Cell structure of a double tetrahedron

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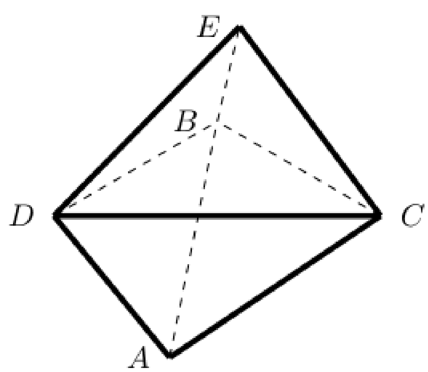
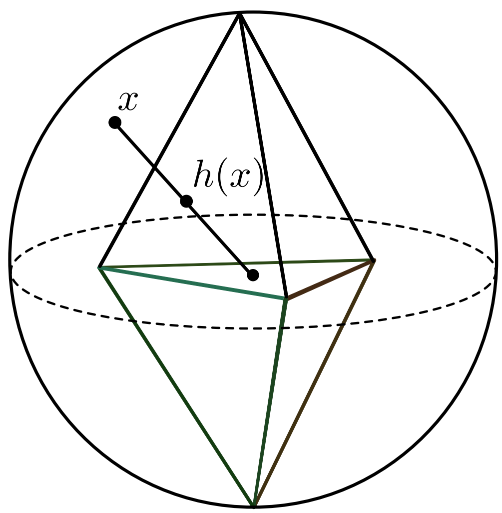
Consider a (filled) double tetrahedron as depicted. below (which we consider to be filled, i.e. not its surface alone). Glue each triangle in the upper half to the mirrored one in the lower half, rotated by 120 degrees. This means we glue DCE to CBA, CBE to BDA and BDE to DCA. Call the resulting space L.



1. *Describe how the depicted double tetrahedron gives a cell structure on D^3 with five 0-cells, nine 1-cells, six 2-cells and one 3-cell.*

   We start with five 0-cells: define $X^{(0)} = \{A, B, C, D, E\}$.
   Attach nine 1-cells by defining $X^{(1)} = X^{(0)} \cup_f (\coprod_{i=1}^{9} D_i^1)$, with $f_i \colon \partial D_i^1 = \{-1, 1\} \to X^{(0)}$ attaching: A and B for $i = 1$; A and C for $i = 2$; A and D for $i = 3$; B and C for $i = 4$; B and D for $i = 5$; B and E for $i = 6$; C and D for $i = 7$; C and E for $i = 8$; D and E for $i = 9$. So for example $f_3(-1) = A$ and $f_3(1) = D$.
   Next, we attach six 2-cells by defining $X^{(2)} = X^{(1)} \cup_g (\coprod_{i=1}^{6} D_i^2)$, with

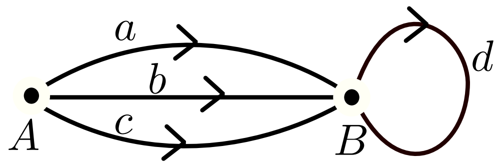
$g_i\colon \partial D_i^2 = S^1 \to X^{(1)}$ gluing each circle to a triangle, namely: g_1 glues ∂D_1^2 to the triangle ABC; g_2 glues ∂D_2^2 to the triangle ABD; g_3 glues ∂D_3^2 to the triangle ACD; g_4 glues ∂D_4^2 to the triangle BDE; g_5 glues ∂D_5^2 to the triangle BCE; g_6 glues ∂D_6^2 to the triangle CDE.
Finally, we attach one 3-cell by defining $X^{(3)} = X^{(2)} \cup_h D^3$, with $h\colon \partial D^3 = S^2 \to X^{(2)}$ the projection of the sphere onto the double tetrahedron, obtained by: embedding the double tetrahedron inside the sphere; for each $x \in S^2$ taking the line trough x and the center of S^2; setting $h(x)$ to be the intersection point of that line and the double tetrahedron. This is illustrated in the figure below.



Continuity of such a projection is clear.
So we constructed a cell structure for the double tetrahedron (which is homeomorphic to D^3).

2. *Find a cell structure on L (with only two 0-cells) and use it to compute the fundamental group of L. (I will denote X = L).*

   Start with $X^{(0)} = \{A, B\}$.
   Attach four 1-cells by defining $X^{(1)} = X^{(0)} \cup_{\phi^1} (\coprod_{i=1}^4 D_i^1)$, with the $\phi_i^1 : \partial D_i^1 = S^0 \to X^{(0)}$ gluing the endpoints of I_i to A and B, for $i = 1, 2, 3$, and ϕ_4^1 gluing both endpoints of I_4 to B. The result is a graph G as depicted below.



The fundamental group of G has $1-\chi(G) = 1-(2-4) = 3$ generators, which are loops. Namely, taking B as the basepoint for the loops, we can choose the generators ab^{-1}, cb^{-1}, d. Then $\pi_1(X^{(1)}) \cong \pi_1(G) \cong \langle ab^{-1}, cb^{-1}, d\rangle$. We will use this to determine $\pi_1(X^{(2)})$.
Attach three 2-cells by defining $X^{(2)} = X^{(1)} \cup_{\phi^2} (\coprod_{i=1}^{3} D_i^2)$, with the $\phi_i^2 : \partial D_i^2 = S^1 \to X^{(1)}$ gluing ∂D_1^2 to the loop in $X^{(1)}$ corresponding to the triangle ABC, ∂D_2^2 to the loop in $X^{(1)}$ corresponding to the triangle ABD and ∂D_3^2 to the loop in $X^{(1)}$ corresponding to the triangle ACD. These loops are adb^{-1}, $ad^{-1}c^{-1}$ and bdc^{-1}, respectively. Hence:

$$\begin{aligned}\pi_1(X^{(2)}) &\cong \langle ab^{-1}, cb^{-1}, d \,|\, adb^{-1}, ad^{-1}c^{-1}, bdc^{-1}\rangle \\ &= \langle ab^{-1}, cb^{-1}, d \,|\, db^{-1}a, d^{-1}c^{-1}a, dc^{-1}b\rangle\end{aligned}$$

We want to express this group in a more familiar way. To do that, we start by defining $l_1 = ab^{-1}$, $l_2 = cb^{-1}$ and $l_3 = d$. Then:

$$\begin{aligned}\pi_1(X^{(2)}) &\cong \langle l_1, l_2, l_3 \,|\, l_3 l_1, l_3 l_2^{-1} l_1, l_3 l_2^{-1}\rangle \\ &= \langle l_1 \,|\, l_1^3\rangle\end{aligned}$$

where the second equality can be understood by considering the following: the first relation on the first line implies that $l_3 = l_1^{-1}$, and the third relation implies $l_3 = l_2$, so that the second relation can be written $(l_1)^3 = e$. So we are reduced to one generator with one relation only.
Now, $\langle l_1 \,|\, l_1^3\rangle$ is a finite cyclic group of order three, and so it is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Hence: $\pi_1(X^{(2)}) \cong \mathbb{Z}/3\mathbb{Z}$.
Now, to obtain X we would have to attach 3-cells, but this is unnecessary because we know that $\pi_1(X) \cong \pi_1(X^2)$. Hence $\pi_1(X) \cong \mathbb{Z}/3\mathbb{Z}$.