2. If $\forall \alpha \in A$, $U_\alpha \in \tau$, then $\bigcup\limits_\alpha U_\alpha \in \tau$, with A any set
3. If $U_1, ..., U_n \in \tau$, then $U_1 \cap ... \cap U_n \in \tau$

Example 16. $\mathbb{R}^n$ is a topological space.

Example 17. Any metric space is a topological space.

Definition 18. Let X, Y be topological spaces. $f : X \to Y$ is *continuous at $x \in X$* if for any open $V \subseteq Y$ with $y = f(x) \in V$, there exists an open $U \subseteq X$ with $x \in U$ such that $f(U) \subseteq V$.

f is *continuous* if it is continuous for all $x \in X$.

Theorem 19. *$f : X \to Y$ is continuous iff for all open V in Y, $f^{\dashv}(V) \in \tau_X$.*

Proof. Omitted. □

Definition 20. $F \subseteq X$ is *closed* if $X \backslash F$ is open.

The following can be taken as the definition of continuity. It is a straightforward generalization of proposition 12.

Proposition 21. *$f : X \to Y$ is continuous iff for all closed V in Y, $f^{\dashv}(V)$ is closed in X.*

Proof. Suppose f is continuous. Let V be a closed set in Y. Then, $Y \backslash V \in \tau_Y$. We have $X \backslash f^{\dashv}(V) = f^{\dashv}(Y \backslash V)$[1], which is open by continuity of f, and hence $f^{\dashv}(V)$ is closed.

Suppose $f^{\dashv}(V)$ is closed for all closed V. Now, let U be an open set in Y. We know that $Y \backslash U$ is closed, so $f^{\dashv}(Y \backslash U) = X \backslash f^{\dashv}(U)$ is also closed. Thus, $f^{\dashv}(U)$ is open. this proves that f is continuous. □

[1] This is very simple to prove:
Let $x \in X \backslash f^{\dashv}(V)$. Then, $f(x) \notin V$. But $f(x) \in Y$. Hence, $f(x) \in Y \backslash V$, so $x \in f^{\dashv}(Y \backslash V)$. This proves that $X \backslash f^{\dashv}(V) \subseteq f^{\dashv}(Y \backslash V)$.
Let $x \in f^{\dashv}(Y \backslash V)$. If $x \in f^{\dashv}(V)$, then $f(x) \in V$, which is absurd. Then$x \in X \backslash f^{\dashv}(V)$. We conclude that $X \backslash f^{\dashv}(V) = f^{\dashv}(Y \backslash V)$.

Definition 22. Let (X, τ) be a topological space. A *basis for the topology* τ is a collection/set $\mathcal{B} \subseteq \tau$ such that any $U \in \tau$ is the union of sets from $\mathcal{B}$.

Example 23. $\{B(x, \epsilon) : x \in \mathbb{R}^n,\ \epsilon > 0\}$ is a basis for the usual topology in $\mathbb{R}^n$.

Definition 24. A topological space is said to be *second countable* if it has a countable basis for its topology.

Example 25. $\{B(x, \epsilon) : x \in \mathbb{Q}^n,\ \epsilon \in \mathbb{Q}^+\}$ is a basis for the usual topology in $\mathbb{R}^n$. Notice that it is countable! $\mathbb{R}^n$ is therefore second countable.

Definition 26. A topological space is *locally Euclidean of dimension* m if every point p in M has a neighborhood which is homeomorphic to $\mathbb{R}^m$ (or, equivalently, to an open set of $\mathbb{R}^m$).

Let U be a neighborhood of p and ϕ be an homomorphism between U and an open set V of $\mathbb{R}^m$. We call (U, ϕ) a *chart*, U a *coordinate neighborhood* and ϕ a *coordinate map on U.*

We say that (U, ϕ) *is centered at* p if $\phi(p) = 0$.

Definition 27. A topological space X is an *Hausdorff space* if for any two points $x, y \in X$, there are disjoint open sets U, V of X such that $x \in U \ \wedge \ y \in V$.

Definition 28. A *topological manifold of dimension* m is a second countable, Hausdorff, locally euclidean space of dimension m topological space.

References

[1] Sutherland's book on topology and metric spaces

[2] My first class of differential manifolds

[3] Tu's book on manifolds