A quick mathematical take on tensors, for physicists

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Proposition 1. *Let V and W be vector spaces over the field K. The set $\mathcal{L}(V,W)$ of all linear maps $f : V \to W$ together with the usual addition of maps and the usual multiplication of a map by a scalar (*i.e.* element of K) is a vector space over K.*

Proof. One has to check all properties of vector spaces. This is well done here: Proof that L(V,W) is a vector space ☐

Corollary 2. *Let V be a vector space over the field K. The set V^* of all linear maps $f : V \to K$ (also called functionals) together with the usual addition of maps and the usual multiplication of a map by a scalar (element of K) is a vector space over K.*

Proof. Comes straight from proposition 1, with $W = K$ (it is known that all fields are vector spaces over themselves). ☐

Definition 3. Let V be a vector space.

The *dual space* of V is the vector space V^* described above. The elements of V^* are called *functionals*, *one-forms* or *dual vectors*.

Proposition 4. *Let V be a finite-dimensional vector space of dimension n, and $\{e_j\}$ a basis for V. The set*

$$\{\alpha^i : V \to K \text{ such that} \forall v = \sum_{i=1}^{n} v^i e_i \in V, \alpha^i(v) = v^i\} \tag{1}$$

is a basis for V^*. In particular, V^* is n-dimensional.

Proof. Recall that any linear map is completely defined by its action on the basis of the domain. In this case, we see that

$$\forall j, \alpha^i(e_j) = \delta^i_j \tag{2}$$

Let us see that $\{\alpha^i\}$ spans V^*.

Let $f \in V^*$ and $v \in V$. Then,

$$f(v) = \sum v^i f(e_i) = \sum f(e_i)\alpha^i(v) = \left(\sum f(e_i)\alpha^i\right) v \tag{3}$$

So $f = \sum f(e_i)\alpha^i$, and thus $\{\alpha^i\}$ spans V^*.

We still have to prove linear independence.

Suppose $\sum c_i\alpha^i = 0$, for some $c_i \in K$. Let $j \in \{1, 2, ..., n\}$.We apply both sides to e_j and get

$$\sum c_i \delta^i_j = c_j = 0 \tag{4}$$

Done. ☐

Corollary 5. *V and V^* are isomorphic (but this isomorphism may depend on a choice of basis).*

Proof. This is true simply because all n-dimensional vector spaces are isomorphic, and this isomorphism may depend on a choice of basis. ☐

Lemma 6. *Let V be a finite-dimensional vector space. If $v \in V$ and $\forall w \in V^*, w(v) = 0$, then $v = 0$.*

Proof. We shall prove the contrapositive.

Suppose $v \neq 0$. We can explicitly build a functional w with $w(v) \neq 0$:

The set $\{v\}$ is obviously linearly independent. We can therefore extend it to a basis (well-known result of linear algebra) $\{v, v_1, ..., v_{n-1}\}$. We can define $w \in V^*$ by $w(v) = 1$ and $w(v_i) = 0$ (we know that a *linear* map is completely specified by its action on a basis, so this defines w completely). $\square$

Theorem 7. *Let V be a finite-dimensional vector space. The double dual V^{**} is naturally isomorphic to V (*i.e.* there is a basis-independent isomorphism between them).*

Proof. Consider the map

$$\begin{array}{rcl} T : V & \to & V^{**} \\ v & \mapsto & T(v) \end{array}$$

where

$$\begin{array}{rcl} T(v) : V^* & \to & K \\ w & \mapsto & w(v) \end{array}$$

We just have to check that T is both linear and bijective.

Let $w \in V^*$, $v, u \in V$ and $a \in K$. We have

$$T(v+u)(w) = w(v+u) = w(v) + w(u) = T(v)(w) + T(u)(w) = \left(T(v) + T(u)\right)(w) \quad (5)$$

and

$$T(av)(w) = w(av) = aw(v) = aT(v)(w) \quad (6)$$

Hence T is linear.

Now, since T is linear, it is injective iff $\ker T = \{0\}$.

Let $v \in \ker T$. Let $w \in V^*$. Then

$$\begin{array}{rrcl} & T(v)(w) & = & 0 \\ \iff & w(v) & = & 0 \end{array} \quad (7)$$

Since this is valid for any $w \in V^*$, then $v = 0$ from lemma 6. This shows that T is injective.

Now, the dimensionality theorem tells us that

$$\dim V - \dim \ker T = \dim \mathrm{Im} T \quad (8)$$

So $\dim \mathrm{Im} T = \dim V = \dim V^{**}$. since the only vector subspace of V^{**} with dimension $\dim V^{**}$ is V^{**} itself. Hence T is surjective. $\square$

Definition 8. A *pre-Hilbert space* (also called *inner product space* or *Euclidean space*) is a vector space V together with an *inner product*, which is a map $\langle \cdot, \cdot \rangle : V \times V \to K$ such that for all $v, w, u \in V$ and $a \in K$,

1. $\langle v, v \rangle \geq 0$
2. $\langle v, v \rangle = 0 \iff v = 0$
3. $\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$
4. $\langle av, w \rangle = a \langle v, w \rangle$
5. $\langle v, w \rangle = \langle w, v \rangle^*$

Theorem 9. *Let V be a pre-Hilbert space. Then, V and V^* are naturally isomorphic.*

Proof. Consider the map

$$\begin{aligned} T : V &\to V^* \\ v &\mapsto T(v) \end{aligned}$$

where $\forall u \in V$, $T(v)(u) = \langle v, u \rangle$.

I'll leave it to you to check that T is linear and bijective. □

Remark 10. The process of going from an element of a pre-Hilbert space V to the corresponding element of V^* is usually called "lowering of indices" (and the reverse is called "raising of indices"). I shall not delve into this, but it should be clear that the existence of an inner product (which may be the metric tensor) is essential for this connection between V and V^* to exist.

Example 11. In quantum mechanics, the state of our system is described by a *ket* $|\psi\rangle$, which is a vector of a vector space called the *state space* $\mathcal{E}$. This space is actually a pre-Hilbert space, possessing an inner product $(\cdot, \cdot)$ defined by $(|\psi\rangle, |\phi\rangle) = \int \psi(x)\phi(x)dx$. Take a map T as in the proof of theorem 9. We see that $T(|\psi\rangle)(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$. Physicists usually write $T(|\psi\rangle) = \langle\psi|$, called a *bra*, and $T(|\psi\rangle)(|\phi\rangle) = \langle\psi|\phi\rangle$.

Definition 12. Let V be a finite-dimensional vector space.

A *tensor of type* (k, l) *over* V *is a multilinear map*

$$T : \underbrace{V \times ... \times V}_{k} \times \underbrace{V^* \times ... \times V^*}_{l} \to K \tag{9}$$

Remark 13. A $(1, 0)$-tensor is a one-form. A $(0, 1)$-tensor is an element of V^{**}. Since we identify naturally V^{**} with V, we can think of it as a vector of V.

It turns out that the set of all (k, l)-tensors is a vector space. In a change of basis, the components of one of the tensors of this space change in a certain way. Some physicists define tensors by the way these components change under a change of basis.

References

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