# Axiomatic Conformal Field Theory

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**Abstract**

This text presents an axiomatic treatment of conformal field theories following [5],[3] and [4] closely. After defining what a conformal field theory is (by giving a long list of data together with a long list of axioms), we prove some first results about them, discuss a simple example in detail and follow with a discussion about how primary fields and the Virasoro algebra(s) show up in this axiomatic approach to conformal field theory. We end with a few important restrictions that the correlators of a conformal field theory are subject to.

# Contents

# 1 Motivation

In the physics literature, a conformal field theory (CFT) is a QFT whose action is invariant under conformal transformations. There are many important conformal field theories in physics. Namely, the bosonic string theory is a CFT [1].

This led mathematicians to explore conformal transformations in detail. See [2], for example. These approaches are conceptually very close to what physicists do, and try to make rigorous concepts that physicists use non-rigorously like "quantum field", using heavy mathematical machinery like "operator-valued distributions". Besides being quite technical, this approach has another downside: the QFT correlators - which give us the all-important scattering amplitudes - are very hard to compute in most cases.

The way CFT is treated in this text is an axiomatization of the CFT discussed above in the sense that we define CFT using mathematical objects and axioms which try to capture the essential properties of the objects of a "standard" CFT. For example, instead of delving into the theory of operator-valued distributions and other complicated stuff in order to describe what "quantum fields" are in a rigorous way, we will just say that the "space of fields" of our system is a complex vector space with certain properties. Also, in this new axiomatized light, we won't need the notion of action at all, since its properties are (due to conformal symmetry) encapsulated in the components $T, \bar{T}$ of the stress tensor [1][5], so that we will take those components as ingredients of a CFT and their key properties as axioms.

As we will see, with this axiomatic approach, we will recover important structures that one gets when taking the "standard" approach [2] to CFT, like two Virasoro algebras. Very importantly, we get contraints in our correlators which can be useful when the standard treatment (using the framework of QFT and "standard" CFT) does not work or is too complicated.

# 2 Ingredients for a CFT

**Definition 1.** A *(closed string) worldsheet* $X$ is either a closed disc $D \subset \mathbb{R}$ or the plane $\mathbb{R}^2$, minus $d_1 \cup ... \cup d_n \cup \{p_1, ..., p_m\}$ with $(d_1, ..., d_n)$ an ordered set of $n$ open discs (called *marked disks*) in $\mathring{X}$ whose closures are mutually disjoint, and $(p_1, ..., p_m)$ an ordered set of distinct points (called *marked*

*points of* $X$) in $\mathring{X}$.
We denote such a worldsheet by $D(p_1, ..., p_m; d_1, ..., d_n)$ or $\mathbb{R}^2(p_1, ..., p_m; d_1, ..., d_n)$.

**Definition 2.** An *allowed circle* in the worldsheet $X(p_1, ..., p_m; d_1, ..., d_n)$ is a circle $b \subset \mathring{X} - \{p_1, ..., p_m\}$.
Such a circle splits $X$ into the so called *inner worldsheet* $X_i(b) := D(p_I; d_J)$ with $\partial D = b$, where the index sets $I, J$ refer to the marked points and discs inside $b$; and the so called *outer worldsheet* $X_o(b) := X(p_K; d_L, b)$, where the index sets $I, J$ refer to the marked points and discs outside $b$.

A conformal field theory (CFT) will be a bunch of stuff respecting a bunch of axioms. The ingredients are the following:

(i) A countable set $S \subset \mathbb{R}$ with $2 \in S$. We call it the *spectrum*.

(ii) An $S$-graded complex vector space

$$\mathcal{F} = \bigoplus_{\Delta \in S} \mathcal{F}_\Delta$$

with each $\mathcal{F}_\Delta$ finite dimensional. $\mathcal{F}$ is called the *space of fields*.
**Notation:**

$$\bar{\mathcal{F}} = \prod_{\Delta \in S} \mathcal{F}_\Delta \qquad\qquad \check{\mathcal{F}} = \bigoplus_{\Delta \in S} (\mathcal{F}_\Delta)^* \subseteq \mathcal{F}^*$$

and $P_\Delta$ is the projection $\mathcal{F} \to \mathcal{F}_\Delta$.

(iii) A linear map $d : \mathcal{F} \to \mathcal{F}$ compatible with the $S$-grading and such that the $\Delta \in S$ are generalized eigenvalues of $d|_{\mathcal{F}_\Delta}$ with corresponding generalized eigenspaces $\mathcal{F}_{[\Delta]} = \mathcal{F}_\Delta$. Explicitly:

- $\forall \Delta \in S,\ d(F_\Delta) \subset \mathcal{F}_\Delta$
- $\mathcal{F}_\Delta = \mathcal{F}_{[\Delta]} := \{v \in \mathcal{F} |\ \exists n \in \mathbb{N}^+ : (d - \Delta \cdot I)^n|_{\mathcal{F}_\Delta} v = 0\}$

The map $d$ is called *generator of scale transformations*.

(iv) Two vectors (fields) $T, \bar{T} \in \mathcal{F}$. These are called the *holomorphic component of the stress tensor* and the *anti-holomorphic component of the stress tensor*, respectively.

(v) For each worldsheet $X(p_1, ..., p_m; d_1, ..., d_n)$ with at least one marked point or marked disk, a map

$$\mathcal{A}_{X(p_1,...,p_m;d_1,...,d_n)} : \mathcal{F}^{m+n} \to \mathbb{C}$$

if $X(p_1, ..., p_m; d_1, ..., d_n) = \mathbb{R}^2(p_1, ..., p_m; d_1, ..., d_n)$ or a map

$$\mathcal{A}_{X(p_1,...,p_m;d_1,...,d_n)} : \mathcal{F}^{m+n} \to \bar{\mathcal{F}}$$

if $X(p_1, ..., p_m; d_1, ..., d_n) = D(p_1, ..., p_m; d_1, ..., d_n)$, with $D \subset \mathbb{R}^2$ a closed disc.

The maps $\mathcal{A}_{X(p_1,...,p_m;d_1,...,d_n)}$ are multilinear and smooth. By smooth here we mean that:

- If $X(p_1, ..., p_m; d_1, ..., d_n) = \mathbb{R}^2(p_1, ..., p_m; d_1, ..., d_n)$: smoothness means $\forall \phi_1, ..., \phi_{m+n} \in \mathcal{F}$, $\mathcal{A}_{X(p_1,...,p_m;d_1,...,d_n)}(\phi_1, ..., \phi_{m+n})$ is smooth in $p_1, ..., p_m$.
- If $X(p_1, ..., p_m; d_1, ..., d_n) = D(p_1, ..., p_m; d_1, ..., d_n)$: smoothness means $\forall u^* \in \check{\mathcal{F}}, \forall \phi_1, ..., \phi_{m+n} \in \mathcal{F}$, $\langle u^*, \mathcal{A}_{X(p_1,...,p_m;d_1,...,d_n)}(\phi_1, ..., \phi_{m+n})\rangle$ is smooth in $p_1, ..., p_m$.

If $X$ is a worldsheet with no marked points, we set $\mathcal{A}_X \in \mathbb{C}$ if $X$ is the plane and $\mathcal{A}_X \in \bar{\mathcal{F}}$ if $X$ is a disk

**Remark 3.** $S$ is the spectrum of $d$.

# 3 Axioms for a CFT

The first axiom creates a bridge between the objects we are calling fields (elements of $\mathcal{F}$) and the objects that physicists call fields (roughly speaking, maps from a worldsheet to a space of operators which act on a Hilbert space). We will refer to fields of this second type as *operator fields*. As we will discuss in Remark 42, we interpret $\mathcal{A}_{D(o_D)}(\phi)$ as $\tilde{\phi}(o_D)$, where $\tilde{\phi}$ is the operator field corresponding (in the physics side) to the field $\phi$. But how exactly does this $\tilde{\phi}$ "corresponds" to $\phi$? The first axiom answers this question (notice that it amounts to saying that $\tilde{\phi}(o_D) = \phi$):

**Axiom 1.** ***State-Field correspondence***
*Let $D$ be a disc with center $0_D$ and $\phi \in \mathcal{F}$. Then $\mathcal{A}_{D(0_D)}(\phi) = \phi$.*

Notice that on the RHS we are implicitly thinking of $\phi$ as $(0, ..., 0, \phi, 0, ...) \in \bar{\mathcal{F}}$.
The next axiom correspond to the fact that we are trying to model bosons, so that permutations do not affect amplitudes.

**Axiom 2.** ***Permutation Invariance***
*For $\sigma \in S_m$ and $\rho \in S_n$, we have*

$$\mathcal{A}_{D(p_1,...,p_m;d_1,...,d_n)}(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$$

$$= \mathcal{A}_{D(p_{\sigma(1)},...,p_{\sigma(m)};d_{\rho(1)},...,d_{\rho(n)})}(\phi_{\sigma(1)}, ..., \phi_{\sigma(m)}, \psi_{\rho(1)}, ..., \psi_{\rho(n)})$$

**Definition 4.** For $(v_i)$ a countable sequence in $\bar{F}$, we say that $\sum_i v_i$ *converges absolutely* if $\forall u^* \in \check{\mathcal{F}}$, $\sum_i \langle u^*, v_i \rangle$ converges absolutely in $\mathbb{C}$.

**Axiom 3.** ***Factorization***
*Let $X = D(p_1, ..., p_m; d_1, ..., d_n)$ be a worldsheet and $b$ be an allowed circle such that the marked points and disks in $D$ that lie outside $b$ are $p_1, ..., p_k$ and $d_1, ..., d_l$, respectively. Then*

$$\sum_{\Delta \in S} \mathcal{A}_{X_o(b)}(\phi_1, ..., \phi_k, \psi_1, ..., \psi_l, P_\Delta \mathcal{A}_{X_i(b)}(\phi_{k+1}, ..., \phi_m, \psi_{l+1}, ..., \psi_n))$$

*converges absolutely to $\mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$.*

**Remark 5.** This axiom would not make sense for $X$ a plane worldsheet, since then $\mathcal{A}_{X_i(b)}(\phi_{k+1}, ..., \phi_m, \psi_{l+1}, ..., \psi_n)$ is a number and cannot be inserted as a field in the other amplitude.

Notice that $\mathcal{A}_{X_o(b)} : \mathcal{F}^{k+l+1} \to (\mathbb{C} \text{ or } \bar{\mathcal{F}})$, taking as arguments 1 field for each point and one field for each disk in $X_0(b)$, including one field for the disk $b$ itself. So this axiom gives us a way to plug the amplitude coming from the inner worldsheet (that we cut out) back in and get something converging absolutely to the full amplitude.

**Remark 6.** Since the sum in Axiom 3 will always be finite (because there can only be a finite number of nonzero spaces $\mathcal{F}_\Delta$ in the direct sum $\sum_\Delta \mathcal{F}_\Delta$), we could write equality instead of absolute convergence. But this way of writing this axiom is more easily generalizable to other approaches where we may want to let $S$ be uncountable, in which case the sum above need not be finite.

**Axiom 4.** ***Translation invariance***
*Let $a \in \mathbb{R}^2$ and $X(p_1, ..., p_m; d_1, ..., d_n)$ be a worldsheet. Then*

$$\mathcal{A}_X = \mathcal{A}_{X'}$$

*where $X'$ is the translated worldsheet $X' = (X+a)(p_1+a, ..., p_m+a; d_1+a, ..., d_n+a)$.*

**Axiom 5.** ***Scale covariance***
*Let $\mathbb{R}^2(p_1, ..., p_m; d_1, ..., d_n)$ be a worldsheet and $\lambda \in \mathbb{R}^+$. Then*

$$\mathcal{A}_{\mathbb{R}^2(p_1,...,p_m;d_1,...,d_n)}(\phi_1, ..., \phi_{m+n}) = \mathcal{A}_{\mathbb{R}^2(\lambda p_1,...,\lambda p_m;\lambda d_1,...,\lambda d_n)}(\lambda^d\phi_1, ..., \lambda^d\phi_{m+n})$$

*where $d : \mathcal{F} \to \mathcal{F}$ is the generator of scale transformations and $\lambda^d := e^{\log(\lambda)d} : \mathcal{F} \to \mathcal{F}$.*
*Now let $D(p_1, ..., p_m; d_1, ..., d_n)$ be a worldsheet. Then*

$$\lambda^d \cdot \mathcal{A}_{D(p_1,...,p_m;d_1,...,d_n)}(\phi_1, ..., \phi_{m+n}) = \mathcal{A}_{(\lambda D)(\lambda p_1,...,\lambda p_m;\lambda d_1,...,\lambda d_n)}(\lambda^d\phi_1, ..., \lambda^d\phi_{m+n})$$

*with $\lambda^d \cdot : \bar{\mathcal{F}} \to \bar{\mathcal{F}}$ given by acting component-wise: $\lambda^d \cdot (\phi_{\Delta_1}, \phi_{\Delta_2}, ...) = (\lambda^d\phi_{\Delta_1}, \lambda^d\phi_{\Delta_2}, ...)$.*

Before introducing the last axiom, we need to introduce some definitions and notation regarding complex functions, derivatives of fields and the identity field.

We denote contour integrals along a circle $b$ parametrized smoothly by $\gamma : I \to \mathbb{C}$ as follows:

$$\oint_b f dz := \frac{1}{2\pi i}\int_0^1 f(\gamma(t))\gamma'(t)dt$$

$$\oint_b f d\bar{z} := \frac{1}{2\pi i}\int_0^1 f(\gamma(t))\overline{\gamma'(t)}dt$$

Also:

$$\partial := \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - i\frac{1}{2}\frac{\partial}{\partial y}$$

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + i\frac{1}{2}\frac{\partial}{\partial y}$$

**Definition 7.** Let $\phi \in \mathcal{F}$. A *derivative field* for $\phi$ is a field $\partial\phi \in \mathcal{F}$ such that $\forall u^* \in \check{\mathcal{F}}, \forall X = D(z, p_1, ..., p_m; d_1, ..., d_n)$,

$$\frac{\partial}{\partial z}\langle u^*, \mathcal{A}_X(\phi, \phi_1, ..., \phi_{m+n})\rangle = \langle u^*, \mathcal{A}_X(\partial\phi, \phi_1, ..., \phi_{m+n})\rangle$$

Similarly, a *conjugate derivative field* for $\phi$ is a field $\bar{\partial}\phi \in \mathcal{F}$ such that $\forall u^* \in \check{\mathcal{F}}, \forall X = D(z, p_1, ..., p_m; d_1, ..., d_n)$,

$$\frac{\partial}{\partial \bar{z}}\langle u^*, \mathcal{A}_X(\phi, \phi_1, ..., \phi_{m+n})\rangle = \langle u^*, \mathcal{A}_X(\bar{\partial}\phi, \phi_1, ..., \phi_{m+n})\rangle$$

**Definition 8.** Let $D_1$ be the closed unit disk in $\mathbb{R}^2$. The *identity field* is $\mathbf{1} := \mathcal{A}_{D_1} \in \bar{\mathcal{F}}$. The *out-vacuum* is $\Omega^* := \mathcal{A}_{\mathbb{R}^2(D_1)} \in \mathcal{F}^*$.

Notice that $\mathcal{A}_{D_1} \in \bar{\mathcal{F}}$ because, since there are no marked points or discs in the worldsheet $D_1$, then $\mathcal{A}_{D_1}$ is a map $\mathcal{F}^0 \to \bar{\mathcal{F}}$.

Let's see how to integrate over $\mathcal{F}$:

**Definition 9.** A map $f : \mathbb{C} \to \bar{\mathcal{F}}$ can be written

$$f(z) = \bigoplus_{i\in I} f_i(z)\phi_i$$

with $\{\phi_i\}_{i\in I}$ a graded basis for $\mathcal{F}$ and the $f_i$ maps $\mathbb{C} \to \mathbb{C}$. The function $f$ is said to be *holomorphic* if all $f_i$ are holomorphic. To *integrate* or *differentiate* $f$, we simply integrate or differentiate each component $f_i$.

Finally:

**Definition 10.** We define, for $m \in \mathbb{Z}$ and $D$ a disk centered at 0, functions $L_m^D : \mathcal{F} \to \bar{\mathcal{F}}$ and $\bar{L}_m^D : \mathcal{F} \to \bar{\mathcal{F}}$ given by

$$L_m^D(\phi) = \oint_{\partial D} z^{m+1}\mathcal{A}_{D(z,0)}(T, \phi)dz$$

$$\bar{L}_m^D(\phi) = \oint_{\partial D} \bar{z}^{m+1} \mathcal{A}_{D(z,0)}(\bar{T}, \phi) d\bar{z}$$

These are called the *Virasoro generators.*

**Remark 11.** It will turn out that he radius of the disk $D$ in the above definition does not matter (see Proposition 15), so we can omit the label $D$ and write simply $L_m, \bar{L}_m$.

We are now ready to state the final axiom:

**Axiom 6.** ***Stress tensor***

*(i) Either $T = \bar{T} = 0$, or $2 \in S$ and $T, \bar{T} \in \mathcal{F}_2$.*

*(ii) The (conjugate) field derivatives $\partial T, \partial \bar{T}, \bar{\partial} T, \bar{\partial} \bar{T}$ exist and $\partial \bar{T} = \bar{\partial} T = 0$.*

*(iii) Take a worldsheet $\mathbb{R}^2(z, p_1, ..., p_m; d_1, ..., d_n)$ and let $R > |p_i|$ $(\forall p_i)$. Then the function*

$$f(z) := |z|^4 |\mathcal{A}_{\mathbb{R}^2(z,p_1,...,p_m;d_1,...,d_n)}(T, \phi_1, ..., \phi_{m+n})|$$

*is bounded for $|z| > R$. And similarly for $\bar{T}$.*

*(iv) There are constants $c, \tilde{c} \in \mathbb{C}$ (called central charges) such that for all $p \neq q \in \mathbb{R}^2 \cong \mathbb{C}$ and for all worldsheets $D(p, q)$ with $q$ in the center of the disk $D$:*

$$\mathcal{A}_{D(p,q)}(T, T) = \frac{\frac{c}{2}}{(p-q)^4} \mathbf{1} + \frac{2}{(p-q)^2} T + \frac{1}{p-q} \partial T + f_1(p-q)$$
$$\mathcal{A}_{D(p,q)}(\bar{T}, \bar{T}) = \frac{\frac{\tilde{c}}{2}}{(\bar{p}-\bar{q})^4} \mathbf{1} + \frac{2}{(\bar{p}-\bar{q})^2} \bar{T} + \frac{1}{\bar{p}-\bar{q}} \bar{\partial} \bar{T} + f_2(\bar{p}-\bar{q})$$
$$\mathcal{A}_{D(p,q)}(T, \bar{T}) = f_3(p-q)$$

*with $f_1, f_2, f_3$ holomorphic functions.*

*(v) Let $\phi \in \mathcal{F}$. Then*

$$\partial\phi = L_{-1}(\phi) \qquad \bar{\partial}\phi = \bar{L}_{-1}(\phi) \qquad d\phi = (L_0 + \bar{L}_0)\phi$$

We are finally ready to put everything together and define a CFT:

**Definition 12.** A *conformal field theory (CFT)* is a collection of objects as described in Section 2 respecting the axioms in Section 3.

As we will see, we can instead take the list of objects in Section 5 respecting the axioms in Section 3 (adapted to the new objects).

# 4 First results

The state-field correspondence guarantees the uniqueness of the derivative of a field:

**Proposition 13.** *Let $\phi \in \mathcal{F}$. Its derivative $\partial\phi$ is unique. Its conjugate derivative $\bar{\partial}\phi$ is also unique.*

*Proof.* Let $\partial\phi$ and $\partial\phi'$ be two derivative fields of $\phi$, and $u^* \in \check{\mathcal{F}}$. Consider also a worldsheet $D(z)$, with $D$ centered at zero. Then

$$\langle u^*, \partial\phi\rangle = \langle u^*, \mathcal{A}_{D(0)}(\partial\phi)\rangle = \langle u^*, \mathcal{A}_{D(z)}(\partial\phi)\rangle\big|_{z=0} = \frac{\partial}{\partial z}\Big|_{z=0}\langle u^*, \mathcal{A}_{D(z)}(\phi)\rangle$$

where I used the state-field correspondence and the definition of derivative operator. But since $\partial\phi'$ is also a derivative operator of $\phi$, then

$$\frac{\partial}{\partial z}\Big|_{z=0}\langle u^*, \mathcal{A}_{D(z)}(\phi)\rangle = \langle u^*, \mathcal{A}_{D(z)}(\partial\phi')\rangle\big|_{z=0} = \langle u^*, \mathcal{A}_{D(0)}(\partial\phi')\rangle = \langle u^*, \partial\phi'\rangle$$

Since this argument is valid for any $u^* \in \check{\mathcal{F}}$, then $\partial\phi = \partial\phi'$. □

It turns out that replacing a marked point of a worldsheet with a disk centered at that point does not affect the amplitude:

**Proposition 14.** *Let $X(p_1, ..., p_m; d_1, ..., d_n, d)$ and $X'(p_1, ..., p_m, o_d; d_1, ..., d_n)$ be two worldsheets, with $o_d$ the center of $d$. Then for all $\phi_1, ..., \phi_m, \phi, \psi_1, ..., \psi_n \in \mathcal{F}$ we have*

$$\mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n, \phi) = \mathcal{A}_{X'}(\phi_1, ..., \phi_m, \phi, \psi_1, ..., \psi_n)$$

*Proof.* By definition of the projections $P_\Delta$, we can write

$$\phi = \bigoplus_{\Delta\in S} P_\Delta \phi = \bigoplus_{\Delta\in S} P_\Delta \mathcal{A}_{d(o_d)}(\phi)$$

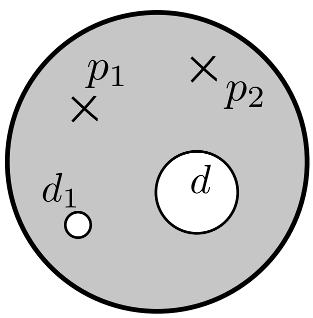
where in the last equality I used the state-field correspondence axiom. Making now the usual identification $\mathcal{F}_\Delta \cong \{0\} \oplus ... \oplus \{0\} \oplus \mathcal{F}_\Delta \oplus \{0\} \oplus ...$, we write

$$\phi = \sum_{\Delta\in S} P_\Delta \mathcal{A}_{d(o_d)}(\phi)$$

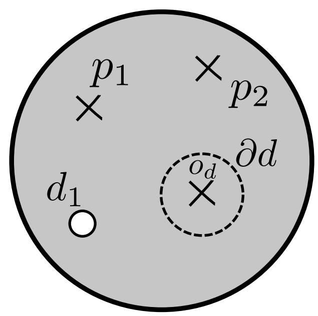
Notice that, since only a finite number of $\mathcal{F}_\Delta$ are nonzero, then only a finite number of terms in this sum will be nonzero, so that we can take it to be finite. Hence we can use the linearity of the amplitude to get

$$\mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n, \phi) = \sum_{\Delta\in S} \mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n, P_\Delta \mathcal{A}_{d(o_d)}(\phi))$$

Now, it is clear by the definition of marked disk that $\partial d$ is an allowed circle for $X' = X \cup d(o_d)$. Notice that $X = X'_o(\partial d)$ and $d(o_d) = X'_i(\partial d)$.




$X = D(p_1, p_2; d_1, d)$




$X' = D(p_1, p_2, o_d; d_1)$

Hence we can use the factorization axiom to conclude that

$$\sum_{\Delta\in S} \mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n, P_\Delta \mathcal{A}_{d(o_d)}(\phi))$$

converges absolutely to

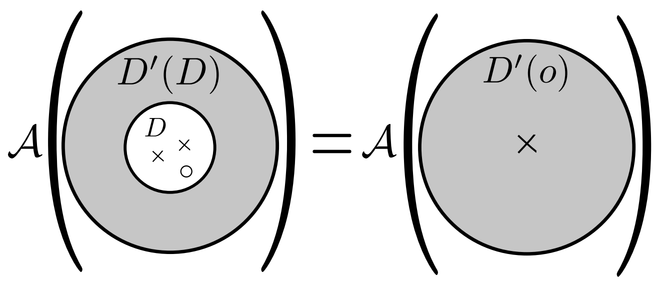
$$\mathcal{A}_X(\phi_1, ..., \phi_m, \psi_1, ..., \psi_n, \phi) = \mathcal{A}_{X'}(\phi_1, ..., \phi_m, \phi, \psi_1, ..., \psi_n)$$

and since the sum is finite we actually have equality. This proves the assertion. $\square$

We also don't affect the amplitude if we replace a disk worldsheet with a second disk worldsheet with the same marked disks and points as the first, as long as the disk worldsheets have the same center:

**Proposition 15.** *Let $D, D' \subset \mathbb{R}^2$ be disks with the same center but possibly different radius. Consider the worldsheets $X = D(p_1, ..., p_m; d_1, ..., d_n)$ and $X' = D'(p_1, ..., p_m; d_1, ..., d_n)$. Then $\mathcal{A}_X = \mathcal{A}_{X'}$.*

*Proof.* It suffices to show that all components are equal, *i.e.* $\forall \Delta \in S,\ P_\Delta \mathcal{A}_X = P_\Delta \mathcal{A}_{X'}$.
Denote the center of the disks by $o$. By Proposition 14 we have that $\mathcal{A}_{D'(D)}(\phi) = \mathcal{A}_{D'(o)}(\phi) = \phi$, where the last equality comes from the state-field correspondence.




Notice that $\partial D$ is an allowed circle in the worldsheet $X'$. Furthermore, $X'_o(\partial D) = D'(D)$ and $X'_i(\partial D) = X$. We can therefore use the factorization axiom to get:

$$\begin{aligned}
P_\Delta \mathcal{A}_{X'}(\phi_1, ..., \phi_{m+n}) &= P_\Delta \sum_{\Delta'} \mathcal{A}_{D'(D)}(P_{\Delta'} \mathcal{A}_X(\phi_1, ..., \phi_{m+n})) \\
&= \sum_{\Delta'} P_\Delta \mathcal{A}_{D'(0)}(P_{\Delta'} \mathcal{A}_X(\phi_1, ..., \phi_{m+n})) \\
&= \sum_{\Delta'} \delta_{\Delta\Delta'} P_\Delta \mathcal{A}_X(\phi_1, ..., \phi_{m+n}) \\
&= P_\Delta \mathcal{A}_X(\phi_1, ..., \phi_{m+n})
\end{aligned}$$

where in the second equality we used linearity of the projections (together with the fact that the sum is finite, as we saw in previous proofs) and that $\mathcal{A}_{D'(D)} = \mathcal{A}_{D'(o)}$ (as we saw above); and in the third equality we used that $\mathcal{A}_{D'(o)} = id_{\mathcal{F}}$ (by the state-field correspondence) and that the $P_\Delta$ are projections. □

We now will now introduce the all-important operator product expansion.

**Definition 16.** The *operator product expansion (OPE)* is the map

$$\begin{aligned} m : \mathbb{C} \setminus \{0\} \times \mathcal{F} \times \mathcal{F} &\to \bar{\mathcal{F}} \\ (z, \phi, \psi) &\mapsto \mathcal{A}_{D(z,0)}(\phi, \psi) \end{aligned}$$

where $D$ is a disk centered at 0 and with radius $r > |z|$.

**Remark 17.** By Proposition 15 we see that the OPE is well defined (it does not depend on the $r$ chosen).
Notice that, since the amplitudes are smooth, so are OPEs, where smoothness here has the same meaning as it had for amplitudes.

Given a graded basis $\{\phi_i\}_{i \in I}$ of $\mathcal{F}$ (so that $\forall i \in I,\ \exists \Delta \in S:\ \phi_i \in \mathcal{F}_\Delta$), we get an induced basis of $\bar{\mathcal{F}}$ which we still denote $\{\phi_i\}_{i\in I}$ (although a more explicit notation would be $\{(0, \dots, 0, \phi_i, 0, \dots)\}_{i \in S}$). So we can write

$$m_z(\phi, \psi) := m(z, \phi, \psi) = \sum_i f_i(z)\phi_i \in \bar{\mathcal{F}}$$

**Definition 18.** Let $\{\phi_i\}_{i\in I}$ be a graded basis of $\mathcal{F}$. The *Wilson coefficients of* $\phi(p)\psi(q)$ (w.r.t. this graded basis) are the coefficients $f_i$ in

$$m_{p-q}(\phi, \psi) = \sum_i f_i(p-q)\phi_i$$

**Proposition 19.** *Let $D(p, q)$ be a worldsheet. Then $m_{p-q}(\phi, \psi) = \mathcal{A}_{D(p,q)}(\phi, \psi)$.*

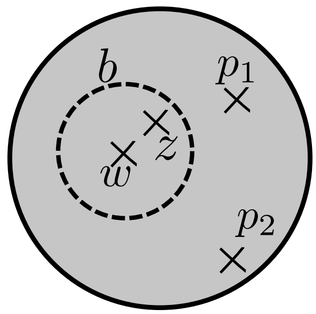
*Proof.* Let $D(p, q)$ be a worldsheet. Notice that the points 0 and $p - q$ are in the translated disk $D' = D - q$. so we can write $m_{p-q}(\phi, \psi) = \mathcal{A}_{D'(0,p-q)}$. But $\mathcal{A}_{D'(0,p-q)} = \mathcal{A}_{(D'+q)(q,p)} = \mathcal{A}_{D(p,q)}$, where in the first equality I used translation invariance and in the second I used permutation invariance. □

**Theorem 20.** *Let $f_i$ be the Wilson coefficients of $\phi(p)\psi(q)$ w.r.t. the graded basis $\{\phi_{i\in I}\}$ of $\mathcal{F}$. Let $p_1, ..., p_m, z, w$ be distinct points in $\mathbb{R}^2 \cong \mathbb{C}$ such that $\forall i \in I,\ |z - w| < |p_i - w|$. Then for all $v_1, ..., v_m, \phi, \psi \in \mathcal{F}$ we have*

$$\mathcal{A}_{D(p_1,\dots,p_m,z,w)}(v_1, ..., v_m, \phi, \psi) = \sum_{i\in I} f_i(z-w)\mathcal{A}_{D(p_1,\dots,p_m,w)}(v_1, ..., v_m, \phi_i)$$

*and the sum on the RHS converges absolutely.*

*Proof.* By definition, $\mathcal{A}_{d(p,q)}(\phi,\psi) = \sum_i f_i(p-q)\phi_i$, with $d$ any disk containing the distinct points $p, q$. We will use this later.
Let $b$ be an allowed circle centered at $w$ and enclosing only $z$. Such an allowed circle exists by virtue of the condition $|z-w| < |p_i - w|$. We denote by $d_b$ the open disc such that $\partial d_b = b$.




We cut the worldsheet using $b$, so that we find ourselves in a position to use the factorization axiom (with equality instead of just absolute convergence since the sum on the RHS is finite):

$$\begin{aligned}
\mathcal{A}_{D(p_1,\ldots,p_m,z,w)}(v_1,\ldots,v_m,\phi,\psi) &= \sum_\Delta \mathcal{A}_{D(p_1,\ldots,p_2;d_b)}(v_1,\ldots,v_m,P_\Delta \mathcal{A}_{d_b(z,w)}(\phi,\psi)) \\
&= \sum_\Delta \mathcal{A}_{D(p_1,\ldots,p_2;d_b)}(v_1,\ldots,v_m,P_\Delta \sum_{i\in I} f_i(z-w)\phi_i) \\
&= \sum_\Delta f_i(z-w) \sum_{i\in I} \mathcal{A}_{D(p_1,\ldots,p_2;d_b)}(v_1,\ldots,v_m,P_\Delta \phi_i)
\end{aligned}$$

where in the last equality I used the linearity of the amplitude and the finiteness of the sum over $i$. Now each $\phi_i$ is in some $\mathcal{F}_{\Delta_i}$ since $\{\phi_{i\in I}\}$ is a graded basis. Hence $P_\Delta \phi_i = \delta_{\Delta\Delta_i}\phi_i$. Thus

$$\mathcal{A}_{D(p_1,\ldots,p_m,z,w)}(v_1,\ldots,v_m,\phi,\psi) = \sum_{i\in I} f_i(z-w)\mathcal{A}_{D(p_1,\ldots,p_2;d_b)}(v_1,\ldots,v_m,\phi_i)$$

We get the result by using Proposition 14. □

As we will see later, this theorem will allow us to extract all amplitudes from the OPE.
In the rest of the text, it will be useful to know that:

**Lemma 21.** *$\mathbf{1} \in \mathcal{F}_0$ and $\Omega^* \in \mathcal{F}_0^*$.*

*Proof.* Omitted. □

## The Virasoro generators

We defined in Definition 10 the Virasoro generators $L_m, \bar{L}_m$. We will show that $L_m\phi, \bar{L}_m\phi$ play a role analogous to Laurent coefficients of the product of $T$ and $\phi$ (or $\bar{T}$ and $\phi$). This should come as no surprise, since we defined the Virasoro generators with an expression that is analogous to the expression that we use to extract the Laurent coefficients of a complex function – although $m_{(-)}(T,\phi) : \mathbb{C} \to \bar{\mathcal{F}}$ is not a complex function, and the integral $\oint$ is not common integral of a complex function, but is as defined in Definition 9.

**Lemma 22.** *Let $\phi \in \mathcal{F}_\Delta, z \in \mathbb{C}\setminus\{0\}$ and $\Delta' \in S$. Then:*

*(i)* $$P_{\Delta'} m_z(T,\phi) = \begin{cases} z^{\Delta'-\Delta-2} P_{\Delta'} m_1(T,\phi) & \text{if } \Delta' - \Delta \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

*(ii)* $$P_{\Delta'} L_m(\phi) = \begin{cases} P_{\Delta'} m_1(T,\phi) & \text{if } m = \Delta - \Delta' \\ 0 & \text{otherwise} \end{cases}$$

*The same holds for $\bar{L}_m$.*

*Proof.* Omitted. □

**Corollary 23.** *The Virasoro generators $L_m|_{\mathcal{F}_\Delta}, \bar{L}_m|_{\mathcal{F}_\Delta}$ are graded maps of degree $-m$. (So if $\Delta - m \notin S$, then $L_m\phi = 0$ for every $\phi \in \mathcal{F}_\Delta$).*

*Proof.* Just notice that $L_m|_{\mathcal{F}_\Delta}, \bar{L}_m|_{\mathcal{F}_\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_{\Delta-m}$ by the above Lemma. □

**Remark 24.** When we write $L_m|_{\mathcal{F}_\Delta}, \bar{L}_m|_{\mathcal{F}_\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_{\Delta-m}$, it is to be understood that we are identifying $\mathcal{F}_{\Delta-m}$ with $0 \oplus \ldots \mathcal{F}_{\Delta-m} \oplus 0 \oplus \ldots$ if $\Delta - m \in S$, or $\bigoplus 0$ if $\Delta - m \notin S$. In particular, if $\Delta - m \notin S$, this does not mean that $L_m$ is ill-defined! It just means that $L_m\phi = 0$ for $\phi \in \mathcal{F}_\Delta$

**Proposition 25.** *Let $\phi \in \mathcal{F}$. Then*

$$m_z(T,\phi) = \sum_{m\in\mathbb{Z}} z^{-m-2} L_m\phi \qquad m_z(\bar{T},\phi) = \sum_{m\in\mathbb{Z}} \bar{z}^{-m-2} \bar{L}_m\phi$$

*Proof.* We will show the first expression. The second is proved analogously. We will prove it for $\phi \in \mathcal{F}_\Delta$ and projecting onto $\mathcal{F}_{\Delta'}$. Then, by linearity ($m_z(T,-)$ and $L_m$ are linear and any field is a finite sum of fields from each $\mathcal{F}_\Delta$) and since the RHS and the LHS agree on every component (in $\mathcal{F}'_\Delta$), they are equal.
Have:

$$\begin{aligned}
P_{\Delta'}\left(\sum_{m\in\mathbb{Z}} z^{-m-2}L_m\phi\right) &= \sum_{m\in\mathbb{Z}} z^{-m-2}P_{\Delta'}L_m\phi \\
&= \begin{cases} \sum_{m\in\mathbb{Z}} z^{-m-2}\delta_{m,\Delta-\Delta'}P_{\Delta'}m_1(T,\phi) & ,\Delta-\Delta'\in\mathbb{Z} \\ 0 & ,\text{otherwise} \end{cases} \\
&= \begin{cases} z^{-\Delta+\Delta'-2}P_{\Delta'}m_1(T,\phi) & ,\Delta-\Delta'\in\mathbb{Z} \\ 0 & ,\text{otherwise} \end{cases} \\
&= P_{\Delta'}m_z(T,\phi)
\end{aligned}$$

where in the the first equality I used linearity of the projection and that, since only a finite number of $\mathcal{F}_\Delta$'s are nonzero and by Corollary 23 $L_m\phi \in \mathcal{F}_{\Delta-m}$, then only a finite number of $L_m\phi$ are nonzero, and hence the sum is finite; in the second and fourth equalities I used Lemma 22. □

**Corollary 26.**

$$\begin{aligned}
&L_0T = 2T \qquad L_1T = 0 \qquad L_2T = \frac{c}{2}\mathbf{1} \qquad \forall m\geq 3, L_mT = 0 \\
&\bar{L}_0\bar{T} = 2\bar{T} \qquad \bar{L}_1\bar{T} = 0 \qquad L_2\bar{T} = \frac{\tilde{c}}{2}\mathbf{1} \qquad \forall m\geq 3, \bar{L}_mT = 0 \\
&\qquad\qquad \textit{and } \forall m, \bar{L}_mT = L_m\bar{T} = 0
\end{aligned}$$

*Proof.* Follows directly from the above Proposition together with point (iv) of Axiom 6, by taking $\phi = T$ and $z = p - q$. □

We can use this to get an expression for any amplitude containing the stress tensor:

**Theorem 27.** *Let $X = D(z, p_1, ..., p_n)$ be a worldsheet and $\phi_1, ..., \phi_n \in \mathcal{F}$. Then for all $p_i, \in \{1, ..., n\}$ such that $\forall k \neq i, |z - p_i| < |p_k - p_i|$, we have*

$$\mathcal{A}_X(T, \phi_1, ..., \phi_n) = \sum_{m\in\mathbb{Z}} (z - p_i)^{-m-2}\mathcal{A}_{X\cup\{z\}}(\phi_1, ..., L_m\phi_i, ..., \phi_n)$$

*And similarly for $\bar{T}$ and $\bar{L}_m$.*

*Proof.* Since any field can be written as a finite sum of fields belonging to a specific $\mathcal{F}_\Delta$ and furthermore $\mathcal{A}_X(\phi_1, ..., -, ..., \phi_n)$ and $L_m$ are linear maps, then it suffices to prove the result for $\phi_i \in \mathcal{F}_\Delta$ for some $\Delta \in S$.
Notice that by our imposition on the distances between the points, there is an allowed circle $b$ centered at $p_i$ containing $z$ and no other marked point of $X$. Using the factorization axiom (and the finiteness of the sum):

$$\begin{aligned}
\mathcal{A}_X(T, \phi_1, ..., \phi_m) &= \sum_{\Delta' \in S} \mathcal{A}_{X_o(b)}(\phi_1, ..., P_{\Delta'}\mathcal{A}_{X_i(b)}(T, \phi_i), ..., \phi_n) \\
&= \sum_{\Delta' \in S} \mathcal{A}_{X_o(b)}(\phi_1, ..., P_{\Delta'} m_{z-p_i}(T, \phi_i), ..., \phi_n) \\
&= \sum_{\Delta' \in S} \mathcal{A}_{X_o(b)}(\phi_1, ..., (z-p_i)^{\Delta'-\Delta-2} P_{\Delta'} m_1(T, \phi_i), ..., \phi_n)\theta(\Delta' - \Delta)
\end{aligned}$$

where I used Lemma 22 and $\theta(s) := \begin{cases} 1 & , s \in \mathbb{Z} \\ 0 & , \text{otherwise} \end{cases}$. We can rewrite the sum using a variable $m \in \mathbb{Z}$ (and setting $\Delta - \Delta' = m$ whenever $\Delta - \Delta' \in \mathbb{Z}$) and linearity:

$$\begin{aligned}
\mathcal{A}_X(T, \phi_1, ..., \phi_m) &= \sum_{m \in \mathbb{Z}} (z-p_i)^{-m-2} \mathcal{A}_{X_o(b)}(\phi_1, ..., P_{\Delta-m} m_1(T, \phi_i), ..., \phi_n) \\
&= \sum_{m \in \mathbb{Z}} (z-p_i)^{-m-2} \mathcal{A}_{X_o(b)}(\phi_1, ..., P_{\Delta-m} L_m \phi_i, ..., \phi_n) \\
&= \sum_{m \in \mathbb{Z}} (z-p_i)^{-m-2} \mathcal{A}_{X_o(b)}(\phi_1, ..., L_m \phi_i, ..., \phi_n)
\end{aligned}$$

where I used the second part of Lemma 22 (notice that we never get the case 0 in this case), and the fact that $L_m \phi \in \mathcal{F}_{\Delta - m}$ (by Corollary 23).
We now simply notice that $X_o(b) = D(p_1, ..., \hat{\phi}_i, ..., p_n; d_b)$ and thus $\mathcal{A}_{X_o(b)} = \mathcal{A}_{(D(p_1,...,p_n)=X \cup \{z\})}$. $\square$

Let's see what happens if we act with the Virasoro generators on an amplitude.

**Lemma 28.** *Let $D(p_1, ..., p_n)$ be a worldsheet. Then*

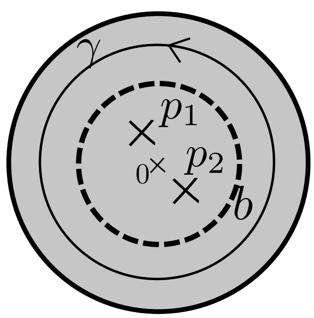
$$\begin{aligned}
L_m \mathcal{A}_{D(p_1,...,p_n)}(\phi_1, ..., \phi_n) &= \oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,...,p_n)}(T, \phi_1, \ldots, \phi_n) dz \\
\bar{L}_m \mathcal{A}_{D(p_1,...,p_n)}(\phi_1, ..., \phi_n) &= \oint_\gamma \bar{z}^{m+1} \mathcal{A}_{D(\bar{z},p_1,...,p_n)}(\bar{T}, \phi_1, \ldots, \phi_n) d\bar{z}
\end{aligned}$$

*with $\gamma$ a curve whose trace is a circle in $D$ containing $p_1, ..., p_n$ and centered at $o_D$.*

*Proof.* We will prove the first equation. The proof of the second is identical. As usual, we can prove that the projections agree. by translation invariance, we can assume that $D$ is centered at the origin 0. We have

$$\begin{aligned}
P_\Delta \oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz &= \oint_\gamma z^{m+1} P_\Delta \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz \\
&= \oint_\gamma z^{m+1} P_\Delta \sum_{\Delta' \in S} \mathcal{A}_{D(z,d_b)}(T, P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz \\
&= \oint_\gamma z^{m+1} \sum_{\Delta' \in S} P_\Delta \mathcal{A}_{D(z,0)}(T, P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz \\
&= \oint_\gamma z^{m+1} \sum_{\Delta' \in S} P_\Delta m_z(T, P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz
\end{aligned}$$

where I used: the linearity and continuity of $P_\Delta$; the factorization axiom; Proposition 14 and Proposition 19, in this order. The factorization axiom was used by cutting the worldsheet using an allowed circle $b$ containing all $p_i$ but smaller than the trace of $\gamma$ (and thus not containg $z$):




So:

$$\begin{aligned}
&P_\Delta \oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz \\
&\qquad = \oint_\gamma z^{m+1} P_\Delta m_z(T, \sum_{\Delta' \in S} P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz \\
&\qquad = \oint_\gamma z^{m+1} P_\Delta m_z(T, \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz
\end{aligned}$$

$$
\begin{aligned}
P_\Delta \oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz \\
&= \sum_{\Delta' \in S} P_\Delta \oint_\gamma z^{m+1} m_z(T, P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n)) dz \\
&= \sum_{\Delta' \in S} P_\Delta L_m \left( P_{\Delta'} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n) \right) \\
&= \sum_{\Delta' \in S} P_\Delta P_{\Delta'-m} L_m \left( \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n) \right) \\
&= P_\Delta L_m \left( \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, \ldots, \phi_n) \right)
\end{aligned}
$$

where we pulled the sum out of the integral (since it is finite); used the definition of the Virasoro generators, followed by $L_m P_\Delta = P_{\Delta-m} L_m$ (direct consequence of Corollary 23); and finally we used that $P_\Delta P_{\Delta'-m} = P_\Delta \delta_{\Delta,\Delta'-m}$. □

It will also be relevant in the rest of our discussion how the Virasoro generators interact with the identity field and the out-vacuum:

**Proposition 29.** *We have*

*(i)* $\forall m \geq -1,\ L_m \boldsymbol{1} = \bar{L}_m \boldsymbol{1} = 0$

*(ii)* $\forall m \leq 1,\ \Omega^* \circ L_m = \Omega^* \circ \bar{L}_m = 0$

*Proof.* Omitted. □

# 5 Alternative recipe for a CFT

Theorem 20 actually allows us to use $m$ to get any amplitude, if one uses the identity field and the out-vacuum field.

To see how, it is essential to understand how the identiy field and the out-vacuum interact with amplitudes. The words in the next theorem make sense if one sees amplitudes of plane worldsheets as expectaction values for the vacuum state - see Section 9 for a discussion on this.

**Theorem 30.** *We have:*

$$
\begin{aligned}
\mathcal{A}_{D(p_1,\ldots,p_m,z)}(\phi_1, \ldots, \phi_m, \boldsymbol{1}) &= \mathcal{A}_{D(p_1,\ldots,p_m)}(\phi_1, \ldots, \phi_m) \\
\Omega^* \left( (\mathcal{A}_{D(p_1,\ldots,p_m)}(\phi_1, \ldots, \phi_m) \right) &= \mathcal{A}_{\mathbb{R}^2(p_1,\ldots,p_m)}(\phi_1, \ldots, \phi_m)
\end{aligned}
$$

*In other words, the identity field acts as a multiplicative identity and the out-vacuum can be viewed as giving the expectatian value in the vacuum state.*

*Proof.* Let $b$ be an allowed circle centered at $z$ and containing no other marked point of $X = D(p_I)$. Then by the factorization axiom we have

$$\mathcal{A}_X(\phi_I) = \sum_{\Delta} \mathcal{A}_{D(p_I;d_b)}(\phi_I, P_\Delta \mathcal{A}_{X_i(b)})$$

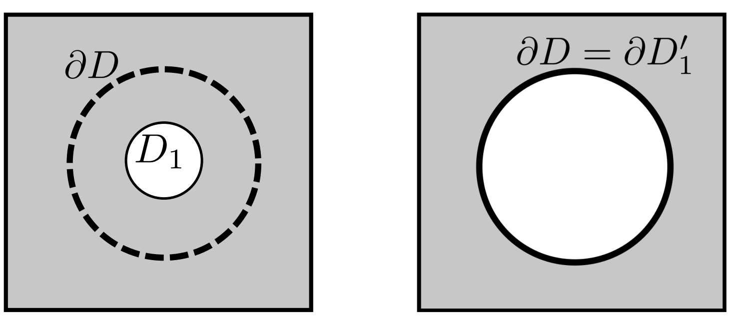
where $d_b$ is the open disk whose boundary is $b$. Now, using Proposition 14 we have $\mathcal{A}_{D(p_I;d_b)} = \mathcal{A}_{D(p_I,z)}$. Also, translation invariance together with Proposition 15 allows us to write $\mathcal{A}_{X_i(b)} = \mathcal{A}_{d_b} = \mathcal{A}_{D_1} = \mathbf{1}$. Hence:

$$\begin{aligned}
\mathcal{A}_X(\phi_I) &= \sum_{\Delta} \mathcal{A}_{D(p_I,z)}(\phi_I, P_\Delta \mathbf{1}) \\
&= \sum_{\Delta} \mathcal{A}_{D(p_I,z)}(\phi_I, \delta_{\Delta,0} \mathbf{1}) \\
&= \mathcal{A}_{D(p_I,z)}(\phi_I, \mathbf{1})
\end{aligned}$$

where in the second equality I used Lemma 21. This proves the first part of the Theorem.
For the second part:

$$\begin{aligned}
\Omega^* \left(\mathcal{A}_{D(p_I)}(\phi_I)\right) &= \mathcal{A}_{\mathbb{R}^2(D_1)} \left(\mathcal{A}_{D(p_I)}(\phi_I)\right) \\
&= \mathcal{A}_{[\mathbb{R}^2(p_I)]_o(\partial D)} \left(\mathcal{A}_{[\mathbb{R}^2(p_I)]_i(\partial D)}(\phi_I)\right)
\end{aligned}$$

where I used the definition of the out-vacuum followed by scale invariance, choosing a disk $D_1'$ enclosing all the marked points (so we can just take $D_1' = D$).




$\mathbb{R}^2(D_1)$ $\quad\quad$ $\mathbb{R}^2(D_1') = \left(\mathbb{R}^2(p_I)\right)_o(\partial D)$

We now use the factorization axiom (in "reverse", and noticing that the sum is finite as thus we actually have equality, not just absolute convergence):

$$\begin{aligned}\Omega^*\left(\mathcal{A}_{D(p_I)}(\phi_I)\right) &= \sum_{\Delta} \mathcal{A}_{[\mathbb{R}^2(p_I)]_o(\partial D)}\left(P_\Delta \mathcal{A}_{[\mathbb{R}^2(p_I)]_i(\partial D)}(\phi_I)\right) \\ &= \mathcal{A}_{\mathbb{R}^2(p_I)}(\phi_I)\end{aligned}$$

where the first equality holds by linearity and finiteness of the sum. □

If we know $m, \mathbf{1}$ and $\omega^*$, we know all amplitudes: we can use Theorems 30 and 20 together with Proposition 14 to obtain them from $m$. We will see this in action in Section 6. So we see that we can restate what are the ingredients we need to specify a CFT:

(i) Space of fields $\mathcal{F} = \bigoplus_{\Delta \in S} \mathcal{F}_\Delta$.

(ii) Generator of scale transformations $d : \mathcal{F} \to \mathcal{F}$.

(iii) Components $T, \bar{T}$ of the stress tensor, identity field $\mathbf{1}$ and out-vacuum $\Omega^*$.

(iv) OPE $m : \mathbb{C} \setminus \{0\} \times \mathcal{F} \times \mathcal{F} \to \bar{\mathcal{F}}$.

# 6 A simple example of a CFT

Let $(A, \cdot, 1)$ be a finite dimensional associative commutative unital algebra over $\mathbb{C}$, where $A$ is a vector space. Let $\epsilon \in A^*$.
Set $S = \{0\}$; $\mathcal{F} = \mathcal{F}_0 = \bar{\mathcal{F}} = A$; $d = 0$; $T = \bar{T} = 0$; $\mathbf{1} = 1$; $\Omega^* = \epsilon$ and $m_z(a_1, a_2) = a_1 \cdot a_2$ $(\forall z \in \mathbb{C})$.
So we have all the ingredients that we need. But do they satisfy the axioms? To see if they do, we must find the amplitudes corresponding to the given data.

**Claim 31.** *The amplitudes corresponding to the CFT data above are given by*

$$\mathcal{A}_{D(p_1,\ldots,p_m;d_1,\ldots,d_n)}(a_1, \ldots, a_{m+n}) = \begin{cases} a_1 \cdot \ldots \cdot a_{m+n} & \textit{if } D \textit{ is an (appropriate) disk} \\ \epsilon(a_1 \cdot \ldots \cdot a_{m+n}) & \textit{if } D = \mathbb{R}^2 \end{cases}$$

*Proof.* We will assume for now that $D$ is a disk. The case when $D$ is the whole plane follows easily by using Theorem 30.
For $m = 1, n = 0$, we use Theorem 30 to see that $\mathcal{A}_{D(p)}(a) = \mathcal{A}_{D(p,z)}(a, \mathbf{1}) = m_{p-z}(a, \mathbf{1}) = a \cdot 1 = a$.
For $m = 2, n = 0$, the result follows directly since $\mathcal{A}_{D(p_1,p_2)}(a_1, a_2) = m_{p_1-p_2}(a_1, a_2) = a_1 \cdot a_2$.
We now show for $m = 3, n = 0$ and by induction the result will be establish for arbitrary $m$ and $n = 0$.
Take three distinct points $p_1, p_2, q$. Assume for now that $|q - p_2| < |p_1 - p_2|$. We then have

$$\mathcal{A}_{D(p_1,p_2,q)}(a_1, a_2, \phi) = \sum_{i\in I} f_i(q - p_2)\mathcal{A}_{D(p_1,p_2)}(a_1, e_i)$$

by virtue of Theorem 20, with $\{e_i\}_{i\in I}$ a basis for $A$ and $\sum_{i\in I} f_i(q - p_2)e_i = a_2 \cdot \phi$. Hence

$$\begin{aligned}
\mathcal{A}_{D(p_1,p_2,q)}(a_1, a_2, \phi) &= \sum_{i\in I} f_i(q - p_2)\mathcal{A}_{D(p_1,p_2)}(a_1, e_i) \\
&= \sum_{i\in I} f_i(q - p_2)a_1 \cdot e_i \\
&= a_1 \cdot (\sum_{i\in I} f_i(q - p_2)e_i) \\
&= a_1 \cdot a_2 \cdot \phi
\end{aligned}$$

If $q$ and $p_1$ are at the same distance from $p_2$, then we use Theorem 30 to introduce a point $z$ that is closer to $p_2$ than the other two points, so that we can still apply Theorem 20 and go back to the ptrevious case:

$$\begin{aligned}
\mathcal{A}_{D(p_1,p_2,q)}(a_1, a_2, \phi) &= \mathcal{A}_{D(p_1,p_2,q,z)}(a_1, a_2, \phi, \mathbf{1}) \\
&= \sum_{i\in I} f_i(z - p_2)\mathcal{A}_{D(p_1,q,p_2)}(a_1, \phi, e_i) \\
&= \mathcal{A}_{D(p_1,q,p_2)}(a_1, \phi, \sum_{i\in I} f_i(z - p_2)e_i) \\
&= \mathcal{A}_{D(p_1,q,p_2)}(a_1, \phi, a_2) \\
&= a_1 \cdot a_2 \cdot \phi
\end{aligned}$$

where in the second equality I used the permutation invariance of the amplitude and in the fourth that $\sum_{i\in I} f_i(z - p_2)e_i = a_2 \cdot \mathbf{1} = a_2$.

The induction step is clear: notice that the same argument that we used for $m = 3$ would work if instead of just $a_1$ we had a bunch of other points. So by induction the result is established for arbitrary $m$ and $n = 0$.
But this also establishes the result for arbitrary $m$ and arbitrary $n$, since when we have disks we can always use Proposition 14 to go back to the case with only marked points.
This establishes the result for $D$ a disk. Using Theorem 30 and $\Omega^* = \epsilon$, the result for the case $D = \mathbb{R}^2$ is also established. □

**Remark 32.** Notice that the OPE $m$ does not depend on $z$. In terms of amplitudes, this means that it does not matter what marked points and disks the worldsheet has, but only how many. Since marked points can be replaced by disks using Proposition 14, then we can interpret this as follows: the amplitudes of this CFT only depend on the topology of the worldsheets. We thus say that it is a *topological field theory*.

So we can now check that this data indeed satisfies the axioms for a CFT.

**Claim 33.** *Let $(A, \cdot, 1)$ be a finite dimensional associative commutative unital algebra over $\mathbb{C}$. Then the collection $(S = \{0\}, \mathcal{F} = A, d = 0, T = 0, \bar{T} = 0, \boldsymbol{1} = 1, \Omega^*)$ is a CFT.*

*Proof.* Let's check the axioms:

(A1) (State-field correspondence) Using Claim 31 we get immediately that $\mathcal{A}_{D_{o_D}}(a) = a$.

(A2) (Permutation Invariance) Immediate from Claim 31 together with commutativity of $\cdot$.

(A3) (Factorization) Clear using the associativity of $\cdot$. In fact:

$$\begin{aligned} \mathcal{A}_{X_o(b)}(a_1, ..., a_l, \mathcal{A}_{X_i(b)(a_{l+1},...,a_{m+n})}) &= a_1 \cdot \ldots \cdot a_l \cdot (a_{l+1} \cdot \ldots \cdot a_{m+n}) \\ &= a_1 \cdot \ldots \cdot a_{m+n} \\ &= \mathcal{A}_X(a_1, ..., a_{m+n}) \end{aligned}$$

(using the notation of Axiom 3).

(A4) (Translation invariance) Immediate from the fact that the amplitudes do not depend on the position of the marked points and disks.

(A5) (Scale covariance) Since in this case $\lambda^d = id_{\mathcal{F}}$ because $d = 0$, then scale covariance comes immediately from the fact that the amplitudes do not depend on the position of the marked points and disks.

(A6) (Stress Tensor) All the parts of this axiom hold trivially because $T = \bar{T} = 0$.

□

# 7 Primary Fields

**Definition 34.** A field $\phi \in \mathcal{F}_\Delta$ is *quasi-primary* if

$$L_1\phi = \bar{L}_1\phi = 0$$

and it is *primary* if

$$\forall m \geq 1,\ L_m\phi = \bar{L}_m\phi = 0$$

**Example 35.** The identity field $\mathbf{1}$ is primary (by Proposition 29).
By Corollary 26, $T$ and $\bar{T}$ are quasi-primary and, if $c = \tilde{c} = 0$, they are primary.

**Definition 36.** We define the operators

$$\mathcal{D}_{m,z} := (m+1)z^m L_0 + z^{m+1}L_{-1}$$
$$\bar{\mathcal{D}}_{m,\bar{z}} := (m+1)\bar{z}^m \bar{L}_0 + \bar{z}^{m+1}\bar{L}_{-1}$$

with $m \in \mathbb{Z}$ and $z \in \mathbb{C}\setminus\{0\}$.

We saw in Lemma 28 how the Virasoro generators act on amplitudes. If we are dealing with quasi-primary or primary operators, we get an even simpler expression:

**Proposition 37.** *Let $X = D(p_1, ..., p_n)$ be a disk worldsheet, $m \in \{-1, 0, 1\}$ and $\phi_1, ..., \phi_n$ be quasi-primary fields. Then*

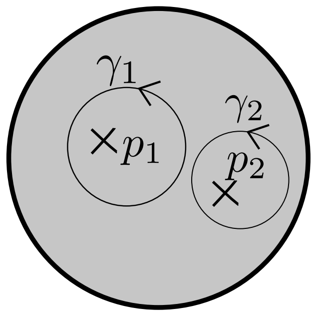
$$L_m\mathcal{A}_X(\phi_1, ..., \phi_n) = \sum_{k=1}^{n} \mathcal{A}_X(\phi_1, ..., \mathcal{D}_{m,p_k}\phi_k, ..., \phi_n)$$
$$\bar{L}_m\mathcal{A}_X(\phi_1, ..., \phi_n) = \sum_{k=1}^{n} \mathcal{A}_X(\phi_1, ..., \bar{\mathcal{D}}_{m,\bar{p}_k}\phi_k, ..., \phi_n)$$

*If $\phi_1, ..., \phi_n$ be primary field, then this holds for $m \in \mathbb{Z}$.*

*Proof.* First, we will prove the result for $\phi_1 \ldots, \phi_n$ primary fields and $m \in \mathbb{Z}$. Recall that from Lemma 28 we have

$$L_m \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, ..., \phi_n) = \oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz$$

with $\gamma$ a curve containing all the $p_i$. Notice that $\mathcal{A}_{D(-,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n)$ is a holomorphic function $\mathcal{C} \backslash \{p_1, \ldots, p_n\} \to \bar{\mathcal{F}}$ (in the sense of Definition 9), so that the integral above can be rewritten using $n$ mutually disjoint curves $\gamma_i$ each containing only the point $p_i$.




Hence

$$\oint_\gamma z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz = \sum_{i=1}^{n} \oint_{\gamma_i} z^{m+1} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) dz$$

Now, by Theorem 27 we can write

$$\begin{aligned} \mathcal{A}_{D(z,p_1,\ldots,p_n)}(T, \phi_1, \ldots, \phi_n) &= \sum_{s \in \mathbb{Z}} (z - p_i)^{-s-2} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, ..., L_s \phi_i, ..., \phi_n) \\ &= \sum_{s=0}^{\infty} (z - p_i)^{s-2} \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1, ..., L_{-s} \phi_i, ..., \phi_n) \end{aligned}$$

Because we are dealing with primary operators. Thus:

$$\begin{aligned}
&\oint_\gamma z^{m+1}\mathcal{A}_{D(z,p_1,\ldots,p_n)}(T,\phi_1,\ldots,\phi_n)dz \\
&=\sum_{i=1}^n \oint_{\gamma_i} z^{m+1}\sum_{s=0}^\infty (z-p_i)^{s-2}\mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,L_{-s}\phi_i,...,\phi_n)dz \\
&=\sum_{i=1}^n \mathrm{Res}\left[\left(p_i^{m+1}+(m+1)p_i^m(z-p_i)+O[(z-p_i)^2]\right)\left(\sum_{s=0}^\infty (z-p_i)^{s-2}\mathcal{A}_{D(p_1,\ldots,p_n)}(\ldots)\right)\right] \\
&=\sum_{i=1}^n \mathrm{Res}[p_i^{m+1}(z-p_i)^{-1}\mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,L_{-1}\phi_i,...,\phi_n) \\
&\qquad\qquad\qquad + (m+1)p_i^m(z-p_i)^{-1}\mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,L_0\phi_i,...,\phi_n)] \\
&=\sum_{i=1}^n p_i^{m+1}\mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,L_{-1}\phi_i,...,\phi_n)+(m+1)p_i^m\mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,L_0\phi_i,...,\phi_n) \\
&=\sum_{i=1}^n \mathcal{A}_{D(p_1,\ldots,p_n)}(\phi_1,...,\mathcal{D}_{m,p_i}\phi_i,...,\phi_n)
\end{aligned}$$

Where we used the Residue theorem and plugged in the Taylor expansion for $z^{m+1}$. Notice that only the terms with $s=0,1$ contributed.
For $\bar{L}$ the proof is similar. For quasi-primary fields, the same argument holds, except that we only use $m+1\in\{0,1,2\}$ from the start, and therefore we again have that only the terms with $s=0,1$ contribute to the residues, so that we get back to the same case. □

# 8 The Virasoro Algebra

**Definition 38.** A *Virasoro algebra Vir* is a complex vector space spanned by a basis $\{L_m:\ m\in\mathbb{Z}\}\cup\{C\}$, with $C:=c\,id_{\mathcal{F}},\ c\in\mathbb{C}$ a constant operator and $L_m$ the familiar Virasoro generators, together with a binary operation $[\cdot,\cdot]:Vir\times Vir\to Vir$ specified by

$$\begin{aligned}
[L_m,L_n]&=(m-n)L_{m+n}+\frac{1}{12}(m^3-m)\delta_{m+n,0}C \\
[L_m,C]&=0
\end{aligned}$$

for all $m,n\in\mathbb{Z}$.

**Remark 39.** It is straightforward to check that the bracket operation of a Virasoro algebra satisfies the Lie bracket conditions, and thus a Virasoro algebra is a Lie algebra.

**Theorem 40.** *The Virasoro generators $L_m$ and $\bar{L}_m$ together with the operators $c\,id_{\mathcal{F}}$ and $\tilde{c}\,id_{\mathcal{F}}$, respectively, span a complex vector space which, together with the standard commutator brackets, is a Virasoro algebra. Explicitly, this means that:*

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} id_{\mathcal{F}} \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{m+n,0} id_{\mathcal{F}} \\
[L_m, c\,id_{\mathcal{F}}] &= [\bar{L}_m, \tilde{c}\,id_{\mathcal{F}}] = 0
\end{aligned}$$

*for all $m, n \in \mathbb{Z}$*

*Proof.* Omitted. □

# 9 Correlators

An $n$-point correlator (or $n$-point function) is an object from QFT that, in our axiomatic formulation, corresponds to (is modelled by) an amplitude of a plane worldsheet with $n$ marked points:

**Definition 41.** An *n-point (normalized) correlator* is the result of evaluating the amplitude of a plane worldsheet with $n$ marked points $p_1, ..., p_n$ on $n$ fields $\phi_1, ..., \phi_n$, divided by the amplitude of the plane worldsheet with no marked points or disks. *i.e.* an object of the type

$$\frac{1}{\mathcal{A}_{\mathbb{R}^2}} \mathcal{A}_{\mathbb{R}^2(p_1,...,p_n)}(\phi_1(p_1)...\phi_n(p_n)) \in \mathbb{C}$$

Also, when $\mathcal{A}_{\mathbb{R}^2} = 1$ we say that the CFT is *normalized*.

**Notation:** Because of its connection to QFT, we can denote such correlator by

$$\langle 0|\phi_1(p_1)...\phi_n(p_n)|0\rangle := \frac{1}{\mathcal{A}_{\mathbb{R}^2}} \mathcal{A}_{\mathbb{R}^2(p_1,...,p_n)}(\phi_1...\phi_n)$$

**Remark 42.** While we are at it, we may as well say what is the physical interpretation of amplitudes of disk worldsheets: they correspond to products of operators:

$$\tilde{\phi}_1(p_1)...\tilde{\phi}_n(p_n) := \mathcal{A}_{D(p_1,...,p_n)}(\phi_1, ..., \phi_n)$$

where the $\sim$ serves to distinguish fields from operator fields (in the language used in the beginning of Section 3).
Notice how this makes consistent our discussion of operator product expansions, and that they coincide with the OPEs in "standard" CFT and string theory.

Our axioms will constrain what kind of one-point and two-point correlators we can have in a CFT. The propositions below tell us how.

**Proposition 43.** *Let $\phi \in \mathcal{F}$ and $z \in \mathbb{C}$. Then*

*(i)* $\mathcal{A}_{\mathbb{R}^2(z)}(L_0\phi) = \mathcal{A}_{\mathbb{R}^2(z)}(\bar{L}_0\phi) = 0$.

*(ii) If* $P_0\phi = 0$, *then* $\mathcal{A}_{\mathbb{R}^2}(z)(\phi) = 0$.

*Proof.* (i): By the translation invariance axiom, $\mathcal{A}_{\mathbb{R}^2(z)}(L_0\phi) = \mathcal{A}_{\mathbb{R}^2(0)}(L_0\phi)$. Using Theorem 30, this can be written $\Omega^*\left(\mathcal{A}_{D(0)}(L_0\phi)\right)$, with $D$ a with center 0. Using the state-field correspondence axiom and Proposition 29, we get $\Omega^*\left(\mathcal{A}_{D(0)}(L_0\phi)\right) = \Omega^*(L_0\phi) = 0$. For $\bar{L}_0$ the proof is similar.

(ii): Exactly the same argument gives us $\mathcal{A}_{\mathbb{R}^2}(z)(\phi) = \Omega^*(\phi)$. But from Lemma 21 we know that $\Omega^* \in \mathcal{F}_0^*$, so that (since the $\mathcal{F}_0$ part of $\phi$ is zero) we have $\Omega^*(\phi) = 0$. □

**Proposition 44.** *Let $\phi_1, \phi_2$ be quasi-primary fields. If $\phi_1$ and $\phi_2$ are eigenvectors of both $L_0$ and $\bar{L}_0$ with eigenvalues $h_1, \bar{h}_1$ and $h_2, \bar{h}_2$, respectively, then*

$$\mathcal{A}_{\mathbb{R}^2(p_1,p_2)}(\phi_1, \phi_2) = \frac{C\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{(p_1 - p_2)^{2h_1}(\bar{p}_1 - \bar{p}_2)^{2\bar{h}_1}}$$

*with $C \in \mathbb{C}$.*

*Proof.* Omitted. □

To illustrate the usefulness of this result, and thus in some sense the usefulness of this axiomatic formulation of CFT, we will apply this proposition to the correlator $\langle 0|T(z)T(0)|0\rangle$:

**Corollary 45.** *In a normalized CFT, we have* $\forall z \in \mathbb{C}\setminus\{0\}$, $\langle 0|T(z)T(0)|0\rangle = \frac{c}{2}z^{-4}$*, with* $c$ *the central charge.*

*Proof.* Since the CFT is normalized, $\langle 0|T(z)T(0)|0\rangle = \mathcal{A}_{\mathbb{R}^2(z,0)}(T,T)$. By Corollary 26, $T$ is a quasi-primary field and has eigenvalues $h = 2$ and $\bar{h} = 0$ w.r.t. $L_0$ and $\bar{L}_0$, respectively. So we can use Proposition 44 to conclude that, for $z \in \mathbb{C}$:

$$\mathcal{A}_{\mathbb{R}^2(z,0)}(T,T) = \frac{C}{z^4}$$

with $C$ a complex number. Finally, using Axiom 6 we see that we must have $C = \frac{c}{2}$. $\square$

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