Cobordisms, 2TQFTs and Frobenius Algebras

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1 Some Categorical Mumbo Jumbo

1.1 Foundational remarks

We leave *collection* undefined.
A *class* is a collection satisfying certain axioms (which are a modification of the ZFC axioms). In particular, the axioms of extensionality and of pairing from ZFC theory are modified in the obvious way to apply to classes.
A *set* is a class that is an element of some other class. A *proper class* is a class that does not belong to any other class.
One of the cool things about classes is that we do not get a paradox by considering the class of all sets. This class must clearly be proper though, otherwise we would get Russel's paradox again.
Statements defining a class by some common property of its elements are not allowed to have quantifiers running over classes, only over sets (*vide* the *class existence theorem*). In particular, we cannot take the class of all classes, and thus we do not get Russel's paradox for classes.
Thus, we see that we can take the category of all sets, whose objects form a proper class $Obj(\mathbf{Set})$. However, we can not take the class of all categories, since some of these categories will be proper classes. The notion of a *conglomerate* comes to the rescue: it is a generalization of class, just like class is a generalization of set. We can therefore talk of the category of all categories (which will be a conglomerate).
The above is of course a very incomplete version of the story. To know a bit more (and definitely enough for understanding most applications of category theory), see for example [1][pp.13-17].

1.2 Products

We first recall what is the product of two categories [2][p.19], which will play a role in the definition of monoidal category:

Definition 1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is the category whose objects are the ordered pairs (C, D) with $C \in \mathcal{C}, D \in \mathcal{D}$ and whose arrows are pairs of arrows $f \times g$ (or (f, g)) where $f : A \to B$ is an arrow of $\mathcal{C}$ and $g : R \to S$ is an arrow of $\mathcal{D}$:

$$(A, R) \xrightarrow{f \times g} (B, S)$$

The composition of arrows is the obvious one: $(f \times g)(f' \times g') = (ff') \times (gg')$.

Remark 2. Notice that we can take ordered pairs since the product of classes (and of conglomerates) is well-defined.

We now turn to the product of two objects of a category. This can be defined by generalizing the cartesian product of objects of **Set** (for details, see [3][pp. 107-108]), resulting in the following definition:

Definition 3. Let $\mathcal{D}$ be a category and $X, Y \in \mathcal{D}$.
A *product* of X and Y is a tuple (P, p_1, p_2) where $P \in \mathcal{D}_0$ and

$$P \xrightarrow{p_1} X \qquad P \xrightarrow{p_2} Y$$

such that, for any diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

there is an unique arrow $A \xrightarrow{u} P$ of $\mathcal{D}$ such that

$$\begin{tikzcd} & A \arrow[ddl, "f"'] \arrow[d, dashed, "u"] \arrow[ddr, "g"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

commutes. p_1, p_2 are called *projections*.

Remark 4. One can show that products are limiting cones. We shall not do that here. But since limiting cones are terminal objects and terminal objects are unique up to isomorphism, then *products are unique up to isomorphism.* We thus speak of *the* product of X and Y.
The product of categories is a particular case of such a product, with $\mathcal{D} =$ **Cat**.

1.3 Some important Categories

We shall list some categories which will be relevant for our discussion of Monoidal categories and the like. (more details can be read in the pages dedicated to these categories in [11]).

(a) $\mathbf{Vect}_{\mathbb{K}}$ is the category whose:

- objects are vector spaces over $\mathbb{K}$.
- arrows are linear maps.

Sometimes we omit $\mathbb{K}$ if it is clear from context.

(b) **Ban** is the category whose:

- objects are Banach spaces (over some normed field $\mathbb{K}$, usually $\mathbb{R}$).
- arrows are short linear maps, also called linear contractions: bounded linear maps A s.t. $\| A \| \leq 1$.

This is the so called "isometric category", since it uses linear surjective isometries as arrows. There is another natural notion of arrow between Banach spaces - bounded bijective linear maps whose inverse is also bounded. This gives rise to a category of Banach spaces called "isomorphic category".

(c) **Hilb** is the category whose:

- objects are Hilbert spaces (over some normed field $\mathbb{K}$, usually $\mathbb{R}$ or $\mathbb{C}$).
- arrows are short linear maps.

(d) **nCob** is the category whose:

- objects are $(n-1)$-dimensional compact oriented manifolds.
- arrows are n-dimensional oriented (equivalence classes of) cobordisms.

where an n-dimensional *cobordism* is a compact manifold M with boundary together with two closed (*i.e.* compact and without boundary) manifolds Σ and Γ such that $\partial M = \Sigma \sqcup \Gamma$. (M, Σ, Γ) is an *oriented* *cobordism* if Σ, Γ are oriented and M has the orientation agreeing with the orientation of Σ (called the *ingoing boundary component*) and opposite to the one of Γ (called the *outgoing boundary component*. More about cobordisms in Section 4.

2 Monoidal Categories

2.1 Motivating Monoidal Categories

We now want to give some extra structure to categories. For example, we can give our category some kind of multiplication that mimics the multiplication in a *monoid*, which is a set M together with an associative map $\cdot : M \times M \to M$ and an (identity) element e such that $me = em = m$ $(\forall m \in M)$. In other words, a monoid is a group whose elements need not have inverses. A category with such a monoid-like structure will be called a monoidal category.
A first guess would be to define a monoidal category as a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that $\forall A, B, C \in \mathcal{C},\ (A \otimes B) \otimes C = A \otimes (B \otimes C)$ such that there is an object 1 such that $\forall A \in \mathcal{C}, A \otimes 1 = 1 \otimes A = A$. This, however, is not a satisfactory definition: to demand equality in these statements is too much, as is in category theory. To illustrate this problem, consider the category **FinSet** of finite sets. Together with the cartesian product $\times$ it is a monoidal category in the sense described above, right? Not really, since $A \times (B \times C) \neq (A \times B) \times C$. However, it is clear that $A \times (B \times C) \cong (A \times B) \times C$. Similarly, we have $A \times 1 \cong 1 \times A \cong A$ instead of equalities, where 1 is a singleton set (choose any one to play the role of the identity element). If we want **FinSet** to be a monoidal category, then we should replace our equalities by isomorphism equivalences - and *we do* want $(\mathbf{FinSet}, \times)$ to be a monoidal category, since $(\mathbb{N}, \cdot)$ is a monoid and $(\mathbf{FinSet}, \times)$ is its categorification (more about this in [4] and [5]).

Now, to say that

$$\forall A, B, C \in \mathcal{C},\ (A \otimes B) \otimes C \overset{a_{A,B,C}}{\cong} A \otimes (B \otimes C)$$

is to say that there exists a natural isomorphism

$$((-) \otimes (-)) \otimes (-) \xRightarrow{a} (-) \otimes ((-) \otimes (-))$$

(you may recall the many equivalent definitions of natural isomorphism in [8]

). To see this, just notice that a is a natural transformation (it is not hard to check that the maps involved are indeed functors - see Remark 5)

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[rr, bend left, "((-)\otimes(-))\otimes(-)"] \arrow[rr, bend right, "(-)\otimes((-)\otimes(-))"'] \arrow[rr, phantom, "a\Downarrow"] & & \mathcal{C} \end{tikzcd}$$

with components $a_{A,B,C}$ which are isomorphisms, meaning that, if a is a natural transformation, then it is a natural isomorphism (see Remark 6). We call it the *associator*.

Remark 5. Let us check for example that $F = ((-)\otimes(-))\otimes(-)$ is a functor. Consider three arrows $f : A \to A'$, $g : B \to B'$ and $h : C \to C'$ of $\mathcal{C}$.

F1) From the functoriality property of $\otimes$, we have $f\otimes g : A\otimes B \to A' \to B'$. Using the functoriality property twice, we get $F(f, g, h) = (f \otimes g) \otimes h : (A \otimes B) \otimes C \to (A' \otimes B') \otimes C'$, *i.e.* $F(f, g, h) : F(A, B, C) \to F(A', B', C')$. $\triangle$

F2) $F((f, g, h)(x, y, z)) = F(fx, gy, hz) = (fx \otimes gy) \otimes hz = ((f \otimes g)(x \otimes y)) \otimes hz = [(f \otimes g)h][(x \otimes y)z] = F(f, g, h)F(x, y, z)$, where in the last two equalities I used that $\otimes$ is a functor[1] . $\triangle$

F3) Once more using that $\otimes$ is a functor, it is easy to see that

$$\begin{aligned} F(id_{(A,B,C)}) &= F(id_A, id_B, id_C) = (id_A \otimes id_B) \otimes id_C \\ &= id_{(A\otimes B)\otimes C} = id_{F(A,B,C)} \end{aligned}$$

(just notice that $\otimes(id_A, id_B) = \otimes(id_{(A,B)}) = id_{A\otimes B}$). $\square$

Remark 6. It is an *extra assumption* of our definition of monoidal category that the associator a is indeed a natural transformation: we do not prove that from the other parts of the definition.
Let's see explicitly what it means for a to be a natural transformation, for future reference (it is simply a matter of untangling definitions). It means that the following diagram commutes:

[1] It is easy to see that we must have $fx \otimes gy = (f \otimes g)(x \otimes y)$:

$$\otimes(fx, gy) = \otimes((f, g)(x, y)) = [\otimes(f, g)][\otimes(x, y)] = (f \otimes g)(x \otimes y)$$

where I simply used that $\otimes$ is a functor $\mathcal{C}^2 \to \mathcal{C}$.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "a_{A,B,C}"'] & (A' \otimes B') \otimes C' \arrow[d, "a_{A',B',C'}"] \\
A \otimes (B \otimes C) \arrow[r, "f\otimes(g\otimes h)"'] & A' \otimes (B' \otimes C')
\end{tikzcd}$$

So we saw how the associator arises from our efforts to define monoidal category in a sensible way. In a similar fashion, we have two other natural isomorphisms coming from demanding that

$$\forall A \in \mathcal{C}, A \otimes 1 \overset{r_A}{\cong} A \overset{l_A}{\cong} 1 \otimes A$$

namely,

$$r : (-) \otimes 1 \Rightarrow id_{\mathcal{C}}$$

$$l : 1 \otimes (-) \Rightarrow id_{\mathcal{C}}$$

called the *right unitor* and the *left unitor*, respectively.

So far, so good. But there is something else that we shall demand from our monoidal category: two objects of the type $A_1 \otimes A_2 \otimes ... \otimes A_n$ but with possibly different positioning of parenthesis must have a *unique* isomorphism between them. To illustrate this, consider the following examples:

Example 7. Take $A \otimes (B \otimes (C \otimes D))$ and $((A \otimes B) \otimes C) \otimes D$. It is not hard to think of two (possibly distinct) isomorphisms between these two objects constructed using the associator. These two isomorphisms correspond to the two different paths in the following diagram:

$$\begin{tikzcd}
 & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "a_{A\otimes B,C,D}"'] \arrow[dr, "a_{A,B,C}\otimes id_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "a_{A,B,C\otimes D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "a_{A,B\otimes C,D}"] \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "id_A\otimes a_{B,C,D}"]
\end{tikzcd}$$

(Notice that f iso implies $f \otimes id_A$ iso - just take $(f \otimes id_A)^{-1} = f^{-1} \otimes id_A$). In fact, these two "paths" are the only two isomorphisms induced by the associator (I won't show this). Hence, to demand that there is a unique isomorphism is to demand that this diagram commutes. This commutative diagram is called the *pentagon identity*.

Example 8. It is clear that we can construct two (possibly different) isomorphisms from $(A \otimes 1) \otimes B$ and $A \otimes B$, using the associator, the left unitor and the right unitor. These are the two different paths in the following diagram:

$$
\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "a_{A,1,B}"] \arrow[dr, "r_A \otimes id_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "id_A \otimes l_B"] \\
& A \otimes B &
\end{tikzcd}
$$

We shall demand that these two isomorphisms are one and the same. In other words, we demand that the above diagram commutes. This commutative diagram is called the *triangle identity*.

The examples above are not just random particular cases that illustrate the issue at hand. They are in fact very special: if both the pentagon identity and the triangle identity hold, then every "formal" diagram built up from instances of the associator and the unitors commutes. This result is called the *coherence theorem* - see [7] and [10] for details.

2.2 Defining Monoidal Categories

Let's put everything from the last section together in a definition.

Definition 9. A *monoidal category* is a category M together with a functor $\otimes : M \times M \to M$ called *tensor product* and an object $1 \in M$ called the *identity object* such that:

M1) There is a natural isomorphism

$$a : ((-) \otimes (-)) \otimes (-) \Longrightarrow (-) \otimes ((-) \otimes (-))$$

called the *associator*

M2) There are natural isomorphisms

$$r : (-) \otimes 1 \Longrightarrow id_M$$

and

$$l : 1 \otimes (-) \Longrightarrow id_M$$

called *right unitor* and *left unitor* respectively.

M3) The *pentagon identity* holds. *i.e.* the diagram

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl, "a_{A\otimes B,C,D}"'] \arrow[dr, "a_{A,B,C}\otimes id_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "a_{A,B,C\otimes D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "a_{A,B\otimes C,D}"] \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "id_A \otimes a_{B,C,D}"]
\end{tikzcd}$$

commutes

M4) The *triangle identity* holds. *i.e.* the diagram

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "a_{A,1,B}"] \arrow[dr, "r_A \otimes id_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "id_A \otimes l_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes.

Example 10. Vect with the Vector Space Tensor Product

The associator components are the "obvious" linear maps given by $a_{U,V,W}((u\otimes v)\otimes w) = u \otimes (v \otimes w)$, with u, v, w basis elements of U, V, W, respectively. In fact, $a_{U,V,W}$ is clearly invertible (with the linear map taking $u \otimes (v \otimes w)$ to $(u \otimes v) \otimes w$ its inverse), hence an isomorphism; also, a is natural: for

$$f : U \to U'$$

$$g : V \to V'$$

$$h : W \to W'$$

we have

$$f \otimes (g \otimes h)(a_{U,V,W}[(u \otimes v) \otimes w)]) = f(u) \otimes (g(v) \otimes h(w))$$
$$= a_{U'\otimes(V'\otimes W')}((f(u) \otimes g(v)) \otimes h(w)) = a_{U'\otimes(V'\otimes W')}((f \otimes g) \otimes h)[(u \otimes v) \otimes w)]$$

and, by linearity, this implies naturality. *i.e.* the naturality square

$$\begin{array}{ccc}
(U \otimes V) \otimes W & \xrightarrow{(f\otimes g)\otimes h} & (U' \otimes V') \otimes W' \\
{\scriptstyle a_{U,V,W}}\downarrow & & \downarrow{\scriptstyle a_{U',V',W'}} \\
U \otimes (V \otimes W) & \xrightarrow[f\otimes(g\otimes h)]{} & U' \otimes (V' \otimes W')
\end{array}$$

commutes. The rest of the properties of the monoidal structure are also easy to check.

Example 11. Hilb with the Tensor Product of Hilbert Spaces

The *tensor product of Hilbert spaces* H_1, H_2, sometimes denoted by $H_1 \hat{\otimes} H_2$, is the completion (see [13]) of the inner product space $(H_1 \otimes H_2, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is given by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle_1 \langle v_2, w_2 \rangle_2$ and linearity [12]. The tensor product of the arrows (short linear maps) $f : H_1 \to H_2$, $g : G_1 \to G_2$ is the short linear map $f \otimes g : H_1 \otimes G_1 \to H_2 \otimes G_2$ given by $f \otimes g(x \otimes y) = f(x) \otimes g(y)$.

Example 12. nCob with the Disjoint Union

The tensor product of objects will simply be their disjoint union. For example, $S^1 \otimes (S^1 \sqcup S^1) := S^1 \sqcup (S^1 \sqcup S^1)$. The tensor product of arrows (cobordisms) will also be the disjoint union of manifolds.

Remark 13. The tensor product may or may not be a produt in the categorical sense. For example, in **Set** it is, while in **Hilb** it is not.

Definition 14. A functor $F : M \to M'$ between monoidal categories is said to be *(strict) monoidal* if $\forall x, y \in M,\ F(x \otimes_M y) = F(x) \otimes_{M'} F(y)$ and $\forall f, g \in \mathrm{Hom}(M),\ F(f \otimes_M g) = F(f) \otimes_{M'} F(g)$.

2.3 Braided and Symmetric Monoidal Categories

Many of the categories that we will be working with are not just monoidal but actually *symmetric monoidal*, meaning that they are "essentially" commutative. There is a still weaker version of commutativity that gives rise to the concept of *braided monoidal category*, which we define by demanding that for all x, y in our monoidal category M, there is an isomorphism $B_{x,y} : x \otimes y \longrightarrow y \otimes x$. We furthermore demand these $B_{x,y}$ to be components of a natural transformation B. Finally, we shall demand that it does not matter in what order we switch stuff around. This gives rise to the *hexagon identities* and is analogous to our discussion of the pentagon identities. *vide* [14],[6][pp. 21-22]. Let us write this down succintly:

Definition 15. A *braided monoidal category* is a monoidal category M with natural isomorphism $B : \otimes \Longrightarrow \otimes \circ t$ (called *braiding*), with t the "twist" in $M \times M$ given by $t(x, y) = t(y, x)$, such that the following diagrams commute:

$$\begin{tikzcd}
 & (x \otimes y) \otimes z \arrow[r, "B_{x,y} \otimes id_z"] & (y \otimes x) \otimes z \arrow[dr, "a_{y,x,z}"] & \\
x \otimes (y \otimes z) \arrow[ur, "a^{-1}_{x,y,z}"] \arrow[dr, "B_{x,y\otimes z}"'] & & & y \otimes (x \otimes z) \arrow[dl, "id_y \otimes B_{x,z}"] \\
 & (y \otimes z) \otimes x \arrow[r, "a_{y,z,x}"'] & y \otimes (z \otimes x) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x \otimes (y \otimes z) \arrow[r, "id_x \otimes B_{y,z}"] & x \otimes (z \otimes y) \arrow[dr, "a^{-1}_{x,z,y}"] & \\
(x \otimes y) \otimes z \arrow[ur, "a_{x,y,z}"] \arrow[dr, "B_{z,x\otimes y}"'] & & & (x \otimes z) \otimes y \arrow[dl, "B_{x,z} \otimes id_y"] \\
 & z \otimes (x \otimes y) \arrow[r, "a^{-1}_{z,x,y}"'] & (z \otimes x) \otimes y &
\end{tikzcd}$$

These are called the *hexagon identities*.

A braided symmetric monoidal category will have the components of its braiding having the "natural" inverse:

Definition 16. A *braided symmetric monoidal category* is a braided monoidal category M such that, for all $x, y \in M$, the inverse of $B_{x,y}$ is $B_{y,x}$. In that case, we call B a *twist*.

Definition 17. A (strict) monoidal functor $F : M \to M'$ between braided monoidal categories is said to be *(strict) braided* if $\forall x, y \in M$, $FB_{x,y} = B_{Fy,Fx}$.

If M, M' are symmetric, then F is called a *(strict) symmetric monoidal functor*.

Remark 18. We will drop the "strict" from now on, but it was there for a reason: to distinguish with other (more general) variants of these definitions (*strong monoidal*, *lax monoidal*,...). However, we will not need these.

3 Boundaries and Orientations

Our next goal is to analyze some details of the category of n-cobordisms. We will follow [15] for that. But first, we must revise how to deal with manifolds with boundary and oriented manifolds (following [16] and [15]).

3.1 Manifolds with Boundary

The main example of a manifold with boundary is the closed half plane $\mathbb{H}^n$ with the metric induced by $\mathbb{R}^n$. It will serve as the basis for our definitions.

The points x with $x_n > 0$ are called *interior points* while the others are called *boundary points*. The sets of these points are denoted $\mathrm{int}\mathbb{H}^n$ and $\partial\,\mathbb{H}^n$, respectively.

A topological space X is *locally* $\mathbb{H}^n$ if $\forall p \in X$ there is a neighbourhood U of p which is homeomorphic to an open subset of $\mathbb{H}^n$.

Notice that there are open subsets of $\mathbb{H}^n$ that have elements of $\partial\,\mathbb{H}^n$ - they result of the intersection of an open subset of $\mathbb{R}^n$ (which intersects the $x^n = 0$ hyperplane) with $\mathbb{H}^n$. These are not open subsets of $\mathbb{R}^n$, of course.

We can thus generalize the definition of manifold as follows:

Definition 19. A *topological manifold with boundary of dimension* n is a second countable Hausdorff locally $\mathbb{H}^n$ topological space.

A *chart* on a topological manifold M of dimension $n \geq 2$ is a homeomorphism $\phi : U \longrightarrow \phi(U) \subseteq \mathbb{H}^n$, where U is an open subset of M. If $n = 1$, we allow $\phi(U)$ to be a subset of $\mathbb{H}^1 = [0, +\infty)$ or of the half line $\mathbb{L}^1 = (-\infty, 0]$.

An *atlas* $\mathcal{A}$ for M is a set of charts such that the change of coordinates functions $\phi \circ \psi^{-1}$ are smooth (for any $\phi, \psi \in \mathcal{A}$).

M together with a maximal atlas is a *smooth manifold with boundary of dimension* n.

So no surprises here, except for the special case where $n = 1$. If we did not allow for the image of the charts to be subsets of the half line, we would not be able to give an orientation to the closed interval and its boundary. We will explore this a bit in Example 29.

Let M be a manifold with boundary. $p \in M$ is an *interior point* if $\phi(p) \in \mathrm{int}\,\mathbb{H}^n$ for some chart ϕ around p. It is a *boundary point* if $\phi(p) \in \partial\,\mathbb{H}^n$. (It turns out that this does not depend on the chart chosen).

Remark 20. ∂M may differ from the topological boundary $\mathrm{bd}(M)$. For example, the open unit disk D in $\mathbb{R}^2$ has $\partial D = \emptyset$ and $\mathrm{bd}(D) = S^1$. also, $\partial \mathbb{H}^n = \{x \in \mathbb{H}^n \,|x_n = 0\}$ but $\mathrm{bd}(\mathbb{H}^n) = \emptyset$ (taking $\mathbb{H}^n$ as a subset of itself).

It turns out that the atlas of M induces an atlas on ∂M with charts of the form $\phi' = \phi|_{U \cap \partial M} : U \cap \partial M \longrightarrow \partial \mathbb{H}^n \cong \mathbb{R}^{n-1}$. Hence ∂M is an $(n-1)$-manifold withour boundary.

3.2 Oriented Manifolds

We first discuss orientations in finite-dimensional vector spaces.

A motivational example is $\mathbb{R}^2$, where intuitively we have two orientations - clockwise and counterclockwise. We see that each one corresponds to a choice of the order of the elements of a basis. Taking the standard basis $\{e_1, e_2\}$, we can assign to the par (e_1, e_2) the counterclockwise orientation and to the par (e_2, e_1) the clockwise orientation. But it clearly does not matter that we took the standard basis: any basis $\{a, b\}$ would do, with (a, b) corresponding to one orientation and (b, a) to the other.

Now, given two ordered bases $u = (u_1, u_2)$ and $v = (v_1, v_2)$, there exists a unique invertible matrix A such that $u = A_{u,v}v$. The relation $u \sim v \Leftrightarrow \det A_{u,v} > 0$ is an equivalence relation. It is easy to see that there are exactly two equivalence classes. Each equivalence class is an *orientation* on $\mathbb{R}^2$. Notice how this fits well with the image we were constructing: proper rotations for example have positive determinant and indeed they do not "switch" the vectors when changing basis, while improper rotations do. More generally:

Definition 21. An *orientation* on a finite dimensional vector space V is an equivalence class of ordered basis w.r.t. the equivalence relation $(u_1, ..., u_n) \sim (v_1, ..., v_n) \Leftrightarrow \det A_{u,v} > 0$, where $A_{u,v}$ is the (unique) matrix such that $\forall j, u_j = \sum_i A_{u,v_{ji}} v_i$.

Now that we know how to orient a vector space, we can orient manifolds by orienting the tangent spaces. It turns out that the good way to do this is using n-covectors. So first we still have to see how to use n-covectors to get an orientation in a finite dimensional vector space.

First,recall that an *k-covector* ω on an n-dimensional real vector space V is an alternating[2], k-tensor on V ($\omega \in \mathrm{Hom}(V^k, \mathbb{R})$ and is alternating). We write $\omega \in \bigwedge^k(V^*)$. Also, $\bigwedge^k(V^*)$ is an $\binom{n}{k}$-dimensional vector space.

[2] ω being alternating means that $\omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \mathrm{sign}(\sigma)\omega(v_1, ..., v_k)$, where $\sigma \in S_n$.

The following result is easily checked:

Proposition 22. *Let V be an n-dimensional vector space and ω be a k-covector on V. Let $u_1, ..., u_n, v_1, ..., v_n$ be vectors of V, such that $u = Av$ (i.e. $u_i = \sum_j A_{ij} v_j$) with $A \in M_n(\mathbb{R})$. Then, $\omega(u_1, ..., u_n) = \det(A)\, \omega(v_1, ..., v_n)$.*

Proof. Omitted. □

In particular, this means that for ordered basis $(u_1..., u_n), (v_1..., v_n)$ of V (and nonzero ω), $\operatorname{sgn} \omega(u_1..., u_n) = \operatorname{sgn} \omega(v_1..., v_n)$ if and only if $\det A > 0$, and we know this to be true if and only if $(u_1..., u_n) \sim (v_1..., v_n)$.

Thus, ω conserves its sign throughout each of the two equivalence classes. We say that ω *represents* the orientation $[(u_1..., u_n)]$ if $\omega(u_1..., u_n) > 0$.

Now, $\bigwedge^n(V^*)$ is one-dimensional, and thus given two nonzero n-covectors ω, θ, we can write $\omega = a\theta$, with $a \in \mathbb{R} - \{0\}$. Notice that $\operatorname{sgn}(\omega(u_1..., u_n)) = \operatorname{sgn}(a)\operatorname{sgn}(\theta(u_1..., u_n))$, so that, if ω represents $(u_1..., u_n)$, then θ represents $(u_1..., u_n)$ iff $a > 0$.

Motivated by this, we define an equivalence relation in $\bigwedge^n(V^*) - \{0\}$ given by

$$\omega \sim \theta \;\text{ iff }\; \omega = a\theta \text{ with } a > 0$$

We see that this divides $\bigwedge^n(V^*) - \{0\}$ in two equivalence classes, each one corresponding to a different orientation of V. We will use this to orient manifolds.

We now turn to manifolds. The idea is: *to orient an m-dimensional manifold M is to give an orientation to each tangent space T_pM, in a smooth way.* Because of the above observation, we can do this simply by smoothly assigning a nonzero m-covector to each T_pM. This is precisely what it means to construct a C^∞ nowhere-vanishing m-covector field (also called (differential) m-form). We thus define orientability as follows:

Definition 23. A manifold M of dimension m is *orientable* if it has a C^∞ nowhere-vanishing m-differential form.

Now, we know that given two (nowhere-vanishing) smooth m-forms ω, θ, we can write $\omega = f\theta$ with $f \in C^\infty(M)$. Since they don't vanish, this function f must be always positive or always negative in each connected component of M. Thus, similarly to what happened with the n-covectors above, we get an equivalence relation on the space of nowhere-vanishing smooth m-forms given by

$$\omega \sim \theta \text{ iff } \omega = f\theta, \text{ with } f > 0$$

It is clear that we will have 2 equivalence classes on each connected component of the manifold (sometimes denoted $\{+,-\}$), and therefore a manifold M with k connected components has 2^k equivalence classes. In the case of a connected oriented manifold (for which we fixed an orientation $[\omega]$), we may call *positive basis* to the elements of the equivalence class of ordered bases $[e_1, ..., e_m]$ of T_pM which is represented by the equivalence class $[\omega_p]$ of m-covectors.

A smooth map between oriented manifolds $\phi : M \to N$ is *orientation-preserving* if ϕ_* takes positive bases of M to positive bases of N. It is *orientation-reversing* if it takes positive bases of M to negative bases of N.

Definition 24. An *orientation* on M is an equivalence class of nowhere-vanishing smooth differential m-forms.

Remark 25. The empty manifold $\emptyset$ has one orientation only, since giving an orientation to the manifold is assigning an orientation to each tangent space. In this case there are no tangent spaces, hence we want to count how many functions $\emptyset \longrightarrow \{+,-\}$, and there is exactly one such function (recall the set theory definition of function!).

There is still another way of accessing if a manifold is orientable: by looking at the atlases it admits:

Definition 26. An atlas $\mathcal{A}$ on M is *oriented* if for any two overlapping charts $(U, x^1, ..., x^n)$ and $(V, y^1, ..., y^n)$ of $\mathcal{A}$, the determinant of the Jacobian $[\frac{\partial y^i}{\partial x^j}]$ is positive on $U \cap V$.

Proposition 27. *M has a smooth nowhere-vanishing top form iff M has an oriented atlas.*

Proof. Omitted. □

Remark 28. Given an oriented atlas $\mathcal{A}$ of a manifold M, there is an oriented atlas $\mathcal{A}_\partial$ for ∂M obtained by restricting the charts of $\mathcal{A}$.

Assume M has dimension $n \geq 2$. If M is even dimensional, then we take ∂M to have the orientation induced by the oriented atlas $\mathcal{A}_\partial$. If M is odd dimensional, we impose on ∂M the orientation induced by the atlas "opposite" to ∂, namely the atlas whose charts are precisely the charts $(z^1, ..., z^n)$ of the (maximal) atlas $\mathcal{A}_M$ of M which give $\det[\frac{\partial z^i}{\partial x^j}] < 0$ for $(x^1, ..., x^n) \in \mathcal{A}$.

If M is one-dimensional, we see that it is always possible to give it an orientation, since $\Omega^0(M) = \mathbb{C}^\infty(M)$ clearly has non-vanishing elements (functions $f : M = \{p_1, ..., p_k\} \to \{a_1, ..., a_l\}$ with $a_i \neq 0$ for at least one i). Exactly like for the $n \neq 0$ case, we can divide $\Omega^0(M)$ in two classes $\{+, -\}$ corresponding to the two connected components of $\mathbb{R} - \{0\}$. The standard choice is to give orientation $-$ to the point $0 \in \partial \mathbb{H}^1$, in accordance with what we do for dimension $n > 1$ (since 1 is odd). Now, let $p \in \partial M$. If there is an orientation-preserving(reversing) chart $\phi : U \to \mathbb{H}^1$ around p in the atlas that orients M, then we assign the orientation $-(+)$ to p.

The closed interval will be an important object in our study of cobordisms, so let's take a detailed look at its structure and boundary:

Example 29. Take $I = [0, 1]$, which is a manifold with boundary.

A possible atlas is $\{(U_1, \phi_1), (U_2, \phi_2)\}$, where $U_1 = [0, 1)$, $U_2 = (0, 1]$, $\phi_1(x) = x$ and $\phi_2(x) = 1 - x$. (Notice that we coul not take simply $U = [0, 1]$: although U is open, we cannot construct a homeomorphism $\phi : U \to \mathbb{H}^1$, since ϕ must map $\mathrm{bd}(I)$ to $\mathrm{bd}(\mathbb{H}^1)$ we know that we would have to have $\phi(0) = \phi(1) = 0$).

To give I an orientation, we choose (the equivalence class of) a nowhere-vanishing differential 1-form. We simply choose the orientation $[dx]$.

Following our discussion in Remark 28, in order to give an orientation to $\partial I = \{0, 1\}$ we simply assign an orientation $+$ or $-$ to each one of its points, using the charts of the oriented atlas associated with the orientation $[dx]$

Another standard choice for this case is to assign the orientation $+$ to both points 0 and 1.

3.3 In-boundaries and Out-boundaries

Definition 30. Let Σ be an oriented closed submanifold of codimension 1 of an oriented manifold M. Let $x \in \Sigma$ and $(v_1, ..., v_{n-1})$ be a positive basis for $T_x\Sigma$. $u \in T_xM$ is a *positive normal* if $(v_1, ..., v_{n-1}, u)$ is a positive basis of T_xM.

It is possible to show that if we can pick any $(v_1, ..., v_n)$: $(v_1, ..., v_{n-1}, u)$ is either a positive basis for all of them or for none. This is clear intuitively in a submanifold of $\mathbb{R}^n$: all we want is for u to be outside the hyperplane T_xM.

We now take Σ to be a connected component of ∂M. Let $x \in \Sigma$ and $u \in T_xM$ a positive normal.

We can use a chart ϕ around x to make sense of the statement "u points *inwards/outwards*": this will correspond to $\phi_*(u)$ pointing to the interior of $\mathbb{H}^n$ or not. We will not be more formal than this.

It turns out that, if a certain positive normal u points inwards (outwards), then all positive normals in Σ point inwards (outwards). This is again well understood intuitively if one takes the time to draw some pictures.

If they point inwards, Σ is called an *in-boundary*. If they point outwards, Σ is called an *out-boundary*.

4 The Category of n-Cobordisms

4.1 Cobordisms - First Examples and Equivalence Classes.

There are unoriented and oriented cobordisms. We will jump straight into the ones that interest us: the oriented cobordisms.

The following provisional definition captures the concept of oriented cobordism: an *oriented n-cobordism* from a closed oriented manifold Σ_0 to another Σ_1 is an oriented n-manifold M whose in-boundary is Σ_0 and out-boundary is Σ_1.

This is not the final definition because it does not allow for cobordisms from a manifold to itself, since it cannot be an in-boundary and an out-boundary simultaneously. And we need this to be possible if we want to construct a category whose arrows are cobordisms!

The trick here is to take the target (source) of the cobordism to be a manifold which is diffeomorphic to the out-boundary (in-boundary) of M, instead of being exactly the out-boundary (in-boundary) of M. This will allow us to take both the source and target to be Σ, as long as we can find a (orientation-preserving) diffeomorphism from Σ to the in-boundary and another one from Σ to the out-boundary.

Definition 31. An *oriented n-cobordism* from a closed oriented manifold Σ_0 to another Σ_1 is an oriented n-manifold M whose in-boundary is diffeomorphic to Σ_0 and out-boundary is diffeomorphic to Σ_1. The diffeomorphisms involved must be orientation-preserving.

The power of this definition can be understood by analysing this example:

Example 32. Take $I = [0, 1]$ with its standard orientation and assign the orientation $+$ to both boundary points 0 and 1. Thus 0 is an in-boundary and 1 is an out-boundary. It is therefore trivial that I is a cobordism from $\{0\}$ to $\{1\}$.

Given manifolds $\{p_1\}$ and $\{p_2\}$, I is a cobordism between them! This is true because $\{p_1\}$ is trivially diffeomorphic to $\{0\}$ and $\{p_2\}$ is trivially diffeomorphic to $\{1\}$.

So we see that this definition allows us to extract many different cobordisms from a single manifold I.

Furthermore, if we want a cobordism from $\{p_1\}$ to $\{p_2\}$ realized by a different manifold M we can easily get it - assuming that M is diffeomorphic to I in an orientation-preserving way: simply compose the diffeomorphisms:

$$p_1 \to \partial I_{in} \to \partial M_{in}$$

$$p_2 \to \partial I_{out} \to \partial M_{out}$$

Thus obtaining a cobordism $p_1 \xrightarrow{M} p_2$.

Example 33. Take the cobordism $p_1 \xrightarrow{I} p_2$ of the example above. If we take a step back and change the orientations of the boundary points 0 and 1 (while maintaining the orientation of I), we obtain distinct cobordisms.

For example, assign $+$ to 0 and $-$ to 1. Then 0 and 1 are both in-boundaries. The cobordism is now $\{p_1, p_2\} \xrightarrow{I} \emptyset$.

For the other two cases it is similar, and we get the cobordisms $\emptyset \xrightarrow{I} \{p_1, p_2\}$ for $0 : -$,$1 : +$ and $p_2 \xrightarrow{I} p_1$ for $0 : -$,$1 : -$.

The closed interval can be used to construct cylinders, and to arrive at the following result:

Proposition 34. *Let Σ_0, Σ_1 be $(n-1)$ manifolds, both diffeomorphic to a closed oriented manifold Σ. Then Σ_0 and Σ_1 are cobordant.*

Proof. Let I be the unit interval, with the standard orientation and with orientation $+, +$ for its boundary points.

Take the manifold $\Sigma\times I$, which has in-boundary $\Sigma\times\{0\}$ and out-boundary $\Sigma \times \{1\}$. it is clear that we have a cobordism $\Sigma_0 \xrightarrow{\Sigma\times I} \Sigma_1$: just take the composite diffeomorphisms

$$\Sigma_0 \xrightarrow{\cong} \Sigma \xrightarrow{\cong} \Sigma \times \{0\}$$

$$\Sigma_1 \xrightarrow{\cong} \Sigma \xrightarrow{\cong} \Sigma \times \{1\}$$

(So the cobordism is just a (disjoint union of) cylinder(s)). $\square$

Remark 35. In the conditions of the above proposition, just like in Example 32, if we have a manifold $M \cong \Sigma\times I$ then we get another cobordism $\Sigma_0 \xrightarrow{M} \Sigma_1$.

More generally, if $\Sigma_0 \xrightarrow{M} \Sigma_1$ is an oriented cobordism and N is diffeomorphic (with preserved orientation) to M, then $\Sigma_0 \xrightarrow{N} \Sigma_1$ is an oriented cobordism.

We will say that cobordisms related in this way are equivalent.

Definition 36. Let $\Sigma_0 \underset{N}{\overset{M}{\rightrightarrows}} \Sigma_1$ be a oriented cobordisms. This means that we have diffeomorphisms onto the in/out-boundaries of M and N as depicted:

$$\begin{tikzcd}
 & M & \\
\Sigma_0 \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & \Sigma_1 \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\
 & N &
\end{tikzcd}$$

We say that M and N are *equivalent* if there is a (orientation-preserving) diffeomorphism Ψ such that the diagram

$$\begin{tikzcd}
 & M \arrow[dd, "\Psi"] & \\
\Sigma_0 \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & \Sigma_1 \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\
 & N &
\end{tikzcd}$$

commutes.

It is easy to see that this gives us an equivalence relation for cobordisms. The arrows of the category of cobordisms will be these equivalence classes.

4.2 Problems with the construction of nCob

The idea of the category of n cobordisms is to have cobordisms as arrows and their boundaries as objects. What would be a natural composition of arrows in this case? Gluing (I will make this more precise later) the out-boundary of the first cobordism to the in-boundary of the second seems to be a good idea: we obtain a new manifold whose in-boundary is the in-boundary of the first cobordism and out-boundary is the out-boundary of the second... or do we? We will see that things are not that simple, and two problems arise when one tries to formulate things in this way.

First Problem:

Given n-cobordisms $\Sigma_0 \xrightarrow{M} \Sigma_1$, $\Sigma_1 \xrightarrow{N} \Sigma_2$, it is not always true that gluing N with M (at Σ_1) gives a smooth manifold: the problem is in the glued boundary, which may have points that do not admit charts compatible with the atlases in M and N. We will make this more concrete later.

Second Problem:

The identity should clearly be (isomorphic to) a cylinder of height 0 like $S^1 \times \{0\}$. But this is an $(n-1)$-manifold, not an n-manifold and thus it cannot be an arrow.
In other words, the only reasonable candidate for identity is not an arrow in the first place.

In order to solve these issues, we will make precise the notion of gluing.

4.3 Glue

Our attempt to glue smooth manifolds will lead us to consider gluing equivalence classes instead of the manifolds themselves. We first start by studying how to glue topological spaces, then topological manifolds, and finally smooth manifolds.

4.3.1 Gluing Topological Spaces

We can always glue any two topological spaces M_0 and M_1. We just need to have a topological space Σ and continuous maps $\Sigma \xrightarrow{f_0} M_0$ and $\Sigma \xrightarrow{f_1} M_1$, but it is always possible to come up with such objects, since for any two topological spaces X, Y one can easily construct a continuous map $X \xrightarrow{f} Y$ (just take $\forall x, f(x) = y'$, with y' a certain elenent of Y).

Roughly speaking, in order to glue M_0 and M_1, we identify the points in M_0 and M_1 which are obtained using f_0, f_1 from a same point in Σ. This is achieved by using the following equivalence relation in $M_0 \sqcup M_1$:

$$(0, m_0) \sim (1, m_1) \text{ iff } \exists x \in \Sigma : f_i(x) = m_i \ (\forall i \in \{0, 1\})$$

and $(0, m_0) \sim (0, m_0), (1, m_1) \sim (1, m_1)$, of course.

We can now define the gluing of topological spaces.

Definition 37. Given continuous maps between topological spaces

$$M_0 \xleftarrow{f_0} \Sigma \xrightarrow{f_1} M_1$$

the *gluing* of M_0 and M_1 over Σ is the topological space

$$M_0 \sqcup_\Sigma M_1 := {M_0 \sqcup M_1} /_\sim$$

I said that the gluing is a topological space but I did not mention yet what topology we use for this set. It is fairly natural: $U \subseteq M_0 \sqcup_\Sigma M_1$ is open iff $j_0^{\dashv}(U) \subseteq M_0$ and $j_1^{\dashv}(U) \subseteq M_1$ are open, where

$$M_0 \xrightarrow{j_0} M_0 \sqcup_\Sigma M_1 \xleftarrow{j_1} M_1$$

are the standard injections into the disjoint union composed with the quotient map. *i.e.* $j_0 = [-] \circ i_0$ and $j_1 = [-] \circ i_1$.

Let's analyse our gluing from a more abstract/categorical point of view. First, it is easy to check that the diagram

$$\begin{tikzcd}
\Sigma \arrow[r, "f_1"] \arrow[d, "f_0"'] & M_1 \arrow[d, "j_1"] \\
M_0 \arrow[r, "j_0"'] & M_0 \sqcup_\Sigma M_1
\end{tikzcd}$$

commutes.

It turns out that $(M_0 \sqcup_\Sigma M_1, j_0, j_1)$ is universal w.r.t. this commutative diagram. That is:

Proposition 38. *Given a commutative diagram*

$$\begin{tikzcd} \Sigma \arrow[r, "g_1"] \arrow[d, "g_0"'] & M_1 \arrow[d, "j_1"] \\ M_0 \arrow[r, "j_0"'] & X \end{tikzcd}$$

there is a unique continuous map $u : M_0 \sqcup_\Sigma M_1 \to X$ *such that*

$$\begin{tikzcd} \Sigma \arrow[r, "f_1"] \arrow[d, "f_0"'] & M_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "g_1"] & \\ M_0 \arrow[r, "j_0"'] \arrow[drr, bend right, "g_0"'] & M_0 \bigsqcup_\Sigma M_1 \arrow[dr, dashed, "u"] & \\ & & X \end{tikzcd}$$

commutes.

(In other words, $(M_0 \sqcup_\Sigma M_1, j_0, j_1)$ *is the pushout of* $M_0 \xleftarrow{f_0} \Sigma \xrightarrow{f_1} M_1$ *in* ***Top****).*

Remark 39. Being a universal construction, the pushout is unique up to isomorphism.

In a pictorial,informal way, we can think of the pushout square as being the smallest commutative square with $M_0 \xleftarrow{f_0} \Sigma \xrightarrow{f_1} M_1$, since it fits "inside" all other commutative squares with $M_0 \xleftarrow{f_0} \Sigma \xrightarrow{f_1} M_1$.

4.3.2 Gluing Smooth Manifolds

From what we saw before, we cannot get a unique smooth gluing. But it *is* true that at least *some* smooth gluing exists. To execute this gluing of manifolds M_0, M_1, we search for manifolds N_0, N_1 which are diffeomorphic to M_0, M_1, respectively, and such that N_0, N_1 can naturally be glued smoothly. It turns out that the resulting smooth manifold is unique up to diffeomorphism, and that it does not depend on the specific cobordisms chosen to glue, but only on the cobordism class to which they belong. Also, it is not hard to check that this gluing is associative, as it should.

4.4 nCob

(Missing: Some important theorems, final def. of nCob, proving that nCob is monoidal, Disjoint union and monoidal structure... See the references.)

Proposition 40. *The monoidal category* $(\boldsymbol{nCob}, \bigsqcup)$ *is symmetric.*

Proof. The twist map between $\Sigma \bigsqcup \Sigma'$ and $\Sigma' \bigsqcup \Sigma$ will be the cobordism induced by the twist diffeomorphism of manifolds $\Sigma \bigsqcup \Sigma' \cong \Sigma' \bigsqcup \Sigma$: we can take Γ a manifold diffeomorphic to $\Sigma \bigsqcup \Sigma'$ and $\Sigma' \bigsqcup \Sigma$ and get a cobordism $\Sigma \bigsqcup \Sigma' \cong \Gamma \times 0 \xrightarrow{T_{\Sigma,\Sigma'} = \Gamma \times I} \Gamma \times 1 \cong \Sigma' \bigsqcup \Sigma$. It is clear that if we compose this with $T_{\Sigma',\Sigma}$ we get a cylinder from Σ to itself *i.e.* the identity.

We still have to check that T is a braiding, so that **nCob** is a braided monoidal category in the first place. This is to say that $T : \otimes \Rightarrow \otimes \circ t$ is a natural transformation and respects the hexagon identities.

We first notice that T is a natural transformation iff $(M \bigsqcup N) \circ T_{\Sigma,\Sigma'} = T_{\Gamma,\Gamma'} \circ (N \bigsqcup N)$, for all arrows $\Sigma \xrightarrow{M} \Gamma$, $\Sigma' \xrightarrow{N} \Gamma'$. This is clearly true: since $T_{\Sigma,\Sigma'}, T_{\Gamma,\Gamma'}$ are cylinders, then $(M \bigsqcup N) \circ T_{\Sigma,\Sigma'} \cong M \bigsqcup N \cong N \bigsqcup M \cong T_{\Gamma,\Gamma'} \circ (N \bigsqcup N)$, and clearly $(M \bigsqcup N) \circ T_{\Sigma,\Sigma'}$ and $T_{\Gamma,\Gamma'} \circ (N \bigsqcup N)$ have the same in and out boundaries. This is all we need.

We now turn to the hexagon identities. Recall that the associator $a_{A,B,C}$ is simply the cylinder associated with the standard diffeomorphism $(A \bigsqcup B) \bigsqcup C \cong A \bigsqcup (B \bigsqcup C)$. This means that composing (gluing) with components of a or a^{-1} does not change the diffeomorphism class, so we only need to worry with the arrows of the hexagons with T. But the components of T are also cylinders, and thus they also do not change the diffeomorphism class. It is thus clear that both hexagon diagrams commute (both compositions are the same cobordism (class)). $\square$

4.5 Generators for 2Cob

4.5.1 Generating sets and Skeletons of a Category

One way of studying a mathematical object is to study its building blocks. We do this with groups when we study their generating sets. Recall that a group can be seen as a category with a single object and whose arrows are all invertible. A generating set for a group G can thus be seen as a collection H of arrows of the category G such that any arrow (group element) can be seen as a composition (group multiplication) of arrows (group elements) of H. This notion generalizes neatly to categories

Definition 41. A *generating set for a category* $\mathcal{C}$ is a collection $S \subseteq \mathrm{Hom}(\mathcal{C})$ such that every arrow in $\mathrm{Hom}(\mathcal{C})$ can be written as a composition of arrows in S.

If $\mathcal{C}$ is a large category (*i.e.* $\mathrm{Ob}(\mathcal{C})$ is not a set, but a class, conglomerate, or other generalization of class), then, since there are at least as many arrows in a category as there are objects, we have way too many arrows to hope that we can find a generating set for $\mathcal{C}$. **nCob** is one such category.

The usual procedure to avoid this problem is to work with the *skeleton of the theory*: we partition $\mathrm{Ob}(\mathcal{C})$ in isomorphism classes, take one element of each isomorphism class and all arrows between those objects - this gives us a skeleton:

Definition 42. A *skeleton of a category* $\mathcal{C}$ is a full subcategory Z of $\mathcal{C}$ which has exactly one object of each isomorphism class of $\mathcal{C}$.

(Recall that Z is a full subcategory of $\mathcal{C}$ if $\forall X, Y \in Z, Z(X,Y) = \mathcal{C}(X,Y)$. So we are indeed taking all the arrows of $\mathcal{C}$ between all the objects of Z.)

Remark 43. Z contains the structural information about $\mathcal{C}$ because Z is equivalent to $\mathcal{C}$ (recall that this means that there is a functor $Z \to \mathcal{C}$ with pseudo-inverse), and we know that this means that Z has all the categorical properties of $\mathcal{C}$.

In fact, the embedding $E : Z \hookrightarrow \mathcal{C}$ is always a faithful functor (*i.e.* injective on arrows). But we also assume it to be full, and it is clearly surjective on objects, because given $y \in \mathcal{C}$ we can by construction take $y' \in [y]$ such that $\exists x \in X | E(x) = y'$ (just take $x = y'$) and of course $y \cong y'$. Therefore (using a basic result of category theory) E is an equivalence.

Z is the minimal category equivalent to $\mathcal{C}$: a subcategory with less objects wouldn't have essentially surjective E, and with less arrows would not have full E. Of course there are many different skeletons, corresponding to different choices of representatives of the equivalence classes, but it is easy to check that they are all isomorphic.

4.5.2 A skeleton of 2Cob

It is a fact from the theory of smooth manifolds that every closed 1-manifold with n connected components is diffeomorphic to the disjoint union of n

circles.

Proposition 44. *Two oriented* 1*-manifolds* Σ_0 *and* Σ_1 *are diffeomorphic iff there is an invertible cobordism* $\Sigma_0 \xrightarrow{M} \Sigma_1$*.*

Proof. Suppose $\Sigma_0 \cong \Sigma_1$. Then we can define a cobordism (cylinder) $\Sigma_0 \xrightarrow{\Sigma_0 \times I} \Sigma_1$ as in the contruction of Proposition 34. Repeating such construction but now taking $\Sigma_0 \cong \Sigma_1 \times \{1\}$ instead of $\Sigma_0 \cong \Sigma_1 \times \{0\}$, we get a cobordism $\Sigma_0 \xrightarrow{\Sigma_1 \times I} \Sigma_0$, which is also a cylinder, of course. Gluing will give (the class of) a cylinder, which is the identity.

Conversely, suppose there is an invertible cobordism $\Sigma_0 \xrightarrow{M} \Sigma_0$. Then (using Proposition Σ_0 and Σ_1 have the same number of connected components, and thus $\Sigma_0 \cong \Sigma_1$. $\square$

Corollary 45. *Two objects of* ***nCob*** *are in the same isomorphism class iff they have the same number of connected components.*

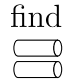
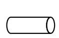
Proposition 46. *Let* **0** *be the empty* 1*-manifold, and* **1** *be a certain circle* Σ *(say* S^1*).*

Let $\boldsymbol{n} := \bigsqcup_{i=1}^{n} \Sigma$*. Then the subcategory* $\{0, 1, 2, ...\}$ *of* ***2Cob*** *is a skeleton of* ***2Cob****.*

Proof. Clear using the above corollary and the definition of skeleton! $\square$

Notation: We abusively denote this skeleton by **2Cob**. Other times we are more careful and denote it by **<u>2Cob</u>**.

4.5.3 The generators of <u>2Cob</u>

If we try to find a generating set for the skeleton of **2Cob** we still find ourselves with too much stuff to deal with: by composing, I cannot get  from  for example, so I would need an infinitely big generating set - at least one element for each cobordism with n disjoint cylinders - which we would like to avoid if possible. So we will use the fact that we have a tensor product $\otimes = \bigsqcup$ at our disposal to get a more useful definition of generating set.

Definition 47. A *generating set for a monoidal category* $\mathcal{C}$ is a collection $S \subseteq \mathrm{Hom}(\mathcal{C})$ such that every arrow of $\mathcal{C}$ can be written in terms of arrows of S using composition and tensor product.

The following result is the main result of this part of the text. We will discuss the main ingredients of its proof, and motivate them in a hopefully intuitive and convincing way.

Theorem 48. *The (skeleton of the) monoidal category **2Cob** has the generating set*

$\{$  $\}$

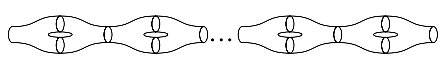
whose elements are called cylinder, pair of pants, reverse pair of pants, twist, birth of circle and death of circle, respectively.

Proof. (Sketch) We will start by looking at connected 2-cobordisms. One of the main results of differential topology is the following:

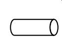
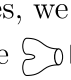
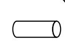
Lemma 49. *Two connected, compact, oriented surfaces with oriented boundaries are diffeomorphic iff they have the same genus and the same number of in-boundaries and out-boundaries.*

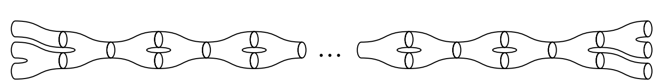
We then see that a connected 2-cobordism M of genus g with an in-boundary with n connected components and an out-boundary with m connected components can be constructed (up to diffeomorphism) as follows:

First, take g pairs of pants and g reversed pairs of pants and glue them alternately to get

$\tilde{M} =$ 

which is a a 2-cobordism of genus g with one in-boundary and one out-boundary.

We now glue pairs of pants, reversed pairs of pants and cylinders until we get the desired number of in-boundaries and out-boundaries. More concretely, we need to precompose M' with $m-1$ cobordisms of the type  $\sqcup \left(\bigsqcup_{i=1}^{k}$  $\right)$, with $k = 0$ for the first gluing, $k = 1$ for the second, and so on up to $k = m-1$. This gives us a cobordism $\check{M}$ with one in-boundary and m out-boundaries. To get n in-boundaries, we do something similar: we compose $\check{M}$ with $n-1$ cobordisms of the type  $\sqcup \left(\bigsqcup_{i=1}^{k}$  $\right)$, with $k = 0$ for the first gluing, $k = 1$ for the second, and so on up to $k = n-1$. We end up with a cobordism M' with n in-boundaries and m out-boundaries, as we wanted:

$M' = \ldots$  $\ldots$ (1)

We still have to discuss the case when $m = 0$ or $n = 0$; But that is easy: we just glue a birth of circle or a death of circle to $\tilde{M}$. Notice that what we just did was show the following result:

Lemma 50. *Every connected 2-cobordism (better: element of the* ***2Cob*** *skeleton) can be ontained by composition and disjoint union of the elements* , , , , .

We could think that this gives us the theorem as an immediate corollary, and without even needing the twist for anything! But it does not, since a cobordism need not be the disjoint union of its connected components. Let's see a couple of examples.

Example 51. The twist is, as a manifold, simply $\sqcup$, but as a cobordism it is not, since connects different boundaries than .

Example 52. Notice that $M =$ is not the same cobordism as the disjoint union , and in fact it cannot be constructed in the form of M' above. But if we compose M with $S =$, then clearly $SM = M'$, with M' a disjoint union of cobordisms of the form (1). Notice how we had to use the twist.

It should be clear by now (although we did not and will not prove it) that:

Lemma 53. *Every 2-cobordism M can be written as*

$$M = S \left(\bigsqcup_i M'_i \right) T$$

where each M'_i is of the form (1), and S and T are permutation cobordisms: 2-cobordisms which are compositions and disjoint unions of twists and cylinders.

(in the example above, $M'_1 =$ and $M'_2 =$).

Notice that the theorem we are trying to prove is a corollary of this lemma. **[End of the proof of the theorem.]**

$\square$

4.6 Relations in <u>2Cob</u>

Definition 54. Let S be a generating set for a category $\mathcal{C}$. A *relation* of S is a pair $s_1 \circ ... \circ s_n$ and $s'_1 \circ ... \circ s'_m$ of compositions of the elements of S which yield the same arrow (so the relations are in bijection with the commutative and "2-branch diagrams" – diagrams exactly one object with two incoming arrows and one object with two outgoing arrows, and all others with one incoming and one outgoing – that can be constructed using arrows)

A set R of relations of S is *complete* if any relation of S can be obtained by using the relations in R.

I will not draw the relations part – see them in the references. But here is a complete set of relations:

1. Identity relations
2. disc-sewing relations
3. Frobenius relations
4. associativity and coassociativity relations
5. commutativity and cocommutativity relations

5 Frobenius Algebras

A Frobenius algebra is an algebra equipped with a functional whose kernel contains no nontrivial left ideals. A standard example is $(M_n(\mathbb{R}), \mathrm{Tr})$. There is another useful definition using a nondegenerate associative linear pairing instead of the functional. All of this will be explained below.

This section has two main goals.

First, we will show that every Frobenius algebra (A, ϵ) has a unique coalgebra structure for which the counit is exactly ϵ and the Frobenius condition (to define below) holds. We will use this to characterize a Frobenius algebra as an algebra which is also a coalgebra in a "compatible" way. This point of view is generalized to category theory under the name of Frobenius monoid [17].

Our second goal is to introduce graphical calculus. This will give rise to a pictorial representation of Frobenius algebras that resemble our treatment of **2Cob**. This is not a coincidence, as we will explore in section 6.

We will follow [15] closely in this section.

5.1 Frobenius Algebras, Algebras and Coalgebras

Definition 55. A *Frobenius Algebra* is a finite dimensional $\mathbb{K}$-algebra A equipped with a linear functional $\epsilon : A \to \mathbb{K}$ whose kernel contains no nontrivial left ideals.

Equivalently: $\forall y \in A, \quad [\epsilon(A \cdot y) = \{0\} \;\Rightarrow\; y = 0]$.

The functional ϵ is called a *Frobenius form.*

Remark 56. Given a functional $\epsilon \in A^*$ with A an algebra, we have a canonic linear associative pairing $\beta_\epsilon : A \otimes A \to \mathbb{K}$ given by $\beta(x \otimes y) = \epsilon(xy)$. Conversely, given an associative linear pairing β on A, we get a canonic functional $\epsilon_\beta(x) = \beta(x \otimes 1_A) = \beta(1_A \otimes x)$.

It is easy to check that if we construct β from ϵ, and from that pairing we construct a new functional, this new functional is ϵ again.

The next Lemma will allow us to get an alternative definition of Frobenius algebra. But first we have to introduce the notion of nondegenerate pairing:

Definition 57. Let V and W be vector spaces over $\mathbb{K}$. A linear pairing $\beta : V \otimes W \to \mathbb{K}$ is *nondegenerate* if there exists $\gamma : \mathbb{K} \to W \otimes V$ such that the diagrams

$$\begin{tikzcd} V \otimes \mathbb{K} \arrow[r, "id_V \otimes \gamma"] \arrow[d, "\cong"'] & V \otimes W \otimes V \arrow[d, "\beta \otimes id_V"] \\ V & \mathbb{K} \otimes V \arrow[l, "\cong"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes W \arrow[r, "\gamma \otimes id_W"] \arrow[d, "\cong"'] & W \otimes V \otimes W \arrow[d, "id_W \otimes \beta"] \\ W & W \otimes \mathbb{K} \arrow[l, "\cong"] \end{tikzcd}$$

commute.

Lemma 58. *Let A be a $\mathbb{K}$-algebra and $\epsilon \in A^*$. Let β_ϵ be the associative linear pairing defined as in the Remark above. Then β_ϵ is degenerate iff* $\ker \epsilon$ *has no nontrivial left ideals iff* $\ker \epsilon$ *has no nontrivial right ideals.*

This gives us a second characterization/definition for Frobenius algebras:

Theorem 59. *Let A be a finite dimensional $\mathbb{K}$-algebra and $\epsilon \in A^*$. Then, (A, ϵ) is a Frobenius algebra if and only if the associative linear pairing β induced by ϵ is nondegenerate.*

In particular, we can view a Frobenius algebra as a $\mathbb{K}$-algebra equipped with a nondegenerate associative linear pairing β (called the Frobenius pairing).

In order to define coalgebra, we will first look at a definition of algebra using diagrams. Inverting the arrows of these diagrams will give us the notion of coalgebra.

Definition 60. A *(unital)* $\mathbb{K}$*-algebra* is a $\mathbb{K}$ vector space A together with $\mathbb{K}$-linear maps

$$\mu : A \otimes A \to \mathbb{K} \qquad \eta : \mathbb{K} \to A$$

called *multiplication* and *unit* respectively, such that the following diagrams commute:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "\mu \otimes id_A"'] & & A \otimes (A \otimes A) \arrow[d, "id_A \otimes \mu"] \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "\eta \otimes id_A"] \arrow[dr, "\cdot"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K} \arrow[l, "id_A \otimes \eta"'] \arrow[dl, "\cdot"] \\
& A &
\end{tikzcd}$$

with $\cdot$ the scalar multiplication on A and $a_{A,A,A}$ the standard isomorphism expressing the associativity of $\otimes$ (a is the associator of $\mathbf{Vect}_{\mathbb{K}}$). These diagrams are called *associativity axiom* and *unit axiom*, respectively.

Remark 61. Notice that this definition is equivalent to the usual definition of unital algebra if we identify μ with the algebra multiplication and the unit 1_A with the image of $1_{\mathbb{K}}$ under η. (Note that, if A is a $\mathbb{K}$-algebra with unit 1_A, a map $\eta : \mathbb{K} \to A$ is completely defined if we just specify $\eta(1_{\mathbb{K}})$. Fixing $\eta(1_{\mathbb{K}}) = 1_A$ ensures the unit axiom).

Remark 62. You may be wondering why we used $\mu : A \otimes A \to A$ instead of $\mu : A \times A \to A$, and also how are we ensuring that the multiplication distributes over the vector space addition (one of the axioms present in the usual definition of algebra over a field $\mathbb{K}$).

The answer to the first question is the second. Indeed, to have a linear map $\mu : A \otimes A \to A$ is to have a bilinear map $\tilde{\mu} : A \times A \to A$ given by $\tilde{\mu}(a, b) = \mu(a \otimes b)$ (μ and $\otimes$ are both bilinear) and bilinearity of $\tilde{\mu}$ ensures distributivity of $\tilde{\mu}$.

We now turn to coalgebras:

Definition 63. A *(counital)* $\mathbb{K}$*-coalgebra* is a $\mathbb{K}$ vector space A together with $\mathbb{K}$-linear maps

$$\delta : \mathbb{K} \to A \otimes A \qquad \epsilon A \to \mathbb{K}$$

called *comultiplication* and *counit* respectively, such that the following diagrams commute:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] & & A \otimes (A \otimes A) \\
A \otimes A \arrow[u, "\delta \otimes id_A"] & & A \otimes A \arrow[u, "id_A \otimes \delta"'] \\
& A \arrow[ul, "\mu"] \arrow[ur, "\mu"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \otimes A & A \otimes A \arrow[l, "\epsilon \otimes id_A"'] \arrow[r, "id_A \otimes \epsilon"] & A \otimes \mathbb{K} \\
& A \arrow[u, "\mu"'] \arrow[ul, "\cdot^{-1}"] \arrow[ur, "\cdot^{-1}"'] &
\end{tikzcd}$$

with $\cdot$ the scalar multiplication on A (which is a linear isomorphism) and $a_{A,A,A}$ the standard isomorphism expressing the associativity of $\otimes$ (a is the associator of $\mathbf{Vect}_{\mathbb{K}}$). These diagrams are called *coassociativity axiom* and *counit axiom*, respectively.

5.2 Characterizing Frobenius Algebras and Graphical Calculus

Our first goal is to show that a Frobenius algebra has a natural coalgebra structure for which the Frobenius form is the counit.

We will now try to construct a comultiplication for the Frobenius algebra (A, ϵ) in a natural way.

The tools we have to work with are:

$$\begin{aligned}
&\mu : A \otimes A \to A \\
&\eta : \mathbb{K} \to A \\
&\epsilon : A \to \mathbb{K} \\
&\beta : A \otimes A \to \mathbb{K} \\
&id_A : A \to A
\end{aligned}$$

(although technically we did not need β explicitly, since it can be constructed from ϵ, as we know).

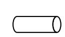
In order to handle our "diagrammatic computations" easily, we will use a graphical notation to represent our maps. The correspondence is as follows:

$\mu : A \otimes A \to A$ =: 

$\eta : \mathbb{K} \to A$ =: 

$\epsilon : A \to \mathbb{K}$ =: 

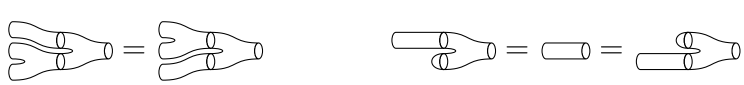
$\beta : A \otimes A \to \mathbb{K}$ =: 

$id_A : A \to A$ =: 

Composition will be represented by "gluing" and $\otimes$ will be represented by "justaposition".

Let's represent in this graphical language some important relations that we will need later on, and will be essential in the construction of the coalgebra structure.

Example 64. Let's write the axioms for algebra in Definition 60 using graphical notation:

 (2)

These are the associativity axiom and the unit axiom, respectively.

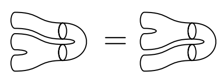
Example 65. We have seen in Remark 56 that the pairing and the counit are related by $\beta = \epsilon \circ \mu$ and $\beta \circ (\eta \otimes id_A) = \epsilon = \beta \circ (id_A \otimes \eta)$. Graphically:


(3)

It turns out that the axiom for the Frobenius form is not expressible in graphical language because we cannot represent ideals using diagrams... So it is useful to use the definition of Frobenius algebra using the Frobenius pairing instead:

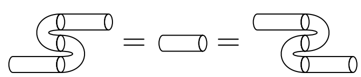
Example 66. We will express in graphical language that a linear pairing β is associative and nondegenerate (and thus a Frobenius pairing).

Associativity:


(4)

Notice that it is a consequence of the associativity of μ.

Nondegeneracy:

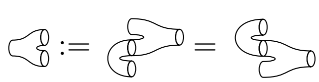

(5)

It will be a consequence of the Frobenius condition, which we will see later.

5.2.1 The comultiplication

We now have the necessary tools to construct a comultiplication:

Definition 67. Let ($A, \epsilon =$ ) be a Frobenius algebra. We define a map $\delta =$  $: A \to A \otimes A$ by:



(Omitting the parallel cylinders, as we will do from now on).

Of course, in order to be sure that this is a possible definition, we should confirm that the equality in the definition truly holds. We shall do that now:

Lemma 68. *See the lemma with the three point function in [15].*

So all is fine. We now confirm that this comultiplication is exactly the counit, as we wanted.

Proposition 69. *ϵ is the counit for δ [15].*

Proposition 70. *Frobenius condition...[15]*

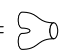
Proposition 71. *Associativity of delta... [15]*

We now reach our first main result of this section:

Theorem 72. *Given a Frobenius algebra (A, ϵ), there exists a unique comultiplication $\delta =$  whose counit is ϵ and which satisfies the Frobenius condition. Furthermore,*  *is coassociative.*

5.2.2 Characterization of Frobenius Algebras

We are now in a position to give a characterization of Frobenius algebras. In fact, the Frobenius relation characterizes Frobenius algebras among general vector spaces equipped with a multiplication with unit and a comultiplication with counit. Concretely:

Theorem 73. *Let A be a vector space equipped with maps $\mu =$  $: A \otimes A \to A$, $\eta =$  $: \mathbb{K} \to A$, $\delta =$  $: A \to A \otimes A$ and $\epsilon =$  $: A \to \mathbb{K}$, about which we only assume that η respects the unit axiom w.r.t. μ and ϵ respects the counit axiom w.r.t. δ. Suppose that the Frobenius condition holds.*
Then, $((A, \mu, \eta)\epsilon)$ is a Frobenius algebra. Also, δ is coassociative (and hence (A, δ, ϵ) is a coalgebra).

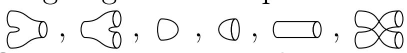
6 Frobenius Algebras and 2TQFTs

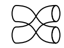
The similarities between our treatment of 2-cobordisms and the graphical calculus of Frobenius algebras are not a coincidence. They motivate us to try and relate **2Cob** with $\mathbf{cFA}_{\mathbb{K}}$. Or, as we shall see, **2TQFT** with $\mathbf{cFA}_{\mathbb{K}}$.

Definition 74. An *n-topological quantum field theory (TQFT)* is a symmetric monoidal functor $Z : \mathbf{nCob} \to \mathbf{Vect}_{\mathbb{K}}$.

The category ***nTQFT*** is the category whose objects are n-TQFTs and arrows are monoidal natural transformations.

We first consider the category $\overline{\mathbf{2TQFT}}$ of functors which are in everything like the objects of **2TQFT** but whose domain is the skeleton $\underline{\mathbf{2Cob}}$ of **2Cob**.

It is clear that a monoidal functor $Z : \underline{\mathbf{2Cob}} \to \mathbf{Vect}_{\mathbb{K}}$ is completely determined by its action on the generators of $\underline{\mathbf{2Cob}}$ and on $1 = \Sigma \cong S^1$. This amounts to assigning a vector space A to $1 \in \underline{\mathbf{2Cob}}$ and a linear map to each generator  .

Since $Z \in \underline{\mathbf{2TQFT}}$ is symmetric, the image of  has to be the twist σ of $\mathbf{Vect}_{\mathbb{K}}$.

Recall that a (non-minimal) complete set of relations of the standard generating set of $\underline{\mathbf{2Cob}}$ is composed by:

1. Identity relations
2. disc-sewing relations
3. Frobenius relations
4. associativity and coassociativity relations
5. commutativity and cocommutativity relations

But it is not hard to check that the associativity and coassociativity conditions follow from the Frobenius condition together with the disc-sewing relations – just do what we did in graphical language in the proof of Theorem 73. Thus, a complete set of relations for the standard generating set of $\underline{\mathbf{2Cob}}$ is:

1. Identity relations
2. Disc-sewing relations
3. Frobenius relations
4. Commutativity and cocommutativity relations

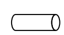
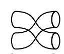
Recall also that, from the characterization that we saw of Frobenius algebras, we know that:

Given a vector space A and maps

$$\begin{aligned} &\mu : A \otimes A \to A \\ &\eta : \mathbb{K} \to A \\ &\delta : A \to A \otimes A \\ &\epsilon : A \to \mathbb{K} \end{aligned} \tag{6}$$

about which we assume nothing except that the unit and counit axioms and the Frobenius condition hold, then (A, ϵ) is a Frobenius algebra (and this is a iff). And if we add the commutativity axiom, then we get a commutative Frobenius algebra.
This characterization means that the maps in (6) respect the conditions/axioms:

1. Identity conditions (trivially)
2. Unit and counit axioms
3. Frobenius condition
4. Commutativity and cocummutativity axiom

The fundamental realization is: if we assign a vector space and maps $A, \mu, \delta, \epsilon, \eta, id_A, \sigma$ to $1 = \Sigma$,  ,  ,  ,  ,  ,  , respectively, then A is a commutative Frobenius algebra – notice how the lists of relations and axioms above coincide when one makes this identification!
This amounts to taking an element Z of $\underline{\mathbf{2TQFT}}$ and defining $\mathcal{F}(Z) = (A, \mu, \delta, \epsilon, \eta)$ by $A := Z(1)$ with $\mu := Z($$)$ and so on.
Conversely, if we take a Frobenius algebra $(A, \mu, \delta, \epsilon, \eta) \in \mathbf{cFA}_{\mathbb{K}}$ and construct a functor $\mathcal{G}((A, \mu, \delta, \epsilon, \eta)) = Z : \underline{\mathbf{2Cob}} \to \mathbf{Vect}_{\mathbb{K}}$ given by $Z(1) = A$, $Z($$) = \mu$ and so on, then we get a symmetric monoidal functor $Z \in \underline{2TQFT}$, and it is clear that $\mathcal{G} = \mathcal{F}^{-1}$ (on objects).

All in all, we proved the following result:

Theorem 75. *$\underline{\mathbf{2TQFT}}_{\mathbb{K}}$ is isomorphic to $\mathbf{cFA}_{\mathbb{K}}$.*

Now, since $\underline{\mathbf{2Cob}}$ is equivalent to $\mathbf{2Cob}$, then $\underline{\mathbf{2TQFT}}_{\mathbb{K}}$ is equivalent to $\mathbf{2TQFT}_{\mathbb{K}}$. This gives us the following corollary:

Corollary 76. *$\mathbf{2TQFT}_{\mathbb{K}}$ is equivalent to $\mathbf{cFA}_{\mathbb{K}}$.*

References

[1] Jiri Adámek, Horst Herrlich, George E. Strecker. *Abstract and Concrete Categories - The Joy of Cats*. John Wiley and Sons, 2004.

[2] Harold Simmons. *An introduction to Category Theory*. Cambridge University Press, 2011.

[3] Tom Leinster. *Basic Category Theory*. Cambridge University Press, 2014.

[4] "Unapologetic Mathematician". https://unapologetic.wordpress.com/2007/06/28/monoidal-categories/

[5] "Unapologetic Mathematician". https://unapologetic.wordpress.com/2007/06/27/categorification/

[6] John C. Baez, Aaron Lauda. *A Prehistory of n-Categorical Physics*. pp. 19-21

[7] Mac Lane. *Categories for the Working Mathematician*, pp. 157-160.

[8] "The nLab". https://ncatlab.org/nlab/show/natural+isomorphism

[9] "The nLab". https://ncatlab.org/nlab/show/monoidal+category

[10] "The nLab". https://ncatlab.org/nlab/show/Mac+Lane%27s+proof+of+the+coherence+theorem+for+monoidal+categories

[11] "The nLab". https://ncatlab.org/nlab/show/HomePage

[12] "The nLab". https://ncatlab.org/nlab/show/tensor+product+of+Banach+spaces

[13] British Columbia Lecture Notes. http://www.math.ubc.ca/~feldman/m511/completion.pdf

[14] "The nLab". https://ncatlab.org/nlab/show/braided+monoidal+category

[15] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*

[16] Loring Tu. *An introduction to Manifolds*

[17] "The nLab". https://ncatlab.org/nlab/show/Frobenius+algebra