Towards Modular Categories

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Abstract

The goal of these notes is to take the shortest route towards modular categories. A big part of the text is dedicated to Abelian Categories, since after understanding the basics of abelian categories we are just a bunch of definitions away from the concept of modular category.
We assume knowledge of basic category theory and the basics of monoidal categories.

Contents

1 Abelian Categories

Abelian categories generalize the category **Ab** of abelian groups, and, more generally, the category $R-\mathbf{Mod}$ of left-modules over the ring R, extracting their "nice" properties. In particular, we will be able to develop homological algebra inside abelian categories.

We will start by exploring the important concepts of Kernels and Cokernels in a category. We will need these throughout the rest of this text. We then go on to discuss **Ab**-enriched categories,whose hom-sets are abelian groups. If such a category has a zero object and finite coproducts (direct sums), we get ourselves an additive category. A particular important example are the linear categories, whose hom-sets are actually vector spaces. An additive category which not only has all finite coproducts, but all finite limits and colimits is called pre-abelian. As we will see, this is equivalent to every morphism having "a kernel and a cokernel". If a pre-abelian category is such that every mono is a "kernel" and every epi is a "cokernel", then we are in the presence of an abelian category. I will follow mostly [2] and [3] , but also [1], [4], [5] and [6]. We will assume all categories to be locally small. Recall that **Set** is locally small (clear if one thinks of a map $f : A \to B$ as a relation $f \subseteq A \times B$ with the usual conditions for being a function – then $\mathrm{Hom}(A, B) \subseteq \mathcal{P}(A \times B)$, which is a proper set), and therefore any other category whose arrows are just maps between proper sets with possibly some other extra conditions (like continuity) is also locally small. Hence **Mon**, **Grp**, **Ab**, **Top**, **Man**, **Ring**, **R-Mod**, $\mathbf{Vect}_{\mathbb{K}}$, **Pos** and $\mathbf{cFA}_{\mathbb{K}}$ (and thus **2TQFT**) are all locally small.

1.1 Kernels and Cokernels

Definition 1. A *zero object* 0 is an object that is both terminal and initial. Given two objects A, B, the *zero arrow* $0 = 0_{AB} : A \to B$ between them is the composition of the unique arrows $A \to 0 \to B$.

Notice that the composition of two arrows with one of them a zero arrow is a zero arrow: clear using uniqueness of the arrows with domain 0 or codomain 0.

Definition 2. Let $\mathcal{C}$ be a category with a zero object. A *kernel* of $A \xrightarrow{f} B$ is an equalizer of $A \underset{0}{\overset{f}{\rightrightarrows}} B$.

Dually, the *cokernel* of f is a coequalizer of $A \underset{0}{\overset{f}{\rightrightarrows}} B$.

Remark 3. (i) Explicitly, $K \xrightarrow{k} A$ is a kernel of $A \xrightarrow{f} B$ if $fk = 0k = 0$ and if $fu = 0$ then u factors uniquely through k:

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[rr, bend left, "0{=}0_{KB}"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \\
C \arrow[u, dashed, "\bar{u}"] \arrow[ur, "u"'] & &
\end{tikzcd}$$

(ii) Being an equalizer, a kernel is unique up to isomorphism and is a mono. Notice that this holds precisely because two kernels k, k' factor through each other. We can therefore think of a kernel $k : K \to A$ as a subobject of A *i.e.* an equivalence class of monos with codomain A, two monos being equivalent iff they factor through each other.

Example 4. Let $f : G \to H$ be an arrow in **Grp**.

(i) Consider the inclusion $i : \ker f \to G$, where $\ker f$ is the usual kernel of f. Then $i : \ker f \to G$ is a (categorical) kernel of f.
Indeed: $(fi)x = f(ix) = fx = e_H$, and $(0_{GH}i)x = (0_{0H}0_{G0}i)x = 0_{0H}0_{G0}x = 0_{0H}(*) = e_H$, where I used that $0_{G0_{0H}0_{G0}}$ is the unique arrow from G to the zero object $0 = \{*\}$ of **Grp**. All in all, we see that

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H
\end{tikzcd}$$

is a fork.We now take another fork

$$\begin{tikzcd}
J \arrow[r, "j"] & G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H
\end{tikzcd}$$

Since for $y \in J$ we have $(fj)y = e_H$, then $jy \in \ker f$. We can therefore define $\bar{j} : J \to \ker f$ by $\bar{j}y = jy$. Clearly $j = i\bar{j}$. Finally, $\bar{j}$ is the unique arrow making

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H \\
J \arrow[u, dashed, "\bar{j}"] \arrow[ur, "j"'] & &
\end{tikzcd}$$

commute: if $i\bar{j} = j = i\tilde{j}$, then (since i is clearly an injective homomorphism and thus monic) $\bar{j} = \tilde{j}$. This shows that $i : \ker f \to G$ is a kernel of f. ■

(ii) The quotient map $p : H \to {}^{H}\!/_{\operatorname{im} f}$ is a (categorical) cokernel of f. **Indeed:** notice that for $x \in G$, $(pf)x = p(fx) = [fx] = [e_H \cdot fx] = [e_H] = p(0x)$. Hence

$$\begin{tikzcd}
G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H \arrow[r, "p"] & {}^{H}\!/_{\operatorname{im} f}
\end{tikzcd}$$

is a cofork. Now, given another cofork

$$\begin{tikzcd}
G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H \arrow[r, "j"] & J
\end{tikzcd}$$

we see that $jfx = j0x = j(e_H) = e_J$, and thus for $y \in H$, $j(y \cdot fx) = (jy) \cdot (jfx) = jy$, meaning that j is constant on each class of ${}^{H}\!/_{\operatorname{im} f}$. So if we define $\bar{j} : {}^{H}\!/_{\operatorname{im} f} \to J$ by picking a representative of the class $(\bar{j}([y]) := j(y + fx) = j(y))$ then by construction $\bar{j}p = j$. Furthermore, if $\tilde{j}$ is another arrow such that $\tilde{j}p = j$, then $\tilde{j} = \bar{j}$ since p is epic (it is clearly a surjective homomorphism). This shows uniqueness of $\bar{j}$.

$$\begin{tikzcd}
G \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & H \arrow[r, "p"] \arrow[dr, "j"'] & H/\operatorname{im} f \arrow[d, dashed, "\bar{j}"] \\
& & J
\end{tikzcd}$$

Thus, p is a cokernel for f. ■

1.2 Ab-enriched Categories

Definition 5. An ***Ab****-enriched* category is a category $\mathcal{C}$ such that every hom-set has the structure of an abelian group $(\mathrm{Hom}(A,B),+)$ such that the composition of morphisms is biadditive ($i.e.$ homomorphism):
for

$$A \underset{g'}{\overset{g}{\rightrightarrows}} B \qquad B \underset{f'}{\overset{f}{\rightrightarrows}} C$$

we have

$$(f+f')\circ(g+g') = (f\circ g)+(f\circ g')+(f'\circ g)+(f'\circ g')$$

Remark 6. (i) We denote by 0_{AB} the identity element of the group $\mathrm{Hom}(A,B)$. We have $0_{AC} = 0_{BC}0_{AB}$ for any objects A, B, C: In particular, $f0_{BA} = 0_{BC}$ and $0_{CB}f = 0_{AB}$

(ii) For all A, B objects of an **Ab**-enriched category, $(\mathrm{Hom}(A,B),+,\circ)$ is a ring.

Example 7. (i) The category **Ab** of abelian groups is an **Ab**-enriched category. For objects G and H, the group structure in $\mathrm{Hom}(G,H)$ is induced by the group structure in H: $(f+g)(x) := f(x)+g(x)$. It is easy to check that the composition is biadditive: $[(f+f')\circ(g+g')](x) = (f+f')(g(x)+g'(x)) = f(g(x)+g'(x)) + f'(g(x)+g'(x)) = fg(x)+fg'(x)+f'g(x)+f'g'(x)$ ■

(ii) Let $(R,+,\cdot)$ be a ring with unit. Since $(R,\cdot)$ is a monoid, we can see it as a category with a single object $*$, and whose arrows are $\mathrm{Hom}(*,*) = R$ and composition is $\cdot$. Also $(\mathrm{Hom}(*,*),+)$ is an abelian group, and we know that $\cdot$ is biadditive (by the definition of ring). Hence, R is an **Ab**-enriched category. ■

Definition 8. A *biproduct diagram* for the objects A, B of an **Ab**-enriched category $\mathcal{C}$ is a diagram

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} C \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

such that

$$p_1i_1 = id_A, \quad p_2i_2 = id_B, \quad i_1p_1 + i_2p_2 = id_C$$

The justification for the name "biproduct diagram" is the next Theorem, which encompasses the most important property of **Ab**-enriched categories [3].

Theorem 9. *Let $\mathcal{C}$ be an* ***Ab****-enriched category. Two objects $A, B \in \mathcal{C}$ have a product if and only if they have a biproduct diagram in $\mathcal{C}$. Furthermore, $A \xleftarrow{p_1} C \xrightarrow{p_2} B$ is a product of A, B and $A \xrightarrow{i_1} C \xleftarrow{i_2} B$ is a coproduct of A, B.*

Proof. Suppose A, B have a biproduct diagram. Consider a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ A & & B \end{tikzcd}$$

and define $f = i_1 f_1 + i_2 f_2$. Then $p_1 f = p_1(i_1 f_1 + i_2 f_2) = id_A f_1 + 0 f_2 = f_1$, where I used that $p_2 i_1 = 0_{BA} = 0$ (indeed: $p_2 i_1 = p_2(i_1 p_1 + i_2 p_2) i_1 = p_2 i_1 id_A + id_B p_2 i_1 = p_2 i_1 + p_2 i_1$ so that $p_2 i_1 = 0$). Similarly, $p_2 f = f_2$. Hence the diagram

$$\begin{tikzcd} & Z \arrow[ddl, "f_1"'] \arrow[d, "f"] \arrow[ddr, "f_2"] & \\ & C \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ A & & B \end{tikzcd}$$

commutes. To see that f is unique, let g be another arrow $Z \to C$ such that the diagram above commutes (with g instead of f). Then $g = (i_1 p_1 + i_2 p_2) g = i_1 f_1 + i_2 f_2 = f$. This asserts uniqueness. Hence $A \xleftarrow{p_1} C \xrightarrow{p_2} B$ is a product of A, B. (By a very similar reasoning, one can prove that $A \xrightarrow{i_1} C \xleftarrow{i_2} B$ is a coproduct of A, B).
Conversely, assume that A, B have a product $A \xleftarrow{p_1} C \xrightarrow{p_2} B$. We want to construct i_1 and i_2 so that we obtain a biproduct diagram. We just use the product property and define i_1 as the unique morphism satisfying the commutative diagram

$$\begin{tikzcd}
 & A \arrow[ddl, "id_A"'] \arrow[d, dashed, "i_1"] \arrow[ddr, "0_{AB}"] & \\
 & C \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
A & & B
\end{tikzcd}$$

and similarly for i_2. We see that the first two conditions for $A \underset{p_1}{\overset{i_1}{\rightleftarrows}} C \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$ to be a biproduct diagram are immediately satisfied. We just need to check the third:

$$p_1(i_1p_1 + i_2p_2) = id_A p_1 + 0p_2 = p_1$$

$$p_2(i_1p_1 + i_2p_2) = 0p_1 + id_B p_2 = p_2$$

By the product property, the projections uniquely define $i_1p_1 + i_2p_2 =: f$:

$$\begin{tikzcd}
 & C \arrow[ddl, "p_1"'] \arrow[d, "f"] \arrow[ddr, "p_2"] & \\
 & C \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
A & & B
\end{tikzcd}$$

Since $f = id_C$ works, then by uniqueness $f = i_1p_1 + i_2p_2 = id_C$.
(notice that again, by the same token as in the first part of the proof, $A \xrightarrow{i_1} C \xleftarrow{i_2} B$ is a coproduct of A, B). $\square$

Corollary 10. *In* $\boldsymbol{Ab}$*-enriched categories, two objects* A, B *have a product if and only if they have a coproduct, and they coincide (up to isomorphism). As a consequence, all finite products coincide with finite coproducts.*

Corollary 11. *Let* $\mathcal{C}$ *be an* $\boldsymbol{Ab}$*-enriched category and* O *be an initial object. Then* O *is also a terminal object (and thus a zero object). Conversely, if* O *is a final object, then it is also an initial object (and thus a zero object).*

Proof. Just recall that a terminal object is the product of the empty diagram, and that an initial object is the coproduct of the empty diagram. Then Theorem 9 gives the result. $\square$

Definition 12. A *biproduct* is an object that is both (the object part of) a product and (the object part of) a coproduct. Notice that in **Ab**-enriched categories these three concepts coincide.

Another important property of **Ab**-enriched categories is that, in such categories, all equalizers are kernels. This means that equalizers and kernels are "the same thing".

Proposition 13. *In an **Ab**-enriched category, an arrow $A \xrightarrow{e} B$ is an equalizer iff it is a kernel*

Proof. We know by definition of kernel that all kernels are equalizers. So we just have to show the converse.
Let

$$A \xrightarrow{e} B \overset{f}{\underset{g}{\rightrightarrows}} C$$

be an equalizer diagram in an **Ab**-enriched category $\mathcal{C}$. In particular, $fe = ge \iff fe - ge = 0_{AC} \iff (f-g)e = 0_{BC}e$. Hence,

$$A \xrightarrow{e} B \overset{f-g}{\underset{0}{\rightrightarrows}} C$$

is a fork. Now consider another fork

$$J \xrightarrow{j} B \overset{f-g}{\underset{0}{\rightrightarrows}} C$$

By the same argument, we get another fork

$$J \xrightarrow{j} B \overset{f}{\underset{g}{\rightrightarrows}} C$$

so that we get an unique $\bar{j} : J \to A$ such that $j = e\bar{j}$. This same $\bar{j}$ ensures that e is an equalizer for $(f-g, 0)$ and hence a kernel for $f-g$. □

Lemma 14. *Let $f : A \to B$ be an arrow in an **Ab**-enriched category $\mathcal{C}$. If* $\ker f = 0$, *then f is a mono.*

Proof. Suppose $fg = fg'$ for composable g, g', and let $\ker f = 0$.
Then

$$Y \overset{g}{\underset{g'}{\rightrightarrows}} B \xrightarrow{b} C$$

is cofork, and thus also is

$$Y \underset{0'}{\overset{g-g'}{\rightrightarrows}} B \xrightarrow{b} C$$

So clearly

$$Y \xrightarrow{g-g'} A \underset{0}{\overset{b}{\rightrightarrows}} C$$

is a fork. Now, since

$$A \xrightarrow{0} B \underset{0}{\overset{b}{\rightrightarrows}} C$$

is an equalizer diagram, and therefore $\exists!\ u : Y \to A$ such that $0u = g - g'$. Hence $g - g' = 0$, so that $g = g'$. $\square$

1.3 Additive and Linear Categories

Definition 15. A category $\mathcal{C}$ is said to be an *additive category* if:

(A1) $\mathcal{C}$ is an **Ab**-enriched category.

(A2) $\mathcal{C}$ has a zero object 0.

(A3) For any objects $A, B \in \mathcal{C}$, there exists their coproduct $A \oplus B$. This is called the *direct sum* of A and B.

Remark 16. (i) Clearly (A3) implies the existence of all finite coproducts, and Theorem 9 tells us that we also have all finite products. Notice that because of Theorem 9 we could write "$\mathcal{C}$ has a terminal(initial) object 0" instead of (A2).

(ii) Explicitly, (A3) says that there are morphisms $A \xrightarrow{i_1} A \oplus B \xleftarrow{i_2} B$ such that, given any other pair of arrows $A \xrightarrow{f_1} C \xleftarrow{f_2} B$, there is a unique morphism $A \oplus B \xrightarrow{f} C$ such that

$$\begin{tikzcd}
 & C & \\
 & A \oplus B \arrow[u, dashed, "f"] & \\
A \arrow[uur, "f_1"] \arrow[ur, "i_1"'] & & B \arrow[uul, "f_2"'] \arrow[ul, "i_2"]
\end{tikzcd}$$

commutes. Another alternative (adopted in [2]) is given by using Theorem 9: for any objects $A, B \in \mathcal{C}$, there is a product diagram for them.

(iii) We know that all coproducts are unique up to isomorphism. As usual, a choice of a coproduct $A \oplus B$ for each pair of objects A, B defines a functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which acts on arrows $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$ by giving the unique arrow $A \oplus B \xrightarrow{f \oplus g} A' \oplus B'$ such that the diagrams

$$\begin{tikzcd} A \oplus B \arrow[r, "f\oplus g"] & A' \oplus B' \\ A \arrow[u, "i_1"] \arrow[r, "f"'] & A' \arrow[u, "i'_1"'] \end{tikzcd} \qquad \begin{tikzcd} A \oplus B \arrow[r, "f\oplus g"] & A' \oplus B' \\ B \arrow[u, "i_2"] \arrow[r, "g"'] & B' \arrow[u, "i'_2"'] \end{tikzcd}$$

commute. (Note that uniqueness comes from the coproduct (universal) property - just like we proved uniqueness of f in the proof of Theorem 9). Equivalently, since $A \oplus B$ is not only a coproduct $A \sqcup B$ but also a product $A \times B$, we can always define $f \oplus g$ as the unique arrow such that the diagrams

$$\begin{tikzcd} A \oplus B \arrow[r, "f\oplus g"] \arrow[d, "p_1"'] & A' \oplus B' \arrow[d, "p'_1"] \\ A \arrow[r, "f"'] & A' \end{tikzcd} \qquad \begin{tikzcd} A \oplus B \arrow[r, "f\oplus g"] \arrow[d, "p_2"'] & A' \oplus B' \arrow[d, "p'_2"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

commute.

Example 17. The category **Ab** of abelian groups is an additive category: we already saw that it is **Ab**-enriched. Also, we know that the direct product of abelian groups G, H is again an abelian group, and is the product (and hence the coproduct) of G, H in **Ab**. So we have coproducts. Finally, the zero object is clearly just the trivial group $\{*\}$, just like in **Grp**.
Clearly, by an identical argument $\mathbf{Vect}_{\mathbb{K}}$ and $\mathbf{FinVect}_{\mathbb{K}}$ are also additive categories.

We now allow the hom-sets to be not only abelian groups but actually vector spaces.

Definition 18. Let $\mathbb{K}$ be a field. An additive category $\mathcal{C}$ is said to be $\mathbb{K}-$*linear* if every hom-set $\mathrm{Hom}(A, B)$ is equipped with a scalar composition turning $\mathrm{Hom}(A, B)$ into a $\mathbb{K}$-vector space, and such that composition of arrows is linear.

Example 19. The additive categories $\mathbf{Vect}_{\mathbb{K}}$ and $\mathbf{FinVect}_{\mathbb{K}}$ are $\mathbb{K}$-linear. The scalar multiplication of the morphisms in a hom-set $\mathrm{Hom}(V, W)$ is induced by the scalar multiplication in W: $(\lambda \cdot f)(v) := \lambda \cdot (f(v)), \quad \lambda \in \mathbb{K}$. Since we already know that composition is biadditive, we just have to check that $[(\lambda f) \circ (\gamma g)](v) = (\lambda f)((\gamma g)(v)) = (\lambda f)(\gamma g(v)) = \lambda f(\gamma g(v)) = \lambda\gamma f(g(v)) = [\lambda\gamma f \circ g](v)$ (where I used linearity of f). This shows that composition is linear. In $\mathbf{FinVect}_{\mathbb{K}}$, the hom-sets are finite dimensional, with dimension $\dim \mathrm{Hom}(V, W) = \dim(V^* \otimes W) = \dim V \dim W$.

As usual, functors that preserve these structures will be given a special name.

Definition 20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\mathbf{Ab}$-enriched categories. Then F is *additive* if $\forall A, B \in \mathcal{C}$, the map

$$\begin{gathered}\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(FA, FB)\\ f \mapsto Ff\end{gathered}$$

is a homomorphism of groups.
If $\mathcal{C}, \mathcal{D}$ are actually $\mathbb{K}$-linear, then F is $\mathbb{K}$-*linear* if the above homomorphisms are $\mathbb{K}$-linear.

Example 21. (i) Let $\mathcal{C}$ be an $\mathbf{Ab}$-enriched category (it particular, $\mathcal{C}$ has small hom-sets). The hom-functors

$$\mathcal{C}^{op} \xrightarrow{\mathrm{Hom}(-,A)} \mathbf{Ab} \qquad \mathcal{C} \xrightarrow{\mathrm{Hom}(A,-)} \mathbf{Ab}$$

are additive functors. Let us check this for the first hom-functor. Given $B \underset{f'}{\overset{f}{\rightrightarrows}} B' \xrightarrow{g} A$ in $\mathcal{C}$ and denoting $F = \mathrm{Hom}(-, A)$, we have:

$$\begin{aligned} F(f^{op} + f'^{op})(g) &= g \circ (f + f') \\ &= gf + gf' \\ &= F(f^{op})(g) + F(f'^{op})(g) \\ &= (F(f^{op}) + F(f'^{op}))(g) \end{aligned}$$

Hence F is additive. ■

(ii) If $\mathcal{C}$ and $\mathcal{D}$ are **Ab**-enriched categories, so is $\mathcal{C} \times \mathcal{D}$. Indeed, it is easy to see that the hom-set $\operatorname{Hom}((A, D), (B, E))$ will be the set $\operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{D}}(C, D)$, which has the structure of the product of abelian groups, thus being an abelian group.
The projections $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ are additive functors, and so is the tensor product of abelian groups $\mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$.

1.4 Abelian Categories

Definition 22. An *abelian category* is an additive category $\mathcal{C}$ such that:

(B1) Every arrow has a kernel and a cokernel.

(B2) Every mono is a kernel and every epi is a cokernel.

Remark 23. Because of Proposition 13, (B2) can be restated as "every mono is regular and every epi is regular" (*i.e.* every mono is an equalizer and every epi is a coequalizer).

Example 24. The categories **Ab**, **R-Mod**, **Vect** and **FinVect** are abelian. Let's focus on **Ab**: just as in Example 4, the kernel of an arrow $f : G \to G$ are $\ker f \xrightarrow{i} G$ with $\ker f$ the usual kernel and i the inclusion, and the cokernel is the quotient map $H \xrightarrow{p} H /_{\operatorname{im} f}$. This establishes (B1). For (B2): let $G \xrightarrow{m} H$ be a mono. Then m is a cokernel for the quotient map $H \xrightarrow{p} H /_{\operatorname{im} f}$. Indeed, for $x \in G$ we have $pfx = [fx] = [e_H] = 0(fx)$, so that we get a fork diagram. We now consider another fork diagram with first arrow $J \xrightarrow{j} H$:

$$\begin{tikzcd}
J \arrow[d, dashed, "\bar{j}"'] \arrow[dr, "j"] & & \\
G \arrow[r, hook, "f"] & H \arrow[r, shift left, "p"] \arrow[r, shift right, "0"'] & H /_{\operatorname{im} f}
\end{tikzcd}$$

Now, since $pj = 0$, then $\operatorname{im} j \subseteq \operatorname{im} f$, so that we can simply take $\bar{j} = \tilde{f}^{-1} j$, with $\tilde{f} : G \to \operatorname{im} f,\ x \mapsto fx$. Notice that any other arrow j' must be equal to $\bar{j}$ because $fj' = j = f\bar{j}$ and f is mono. This shows that all monos in **Ab** are kernels.
In a similar way, one can show that all epis in **Ab** are cokernels. ■
The **R-Mod**, **Vect** and **FinVect** cases are actually very similar to this one.

Proposition 25. *Abelian categories have all finite limits and colimits.*

Proof. Let $\mathcal{C}$ be an abelian category. We know that a category with finite products and equalizers also has all finite limits.
Notice that (B1) implies the existence of equalizers (using Proposition 13): the equalizer of (f, g) is the kernel of $f - g$. Also, $\mathcal{C}$ has all finite products by virtue of being additive.
The discussion for colimits is dual. $\square$

Proposition 26. *In an abelian category, an arrow that is both epic and monic is an isomorphism.*

Proof. This is a consequence of (B2):
An arrow $A \xrightarrow{i} B$ that is both epic and monic is, by (B2), a kernel and a cokernel. In particular, there is an equalizer diagram

$$A \xrightarrow{i} B \underset{0}{\overset{f}{\rightrightarrows}} D$$

In particular, $fi = 0i$. But i is an epi, so that $f = 0 = 0_{BD}$. Hence i is a kernel of 0. But id_B is (trivially) also a kernel for 0. By uniqueness (up to isomorphism) of the kernel, i is an isomorphism. Explicitly:

$$\begin{tikzcd} A \arrow[r, "i"] & B \\ B \arrow[u, "\exists\phi"', "\cong"] \arrow[ur, "id_B"'] & \end{tikzcd}$$

so that $i = id_B\phi^{-1}$, and therefore $\phi\, id_B = i^{-1}$. $\square$

Proposition 27. *In an abelian category, every kernel is the kernel of its cokernel. Dually, every cokernel is the cokernel of its kernel.*

Proof. We will only prove the first part. The second part is analogous.
Consider a kernel of an arrow $f : A \to B$

$$K \xrightarrow{k} A \underset{0}{\overset{f}{\rightrightarrows}} B$$

and consider a cokernel of k:

$$K \underset{0}{\overset{k}{\rightrightarrows}} A \xrightarrow{c} C$$

Notice that then

$$K \xrightarrow{k} A \underset{0}{\overset{c}{\rightrightarrows}} C$$

is a fork. Now let $J \xrightarrow{j} A$ be such that

$$J \xrightarrow{j} A \underset{0}{\overset{c}{\rightrightarrows}} C$$

is a fork. We want $\bar{j}$ as in

$$\begin{tikzcd} J \arrow[d, dashed, "\bar{j}"'] \arrow[dr, "j"] & & \\ K \arrow[r, "k"] & A \arrow[r, shift left, "c"] \arrow[r, shift right, "0"'] & C \end{tikzcd}$$

Since k equalizes $(f, 0)$, it suffices to show that

$$J \xrightarrow{j} A \underset{0}{\overset{f}{\rightrightarrows}} B$$

is a fork. And this is true. Indeed: notice that $fk = 0k = 0 = f0$. Since c coequalizes $(k, 0)$, then we can factor c uniquely through f, and we get a diagram

$$\begin{tikzcd} & & C \arrow[d, dashed, "\bar{f}"] \\ K \arrow[r, "k"] & A \arrow[ur, "c"] \arrow[r, "f"] & B \\ J \arrow[ur, "j"] & & \end{tikzcd}$$

so that $fj = \bar{f}cj = \bar{f}0 = 0 = 0j$. Done. □

Remark 28. Notice that, using (B2) and Proposition 13, this means that every mono is the kernel of its cokernel and every epi is the cokernel of its kernel.

Lemma 29. *In an abelian category, if $f = me$ with $m = \ker(\operatorname{coker} f)$ and $e = \operatorname{coker}(\ker f)$, and also $f = m'e'$ with m' a kernel, then there is a unique t such that $m = m't$.*

Proof. Take $f = me = m'e'$ as above. We know that $m' = \ker p'$ for some arrow p'. By Proposition 27, one such p' is precisely $\operatorname{coker} m'$. So we choose $p' = \operatorname{coker} m'$ Now take $p = \operatorname{coker} m$. Clearly $\operatorname{coker} m = \operatorname{coker} f$ by definition of cokernel and because e is epic.
Now, notice that $p'f = p'm'e' = 0q' = 0$, so that p' coforks $(f, 0)$. Since $p = \operatorname{coker} f$, this implies that $\exists! w : p' = wp$.

$$\begin{tikzcd}
A \arrow[r, "e'"] \arrow[dr, "e"'] \arrow[rr, bend left, "f"] & M' \arrow[r, "m'"] & B \arrow[r, "p=\operatorname{coker} f"] \arrow[dr, "p'"'] & P \arrow[d, dashed, "w"] \\
& M \arrow[u, dashed, "t"] \arrow[ur, "m"'] & & P'
\end{tikzcd}$$

Hence $p'm = wpm = w0 = 0$, meaning that m forks $(p', 0)$. Since $m' = \ker p'$, we get a unique t such that $m = m't$. $\square$

Theorem 30. *In an abelian category, every arrow f has a decomposition $f = me$ with m monic and e epic. Furthermore, $m = \ker(\operatorname{coker} f)$ and $e = \operatorname{coker}(\ker f)$.*

Proof. Consider the diagram

$$K \xrightarrow{\ker f} A \xrightarrow{f} B \xrightarrow{\operatorname{coker}} C$$

(exists because the category is abelian).
Let $m = \ker(\operatorname{coker} f)$. So m is mono. Now notice that

$$A \xrightarrow{f} B \underset{0}{\overset{\operatorname{coker}}{\rightrightarrows}} C \qquad \text{and} \qquad M \xrightarrow{m} B \underset{0}{\overset{\operatorname{coker}}{\rightrightarrows}} C$$

are forks: the one on the right is obvious by definition and the one of the left simply follows from $\operatorname{coker} f \circ f = \operatorname{coker} 0 = 0 = 0f$. Since m is equalizer, we get a unique $e : A \to M$ such that $f = me$.
We now check that e is epic. Suppose $ge = he$, and let $q : X \to M$ be an equalizer of g, h. Notice that if we prove that q is an iso, it follows that e is epi. So that is what we will do. We have a diagram

$$\begin{tikzcd}
X \arrow[r, "q"] & M \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Y \\
A \arrow[u, dashed, "\bar{e}"] \arrow[ur, "e"'] & &
\end{tikzcd}$$

So $f = me = mq\bar{e}$. Now, m is monic and q is monic, so that mq is monic, and hence a kernel. Using also the fact that $f = me = mq\bar{e}$, we see that we are in the conditions of Lemma 1.4, so that $\exists! t : mqt = m$. Hence $qt = id$, and thus $q^{-1} = t$. Hence q is an iso and thus e is epic.
The last thing we have to check is that $e = \operatorname{coker}(\ker f)$. Since e is epic, then by Proposition 27 $e = \operatorname{coker}(\ker e)$. Now, it is easy to see that, since (for any appropriate composable arrow t) $ft = 0 \iff ek = 0$, then $\ker e = \ker f$. Thus $e = \operatorname{coker}(\ker f)$. □

Proposition 31. *In an abelian category, for every arrow f, a decomposition $f = me$ (with m monic and e epic) is unique up to unique isomorphism k. i.e. the diagram*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[rr, bend left, "f"] \arrow[dr, "e'"'] & C \arrow[r, "m"] \arrow[d, dashed, "k"] & B \\ & C' \arrow[ur, "m'"'] & \end{tikzcd}$$

commutes.

Proof. See Proposition 1 in VIII.3 of [3] and comments below its proof. □

Using this result, we can define (uniquely up to isomorphism) the concepts of image and coimage.

Definition 32. Let f be an arrow of an abelian category and write its standard decomposition as $f = me$. We call m the *image of f* and e the *coimage of f*, and denote them

$$m = \operatorname{im} f \qquad e = \operatorname{coim} f$$

We can define these more generally for a decomposition $f = mte$ with m monic, e epic and t an iso. Again m is the *image* and e the *coimage of f*.

Lemma 33. *Consider the arrows $A \xrightarrow{f} B \xrightarrow{g} C$. If $\ker g = \operatorname{im} f$, then $gf = 0$.*

Proof. We can write $f = \operatorname{im}(f)e = \ker(g)e$. Hence $gf = g\ker(f)e = 0e = 0$. □

Remark 34. Using Theorem 30, we see that we can write

$$\operatorname{im} f = \ker(\operatorname{coker} f) \qquad \operatorname{coim} f = \operatorname{coker}(\ker f)$$

Claim 35. *Consider* $0 \xrightarrow{0} A \xrightarrow{f} B$ *and* $A \xrightarrow{f} B \xrightarrow{0} 0$. *Then* $\operatorname{im} 0 = \ker f$ *iff* f *is monic. Dually,* $\operatorname{coim} 0 = \operatorname{coker} f$ *iff* f *is epi.*

Proof. I will only show the first part. The second part is analogously shown. Suppose f is monic. We have $f \ker f = 0_{KB} = f 0_{KA}$. Since f is monic, $\ker f = 0_{KA}$. Now, $\operatorname{im} 0 = \ker(\operatorname{coker} 0) = 0_A$ (where in the last equality I used the fact that 0 is initial object). 0_K gives a unique isomorphism between $\operatorname{im} 0$ and $\ker f$.

$$\begin{tikzcd}
\bullet \arrow[r, "\operatorname{coker} 0"] & 0 \arrow[d, dashed, "0_K"'] \arrow[dr, dashed, "0=\operatorname{im} 0"] & & \\
& K \arrow[r, "\ker f"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Conversely, suppose that $\operatorname{im} 0 = \ker f$.

□

1.5 Exact Sequences

Notice that, given two arrows $A \xrightarrow{f} B \xrightarrow{g} C$, both $\operatorname{im} f = \ker(\operatorname{coker} f) : M \to B$ and $\ker g : K \to B$ are monic. When these two monos belong to the same subobject of b (equivalence class of monos with codomain B), we give the situation a special name:

Definition 36. A pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *exact* if $\operatorname{im} f \sim \ker g$, with $\sim$ the usual equivalence of monos with codomain B. Equivalently, when $\operatorname{coim} f \sim \operatorname{coker} g$.

Definition 37. A diagram

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *exact* if all consecutive pairs of arrows are exact.

Remark 38. Using Claim 35, it is clear that exactness at A and C is equivalent to f being monic and epic.

1.6 Finite and Locally Finite Categories

1.6.1 Direct sum of categories

Definition 39. Let $\{\mathcal{C}_\alpha, \alpha \in I\}$ be a family of additive categories.
The *direct sum* $\mathcal{C} = \bigoplus\limits_{\alpha\in I}\mathcal{C}_\alpha$ is the category whose objects are

$$X = \bigoplus_{\alpha\in I} X_\alpha, \quad X_\alpha \in \mathcal{C}_\alpha$$

such that only a finite number of X_α are $\neq 0_\alpha$. The arrows are

$$\mathrm{Hom}_{\mathcal{C}}(\bigoplus_{\alpha\in I} X_\alpha, \bigoplus_{\alpha\in I} Y_\alpha) = \bigoplus_{\alpha\in I} \mathrm{Hom}_{\mathcal{C}_\alpha}(X_\alpha, Y_\alpha)$$

Remark 40. Strictly speaking, it makes no sense to take the direct sum $\bigoplus_{\alpha\in I} X_\alpha$, since the objects being summed over belong to different additive categories. So this definition should be read (as sometimes happens in mathematics) as a definition of both a new object *and* of notation for that object. Namely, we are defining the elements of the direct sum of categories to be an I-indexed class of objects from the categories being sumed over, with only a finite number of these objects different from the zero object.

Remark 41. $\mathcal{C} = \bigoplus\limits_{\alpha\in I}\mathcal{C}_\alpha$ is an additive category. Also, $\mathcal{C}$ is abelian iff all $\mathcal{C}_\alpha$ are abelian .

Definition 42. Let $\mathcal{C}$ be an abelian category.
$0 \neq X \in \mathcal{C}$ is *simple* if 0 and X are its only subobjects. X is *semisimple* if it is a direct sum of simple objects.
$\mathcal{C}$ is *semisimple* if all objects in $\mathcal{C}$ are semisimple.

1.6.2 Length of an object and locally finite categories

Definition 43. Let $X \in \mathcal{C}$ with $\mathcal{C}$ an abelian category. X has *finite length* if there is a filtration (*i.e.* sequence of monos of X ending in X) of the form

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow ... \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

such that $\forall i, \quad X_i \big/ X_{i-1}$ is simple. This filtration is called a *Jordan-Molder series of* X. The *length of* X is the length of its Jordan-Molder series.

Remark 44. All Jordan-Molder series of an object have the same length.

Definition 45. A $\mathbb{K}$-linear abelian category $\mathcal{C}$ is *locally finite* if the following hold:

(i) $\forall A, B \in \mathcal{C}$, the vector space $\mathrm{Hom}(A, B)$ is finite dimensional.

(ii) Every object in $\mathcal{C}$ has finite length.

1.6.3 Projective Covers and Finite Categories

Definition 46. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor of abelian categories. F is *left* (*right*) *exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$ we that

$$0 \to FA \to FB \to FC \quad (FA \to FB \to FC \to 0)$$

is exact in $\mathcal{D}$. If F is both left and right exact, we say simply that F is *exact.* if F is a contravariant functor, then we say that F is *left exact, right exact* or *exact* if the corresponding covariant functor $\tilde{F} : \mathcal{C} \to \mathcal{D}^{op}$ is *left exact, right exact* or *exact*, respectively.

Example 47. Let $\mathcal{C}$ be an abelian category. The (covariant) functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Ab}$ is left exact. Indeed: take $0 \xrightarrow{0} A \xrightarrow{b} B \xrightarrow{c} C \xrightarrow{0} 0$ short exact. We want to show that

$$\{0\} \xrightarrow{0} \mathrm{Hom}(X, A) \xrightarrow{b\circ} \mathrm{Hom}(X, B) \xrightarrow{c\circ} \mathrm{Hom}(X, C)$$

is exact. (Recall that $\mathrm{Hom}(X, f) = f\circ$, and notice that $\mathrm{Hom}(X, 0) = \{0\}$).
Exactness at $\mathbf{Hom}(X, A)$: We want to show that $\ker b\circ = \mathrm{im}\, 0 = \{0\}$. Since we are working in $\mathbf{Ab}$, this is equivalent to $b\circ$ being injective *i.e.* for all composable g, g', $b \circ g = b \circ g' \Rightarrow g = g'$. But this holds since, from Lemma 14, b is monic.
Exactness at $\mathbf{Hom}(X, B)$: We want to show that $\mathrm{im}(b\circ) = \ker(c\circ)$. Again, since we are in $\mathbf{Ab}$ these are just the usual kernel and image.
We first prove that $\mathrm{im}(b\circ) \subseteq \ker(c\circ)$: let $g \in \mathrm{im}(b\circ)$. Hence $\exists f \in \mathrm{Hom}(X, B)$: $b \circ f = g$. Then $c \circ g = c \circ b \circ f = 0 \circ f = 0$, where $c \circ b = 0$ because $\ker c = \mathrm{im}\, b$ (and using Lemma 33). Hence $\mathrm{im}(b\circ) \subseteq \ker(c\circ)$.

We still have to prove that $\operatorname{im}(b\circ) \supseteq \ker(c\circ)$: let $g \in \ker(c\circ)$. Then $c \circ g = 0$. So g factors through $\ker c$: write $g = \ker(c)u$. But now $\ker c = b$. Indeed: $e = \operatorname{coker}\ker(e) = \operatorname{coker}\ker(b) = \operatorname{coker}(\operatorname{im} 0)$, where I used that $\ker e = \ker b$ (easy to check) and that $\ker b = \operatorname{im} 0$ (by exactness). Now, $\operatorname{im} 0 = 0$ (just consider the diagram

$$\begin{tikzcd} 0 \arrow[rr, dashed, "0_{0A}"] \arrow[dr, dashed, "0_{0\bullet}"'] & & A \\ & \bullet \arrow[ur, dashed, "0_{\bullet A}"'] & \end{tikzcd}$$

and notice that $\operatorname{im} 0 = 0_{\bullet A} =: 0$), so that we get $\ker e = \ker \operatorname{coker}(\operatorname{im} 0) = \operatorname{im}(\operatorname{im} 0) = \operatorname{im} 0$, and then by Claim 35 we get that e is monic. Hence e is both epic and monic, and thus by Proposition 26 e is an isomorphism. ■

Definition 48. let $\mathcal{C}$ be an abelian category. $P \in \mathcal{C}$ is said to be *projective* if $\operatorname{Hom}(P, -)$ is exact. $I \in \mathcal{C}$ is *injective* if $\operatorname{Hom}(-, I)$ is exact.

Definition 49. Let $\mathcal{C}$ be an abelian category whose objects all have finite length. A *projective cover* of $A \in \mathcal{C}$ is a projective object $P(A)$ together with an epi $p : P(A) \to A$ such that if $g : P \to A$ is an epi with P a projective object, then there is an epi $h : P \to P(A)$ such that the diagram

$$\begin{tikzcd} P(A) \arrow[r, "p"] & A \\ P \arrow[u, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. *i.e.* we can factorize g through p.

We can finally define finite categories.

Definition 50. A $\mathbb{K}$-linear abelian category $\mathcal{C}$ is *finite* if the following hold:

(i) The Hom-sets are finite dimensional (vector spaces).

(ii) Every object in $\mathcal{C}$ has finite length.

(iii) Every simple object in $\mathcal{C}$ has a projective cover. We then say that $\mathcal{C}$ *has enough projectives*.

(iv) There are finitely many isomorphism classes of simple objects.

2 Rigid Monoidal Categories

Definition 51. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category and $X \in \mathcal{C}$. An object $X^* \in \mathcal{C}$ is called a *left dual of* X if there are morphisms

$$\mathrm{ev}_X : X^* \otimes X \to 1 \qquad \mathrm{coev}_X : 1 \to X \otimes X^*$$

called *evaluation* and *coevaluation*, such that

$$\left[X \xrightarrow{\mathrm{coev}_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes \mathrm{ev}_X} X\right] = id_X \tag{1}$$

and

$$\left[X^* \xrightarrow{id_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^*) \xrightarrow{\mathrm{ev}_{X^*} \otimes id_X} X^*\right] = id_{X^*} \tag{2}$$

where I identified X with $1 \otimes X$ and $X \otimes 1$, and X with $1 \otimes X$ and $X \otimes 1$ in the first and last objects of the diagrams. Similarly, $X^\vee$ is called *right dual of* X if there are morphisms

$$\mathrm{ev}'_X : X \otimes X^\vee \to 1 \qquad \mathrm{coev}'_X : 1 \to X^\vee \otimes X$$

such that

$$\left[X \xrightarrow{id_X \otimes \mathrm{coev}'_X} X \otimes (X^\vee \otimes X) \xrightarrow{a^{-1}_{X,X^\vee,X}} (X \otimes X^\vee) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes id_X} X\right] = id_X \tag{3}$$

and

$$\left[X^\vee \xrightarrow{\mathrm{coev}'_X \otimes id_{X^\vee}} (X^\vee \otimes X) \otimes X^\vee \xrightarrow{a_{X^\vee,X,X^\vee}} X^\vee \otimes (X \otimes X^\vee) \xrightarrow{id_{X^\vee} \otimes \mathrm{ev}'_X} X^\vee\right] = id_X \tag{4}$$

Proposition 52. *(i) If X^* is left dual of X, then X is right dual of X^*, and vice versa.*

(ii) 1 is both left and right dual of itself.

Proof. (i): Suppose X^* is left dual of X. just take $\mathrm{ev}'_{X^*} = \mathrm{ev}_X : X^* \otimes X \to 1$ and $\mathrm{coev}'_{X^*} = \mathrm{coev}_X : 1 \to X \otimes X^*$. Notice that (3) and (4) correspond precisely to (1) and (2) in this case. Since we know that (1) and (2) hold,

then (3) and (4) also hold. Hence X is right dual of X^*.
Conversely, if $X^\vee$ is right dual of X, then if we take $\mathrm{ev}_{X^\vee} = \mathrm{ev}'_X : X \otimes X^\vee \to 1$ and $\mathrm{coev}_{X^\vee} = \mathrm{coev}'_X : 1 \to X^\vee \otimes X$ we again see immediately that X is left dual of $X^\vee$.
(ii): Take $\mathrm{ev}_1 = \mathrm{ev}'_1 = l : 1 \otimes 1 \to 1$ and $\mathrm{coev}_1 = \mathrm{coev}'_1 = r^{-1} : 1 \to 1 \otimes 1$. Then it is easy to see that all equations (1)-(4) hold, if we only use the triangle identities. For example:

$$1 \cong 1 \otimes 1 \xrightarrow{r^{-1} \otimes id_1} (1 \otimes 1) \otimes 1 \xrightarrow{a} 1 \otimes (1 \otimes 1) \xrightarrow{id_1 \otimes l} 1 \otimes 1 \cong 1$$

is indeed the identity because $(id_1 \otimes l)a(r^{-1} \otimes id_1) = id_1$ by the triangular identity. □

Proposition 53. *If $X \in \mathcal{C}$ has a left (right) dual object X^* ($X^\vee$), then it is unique up to a unique isomorphism. (Hence, dual objects are universal objects.)*

Proof. I will omit this proof. See 2.10.5 in [2]. □

Using 53 and (i) of 52, we get:

Corollary 54. $(X^*)^\vee \cong X \cong (X^\vee)^*$

When every object has left and right duals, we give our category a special name.

Definition 55. An object of a monoidal category is said to be *rigid* if it has both left and right duals. A monoidal category is *rigid* if every object is rigid.

Example 56. The monoidal category **FinVect** is rigid. Both the right and the left duals of $V \in$ **FinVect** are the usual dual V^*, and we can take the evaluation map to be

$$\begin{aligned} \mathrm{ev}_V : V^* \otimes V &\to \mathbb{K} \\ \epsilon \otimes v &\mapsto \epsilon(v) \end{aligned}$$

and the coevaluation map to be

$$\begin{aligned} \mathrm{coev}_V : \mathbb{K} &\to V \otimes V^* \\ \lambda &\mapsto \lambda \sum_i e_i \otimes e_i^* \end{aligned}$$

with $\{e_i\}$ a basis for V and $\{e_i^*\}$ its dual basis. (The coevaluation map turns out not to depend on the choice of bases). Let's check (1) from Definition 51 explicitly (omitting the associator):

$$\begin{aligned}
(id_V \otimes \mathrm{ev}_V)(\mathrm{coev}_V \otimes id_V)(1_{\mathbb{K}} \otimes v) &= (id_V \otimes \mathrm{ev}_V)\left((1_{\mathbb{K}} \sum_i e_i \otimes e_i^*) \otimes v\right) \\
&= (id_V \otimes \mathrm{ev}_V)\left(\sum_i e_i \otimes (e_i^* \otimes v)\right) \\
&= \sum_i (e_i \otimes e_i^*(v)) \\
&= \sum_i (e_i \otimes v^i) \\
&= \sum_i (v^i e_i \otimes 1_{\mathbb{K}}) = v \otimes 1_{\mathbb{K}}
\end{aligned}$$

Conditions (2), (3) and (4) are proved in a similar way (and with ev' and coev' given by $v \otimes \epsilon \mapsto \epsilon(v)$ and $\lambda \mapsto \lambda \sum_i e_i^* \otimes e_i$, respectively). ■

We also have a notion of dual of an arrow:

Definition 57. Let A, B be objects of a monoidal category $\mathcal{C}$, with duals A^*, B^*. The *left dual of an arrow* $f : A \to B$ is the arrow $f^* : B^* \to A^*$ given by the composition

$$f^* :$$

[See the rest of the def. on p41 of [2].]

3 Tensor and Fusion Categories

From now on, the field $\mathbb{K}$ is algebraically closed.

Definition 58. Let $\mathcal{C}$ be a locally finite $\mathbb{K}$-linear abelian rigid monoidal category.
$\mathcal{C}$ is a *multitensor category over* $\mathbb{K}$ if $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms.
A multitensor category $\mathcal{C}$ is *decomposable* if it is equivalent to a direct sum of nonzero multitensor categories.
If $\mathcal{C}$ is an indecomposable multitensor category and $\mathrm{End}(1) \cong \mathbb{K}$ (as vector spaces), then $\mathcal{C}$ is a *tensor category*.

Fusion categories are just finite semisimple tensor categories:

Definition 59. A *multifusion category* is a finite semisimple multitensor category.
A *fusion category* is a finite semisimple tensor category.

Example 60. The category $\mathbf{FinVect}_{\mathbb{K}}$ of finite dimensinal $\mathbb{K}$-vector spaces is a fusion category. Let's check this in detail:

- $\mathbf{FinVect}_{\mathbb{K}}$ is finite:

  (i) From Example 19, the hom-sets are finite dimensional vector spaces.

  (ii) Let $V \in \mathbf{FinVect}$. Write $V_k = span(v_1, ..., v_k)$ with $(v_1, ..., v_n)$ a basis for V. The sequence

  $$0 = V_0 \hookrightarrow V_1 \hookrightarrow ... \hookrightarrow V_n = V$$

  is a filtration with all quotients simple. Indeed, ${V_k}/{V_{k-1}} \cong {\mathbb{K}^k}/{\mathbb{K}^{k-1}} \cong \mathbb{K}$, and this is a one-dimensional vector space and hence simple. Thus V has finite length.

  (iii) [Couldn't show the projective part. pp619,620 of Paolo may help? Contact me if you figure it out.]

  (iv) A finite dimesnional vector space is simple iff it has no proper subspaces. Hence V is simple iff V is one-dimensional. Hence all simple finite dimensional vector spaces are isomorphic to $\mathbb{K}$.

- $\mathbf{FinVect}_{\mathbb{K}}$ is a locally finite $\mathbb{K}$-linear abelian rigid monoidal category: From Examples 24 and 19, we know that $\mathbf{FinVect}$ is $\mathbb{K}$-linear abelian. We also saw in Example 56 that $\mathbf{FinVect}$ is rigid monoidal. Local finiteness is a consequence of finiteness: (i) and (ii) above show that $\mathbf{FinVect}$ is locally finite.

- $\mathbf{FinVect}_{\mathbb{K}}$ is semisimple:
  We know that every finite dimensional vector space can be written as the direct sum of simple (and thus one-dimensional) vector spaces. Hence every object of $\mathbf{FinVect}$ is semisimple.

- $\mathbf{FinVect}_{\mathbb{K}}$ is a multitensor category:
  We just have to argue that $\otimes$ is bilinear on arrows. But we know this to be true by the basic properties of the tensor product of linear maps.

- **FinVect**$_{\mathbb{K}}$ is indecomposable:

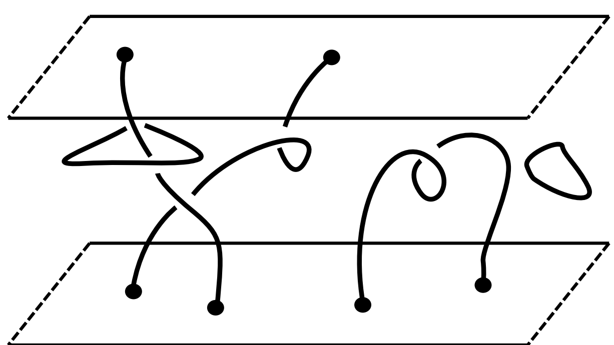
- **FinVect**$_{\mathbb{K}}$ has $\mathrm{End}(1) \cong \mathbb{K}$:
  We have precisely $1 = \mathbb{K}$, and we know that $\mathrm{End}\,\mathbb{K}$ is a one dimensional $\mathbb{K}$-vector space, and thus $\mathrm{End}(\mathbb{K}) \cong \mathbb{K}$. $\blacksquare$

4 Ribbon Categories

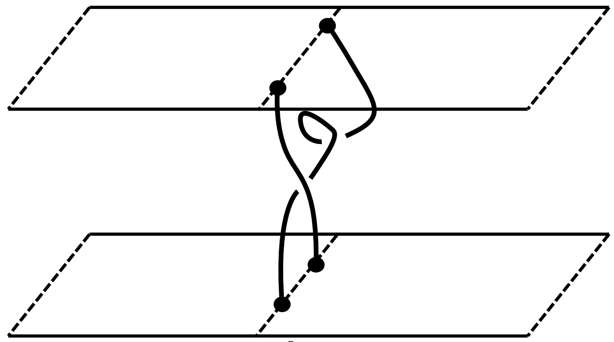
We start this section with a motivating example.

4.1 The Category of Tangles

Denote by S_{mn} the disjoint union of m circles S^1 and n unit intervals $I = [0,1]$. Notice that the boundary of S_{mn} is just the disjoint union of the boundaries of the n unit intervals. A *tangle* is a smooth embedding (immersion which is topological embedding) $f : S_{mn} \to \mathbb{R}^2 \times I$, such that the boundary of S_{mn} is mapped to the boundary of $\mathbb{R}^2 \times I$. The *inputs* of a tangle are the points in $f(S_{mn}) \cap \mathbb{R}^2 \times \{0\}$ and the *outputs* are the points in $f(S_{mn}) \cap \mathbb{R}^2 \times \{1\}$. For example, if $m = 1$ we can have either 2 inputs and 0 outputs, 1 input and 1 output, or 0 inputs and 2 outputs (irrespective of the value of n). We also use the word tangle for $f(S_{mn})$.



We now focus on the tangles whose inputs and outputs lie on the $y = 0$ (and $z = 0, 1$) lines.



Denote by $\tilde{T}_{pq}$ the set of all tangles with p inputs and q outputs. Let T_{pq} be the quotient $\tilde{T}_{pq}/_\sim$, with $\sim$ the isotopy equivalence relation.
We can define the composition of an element t of T_{pq} with an element s of T_{qr} by an appropriate concatenation. More concretely, we take a representative $\tilde{t}$ of t and a representative $\tilde{s}$ of s such that we can glue t and s in a smooth way. After gluing, we rescale $z \to z/2$, obtaining a tangle that we denote by $\tilde{s} \circ \tilde{t}$, and declare $s \circ t = [\tilde{s} \circ \tilde{t}]$. (We won't show here that this is indeed well defined). Notice the similarities between this and the composition of cobordisms in **nCob**.

This picture gives us a natural way of defining the category $\mathcal{T}$ of tangles, with natural numbers as objects and the equivalence classes of tangles in T_{pq} as arrows $p \to q$ (so that the hom-sets are $\mathrm{Hom}(p, q) = T_{pq}$). The composition is the concatenation described above.
We also define a tensor product in this category, given on objects by $p \otimes q = p + q$ and on arrows simply by union of tangles as follows: given two arrows $s \in T_{p,q}$ and $t \in T_{m,n}$, take two representatives $\tilde{s} \in \tilde{T}_{p,q}$ and $\tilde{t} \in \tilde{T}_{m,n}$ such that all points of $\tilde{t}$ are to the left of all points in $\tilde{s}$. We declare $t \otimes s = [\tilde{t} \cup \tilde{s}]$. Again, we will not show here that this is indeed well-defined.
This tensor product makes $\mathcal{T}$ a monoidal category. There is also a natural braiding in $\mathcal{T}$, turning it into a braided monoidal category. This braiding B has components B_{pq}, with B_{pq} the equivalence class of a tangle $f : S_{0m+n} \to \mathbb{R}^2 \times I$ of $\tilde{T}_{p+q,q+p}$ with "p strings passing over q strings".

4.2 The Category of Braids

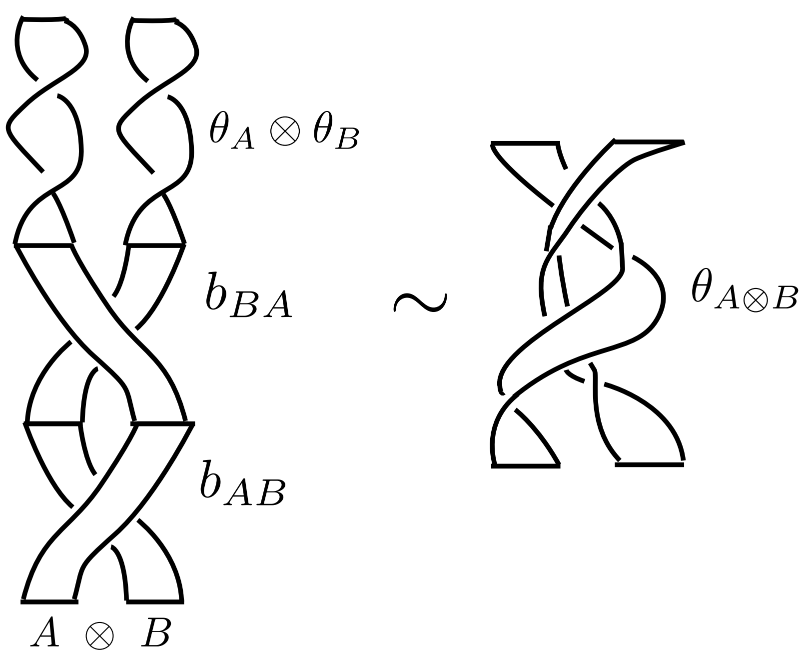
A *braid on n strands* is a tangle $f : S_{0,n} = I_1 \sqcup ... \sqcup I_n \to \mathbb{R}^2 \times I$ such that

$$\forall t \in [0,1], \forall i \in \{1, ..., n\}, \exists! x \in I_i : x \in \mathbb{R}^2 \times \{t\}$$

or, in other words, "there are no circles, and the strings of the tangle don't go back".
Using the same composition as in $\mathcal{T}$, we get a (braided monoidal) category $\mathcal{B}$ of braids, which is a subcategory of $\mathcal{T}$.
Notice that the components of the braiding in $\mathcal{T}$ are braids.

4.3 The Category of Framed Tangles

This is the same as $\mathcal{T}$, but with "ribbons" instead of intervals and circles.
The category of framed tangles $\mathcal{FT}$ has something very different from $\mathcal{T}$: since we now have ribbons instead of strings, we can twist them to get other (non-isotopic) ribbons. Given an object A of $\mathcal{FT}$, its *twist* θ_A will be the a ribbon with a double twist and it turns out that $\theta_{A\otimes B} = (\theta_A \otimes \theta_B) \circ b_{BA} \circ b_{AB}$.
This is easy to visualize in some simple cases. A category with such a twist will be called a ribbon category.



4.4 Ribbon Categories

Definition 61. Let $\mathcal{C}$ be a braided rigid monoidal category with braiding b. A *twist* in $\mathcal{C}$ is a natural transformation $id_{\mathcal{C}} \overset{\theta}{\Rightarrow} id_{\mathcal{C}}$ such that

$$\theta_{A\otimes B} = (\theta_A \otimes \theta_B) \circ b_{BA} \circ b_{AB}$$

Notice that naturality in this case just means that for any arrow $f : A \to B$, we have $\theta_B f = f\theta_A$.

Definition 62. A twist θ is a *ribbon structure* if $\forall A \in \mathcal{C},\ (\theta_A)^* = \theta_{A^*}$.
A rigid monoidal category with a ribbon structure is called a *ribbon category*.

5 Modular Categories

5.1 Quantum Trace

Definition 63. Let A be an object of a rigid monoidal category $\mathcal{C}$ and $a \in \mathrm{Hom}(A, A^{**})$. The *left quantum trace* of a is the composition

$$\mathrm{Tr}^L(a) : 1 \xrightarrow{\mathrm{coev}_A} A \otimes A^* \xrightarrow{a\otimes id_{A^*}} A^{**} \otimes A^* \xrightarrow{\mathrm{ev}_{A^*}} 1$$

Let instead $a \in \mathrm{Hom}(A, A^{\vee\vee})$. The *right quantum trace* of a is the composition

$$\mathrm{Tr}^R(a) : 1 \xrightarrow{\mathrm{coev}_{A^\vee}} A^\vee \otimes A \xrightarrow{id_{A^\vee}\otimes a} A^{\vee\vee} \otimes A^\vee \xrightarrow{\mathrm{ev}_{A^{\vee\vee}}} 1$$

Notice that $\mathrm{Tr}^L(a), \mathrm{Tr}^R(a) \in \mathrm{End}(1)$. Since in tensor categories we have $\mathrm{End}(1) \cong \mathbb{K}$, then in particular we can regard the quantum traces as elements of $\mathbb{K}$.

The following proposition justifies why we call this a trace:

Proposition 64. *Let $a \in Hom(A, A^{**})$ and $b \in Hom(B, B^{**})$. Then:*

(i) $\mathrm{Tr}^L(a) = \mathrm{Tr}^R(b)$

(ii) $\mathrm{Tr}^L(a \oplus b) = \mathrm{Tr}^L(a) + \mathrm{Tr}^L(b))$

(iii) $\mathrm{Tr}^L(a \otimes b) = \mathrm{Tr}^L(a)\,\mathrm{Tr}^R(b)$

(iv) *For $c \in Hom(A, A)$,* $\mathrm{Tr}^L(ac) = \mathrm{Tr}^L(c^{**}a)$ *and* $\mathrm{Tr}^R(ac) = \mathrm{Tr}^R(ca^{\vee\vee})$.

Proof. Omitted. □

5.2 Pivotal and Spherical structures

Given an arrow $f : A \to B$ in a rigid monoidal category, its dual is an arrow $f^* : B^* \to A^*$. Taking the dual again, we get an arrow $f^{**} : A^{**} \to B^{**}$. So $(-)^{**}$ smells like a functor. Indeed it is.

Definition 65. Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is a natural isomorphism $p : id_{\mathcal{C}} \Rightarrow (-)^{**}$ such that $\forall A, B \in \mathcal{C}, p_{A\otimes B} = p_A \otimes p_B$.

Definition 66. Let A be an object in a rigid monoidal category $\mathcal{C}$ with a pivotal structure p. The *dimension of A with respect to p* is

$$\dim_p(A) := \mathrm{Tr}_L(p_A)$$

Definition 67. A *spherical structure* on a tensor category $\mathcal{C}$ is a pivotal structure p such that $\forall A \in \mathcal{C},\ \dim_p(A) = \dim_p(A^*)$.

Theorem 68. *Let $\mathcal{C}$ be a spherical category with spherical structure p. Let $A \in \mathcal{C}$ and $x \in Hom(A, A)$. Then*

$$\mathrm{Tr}^L(p_A x) = \mathrm{Tr}^R(x p_A^{-1})$$

We denote this by $\mathrm{Tr}(x)$*.*

Proof. Omitted. You may want to see thm 4.7.15 of [2]. □

Remark 69. Notice that $\dim_p(A) = \mathrm{Tr}^L(p_A) = \mathrm{Tr}(id_A)$.

5.3 The canonical pivotal structure

Let $\mathcal{C}$ be a braided rigid monoidal category. Denote its braiding by b. We define a natural transformation $u : id_{\mathcal{C}} \Rightarrow (-)^{**}$ by declaring its components $u_X : X \to X^{**}$ to be the composition

$$X \xrightarrow{id_X \otimes \mathrm{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{b_{X\otimes X^*} \otimes id_{X^{**}}} X^* \otimes X \otimes X^{**} \xrightarrow{\mathrm{ev}_X \otimes id_{X^{**}}} X^{**}$$

Proposition 70. *With u the natural transformation defined above, we have*

$$u_A \otimes u_B = u_{A\otimes B} \circ b_{BA} \circ b_{AB}$$

Definition 71. The natural transformation u above is called the *Drinfeld morphism.*

Proposition 72. *The Drinfeld morphism is a natural isomorphism.*

Proof. Omitted. See [2]. $\square$

We thus see that the components of any natural isomorphism $\psi : id_{\mathcal{C}} \Rightarrow (-)^{**}$ can be written $\psi_X = u_X\theta_X$, with $\theta \in \mathrm{Aut}(id_{\mathcal{C}})$. (Just have to take $\theta_X = u_X^{-1}\psi_X$).

Proposition 73. *A natural isomorphism $\psi = u\theta : id_{\mathcal{C}} \Rightarrow (-)^{**}$ is a pivotal structure in $\mathcal{C}$ if and only if θ is a twist.*

Proof. Suppose ψ is a pivotal structure. Then

$$\begin{aligned}\psi_{A\otimes B} &= \psi_A \otimes \psi_B \\ &= u_A\theta_A \otimes u_B\theta_B \\ &= (u_A \otimes u_B)(\theta_A \otimes \theta_B) \\ &= u_{A\otimes B}b_{BA}b_{AB}(\theta_A \otimes \theta_B)\end{aligned}$$

where I used Proposition 70. So we see that $u_{A\otimes B}\theta_{A\otimes B} = u_{A\otimes B}b_{BA}b_{AB}(\theta_A \otimes \theta_B)$. But $u_{A\otimes B}$, being an isomorphism, is also monic. Thus $\theta_{A\otimes B} = b_{BA}b_{AB}(\theta_A \otimes \theta_B) = (\theta_A \otimes \theta_B)b_{BA}b_{AB}$ (we can pull the components of the braiding to the right usoing the naturality of θ).
Now suppose that θ is a twist. Then

$$\begin{aligned}\psi_{A\otimes B} &= u_{A\otimes B}\theta_{A\otimes B} \\ &= u_{A\otimes B}(\theta_A \otimes \theta_B)b_{BA}b_{AB} \\ &= u_{A\otimes B}b_{BA}b_{AB}(\theta_A \otimes \theta_B) \\ &= (u_A \otimes u_B)(\theta_A \otimes \theta_B) \\ &= u_A\theta_A \otimes u_B\theta_B \\ &= \psi_A \otimes \psi_B\end{aligned}$$

Hence ψ is a pivotal structure. $\square$

Definition 74. Let θ be a twist on a braided category $\mathcal{C}$. The pivotal structure $\psi = u\theta$ (where u is the Drinfeld morphism) is called the *canonical pivotal structure associated to θ.*

Proposition 75. *Let $\mathcal{C}$ be a braided rigid monoidal category with a twist θ. Let $\psi = u\theta$ be the canonical pivotal structure associated to θ. Then ψ is a spherical structure on $\mathcal{C}$ if and only if θ is a ribbon structure.*

Proof. See [2]. □

5.4 Pre-modular categories and the S-matrix

From now on, $\mathbb{K}$ is an algebraically closed field of characteristic 0 (*e.g.* $\mathbb{C}$). We explored the concepts of fusion category and ribbon category. When we merge these, we get a pre-modular category.

Definition 76. A *pre-modular category* is a fusion ribbon category.

Remark 77. Notice that Proposition 75 tells us that a pre-modular category is precisely a braided fusion category with a spherical structure.

Denote by $\mathcal{O}(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$. We will write Tr for the trace associated with the pivotal structure ψ: $\mathrm{Tr}(x) = \mathrm{Tr}^L(\psi_A x) = \mathrm{Tr}^R(x\psi_A^{-1})$. Also, $\dim := \dim_\psi$.

Notice that $b_{AB}b_{BA} \in \mathrm{End}(A \otimes B)$, so that it makes sense to write $\mathrm{Tr}(b_{AB}b_{BA}) \in \mathbb{K}$.

Definition 78. Let $\mathcal{C}$ be a pre-modular category. Denote its braiding by b. The *S-matrix* of $\mathcal{C}$ is

$$S = (s_{AB})_{A,B\in\mathcal{O}(\mathcal{C})}, \quad \text{with} \quad s_{A,B} := \mathrm{Tr}(b_{BA}b_{AB})$$

Proposition 79. *The elements s_{AB} of the S-matrix satisfy:*

(i) $\forall A, B \in \mathcal{O}(\mathcal{C}),\ s_{A^*B^*} = s_{AB}$

(ii) $\forall A \in \mathcal{O}(\mathcal{C}),\ s_{A1} = s_{1A} = \dim(A)$

5.5 Modular Categories

Definition 80. A *modular category* is a pre-modular category whose S-matrix is non-degenerate.

6 Appendix

6.1 The categories Grp and Ab

Proposition 81. *The direct product $G \times H$ of two groups G, H is a (categorical) product in* ***Grp****.*

Proof. This follows quickly from what we know from **Set**. Given $G \xleftarrow{f_1} W \xrightarrow{f_2} H$ in **Ab**, we want to have a diagram

$$\begin{tikzcd} & W \arrow[d, dashed, "f"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\ & G \times H \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G & & H \end{tikzcd}$$

Since, as a set, $G \times H$ is just the cartesian product, we do have such a diagram in **Set**, with $f = f_1 \times f_2$. Since the projections are already homomorphisms, all we need to show is that f is indeed a homomorphism. This is easy:

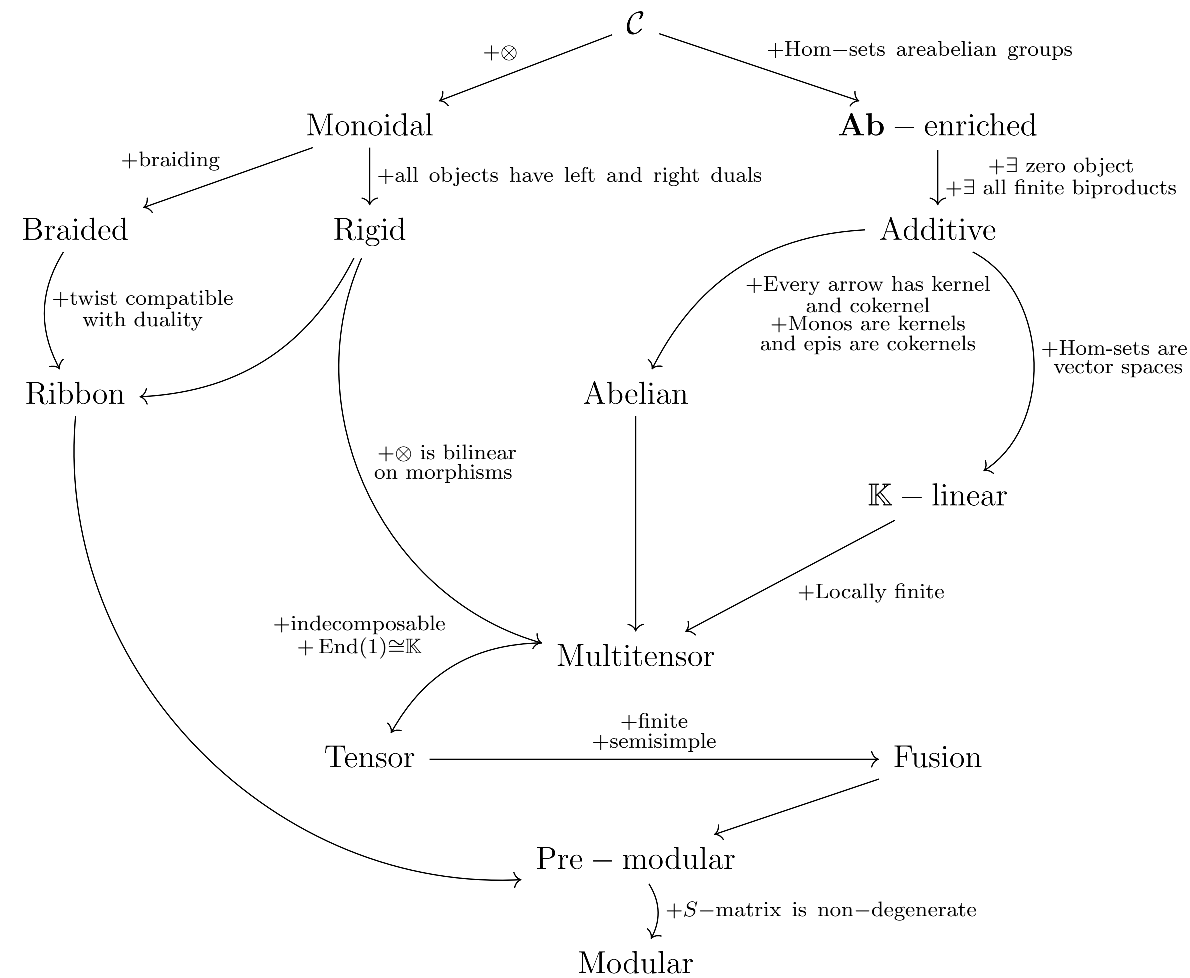
$$f(ab) = (f_1(ab), f_2(ab)) = (f_1(a), f_2(a))(f_1(b), f_2(b)) = f(a)f(b)$$

□

Proposition 82. *The direct product $G \times H$ of two abelian groups G, H is a (categorical) product in* ***Ab****.*

Proof. We know that $G \times H$ is abelian iff G and H are abelian. The result follows. □

6.2 Roadmap



$\mathcal{C}$
$+\otimes$
+Hom−sets areabelian groups
Monoidal
Ab − enriched
+braiding
+all objects have left and right duals
+∃ zero object
+∃ all finite biproducts
Braided
Rigid
Additive
+twist compatible with duality
+Every arrow has kernel and cokernel
+Monos are kernels and epis are cokernels
+Hom-sets are vector spaces
Ribbon
Abelian
$+\otimes$ is bilinear on morphisms
$\mathbb{K}$ − linear
+Locally finite
+indecomposable
+ End(1)≅$\mathbb{K}$
Multitensor
Tensor
+finite
+semisimple
Fusion
Pre − modular
+S−matrix is non−degenerate
Modular

References

[1] Paolo Aluffi. *Algebra: Chapter 0*. American Mathematical Society, 2009.

[2] Pavel Etingof et al. *Tensor Categories*. American Mathematical Society, 2015.

[3] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, Second Edition.

[4] "The nLab". https://ncatlab.org/nlab/show/HomePage

[5] Peter Freyd. *Abelian Categories*. Harper and Row, 1964.

[6] N. Popescu. *Abelian Categories with Applications to Rings and Modules*. Academic Press, 1973.