The de Rham Cohomology for the n-Sphere

I already proved that

$$H_p(S^1) = \begin{cases} \mathbb{R} & , p = 0, 1 \\ 0 & , \text{otherwise} \end{cases} \tag{1}$$

I now want to prove that

$$\boxed{H_p(S^2) = \begin{cases} \mathbb{R} & , p = 0, 2 \\ 0 & , \text{otherwise} \end{cases}} \tag{2}$$

Proof. Consider the 2-sphere S^2 with poles $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Let $U = S^2 \backslash \{N\}$ and $V = S^2 \backslash \{S\}$. Notice that

$$U \cap V = S^2 \backslash \{S, N\} \cong S^1 \times \mathbb{R} \simeq S^1 \tag{3}$$

where $\cong$ denotes the homeomorphism relation and $\simeq$ denotes the homotopy equivalence relation.

The Mayer-Vietoris Sequence is

$$\begin{array}{ccccccccc} 0 & \to & H_0(S^2) & \to & H_0(U) \oplus H_0(V) & \to & H_0(U \cap V) & \to \\ & \to & H_1(S^2) & \to & H_1(U) \oplus H_1(V) & \to & H_1(U \cap V) & \to \\ & \to & \dots & & & & & \end{array} \tag{4}$$

Now, $H_0(S^2) = \mathbb{R}$ because S^2 has one connected component. For the same reason, $H_0(U) = H_0(V) = H_0(U \cap V) = \mathbb{R}$.

Also, U and V are homotopically equivalent to $\mathbb{R}^2$. Hence,

$$H_p(U) = H_p(V) = H_p(\mathbb{R}^2) = \begin{cases} \mathbb{R} & , p = 0 \\ 0 & , \text{otherwise} \end{cases} \tag{5}$$

and, since $U \cap V \simeq S^1$, we have $H_p(U \cap V) = H_p(S^1)$.

Thus, we can rewrite the MVS as follows:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \to & \mathbb{R} & \to \\ & \to & H_1(S^2) & \to & 0 & \to & H_1(S^1) = \mathbb{R} & \to \\ & \to & H_2(S^2) & \to & 0 & \to & H_2(S^1) = 0 & \to \\ & \to & 0 & \to & 0 & \to & \dots & \end{array} \tag{6}$$

And now I use a theorem that I read in Tu's book:

Lemma 1. *Let $0 \to A^0 \to A^1 \to \dots \to A^m \to 0$ be an exact sequence of finite-dimensional vector spaces.*

Then,

$$\sum_{k=0}^{m} (-1)^k \dim A^k = 0 \tag{7}$$

In our case, if we restrict ourselves to the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \to & \mathbb{R} & \to \\ & \to & H_1(S^2) & \to & 0 & \to & H_1(S^1) = \mathbb{R} & \to \\ & \to & H_2(S^2) & \to & 0 & & & \end{array} \tag{8}$$

we get

$$1 - 2 + 1 - x_1 + 0 - 1 + x_2 = 0 \tag{9}$$

with $x_1 = \dim H_1(S^2)$ and $x_2 = \dim H_2(S^2)$.
So

$$x_2 - x_1 = 1 \tag{10}$$

Now, part of my exact sequence is

$$... \to 0 \overset{\partial}{\to} \mathbb{R} \overset{\partial'}{\to} H_2(S^2) \overset{\partial''}{\to} 0 \to ... \tag{11}$$

and therefore

$$\ker \partial' = \mathrm{Im}\partial = 0 \tag{12}$$

because ∂ is an homomorphism. Also,

$$\mathrm{Im}\partial' = \ker \partial'' = H_2(S^2) \tag{13}$$

and therefore ∂' is bijective and hence an isomorphism. We conclude that $\dim H_2(S^2) = \dim \mathbb{R} = 1$, so $H_2(S^2) = \mathbb{R}$ (up to vector space isomorphism).
Using (10), we see that $H_1(S^2) = 0$. □

The expressions for $H_p(S^1)$ and $H_p(S^2)$ suggest that

$$\boxed{H_p(S^n) = \begin{cases} \mathbb{R} & , p = 0, n \\ 0 & , \text{otherwise} \end{cases}} \tag{14}$$

Let us prove that this is indeed true.

Proof. We shall prove by induction.

We already know that it is true for $n = 1$. Now, suppose it is true for $n = k - 1$. In a first moment, we proceed exactly as in the proof for the $n = 2$ case: where there is a 2, we replace it by an n. Things start to get little different in (6). In fact, in this case our MVS is

$$\begin{array}{cccccccc} 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \to & \mathbb{R} & \to \\ & \to & H_1(S^k) & \to & 0 & \to & H_1(S^{k-1}) \overset{\text{I.H.}}{=} 0 & \to \\ & \to & H_2(S^k) & \to & 0 & \to & H_2(S^{k-1}) \overset{\text{I.H.}}{=} 0 & \to \\ & \to & ... & & & & & \\ & \to & H_{k-1}(S^k) & \to & 0 & \to & H_{k-1}(S^{k-1}) \overset{\text{I.H.}}{=} \mathbb{R} & \to \\ & \to & H_k(S^k) & \to & 0 & \to & 0 & \to \\ & \to & 0 & \to & ... & & & \end{array} \tag{15}$$

Again, using lemma 1, we can easily see that

$$\begin{cases} 1 - 2 + 1 - x_1 - 1 + x_2 = 0 \iff x_2 - x_1 = 1 & , \text{if } 3(k-1) \text{ is even} \\ 1 - 2 + 1 - x_1 + 1 - x_2 = 0 \iff x_1 + x_2 = 1 & , \text{if } 3(k-1) \text{ is odd} \end{cases} \tag{16}$$

with $x_1 = \dim H_1(S^k)$ and $x_2 = \dim H_k(S^k)$.
Now, part of my exact sequence is

$$... \to 0 \overset{\partial}{\to} \mathbb{R} \overset{\partial'}{\to} H_k(S^k) \overset{\partial''}{\to} 0 \to ... \tag{17}$$

and therefore

$$\ker \partial' = \mathrm{Im}\partial = 0 \tag{18}$$

because ∂ is an homomorphism. Also,

$$\mathrm{Im}\partial' = \ker \partial'' = H_k(S^k) \tag{19}$$

and therefore ∂' is bijective and hence an isomorphism. We conclude that $\dim H_2(S^2) = \dim \mathbb{R} = 1$, so $H_2(S^2) = \mathbb{R}$ (up to vector space isomorphism).

Using (16), we conclude that $H_1(S^k) = 0$. $\square$