The GKO construction

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Abstract

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Contents

1 Overview of semisimple Lie algebra representation theory **3**

1.1 Overview of the representation theory of $\mathfrak{sl}_2\mathbb{C}$ 3

1.2 Overview of the representation theory of semisimple Lie algebras 3

1.2.1 The case of $\mathfrak{sl}_4\mathbb{C}$ 3

2 Kac-Moody Algebras **3**

2.1 The gist of it 3

2.2 Explicit construction of an affine Kac-Moody algebra (a loop algebra to be exact) 4

2.3 Central extension of a loop algebra 4

2.4 Current algebra from affine Kac-Moody algebra 4

3 Highest weight irreducible representations of the Virasoro algebra **4**

3.1 Representations of *Vir* from representations of loop algebras . 7

3.2 Representations of *Vir* from representations of a loop algebra and a subalgebra 9

4 Highest weight irreducible representations of the super-Virasoro algebra **16**

1 Overview of semisimple Lie algebra representation theory

The GKO (or coset) construction is a means to construct the highest weight unitary irreducible representations of the Virasoro algebra and its supersymmetric extensions: the Ramond algebra and the Neveu-Schwarz algebra. This section is mostly a discussion on some of the most important aspects of the original papers [?],[?] where this construction first appeared. But since the motivation for it comes from the representation theory of semisimple Lie algebras, we will tackle those first.

1.1 Overview of the representation theory of $\mathfrak{sl}_2\mathbb{C}$

1.2 Overview of the representation theory of semisimple Lie algebras

Note: The dimension of the Cartan subalgebra of the Lie algebra (so its rank) of a system equals the number of quantum numbers we need, because it is precisely the number of simultaneously diagonalizable operators, and thus it tells us how many eigenvalues we need to specify a state. See p178 [Blum]

1.2.1 The case of $\mathfrak{sl}_4\mathbb{C}$

2 Kac-Moody Algebras

In this section, I give an overview of the construction of Kac-Moody algebras, based on [?]. Affine Kac-Moody algebras appear as symmetry algebras of important models in physics. Of special importance to us will be the symmetry algebra of coset algebras, which is indeed an affine Kac-Moody algebra.

2.1 The gist of it

Given a semisimple Lie algebra we construct a matrix with entries $A_{ij} = \frac{2(\alpha_j|\alpha_i)}{(\alpha_i|\alpha_i)}$, where the α_i are simple roots and $(\cdot|\cdot)$ is the inner product induced by the Killing form [?].
Finite-dimensional Lie Algebras can

Finish!! 151 handwritten notes

2.2 Explicit construction of an affine Kac-Moody algebra (a loop algebra to be exact)

An approach that is arguably more useful for physicists is to construct an affine Lie algebra by constructing a *loop algebra* which is an affine Kac-Moody algebra. This is based on [?].

Do as in p.322 [BLT]

2.3 Central extension of a loop algebra

expand on p.323 [BLT]

2.4 Current algebra from affine Kac-Moody algebra

Do as in p.324 [BLT]

3 Highest weight irreducible representations of the Virasoro algebra

The Virasoro algebra is not semisimple. [[See p.199 [Me]. Couldn't show this, but probably waste of time anyway.]] However, we can draw some lessons from the representation theory of semisimple Lie algebras in order to find representations of the Virasoro algebra. As a first step, we find a Cartan subalgebra of the Virasoro algebra.

Claim 3.1. *(OG)*$\mathbb{C}L_0$ *is a Cartan subalgebra of Vir.*

Proof. Notice that

$$\forall n \in \mathbb{Z}, \quad [L_0, L_n] = -nL_n \tag{1}$$

so that $\mathrm{ad}(L_0)$ is diagonalized by choosing the basis $\{L_n\}_{n\in\mathbb{Z}}$ of *Vir*. Hence L_0 is ad-diagonalizable. We also see that there is no element in this basis other than L_0 which commutes with L_0. Thus $\{L_0\}$ spans a maximal commutative subalgebra of ad-diagonalizable elements of *Vir*, *i.e.* a Cartan subalgebra of *Vir*. □

Hence we can choose for example $Vir_0 = \mathbb{C}(-L_0)$. So the weights of a representation π of *Vir* are simply the eigenvalues of $-\pi(L_0)$. In physics, we want representations of the Virasoro algebra with representation space a possibly infinite-dimensional Hilbert space. Since we cannot assume finite-dimensionality of the representation space anymore (like we did for the representation theory of semi-simple Lie algebras [[Again: why did we do this?]]), then we are not sure that a highest weight vector exists for an arbitrary representation of *Vir*. However, the Hamiltonian can be written $H = L_0 + \bar{L}_0$ [[Saw this in [Tong], did not go deeper. Does this mean that the energy will be twice the eigenvalue of L_0? Or can we have different reps for the chiral and non chiral parts of the virasoro algebra?]], so that $\pi(L_0)$ must have its spectrum bounded below, since the system has a minimum possible energy corresponding to the ground-state, which we write as the eigenvector $|h\rangle$ of $\pi(L_0)$ whose eigenvalue we denote by h. In conclusion, the representations of *Vir* which are physically meaningful must have a highest weight state $|h\rangle$ with weight $-h$. It would probably be more natural from a physics point of view to set $Vir_0 = \mathbb{C}L_0$ and demand the existence of a *lowest* weight state corresponding to the lowest energy of the physical system, but in this case the analogy with semisimple representation theory would be less clear.

Since in physics we also want representations which are unitary (so that probabilities are preserved under the action of observables) and we know that we can construct all unitary representations of a compact Lie algebra as a direct sum of irreducible representations, [[Can we? I read on wikipedia that this holds for finite dim. reps. But in general we want infinite dim. ones!! Does it still hold?]] then we can just focus on finding all the *irreducible unitary highest weight representations* of *Vir*.

Remark 3.2. (OG)Notice that, since this h is the minimum of the spectrum of (the representation of) L_0 and from the commutation relations of *Vir* we have $L_0 L_n|h\rangle = (h-n)L_n|h\rangle$, then $L_n|h\rangle = 0$ for $n > 0$.

Now, by the same argument as in the representation theory of $\mathfrak{sl}_2\mathbb{C}$, in a irreducible representation all the eigenvectors of $\pi(L_0)$ are (up to rescaling) in the string $\{\pi(L_{k_1})\dots\pi(L_{k_m})|h\rangle : k_1 \leq \dots \leq k_m\}_{m\in\mathbb{N}}$ (called the *Verma module* of $|h\rangle$). So all elements of the representation space can be written as linear combinations of elements of the Verma module. See [**?**] for more details.

Notation: In the physics literature, it is customary to omit the representation symbol, so that one writes L_k instead of $\pi(L_k)$, for example. From now on I will adopt this convention unless it is not clear from the context what representation I am referring to.

The next claim reconciles the physicists' characterization of a highest weight vector with the usual definition from representation theory.

Claim 3.3. *(OG)Let $|h\rangle$ be an eigenvector of L_0 with eigenvalue h. It is a maximum weight state if and only if $\forall n > 0, L_n|h\rangle = 0$.*

Proof. The "if" part of the claim was discussed in Remark 3.2. Now assume that $\forall n > 0, L_n|h\rangle = 0$. Let $|h'\rangle \in H$ be the highest weight vector, with weight $-h'$. We can write $|h\rangle$ as a linear combination of elements $L_{k_1} \dots L_{k_m}|h'\rangle$ of the Verma module, and all must have the same weight $-h$. Using the commutation relations for L_0, L_k, we can write the weight of one such state as $-h = -h' + k_1 + ... + k_m$. But $k_1 + \dots k_m > 0$. Indeed: [[see p213. How to show this?]] Then $-h > -h'$. This proves the result by contradiction. □

It turns out that an irreducible highest weight representation of *Vir* is completely determined by the maximum weight h and the central charge c [?]. If the representation is unitary, then $c \geq 0, h \geq 0$ [?], and there are irreducible highest weight representations for each pair $(c \geq 1, h \geq 0)$; but the only values of c and h for which we *may* have a unitary irreducible highest weight representation when $c < 1$ are [?]

$$c = 1 - \frac{6}{m(m-1)}, \quad m = 2, 3, ... \tag{2}$$

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \quad p = 1, ..., m-1; \quad q = 1, ..., p \tag{3}$$

These results come from imposing the non-existence of ghosts. See [?] and §2.10 of [?] for more details.

Two important supersymmetric extensions of the Virasoro algebra which arise in Superstring Theory [?] are the Ramond and Neveu-Schwarz algebras, which have generators $\{L_m, G_r\}$ obeying the commutation relations

[Maybe should talk about supersymmtery before this section, since it is referred here]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{4}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \tag{5}$$

$$[L_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r,-s} \tag{6}$$

with $m \in \mathbb{Z}$ and $r, s \in \mathbb{Z}$ for the Ramond algebra and $r, s \in \mathbb{Z} + \frac{1}{2}$ for the Neveu-Schwarz algebra. Like with the non-super case, the irreducible unitary highest weight representations are given by a pair (c, h), where[**?**]: $c \geq \frac{3}{2}$ and $h \geq 0$, or

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right), \quad m = 2, 3, ... \tag{7}$$

and

$$h = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{\epsilon}{8}, \quad p = 1, ..., m-1; \quad q = 1, ..., m+1 \tag{8}$$

with $p - q$ odd and $\epsilon = \frac{1}{2}$ for the Ramond algebra, and $p - q$ even and $\epsilon = 0$ for the Neveu-Schwarz algebra.

3.1 Representations of *Vir* from representations of loop algebras

Recall that given a simple Lie algebra $\mathfrak{g}$ with commutation relations $[T_a, T_b] = if^{abc}T^c$ (where if $\mathfrak{g}$ is compact, the generators $\{T_a\}$ can be chosen such that the structure constants are completely antisymmetric - we assume this to be the case) [[Actually, they do not say the compactness part in the paper! But it is needed!]], the (unique non-trivial central extension of) the associated loop algebra $\hat{\mathfrak{g}}$ has commutation relations [See Fuchs (12.20)]

$$[T^a_m, T^b_n] = f^{abc}T^c_{m+n} + mK\delta^{ab}\delta_{m,-n} \tag{9}$$
$$[T^a_m, K] = 0 \tag{10}$$

As an abuse of notation, we also refer to this central extension of the loop algebra by $\hat{\mathfrak{g}}$.
By Schur's Lemma the central element K is simply a scalar (times the identity) in any irreducible representation. Also, it is always a multiple of $\frac{1}{2}\psi^2$, where ψ is a long root of $\mathfrak{g}$ [**?**]. The integer $x = \frac{2k}{\psi^2}$ is the *level* of the representation.

Notation: The *normal ordering* is denoted by $:(-):$, and is defined by $: T^a_n T^a_{-n} := T^a_{-n} T^a_n$ for $n \geq 0$. [[what about smth like : TTTT :? If we were dealing with normal ordering of oscillators α like in ST, there would be no ambiguities since they commute if n and m are both positive, so we can just say that the normal ordering puts all creation ops on the left and all annihilation ops on the right, because these commute between themselves. !!??]]

β will denote $K + \frac{c_{\mathfrak{g}}}{2}$, with $c_{\mathfrak{g}}$ the quadratic Casimir element for the adjoint representation of $\mathfrak{g}$. Recall that this $c_{\mathfrak{g}}$ is the eigenvalue of the quadratic Casimir operator which is defined by $C = \kappa_{ab} T^a T^b$, with κ the Killing form matrix. In the adjoint representation, we have [**?**]:

$$f^{abc} f^{abd} = c_{\mathfrak{g}} \delta^{cd} \tag{11}$$

Proposition 3.4. *Let $\{T^a_m\}$ be (a representation of) the generators of a loop algebra $\hat{\mathfrak{g}}$. Defining $L^{\mathfrak{g}}_n = \frac{1}{2\beta} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m \in \mathbb{Z}} : T^a_{m+n} T^a_{-m} :$ we get a Virasoro algebra $Vir(\mathfrak{g})$ with generators $\{L^{\mathfrak{g}}_n\}$ and central element $k \frac{\dim \mathfrak{g}}{\beta}$.*

Proof. (OG)

[They refer this in [GKO1], but did not prove it. I tried once but it was a lot of stuff. I'll try again.]

□

Lemma 3.5. *An unitary representation of $\hat{\mathfrak{g}}$ induces an unitary representation of $Vir(\mathfrak{g})$.*

Proof. (OG)By assumption, we have $(T^a_n)^\dagger = T^a_{-n}$. We want to show that $(L^{\mathfrak{g}}_n)^\dagger = L^{\mathfrak{g}}_{-n}$.
We can write

$$(L^{\mathfrak{g}}_n)^\dagger = \frac{1}{2\beta} \sum_{a,m} \left(: T^a_{m+n} T^a_{-m} : \right)^\dagger \tag{12}$$

$$= \begin{cases} \frac{1}{2\beta} \sum_{a,m} \left(T^a_{-m} T^a_{m+n} \right)^\dagger, & -m \geq 0 \\ \frac{1}{2\beta} \sum_{a,m} \left(T^a_{m+n} T^a_{-m} \right)^\dagger, & -m < 0 \end{cases} \tag{13}$$

$$= \begin{cases} \frac{1}{2\beta} \sum_{a,m} T^a_{-m-n} T^a_{m}, & m \leq 0 \\ \frac{1}{2\beta} \sum_{a,m} T^a_{m} T^a_{-m-n}, & m > 0 \end{cases} \tag{14}$$

$$= \frac{1}{2\beta} \sum_{a,m} : T^a_{-m-n} T^a_{m} : \tag{15}$$

$$= \frac{1}{2\beta} \sum_{a,m} : T^a_{m-n} T^a_{-m} : \tag{16}$$

$$= L^{\mathfrak{g}}_{-n} \tag{17}$$

□

Recall that this was all for simple $\mathfrak{g}$. If instead $\mathfrak{g}$ is semisimple with N simple factors, then each simple factor gives us a Virasoro algebra. Adding the generators of these, we get a Virasoro algebra generated by:

$$L_n^{\mathfrak{g}} := \sum_{i=1}^{N} L_n^{\mathfrak{g}_i} \tag{18}$$

It is not hard to see that this indeed generates a Virasoro algebra, with central element

$$c^{\mathfrak{g}} = \sum_{i=1}^{N} c^{\mathfrak{g}_i} = \sum_{i=1}^{N} \frac{2k_i \dim \mathfrak{g}_i}{c_{\mathfrak{g}} + 2k_i} \tag{19}$$

Indeed: .

[Write proof from [p172 - Me]]

3.2 Representations of *Vir* from representations of a loop algebra and a subalgebra

We now come to the fundamental idea of this section: we can construct a Virasoro Lie algebra from a compact simple Lie algebra $\mathfrak{g}$ *and* a Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$. We can order the basis of $\mathfrak{g}$ such that the first $\dim \mathfrak{h}$ generators are a basis of $\mathfrak{h}$. Notice that $\mathfrak{h}$ is also compact and simple. Indeed: if $\mathfrak{h}$ were not simple, it would have a non-trivial ideal $\mathfrak{i} \leq \mathfrak{h}$, which would also be a non-trivial ideal of $\mathfrak{g}$ [[No it wouldn't I think!! ??]]; if $\mathfrak{h}$ were not compact, then its Killing form would not be negative-definite, meaning there would be $X \in \mathfrak{h}$ with $k(X, X) \geq 0$, and since the Killing form in $\mathfrak{h}$ is induced from the one in $\mathfrak{g}$, $\mathfrak{g}$ would also be non-compact. We can thus form two Virasoro algebras $Vir(\mathfrak{g})$ and $Vir(\mathfrak{h})$. We use these to form a new Virasoro algebra $Vir(\mathfrak{g}, \mathfrak{h})$. The following Proposition is proved in §3 of [**?**]:

Proposition 3.6. *Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Denote by $\{L_n^{\mathfrak{g}}\}$, $\{L_n^{\mathfrak{h}}\}$ the standard generators of $Vir(\mathfrak{g})$, $Vir(\mathfrak{h})$, respectively. Then $\{K_n := L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}}\}$ generates a Virasoro algebra $Vir(\mathfrak{g}, \mathfrak{h})$ with central element $c = c^{\mathfrak{g}} - c^{\mathfrak{h}}$.*

Claim 3.7. *(OG)A unitary highest weight representation π of a loop algebra $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{h}$ induces a unitary highest weight representation of $Vir(\mathfrak{g}, \mathfrak{h})$.*

Proof. Unitarity follows from Lemma 3.5:

$$K_n^\dagger = L_n^{\mathfrak{g}\,\dagger} - L_n^{\mathfrak{h}\,\dagger} \tag{20}$$
$$= L_{-n}^{\mathfrak{g}} - L_{-n}^{\mathfrak{h}} \tag{21}$$
$$= K_{-n} \tag{22}$$

This establishes unitarity. Now, if $|\Psi\rangle$ is a highest weight vector of the representation of $\hat{\mathfrak{g}}$ with highest weights h_a for each T_0^a, then

△ I had to assume that what we mean by "highest weight state of a loop algebra" is one for which $T_0^a |\Psi\rangle = h_a |\Psi\rangle$... Is this it?? In §8.3 of [Fuchs] it seems to be more complex than this.

$$L_0^{\mathfrak{g}} |\Psi\rangle = \frac{1}{2\beta} \sum_{a,m} : T_m^a T_{-m}^a : |\Psi\rangle \tag{23}$$
$$= \frac{1}{2\beta} \sum_a : T_0^a T_0^a : |\Psi\rangle \tag{24}$$
$$= \left(\frac{1}{2\beta} \sum_a h_a^2 \right) |\Psi\rangle \tag{25}$$

and, for $n > 0$, we have:

$$L_n^{\mathfrak{g}} |\Psi\rangle = \frac{1}{2\beta} \sum_a \sum_{m\in\mathbb{Z}} : T_{m+n}^a T_{-m}^a : |\Psi\rangle \tag{26}$$
$$= \frac{1}{2\beta} \sum_a : T_n^a T_0^a : |\Psi\rangle + \frac{1}{2\beta} \sum_a \sum_{m\in\mathbb{N}^+} : T_{m+n}^a T_{-m}^a : |\Psi\rangle \tag{27}$$
$$= 0 \tag{28}$$

where the second equality holds because for $m > 0$ we have $T_{-m}^a |\Psi\rangle = 0$, and the third because $m+n > 0$ in the second sum, and the normal ordering operation exchanges T_{m+n}^a and T_{-m}^a. □

Since the central charge of a unitary highest weight irreducible representation of the Virasoro algebra has central charge $c \geq 0$, then $c^{\mathfrak{g}} \geq c^{\mathfrak{h}}$.

△ $\mathfrak{g}$ simple implies $\hat{\mathfrak{g}}$ simple? See p211 of my notes.

△ Def of character does not match with expression on page 110.

Recall that we are trying to find highest weight unitary irreducible representations of the Virasoro algebra. We just saw that we can get representations $Vir(\mathfrak{g},\mathfrak{h})$ of the Virasoro algebra. Notice that $Vir(\mathfrak{g},\mathfrak{h})$ depends not only on the choice of $\mathfrak{g}$ and $\mathfrak{h}$, but also on the choice of representation of $\mathfrak{g}$. We will see that, if we choose $\mathfrak{g}$ (and its representation) and $\mathfrak{h}$ cleverly, then the representations $Vir(\mathfrak{g},\mathfrak{h})$ will have central charges reproducing the entire sequence in (2), and all h in (3). But in order to do all this, we first have to study the highest weight unitary irreducible representations of loop algebras $\hat{\mathfrak{g}}$, since these induce the representations $Vir(\mathfrak{g},\mathfrak{h})$.

This is the main result we need from the representation theory of loop algebras:

Proposition 3.8. *Let $\mathfrak{g}$ be a compact simple Lie algebra. There exists an unitary irrep of (the central extension of) $\hat{\mathfrak{g}}$ with K its central element iff there is a finite-dimensional irrep of $\mathfrak{g}$ with representation space spanned by the states $|\Psi\rangle$ satisfying $T^a_n|\Psi\rangle = 0, n > 0$ (called vacuum states), with highest weight λ satisfying $\forall\alpha \in R, \quad |\alpha\lambda| \leq K$, and also $\frac{2K}{\psi^2} \in \mathbb{Z}$, where ψ is a long root of the representation of $\mathfrak{g}$. Furthermore, such representation is completely determined by the pair $(\frac{2K}{\psi^2}, \lambda)$.*

Find good reference for this.

Our "smart choice" of $\mathfrak{g}$ and $\mathfrak{h}$ is $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h}$ its diagonal $\mathfrak{su}(2)$ subalgebra. We know that $\mathfrak{su}(2)$ is simple and compact, so that we can construct its loop algebra. Recall that the structure constants for $\mathfrak{su}(2)$ with respect to its usual basis are $f_{abc} = i\epsilon_{abc}$.Then (9) in the case of $\mathfrak{su}(2)$ reads:

$$[T^a_m, T^b_n] = i\epsilon^{abc}T^c_{m+n} + mK\delta^{ab}\delta_{m,-n} \tag{29}$$

Since the matrices in (the fundamental representation of) $\mathfrak{su}(2)$ are hermitian, then it's easy to show that $(T^a_n)^\dagger = T^a_{-n}$ (see section 11.1 of [?]). Furthermore, fundamental representations are irreducible. Hence (29) describes a unitary irrep of $\hat{\mathfrak{su}}(2)$.
Hence we can use Proposition 3.8. The roots of (the fundamental representation of) $\mathfrak{su}(2) \cong \mathfrak{sl}(2)$ are ± 1, so that both are long roots, and we can take for example $\psi = 1$. Hence the central element K is a multiple of $\frac{1}{2}$. Write $K = \frac{N}{2}$ so that N is the level of the representation. Furthermore, this representation can be labelled (N, l), with l the largest eigenvalue of $T = T_0^{3=\dim\mathfrak{su}(2)}$ on the vacuum states $\{|\Psi\rangle\}$ (choosing T as the generator of a Cartan subalgebra of $\mathfrak{su}(2)$).

[To explain why we can do this: can put in ap-

Then we have:

$$[T^a_m, T^b_n] = i\epsilon^{abc}T^c_{m+n} + m\frac{N}{2}\delta^{ab}\delta_{m,-n} \tag{30}$$

Again by Proposition 3.8, we know that $|\pm 1l| \leq K = \frac{N}{2}$ and also l is positive

△ Actually, T3 has eigenvalues (weights) +- 1/2, so it can be negative! So why do they write this?

so that $0 \leq 2l \leq N$.

Now, for $\mathfrak{su}(2)$ the quadratic Casimir element for the adjoint representaion is 2. Indeed: in this case, (11) reads

$$\begin{aligned}
c_{\mathfrak{su}(2)}\delta^{cd} &= \epsilon^{abc}\epsilon^{abd} && (31)\\
&= \epsilon^{12c}\epsilon^{12d} + \epsilon^{13c}\epsilon^{13d} + \epsilon^{21c}\epsilon^{21d} + \epsilon^{23c}\epsilon^{23d} + \epsilon^{31c}\epsilon^{31d} + \epsilon^{32c}\epsilon^{32d} && (32)\\
&= \begin{cases} 2 & \text{if } c = d \\ 0 & \text{otherwise} \end{cases} && (33)\\
&= 2\delta^{cd} && (34)
\end{aligned}$$

Using this fact, we can use Proposition 3.4 to determine the central element of the representation $Vir(\hat{\mathfrak{su}}(2))$ of the Virasoro algebra, induced by a level N representation of $\hat{\mathfrak{su}}(2)$:

$$\begin{aligned}
c &= \frac{K \dim \mathfrak{su}(2)}{\beta} && (35)\\
&= \frac{N \dim \mathfrak{su}(2)}{2K + c^{\mathfrak{su}(2)}} && (36)\\
&= \frac{3N}{N+2} && (37)
\end{aligned}$$

Claim 3.9. *Let* $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. *Consider the representation of* $\hat{\mathfrak{g}}$ *which is the direct sum of a level* N *representation and a level* 1 *representation of* $\mathfrak{su}(2)$. *This induces a representation of level* $N+1$ *of the diagonal subalgebra* $\hat{\mathfrak{h}} \cong \mathfrak{su}(2)$.

Proof. (OG)

See pages 206,207 [Me]. (As a lemma, also show that the loop algebra of a direct sum is a direct sum of the loop algebras - see p208)

□

Choosing a representation of $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as in Claim 3.9, the corresponding (representation of the) Virasoro algebra $Vir(\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{h} \cong \mathfrak{su}(2))$ has central element

$$c = c^{\mathfrak{g}_1} + c^{\mathfrak{g}_2} - c^{\mathfrak{h}} \tag{38}$$

$$= \frac{3N}{N+2} + \frac{3}{3} - \frac{3(N+1)}{N+3} \tag{39}$$

$$= 1 - \frac{6}{(N+2)(N+3)} \tag{40}$$

where in the first equality I used Proposition 3.6 and in the second equality I used (37).
Notice that (40) gives us the sequence (2) if we take $N \in \mathbb{N}$. (The cases with $N \in \mathbb{Z}^-$ do not give us useful information, since then $c \geq 1$, and we already knew that there were Virasoro representations for all $c \geq 1$). This allows us to draw the following conclusion:

Corollary 3.10. *For every $m \in \{2, 3, ...\}$, there are unitary highest weight irreducible representations of the Virasoro algebra with central element*

$$c = 1 - \frac{6}{m(m-1)} \tag{41}$$

What is more, every unitary highest weight irreducible representations of the Virasoro algebra either has central element of this type, or $c \geq 1$.

Proof. By the discussion above, the result follows from considering the representations $Vir(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h}$ the diagonal $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$. $\square$

The next step is to investigate what values of h can unitary highest weight irreducible representations of the Virasoro algebra have. We will see that, again with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h} \cong \mathfrak{su}(2)$ its diagonal subalgebra, we get representations for each h in (3), with c of the form (41).
Since $\mathfrak{h}$ is semisimple, then in particular it is reductive. Hence there is a branching rule [?] of the type

△ how does this make sense? The domain of the LHS is larger than the domain on the RHS.

$$\pi^{\mathfrak{g}}_{\Lambda} = \bigoplus_{\lambda \in W_{\mathfrak{h}}} \pi^{\mathfrak{h}}_{\lambda} \tag{42}$$

with $W_{\mathfrak{h}}$ a set of values λ such that each $\pi_\lambda^{\mathfrak{h}}$ is a unitary highest weight irreducible representation of $\mathfrak{h}$. In other words, any unitary highest weight irreducible representation $\pi_\Lambda^{\mathfrak{g}}$ of $\mathfrak{g}$ can be written as a direct sum of unitary highest weight irreducible representations of $\mathfrak{h}$. Now, we are actually interested in using the unitary highest weight irreducible representation of $\hat{\mathfrak{g}}$ to get information about the unitary highest weight irreducible representation of $\hat{\mathfrak{g}} \big/ \hat{\mathfrak{h}} := Vir(\mathfrak{g}, \mathfrak{h})$. Since $\hat{\mathfrak{g}} \big/ \hat{\mathfrak{h}}$ is semisimple

I think I was able to show on page 211 [Me] that loop algebras of simple algebras are simple! - write here the proof

, then $\mathfrak{s} := \hat{\mathfrak{h}} \oplus \frac{\hat{\mathfrak{g}}}{\hat{\mathfrak{h}}}$ is also a semisimple Lie subalgebra of $\mathfrak{g}$. we again get a branching rule

△ Is this true? Should prove. Just don't have time for all these details.

$$\pi_\Lambda^{\hat{\mathfrak{g}}} = \bigoplus_{\lambda \in W_{\mathfrak{s}}} \pi_\lambda^{\mathfrak{s}} \tag{43}$$

$$= [see(around)page208[Me]] \tag{44}$$

$$= \bigoplus_{(\lambda_{\mathfrak{h}}, \lambda_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}) \in W_{\hat{\mathfrak{h}}} \times W_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}} \pi_{\lambda_{\hat{\mathfrak{h}}}}^{\hat{\mathfrak{h}}} \otimes \pi_{\lambda_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}}^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}} \tag{45}$$

Now, this equality holds (up to isomorphism of representations) if and only if the characters on both sides agree, *i.e.*

This must be wrong: the sum should only be over the weights of h...

$$\chi_\Lambda^{\hat{\mathfrak{g}}} = \sum_{(\lambda_{\mathfrak{h}}, \lambda_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}) \in W_{\hat{\mathfrak{h}}} \times W_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}} \chi_{\lambda_{\hat{\mathfrak{h}}}}^{\hat{\mathfrak{h}}} \chi_{\lambda_{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}}^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}} \tag{46}$$

With this in mind, we will seek an expression of the type (46).

Proposition 3.11. *Consider the representations (N, l) of the first $\hat{\mathfrak{su}}(2)$ in $\mathfrak{g}$ and $(1, \epsilon)$ of the second $\hat{\mathfrak{su}}(2)$ in $\hat{\mathfrak{g}}$, where $\epsilon = 0$ or $\epsilon = 1/2$. These induce representations of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}} \big/ \hat{\mathfrak{g}}$. For this choice of representations, we have*

$$\chi_{(N,l)}^{\hat{\mathfrak{su}}(2)}(z,\theta)\chi_{(1,\epsilon=0,1/2)}^{\hat{\mathfrak{su}}(2)}(z,\theta) = \sum_{\substack{q \\ p-q \in 2(\mathbb{Z}+\epsilon) \\ 1 \le q \le N+2}} \chi_{(N+1,\frac{1}{2}[q-1])}^{\hat{\mathfrak{h}}}(z,\theta)\chi_{(c,h_{p,q}(c))}^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}(z) \tag{47}$$

Proof. From the Kac-Weyl character formula, we get that the character for a (N, l) representation of $\hat{\mathfrak{su}}(2)$ is [?]: [Find better reference]

$$\chi_{(N,l)}^{\hat{\mathfrak{su}}(2)} = \Delta_{N,l}(z,\theta) \prod_{n=1}^{\infty} (1-z^n)^{-1}(1-z^n e^{i\theta})^{-1}(1-z^{n-1}e^{-i\theta})^{-1} \tag{48}$$

with

$$\Delta_{N,l}(z,\theta) = z^{l(l+1)/\lambda} \sum_{n\in\mathbb{Z}} z^{\lambda n^2+(2l+1)n}(e^{i(l+\lambda n)\theta} - e^{-i(l+1+\lambda n)\theta}) \tag{49}$$

and $\lambda = N+2$.
In particular, for the case with $N = 1$ and $l = \epsilon = 0, 1/2$, we have

$$\chi_{(1,\epsilon)}^{\hat{\mathfrak{su}}(2)}(z,\theta) = \sum_{m\in\mathbb{Z}+\epsilon} z^{m^2} e^{im\theta} \prod_{n=1}^{\infty}(1-z^n)^{-1} \tag{50}$$

The other relevant character for the proof is the character of the representation (c, h) of the coset Virasoro algebra. These are also known [?] (page 467):

$$\chi_{(c,h)}^{Vir}(z) = \Delta_{p,q}^{m}(z) \prod_{n=1}^{\infty}(1-z^n)^{-1} \tag{51}$$

with

$$\Delta_{p,q}^{m}(z) = \sum_{n\in\mathbb{Z}} (z^{\alpha_{p,q}^{m}(n)} - z^{\beta_{p,q}^{m}(n)}) \tag{52}$$

where

$$\alpha_{p,q}^{m}(n) = \frac{[2m(n+1)n - qm + p(m+1)]^2 - 1}{4m(m+1)} \tag{53}$$

$$\beta_{p,q}^{m}(n) = \frac{[2m(m+1)n + qm + p(m+1)]^2 - 1}{4m(m+1)} \tag{54}$$

We will now prove the Proposition by simply comparing both sides of (47). Plugging (51), (50) and (51) in (47), it is immediate that (47) is equivalent to

$$\left(\sum_{m\in\mathbb{Z}+\epsilon} z^{m^2} e^{im\theta}\right) \Delta_{N,l}(z,\theta) = \sum_{q} \Delta_{N+1,\frac{1}{2}[q-1]}(z,\theta) \tag{55}$$

and this equality is proved using symmetry properties of $\Delta_{N,L}$. The details for this step can be found in [?]. □

Corollary 3.12. *A highest weight representation* $(N, l) \otimes (1, \epsilon = 0, 1/2)$ *of* $\hat{\mathfrak{g}} = \hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{su}}(2)$ *can be decomposed in terms of highest weight representations of* $\hat{\mathfrak{h}}$ *and* $\hat{\mathfrak{g}} / \hat{\mathfrak{h}}$ *as follows:*

$$(N, l) \otimes (1, \epsilon) = \bigoplus_{\substack{q \\ p-q \in 2(\mathbb{Z}+\epsilon) \\ 1 \leq q \leq N+2}} (N+1, \frac{1}{2}[q-1]) \otimes (c, h_{p,q}(c)) \tag{56}$$

with $p = 2l + 1$ *and* $h_{p,q}(c)$ *as in* (3)*.*

Proof. This follows directly from the discussion above.

△ The part of page 111 under (2.34) seems unecessary!!

□

4 Highest weight irreducible representations of the super-Virasoro algebra

We will study a method to construct a (representation of the) Super-Virasoro algebra starting from a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of specific types. Afterwards, we will see that, if one takes $\mathfrak{h} = \mathfrak{su}(2)$, this method yields all highest weight unitary irreducible representations of the Super-Virasoro algebra with $c < \frac{3}{2}$, and that representation with all the values of c in the series (7) and all the values of h in (8) are obtained by this method.

To construct a Super-Virasoro algebra, start by taking a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ is of the form $\mathfrak{h}_T \oplus \mathfrak{h}_\nu$ with $\mathfrak{h}_T \cong \mathfrak{h}_\nu$ and $\mathfrak{h}$ is the diagonal subalgebra of $\mathfrak{g}$. (This mimics what we did in the last subsection with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$). Furthermore, the chosen representation of $\hat{\nu}$ will be constructed using fermionic fields in the adjoint representation of $\mathfrak{h}$

have to explain what this means in the previous section

△ if it is the adjoint rep, it should act on h itself, not on the Fock space... ??? Ask Ana how to understand this mathematically. From a discussion with Yannik, it seems that the b^a are coefficients of the adjoint representation: $ad(b = b^a t_a) = b^a ad(t_a) \in Hom(F) \otimes \mathfrak{gl}(\mathfrak{g})$

and act on fermionic Fock spaces.

Define for $z \in S^1$

I think

$$H^a(z) = \sum_r b_r^a z^{-r}, \quad a = 1, 2, ..., \dim \mathfrak{h} \tag{57}$$

with $r \in \mathbb{Z}$ for the Ramond (R) case and $r \in \mathbb{Z} + \frac{1}{2}$ for the Neveu-Schwarz (NS) case, and the fermionic operators b^a_r satisfying

$$\left\{b^a_r, b^b_s\right\} = \delta^{ab}\delta_{r,-s} \tag{58}$$

$$b^a_r|0\rangle = 0, \quad r > 0 \tag{59}$$

$$b^{a\dagger}_r = b^a_{-r} \tag{60}$$

Fermionic normal ordering will be given by

$$: b^a_r b^b_s : = \begin{cases} b^a_r b^b_s, & r < 0 \\ -b^b_s b^a_r, & r > 0 \\ \frac{1}{2}\left[b^a_r, b^b_s\right], & r = 0 \end{cases} \tag{61}$$

Now, using the fields H^a, we construct a (representation of a) basis of $\hat{\mathfrak{h}}_\nu$ defined by

$$\sum_n V^a_n z^{-n} := V^a(z) := -\frac{1}{2} f^{abc} : H^b(z) H^c(z) : \tag{62}$$

i.e. the V^a_n are the coefficients of the Laurent expansion of V^a around zero. Also [?]:

$$[V^a_m, V^b_n] = \left[V^a_m, V^b_m\right] = i f^{abc} V^c_{m+n} + \frac{c^{\mathfrak{h}}}{2} m \delta^{ab}\delta_{m,-n} \tag{63}$$

So that we get a representation of $\hat{\mathfrak{h}}_\nu$ of level $\frac{c^{\mathfrak{h}}}{\psi^2}$ (ψ^2 will be 1 for $\mathfrak{h} \cong \mathfrak{su}(2)$). Now, we take a highest weight irreducible representation $(\frac{2k}{\psi^2}, \lambda)$ of $\hat{\mathfrak{h}}_T$ with generators T^a_n and define

$$T^a(z) := \sum_n T^a_n z^{-1} \tag{64}$$

We can construct a Virasoro algebra with generators K_n as before, with [?]

$$L^g(z) = L^{h_T}(z) + L^{h_v}(z) = \frac{1}{2(k+v)} : T^a(z) T^a(z) : + \frac{1}{4v} : V^a(z) V^a(z) : \tag{65}$$

$$L^h(z) = \frac{1}{2(k+2v)} : \left[T^a(z) + V^a(z)\right]\left[T^a(z) + V^a(z)\right] : \tag{66}$$

where $v = \frac{c^{\mathfrak{h}}}{2}$. This algebra has central element [?]

It's not clear for me why. See p208[Me]

$$c = \frac{k(k+3v)\dim\mathfrak{h}}{2(k+v)(k+2v)} \tag{67}$$

Defining

$$G(z) = \sum_r G_r z^{-n} = [v(k+v)(k+2v)]^{-1/2}\left(kT^a(z)H^a(z) - \frac{v}{3} : V^a(z)H^a(z) :\right) \tag{68}$$

we get a super Virasoro algebra generated by $\{K_n, G_r\}$ [?].

We now do the procedure described above for $h \cong \mathfrak{su}(2)$, with the (N,l) irrep of $\hat{\mathfrak{h}}_T$ and the adjoint fermionic representation as before for $\hat{\mathfrak{h}}_\nu$. By the above argument, this induces a Super Virasoro algebra $SVir(\mathfrak{g},\mathfrak{h})$. An analogous treatment as for the Virasoro algebra asserts that we can have c in the series

$$(4.1) \tag{69}$$

confirming the first part of the claim.

△ I don't see how to extend the argument to the super case. Alternatively, how is the last sentence of the paragraph before (4.1) of [GKO2] justified??

When it comes to the highest weights that we can obtain using this construction, a result analogous to Proposition 3.11 holds, and is proved in a similar manner. We can use one of the two adjoint fermionic representations of $\hat{\mathfrak{h}}_\nu$ which were constructed above (with generators VV_n^a) corresponding to the R and NS cases in (57). These will be denoted $(2,R)$ and $(2,NS)$, respectively, since they have level $c^{\mathfrak{h}} = c^{\mathfrak{su}(2)} = 2$.

Proposition 4.1. *In the following, F stands for either NS or R. Consider the representations $(N, l = \frac{1}{2}(p-1))$ of the first $\hat{\mathfrak{su}}(2)$ in $\mathfrak{g}$ and $(2,F)$ of the second $\hat{\mathfrak{su}}(2)$ in $\hat{\mathfrak{g}}$. These induce representations of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}/\hat{\mathfrak{g}}$. For this choice of representations, we have*

$$\chi^{\hat{\mathfrak{su}}(2)}_{(N,\frac{1}{2}[p-1])}(z,\theta)\chi^{\hat{\mathfrak{su}}(2)}_{(2,F)}(z,\theta) = \sum_{\substack{q \\ p-q\in 2(\mathbb{Z}+\epsilon) \\ 1\le q\le N+2}} \chi^{\hat{\mathfrak{h}}}_{(N+2,\frac{1}{2}[q-1])}(z,\theta)\chi^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}}_{(c,h_{p,q}(c))_F}(z) \tag{70}$$

Corollary 4.2. *A highest weight representation $(N,\frac{1}{2}[p-1])\otimes(2,F)$ of $\hat{\mathfrak{g}} = \hat{\mathfrak{su}}(2)\oplus\hat{\mathfrak{su}}(2)$ can be decomposed in terms of highest weight representations* *of* $\hat{\mathfrak{h}}$ *and* $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$ *as follows:*

$$(N, \frac{1}{2}[p-1]) \otimes (2, F) = \bigoplus_{\substack{q \\ p-q \in 2(\mathbb{Z}+\epsilon) \\ 1 \leq q \leq N+2}} (N+2, \frac{1}{2}[q-1]) \otimes (c, h_{p,q}(c))_F \tag{71}$$

with $p = 2l + 1$ *and* $h_{p,q}(c)$ *as in* (3).

Remark 4.3. The representations $(2, NS)$ and $(2, R)$ of $\hat{\mathfrak{su}}(2)$ can be written in terms of the (N, l) representations as follows: $(2, NS) = (2, 0) \oplus (2, 1)$ and $(2, R) = (2, \frac{1}{2})$.

Notes and Questions about GKO

■ Finish!! 151 handwritten notes 3
■ Do as in p.322 [BLT] 4
■ expand on p.323 [BLT] 4
■ Do as in p.324 [BLT] 4
■ [Maybe should talk about supersymmtery before this section, since it is referred here] 6
■ [See Fuchs (12.20)] 7
■ [They refer this in [GKO1], but did not prove it. I tried once but it was a lot of stuff. I'll try again.] 8
■ [Write proof from [p172 - Me]] 9
■ △ I had to assume that what we mean by "highest weight state of a loop algebra" is one for which $T_0^a |\Psi\rangle = h_a |\Psi\rangle$... Is this it?? In §8.3 of [Fuchs] it seems to be more complex than this. 10
■ △ $\mathfrak{g}$ simple implies $\hat{\mathfrak{g}}$ simple? See p211 of my notes. 10
■ △ Def of character does not match with expression on page 110. . . 10
■ Find good reference for this. 11
■ [To explain why we can do this: can put in appendix my "Cartansu2.pdf" file...] 11
■ △ Actually, T3 has eigenvalues (weights) +- 1/2, so it can be negative! So why do they write this? 12
■ See pages 206,207 [Me]. (As a lemma, also show that the loop algebra of a direct sum is a direct sum of the loop algebras - see p208) 12
■ △ how does this make sense? The domain of the LHS is larger than the domain on the RHS. 13
■ I think I was able to show on page 211 [Me] that loop algebras of simple algebras are simple! - write here the proof 14
■ △ Is this true? Should prove. Just don't have time for all these details. 14
■ This must be wrong: the sum should only be over the weights of h... 14
■ △ The part of page 111 under (2.34) seems unecessary!! 16
■ have to explain what this means in the previous section 16

△ if it is the adjoint rep, it should act on h itself, not on the Fock space... ??? Ask Ana how to understand this mathematically. From a discussion with Yannik, it seems that the b^a are coefficients of the adjoint representation: $ad(b = b^a t_a) = b^a ad(t_a) \in Hom(F) \otimes \mathfrak{gl}(\mathfrak{g})$ 16
I think 16
It's not clear for me why. See p208[Me] 17
△ I don't see how to extend the argument to the super case. Alternatively, how is the last sentence of the paragraph before (4.1) of [GKO2] justified?? 18

References